Derived equivalences for cluster-tilted algebras of types $\tilde{A}_n$ and D_n

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Kurzzusammenfassung

Die Klassifikation von Cluster-Kippalgebren bis auf derivierte Äquivalenz ist ein wichtiges Problem in der Darstellungstheorie. Solche Klassifikationen gibt es bereits für die Dynkin Typen A_n (Buan und Vatne) und $E_{6,7,8}$ (B., Holm und Ladkani). Als nächsten Schritt sollte man diese Klassifikation für die Typen $\tilde{A}_n$ und D_n angehen; das ist der Hauptgegenstand dieser Arbeit.

Ausgehend von einer "Kategorifizierung" der Cluster Algebren von Fomin und Zelevinsky haben Buan, Marsh und Reiten die sogenannten Cluster-Kippalgebren eingeführt. Allgemein werden Cluster-Kippalgebren, die zu einem azyklischen Köcher Q korrespondieren, als Endomorphismenalgebren von Cluster-Kippobjekten in der Cluster Kategorie $\mathcal{C}_Q$ definiert. Für Cluster-Kippalgebren gibt es den Begriff der "Mutation". Hierbei wird eine Cluster-Kippalgebra in eine andere mutiert. Einige wohlbekannte klassische Konstruktionen der Darstellungstheorie, wie zum Beispiel Spiegelungsfunktoren, werden durch diesen Begriff verallgemeinert. Diese Mutationen lassen sich durch sogenannte "Köchermutation" auf Köcher mit Relationen übertragen. Insbesondere kann man bei algebraisch abgeschlossenem Körper K eine Cluster-Kippalgebra in der Form KQ'/I' als Köcher mit Relationen angeben. Der Köcher Q' ist dabei mutationsäquivalent zum azyklischen Köcher Q und die Relationen sind durch Q' bestimmt, wie Buan, Iyama, Reiten und Smith gezeigt haben.

In dieser Arbeit erzielen wir eine explizite Beschreibung der Klasse aller Köcher, die zu irgendeinem Köcher vom Typ $\tilde{A}_n$ mutationsäquivalent sind. Dann wird diese Klasse in ihre verschiedenen Mutationsklassen unterteilt. Zu diesem Zweck führen wir für jeden Köcher vier kombinatorische Parameter ein und beweisen, dass die unterschiedlichen Mutationsklassen von gewissen Summen dieser Parameter abhängig sind. Anschließend konstruieren wir explizite Kipp-Komplexe für Cluster-Kippalgebren vom Typ $\tilde{A}_n$, mit deren Hilfe wir, unter Verwendung eines Algorithmus von Avella-Alaminos und Geiß, eine vollständige Klassifikation von Cluster-Kippalgebren vom Typ $\tilde{A}_n$ bis auf derivierte Äquivalenz erreichen. Tatsächlich können wir zeigen, dass die beschränkte derivierte Kategorie solch einer Algebra von den vier einzelnen kombinatorischen Parametern abhängt.

Für Cluster-Kippalgebren vom Typ D_n können wir nur eine etwas gröbere Klassifikation erzielen. Hierbei greifen wir auf den Begriff der "guten Mutationsäquivalenz" zurück, der etwas stärker ist als derivierte Äquivalenz. Wir geben eine komplette Klassifikation für Cluster-Kippalgebren vom Typ D_n bis auf gute Mutationsäquivalenz an. Dabei wird in jeder Klasse eine kanonische Normalform dieser Algebren ausgezeichnet. Darüber hinaus können wir weitere derivierte Äquivalenzen zwischen Cluster-Kippalgebren vom Typ D_n identifizieren, die nicht durch gute Mutationen gegeben sind. Basierend auf der Klassifikation bis auf gute Mutationsäquivalenz und diesen zusätzlichen derivierten Äquivalenzen, schlagen wir Standardformen für die derivierten Äquivalenzklassen vor.

Unter Verwendung bekannter numerischer Invarianten sind wir in der Lage, einige der Standardformen vom Typ D_n bis auf derivierte Äquivalenz zu unterscheiden. Insbesondere werden allgemeine Formeln für die Cartan Determinanten vom Typ D_n hergeleitet. Darüber hinaus berechnen wir die charakteristischen Polynome der sogenannten "Asymmetrie Matrizen" für gewisse Cluster-Kippalgebren vom Typ D_n, unter Zuhilfenahme der entsprechenden Formeln für Typ A_n.

Diese Invarianten führen zu einer weitreichenden derivierten Äquivalenzklassifikation von Cluster-Kippalgebren vom Typ D_n. Um eine vollständige Klassifikation zu erreichen, müssten allerdings noch einige diffizile Fragen geklärt werden.

Schlagworte: Cluster-Kippalgebra, derivierte Äquivalenz, Köchermutation, gute Mutation

Abstract

An important problem in representation theory is obtaining a derived equivalence classification for cluster-tilted algebras. Derived equivalence classifications have been found so far for cluster-tilted algebras of Dynkin type A_n by Buan and Vatne and for Dynkin types $E_{6,7,8}$ by the author together with Holm and Ladkani. A natural next step is to find a classification for types $\tilde{A}_n$ and D_n; this is the concern of this thesis.

Cluster-tilted algebras were introduced by Buan, Marsh and Reiten arising from a 'categorification' of the cluster algebras introduced by Fomin and Zelevinsky. Generally, cluster-tilted algebras corresponding to an acyclic quiver Q are defined as the endomorphism algebras of cluster-tilting objects in the cluster category $\mathcal{C}_Q$. For cluster-tilted algebras there is a theory of 'mutation' where one cluster-tilted algebra is mutated into another generalising some classical concepts in representation theory. This mutation can be seen at the level of quivers with relations in the form of 'quiver mutation'. In particular, these algebras can be constructed explicitly by quivers with relations KQ'/I', where K is an algebraically closed field. Buan, Iyama, Reiten and Smith showed that the quiver Q' is mutation equivalent to the acyclic quiver Q and the relations are uniquely determined by Q'.

We give an explicit description of the class of quivers which are mutation equivalent to any quiver of type $\tilde{A}_n$. We then separate this class into the different mutation classes of $\tilde{A}_n$-quivers. For this we define four combinatorial parameters for every quiver, and prove that the different mutation classes depend on certain sums of these parameters. Subsequently, we provide explicit tilting complexes for cluster-tilted algebras of type $\tilde{A}_n$ and using these together with an algorithm of Avella-Alaminos and Geiß, we get a complete classification of cluster-tilted algebras of type $\tilde{A}_n$ up to derived equivalence. Indeed, we prove that the bounded derived category of such an algebra depends on the four combinatorial parameters mentioned above.

We introduce another notion of equivalence, called 'good mutation' equivalence, which is slightly stronger than derived equivalence, and give a complete classification of cluster-tilted algebras of type D_n up to good mutation equivalence together with canonical forms. We can also find further derived equivalences between cluster-tilted algebras of type D_n which are not given by good mutations. Building on the results of the good mutation equivalence classification and these further derived equivalences, we suggest standard forms for the derived equivalence classes.

Using some known numerical invariants, we distinguish some of the standard forms of type D_n up to derived equivalence. In particular, we derive formulae for the determinants of the Cartan matrices of cluster-tilted algebras of type D_n, and for the characteristic polynomials of the asymmetry matrices of cluster-tilted algebras of type A_n and of certain cluster-tilted algebras of type D_n.

These invariants yield a far reaching derived equivalence classification of cluster-tilted algebras of type D_n, but some subtle questions in this classification remain open.

Keywords: cluster-tilted algebra, derived equivalence, quiver mutation, good mutation

Contents

CHAPTER 1

Introduction

In 2002 Fomin and Zelevinsky introduced *cluster algebras* in order to create a combinatorial framework for canonical bases and total positivity in semisimple algebraic groups. They have since gone on to have important and diverse applications in combinatorics, Lie theory, algebraic geometry, representation theory, integrable systems, Teichmüller theory, Poisson geometry and string theory in physics (via recent work on quivers with (super)potentials, see Derksen, Weyman, Zelevinsky [**28**] and Labardini-Fragoso [**41**])

Cluster algebras (without coefficients) are subrings of the field $\mathbb{Q}(x_1, \ldots, x_n)$ of rational functions in n indeterminates, and defined via a set of generators constructed inductively. These generators are called *cluster variables* and are grouped into overlapping subsets of fixed finite cardinality called *clusters*. The induction process begins with a pair $(\mathbf{x}, B)$, called a *seed*, where $\mathbf{x}$ is an initial cluster and B is an integral skew-symmetrizable $n \times n$-matrix.

An essential ingredient of this process is an operation on the integral $n \times n$-matrices called *matrix mutation*. Since there is a natural one-to-one correspondence between integral skew-symmetric $n \times n$-matrices and finite quivers with n vertices having no loops and no oriented 2-cycles, mutation can be defined directly on the level of quivers.

A cluster algebra is said to be of *finite type* if the number of cluster variables is finite. An important result of Fomin and Zelevinsky in [**32**] classifies the cluster algebras of finite type in terms of Dynkin diagrams. The cluster variables are then in bijection with the *almost positive* roots of the corresponding root system, that is, the roots which are positive or opposite to simple roots. This classification is analogous to the classification of representation-finite quivers by Gabriel in [**33**]. Note that the analogy is not exact because the cluster algebras of finite type are classified by all Dynkin diagrams whereas quivers of finite representation type are classified only by the simply-laced Dynkin diagrams of type A, D and E.

In an attempt to 'categorify' cluster algebras (without coefficients), *cluster categories* were introduced by Buan, Marsh, Reineke, Reiten and Todorov in [**18**]. More precisely, these are orbit categories of the form $\mathcal{C}_Q = D^b(KQ)/\tau^{-1}[1]$ where Q is an *acyclic* quiver, i.e. a quiver without oriented cycles, $D^b(KQ)$ is the bounded derived category of the finite dimensional (left) KQ-modules over an algebraically closed field K and τ and [1] are the Auslander-Reiten translation and shift functor on the triangulated category $D^b(KQ)$, respectively. A more detailed description can be found in Section 2.4. Remarkably, these cluster categories are again triangulated categories by a result of Keller [**39**].

Tilting modules play an important role in the representation theory of finite dimensional algebras. The algebras of the form $\mathrm{End}(T)$ for a tilting module T over a path algebra KQ, called tilted algebras, form a central class of algebras. This motivates studying the *cluster-tilted algebras* more closely, which are those of the form $\mathrm{End}_{\mathcal{C}_Q}(T)$, where T is a *cluster-tilting object* in $\mathcal{C}_Q$ (i.e. a maximal object with no self-extensions, see Buan, Marsh, Reineke, Reiten and Todorov in [**18**] and Section 2.4). It is known from Caldero and Keller in [**27**] (see also Buan, Caldero, Keller, Marsh, Reiten and Todorov in [**16**]) that there is a one-to-one correspondence between the set of indecomposable objects without self-extensions of $\mathcal{C}_Q$ and the set of cluster variables in the corresponding cluster algebra $\mathcal{A}(Q)$. This induces a correspondence between clusters and cluster-tilting objects.

A cluster-tilted algebra corresponding to an acyclic quiver Q can be constructed explicitly by a quiver with relations KQ'/I', where the quiver Q' is mutation equivalent to the quiver Q and the relations are uniquely determined by its quiver (see Buan, Iyama, Reiten and Smith in [**17**]).

Cluster-tilted algebras have several interesting properties, e.g. their representation theory can be completely understood in terms of the representation theory of the corresponding path algebra of a quiver (see Buan, Marsh and Reiten in [**20**]). In particular, the Auslander-Reiten-quiver of a cluster-tilted algebra can be obtained directly from the Auslander-Reiten-quiver of the corresponding path algebra. Cluster-tilted algebras have been studied by various authors, see for instance [**4**], [**5**], [**19**] or [**26**].

In recent years, a focal point in the representation theory of algebras has been the investigation of derived equivalences of algebras. Since a lot of properties and invariants of rings and algebras are preserved by derived equivalences, it is important for many purposes to classify classes of algebras up to derived equivalence, instead of Morita equivalence. For instance, the class of self-injective algebras is closed under derived equivalence (see Al-Nofayee in [**1**]) and for self-injective algebras the representation type is preserved under derived equivalences (see Krause [**40**] and Rickard [**47**]). It has been also proved by Rickard in [**48**] that the class of symmetric algebras is closed under derived equivalences. Additionally, we note that derived equivalent algebras have the same number of pairwise non-isomorphic simple modules and isomorphic centres.

In this work, we are concerned with the problem of derived equivalence classification of cluster-tilted algebras of type $\tilde{A}_n$. Additionally, we are going to address this problem for cluster-tilted algebras of Dynkin type D_n. The quivers of cluster-tilted algebras of type $\tilde{A}_n$ (resp., D_n) are exactly the quivers in the mutation classes of $\tilde{A}_n$ (resp., D_n) and the corresponding relations were determined by Assem, Brüstle, Charbonneau-Jodoin and Plamondon in [**3**] and by Buan, Marsh and Reiten in [**21**], respectively. A derived equivalence classification has been achieved so far for cluster-tilted algebras of Dynkin type A_n by Buan and Vatne [**23**]; see also the work of Murphy on the more general case of m-cluster-tilted algebras of type A_n [**45**]. A complete derived equivalence classification has also been given by the author together with Holm and Ladkani in [**12**] for Dynkin types $E_{6,7,8}$.

Our first aim in Chapter 3 is to give a description of the mutation classes of $\tilde{A}_n$-quivers. These mutation classes are known to be finite (for example see Fomin, Shapiro and Thurston in [**30**] or Buan and Reiten in [**22**], stated as Theorem 2.1.4 in this work). The second purpose of this chapter is to describe when two cluster-tilted algebras of type Q have equivalent derived categories, where Q is a quiver whose underlying graph is $\tilde{A}_n$.

The first step is to compute all quivers which are mutation equivalent to any quiver of type $\tilde{A}_n$, i.e. all quivers which are mutation equivalent to any non-oriented $(n+1)$-cycle. Note that for an oriented $(n+1)$-cycle we get the mutation class of D_{n+1} (see for instance Vatne in [**53**] or Lemma 3.1.1). In order to determine all quivers mutation equivalent to $\tilde{A}_n$, we define a class of quivers $\mathcal{Q}_n$ (see Definition 3.1.5), as follows:

Definition. Let $\mathcal{Q}_n$ be the class of connected quivers with $n+1$ vertices which satisfy the following conditions:

(i) There exists precisely one full subquiver which is a non-oriented cycle of length ≥ 2.

(ii) For each arrow $x \xrightarrow{\alpha} y$ in this non-oriented cycle, there may (or may not) be a vertex z_α which is not on the non-oriented cycle, such that there is an oriented 3-cycle of the form

$$\begin{tikzcd} & z_\alpha \arrow[dl] & \\ x \arrow[rr, "\alpha"'] & & y \arrow[ul] \end{tikzcd} .$$

Apart from the arrows of these oriented 3-cycles there are no other arrows incident to vertices on the non-oriented cycle.

(iii) If we remove all vertices in the non-oriented cycle and their incident arrows, the result is a disjoint union of *rooted quivers of type A* (see Definition 2.5.4), one for each z_α.

To show that this class contains all quivers mutation equivalent to some quiver of type $\tilde{A}_n$ we prove in Lemma 3.1.7 that this class is closed under quiver mutation.

The next step is to separate this class into the different mutation classes of $\tilde{A}_n$-quivers. Since each quiver in $\mathcal{Q}_n$ can be embedded into the plane, we fix one of these embeddings. Thus, we can speak

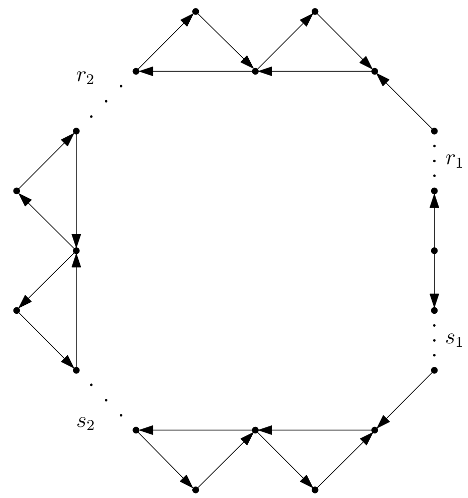


Figure 1.1. Normal form for quivers in $\mathcal{Q}_n$.

of clockwise and anti-clockwise oriented arrows of the (unique) non-oriented cycle. Then we define four combinatorial parameters r_1, r_2, s_1 and s_2 for any quiver $Q \in \mathcal{Q}_n$ in Definition 3.1.9. Roughly speaking, these parameters count the numbers of arrows and oriented 3-cycles in Q according to the different orientations of the arrows in the non-oriented cycle.

Subsequently, we prove that every quiver in $\mathcal{Q}_n$ with parameters r_1, r_2, s_1 and s_2 can be mutated to a *normal form* as in Figure 1.1 without changing the parameters.

Using this result we can show that every quiver in $\mathcal{Q}_n$ with parameters r_1, r_2, s_1 and s_2 is mutation equivalent to some non-oriented cycle with $r_1 + 2r_2$ arrows oriented in one direction and $s_1 + 2s_2$ arrows oriented in the other direction. Hence, we can prove Theorem 3.1.13 which then gives a complete description of the mutation classes of $\tilde{A}_n$-quivers:

Theorem. Let $Q_1, Q_2 \in \mathcal{Q}_n$ with parameters r_1, r_2, s_1 and s_2, respectively $\tilde{r}_1$, $\tilde{r}_2$, $\tilde{s}_1$ and $\tilde{s}_2$. Then Q_1 is mutation equivalent to Q_2 if and only if $r_1 + 2r_2 = \tilde{r}_1 + 2\tilde{r}_2$ and $s_1 + 2s_2 = \tilde{s}_1 + 2\tilde{s}_2$ (or $r_1 + 2r_2 = \tilde{s}_1 + 2\tilde{s}_2$ and $s_1 + 2s_2 = \tilde{r}_1 + 2\tilde{r}_2$).

The second aim of Chapter 3 is to get a complete classification of cluster-tilted algebras of type $\tilde{A}_n$ up to derived equivalence. It turns out that the combinatorial parameters r_1, r_2, s_1 and s_2 are sharp for derived equivalence (see Theorem 3.2.2), i.e.

Theorem. Two cluster-tilted algebras of type $\tilde{A}_n$ are derived equivalent if and only if their quivers have the same parameters r_1, r_2, s_1 and s_2 (up to changing the roles of r_i and s_i, $i \in \{1, 2\}$).

To prove this theorem, we first show that every cluster-tilted algebra of type $\tilde{A}_n$ with parameters r_1, r_2, s_1 and s_2 is derived equivalent to a cluster-tilted algebra which corresponds to a quiver in normal form (see Lemma 3.2.1). We then use an algorithm determined by Avella-Alaminos and Geiß in [**10**] to compute the parameters r_1, r_2, s_1 and s_2 as combinatorial derived invariants for any cluster-tilted algebra of type $\tilde{A}_n$.

In Chapters 4 and 5 we deal with cluster-tilted algebras of Dynkin type D_n. The quivers in the mutation class of D_n were classified by Vatne in [**53**] who divided the mutation class into four types which we shall denote I, II, III and IV. These types are defined as gluing of rooted quivers of type A to certain 'skeleta' (see Section 4.1 later in the thesis for a precise description). In addition, the corresponding relations are known by Buan, Marsh and Reiten in [**21**, Theorem 4.1]. Hence, we are left with the problem of finding a classification of cluster-tilted algebras of type D_n up to derived equivalence.

Chapter 4 deals with so-called good mutations (see below) and proves a result on derived equivalences, whilst Chapter 5 looks at some invariants associated with these derived equivalences. Both chapters are required to provide a partial classification of cluster-tilted algebras of type D_n up to derived equivalence. The classification, while not complete, is far reaching. A major obstacle to obtaining a complete classification is the computation of some polynomial invariants, which is computationally intensive. However, up to D_{14} the computation of these polynomial invariants can be done by brute force; they were computed by a program written by Ladkani, the results of which appear in our joint work (together also with Holm) [**13**, Section 2.3]. The subtlety of the classification beyond D_{14} arises because the polynomial invariants are not perfect, indeed the first example of derived inequivalent cluster-tilted algebras with the same polynomial invariants occurs in type D_{15} (see [**13**, Example 2.26]). Hence, the problem of distinguishing between certain of the cluster-tilted algebras of type D_n is quite delicate.

In [**43**] Ladkani described a procedure to determine when two cluster-tilted algebras whose quivers are related by a mutation are also derived equivalent. Such quiver mutations are called *good mutations* (for more details see Section 4.2.1). Thus, the first aim of Chapter 4 is to find all the good mutations for cluster-tilted algebras of type D_n and we get a complete classification up to good mutation equivalence.

In Sections 4.2.2 and 4.2.3 we determine all the good mutations for cluster-tilted algebras of type A_n and D_n. Using these results we can achieve a complete classification of the cluster-tilted algebras of Dynkin type D_n up to good mutation equivalence and we provide an algorithm for the problem of describing when two cluster-tilted algebras are good mutation equivalent. In particular, we can divide the cluster-tilted algebras of type D_n into eight types of different good mutation equivalence classes in Theorem 4.2.36. The main result of the good mutation equivalence classification can be stated as follows:

Theorem. There is an explicit finite list of canonical forms representing each good mutation equivalence class of cluster-tilted algebras of Dynkin type D_n. See Theorem 4.2.36 for the explicit description.

Following on, we can find further derived equivalences between cluster-tilted algebras of type D_n which are not given by good mutations. In particular, we obtain derived equivalences for algebras related by a *good double mutation*. This derived equivalence consists of two algebra mutations, where the intermediate algebra is not a cluster-tilted algebra. Additionally, we recall a result about derived equivalences for the self-injective cluster-tilted algebras. Note that the determination of self-injective cluster-tilted algebras is due to Ringel in [**51**], but the derived equivalence classification is originally due to Asashiba in [**2**].

Building on the results of the good mutation equivalence classification and these further derived equivalences, we give six types of *standard form* for derived equivalence, this list is stated as Theorem 4.3.9.

In Chapter 5 we distinguish some of the standard forms of type D_n up to derived equivalence. For this we use some known numerical invariants. In particular, we derive formulae for the determinants of the Cartan matrices of cluster-tilted algebras of type D_n in Theorem 5.2.1:

Theorem. Let Q be a quiver which is mutation equivalent to D_n for $n \geq 4$. Let Q' and Q'' be the rooted quivers of type A occurring in the types I, II and III. Let $t(Q')$ and $t(Q'')$ be the numbers of oriented 3-cycles in Q' and Q'', respectively.

(I) If Q is of type I, then $\det C_Q = 2^{t(Q')} = \det C_{Q'}$.
(II) If Q is of type II, then $\det C_Q = 2 \cdot 2^{t(Q')+t(Q'')} = 2 \cdot \det C_{Q'} \cdot \det C_{Q''}$.
(III) If Q is of type III, then $\det C_Q = 3 \cdot 2^{t(Q')+t(Q'')} = 3 \cdot \det C_{Q'} \cdot \det C_{Q''}$.
(IV) For a quiver Q of type IV with central cycle of length $m \geq 3$, let $Q^{(1)}, \ldots, Q^{(r)}$ be the rooted quivers of type A glued to the spikes and let $t(Q^{(j)})$ be the number of oriented 3-cycles in $Q^{(j)}$. In addition, let $c(Q)$ be the number of vertices on the central cycle which are part of two (consecutive) spikes. Then

$$\det C_Q = (m + c(Q) - 1) \cdot \prod_{j=1}^{r} 2^{t(Q^{(j)})} = (m + c(Q) - 1) \cdot \prod_{j=1}^{r} \det C_{Q^{(j)}}.$$

As a consequence, in Corollary 5.2.4 we are able to distinguish any two distinct standard forms of Theorem 4.3.9 except for one standard form. That is, there exists one standard form for which members of this class may be derived equivalent to algebras of the other classes and which, for the sake of the introduction, we shall call the 'exceptional standard form'. This standard form contains many cases of cluster-tilted algebras of type IV. This is the source of the subtlety in obtaining a complete classification.

The second invariant of derived equivalence we use in this chapter is the characteristic polynomial of the so-called *asymmetry matrix* $S = CC^{-T}$ corresponding to the Cartan matrix C. We stress that the asymmetry matrix and its characteristic polynomial are well-defined whenever the Cartan matrix is invertible over $\mathbb{Q}$, as follows from Ladkani in [**42**, Section 3.3]. In the special case when the algebra has finite global dimension, the asymmetry matrix, or better minus its inverse $-C^TC^{-1}$, is related to the Coxeter transformation, and its characteristic polynomial is known as the *Coxeter polynomial* of the algebra. In [**15**] Boldt has determined a method to determine the Coxeter polynomial of certain split finite-dimensional algebras by reducing it to the computation of certain 'smaller' polynomials and we will use this reduction formula for our computations.

We derive formulae for the characteristic polynomials of the asymmetry matrices for the cluster-tilted algebras of type A_n in Proposition 5.3.6 and for the types I, II and III of type D_n, and for certain cluster-tilted algebras of type IV in Propositions 5.3.10, 5.3.11 and 5.3.12:

Proposition. Let Q be the quiver of a cluster-tilted algebra of type A_n. Then the characteristic polynomial of the asymmetry matrix is given by

$$\chi_{S_Q}(x) = (x+1)^{t-1}\Big(x^{s+t+2} + (-1)^{s+1}\Big),$$

where $t = t(Q)$ is the number of oriented 3-cycles in Q and $s = s(Q)$ is the number of arrows in Q which are not part of any oriented 3-cycle.

Proposition. Consider a cluster-tilted algebra of type D_n with quiver Q. Let Q' and Q'' be the rooted quivers of type A occurring in Q. Let $t(Q')$ and $t(Q'')$ be the numbers of oriented 3-cycles in Q' and Q'', respectively, and let $s(Q')$ and $s(Q'')$ be the numbers of arrows in Q' and Q'', which are not part of any oriented 3-cycle.

(I) If Q is of type I, then

$$\chi_{S_Q}(x) = (x+1)^{t}(x-1)\Big(x^{s+t+2} + (-1)^{s}\Big)$$

where $s = s(Q')$ and $t = t(Q')$.

(II/III) If Q is of type II or type III, then

$$\chi_{S_Q}(x) = (x+1)^{t+1}(x-1)\Big(x^{s+t+2} + (-1)^{s+1}\Big)$$

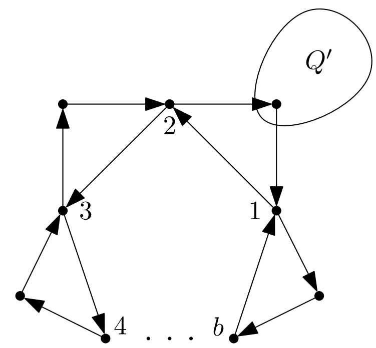
where $s = s(Q') + s(Q'')$ and $t = t(Q') + t(Q'')$.

(IV) If Q is of type IV, then we have the following:

(a) If Q is an oriented cycle of length n without spikes then

$$\chi_{S_Q}(x) = \begin{cases} x^n - 1, & \text{if } n \text{ is odd,} \\ \big(x^{\frac{n}{2}} - 1\big)^2, & \text{if } n \text{ is even.} \end{cases}$$

(b) If Q is a quiver of the form

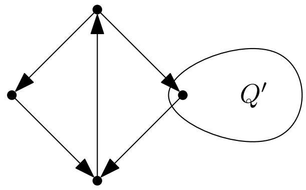


with central cycle of length b, then

$$\chi_{S_Q}(x) = (x+1)^t(x^b-1)\Big(x^{s+t+b} + (-1)^{s+1}\Big)$$

where $s = s(Q')$ and $t = t(Q')$.

(c) If Q is a quiver of the form



then

$$\chi_{S_Q}(x) = (x+1)^{t-1}(x-1)\Big(x^{s+t+4} + 2\cdot x^{s+t+3} + (-1)^{s-1}\cdot 2x + (-1)^{s-1}\Big)$$

where $s = s(Q')$ and $t = t(Q')$.

Using these polynomials, we can distinguish between cluster-tilted algebras of type II and some cluster-tilted algebras with 'exceptional standard form', namely, those of type IV (c). However, this is only a small improvement because it remains to distinguish many cluster-tilted algebras with 'exceptional standard form' from those of the other standard forms.

The thesis is organised as follows. In Chapter 2 we collect some preliminaries about quivers with relations, quiver mutation, derived equivalences, Cartan matrices and cluster-tilted algebras. We also recall the explicit description of cluster-tilted algebras of Dynkin type A_n as quivers with relations as given by Buan and Vatne in [**23**], since these are needed for the definitions of cluster-tilted algebras of types $\tilde{A}_n$ and D_n.

In Chapter 3 we provide the complete classification of cluster-tilted algebras of type $\tilde{A}_n$ up to derived equivalence. In particular, in Section 3.1, we give an explicit description of the mutation classes of quivers of type $\tilde{A}_n$, and we recall the relations for cluster-tilted algebras of type $\tilde{A}_n$ (as determined by Assem, Brüstle, Charbonneau-Jodoin and Plamondon in [**3**]). In Section 3.2 we prove the main result of this chapter, i.e. we show when two cluster-tilted algebras of type $\tilde{A}_n$ are derived equivalent.

In Chapter 4 we establish derived equivalences for cluster-tilted algebras of type D_n. In Section 4.1 we give the quivers in the mutation class of D_n and the corresponding relations (as found by Vatne in [**53**] and by Buan, Marsh and Reiten in [**21**]). In Section 4.2 we collect the basic notions about mutations of algebras (due to Ladkani in [**43**]) and we determine all the good mutations for cluster-tilted algebras of Dynkin types A_n and D_n. In particular, we can obtain a complete classification of the cluster-tilted algebras of type D_n up to good mutation equivalence. In Section 4.3 we present further derived equivalences between cluster-tilted algebras of type D_n which are not given by good mutations. Subsequently, we provide standard forms for derived equivalence, and prove the main result of this chapter.

In Chapter 5 we derive invariants of derived equivalence for cluster-tilted algebras of type A_n and D_n. Section 5.2 contains formulae for the determinants of the Cartan matrices of cluster-tilted algebras of type D_n. This invariant is used in the thesis to distinguish some cluster-tilted algebras of type D_n up to derived equivalence. Continuing this theme we calculate the characteristic polynomials of the asymmetry matrices for cluster-tilted algebras of types A_n and I, II and III of type D_n, and for certain cluster-tilted algebras of type IV in Section 5.3.

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CHAPTER 2

Background and Notation

In this chapter we collect some background material that will be needed later in the thesis. We will also give some examples and fix our notation.

2.1. Quivers with relations and quiver mutation

In this section we present the basic notions about quivers with relations and quiver mutations. By a *quiver* we always mean a finite directed graph Q, consisting of a finite set of vertices Q_0 and a finite set of arrows Q_1 between them. The maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ map each arrow to its starting point and its target point respectively. Special vertices are sinks (i.e. vertices without outgoing arrows) and sources (i.e. vertices without incoming arrows). The *underlying graph* of a quiver Q is the graph obtained from Q by replacing the arrows in Q by undirected edges.

A *path* is an ordered sequence of arrows $\alpha = \alpha_n \dots \alpha_2\alpha_1$ with $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < n$. For such a path α define $s(\alpha) = s(\alpha_1)$ to be its start vertex and $t(\alpha) = t(\alpha_n)$ to be its end vertex. For every vertex $i \in Q_0$ we also have a trivial path e_i of length zero with $s(e_i) = t(e_i) = i$.

A quiver Q' with maps $s', t' : Q'_1 \to Q'_0$ is a *subquiver* of a quiver Q if $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$ and where $t'(\alpha) = t(\alpha) \in Q'_0$, $s'(\alpha) = s(\alpha) \in Q'_0$ for any arrow $\alpha \in Q'_1$. A subquiver is called a *full subquiver* if for any two vertices i and j in the subquiver, the subquiver also will contain all arrows between i and j present in Q. An *oriented cycle* is a subquiver of a quiver whose underlying graph is a cycle, and whose arrows are all oriented in the same direction. By contrast, a *non-oriented cycle* is a subquiver of a quiver whose underlying graph is a cycle, but not all of its arrows are oriented in the same direction. An *acyclic* quiver is a quiver without oriented cycles. Throughout the thesis, unless explicitly stated, we assume that quivers are connected.

Now, let Q be a quiver and K be a field. We can form the *path algebra* KQ of a quiver Q, where the basis of KQ is given by all paths in Q, including the trivial paths e_i at each vertex i of Q. Multiplication in KQ is defined by concatenation of paths. Since our convention is to read paths from right to left, the product of two paths α and β is defined to be the concatenated path $\alpha\beta$ if $s(\alpha) = t(\beta)$ and zero otherwise. The unit element of KQ is the sum of all trivial paths in Q, i.e. $1_{KQ} = \sum\limits_{i \in Q_0} e_i$. The set $\{e_i : i \in Q_0\}$ of all trivial paths e_i is a complete set of primitive orthogonal idempotents for KQ.

Example 2.1.1.

(i) Let Q be the following quiver

$$\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$$

consisting of a single vertex and a single loop. The K-basis of the path algebra KQ is $\{e_1, \alpha, \alpha^2, \dots\}$ and the multiplication is given by

$$e_1\alpha^l = \alpha^l e_1 = \alpha^l, \text{ for all } l \geq 0,$$
$$\alpha^l\alpha^k = \alpha^{l+k}, \text{ for all } l, k \geq 0,$$

where $\alpha^0 = e_1$. Hence, KQ is isomorphic to the polynomial algebra $K[x]$ in one indeterminate x.

(ii) Let Q be an A_3-quiver with the following orientation

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \, .$$

Then the path algebra KQ has as K-basis $\{e_1, e_2, e_3, \alpha, \beta, \beta\alpha\}$. The multiplication is defined by concatenation of paths, i.e. the multiplication table is given by

	e_1	e_2	e_3	α	β	$\beta\alpha$
e_1	e_1	0	0	0	0	0
e_2	0	e_2	0	α	0	0
e_3	0	0	e_3	0	β	$\beta\alpha$
α	α	0	0	0	0	0
β	0	β	0	$\beta\alpha$	0	0
$\beta\alpha$	$\beta\alpha$	0	0	0	0	0

.

Then KQ is isomorphic to the 3×3 lower triangular matrix algebra

$$T_3(K) = \begin{pmatrix} K & 0 & 0 \\ K & K & 0 \\ K & K & K \end{pmatrix}$$

with $e_i \mapsto E_{ii}$, $\alpha \mapsto E_{21}$, $\beta \mapsto E_{32}$ and $\beta\alpha \mapsto E_{31}$ (where the E_{ij} denote the elementary matrices).

Let Q be a quiver and $\mathrm{rad}(KQ)$ be the ideal of KQ generated by all arrows of Q. An ideal $I \subseteq KQ$ is called *admissible* if there exists $m \geq 2$ such that $\mathrm{rad}^m(KQ) \subseteq I \subseteq \mathrm{rad}^2(KQ)$. A *relation* in Q with coefficients in K is a K-linear combination $k_1\alpha_1 + \cdots + k_m\alpha_m$ of paths α_i in Q having the same starting point and the same end point. If $m = 1$, we call the relation a *zero-relation*. A relation of the form $\alpha_1 - \alpha_2$ is called a *commutativity relation*. Let $(\rho_j)_{j\in J}$ be a set of relations for Q such that the ideal I generated by all the ρ_j is admissible. Then the algebra KQ/I is said to be a *quiver with relations*.

We now recall the definition of *quiver mutation* which was introduced by Fomin and Zelevinsky in [**31**].

Definition 2.1.2. Let Q be a quiver without loops and oriented 2-cycles. The *mutation* of Q at a vertex k to a new quiver $\mu_k(Q)$ is obtained as follows:

(1) Add a new vertex k^*.
(2) Suppose that the number of arrows $i \to k$ in Q equals a, the number of arrows $k \to j$ equals b and the number of arrows $j \to i$ equals $c \in \mathbb{Z}$. Then we have $c - ab$ arrows $j \to i$ in $\mu_k(Q)$. Here, a negative number of arrows means arrows in the opposite direction.
(3) For any arrow $i \to k$ (resp., $k \to j$) in Q add an arrow $k^* \to i$ (resp., $j \to k^*$) in $\mu_k(Q)$.
(4) Remove the vertex k and all its incident arrows.

No other arrows are affected by this operation. Note that steps (2) and (3) should be carried out for all possible pairs i, j of vertices in Q.

Example 2.1.3. Now, we compute some mutations of small quivers of type A, D and $\tilde{A}$; for the corresponding (extended) Dynkin diagrams see Figure 2.1.

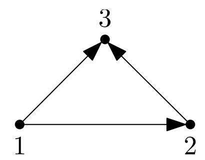
(i) First, we consider the quiver Q of type A_3 again.

$$1 \longrightarrow 2 \longrightarrow 3$$

If we mutate at vertex 2 we have $a = 1$ arrow from vertex 1 to vertex 2, $b = 1$ arrow from vertex 2 to vertex 3 and no arrows from vertex 3 to vertex 1. Thus, we have $c - ab = -1$ arrow from vertex 3 to vertex 1 in $\mu_2(Q)$, i.e. there is one arrow from vertex 1 to vertex 3:

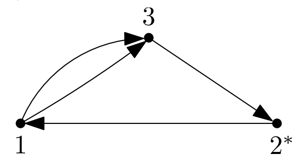
$$\begin{tikzcd} & 2^* \arrow[dl] & \\ 1 \arrow[rr] & & 3 \arrow[ul] \end{tikzcd} \, .$$

(ii) Next we consider the following quiver Q of type $\tilde{A}_2$:



.

If we mutate at vertex 2 there is $a = 1$ arrow $1 \to 2$, $b = 1$ arrow $2 \to 3$ and one arrow $1 \to 3$, i.e. $c = -1$. Thus, we have $c - ab = -2$ arrows $3 \to 1$ in $\mu_2(Q)$, i.e. there are two arrows from vertex 1 to vertex 3:



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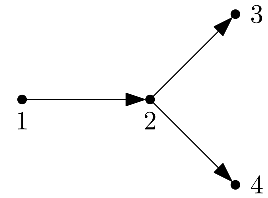
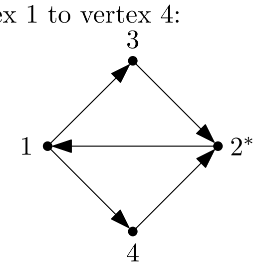
(iii) Finally, consider the quiver Q of type D_4 below:



.

If we mutate at vertex 2 there is $a = 1$ arrow $1 \to 2$, $b_1 = 1$ arrow $2 \to 3$ and no arrows between vertex 3 and vertex 1. Thus, we have $c_1 - ab_1 = -1$ arrow $3 \to 1$ in $\mu_2(Q)$, i.e. there is one arrow from vertex 1 to vertex 3. Moreover, we have $b_2 = 1$ arrow $2 \to 4$ and no arrows between vertex 4 and vertex 1. Thus, we have $c_2 - ab_2 = -1$ arrow $4 \to 1$ in $\mu_2(Q)$, i.e. there is one arrow from vertex 1 to vertex 4:



.

Quiver mutation is an involution, that is, we have $\mu_k(\mu_k(Q)) = Q$. A special case of quiver mutation is the mutation at a sink or a source. In this case, the mutation only reverses the arrows incident with the mutated vertex. Two quivers are called *mutation equivalent* (*sink/source equivalent*) if one can be obtained from the other by a finite sequence of mutations (at sinks and/or sources). The *mutation class* of a quiver Q is the class of all quivers mutation equivalent to Q. The mutation class of a quiver Q can be either finite or infinite. If Q has a finite mutation class, then Q is called a quiver of *finite mutation type*.

The classification of the quivers with a finite mutation class has recently been settled in [**29**]. We point out that the classification of acyclic quivers with finite mutation type has been obtained in [**22**]:

Theorem 2.1.4 (Buan and Reiten [22]). *Let Q be a finite acyclic quiver. Then the mutation class of Q is finite if and only if Q has at most two vertices, or Q is a Dynkin quiver of type A, D, E or an extended Dynkin quiver of type $\tilde{A}$, $\tilde{D}$, $\tilde{E}$.*

See Figure 2.1 for an illustration of the (extended) Dynkin diagrams of types A, D, E, $\tilde{A}$, $\tilde{D}$ and $\tilde{E}$.

Furthermore, we have the following well-known lemma, see for example Fomin and Zelevinsky in Proposition 9.2 in [**32**].

Lemma 2.1.5. *If two quivers Q, Q' have the same underlying graph and that graph is a tree, then Q and Q' are mutation equivalent.*

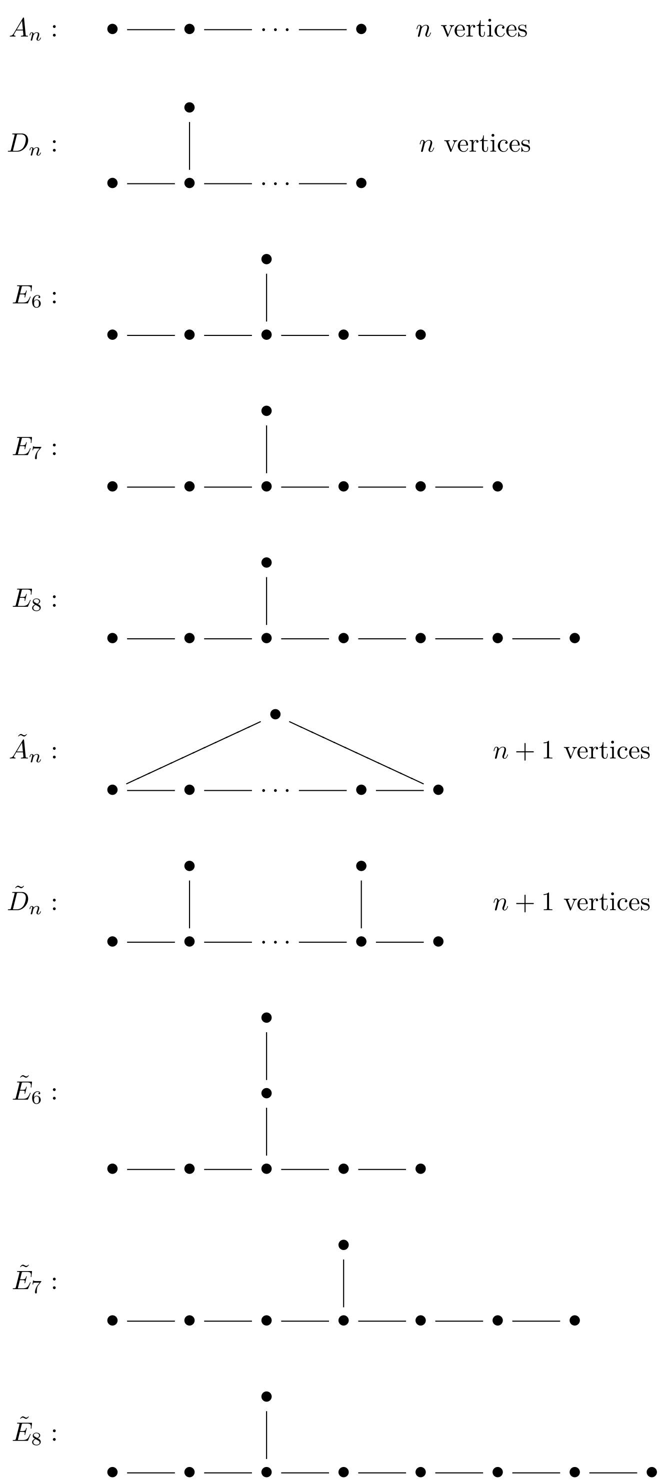


Figure 2.1. The (extended) Dynkin diagrams of types A, D, E, $\tilde{A}$, $\tilde{D}$ and $\tilde{E}$. For each quiver of type A, D, E, $\tilde{D}$ and $\tilde{E}$, the orientation of the edges can be chosen arbitrarily (see Lemma 2.1.5). For a quiver of type $\tilde{A}$ different choices of the orientations may give different mutation classes (see Lemma 3.1.3).

Proof. We will show that any two orientations of a tree are mutation equivalent. More precisely, using induction on the number of vertices of Q, we will show that we can arbitrarily orient the arrows of Q by applying a sequence of sink/source mutations.

If Q consists only of a single vertex, there is nothing to prove. Otherwise, we choose a vertex $l \in Q_0$ of valency 1 (such a vertex exists since Q is a tree, i.e. Q contains no cycles) and apply the induction assumption to the quiver obtained from Q by deleting the vertex l and the incident arrow, which we denote by $Q\backslash\{l\}$. So we are able to arbitrarily reorient the arrows of $Q\backslash\{l\}$ by a sequence of sink/source mutations. The remaining arrow between the unique vertex adjacent to l and l can then be given an arbitrary orientation by a (sink/source) mutation at l. $\square$

Now we give an example of a quiver with infinite mutation class.

Example 2.1.6. Let Q be the following quiver:

$$\bullet_1 \longrightarrow \bullet_2 \rightrightarrows \bullet_3 \,.$$

We claim that doing a mutation first at vertex two and then at vertex three and repeating this m-times, i.e. $(\mu_3 \circ \mu_2)^m(Q)$, will give a quiver of the following form

$$\begin{tikzcd}
& \bullet_2 \arrow[dr, shift left] \arrow[dr, shift right] & \\
\bullet_1 \arrow[ur, "2m+1"] & & \bullet_3 \arrow[ll, "2m"]
\end{tikzcd},$$

i.e. there are $2m+1$ arrows from vertex 1 to vertex 2 and $2m$ arrows from vertex 3 to vertex 1. We shall show this by induction on m.

If we first mutate at vertex 2, and then mutate at vertex 3, we get the following quivers:

$$\mu_2(Q) = \begin{tikzcd}
& \bullet_2 \arrow[dl] & \\
\bullet_1 \arrow[rr, shift left] \arrow[rr, shift right] & & \bullet_3 \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}, \quad \mu_3(\mu_2(Q)) = \begin{tikzcd}
& \bullet_2 \arrow[dr, shift left] \arrow[dr, shift right] & \\
\bullet_1 \arrow[ur] \arrow[ur, shift left=2] \arrow[ur, shift right=2] & & \bullet_3 \arrow[ll, shift left] \arrow[ll, shift right]
\end{tikzcd}.$$

Thus, we have $2m+1 = 3$ arrows from vertex 1 to vertex 2 and $2m = 2$ arrows from vertex 3 to vertex 1.

Now, let Q' be the quiver which we get after applying $(\mu_3 \circ \mu_2)^m$. Suppose, by induction, Q' has the form

$$\begin{tikzcd}
& \bullet_2 \arrow[dr, shift left] \arrow[dr, shift right] & \\
\bullet_1 \arrow[ur, "2m+1"] & & \bullet_3 \arrow[ll, "2m"]
\end{tikzcd}.$$

Then mutation first at vertex 2 and then at vertex 3 leads to

$$\mu_2(Q') = \begin{tikzcd}
& \bullet_2 \arrow[dl, "2m+1"'] & \\
\bullet_1 \arrow[rr, "2m+2"'] & & \bullet_3 \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}, \quad \mu_3(\mu_2(Q')) = \begin{tikzcd}
& \bullet_2 \arrow[dr, shift left] \arrow[dr, shift right] & \\
\bullet_1 \arrow[ur, "2(m+1)+1"] & & \bullet_3 \arrow[ll, "2(m+1)"]
\end{tikzcd}.$$

Hence, by induction, the iterated mutation $(\mu_3 \circ \mu_2)^m(Q)$ has the form as claimed. Note that we have constructed a sequence of mutations where the number of arrows is increasing after each step. Thus, the mutation class of Q is infinite.

2.2. Derived equivalences and tilting complexes

In this section, we briefly review the fundamental results on derived equivalences. First, we recall some background and the definition of a derived category following [**37**]. There are many good references for this material, see for example [**35**] and [**40**] amongst many others.

We start with an abelian category $\mathcal{A}$. The set of all morphisms from objects X to Y in $\mathcal{A}$ is denoted by $\mathrm{Hom}_{\mathcal{A}}(X,Y)$. Then the *category of complexes* $C(\mathcal{A})$ has as objects complexes, i.e. sequences of objects $X^n \in \mathcal{A}$ and morphisms $d^n : X^n \to X^{n+1}$

$$\dots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \dots$$

such that $d^n d^{n-1} = 0$ for all n. The morphisms of complexes $f : X \to Y$ are sequences $f = (f^n)_{n\in\mathbb{Z}}$ of morphisms $f^n : X^n \to Y^n$ in $\mathcal{A}$ such that $d^n f^n = f^{n+1} d^n$ for all n. A complex $X = (X^n, d^n)$ is a *stalk complex*, if there exists n_0 such that $X^{n_0} \neq 0$ and $X^n = 0$ for all $n \neq n_0$. The object X^{n_0} is then called the *stalk*. Note that each object X of $\mathcal{A}$ can be identified with a stalk complex with stalk X in degree zero. Thus, there is a full embedding of $\mathcal{A}$ into the category of complexes $C(\mathcal{A})$.

Morphisms $f, g : X \to Y$ in $C(\mathcal{A})$ are called *homotopic*, denoted $f \sim g$, if there exist maps $s^n \in \mathrm{Hom}_{\mathcal{A}}(X^n, Y^{n-1})$ such that $f^n - g^n = d^{n-1}s^n + s^{n+1}d^n$ for all n. We define the *homotopy category* $K(\mathcal{A})$: its objects are the same as those in $C(\mathcal{A})$ and the morphisms are given by equivalence classes $\mathrm{Hom}_{K(\mathcal{A})}(X,Y) = \mathrm{Hom}_{C(\mathcal{A})}(X,Y)/\mathrm{Ht}(X,Y)$, where $\mathrm{Ht}(X,Y) = \{f : X \to Y :$ f homotopic to zero$\}$.

For any $k \in \mathbb{Z}$ we have the *shift functor* $[k] : K(\mathcal{A}) \to K(\mathcal{A})$ defined on objects by $(X[k])^n = X^{n+k}$, $d^n_{X[k]} = (-1)^k d^{n+k}_X$ and on morphisms by $f[k] : X[k] \to Y[k]$, $f[k]^n = f^{n+k}$ for all $n \in \mathbb{Z}$. In particular, we have an automorphism $[1] : K(\mathcal{A}) \to K(\mathcal{A})$.

For $f : X \to Y$ in $C(\mathcal{A})$ the *mapping cone* $M(f) \in C(\mathcal{A})$ is defined by

$$M(f)^n = X^{n+1} \oplus Y^n \text{ and } d^n_{M(f)} = \begin{pmatrix} d^n_{X[1]} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix},$$

where $d^n_{X[1]} = -d^{n+1}_X$. Furthermore, we have the canonical morphisms $\alpha(f) : Y \to M(f)$, $\alpha(f)^n = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \end{pmatrix}$ and $\beta(f) : M(f) \to X[1]$, $\beta(f)^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \end{pmatrix}$, and the diagram

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1]$$

is called a *standard triangle* in $K(\mathcal{A})$. The homotopy category $K(\mathcal{A})$ is a *triangulated category* (due to Verdier [**54**]), if one defines *distinguished triangles* to be isomorphic (in $K(\mathcal{A})$, i.e. homotopy equivalent) to a standard triangle. For the definition of a triangulated category and formal properties see for instance [**38**] and [**46**].

The n^{th}-*cohomology* of a complex X in $C(\mathcal{A})$ is the quotient module $H^n(X) = \ker(d^n)/\mathrm{im}(d^{n-1})$. In the *derived category* of $\mathcal{A}$, denoted $D(\mathcal{A})$, we are interested in objects up to isomorphism in cohomology, so we formally invert morphisms in $K(\mathcal{A})$ which induce isomorphisms in cohomology, the so-called *quasi-isomorphisms*. More precisely, a morphism $f : X \to Y$ is a quasi-isomorphism if for all n the map $H^n(f) : H^n(X) \to H^n(Y)$, $x + \mathrm{im}(d^{n-1}_X) \mapsto f(x) + \mathrm{im}(d^{n-1}_Y)$, is an isomorphism. In particular, the class of quasi-isomorphisms S in $K(\mathcal{A})$ forms a multiplicative system (which is analogous to the corresponding concept in classical ring theory) which is compatible with triangulation. One then defines the derived category $D(\mathcal{A})$ to be the 'localisation' of $K(\mathcal{A})$ with respect to S:

$$D(\mathcal{A}) := K(\mathcal{A})[S^{-1}].$$

Let $L : K(\mathcal{A}) \to D(\mathcal{A})$ denote the localisation functor. Then it can be shown that there is a unique triangulated structure on $D(\mathcal{A})$ such that L is a triangulated functor and any functor $F : K(\mathcal{A}) \to \mathcal{B}$, where $\mathcal{B}$ is any category, such that $F(s)$ is an isomorphism for all $s \in S$ factors uniquely through L. The details of the construction are quite technical and require some set-theoretic considerations. An exposition of the proof is given in [**35**, Chapter 1, §3], and the set-theoretic considerations are dealt with in, among other places, [**55**, Chapter 10]. Note that in $K(\mathcal{A})$ an object X is quasi-isomorphic to its projective and injective resolutions, so that in $D(\mathcal{A})$, X becomes isomorphic to them and hence can be identified with all its projective and injective resolutions.

From now on let K be an algebraically closed field. All algebras are assumed to be finite dimensional K-algebras. For a K-algebra A the bounded derived category of finite dimensional (left) A-modules, i.e. the derived category of bounded complexes, is denoted by $D^b(A\text{-mod})$, or short-hand by $D^b(A)$. Note that all modules will be assumed to be left modules in the following.

Before we recall the results on derived equivalences we will say a few words on the derived category of a path algebra $A = KQ$ of a Dynkin quiver Q. For the proofs and more details we refer to [**34**].

The indecomposable objects in $D^b(A)$ are determined as follows:

Lemma 2.2.1 (Happel [34]). *Let $A = KQ$ be the path algebra of a Dynkin quiver Q and let X be an indecomposable object in $D^b(A)$. Then X is quasi-isomorphic to a stalk complex with indecomposable stalk.*

A good way of illustrating such a category is to draw the so-called *Auslander-Reiten-quiver/AR-quiver* (when it exists). Note that all the categories occurring in this thesis have Auslander-Reiten-triangles or Auslander-Reiten-sequences and hence, one can, in principle, draw the AR-quiver. The vertices are given by the isomorphism classes of the indecomposable objects and the arrows are the irreducible maps. We can describe the AR-quiver of the bounded derived category of $A = KQ$ (Q a Dynkin quiver) in terms of the AR-quiver of A: let $\Gamma = \Gamma_A$ be the AR-quiver of A. Denote by $\Gamma[i]$ a copy of Γ for $i \in \mathbb{Z}$. Recall that the indecomposable projective modules as well as the indecomposable injective modules are indexed by the vertices of Q. Then we denote by $\tilde{\Gamma}$ the quiver obtained from the disjoint union $\dot{\bigcup}_{i\in\mathbb{Z}}\Gamma[i]$ by adding an arrow from the injective module $I_a[i]$ to the projective module $P_b[i+1]$ whenever there is an arrow from a to b in Q.

Theorem 2.2.2 (Happel [34]). *Let $A = KQ$ be the path algebra of a Dynkin quiver Q. The AR-quiver $\Gamma(D^b(A))$ is $\tilde{\Gamma}$.*

For a Dynkin quiver Q the *infinite translation quiver* $\mathbb{Z}Q$ is constructed as follows (see Riedtmann in [**50**]): $(\mathbb{Z}Q)_0 = \mathbb{Z} \times Q_0 = \{(n, x) : n \in \mathbb{Z}, x \in Q_0\}$. For each arrow $\alpha : x \to y$ in Q_1, there exist two arrows

$$(n, \alpha) : (n, x) \to (n, y) \text{ and } (n, \alpha') : (n-1, y) \to (n, x)$$

in $(\mathbb{Z}Q)_1$, and these are all arrows in $(\mathbb{Z}Q)_1$. Then $\tau : \mathbb{Z}Q \to \mathbb{Z}Q$ is the automorphism of $\mathbb{Z}Q$ taking (n, x) to $(n-1, x)$ and (n, α) to $(n-1, \alpha)$ for all $n \in \mathbb{Z}$, $x \in Q_0$ and $\alpha \in Q_1$. τ is called the *Auslander-Reiten translation*. In the usual way of drawing $\mathbb{Z}Q$ in the literature the AR-translation shifts each copy of Q one to the left (for instance see Example 2.2.4 below).

Corollary 2.2.3 (Happel [34]). *Let $A = KQ$ be the path algebra of a Dynkin quiver Q. Then $\Gamma(D^b(A)) = \mathbb{Z}Q$.*

Example 2.2.4. Let $A = KQ$ be the path algebra of a linearly oriented A_3-quiver:

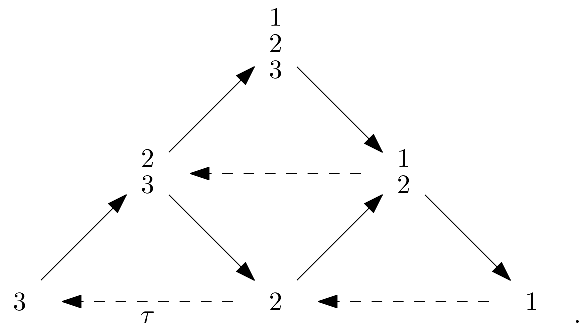
$$\underset{1}{\bullet} \xrightarrow{\ \alpha\ } \underset{2}{\bullet} \xrightarrow{\ \beta\ } \underset{3}{\bullet}\ .$$

In order to draw the Auslander-Reiten quiver we want to describe all the indecomposable A-modules. We have a decomposition $A = A \cdot 1 = \bigoplus_{i\in Q_0} Ae_i$ and hence, the (left) A-modules $P_i = Ae_i$ are the indecomposable projective A-modules. These are spanned by all paths starting at i, that is, $P_1 = \operatorname{span}\{e_1, \alpha, \beta\alpha\} =: \begin{smallmatrix}1\\2\\3\end{smallmatrix}$, $P_2 = \operatorname{span}\{e_2, \beta\} =: \begin{smallmatrix}2\\3\end{smallmatrix}$ and $P_3 = \operatorname{span}\{e_3\} =: 3$.

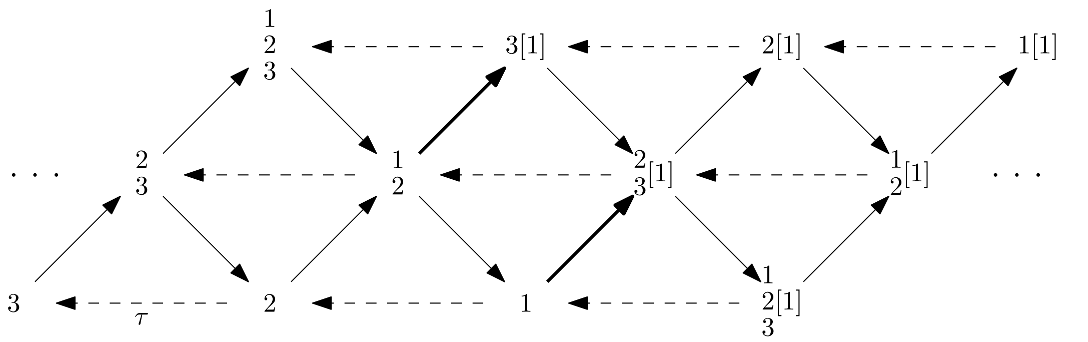
The injective indecomposable A-modules are denoted by $I_i = \operatorname{Hom}_K(e_iA, K)$, $i \in \{1, 2, 3\}$, and we get $I_1 = P_1/\operatorname{span}\{\alpha, \beta\alpha\} =: 1$, $I_2 = P_1/\operatorname{span}\{\beta\alpha\} =: \begin{smallmatrix}1\\2\end{smallmatrix}$ and $I_3 \cong P_1$.

Finally, the simple A-modules are denoted by S_i, $i \in \{1, 2, 3\}$, and we get $S_1 \cong I_1$, $S_2 = \operatorname{span}\{e_2, \beta\}/\operatorname{span}\{\beta\} =: 2$ and $S_3 \cong P_3$.

Thus, the AR-quiver of A is given by



Then the AR-quiver of the bounded derived category $D^b(A)$ is



Note that $\begin{smallmatrix}1\\2\end{smallmatrix}$ is quasi-isomorphic to its projective resolutions and $\begin{smallmatrix}2\\3\end{smallmatrix}$ is quasi-isomorphic to its injective resolutions. Hence, the fat arrows in the picture can be constructed as follows:

$$
\begin{array}{rcccccc}
\text{projective resolution of } \begin{smallmatrix}1\\2\end{smallmatrix}: & 0 & \to & 3 & \to & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \to & 0 \\
 & & & \downarrow \mathrm{id} & & \downarrow 0 & & \\
3[1]: & 0 & \to & 3 & \to & 0 & & \\
 & & & & & & & \\
\text{injective resolution of } \begin{smallmatrix}2\\3\end{smallmatrix}[1]: & 0 & \to & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \to & 1 & \to & 0 \\
 & & & \uparrow 0 & & \uparrow \mathrm{id} & & \\
1: & & & 0 & \to & 1 & \to & 0
\end{array}
$$

Recall that two algebras A and B are called *derived equivalent* if $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories. A famous theorem of Rickard [**49**] characterises derived equivalence in terms of so-called tilting complexes, whose definition is given now.

Definition 2.2.5. A *tilting complex* T over A is a bounded complex of finitely generated projective A-modules satisfying the following conditions:

(i) $\mathrm{Hom}_{D^b(A)}(T, T[i]) = 0$ for all $i \neq 0$, where $[.]$ denotes the shift functor in $D^b(A)$;

(ii) the category $\mathrm{add}(T)$ (i.e. the full subcategory consisting of direct summands of finite direct sums of T) generates the homotopy category $K^b(P_A)$ of projective A-modules as a triangulated category (that is, $K^b(P_A)$ is the smallest triangulated category which contains $\mathrm{add}(T)$).

Example 2.2.6. Let A be the path algebra of the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \; .$$

Recall that the projective indecomposable A-modules $P_i = Ae_i$, $i \in \{1, 2, 3\}$, are spanned by all paths starting at i, i.e. we have $P_1 = \mathrm{span}\{e_1, \alpha, \beta\alpha\}$, $P_2 = \mathrm{span}\{e_2, \beta\}$ and $P_3 = \mathrm{span}\{e_3\}$. Any homomorphism $\varphi : Ae_j \to Ae_i$ of left A-modules is uniquely determined by $\varphi(e_j) \in e_jAe_i$, the K-vector space generated by all paths in Q from vertex i to vertex j that are non-zero in A. Since we deal with left modules and read paths from right to left, a nonzero path from vertex i to j gives a homomorphism $P_j \to P_i$ by right multiplication. Thus, we have homomorphisms $P_2 \xrightarrow{\alpha} P_1$, $e_2 \mapsto e_2\alpha e_1$, $\beta \mapsto \beta\alpha e_1$, $P_3 \xrightarrow{\beta} P_2$, $e_3 \mapsto e_3\beta e_2$ and $P_3 \xrightarrow{\beta\alpha} P_1$, $e_3 \mapsto e_3\beta\alpha e_1$, and scalar multiples of these homomorphisms. In the following we are dealing with a basis of the space of homomorphisms, i.e. we will often ignore the scalars.

Hence, we can form a complex $T = \bigoplus_{i=1}^{3} T_i$ with

$$\begin{array}{lccccccc} T_1 & : & & 0 & \to & P_1 & \to & 0 \\ T_2 & : & & 0 & \to & P_2 & \to & 0 \\ T_3 & : & 0 \to & P_3 & \xrightarrow{\beta} & P_2 & \to & 0 \end{array}$$

where T_1 and T_2 are concentrated in degree zero and T_3 is concentrated in degrees -1 and 0.

We claim that T is a tilting complex over A. First we have to check that $\mathrm{Hom}_{K^b(A)}(T, T[i]) = 0$ for $i \neq 0$ since this then also holds in the localised category $D^b(A)$. This is clear for $|i| \geq 2$ since T is concentrated in two degrees.

We begin with possible maps $T_3 \to T_3[1]$ and $T_3 \to T_3[-1]$:

$$\begin{array}{ccccccccc} & & 0 & \to & P_3 & \xrightarrow{\beta} & P_2 & \to & 0 \\ & & & & \downarrow \beta & & & & \\ 0 & \to & P_3 & \xrightarrow{\beta} & P_2 & \to & 0 & & \\ & & & & \downarrow 0 & & & & \\ & & 0 & \to & P_3 & \xrightarrow{\beta} & P_2 & \to & 0 \end{array}$$

The first homomorphism is homotopic to zero. In the second case there is no non-zero homomorphism $P_2 \to P_3$ since there is no non-zero path from vertex 3 to vertex 2 in the quiver of A.

Next we consider possible maps $T_3 \to T_j[1],\ j \neq 3$. These maps are given by a map of complexes as follows

$$\begin{array}{ccccccc} 0 & \to & P_3 & \xrightarrow{\beta} & P_2 & \to & 0 \\ & & \downarrow & & & & \\ 0 & \to & Q & \to 0 & & & \end{array}$$

where Q could be either P_1 or P_2. There exist non-zero homomorphisms of complexes. But they are all homotopic to zero since every path from vertex 1 or 2 to vertex 3 ends with β. Thus, every homomorphism $P_3 \to Q$ can be factored through the map $\beta : P_3 \to P_2$. Immediately from the definition we see that $\mathrm{Hom}(T, T_j[1]) = 0$ for $j \neq 3$ and thus we have shown that $\mathrm{Hom}(T, T[1]) = 0$. Note that we already have $\mathrm{Hom}(T, T[-1]) = 0$ since there are no non-zero homomorphisms from P_1 or P_2 to P_3.

Secondly we have to show that $\mathrm{add}(T)$ generates $\mathrm{K}^b(P_A)$ as a triangulated category. We denote by $P_k[n]$ the stalk complex with P_k concentrated in degree $-n$. Since $P_1[0]$ and $P_2[0]$ occur as summands of T, $P_1[0]$ and $P_2[0]$ are in $\mathrm{add}(T)$ and therefore $P_1[n]$ and $P_2[n]$ are in the triangulated category generated by $\mathrm{add}(T)$ for all n. Thus, we have to check that $P_3[n]$ also is in the triangulated category generated by $\mathrm{add}(T)$.

There exists a homomorphism of complexes f from $P_2[0]$ to the complex $T_3 : 0 \to P_3 \xrightarrow{\beta} P_2 \to 0$ given by id_{P_2} in degree zero. Its mapping cone is $M(f) : 0 \to P_2 \oplus P_3 \xrightarrow{(\mathrm{id},\beta)} P_2 \to 0$ and we want to check that $M(f)$ is homotopy equivalent to $P_3[1]$. For this, we define maps $g := (0, \mathrm{id}) : M(f) \to P_3[1]$ and $h := (-\beta, \mathrm{id}) : P_3[1] \to M(f)$. We can see that $gh = \mathrm{id}$ and $hg \sim \mathrm{id}$ and so $M(f) \sim P_3[1]$.

Hence, there is a triangle of the form

$$\underbrace{P_2[0]}_{\in \mathrm{add}(T)} \xrightarrow{f} \underbrace{T_3}_{\in \mathrm{add}(T)} \to P_3[1] \to \underbrace{P_2[1]}_{\in \mathrm{add}(T)}$$

It follows that $P_3[1]$ is in the triangulated category generated by $\mathrm{add}(T)$ and therefore also $P_3[n]$ for all n.

Having all these stalk complexes of the projective indecomposable modules in the triangulated category generated by $\mathrm{add}(T)$ implies that $\mathrm{add}(T)$ generates $K^b(P_A)$ as a triangulated category since the smallest triangulated category containing the three indecomposable projectives is precisely $K^b(P_A)$.

Additionally, we note that the mapping cone of a map $f : P_i[0] \to P_j[0]$ is $M(f) : 0 \to P_i[0] \xrightarrow{f} P_j[0] \to 0$ and we can construct all bounded complexes of projective modules.

Theorem 2.2.7 (Rickard [49]). *Two algebras A and B are derived equivalent if and only if there exists a tilting complex T for A such that the endomorphism algebra $\mathrm{End}_{D^b(A)}(T) \cong B$.*

Although Rickard's theorem gives us a criterion for derived equivalence, it does not give a decision process nor a constructive method to produce tilting complexes. Thus, given two algebras A and B in concrete form, it is usually still unknown whether they are derived equivalent or not, as we do not know how to construct a suitable tilting complex or to prove the non-existence of such.

In Section 3.2 we will also need the following Proposition:

Proposition 2.2.8 (Rickard [49]). *If A and B are derived equivalent algebras, then the opposite algebras A^{op} and B^{op} are derived equivalent.*

2.3. Computation of Cartan matrices

Let $A = KQ/I$ be an algebra given by a quiver $Q = (Q_0, Q_1)$ with relations with unit element $\sum_{i\in Q_0} e_i + I$. We recall that we have a decomposition $A = A \cdot 1 = \bigoplus_{i\in Q_0} Ae_i$ and hence, the (left) A-modules $P_i = Ae_i$ are the indecomposable projective A-modules. The *Cartan matrix* $C_A = (c_{ij})$ of A is the $n \times n$-matrix whose entries are $c_{ij} = \dim_K \mathrm{Hom}_A(P_j, P_i)$, where $n = |Q_0|$. Any homomorphism $\varphi : Ae_j \to Ae_i$ of left A-modules is uniquely determined by $\varphi(e_j) \in e_jAe_i$, the K-vector space generated by all paths in Q from vertex i to vertex j that are non-zero in A. In particular, we have $c_{ij} = \dim_K e_jAe_i$ so that computing entries of the Cartan matrix for A reduces to counting paths in the quiver Q which are non-zero in A.

Example 2.3.1. First, we consider two quivers which are in the mutation class of $\tilde{A}_2$. The third quiver is a quiver in the mutation class of D_4.

(i) First, we have a look at the path algebra of the following quiver Q:

$$\begin{tikzcd} & 3 & \\ 1 \arrow[ur, "\alpha_3"] \arrow[rr, "\alpha_1"'] & & 2 \arrow[ul, "\alpha_2"'] \end{tikzcd}.$$

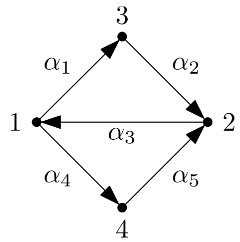
Its Cartan matrix is $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

(ii) Now consider an algebra which corresponds to the quiver below

$$\begin{tikzcd} & 3 \arrow[dl, "\alpha_3"'] & \\ 1 \arrow[rr, "\alpha_1"'] & & 2 \arrow[ul, "\alpha_2"] \arrow[ul, bend right, "\alpha_4"'] \end{tikzcd}$$

where we have three zero-relations $\alpha_1\alpha_3$, $\alpha_2\alpha_1$ and $\alpha_3\alpha_2$. Note that the paths $\alpha_3\alpha_4$ and $\alpha_4\alpha_1$ are not zero. Thus, we can compute the Cartan matrix to be $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$.

(iii) Finally, we consider an algebra which corresponds to the quiver



with four zero-relations $\alpha_1\alpha_3$, $\alpha_3\alpha_2$, $\alpha_4\alpha_3$, $\alpha_3\alpha_5$ and one commutativity relation $\alpha_2\alpha_1 = \alpha_5\alpha_4$. Then we can compute the Cartan matrix to be $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

To calculate the endomorphism algebra $\mathrm{End}_{D^b(A)}(T)$ of a tilting complex T over the algebra A, we can use the following statement which explicitly gives the Cartan matrix of the endomorphism algebra in terms of the tilting complex and the Cartan matrix of A.

Proposition 2.3.2 (Happel [34]). *For an algebra A let $Q = (Q^r)_{r\in\mathbb{Z}}$ and $R = (R^s)_{s\in\mathbb{Z}}$ be bounded complexes of projective A-modules. Then*

$$\sum_i (-1)^i \dim_K \mathrm{Hom}_{D^b(A)}(Q, R[i]) = \sum_{r,s} (-1)^{r-s} \dim_K \mathrm{Hom}_A(Q^r, R^s).$$

In particular, if Q and R are direct summands of the same tilting complex then

$$\dim_K \mathrm{Hom}_{D^b(A)}(Q, R) = \sum_{r,s} (-1)^{r-s} \dim_K \mathrm{Hom}_A(Q^r, R^s).$$

Example 2.3.3. Let A be the path algebra of the following quiver Q

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \quad .$$

We look at the tilting complex of Example 2.2.6 again, i.e. $T = \bigoplus_{i=1}^3 T_i$ with $T_1 = P_1[0]$, $T_2 = P_2[0]$ and $T_3 = 0 \to P_3 \xrightarrow{\beta} P_2 \to 0$ in degrees -1 and 0. We want to calculate $E := \mathrm{End}_{D^b(A)}(T)$. First, we compute the Cartan matrix of A as $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and then we also compute the Cartan matrix of E by the formula given in Proposition 2.3.2 and get $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

Next, we define homomorphisms of complexes between the summands of T which correspond to the arrows of the quiver Q' of E. Note that the homomorphisms are opposite to the arrows of the quiver since we read paths from right to left (and thus, a map $P_i \to P_j$ corresponds to an arrow $j \to i$).

The identity map $\mathrm{id} : P_2 \to P_2$ (in degree zero) gives rise to an embedding $\mathrm{id} : T_2 \to T_3$ and we have the homomorphism $\alpha : T_2 \to T_1$ since there is the arrow α from vertex 1 to vertex 2 in Q. There are no other non-zero homomorphisms between the summands of T and thus, we get the quiver Q' of E as follows

$$1 \xrightarrow{\alpha} 2 \xleftarrow{\mathrm{id}} 3 \quad .$$

According to the Cartan matrix of E there are also no relations in Q'. Hence, by Theorem 2.2.7, A is derived equivalent to E which is the path algebra of the quiver Q'.

2.4. Cluster-tilted algebras

In this section we assume that all quivers are without loops and oriented 2-cycles. We will give the general definition of a cluster-tilted algebra which was introduced by Buan, Marsh and Reiten in [**20**]. However, in this thesis we will only deal with explicit descriptions of cluster-tilted algebras as quivers with relations. We begin with the definition of a cluster category which was given in [**18**].

Definition 2.4.1. Let Q be an acyclic quiver. The *cluster category* $\mathcal{C}_Q$ is the orbit category of the bounded derived category $D^b(KQ)$ modulo the functor $\tau^{-1}[1]$, where τ denotes the Auslander-Reiten translation and $[1]$ is the shift functor on the triangulated category $D^b(KQ)$.

The Auslander-Reiten translation is defined in terms of almost split sequences, see for instance [**7**] and [**9**]. However, in this thesis we will not need this description. We will only need to know its action on the Auslander-Reiten-quiver (for the Dynkin case see Section 2.2).

Example 2.4.2. Let $A = KQ$ be the path algebra of a linearly oriented A_3-quiver as in Example 2.2.4. From the AR-quiver of $D^b(A)$ we can find the AR-quiver of the cluster category $\mathcal{C}_{A_3}$ as

$$\begin{tikzcd}[column sep=small]
 & & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[dr] & & 3[1] \arrow[ll, dashed] \arrow[dr] & & \\
 & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & \begin{smallmatrix}2\\3\end{smallmatrix}[1] \arrow[ll, dashed] \arrow[dr] & \\
3 \arrow[ur] & & 2 \arrow[ll, dashed, "\tau"] \arrow[ur] & & 1 \arrow[ll, dashed] \arrow[ur] & & \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1] \arrow[ll, dashed]
\end{tikzcd}$$

since $2[1] \cong 3$, $\begin{smallmatrix}1\\2\end{smallmatrix}[1] \cong \begin{smallmatrix}2\\3\end{smallmatrix}$ and $3[2] \cong \begin{smallmatrix}1\\2\\3\end{smallmatrix}$.

Objects in $\mathcal{C}_Q$ are the objects in $D^b(KQ)$ and morphisms from X to Y are given by

$$\mathrm{Hom}_{\mathcal{C}_Q}(X,Y) = \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{D^b(KQ)}(X, (\tau^{-1}[1])^p Y).$$

That is, there are more morphisms in $\mathcal{C}_Q$ than in $D^b(KQ)$. Furthermore, one defines $\mathrm{Ext}^1_{\mathcal{C}_Q}(X,Y) = \mathrm{Hom}_{\mathcal{C}_Q}(X, Y[1])$.

Example 2.4.3. We look at the algebra of Example 2.4.2 again. Then there are additional maps:

$$\begin{array}{ccccccccc}
\begin{smallmatrix}2\\3\end{smallmatrix}[1] & \to & 3 & \text{, since} & \begin{smallmatrix}2\\3\end{smallmatrix}[1] & \to & (\tau^{-1}[1])3 & = & 2[1] & \cong & 3 \\
\begin{smallmatrix}1\\2\\3\end{smallmatrix}[1] & \to & \begin{smallmatrix}2\\3\end{smallmatrix} & \text{, since} & \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1] & \to & (\tau^{-1}[1])\begin{smallmatrix}2\\3\end{smallmatrix} & = & \begin{smallmatrix}1\\2\end{smallmatrix}[1] & \cong & \begin{smallmatrix}2\\3\end{smallmatrix}
\end{array}.$$

A crucial role is played by the *cluster-tilting objects* in the cluster category $\mathcal{C}_Q$ which are in bijection with the clusters in the corresponding cluster algebra (see [**27**]).

Definition 2.4.4. An object T of $\mathcal{C}_Q$ is called a *cluster-tilting object* if $\mathrm{Ext}^1_{\mathcal{C}_Q}(T,T) = 0$ and T is maximal with respect to this property, that is, if $\mathrm{Ext}^1_{\mathcal{C}_Q}(T \oplus X, T \oplus X) = 0$, then X is a direct summand of a direct sum of copies of T (i.e. X is in $\mathrm{add}(T)$).

The endomorphism algebras of these cluster-tilting objects are called *cluster-tilted algebras of type Q*.

Example 2.4.5. Let A be the algebra as in Example 2.4.2. Since $\mathrm{Ext}^1_{\mathcal{C}_Q}(X,Y) = \mathrm{Hom}_{\mathcal{C}_Q}(X, Y[1])$, we can define the following two cluster-tilting objects:

$$T' = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \quad \text{and} \quad T'' = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 1 \oplus 3 \ .$$

We now examine the corresponding endomorphism algebras. The quivers Q' and Q'' of the cluster-tilted algebras $\mathrm{End}(T')$ and $\mathrm{End}(T'')$ look as follows:

$$Q' = \begin{tikzcd} \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[r] & 3 \end{tikzcd},$$

$$Q'' = \begin{tikzcd} & 1 \arrow[dl] & \\ \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[rr] & & 3 \arrow[ul] \end{tikzcd}.$$

Since we read paths from right to left the arrows of these quivers are opposite to the homomorphisms, e.g. in Q'' we have an arrow $3 \to 1$ since there is a map $1 \to 2[1] \cong 3$.

In $\mathrm{End}(T')$ all the non-trivial homomorphisms are non-zero, i.e. there are no relations in Q'. In $\mathrm{End}(T'')$ the three paths of length two in the oriented 3-cycle Q'' are zero. These relations are given by the three zero-maps as follows:

$$\begin{array}{ccccccc} 1 & \to & 2[1] & \to & 3[2] & \cong & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \\ \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \to & 1 & \to & 2[1] & \cong & 3 \\ 3 & \to & \begin{smallmatrix}1\\2\\3\end{smallmatrix} & \to & 1 & & \end{array}$$

It is known by [**19**] that for any quiver Q' mutation equivalent to Q, there is a cluster-tilted algebra whose quiver is Q'. Moreover, by [**17**], it is unique up to isomorphism. Hence, there is a bijection between the quivers in the mutation class of an acyclic quiver Q and the isomorphism classes of cluster-tilted algebras of type Q.

When Q is a Dynkin (resp., extended Dynkin) quiver of type A, D or E (resp., $\tilde{A}$, $\tilde{D}$ or $\tilde{E}$) the corresponding cluster-tilted algebras are said to be of Dynkin type (resp., extended Dynkin type). Cluster-tilted algebras of Dynkin type have been investigated in [**21**], where it is shown that they are Schurian and moreover they can be defined by using only zero- and commutativity relations that can be extracted from their quivers in an algorithmic way. Recall that an algebra is *Schurian* if the entries of its Cartan matrix are only 0 or 1. The relations of a cluster-tilted algebra of type $\tilde{A}$ have been determined in [**3**] - they can be defined by using only zero-relations (see Section 3.1.2).

2.5. Cluster-tilted algebras of type A_n

In this section we recall the explicit description of cluster-tilted algebras of Dynkin type A, as quivers with relations. These are important objects in the definitions of cluster-tilted algebras of types $\tilde{A}$ and D.

Recall that a quiver of Dynkin type A_n is a quiver with $n \geq 1$ vertices and underlying graph the Dynkin diagram A_n. Since the orientation of the edges can be chosen arbitrarily (see Lemma 2.1.5), we will start with the following directed graph

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \cdots \longrightarrow \bullet_n \ .$$

The quivers which are mutation equivalent to A_n have been explicitly determined by Buan and Vatne in [**23**]. They can be characterised as follows.

Definition 2.5.1. The *neighbourhood* of a vertex x in a quiver Q is the full subquiver of Q on the subset of vertices consisting of x and the vertices which are targets of arrows starting at x or sources of arrows ending at x.

Proposition 2.5.2 (Buan and Vatne [23]). *Let $n \geq 2$. A quiver is mutation equivalent to A_n if and only if it has n vertices, the neighbourhood of each vertex is one of the nine depicted in Figure 2.2, and there are no cycles in its underlying graph apart from those induced by oriented cycles contained in neighbourhoods of vertices.*

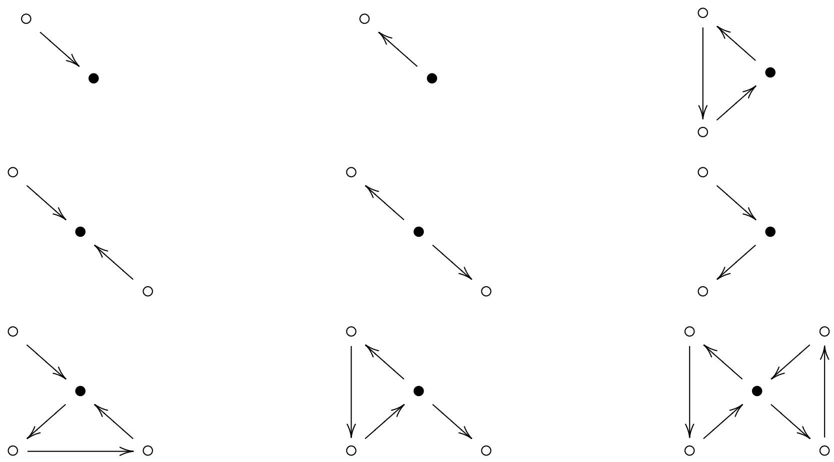
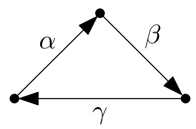


Figure 2.2. The nine possible neighbourhoods of a vertex • in a quiver which is mutation equivalent to A_n, $n \geq 2$. The three in the top row are the possible neighbourhoods of a *root* in a rooted quiver of type A.

Now we present the relations of the quivers of cluster-tilted algebras of type A_n, which were given in [**25**], and more generally for all Dynkin types in [**21**].

Remark 2.5.3. Given a quiver Q mutation equivalent to A_n, the *relations* defining the corresponding cluster-tilted algebra A_Q (which has Q as its quiver) are obtained as follows: any oriented 3-cycle



in Q gives rise to three zero-relations $\alpha\gamma$, $\beta\alpha$, $\gamma\beta$. There are no other relations.

For the definition of cluster-tilted algebras of types $\tilde{A}$ and D we need special kinds of quivers and cluster-tilted algebras of type A, whose definitions we now recall.

Definition 2.5.4. A *rooted quiver of type* A is a pair (Q, z) where Q is a quiver which is mutation equivalent to A_n for some $n \geq 1$, and z is a vertex of Q (the *root*) whose neighbourhood is one of the three appearing in the top row of Figure 2.2 if $n \geq 2$. The rooted quiver of type A is called *attached* to the root z. If the root is clear from context, we do not mention it explicitly.

Remark 2.5.5. All vertices in a rooted quiver of type A have valency at most four. Moreover, the root z has valency at most two and, if z has valency 2, then z is a vertex in an oriented 3-cycle.

CHAPTER 3

Type $\tilde{A}_n$

The results of this chapter appeared in the author's paper [**11**].

3.1. Cluster-tilted algebras of type $\tilde{A}_n$

3.1.1. Mutation classes of $\tilde{A}_n$-quivers. Quivers of type $\tilde{A}_n$ are just cycles with $n+1$ vertices. If the cycle is oriented, then we get the mutation class of D_{n+1} (see for instance Type IV in type D in [**53**] or Lemma 3.1.1 below). If the cycle is non-oriented, we get what we call the *mutation classes of $\tilde{A}_n$*.

Lemma 3.1.1. *The oriented n-cycle is mutation equivalent to D_n.*

Proof. Let Q be the following quiver of type D_n

$$\begin{tikzcd}[column sep=small, row sep=small] \bullet_1 & & & & \\ & \bullet_3 \arrow[ul] \arrow[dl] & \bullet_4 \arrow[l] & \cdots \arrow[l] & \bullet_n \arrow[l] \\ \bullet_2 & & & & \end{tikzcd} .$$

Then we get an oriented cycle by applying the following sequence of mutations: $\mu_2, \mu_3, \mu_4, \ldots, \mu_n$. $\square$

Starting with a non-oriented $(n+1)$-cycle, we first have to fix one drawing of this cycle, i.e. one embedding into the plane. Thus, we can speak of clockwise and anti-clockwise oriented arrows. But we have to consider that this notation is only unique up to reflection of the cycle, i.e. up to changing the roles of clockwise and anti-clockwise oriented arrows.

The following proposition is well-known. Since we could not find a suitable reference, we provide a proof for the convenience of the reader.

Proposition 3.1.2. *Let Q be a non-oriented cycle of length $n+1$. Let s be the number of arrows in Q which are oriented in the clockwise direction, and let r be the number of arrows in Q which are oriented in the anti-clockwise direction. Then Q is sink/source equivalent to a quiver as depicted in Figure 3.1.*

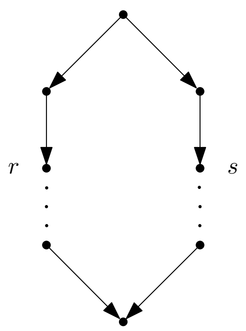
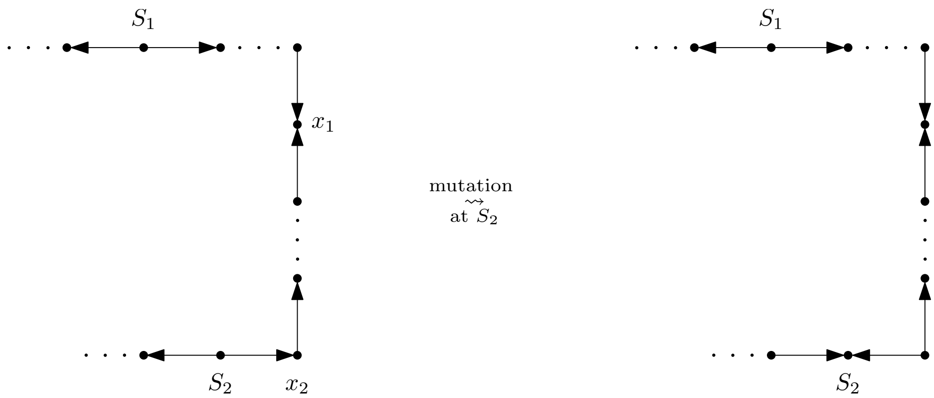


Figure 3.1

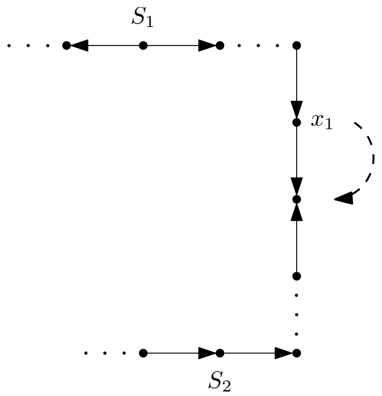
PROOF. First, we note that mutating at sinks or sources does not change the numbers of clockwise and anti-clockwise oriented arrows. Since the cycle is non-oriented there exist at least one sink and one source. Furthermore, the number of sinks equals the number of sources in Q and they appear alternately in the cycle.

We begin with an arbitrary source S_1 and move along the cycle in the clockwise direction until we find the next source S_2. If $S_2 = S_1$, then Q is of the required form.

Thus, let $S_2 \neq S_1$. Then there is one sink x_1 between S_1 and S_2. If we mutate at S_2,



the vertex x_2 which follows S_2 in the anti-clockwise direction is a new source or it is already x_1. If it is x_1, this first step is finished. If it is a new source, we mutate at this source and continue this procedure until the vertex which follows the mutated source in the anti-clockwise direction is not a source, i.e. until this vertex is the sink x_1:



Hence, the sink x_1 moves one arrow in the clockwise direction, i.e. there is one more arrow in the (oriented) path between S_1 and the new sink x_1.

In the next step, we move again from S_1 along the cycle in the clockwise direction and search for the next source. Doing this iteratively, the sink x_1 moves one arrow in the clockwise direction after each step, i.e. this procedure ends with the required form. $\square$

Thus, if two non-oriented cycles of length $n+1$ have the same parameters r and s (up to changing the roles of r and s), then they are mutation equivalent. Lemma 6.8 in [**30**] proves that the converse also holds:

Lemma 3.1.3 (Fomin, Shapiro and Thurston, Lemma 6.8 in [30]). *Let C_1 and C_2 be two non-oriented cycles, so that in C_1 (resp., C_2) there are s (resp., $\tilde{s}$) arrows oriented in the clockwise direction and r (resp., $\tilde{r}$) arrows oriented in the anti-clockwise direction. Then C_1 and C_2 are mutation equivalent if and only if the unordered pairs $\{r, s\}$ and $\{\tilde{r}, \tilde{s}\}$ coincide.*

Thus, two non-oriented cycles of length $n+1$ are mutation equivalent if and only if they have the same parameters r and s (up to changing the roles of r and s).

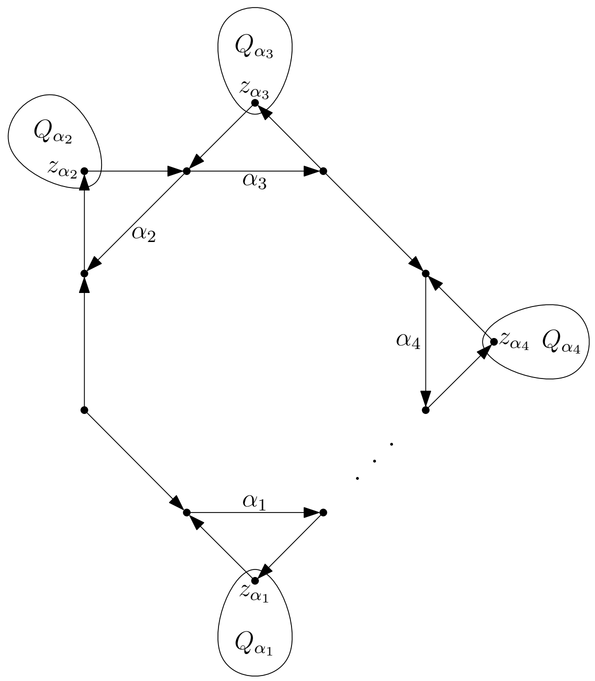


Figure 3.2. Quiver in $\mathcal{Q}_n$.

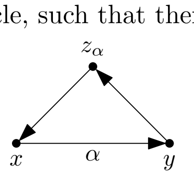
Remark 3.1.4. The 'if' part of Lemma 3.1.3 is exactly Proposition 3.1.2. The 'only if' part is a consequence of some involved results regarding cluster algebras arising from marked Riemann surfaces with triangulations. The precise technical details are beyond the scope of this thesis, the interested reader is directed to [**30**] Lemma 6.8, Theorem 13.3 and Proposition 14.1.

Next we will provide an explicit description of the set of all quivers which are mutation equivalent to any non-oriented cycle. This description is similar to the description of Type IV in type D in [**53**], see also Definition 4.1.4. At the end of this section, we will separate this class into the different mutation classes of $\tilde{A}_n$-quivers.

Definition 3.1.5. Let $\mathcal{Q}_n$ be the class of connected quivers with $n+1$ vertices which satisfy the following conditions (see Figure 3.2 for an illustration):

(i) There exists precisely one full subquiver which is a non-oriented cycle of length ≥ 2. Thus, if the length is two, it is a double arrow.

(ii) For each arrow $x \xrightarrow{\alpha} y$ in this non-oriented cycle, there may (or may not) be a vertex z_α which is not on the non-oriented cycle, such that there is an oriented 3-cycle of the form

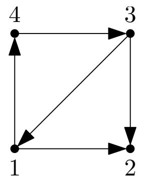


.

Apart from the arrows of these oriented 3-cycles there are no other arrows incident to vertices on the non-oriented cycle.

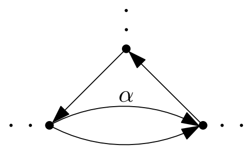
(iii) If we remove all vertices in the non-oriented cycle and their incident arrows, the result is a disjoint union of rooted quivers of type A, one for each z_α (which we call Q_α in the following).

Remark 3.1.6. There can be more than one non-oriented cycle in a quiver of $\mathcal{Q}_n$, but not as a 'full' subquiver, e.g.

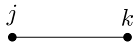


contains exactly one full non-oriented cycle $1 \to 2 \leftarrow 3 \to 1$. The cycle $1 \to 2 \leftarrow 3 \leftarrow 4 \leftarrow 1$ is also non-oriented, but it is not a full subquiver.

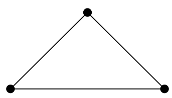
Our convention is to choose only one of the double arrows to be part of the oriented 3-cycle in the following case



and we will always choose the 'upper' arrow denoted by α.

Notation. Whenever we draw an edge 

the direction of the arrow between j and k is not important for this situation; and whenever we draw a cycle



it is an oriented 3-cycle.

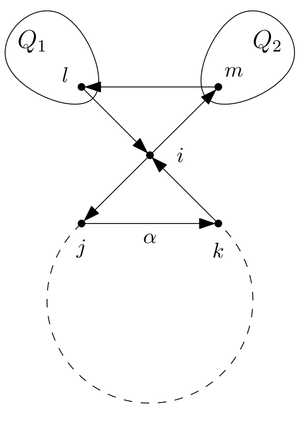
Lemma 3.1.7. *$\mathcal{Q}_n$ is closed under quiver mutation.*

Proof. Let Q be a quiver in $\mathcal{Q}_n$ and let i be some vertex. Note that the subquivers Q_1 and Q_2 highlighted in the pictures are rooted quivers of type A again.

If i is a vertex in one of the rooted quivers Q_α of type A, but not one of the roots z_α connecting this rooted quiver of type A to the rest of the quiver Q, then mutation at i leads to a quiver $\mu_i(Q) \in \mathcal{Q}_n$ since type A is closed under quiver mutation (see Proposition 2.5.2).

It therefore suffices to check what happens when we mutate at the other vertices. We will consider the following four cases.

1) Let i be one of the roots z_α, in particular not on the non-oriented cycle. For the situation where the rooted quiver Q_α of type A consists only of one vertex, we can look at the first mutated quiver in case 2) below since quiver mutation is an involution. Thus, we have the following three cases and their special cases where the non-oriented cycle is a double arrow:



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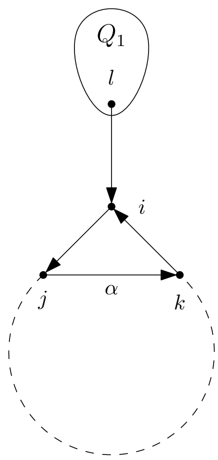

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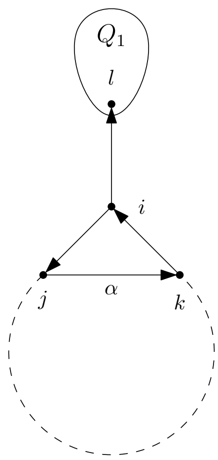

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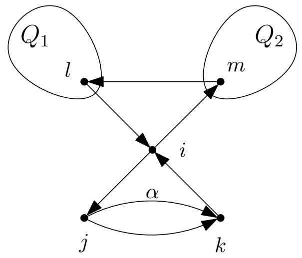

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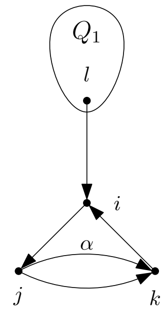

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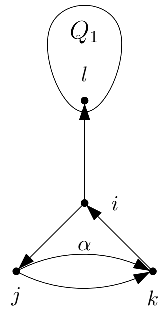
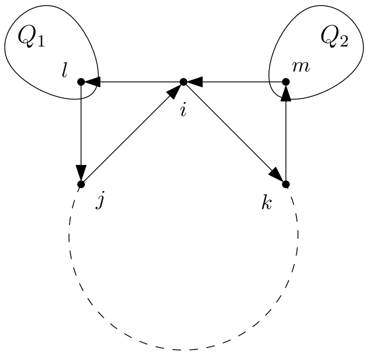


.

Then mutation at i leads to the following six quivers which have a non-oriented cycle one arrow longer than for Q, and this is indeed a non-oriented cycle since the arrows $j \to i \to k$ have the same orientation as α before.



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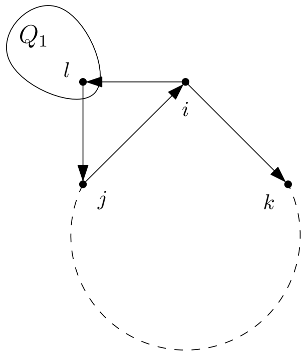

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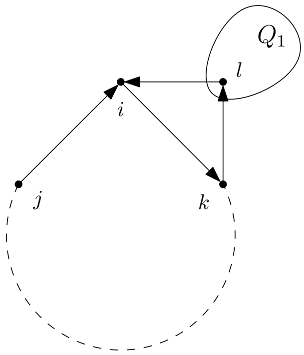

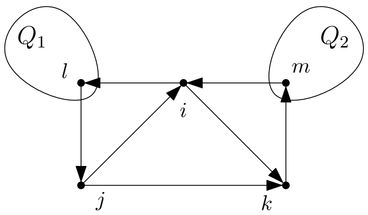

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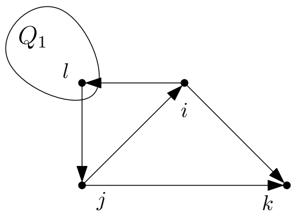

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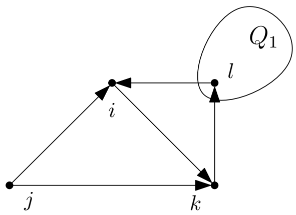
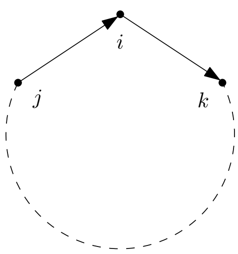


The vertices l and m have at most two incident arrows in the quivers Q_1 and Q_2 since they had at most four (resp., three) incident arrows in Q (see the description of quivers mutation equivalent to quivers of type A in Section 2.5). Furthermore, if l or m has two incident arrows in the quiver Q_1 or Q_2, then these two arrows form an oriented 3-cycle as in Q. Thus, the mutated quiver $\mu_i(Q)$ is also in $\mathcal{Q}_n$.

2) Let i be a vertex on the non-oriented cycle, and not part of any oriented 3-cycle. Then the following three cases can occur:



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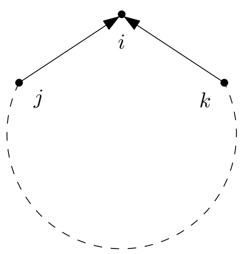

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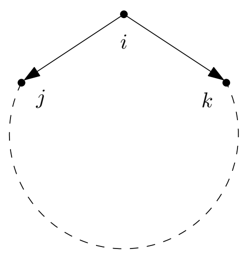

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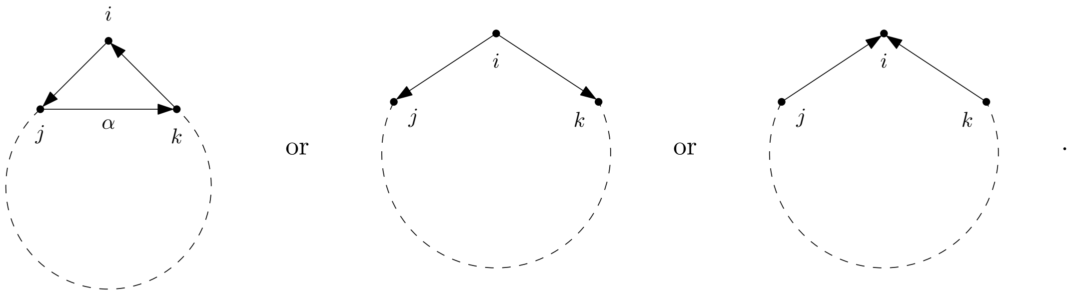
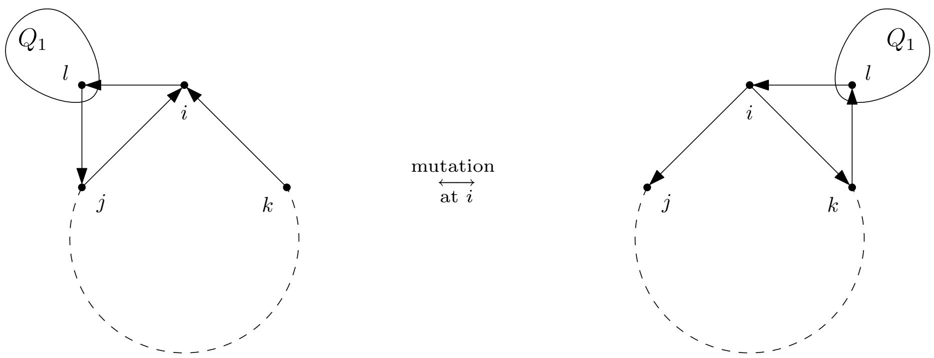
and mutation at i leads to

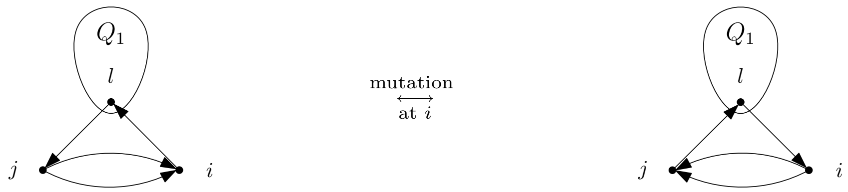


If i is a sink or a source in Q, the non-oriented cycle in $\mu_i(Q)$ is of the same length as before and $\mu_i(Q)$ is in $\mathcal{Q}_n$. If there is a path $j \to i \to k$ in Q, then the mutation at i leads to a quiver $\mu_i(Q)$ which has a non-oriented cycle one arrow shorter than in Q.

Note that in this case the non-oriented cycle in Q consists of at least three arrows and, thus, the non-oriented cycle in $\mu_i(Q)$ has at least two arrows. Thus, the mutated quiver $\mu_i(Q)$ is also in $\mathcal{Q}_n$.

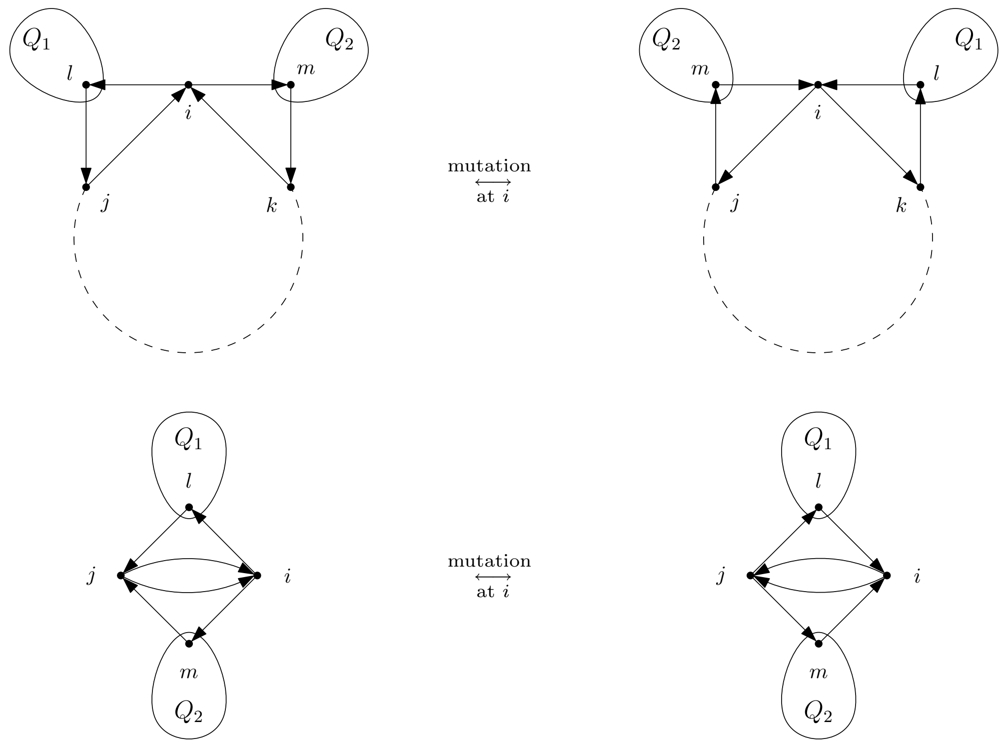
3) Let i be a vertex on the non-oriented cycle which is part of exactly one oriented 3-cycle. Then four cases can occur, but two of them have been dealt with by the second and third mutated quiver in case 1) since quiver mutation is an involution. Thus, we only have to consider the following two situations and their special cases where the non-oriented cycle is a double arrow.





After mutating at vertex i, the non-oriented cycle has the same length as before. Moreover, l has the same number of incident arrows as before. Hence, $\mu_i(Q)$ is in $\mathcal{Q}_n$.

4) Let i be a vertex on the non-oriented cycle which is part of two oriented 3-cycles. Then three cases can occur, but one of them has been dealt with by the first mutated quiver in case 1). Thus, we only have to consider the following two situations and their special cases where the non-oriented cycle is a double arrow.



After mutating at vertex i, the non-oriented cycle has the same length as before. Moreover, l and m have the same numbers of incident arrows as before. Thus, the mutated quiver $\mu_i(Q)$ is in $\mathcal{Q}_n$. $\square$

Remark 3.1.8. It is easy to see that all orientations of a circular quiver of type $\tilde{A}_n$ are in $\mathcal{Q}_n$ (except the oriented case; but this leads to the mutation class of D_{n+1}, see Lemma 3.1.1). Since $\mathcal{Q}_n$ is closed under quiver mutation, every quiver which is mutation equivalent to some quiver of type $\tilde{A}_n$ is in $\mathcal{Q}_n$ too.

Now we fix one drawing of a quiver $Q \in \mathcal{Q}_n$, i.e. one embedding into the plane, without arrow-crossing. Thus, we can again speak of clockwise and anti-clockwise oriented arrows of the non-oriented cycle. But we have to consider that this notation is only unique up to reflection of the non-oriented cycle, i.e. up to changing the roles of clockwise and anti-clockwise oriented arrows. We define four parameters $r_1,\ r_2,\ s_1$ and s_2 for a quiver $Q \in \mathcal{Q}_n$ as follows:

Definition 3.1.9. Let r_1 be the number of arrows which are not part of any oriented 3-cycle and which fulfil one of the following two conditions:

(1) These arrows are part of the non-oriented cycle and they are oriented in the anti-clockwise direction, see the left hand picture of Figure 3.3.
(2) These arrows are not part of the non-oriented cycle, but they are part of a rooted quiver Q_α of type A and the corresponding arrow α is oriented in the anti-clockwise direction, see the right hand picture of Figure 3.3.

Let r_2 be the number of oriented 3-cycles which fulfil one of the following two conditions:

(1) These cycles share one arrow α with the non-oriented cycle and α is oriented in the anti-clockwise direction, see the left hand picture of Figure 3.4.
(2) These cycles are part of a rooted quiver Q_α of type A and the corresponding arrow α is oriented in the anti-clockwise direction, see the right hand picture of Figure 3.4.

Similarly, we define the parameters s_1 and s_2 with 'clockwise' instead of 'anti-clockwise'.

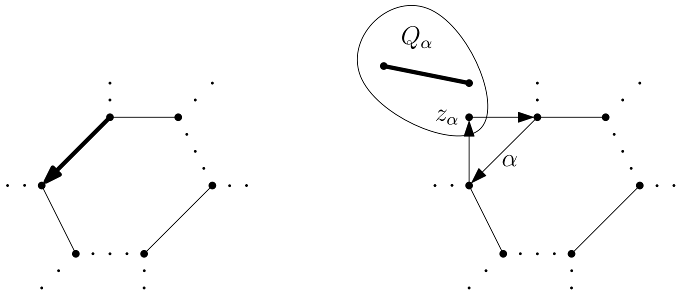


Figure 3.3. Illustration for the parameter r_1.

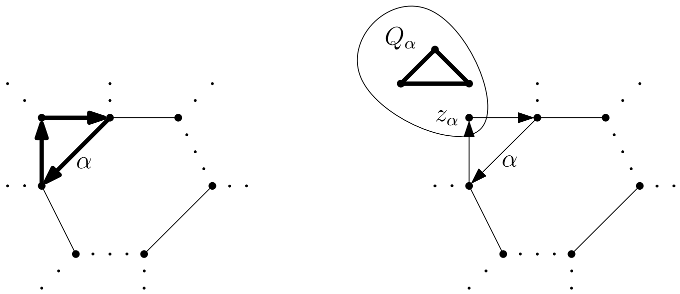
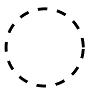
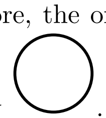
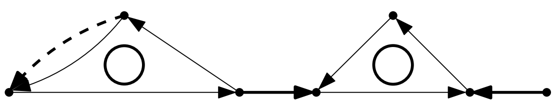


Figure 3.4. Illustration for the parameter r_2.

Example 3.1.10. We denote the arrows which count for the parameter r_1 by and the arrows which count for s_1 by . Furthermore, the oriented 3-cycles of r_2 are denoted by  and the oriented 3-cycles of s_2 are denoted by .

(i) Consider the following quiver $Q_1 \in \mathcal{Q}_6$,

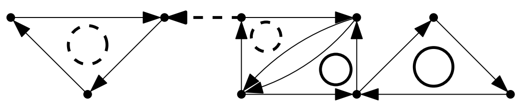


.

Here, we have $r_1 = 1,\ r_2 = 0,\ s_1 = 2$ and $s_2 = 2$.

Note that this example illustrates the affect of the choice made between the two arrows occurring in the double arrow. Taking the other arrow would interchange the roles of r_i and s_i.

(ii) Consider the quiver $Q_2 \in \mathcal{Q}_8$ of the following form



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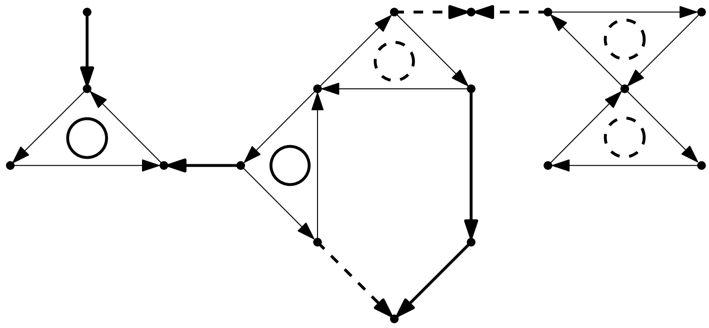
Now, we have $r_1 = 1,\ r_2 = 2,\ s_1 = 0$ and $s_2 = 2$.

(iii) The last quiver $Q_3 \in \mathcal{Q}_{16}$ is of the following form



and we have $r_1 = 3,\ r_2 = 3,\ s_1 = 4$ and $s_2 = 2$.

Lemma 3.1.11. *If Q_1 and Q_2 are quivers in $\mathcal{Q}_n$, and Q_1 and Q_2 have the same parameters $r_1,\ r_2,\ s_1$ and s_2 (up to changing the roles of r_i and s_i, $i \in \{1, 2\}$), then Q_2 can be obtained from Q_1 by iterated mutation, where all the intermediate quivers have the same parameters as well.*

Proof. It is enough to show that all quivers in $\mathcal{Q}_n$ with parameters $r_1,\ r_2,\ s_1$ and s_2 can be mutated to a quiver in *normal form*, see Figure 3.5, without changing the parameters $r_1,\ r_2,\ s_1$ and s_2.

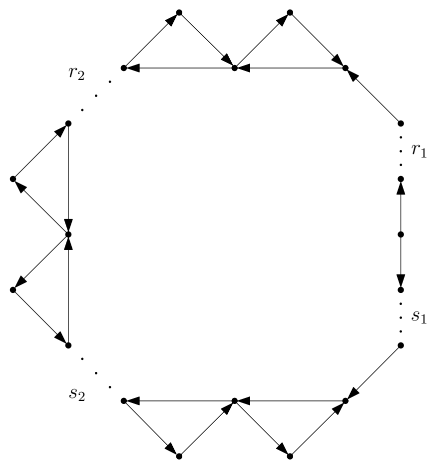


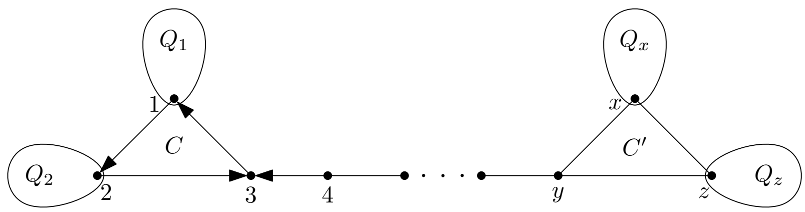
Figure 3.5. Normal form for quivers in $\mathcal{Q}_n$.

In such a quiver, r_1 is the number of arrows in the non-oriented cycle which are not part of an oriented 3-cycle and which are oriented in the anti-clockwise direction, and s_1 is the number of arrows in the non-oriented cycle which are not part of an oriented 3-cycle and which are oriented in the clockwise direction. Furthermore, r_2 is the number of oriented 3-cycles sharing one arrow α with the non-oriented cycle and α is oriented in the anti-clockwise direction and s_2 is the number of oriented 3-cycles sharing one arrow β with the non-oriented cycle and β is oriented in the clockwise direction (see Definition 3.1.9).

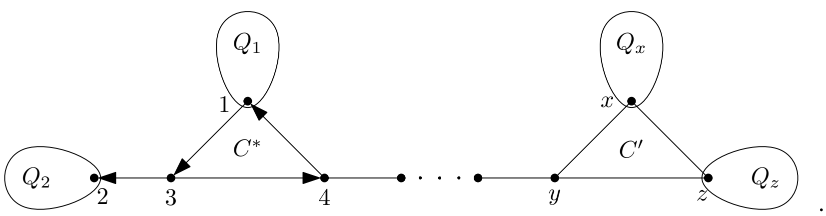
We divide this process into five steps.

Step 1: Let Q be a quiver in $\mathcal{Q}_n$. We move all oriented 3-cycles of Q which are part of a rooted quiver of type A towards the oriented 3-cycle which is attached to them and which shares one arrow with the non-oriented cycle.

Method: Let C and C' be a pair of neighbouring oriented 3-cycles in Q (i.e. no arrow in the (possibly non-oriented) path between them is part of an oriented 3-cycle) such that the length of the unique minimal (possibly non-oriented) path between them is at least one. By a 'non-oriented path' we mean a sequence of arrows $\alpha_l, \ldots, \alpha_1$, where we do not require that $s(\alpha_{i+1}) = t(\alpha_i)$ for all $i \in \{1, \ldots, l-1\}$.
We want to move C and C' closer together by mutation.



In the picture the Q_i are subquivers of Q. Note that the arguments for a quiver with arrow $3 \to 4$ are analogous and that these mutations can also be used if the arrows between 3 and y are part of the non-oriented cycle (see Step 4).
Mutating at vertex 3 will produce a quiver $\mu_3(Q)$ which looks as follows:



Thus, the length of the (possibly non-oriented) path between C^* and C' is one less than the length of the (possibly non-oriented) path between C and C' and there is a path of length one between C^* and Q_2.

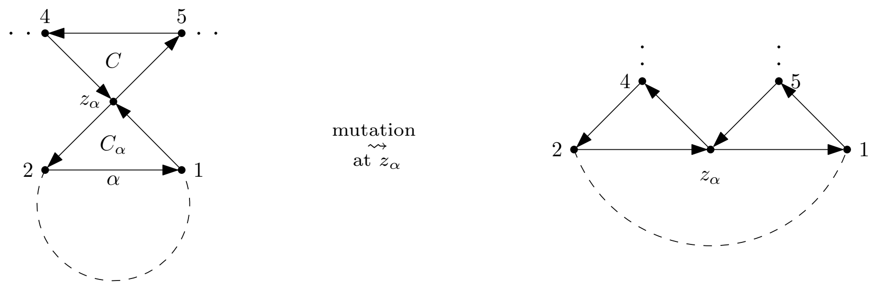
In the situation of Step 1, i.e. if C is part of a rooted quiver Q_α of type A, the parameters r_1, r_2, s_1 and s_2 are left unchanged since mutation at vertex 3 does not change the numbers of arrows and oriented 3-cycles in Q_α.

If the arrows between 3 and y are part of the non-oriented cycle in Q, i.e. we are in the situation of Step 4, we observe the following: suppose that the arrow $2 \to 3$ is part of the non-oriented cycle in Q. Then mutation at vertex 3 just changes the order of the arrows in the non-oriented cycle, but not the number of clockwise and anti-clockwise oriented arrows. Thus, the parameters are left unchanged. Now suppose that the arrow $3 \to 1$ is part of the non-oriented cycle in Q. Then mutation at vertex 3 moves the arrow $4 \to 3$ into the rooted quiver of type A attached to C^*. However, the parameters remain unchanged since $4 \to 3$ and $3 \to 1$ have the same orientation in the non-oriented cycle of Q.

Thus, after Step 1, we get a quiver where all oriented 3-cycles of the rooted quivers of type A are close to the non-oriented cycle, with no arrow interfering.

Step 2: We move all oriented 3-cycles onto the non-oriented cycle.

Method: Let C be an oriented 3-cycle which shares the root z_α with an oriented 3-cycle C_α sharing an arrow α with the non-oriented cycle. Then we mutate at the vertex z_α:



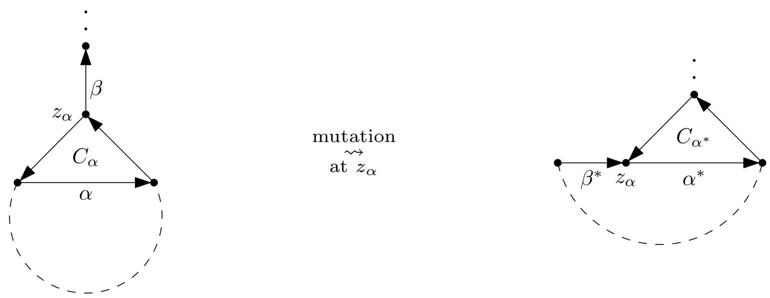
Hence, both of the oriented 3-cycles share one arrow with the non-oriented cycle and these arrows are oriented as α before. Thus, the parameters r_1, r_2, s_1 and s_2 are left unchanged. Furthermore, the length of the non-oriented cycle increases by 1.

By iterated mutation of this kind, we produce a quiver where all the oriented 3-cycles share an arrow with the non-oriented cycle and the rooted quivers of type A are just quivers with underlying graph a Dynkin diagram of type A.

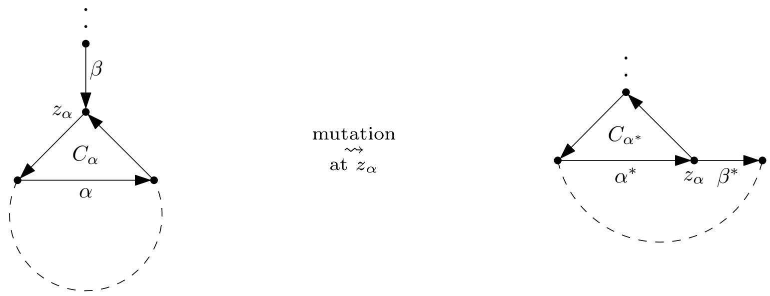
Step 3: We move all arrows onto the non-oriented cycle.

Method: This is a similar process as in Step 2: let C_α be an oriented 3-cycle which shares an arrow α with the non-oriented cycle. All arrows β attached to C_α can be moved into the non-oriented cycle by iteratively mutating at the root z_α. After mutating, all these arrows have the same orientation as α in the non-oriented cycle (see the pictures below). Thus, the parameters $r_1,\ r_2,\ s_1$ and s_2 are left unchanged.

a)



b)

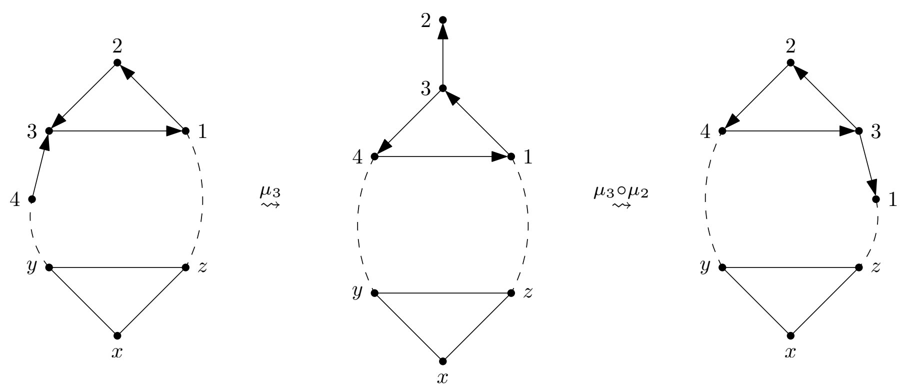


Hence, we produce a quiver where all the rooted quivers of type A consist of just a single vertex.

Step 4: Move oriented 3-cycles along the non-oriented cycle to get the form of Figure 3.6.

Method: First, we number all oriented 3-cycles by $C_1, \dots, C_{r_2+s_2}$ in such a way that C_{i+1} follows C_i in the anti-clockwise direction. As in Step 1 (assuming that the arrows between 3 and y are part of the non-oriented cycle), we can move an oriented 3-cycle C_i towards C_{i+1}, without changing the orientation of the arrows, i.e. without changing the parameters $r_1,\ r_2,\ s_1$ and s_2.

Note that if the arrow $3 \to 1$ (in the first picture of Step 1) is part of the non-oriented cycle, then mutation at vertex 3 moves one arrow into a rooted quiver of type A, i.e. this arrow is no longer part of the non-oriented cycle. However, we can reverse its direction by mutating at a sink or a source, respectively, and insert this arrow into the non-oriented cycle again by mutation as in Step 3:



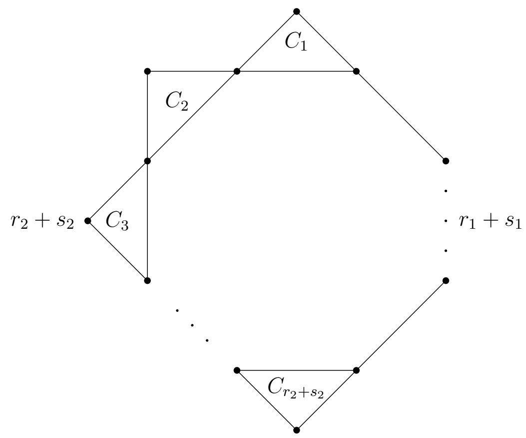


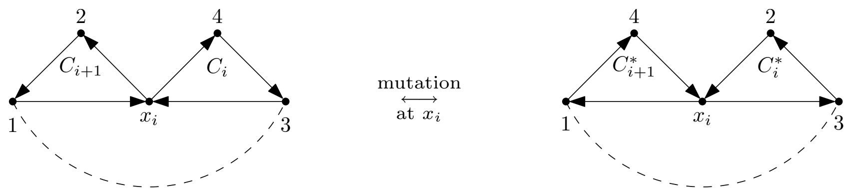
Figure 3.6. Normal form of Step 4.

Doing this iteratively, we produce a quiver as in Figure 3.6, with $r_1 + s_1$ arrows which are not part of any oriented 3-cycle and $r_2 + s_2$ oriented 3-cycles sharing one arrow with the non-oriented cycle.

Step 5: Changing the orientation on the non-oriented cycle to the orientation of Figure 3.5.

Method: The part of the non-oriented cycle without oriented 3-cycles can be moved to the desired orientation of Figure 3.5 via sink/source mutations, without mutating at the 'end' vertices which are attached to oriented 3-cycles. In this process, the parameters r_1 and s_1 are left unchanged.

Each oriented 3-cycle shares one arrow with the non-oriented cycle. If all of these arrows are oriented in the same direction, the quiver is in the required form (with $r_2 = 0$ or $s_2 = 0$). Thus, we can assume that there are at least two arrows of the non-oriented cycle which are part of two neighbouring oriented 3-cycles C_i and C_{i+1}, respectively, and these arrows are oriented in opposite directions. If we mutate at the connecting vertex of C_i and C_{i+1}, the directions of these arrows are changed:

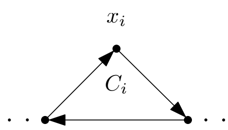


Hence, these mutations act like sink/source mutations at the non-oriented cycle and the parameters r_2 and s_2 are left unchanged. Thus, we can mutate at such connecting vertices as in the part without oriented 3-cycles to reach the desired orientation of Figure 3.5.

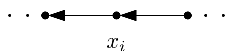
□

Theorem 3.1.12. *Let $Q \in \mathcal{Q}_n$ with parameters $r_1,\ r_2,\ s_1$ and s_2. Then Q is mutation equivalent to a non-oriented cycle of length $n+1$ with parameters $r = r_1 + 2r_2$ and $s = s_1 + 2s_2$.*

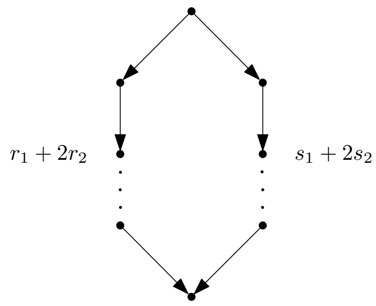
Proof. We can assume that Q is in normal form (see Lemma 3.1.11) and we label the root vertices as in Figure 3.7. Mutation at the vertex x_i of an oriented 3-cycle



leads to two arrows of the following form



Thus, after mutating at all the x_i, the parameter r_2 is zero and we get $r := r_1 + 2r_2$ arrows oriented anti-clockwise. Similarly, we get $s := s_1 + 2s_2$ arrows oriented clockwise. Hence, mutating at all the x_i and y_i leads to a quiver with underlying graph $\tilde{A}_n$ as follows:



.

Since there is a non-oriented cycle in every $Q \in \mathcal{Q}_n$, these parameters r and s are non-zero. Thus, the cycle above is also non-oriented. Hence, Q is mutation equivalent to a non-oriented cycle with $r = r_1 + 2r_2$ arrows oriented anti-clockwise and $s = s_1 + 2s_2$ arrows oriented clockwise. $\square$

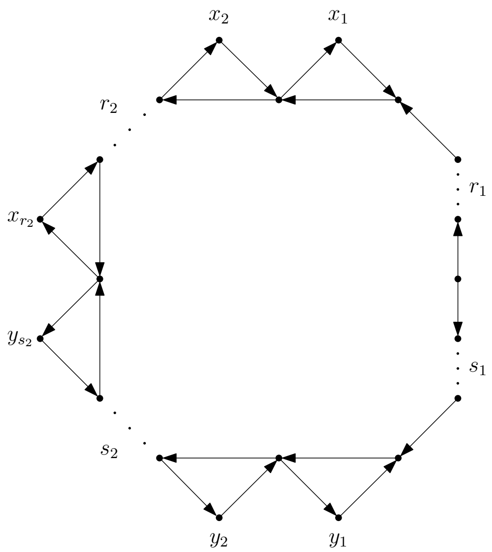


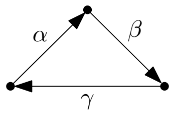
Figure 3.7. Normal form of a quiver in $\mathcal{Q}_n$.

Theorem 3.1.13. *Let $Q_1, Q_2 \in \mathcal{Q}_n$ with parameters $r_1,\ r_2,\ s_1$ and s_2, respectively $\tilde{r}_1,\ \tilde{r}_2,\ \tilde{s}_1$ and $\tilde{s}_2$. Then Q_1 is mutation equivalent to Q_2 if and only if $r_1 + 2r_2 = \tilde{r}_1 + 2\tilde{r}_2$ and $s_1 + 2s_2 = \tilde{s}_1 + 2\tilde{s}_2$ (or $r_1 + 2r_2 = \tilde{s}_1 + 2\tilde{s}_2$ and $s_1 + 2s_2 = \tilde{r}_1 + 2\tilde{r}_2$).*

Proof. The 'if' part follows from Theorem 3.1.12. The 'only-if' part follows from Theorem 3.1.12 and Lemma 3.1.3. $\square$

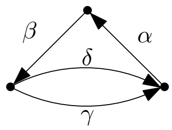
3.1.2. Relations for cluster-tilted algebras of type $\tilde{A}_n$. In general, cluster-tilted algebras are defined as endomorphism algebras of cluster-tilting objects in a cluster category [**20**] (see Section 2.4). However, a cluster-tilted algebra of type $\tilde{A}_n$ can be constructed explicitly by a quiver with relations KQ/I, where the quiver Q is in one of the mutation classes of $\tilde{A}_n$ [**19**] and the relations are uniquely determined by its quiver [**17**].

It is known from [**3**] and [**6**] that the ideal I of relations can be generated by all paths of length two in the oriented 3-cycles, i.e. in each oriented 3-cycle



we have three zero-relations $\alpha\gamma$, $\beta\alpha$ and $\gamma\beta$.

Remark 3.1.14. According to our convention after Remark 3.1.6 there are only three zero-relations in the following quiver



and here, these are $\alpha\delta$, $\beta\alpha$ and $\delta\beta$.

Thus, a cluster-tilted algebra of type $\tilde{A}_n$ is gentle. We recall the definition of gentle algebras:

Definition 3.1.15. We call $A = KQ/I$ a *special biserial algebra* if the following properties hold:

(1) Each vertex of Q is the starting point of at most two arrows and the end point of at most two arrows.
(2) For each arrow α in Q there is at most one arrow β such that $\alpha\beta \notin I$, and at most one arrow γ such that $\gamma\alpha \notin I$.

A is *gentle* if moreover:

(3) The ideal I is generated by paths of length 2.
(4) For each arrow α in Q there is at most one arrow β' with $t(\alpha) = s(\beta')$ such that $\beta'\alpha \in I$, and there is at most one arrow γ' with $t(\gamma') = s(\alpha)$ such that $\alpha\gamma' \in I$.

3.2. Derived equivalence classification of cluster-tilted algebras of type $\tilde{A}_n$

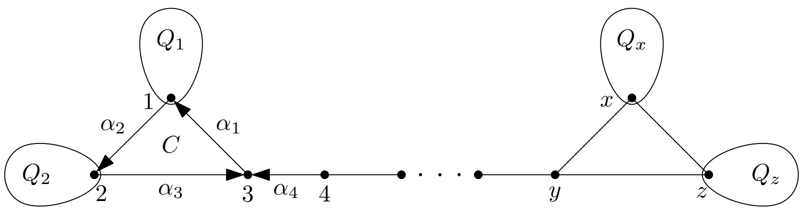
In this section we provide a complete classification of cluster-tilted algebras of type $\tilde{A}_n$ up to derived equivalence.

Lemma 3.2.1. *Let $A = KQ/I$ be a cluster-tilted algebra of type $\tilde{A}_n$. Let r_1, r_2, s_1 and s_2 be the parameters of Q which are defined in Definition 3.1.9. Then A is derived equivalent to a cluster-tilted algebra corresponding to a quiver in normal form as in Figure 3.5.*

Proof. First note that the Cartan determinant of a cluster-tilted algebra A of type $\tilde{A}_n$ is $\det C_A = 2^{r_2+s_2}$ (see [**36**, Theorem 1]). It follows that the number of oriented 3-cycles $r_2 + s_2$ is invariant under derived equivalence. [**10**, Proposition B] implies that the number of arrows is invariant under derived equivalence, i.e. the number $r_1 + s_1$ is also an invariant. Later, we shall show in the proof of Theorem 3.2.2 that even the parameters r_1, r_2, s_1 and s_2 themselves are invariants of derived equivalence.

Our strategy in this proof is to go through the proof of Lemma 3.1.11 and define a tilting complex for each mutation in the five steps. We show that if we mutate at some vertex k of the quiver Q and obtain a quiver $\mu_k(Q)$, then the two corresponding cluster-tilted algebras are derived equivalent.

Step 1 Let A be a cluster-tilted algebra of type $\tilde{A}_n$. We suppose that there is at least one oriented 3-cycle C in a rooted quiver of type A. Then we mutate at vertex 3 in this rooted quiver of type A as in the following situation



.

For a quiver with arrow $3 \to 4$ we refer to the end of this step.

Using the relations described in Section 3.1.2 we can compute the (partial) Cartan matrix of A to be $\begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

Since we deal with left modules and read paths from right to left, a non-zero path from vertex i to j gives a homomorphism $P_j \to P_i$ by right multiplication where $P_i = Ae_i$ are the indecomposable projective A-modules. Hence, two arrows $\alpha : i \to j$ and $\beta : j \to k$ give a path $\beta\alpha$ from i to k and the corresponding homomorphism $P_k \to P_i$ is also denoted by $\beta\alpha$.

In the situation above, we have homomorphisms $P_3 \xrightarrow{\alpha_3} P_2$ and $P_3 \xrightarrow{\alpha_4} P_4$ and we can define a bounded complex of projective A-modules $T = \bigoplus_{i=1}^{n+1} T_i$, where $T_i : 0 \to P_i \to 0$, $i \in \{1, 2, 4, \dots, n+1\}$, are complexes concentrated in degree zero and $T_3 : 0 \to P_3 \xrightarrow{(\alpha_3,\alpha_4)} P_2 \oplus P_4 \to 0$ is a complex concentrated in degrees -1 and 0.

Now we want to show that T is a tilting complex. Since we can show as in Example 2.2.6 that the second condition of Definition 2.2.5 is always fulfilled for all such two-term complexes of indecomposable projective modules we need, it suffices to prove the first condition. That is, we show that $\mathrm{Hom}_{K^b(A)}(T, T[i]) = 0$ for $i \neq 0$ since this then also holds in the localised category $D^b(A)$.

Since condition (i) is obvious for all $|i| \geq 2$ we begin with possible maps $T_3 \to T_3[1]$ and $T_3 \to T_3[-1]$,

$$\begin{array}{ccccccccc} & & 0 & \to & P_3 & \xrightarrow{(\alpha_3,\alpha_4)} & P_2 \oplus P_4 & \to & 0 \\ & & & & \downarrow \psi & & & & \\ 0 & \to & P_3 & \xrightarrow{(\alpha_3,\alpha_4)} & P_2 \oplus P_4 & \to & 0 & & \\ & & & & \downarrow 0 & & & & \\ & & 0 & \to & P_3 & \xrightarrow{(\alpha_3,\alpha_4)} & P_2 \oplus P_4 & \to & 0 \end{array}$$

where $\psi \in \mathrm{Hom}(P_3, P_2 \oplus P_4)$ and $(\alpha_3, 0)$, $(0, \alpha_4)$ is a basis of this space.
The first homomorphism ψ is homotopic to zero. In the second case there is no non-zero homomorphism $P_2 \oplus P_4 \to P_3$ (as we can see in the Cartan matrix of A).

Now consider possible maps $T_3 \to T_j[1]$, $j \neq 3$. These maps are given by a map of complexes as follows

$$\begin{array}{ccccccc} 0 & \to & P_3 & \xrightarrow{(\alpha_3,\alpha_4)} & P_2 \oplus P_4 & \to & 0 \\ & & \downarrow & & & & \\ 0 & \to & Q & \to 0 & & & \end{array}$$

where Q can be any P_j such that there is a non-zero path from vertex j to vertex 3. There exist non-zero homomorphisms of complexes, but they are all homotopic to zero since every path from vertex j to vertex 3 ends with α_3 or α_4. It follows that every homomorphism from $P_3 \to Q$ can be factored through the map (α_3, α_4) : $P_3 \to P_2 \oplus P_4$. Immediately from the definition we see that $\mathrm{Hom}(T, T_j[1]) = 0$ for $j \neq 3$ and thus we have shown that $\mathrm{Hom}(T, T[1]) = 0$.

Finally, we have to consider maps $T_j \to T_3[-1]$ for $j \neq 3$. These are given as follows

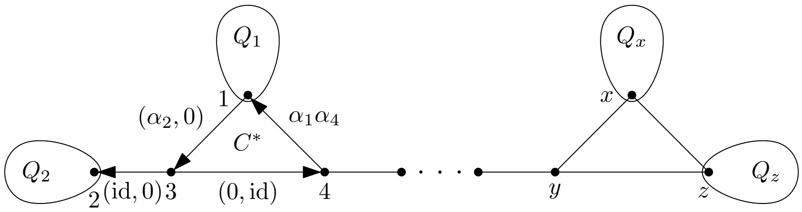
$$\begin{array}{ccccccc} 0 & \to & Q & \to 0 & & & \\ & & \downarrow & & & & \\ 0 & \to & P_3 & \xrightarrow{(\alpha_3,\alpha_4)} & P_2 \oplus P_4 & \to & 0 \end{array}$$

where Q can be any P_j such that there is a non-zero path from vertex 3 to vertex j. However, no non-zero map can be zero when composed with both α_3 and α_4 since the path $\alpha_1\alpha_4$ is not a zero-relation. So the only homomorphism of complexes $T_j \to T_3[-1]$, $j \neq 3$, is the zero map.

It follows that $\mathrm{Hom}_{K^b(A)}(T, T[i]) = 0$ in the homotopy category and then this also holds in the derived category $D^b(A)$. Hence, T is indeed a tilting complex for A.

By Rickard's Theorem 2.2.7, $E := \mathrm{End}_{D^b(A)}(T)$ is derived equivalent to A. Using the alternating sum formula of Happel's Proposition 2.3.2 we can compute the Cartan matrix of E to be $\begin{pmatrix} 1 & 1 & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ 1 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

We define homomorphisms in E as follows



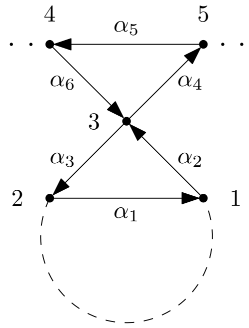
.

Now we have to check the relations, up to homotopy. Clearly, the homomorphism $(\alpha_2\alpha_1\alpha_4, 0)$ in the oriented 3-cycle C^* containing the vertices $1, 3$ and 4 is zero since $\alpha_2\alpha_1$ was zero in A. Furthermore, the composition of $(\alpha_2, 0)$ and $(0, \mathrm{id})$ yields a zero-relation. The last zero-relation in this oriented 3-cycle is the composition of $(0, \mathrm{id})$ and $\alpha_1\alpha_4$: this homomorphism is homotopic to zero

$$\begin{tikzcd} & 0 \arrow[r] & P_1 \arrow[r] \arrow[dl, "\alpha_1"'] \arrow[d, "{(0, \alpha_1\alpha_4)}"] & 0 \\ 0 \arrow[r] & P_3 \arrow[r, "{(\alpha_3, \alpha_4)}"'] & P_2 \oplus P_4 \arrow[r] & 0 \end{tikzcd}$$

since $\alpha_1\alpha_3 = 0$ in A. The relations in all the other oriented 3-cycles of this quiver are the same as in the quiver of A. Thus, we have defined homomorphisms between the summands of T corresponding to the arrows of the quiver which we obtain after mutating at vertex 3 in the quiver of A. We have shown that they satisfy the defining relations of this algebra and the Cartan matrices agree. Thus, A is derived equivalent to E, where the quiver of E is the same as the quiver we obtain after mutating at vertex 3 in the quiver of A. Furthermore, A^{op} is derived equivalent to E^{op} (see Proposition 2.2.8) and this proves the claim in the situation with an arrow $3 \to 4$ in Step 1 of Lemma 3.1.11.

Step 2 Let A be a cluster-tilted algebra of type $\tilde{A}_n$ with corresponding quiver



where the dashed line indicates the non-oriented cycle.

If the arrows between the vertices 2 and 1 along the dashed line of the non-oriented cycle are not all oriented in the same direction, then the (partial) Cartan matrix of A is $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 1 & \dots \\ 0 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

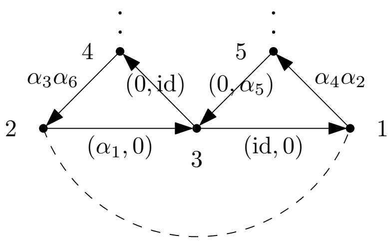
If the arrows between the vertices 2 and 1 along the dashed line of the non-oriented cycle are all oriented in the same direction, i.e. there is a path $2 \rightsquigarrow 1$, then the (partial) Cartan matrix of A is $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & \dots \\ 2 & 1 & 1 & 0 & 1 & \dots \\ 1 & 1 & 2 & 0 & 1 & \dots \\ 1 & 1 & 2 & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

Now let $T = \bigoplus_{i=1}^{n+1} T_i$ be a bounded complex of projective A-modules, where $T_i : 0 \to P_i \to 0,\ i \in \{1, 2, 4, \dots, n+1\}$, are complexes concentrated in degree zero and $T_3 : 0 \to P_3 \xrightarrow{(\alpha_2,\alpha_6)} P_1 \oplus P_4 \to 0$ is a complex concentrated in degrees -1 and 0. We leave it to the reader to verify that this is indeed a tilting complex.

By Theorem 2.2.7 of Rickard, $E := \mathrm{End}_{D^b(A)}(T)$ is derived equivalent to A. Using the alternating sum formula of Happel's Proposition 2.3.2 we can compute the (partial) Cartan matrices of E to be $\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & \dots \\ 1 & 1 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ in the first case and

$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & \dots \\ 2 & 1 & 1 & 0 & 1 & \dots \\ 1 & 0 & 1 & 1 & 0 & \dots \\ 1 & 1 & 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ in the second case.

Then we define homomorphisms in E as follows

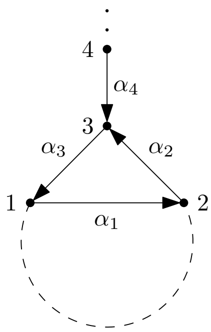


.

Now we have to check the relations, up to homotopy. Clearly, the homomorphisms $(\alpha_1\alpha_3\alpha_6,0)$ and $(0,\alpha_5\alpha_4\alpha_2)$ are zero since $\alpha_1\alpha_3$ and $\alpha_5\alpha_4$ were zero in A. Additionally, the compositions of $(\alpha_1,0)$ and $(0,\mathrm{id})$ and of $(0,\alpha_5)$ and $(\mathrm{id},0)$ yield zero-relations. The path from vertex 3 to vertex 2 is zero since $\alpha_3\alpha_2=0$ and thus, $(0,\alpha_3\alpha_6)$ is homotopic to zero. Similarly, the path from vertex 3 to vertex 5 is zero since $\alpha_4\alpha_6=0$ and hence, $(\alpha_4\alpha_2,0)$ is homotopic to zero. The relations in all the other oriented 3-cycles of this quiver are the same as in the quiver of A.

Hence, A is derived equivalent to E, where the quiver of E is the same as the quiver we obtain after mutating at vertex 3. Moreover, by Proposition 2.2.8, A^{op} is derived equivalent to E^{op}.

Step 3 Let A be a cluster-tilted algebra of type $\tilde{A}_n$ with corresponding quiver



where the dashed line indicates the non-oriented cycle. The other case with an arrow $3\to 4$ can be dealt with Proposition 2.2.8 using opposite algebras.

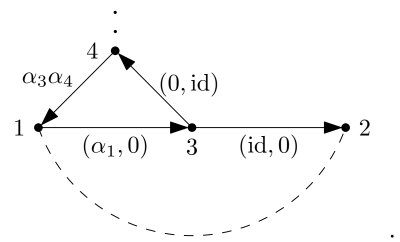
If the arrows between the vertices 1 and 2 along the dashed line of the non-oriented cycle are not all oriented in the same direction, then the (partial) Cartan matrix of A is $\begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

If the arrows between the vertices 1 and 2 along the dashed line are all oriented in the same direction, i.e. there is a path $1 \rightsquigarrow 2$, then the (partial) Cartan matrix of A is $\begin{pmatrix} 1 & 2 & 1 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 1 & 1 & 2 & 0 & \dots \\ 1 & 1 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

We define a tilting complex T as follows: let $T=\bigoplus_{i=1}^{n+1} T_i$ be a bounded complex of projective A-modules, where $T_i: 0\to P_i\to 0,\ i\in\{1,2,4,\dots,n+1\}$, are complexes concentrated in degree zero and $T_3: 0\to P_3 \xrightarrow{(\alpha_2,\alpha_4)} P_2\oplus P_4\to 0$ is a complex concentrated in degrees -1 and 0.

By Theorem 2.2.7, $E := \mathrm{End}_{D^b(A)}(T)$ is derived equivalent to A. Using the alternating sum formula of Happel's Proposition we can compute the (partial) Cartan matrices of E to be $\begin{pmatrix} 1 & 1 & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ 1 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ in the first case and $\begin{pmatrix} 1 & 2 & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ 1 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ in the second case.

Then we define homomorphisms in E as follows



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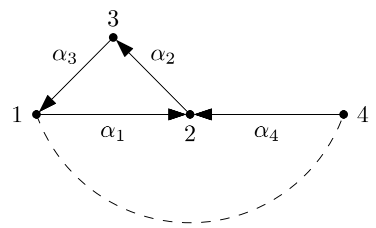
We leave it to the reader to verify that these homomorphisms satisfy the defining relations of the corresponding cluster-tilted algebra.

Hence, A is derived equivalent to E, where the quiver of E is the same as the quiver we obtain after mutating at vertex 3. Additionally, A^{op} is derived equivalent to E^{op} by Proposition 2.2.8.

Step 4 Let A be a cluster-tilted algebra of type $\tilde{A}_n$ with corresponding quiver



where the dashed line indicates the non-oriented cycle. The other case with an arrow $2 \to 4$ has been dealt with by Step 3 in this proof.

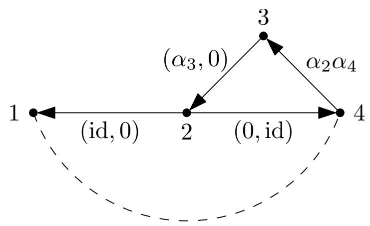
If the arrows between the vertices 1 and 4 along the dashed line are not all oriented in the same direction, then the (partial) Cartan matrix of A is $\begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

If the arrows between the vertices 1 and 4 along the dashed line are all oriented in the same direction, then we consider the following two cases: if there is a path $1 \rightsquigarrow 4$, then the (partial) Cartan matrix of A is $\begin{pmatrix} 1 & 2 & 1 & 1 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 1 & 1 & 2 & 1 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ and if there is a path $4 \rightsquigarrow 1$, then the (partial) Cartan matrix of A is $\begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 1 & 2 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

We define a tilting complex $T = \bigoplus_{i=1}^{n+1} T_i$ where $T_i : 0 \to P_i \to 0$, $i \in \{1, 3, 4, \ldots, n+1\}$, are concentrated in degree zero and $T_2 : 0 \to P_2 \xrightarrow{(\alpha_1,\alpha_4)} P_1 \oplus P_4 \to 0$ is concentrated in degrees -1 and 0.

By Theorem 2.2.7, $E := \mathrm{End}_{D^b(A)}(T)$ is derived equivalent to A. Using the alternating sum formula of Happel's Proposition we can compute the (partial) Cartan matrices of E to be $\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 1 & \ldots \\ 1 & 1 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ in the first case, $\begin{pmatrix} 1 & 0 & 1 & 1 & \ldots \\ 1 & 1 & 1 & 2 & \ldots \\ 1 & 1 & 2 & 1 & \ldots \\ 0 & 0 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ in the second case and $\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 1 & \ldots \\ 1 & 1 & 1 & 0 & \ldots \\ 1 & 0 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ in the third case.

Then we define homomorphisms in E as follows



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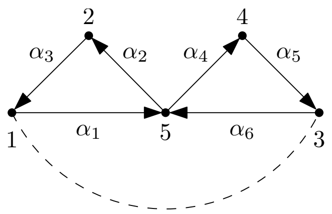
We leave it to the reader to verify that these homomorphisms satisfy the defining relations of the corresponding cluster-tilted algebra.

Hence, A is derived equivalent to E, where the quiver of E is the same as the quiver we obtain after mutating at vertex 2.

Step 5 The first mutations in Step 5 are sink/source mutations. These mutations correspond to APR-tilting and this is a well-known case (see [**8**]).

Now let A be a cluster-tilted algebra of type $\tilde{A}_n$ with corresponding quiver



where the dashed line indicates the non-oriented cycle.

If the arrows between the vertices 1 and 3 along the dashed line are not all oriented in the same direction, then the (partial) Cartan matrix of A is $\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 1 & \ldots \\ 0 & 0 & 1 & 1 & 0 & \ldots \\ 0 & 1 & 0 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

If the arrows between the vertices 1 and 3 along the dashed line are all oriented in the same direction, e.g. there is a path $1 \rightsquigarrow 3$, then the (partial) Cartan matrix of A is $\begin{pmatrix} 1 & 1 & 1 & 1 & 2 & \dots \\ 1 & 2 & 1 & 0 & 1 & \dots \\ 0 & 1 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

We define a tilting complex $T = \bigoplus_{i=1}^{n+1} T_i$ where $T_i : 0 \to P_i \to 0$, $i \in \{1, \dots, 4, 6, \dots, n+1\}$, are concentrated in degree zero and $T_5 : 0 \to P_5 \xrightarrow{(\alpha_1, \alpha_6)} P_1 \oplus P_3 \to 0$ is concentrated in degrees -1 and 0.

By Theorem 2.2.7, $E := \mathrm{End}_{D^b(A)}(T)$ is derived equivalent to A. Using the alternating sum formula of Happel's Proposition we can compute the (partial) Cartan matrices of E to be $\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & \dots \\ 1 & 1 & 0 & 0 & 1 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 1 & \dots \\ 1 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ in the first case and $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & \dots \\ 1 & 2 & 1 & 0 & 1 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 1 & \dots \\ 1 & 1 & 2 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ in the second case.

Then we define homomorphisms in E as follows

$$\begin{tikzcd}
 & 4 \arrow[dr, "{(0,\alpha_5)}"] & & 2 \arrow[dl, "{(\alpha_3,0)}"'] & \\
1 \arrow[ur, "\alpha_4\alpha_1"] \arrow[rrrr, dashed, dash, bend right] & & 5 \arrow[ll, "{(\mathrm{id},0)}"] \arrow[rr, "{(0,\mathrm{id})}"'] & & 3 \arrow[ul, "\alpha_2\alpha_6"']
\end{tikzcd}.$$

We leave it to the reader to verify that these homomorphisms satisfy the defining relations of the corresponding cluster-tilted algebra.

Hence, A is derived equivalent to E, where the quiver of E is the same as the quiver we obtain after mutating at vertex 5.

We have shown that we obtain a quiver of a derived equivalent cluster-tilted algebra by all mutations in the proof of Lemma 3.1.11. Hence, every cluster-tilted algebra $A = KQ/I$ of type $\tilde{A}_n$ is derived equivalent to a cluster-tilted algebra with a quiver in normal form which has the same parameters as Q. □

Our next aim is to prove the main result for the derived equivalence classification:

Theorem 3.2.2. *Two cluster-tilted algebras of type $\tilde{A}_n$ are derived equivalent if and only if their quivers have the same parameters r_1, r_2, s_1 and s_2 (up to changing the roles of r_i and s_i, $i \in \{1, 2\}$).*

First, we recall some background from [**10**]. Let $A = KQ/I$ be a gentle algebra, where $Q = (Q_0, Q_1)$ is a connected quiver. A *permitted path* of A is a non-zero path $C = \alpha_l \dots \alpha_2\alpha_1$, i.e. $\alpha_{i+1}\alpha_i \neq 0$ for all $i \in \{1, 2, \dots, l-1\}$. A maximal permitted path C is called a *non-trivial permitted thread*, i.e. for all $\beta \in Q_1$ neither $C\beta$ nor βC is a permitted path. Similarly a *forbidden path* of A is a path $\Pi = \alpha_l \dots \alpha_2\alpha_1$ formed by pairwise different arrows in Q with $\alpha_{i+1}\alpha_i = 0$ for all

$i \in \{1, 2, \ldots, l-1\}$. A maximal forbidden path Π is called a *non-trivial forbidden thread*, that is, for all $\beta \in Q_1$ neither $\Pi\beta$ nor $\beta\Pi$ is a forbidden path.

We also define trivial threads for some vertices. Let $v \in Q_0$ such that $\#\{\alpha \in Q_1 : s(\alpha) = v\} \leq 1$, $\#\{\alpha \in Q_1 : t(\alpha) = v\} \leq 1$ and if $\beta, \gamma \in Q_1$ are such that $s(\gamma) = v = t(\beta)$ then $\gamma\beta \neq 0$. Then we consider e_v, the trivial paths of length zero, as a *trivial permitted thread* in v and denote it by h_v. Let $\mathcal{H}_A$ be the set of all permitted threads of A, trivial and non-trivial. Similarly, for $v \in Q_0$ such that $\#\{\alpha \in Q_1 : s(\alpha) = v\} \leq 1$, $\#\{\alpha \in Q_1 : t(\alpha) = v\} \leq 1$ and if $\beta, \gamma \in Q_1$ are such that $s(\gamma) = v = t(\beta)$ then $\gamma\beta = 0$, we consider e_v as a *trivial forbidden thread* in v and denote it by p_v. Note that certain paths can be permitted threads and forbidden threads simultaneously, see for instance the second algebra in Example 3.2.3 below.

Example 3.2.3. Consider the following quiver corresponding to a cluster-tilted algebra of type $\tilde{A}_3$

$$\begin{tikzcd}
 & 4 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_3"] & \\
1 \arrow[rr, "\alpha_2"'] & & 2 \arrow[dl, "\beta_{1,2}"] \\
 & 3 \arrow[ul, "\beta_{1,1}"] &
\end{tikzcd}$$

with three zero-relations $\beta_{1,2}\alpha_2$, $\beta_{1,1}\beta_{1,2}$ and $\alpha_2\beta_{1,1}$. The permitted paths are the trivial paths $e_1, \ldots, e_4$, the arrows α_1, α_2, α_3, $\beta_{1,1}$, $\beta_{1,2}$ and the two paths $\alpha_2\alpha_1$ and $\beta_{1,2}\alpha_3$. It follows that the non-trivial permitted threads are $\beta_{1,1}$, $\alpha_2\alpha_1$ and $\beta_{1,2}\alpha_3$. Since there is no trivial permitted thread, $\mathcal{H}_A = \{\beta_{1,1}, \alpha_2\alpha_1, \beta_{1,2}\alpha_3\}$.

The forbidden paths are the trivial paths $e_1, \ldots, e_4$, the arrows α_1, α_2, α_3, $\beta_{1,1}$, $\beta_{1,2}$, the paths $\beta_{1,2}\alpha_2$, $\beta_{1,1}\beta_{1,2}$, $\alpha_2\beta_{1,1}$ and the three paths $\beta_{1,1}\beta_{1,2}\alpha_2$, $\alpha_2\beta_{1,1}\beta_{1,2}$, $\beta_{1,2}\alpha_2\beta_{1,1}$ for the oriented 3-cycle. These are all of them since a forbidden path has pairwise different arrows. Hence, the non-trivial forbidden threads are α_1, α_3, $\beta_{1,1}\beta_{1,2}\alpha_2$, $\alpha_2\beta_{1,1}\beta_{1,2}$ and $\beta_{1,2}\alpha_2\beta_{1,1}$.

There is one vertex $v \in Q_0$ with $\#\{\alpha \in Q_1 : s(\alpha) = v\} \leq 1$ and $\#\{\alpha \in Q_1 : t(\alpha) = v\} \leq 1$, and this is vertex 3. Additionally, there are the two arrows $\beta_{1,2}$ and $\beta_{1,1}$ with $s(\beta_{1,1}) = 3 = t(\beta_{1,2})$ and $\beta_{1,1}\beta_{1,2} = 0$. Hence, $p_3 := e_3$ is a trivial forbidden thread.

Now, consider the following quiver of type $\tilde{A}_1$

$$\begin{tikzcd}
1 \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] & 2
\end{tikzcd}.$$

Here, the two arrows α_1 and α_2 are both, non-trivial permitted threads and non-trivial forbidden threads.

From now on let $A = KQ/I$ be a cluster-tilted algebra of type $\tilde{A}_n$. By Lemma 3.2.1, we can assume that Q is in normal form with notations as in Figure 3.8 (where $r_0 := r_1 + r_2$ and $s_0 := s_1 + s_2$).

Then the set $\mathcal{H}_A$ of all permitted threads is formed by $h_{v_1}, \ldots, h_{v_{r_1-1}}, h_{v_{r_0+1}}, \ldots, h_{v_{r_0+s_1-1}}$, $\gamma_{s_2,2}\alpha_{r_0}\alpha_{r_0-1}\cdots\alpha_2\alpha_1$, $\beta_{r_2,2}\alpha_{r_0+s_0}\alpha_{r_0+s_0-1}\cdots\alpha_{r_0+2}\alpha_{r_0+1}$, $\beta_{1,1}$, $\beta_{1,2}\beta_{2,1}, \ldots, \beta_{r_2-1,2}\beta_{r_2,1}$, $\gamma_{1,1}$, $\gamma_{1,2}\gamma_{2,1}, \ldots, \gamma_{s_2-1,2}\gamma_{s_2,1}$.

Additionally, the forbidden threads of A are $p_{x_1}, \ldots, p_{x_{r_2}}$, $p_{y_1}, \ldots, p_{y_{s_2}}$, $\alpha_1, \ldots, \alpha_{r_1}$, $\alpha_{r_0+1}, \ldots, \alpha_{r_0+s_1}$ and $\alpha_{r_1+j}\beta_{j,1}\beta_{j,2}$, $\beta_{j,2}\alpha_{r_1+j}\beta_{j,1}$, $\beta_{j,1}\beta_{j,2}\alpha_{r_1+j}$ for $j \in \{1, \ldots, r_2\}$, and $\alpha_{r_0+s_1+l}\gamma_{l,1}\gamma_{l,2}$, $\gamma_{l,2}\alpha_{r_0+s_1+l}\gamma_{l,1}$, $\gamma_{l,1}\gamma_{l,2}\alpha_{r_0+s_1+l}$ for $l \in \{1, \ldots, s_2\}$ (i.e. each oriented 3-cycle gives three non-trivial forbidden threads as in Example 3.2.3).

Our next aim is to describe the relations in Q by using two functions $\sigma, \varepsilon : Q_1 \to \{1, -1\}$ as in [**24**, Section 3]. These functions satisfy the following three conditions:

(1) If $\beta_1 \neq \beta_2$ are arrows with $s(\beta_1) = s(\beta_2)$, then $\sigma(\beta_1) = -\sigma(\beta_2)$.
(2) If $\gamma_1 \neq \gamma_2$ are arrows with $t(\gamma_1) = t(\gamma_2)$, then $\varepsilon(\gamma_1) = -\varepsilon(\gamma_2)$.
(3) If β and γ are arrows with $s(\gamma) = t(\beta)$ and $\gamma\beta \neq 0$, then $\sigma(\gamma) = -\varepsilon(\beta)$.

Note that each vertex of Q is the starting point of at most two arrows and the end point of at most two arrows.

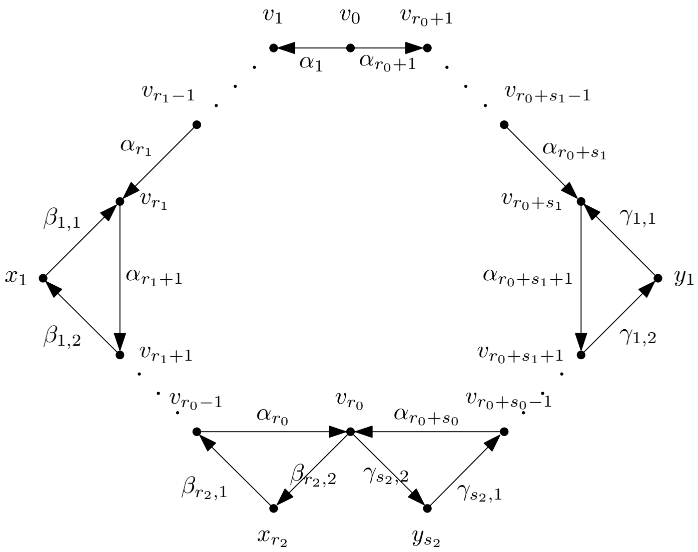


Figure 3.8. A quiver in normal form.

We can extend these functions to threads of A as follows: for a non-trivial thread $H = \alpha_l \dots \alpha_2\alpha_1$ of A define $\sigma(H) := \sigma(\alpha_1)$ and $\varepsilon(H) := \varepsilon(\alpha_l)$. For a trivial permitted thread h_v, $v \in Q_0$, there is some $\alpha_{i+1} \in Q_1$ with $s(\alpha_{i+1}) = v$ and some $\alpha_i \in Q_1$ with $t(\alpha_i) = v$, for $i \in \{1, \dots, r_1 - 1, r_0 + 1, \dots, r_0 + s_1 - 1\}$. We define

$$\sigma(h_v) = -\varepsilon(h_v) := -\sigma(\alpha_{i+1}) \text{ or} \tag{3.2.1}$$

$$\sigma(h_v) = -\varepsilon(h_v) := \varepsilon(\alpha_i), \tag{3.2.2}$$

and these definitions are consistent because of condition (3) above.

For a trivial forbidden thread p_v, $v \in Q_0$, there exists $\beta_{j,1} \in Q_1$ (resp., $\gamma_{l,1} \in Q_1$) with $s(\beta_{j,1}) = v$ (resp., $s(\gamma_{l,1}) = v$) and $\beta_{j,2} \in Q_1$ (resp., $\gamma_{l,2} \in Q_1$) with $t(\beta_{j,2}) = v$ (resp., $t(\gamma_{l,2}) = v$) for $j \in \{1, \dots, r_2\}$ (resp., $l \in \{1, \dots, s_2\}$). We define

$$\sigma(p_v) = \varepsilon(p_v) := -\sigma(\beta_{j,1}) = -\varepsilon(\beta_{j,2}) \tag{3.2.3}$$

$$(\text{resp., } \sigma(p_v) = \varepsilon(p_v) := -\sigma(\gamma_{l,1}) = -\varepsilon(\gamma_{l,2})). \tag{3.2.4}$$

That is, we agree that $\sigma(\beta_{j,1}) = \varepsilon(\beta_{j,2})$ and $\sigma(\gamma_{l,1}) = \varepsilon(\gamma_{l,2})$, respectively. Note that conditions (3.2.3) and (3.2.4) are required in order to ensure Algorithm 3.2.4 works.

Now, we define the functions σ and ε for all arrows in Q. First, we choose

$$\begin{array}{rclrrclcrcl} \sigma(\alpha_i) & = & 1, & \varepsilon(\alpha_i) & = & -1 & \text{for all} & i & = & 1, \dots, r_0 \\ \sigma(\alpha_i) & = & -1, & \varepsilon(\alpha_i) & = & 1 & \text{for all} & i & = & r_0 + 1, \dots, r_0 + s_0 \end{array}.$$

Then condition (1) is satisfied since $\sigma(\alpha_1) = -\sigma(\alpha_{r_0+1})$, condition (2) is satisfied since $\varepsilon(\alpha_{r_0}) = -\varepsilon(\alpha_{r_0+s_0})$, and condition (3) is satisfied since $\sigma(\alpha_{i+1}) = -\varepsilon(\alpha_i)$ for $i \in \{1, \dots, r_0 - 1, r_0 + 1, \dots, r_0 + s_0 - 1\}$.

Applying conditions (1), (2) and (3) we automatically get

$$\begin{array}{rcrlrcl} \varepsilon(\beta_{j,1}) & = & 1 & \text{for all} & j & = & 1, \dots, r_2 \\ \sigma(\beta_{j,2}) & = & -1 & \text{for all} & j & = & 1, \dots, r_2 \\ \varepsilon(\gamma_{l,1}) & = & -1 & \text{for all} & l & = & 1, \dots, s_2 \\ \sigma(\gamma_{l,2}) & = & 1 & \text{for all} & l & = & 1, \dots, s_2 \end{array}.$$

By (3.2.3) and (3.2.4), $\sigma(\beta_{j,1}) = \varepsilon(\beta_{j,2})$ and $\sigma(\gamma_{l,1}) = \varepsilon(\gamma_{l,2})$ for $j \in \{1, \dots, r_2\}$ and $l \in \{1, \dots, s_2\}$, respectively. We will choose

$$\begin{array}{rclclcl} \sigma(\beta_{j,1}) & = & 1 & \text{for all} & j & = & 1,\dots,r_2 \\ \varepsilon(\beta_{j,2}) & = & 1 & \text{for all} & j & = & 1,\dots,r_2 \\ \sigma(\gamma_{l,1}) & = & -1 & \text{for all} & l & = & 1,\dots,s_2 \\ \varepsilon(\gamma_{l,2}) & = & -1 & \text{for all} & l & = & 1,\dots,s_2 \end{array}.$$

According to the conditions (3.2.1), (3.2.2), (3.2.3) and (3.2.4), we define the functions σ and ε for the trivial permitted threads as

$$\begin{array}{rclclclclclcl} \sigma(h_{v_i}) & = & -\varepsilon(h_{v_i}) & = & -\sigma(\alpha_{i+1}) & = & \varepsilon(\alpha_i) & = & -1 & \text{for all} & i & = & 1,\dots,r_1-1 \\ \sigma(h_{v_i}) & = & -\varepsilon(h_{v_i}) & = & -\sigma(\alpha_{i+1}) & = & \varepsilon(\alpha_i) & = & 1 & \text{for all} & i & = & r_0+1,\dots,r_0+s_1-1 \end{array}$$

and for the trivial forbidden threads as

$$\begin{array}{rclclclclclcl} \sigma(p_{x_j}) & = & \varepsilon(p_{x_j}) & = & -\sigma(\beta_{j,1}) & = & -\varepsilon(\beta_{j,2}) & = & -1 & \text{for all} & j & = & 1,\dots,r_2 \\ \sigma(p_{y_l}) & = & \varepsilon(p_{y_l}) & = & -\sigma(\gamma_{l,1}) & = & -\varepsilon(\gamma_{l,2}) & = & 1 & \text{for all} & l & = & 1,\dots,s_2 \end{array}.$$

Now there is a combinatorial algorithm (stated in [**10**]) to produce certain pairs of natural numbers, by using only the quiver with relations which defines a gentle algebra. In the algorithm we are going forward through permitted threads and backwards through forbidden threads in such a way that each arrow and its inverse is used exactly once.

Algorithm 3.2.4 (Avella-Alaminos and Geiß [10]). The algorithm is as follows:

(1) (a) Begin with a permitted thread H_0 of A.
  (b) If H_i is defined, consider Π_i the forbidden thread which ends in $t(H_i)$ and such that $\varepsilon(H_i) = -\varepsilon(\Pi_i)$.
  (c) Let H_{i+1} be the permitted thread which starts in $s(\Pi_i)$ and such that $\sigma(H_{i+1}) = -\sigma(\Pi_i)$.
  The process stops when $H_k = H_0$ for some natural number k. Let $m = \sum_{1\leq i\leq k} l(\Pi_{i-1})$, where $l()$ is the length of a path, i.e. the number of arrows of the path. We obtain the pair (k,m).

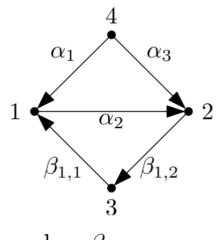
(2) Repeat the first step of the algorithm until all permitted threads of A have occurred as some H_i, or as H_0.

(3) If there are oriented cycles in which each pair of consecutive arrows forms a relation, we get a pair $(0,m)$ for each of those cycles, where m is the length of the cycle.

(4) Define $\phi_A : \mathbb{N}^2 \to \mathbb{N}$, where $\phi_A(k,m)$ is the number of times the pair (k,m) arises in the algorithm.

Notation. For $\alpha \in Q_1$ we define α^{-1} by $s(\alpha^{-1}) := t(\alpha)$, $t(\alpha^{-1}) := s(\alpha)$ and $(\alpha^{-1})^{-1} = \alpha$; and for a path $C = \alpha_l \dots \alpha_2\alpha_1$ we define $C^{-1} := \alpha_1^{-1}\alpha_2^{-1}\dots\alpha_l^{-1}$. Additionally, we define $e_v^{-1} := e_v$ for a trivial path.

Example 3.2.5. Consider the following quiver Q as in Example 3.2.3



with three zero-relations $\beta_{1,2}\alpha_2$, $\beta_{1,1}\beta_{1,2}$ and $\alpha_2\beta_{1,1}$.

First, we use the definitions above to get the functions σ and ε for all arrows in Q:

$$\begin{array}{rclrcl} \sigma(\alpha_1) & = & 1, & \varepsilon(\alpha_1) & = & -1 \\ \sigma(\alpha_2) & = & 1, & \varepsilon(\alpha_2) & = & -1 \\ \sigma(\alpha_3) & = & -1, & \varepsilon(\alpha_3) & = & 1 \\ \sigma(\beta_{1,1}) & = & 1, & \varepsilon(\beta_{1,1}) & = & 1 \\ \sigma(\beta_{1,2}) & = & -1, & \varepsilon(\beta_{1,2}) & = & 1 \end{array}.$$

By (3.2.3) we also get

$$\sigma(p_3) = \varepsilon(p_3) = -\sigma(\beta_{1,1}) = -\varepsilon(\beta_{1,2}) = -1$$

for the trivial forbidden thread p_3.

Let $H_0 = \beta_{1,2}\alpha_3$ and $\Pi_0 = p_3$ with $\varepsilon(\beta_{1,2}\alpha_3) = \varepsilon(\beta_{1,2}) = -\varepsilon(p_3) = 1$. Note that $\beta_{1,2}\alpha_2\beta_{1,1}$ does not qualify for Π_0 since $\varepsilon(\beta_{1,2}\alpha_2\beta_{1,1}) = \varepsilon(\beta_{1,2}) \neq -\varepsilon(H_0)$. Then $H_1 = \beta_{1,1}$ is the permitted thread which starts in $s(\Pi_0) = 3$ and $\sigma(\beta_{1,1}) = -\sigma(\Pi_0) = 1$. Now $\Pi_1 = \alpha_1$ since it is the forbidden thread which ends in vertex $t(H_1) = 1$ and $\varepsilon(\Pi_1) = -\varepsilon(H_1) = -\varepsilon(\beta_{1,1}) = -1$. Note that $\beta_{1,1}\beta_{1,2}\alpha_2$ does not qualify for Π_1 since $\varepsilon(\beta_{1,1}\beta_{1,2}\alpha_2) = \varepsilon(\beta_{1,1})$. Then $H_2 = \beta_{1,2}\alpha_3 = H_0$, $k = 2$ and $m = l(\Pi_0)+l(\Pi_1) = 0+1 = 1$. The corresponding pair is $(2,1) = (r_0, r_1)$, where $r_0 = r_1+r_2$. We can write this as follows:

$$\begin{array}{lllllll} H_0 & = & \beta_{1,2}\alpha_3 & \Pi_0^{-1} & = & p_3 \\ H_1 & = & \beta_{1,1} & \Pi_1^{-1} & = & \alpha_1^{-1} \\ H_2 & = & H_0 & & & \\ & & & \to (2,1) & & \end{array}$$

If we continue with the algorithm we obtain the second pair $(1,1) = (s_0, s_1)$, $s_0 = s_1 + s_2$, in the following way:

$$\begin{array}{llllll} H_0 & = & \alpha_2\alpha_1 & \Pi_0^{-1} & = & \alpha_3^{-1} \\ H_1 & = & H_0 & & & \\ & & & \to (1,1) & & \end{array}$$

Finally, we get one pair $(0,3)$ for the oriented 3-cycle. Thus, we obtain $\phi_A(2,1) = 1$, $\phi_A(1,1) = 1$ and $\phi_A(0,3) = 1$.

The function ϕ_A defined in Algorithm 3.2.4 is invariant under derived equivalence:

Lemma 3.2.6 (Avella-Alaminos and Geiß [10], Theorem A). *Let A and B be gentle algebras. If A and B are derived equivalent, then $\phi_A = \phi_B$.*

Proof of Theorem 3.2.2. Let $A = KQ/I$ be a cluster-tilted algebra of type $\tilde{A}_n$ with parameters r_1, r_2, s_1 and s_2 as defined in Definition 3.1.9. By Lemma 3.2.1, A is derived equivalent to a cluster-tilted algebra $A' = KQ'/I'$ with quiver Q' in normal form, and the same parameters r_1, r_2, s_1 and s_2. We define $r_0 := r_1 + r_2$ to be the number of arrows in the non-oriented cycle of Q' which are oriented in the anti-clockwise direction. Similarly, let $s_0 := s_1 + s_2$ be the number of arrows in the non-oriented cycle of Q' which are oriented in the clockwise direction. We consider the quiver Q' with notations as given in Figure 3.8.

First, we recall the definitions of the functions σ and ε from above:

$$\begin{array}{lrrlrrlcl} \sigma(\alpha_i) & = & 1, & \varepsilon(\alpha_i) & = & -1 & \text{for all } i & = & 1,\dots,r_0 \\ \sigma(\alpha_i) & = & -1, & \varepsilon(\alpha_i) & = & 1 & \text{for all } i & = & r_0+1,\dots,r_0+s_0 \\ \sigma(\beta_{j,1}) & = & 1, & \varepsilon(\beta_{j,1}) & = & 1 & \text{for all } j & = & 1,\dots,r_2 \\ \sigma(\beta_{j,2}) & = & -1, & \varepsilon(\beta_{j,2}) & = & 1 & \text{for all } j & = & 1,\dots,r_2 \\ \sigma(\gamma_{l,1}) & = & -1, & \varepsilon(\gamma_{l,1}) & = & -1 & \text{for all } l & = & 1,\dots,s_2 \\ \sigma(\gamma_{l,2}) & = & 1, & \varepsilon(\gamma_{l,2}) & = & -1 & \text{for all } l & = & 1,\dots,s_2 \\ \sigma(h_{v_i}) & = & -1, & \varepsilon(h_{v_i}) & = & 1 & \text{for all } i & = & 1,\dots,r_1-1 \\ \sigma(h_{v_i}) & = & 1, & \varepsilon(h_{v_i}) & = & -1 & \text{for all } i & = & r_0+1,\dots,r_0+s_1-1 \\ \sigma(p_{x_j}) & = & -1, & \varepsilon(p_{x_j}) & = & -1 & \text{for all } j & = & 1,\dots,r_2 \\ \sigma(p_{y_l}) & = & 1, & \varepsilon(p_{y_l}) & = & 1 & \text{for all } l & = & 1,\dots,s_2 \end{array}$$

Hence, we can apply Algorithm 3.2.4 as follows: let $H_0 = h_{v_1}$ and $\Pi_0 = \alpha_1$ with $\varepsilon(h_{v_1}) = -\varepsilon(\alpha_1) = 1$. Then H_1 is the permitted thread which starts in $s(\Pi_0) = v_0$ and $\sigma(H_1) = -\sigma(\Pi_0) = -1$, that is $\beta_{r_2,2}\alpha_{r_0+s_0}\alpha_{r_0+s_0-1}\dots\alpha_{r_0+2}\alpha_{r_0+1}$. Now $\Pi_1 = p_{x_{r_2}}$ since it is the forbidden thread which ends in x_{r_2} and $\varepsilon(\Pi_1) = -\varepsilon(H_1) = -\varepsilon(\beta_{r_2,2}) = -1$. Note that $\beta_{r_2,2}\alpha_{r_0}\beta_{r_2,1}$ does not qualify for Π_1 since $\varepsilon(\beta_{r_2,2}\alpha_{r_0}\beta_{r_2,1}) = \varepsilon(\beta_{r_2,2}) \neq -\varepsilon(H_1)$. Then $H_2 = \beta_{r_2-1,2}\beta_{r_2,1}$ is the permitted thread starting in x_{r_2} and $\sigma(\Pi_1) = -\sigma(H_2) = -\sigma(\beta_{r_2,1}) = -1$. Thus, $\Pi_2 = p_{x_{r_2-1}}$ with $\varepsilon(H_2) = \varepsilon(\beta_{r_2-1,2}) = -\varepsilon(\Pi_2) = 1$. Note that $\beta_{r_2-1,2}\alpha_{r_0-1}\beta_{r_2-1,1}$ does not qualify for Π_2 since $\varepsilon(\beta_{r_2-1,2}\alpha_{r_0-1}\beta_{r_2-1,1}) = \varepsilon(\beta_{r_2-1,2}) \neq -\varepsilon(H_2)$. Continuing in this way, we get $H_{r_2} = \beta_{1,2}\beta_{2,1}$ and $\Pi_{r_2} = p_{x_1}$. Then $H_{r_2+1} = \beta_{1,1}$ is the permitted thread which starts in $s(\Pi_{r_2}) = x_1$ with $\sigma(H_{r_2+1}) = \sigma(\beta_{1,1}) = -\sigma(\Pi_{r_2}) = 1$. Now $\Pi_{r_2+1} = \alpha_{r_1}$ since it is the forbidden thread which ends in v_{r_1} and $\varepsilon(\Pi_{r_2+1}) = -\varepsilon(H_{r_2+1}) = -\varepsilon(\beta_{1,1}) = -1$. Again $\beta_{1,1}\beta_{1,2}\alpha_{r_1+1}$ does not qualify for Π_{r_2+1} since $\varepsilon(\beta_{1,1}\beta_{1,2}\alpha_{r_1+1}) = \varepsilon(\beta_{1,1}) \neq -\varepsilon(H_{r_2+1})$. Then

$H_{r_2+2} = h_{r_1-1}$ is the permitted thread starting in v_{r_1-1} and $\sigma(\Pi_{r_2+1}) = -\sigma(H_{r_2+2}) = 1$. Thus, $\Pi_{r_2+2} = \alpha_{r_1-1}$ with $\varepsilon(H_{r_2+2}) = -\varepsilon(\Pi_{r_2+2}) = 1$. Continuing in this way, we obtain $H_{r_0-1} = h_{v_2}$ and $\Pi_{r_0-1} = \alpha_2$. Then $H_{r_0} = h_{v_1} = H_0$. We can write this as follows:

$$\begin{array}{lcllcl}
H_0 & = & h_{v_1} & \Pi_0^{-1} & = & \alpha_1^{-1} \\
H_1 & = & \beta_{r_2,2}\alpha_{r_0+s_0}\alpha_{r_0+s_0-1}\ldots\alpha_{r_0+2}\alpha_{r_0+1} & \Pi_1^{-1} & = & p_{x_{r_2}} \\
H_2 & = & \beta_{r_2-1,2}\beta_{r_2,1} & \Pi_2^{-1} & = & p_{x_{r_2-1}} \\
\vdots & & & \vdots & & \\
H_{r_2} & = & \beta_{1,2}\beta_{2,1} & \Pi_{r_2}^{-1} & = & p_{x_1} \\
H_{r_2+1} & = & \beta_{1,1} & \Pi_{r_2+1}^{-1} & = & \alpha_{r_1}^{-1} \\
H_{r_2+2} & = & h_{v_{r_1-1}} & \Pi_{r_2+2}^{-1} & = & \alpha_{r_1-1}^{-1} \\
\vdots & & & \vdots & & \\
H_{r_0-1} & = & h_{v_2} & \Pi_{r_0-1}^{-1} & = & \alpha_2^{-1} \\
H_{r_0} & = & H_0 & & &
\end{array}$$

$$\begin{aligned}
m &= l(\Pi_0) + l(\Pi_{r_2+1}) + l(\Pi_{r_2+2}) + \cdots + l(\Pi_{r_0-1}) \\
&= 1 + \underbrace{1 + 1 + \cdots + 1}_{((r_0-1)-r_2)-\text{times}} \\
&= 1 + (r_0 - 1) - r_2 \\
&= r_0 - r_2 \\
&= r_1
\end{aligned}$$

$$\rightarrow (r_0, r_1)$$

If we continue with the algorithm we obtain the second pair (s_0, s_1) in the following way:

$$\begin{array}{lcllcl}
H_0 & = & h_{v_{r_0+1}} & \Pi_0^{-1} & = & \alpha_{r_0+1}^{-1} \\
H_1 & = & \gamma_{s_2,2}\alpha_{r_0}\alpha_{r_0-1}\ldots\alpha_2\alpha_1 & \Pi_1^{-1} & = & p_{y_{s_2}} \\
H_2 & = & \gamma_{s_2-1,2}\gamma_{s_2,1} & \Pi_2^{-1} & = & p_{y_{s_2-1}} \\
\vdots & & & \vdots & & \\
H_{s_2} & = & \gamma_{1,2}\gamma_{2,1} & \Pi_{s_2}^{-1} & = & p_{y_1} \\
H_{s_2+1} & = & \gamma_{1,1} & \Pi_{s_2+1}^{-1} & = & \alpha_{r_0+s_1}^{-1} \\
H_{s_2+2} & = & h_{v_{r_0+s_1-1}} & \Pi_{s_2+2}^{-1} & = & \alpha_{r_0+s_1-1}^{-1} \\
\vdots & & & \vdots & & \\
H_{s_0-1} & = & h_{v_{r_0+2}} & \Pi_{s_0-1}^{-1} & = & \alpha_{r_0+2}^{-1} \\
H_{s_0} & = & H_0 & & &
\end{array}$$

$$\rightarrow (s_0, s_1)$$

Finally, we get $r_2 + s_2$ pairs $(0, 3)$ for the oriented 3-cycles. Thus, we obtain $\phi_A(r_0, r_1) = 1$, $\phi_A(s_0, s_1) = 1$ and $\phi_A(0, 3) = r_2 + s_2$, where $r_0 = r_1 + r_2$ and $s_0 = s_1 + s_2$.

Now, let A and B be two cluster-tilted algebras of type $\tilde{A}_n$ with parameters r_1, r_2, s_1, s_2 and $\tilde{r}_1$, $\tilde{r}_2$, $\tilde{s}_1$, $\tilde{s}_2$, respectively. From above we can conclude that $\phi_A = \phi_B$ if and only if $r_1 = \tilde{r}_1$, $r_2 = \tilde{r}_2$, $s_1 = \tilde{s}_1$ and $s_2 = \tilde{s}_2$ or $r_1 = \tilde{s}_1$, $r_2 = \tilde{s}_2$, $s_1 = \tilde{r}_1$ and $s_2 = \tilde{r}_2$, which ends up being the same quiver (up to isomorphism).

Hence, if A is derived equivalent to B, we know from Lemma 3.2.6 that $\phi_A = \phi_B$ and thus, that the parameters are the same. Otherwise, if A and B have the same parameters, they are both derived equivalent to the same cluster-tilted algebra with a quiver in normal form. □

CHAPTER 4

Type D_n

The results of this chapter are a joint work with Thorsten Holm and Sefi Ladkani. These results appeared in [**13**], except for the proof of Proposition 4.2.47.

4.1. Cluster-tilted algebras of type D_n

Recall that a quiver of Dynkin type D_n is a quiver with $n \geq 4$ vertices and underlying graph the Dynkin diagram D_n. Since the orientation of the edges can be chosen arbitrarily (see Lemma 2.1.5), we will start with the following directed graph

$$\begin{tikzcd} \bullet_1 \arrow[dr] & & & \\ & \bullet_3 \arrow[r] & \cdots \arrow[r] & \bullet_n \\ \bullet_2 \arrow[ur] & & & \end{tikzcd} .$$

We recall the description by Vatne [**53**] of the quivers which are mutation equivalent to D_n, and the relations defining the corresponding cluster-tilted algebras following [**21**]. It is convenient to use the language of the gluing of rooted quivers of type A (see Definition 2.5.4 for a definition of these quivers).

Definition 4.1.1. Let Q be a quiver, called a *skeleton*, and let $c_1, c_2, \dots, c_k$ be $k \geq 0$ distinct vertices of Q. The *gluing* of k rooted quivers of type A, say $(Q^{(1)}, z_1)$, $(Q^{(2)}, z_2)$, $\dots$, $(Q^{(k)}, z_k)$, to Q at the vertices $c_1, \dots, c_k$ is defined as the quiver obtained from the disjoint union $Q \,\dot{\cup}\, Q^{(1)} \,\dot{\cup} \dots \dot{\cup}\, Q^{(k)}$ by identifying each vertex c_i with the corresponding root z_i, for $1 \leq i \leq k$.

Remark 4.1.2. Given relations (i.e. linear combinations of parallel paths) on the skeleton Q, they induce relations on the gluing, namely, by taking the union of all the relations on $Q, Q^{(1)}, \dots, Q^{(k)}$, where the relations on the rooted quivers of type A are those stated in Remark 2.5.3.

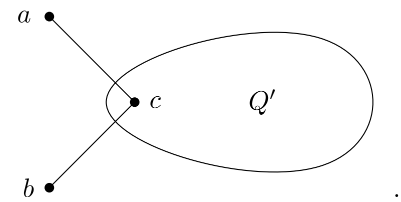
Definition 4.1.3. Let Q be a quiver mutation equivalent to A_n (see Proposition 2.5.2). We denote by $s(Q)$ the number of arrows in Q which are not part of an oriented 3-cycle, and by $t(Q)$ the number of oriented 3-cycles in Q.

A cluster-tilted algebra of Dynkin type D_n belongs to one of the following four families, which are called *types*, and are defined as the gluing of rooted quivers of type A to certain skeleta. Note that in view of Remark 4.1.2, it is enough to specify the relations on the skeleton. For each type we define *parameters* which will be useful when referring to the cluster-tilted algebras of that type.

Type I The gluing of a rooted quiver Q' of type A at the vertex c of one of the three skeleta:

$$\begin{tikzcd} \bullet_a \arrow[dr] & \\ & \bullet_c \arrow[dl] \\ \bullet_b & \end{tikzcd} \qquad \begin{tikzcd} \bullet_a \arrow[dr] & \\ & \bullet_c \\ \bullet_b \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \bullet_a & \\ & \bullet_c \arrow[ul] \arrow[dl] \\ \bullet_b & \end{tikzcd} ,$$

as in the following picture:

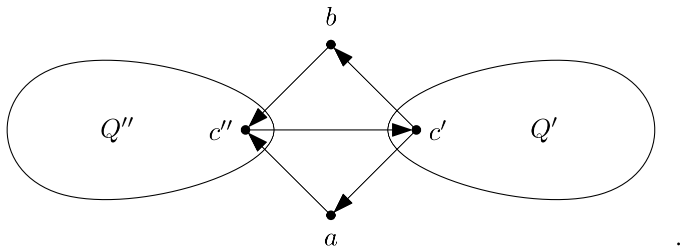


The *parameters* are $\big(s(Q'), t(Q')\big)$.

<u>Type II</u> The gluing of two rooted quivers Q' and Q'' of type A at the vertices c' and c'', respectively, of the following skeleton:

$$\begin{tikzcd} & b \arrow[dl, "\beta"'] & \\ c'' \arrow[rr, "\varepsilon"] & & c' \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & a \arrow[ul, "\delta"] & \end{tikzcd},$$

with the commutativity relation $\beta\alpha - \delta\gamma$, and the zero-relations $\alpha\varepsilon$, $\gamma\varepsilon$, $\varepsilon\beta$, $\varepsilon\delta$ as in the following picture:

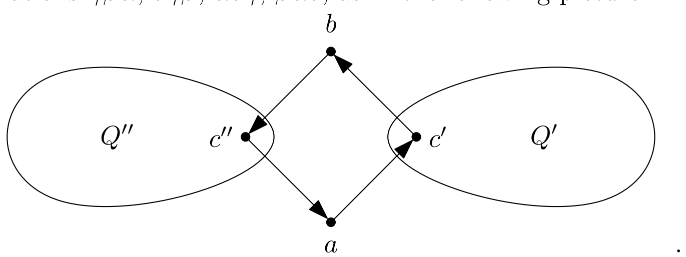


The *parameters* are $\big(s(Q'), t(Q'), s(Q''), t(Q'')\big)$.

<u>Type III</u> The gluing of two rooted quivers Q' and Q'' of type A at the vertices c' and c'', respectively, of the following skeleton:

$$\begin{tikzcd} & b \arrow[dl, "\beta"'] & \\ c'' \arrow[dr, "\gamma"'] & & c' \arrow[ul, "\alpha"'] \\ & a \arrow[ur, "\delta"'] & \end{tikzcd},$$

with the four zero-relations $\gamma\beta\alpha$, $\delta\gamma\beta$, $\alpha\delta\gamma$, $\beta\alpha\delta$, as in the following picture:



Like in type II, the *parameters* are $\big(s(Q'), t(Q'), s(Q''), t(Q'')\big)$.

<u>Type IV</u> The gluing of $r \geq 0$ rooted quivers $Q^{(1)}, \dots, Q^{(r)}$ of type A at the vertices $c_1, \dots, c_r$ of a skeleton $Q(m, \{i_1, \dots, i_r\})$, see Figure 4.1 below.

Definition 4.1.4. Given integers $m \geq 3$, $r \geq 0$ and an increasing sequence $1 \leq i_1 < i_2 < \cdots < i_r \leq m$, we define the quiver with relations $Q(m, \{i_1, \dots, i_r\})$:

(a) $Q(m, \{i_1, \dots, i_r\})$ has $m + r$ vertices, labelled $1, 2, \dots, m$ together with $c_1, c_2, \dots, c_r$, and its arrows are

$$\big\{i \to (i+1)\big\}_{1 \leq i \leq m} \cup \big\{c_j \to i_j, (i_j + 1) \to c_j\big\}_{1 \leq j \leq r},$$

where $i+1$ is considered modulo m, i.e. 1, if $i = m$.

The full subquiver on the vertices $1, 2, \dots, m$ is thus an oriented cycle of length m, called the *central cycle*, and for every $1 \le j \le r$, the full subquiver on the vertices $i_j, i_j + 1, c_j$ is an oriented 3-cycle, called a *spike*.

(b) The *relations* on $Q(m, \{i_1, \dots, i_r\})$ are:
- The paths $i_j, i_j + 1, c_j$ and $c_j, i_j, i_j + 1$ are zero for all $1 \le j \le r$ (note that we indicate a path by the sequence of vertices it traverses);
- For any $1 \le j \le r$, the path $i_j + 1, c_j, i_j$ equals the path $i_j + 1, \dots, 1, \dots, i_j$ of length $m - 1$ along the central cycle;
- For any $i \notin \{i_1, \dots, i_r\}$, the path $i + 1, \dots, 1, \dots, i$ of length $m - 1$ along the central cycle is zero.

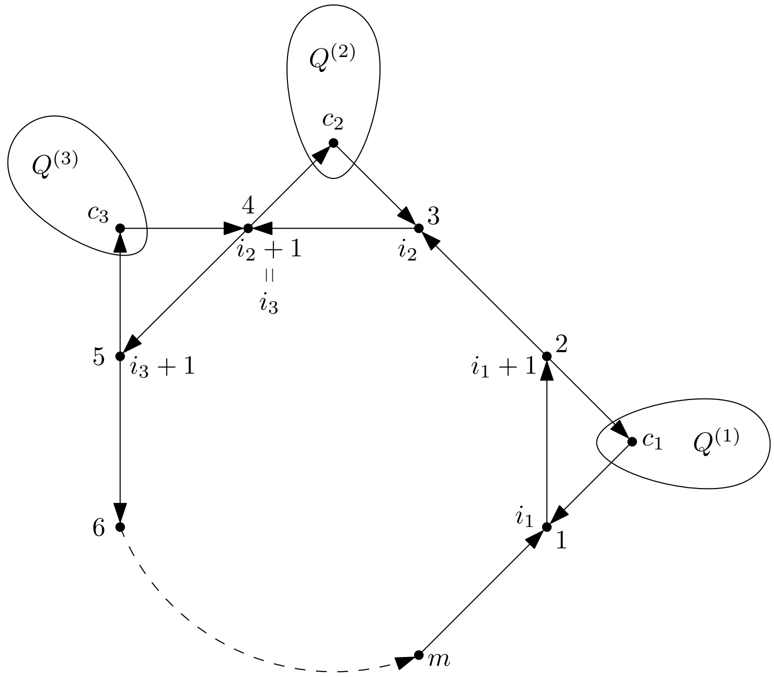


Figure 4.1. A quiver of a cluster-tilted algebra of type D_n in type IV.

The *parameters* are encoded as follows. If $r = 0$, that is, there are no spikes and hence no attached rooted quivers of type A, the quiver is just an oriented cycle, thus parameterized by its length $m \ge 3$.

In all other cases, due to rotational symmetry, we define the *distances* $d_1, d_2, \dots, d_r$ by

$$d_1 = i_2 - i_1, d_2 = i_3 - i_2, \dots, d_{r-1} = i_r - i_{r-1}, d_r = i_1 + m - i_r$$

so that $m = d_1 + d_2 + \cdots + d_r$, and encode the cluster-tilted algebra by the sequence of triples

$$\big((d_1, s_1, t_1), (d_2, s_2, t_2), \dots, (d_r, s_r, t_r)\big) \tag{4.1.1}$$

where $s_j = s(Q^{(j)})$, $t_j = t(Q^{(j)})$ are the numbers of arrows and oriented 3-cycles in the rooted quiver $Q^{(j)}$ of type A glued at the vertex c_j of the j^{th} spike.

Remark 4.1.5. Note that the cluster-tilted algebras of type III can be viewed as a degenerate version of type IV, namely corresponding to the skeleton $Q(2, \{1, 1\})$ with central cycle of length two (hence it is "invisible") with all spikes present. Thus, one can regard type III quivers with parameters $(s(Q'), t(Q'), s(Q''), t(Q''))$ as 'formal' type IV quivers with parameters $((1, s(Q'), t(Q')),$ $(1, s(Q''), t(Q'')))$. It turns out that this point of view is consistent with the constructions of good mutations and double mutations as well as with the computations of determinants presented later.

Notation. Throughout this chapter, for a quiver Q which is mutation equivalent to an acyclic quiver, i.e. to a quiver without oriented cycles, we denote by A_Q the corresponding cluster-tilted algebra and its Cartan matrix by C_Q.

4.2. Good mutation equivalence classification of cluster-tilted algebras of type D_n

4.2.1. Mutations of algebras and good quiver mutations. We recall the notion of mutations of algebras from [**43**]. These are local operations on an algebra A producing new algebras derived equivalent to A.

Let $A = KQ/I$ be an algebra given as a quiver with relations. Let k be a vertex of Q at which there are no loops, i.e. no arrows $k \to k$. Consider the following two complexes of projective A-modules

$$T_k^-(A) = \big(P_k \xrightarrow{f} \bigoplus_{j\to k} P_j\big) \oplus \big(\bigoplus_{i\neq k} P_i\big), \qquad T_k^+(A) = \big(\bigoplus_{k\to j} P_j \xrightarrow{g} P_k\big) \oplus \big(\bigoplus_{i\neq k} P_i\big)$$

where the map f is induced by all the maps $P_k \to P_j$ corresponding to the arrows $j \to k$ ending at k, the map g is induced by the maps $P_j \to P_k$ corresponding to the arrows $k \to j$ starting at k, the term P_k lies in degree -1 in $T_k^-(A)$ and in degree 1 in $T_k^+(A)$, and all other terms are in degree 0.

Definition 4.2.1. Let A be an algebra given as a quiver with relations and k a vertex at which there are no loops.

(a) We say that the negative mutation of A at k is *defined* if $T_k^-(A)$ is a tilting complex over A. In this case, we call the algebra $\mu_k^-(A) := \mathrm{End}_{D^b(A)}(T_k^-(A))$ the *negative mutation* of A at the vertex k.

(b) We say that the positive mutation of A at k is *defined* if $T_k^+(A)$ is a tilting complex over A. In this case, we call the algebra $\mu_k^+(A) := \mathrm{End}_{D^b(A)}(T_k^+(A))$ the *positive mutation* of A at the vertex k.

Remark 4.2.2. By Rickard's Theorem 2.2.7, the negative and positive mutations of an algebra A at a vertex, when defined, are always derived equivalent to A.

There is a combinatorial criterion to determine whether a mutation at a vertex is defined, see [**43**, Prop. 2.3]. Since the algebras we will be dealing with in this chapter are Schurian, i.e. the entries of their Cartan matrices are only 0 or 1, we shall only state the criterion for this case, as it takes a particularly simple form.

Proposition 4.2.3 (Ladkani [43]). *Let A be a Schurian algebra.*

(a) *The negative mutation $\mu_k^-(A)$ is defined if and only if for any non-zero path $k \rightsquigarrow i$ starting at k and ending at some vertex i, there exists an arrow $j \to k$ such that the composition $j \to k \rightsquigarrow i$ is non-zero.*

(b) *The positive mutation $\mu_k^+(A)$ is defined if and only if for any non-zero path $i \rightsquigarrow k$ starting at some vertex i and ending at k, there exists an arrow $k \to j$ such that the composition $i \rightsquigarrow k \to j$ is non-zero.*

Remark 4.2.4. To prove Proposition 4.2.3 one has to determine necessary and sufficient conditions on $T_k^-(A)$ and $T_k^+(A)$ to be tilting complexes, i.e. $T_k^-(A)$ and $T_k^+(A)$ satisfy conditions (i) and (ii) of Definition 2.2.5. As in Example 2.2.6 one can show that condition (ii) is satisfied for all such complexes $T_k^-(A)$ and $T_k^+(A)$. It remains to check that condition (i) is equivalent to conditions (a) and (b) of Proposition 4.2.3, respectively; this is a straightforward calculation.

Remark 4.2.5. When k is a sink or a source in Q, exactly one of the mutations is defined. That is, $\mu_k^-(A)$ is defined for a sink k and $\mu_k^+(A)$ is defined for a source k. This follows immediately from Proposition 4.2.3.

Example 4.2.6.

(1) Let A_Q be the path algebra of the following quiver Q

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \ .$$

Using Remark 4.2.5, we see that at vertex 1, only $\mu_1^+(A_Q)$ is defined, whereas at vertex 3, only $\mu_3^-(A_Q)$ is defined. Now we consider the two possible mutations at vertex 2. The negative mutation $\mu_2^-(A_Q)$ is defined since the combination of the arrow $2 \to 3$ with the arrow $1 \to 2$ is not zero. A standard calculation yields that $\mu_2^-(A_Q)$ is given by the following quiver with zero-relation.

$$\begin{tikzcd} & \bullet_2 \arrow[dl] \arrow[dr, dotted, no head] & \\ \bullet_1 \arrow[rr] & & \bullet_3 \end{tikzcd}$$

By the same argument, the positive mutation $\mu_2^+(A_Q)$ is also defined and it is given by the following quiver with zero-relation:

$$\begin{tikzcd} & \bullet_2 \arrow[dl, dotted, no head] & \\ \bullet_1 \arrow[rr] & & \bullet_3 \arrow[ul] \end{tikzcd} .$$

(2) Let A_Q be the algebra given by the following quiver with full zero-relations

$$\begin{tikzcd} & \bullet_2 \arrow[dl] & \\ \bullet_1 \arrow[rr] & & \bullet_3 \arrow[ul] \end{tikzcd} .$$

It is easy to see that all the mutations $\mu_k^-(A_Q)$ and $\mu_k^+(A_Q)$, $k \in \{1, 2, 3\}$, are not defined since all path of length two are zero in A_Q.

For cluster-tilted algebras of Dynkin type, the statement of Theorem 5.3 in [**43**], linking more generally mutation of cluster-tilting objects in 2-Calabi-Yau categories with mutations of their endomorphism algebras, takes the following form.

Proposition 4.2.7 (Ladkani [43]). *Let Q be mutation equivalent to a Dynkin quiver and let k be a vertex of Q.*

(a) $A_{\mu_k(Q)} \cong \mu_k^-(A_Q)$ *if and only if the two algebra mutations* $\mu_k^-(A_Q)$ *and* $\mu_k^+(A_{\mu_k(Q)})$ *are defined.*

(b) $A_{\mu_k(Q)} \cong \mu_k^+(A_Q)$ *if and only if the two algebra mutations* $\mu_k^+(A_Q)$ *and* $\mu_k^-(A_{\mu_k(Q)})$ *are defined.*

This motivates the following definition.

Definition 4.2.8. When (at least) one of the conditions in the proposition holds, we say that the quiver mutation of Q at k is *good*, since it implies the derived equivalence of the corresponding cluster-tilted algebras A_Q and $A_{\mu_k(Q)}$. When neither of the conditions in the proposition hold, we say that the quiver mutation is *bad*.

Remark 4.2.9. It follows from Propositions 4.2.3 and 4.2.7 that there is an algorithm which decides, given a quiver which is mutation equivalent to a Dynkin quiver, whether or not a mutation at a vertex is good.

Example 4.2.10.

(1) Let A_Q be the same algebra as in Example 4.2.6 (1). We have seen that both algebra mutations $\mu_2^-(A_Q)$ and $\mu_2^+(A_Q)$ are defined. Mutation at vertex 2 yields the algebra $A_{\mu_2(Q)}$ which is given by the oriented 3-cycle with full zero-relations as in Example 4.2.6 (2). We have seen that both mutations $\mu_2^-(A_{\mu_2(Q)})$ and $\mu_2^+(A_{\mu_2(Q)})$ are not defined. It follows that neither of the two conditions in Proposition 4.2.7 hold and the quiver mutation at vertex 2 is bad.

(2) Let A_Q be the algebra of the following quiver Q with full zero-relations in the oriented 3-cycle:

$$\begin{tikzcd} & & \bullet_3 \arrow[dr] & \\ \bullet_1 \arrow[r] & \bullet_2 \arrow[ur] & & \bullet_4 \arrow[ll] \end{tikzcd} .$$

The negative mutation $\mu_2^-(A_Q)$ at vertex 2 is defined since the path $1, 2, 3$ is not zero in A_Q (note that we indicate a path by the sequence of vertices it traverses). Mutation at vertex 2 yields the algebra $A_{\mu_2(Q)}$ which is given by the following quiver with full zero-relations in the oriented 3-cycle:

$$\begin{tikzcd} & \bullet_3 \arrow[dr] & & \\ \bullet_1 \arrow[ur] & & \bullet_2 \arrow[ll] \arrow[r] & \bullet_4 \end{tikzcd} .$$

Since the path $3, 2, 4$ is not zero, the positive mutation $\mu_2^+(A_{\mu_2(Q)})$ at vertex 2 is defined. Thus, condition (a) of Proposition 4.2.7 is fulfilled and the quiver mutation at vertex 2 is good. Indeed, $A_{\mu_2(Q)} \cong \mu_2^-(A_Q)$ and the two algebras A_Q and $A_{\mu_2(Q)}$ are derived equivalent.

Definition 4.2.11. Two cluster-tilted algebras (of Dynkin type) with quivers Q' and Q'' are called *good mutation equivalent* if one can obtain Q'' from Q' by performing a sequence of good mutations. In other words, if there exists a sequence of vertices $k_1, k_2, \ldots, k_l$ such that setting $Q^0 = Q'$ and $Q^j = \mu_{k_j}(Q^{j-1})$ for $1 \leq j \leq l$, and denoting by A_j the cluster-tilted algebra with quiver Q^j, then $Q'' = Q^l$ and for any $1 \leq j \leq l$ we have $A_j = \mu_{k_j}^-(A_{j-1})$ or $A_j = \mu_{k_j}^+(A_{j-1})$.

Remark 4.2.12. Any two cluster-tilted algebras which are good mutation equivalent are also derived equivalent.

Unlike in Dynkin types A and E, where the quivers of any two derived equivalent cluster-tilted algebras are connected by a sequence of good mutations (see Section 4.2.2 for type A and [**12**, Theorem 1.1] for type E), this is no longer the case in type D (see Section 4.3). Therefore, we need also to consider mutations of algebras going beyond the family of cluster-tilted algebras (which is not closed under derived equivalence, see Section 4.3.1).

Definition 4.2.13. Let Q and Q' be quivers with vertices k and k' such that $\mu_k(Q) = \mu_{k'}(Q')$. We call the sequence of the two mutations from Q to Q' (first at k and then at k') a *good double mutation* if both algebra mutations $\mu_k^-(A_Q)$ and $\mu_{k'}^+(A_{Q'})$ are defined and, moreover, they are isomorphic to one another.

By definition, for quivers Q and Q' related by a good double mutation, the cluster-tilted algebras A_Q and $A_{Q'}$ are derived equivalent. Note, however, that we do not require the intermediate algebra $\mu_k^-(A_Q) \cong \mu_{k'}^+(A_{Q'})$ to be a cluster-tilted algebra (see Section 4.3.1 for examples in type III and type IV; in particular, Corollaries 4.3.2 and 4.3.4).

4.2.2. Good mutations for rooted quivers of type A.

Since rooted quivers of Dynkin type A are important building blocks of the quivers of cluster-tilted algebras of type D_n, we first determine all the good mutations for cluster-tilted algebras of type A_n. It turns out that the quivers of derived equivalent cluster-tilted algebras of type A_n are connected by sequences of good mutations. Note that the derived equivalence classification of cluster-tilted algebras of type A_n is originally due to Buan and Vatne [**23**].

Definition 4.2.14. Let Q be a quiver of a cluster-tilted algebra of type A_n. The *standard form* of Q is the following quiver consisting of $s(Q)$ arrows which are not part of an oriented 3-cycle and $t(Q)$ oriented 3-cycles arranged as follows:

(4.2.1)
$$\begin{tikzcd} & & & & \bullet \arrow[dl] & & & & \bullet \arrow[dl] & \\ \bullet_c \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[rr] & & \bullet \arrow[ul] & \cdots & \bullet \arrow[rr] & & \bullet \arrow[ul] \end{tikzcd} .$$

The *standard form* of a rooted quiver (Q, c) of type A is a rooted quiver of type A as in (4.2.1) consisting of $s(Q)$ arrows which are not part of an oriented 3-cycle and $t(Q)$ oriented 3-cycles with the vertex c as the root. If $s(Q) = 0$, then the root c is a vertex of an oriented 3-cycle.

For a rooted quiver (Q, c) of type A, we call a mutation at a vertex other than the root c a mutation *outside the root*.

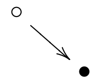
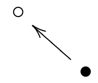
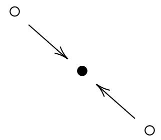
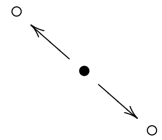
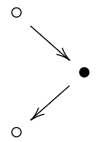
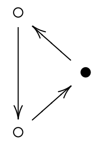
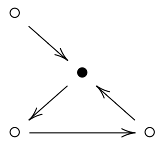
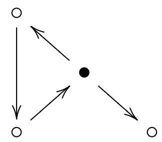
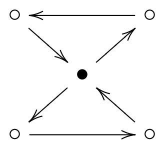
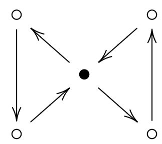
1		$\mu_\bullet^-$		$\mu_\bullet^+$	good
$2a$		$\mu_\bullet^-$		$\mu_\bullet^+$	good
$2b$		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
3		$\mu_\bullet^-$		$\mu_\bullet^+$	good
4		$\mu_\bullet^-, \mu_\bullet^+$		$\mu_\bullet^-, \mu_\bullet^+$	good

Table 4.1. The neighbourhoods in Dynkin type A and their mutations.

Proposition 4.2.15. *Any two rooted quivers of type A with the same numbers of arrows and oriented 3-cycles can be connected by a sequence of good mutations outside the root.*

Remark 4.2.16. It is enough to show that a rooted quiver of type A can be transformed to its standard form via good mutations outside the root.

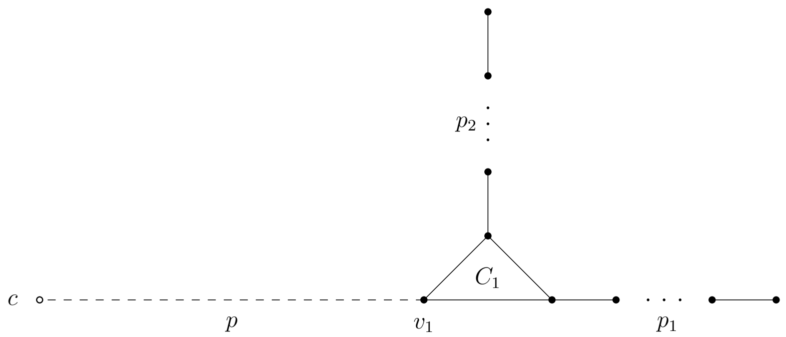
We begin by characterising the good mutations in Dynkin type A_n.

Lemma 4.2.17. *Let Q be a quiver mutation equivalent to A_n. Then a mutation of Q is good if and only if it does not change the number of oriented 3-cycles.*

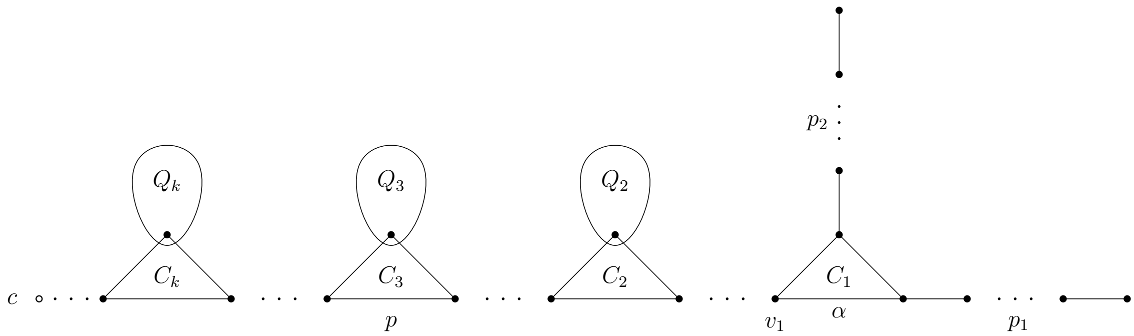
Proof. Each row of Table 4.1 displays a pair of neighbourhoods of a vertex $\bullet$ in such a quiver related by a mutation (at $\bullet$). Using the description of the relations of the corresponding cluster-tilted algebras as in Remark 2.5.3, we can use Proposition 4.2.3 and easily determine, for each entry in the table, which of the negative $\mu_\bullet^-$ or the positive $\mu_\bullet^+$ mutations are defined. Then Proposition 4.2.7 tells us whether or not the quiver mutation is good.

By examining the entries in the table, we see that the only bad mutation occurs in row $2b$, where an oriented 3-cycle is created (or destroyed). □

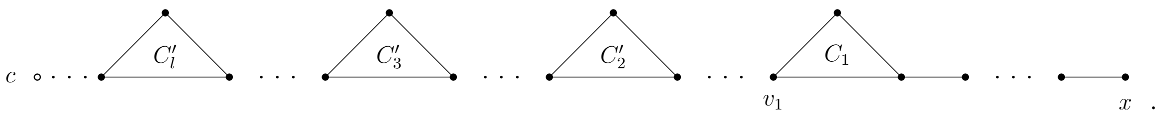
Proof of Proposition 4.2.15. In view of Remark 4.2.16, we provide an algorithm for the mutation to the standard form as in Definition 4.2.14 (similar to the procedures in [**23**] and in Lemma 3.1.11): let Q be a rooted quiver of type A which has at least one oriented 3-cycle (otherwise we get the desired orientation of a standard form by sink/source mutations as in 1 and 2a of Table 4.1). For any oriented 3-cycle C in Q denote by v_C the unique vertex of the oriented 3-cycle having minimal distance to the root c (this distance could be zero, if the root is a vertex of the oriented 3-cycle C). Choose an oriented 3-cycle C_1 in Q such that to the vertices of the oriented 3-cycle $\neq v_1 := v_{C_1}$ only linear parts are attached; denote them by p_1 and p_2, respectively. Note that such an oriented 3-cycle C_1 exists since Q is a finite quiver; and the linear parts p_1 and p_2, respectively, could also consist of just a single vertex.



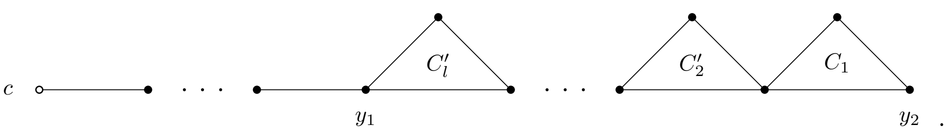
Denote by $C_2, \dots, C_k$ the other oriented 3-cycles along the (possibly non-oriented) path p from vertex v_1 to the root c (if such cycles exist). Recall that by a 'non-oriented path' we mean a sequence of arrows $\alpha_l, \dots, \alpha_1$, where we do not require that $s(\alpha_{i+1}) = t(\alpha_i)$ for all $i \in \{1, \dots, l-1\}$.



Now we move all subquivers $p_2, Q_2, \dots, Q_k$ onto the (possibly non-oriented) path p_1, α, p. For this we use the same mutations as in the Steps 1, 2 and 3 in the proof of Lemma 3.1.11. Note that in Steps 2 and 3 of Lemma 3.1.11 we move oriented 3-cycles and arrows onto a non-oriented cycle. However, these mutations can also be used to move such cycles and arrows onto a (possibly non-oriented) path. All of this can be done with the good mutations presented in Table 4.1. Thus, we get a new complete set of oriented 3-cycles $\{C_1, C'_2, \dots, C'_l\}$ on the (possibly non-oriented) path from vertex x to the root c:



Next we move all the oriented 3-cycles along the (possibly non-oriented) path between vertex x and the root c to the right hand side, i.e. towards x. For this we use the same mutations as in Step 4 in the proof of Lemma 3.1.11. Note that in Step 4 of Lemma 3.1.11 we move oriented 3-cycles along a non-oriented cycle. However, these mutations can also be used to move such cycles along a (possibly non-oriented) path. We then obtain a quiver of the form



To get the desired orientation of a standard form as in (4.2.1) we use the following mutations. If $s(Q) \neq 0$ we orient the part without oriented 3-cycles by sink/source mutations as in 1 and 2a of Table 4.1, without mutating at vertex y_1 which is attached to an oriented 3-cycle. If $t(Q) \neq 0$ we use mutations as in Step 5 in the proof of Lemma 3.1.11 (see also row 4 of Table 4.1) to orient the part consisting only of oriented 3-cycles. Note that these mutations act like sink/source mutations at the (possibly non-oriented) path between vertex y_1 and vertex y_2. The orientation of the oriented 3-cycle C_1 does not matter, since we can just flip it in the picture above. □

As a consequence, we get the following theorem which also follows from the results of [**23**].

Theorem 4.2.18. *Let Q be a quiver of a cluster-tilted algebra of Dynkin type A_n. Then Q can be transformed via a sequence of good mutations to its standard form as in Definition 4.2.14. Moreover, two standard forms are derived equivalent if and only if they coincide.*

4.2.3. Good mutations in type D_n. First we consider quivers of types I, II and III. The good mutations involving these quivers are given in Tables 4.2, 4.3 and 4.4 below. In each row of these tables, we list:

(a) The quiver, where Q, Q', Q'' and Q''' are rooted quivers of type A;
(b) Which of the algebra mutations (negative $\mu_\bullet^-$, or positive $\mu_\bullet^+$) at the distinguished vertex $\bullet$ are defined;
(c) The (Fomin-Zelevinsky) mutation of the quiver at the vertex $\bullet$; and for the corresponding cluster-tilted algebra of the mutated quiver:
(d) Which of the algebra mutations at the vertex $\bullet$ are defined.
(e) Based on these, we determine whether or not the mutation is good, see Proposition 4.2.7.

To check whether or not a mutation is defined, we use the criterion of Proposition 4.2.3. Observe that since the gluing process introduces no new relations, it is enough to assume that each rooted quiver of type A consists of just a single vertex. Since there is at most one arrow between any two vertices, we indicate a path by the sequence of vertices it traverses.

First we observe the following: the distinguished vertices of the first two rows I.1 and I.2 of Table 4.2 are sinks and sources, respectively. Hence, the results follow with Remark 4.2.5.

The correctness of all the other results listed in Tables 4.2, 4.3 and 4.4 is proved in the lemmas below.

Lemma 4.2.19 (I.3a, I.3b). (a) *Consider the two cluster-tilted algebras A and A' with the following quivers*

$$\begin{tikzcd}[column sep=small, row sep=small] \bullet_1 \arrow[dr] & & \\ & \bullet_0 \arrow[r] & \bullet_3 \\ \bullet_2 \arrow[ur] & & \end{tikzcd} \qquad\qquad \begin{tikzcd}[column sep=small, row sep=small] & \bullet_1 \arrow[dl] & \\ \bullet_3 \arrow[rr] & & \bullet_0 \arrow[ul] \arrow[dl] \\ & \bullet_2 \arrow[ul] & \end{tikzcd} .$$

Then the mutations $\mu_0^-(A)$ and $\mu_0^+(A)$ are defined, whereas $\mu_0^-(A')$ and $\mu_0^+(A')$ are not defined.

(b) *Consider the two cluster-tilted algebras A and A' with the following quivers*

$$\begin{tikzcd}[column sep=small, row sep=small] \bullet_1 & & \\ & \bullet_0 \arrow[ul] \arrow[dl] & \bullet_3 \arrow[l] \\ \bullet_2 & & \end{tikzcd} \qquad\qquad \begin{tikzcd}[column sep=small, row sep=small] & \bullet_1 \arrow[dl] & \\ \bullet_0 \arrow[rr] & & \bullet_3 \arrow[ul] \arrow[dl] \\ & \bullet_2 \arrow[ul] & \end{tikzcd} .$$

Then the mutations $\mu_0^-(A)$ and $\mu_0^+(A)$ are defined, whereas $\mu_0^-(A')$ and $\mu_0^+(A')$ are not defined.

Proof. We use the criterion of Proposition 4.2.3.

(a) Since the arrow $1 \to 0$ (or $2 \to 0$) does not appear in any relation in A, its composition with any non-zero path starting at 0 is non-zero. Thus, the negative mutation $\mu_0^-(A)$ is defined. Similarly, since $0 \to 3$ does not appear in any relation of A, its composition with any non-zero path that ends at 0 is non-zero, and the positive mutation $\mu_0^+(A)$ is also defined.

The two algebra mutations $\mu_0^-(A')$ and $\mu_0^+(A')$ are not defined since the two compositions of the arrow $3 \to 0$ with the arrows $0 \to 1$ and $0 \to 2$, respectively, vanish in A'.

(b) Since the arrow $3 \to 0$ does not appear in any relation in A, the negative mutation $\mu_0^-(A)$ is defined. Similarly, the positive mutation $\mu_0^+(A)$ is defined since the arrow $0 \to 1$ (or $0 \to 2$) does not appear in any relation.

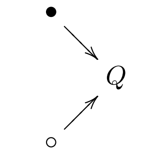
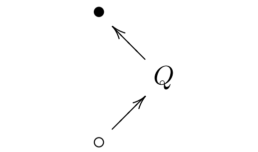
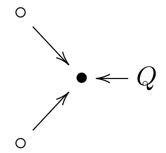
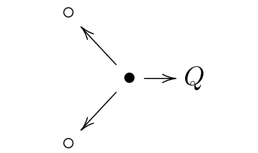
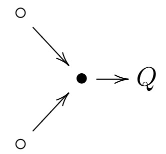
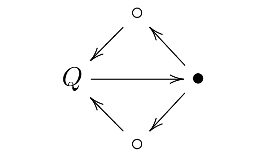
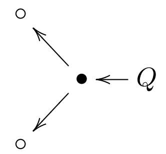
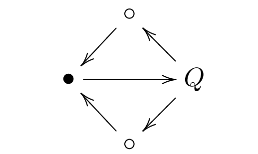
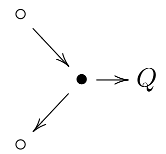
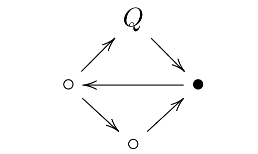
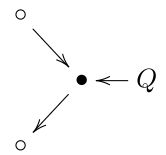
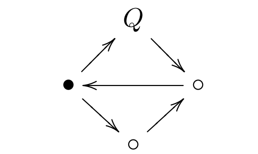
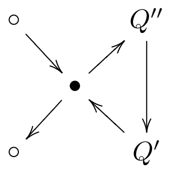
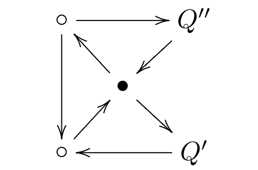
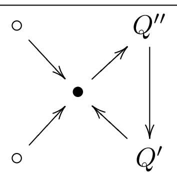
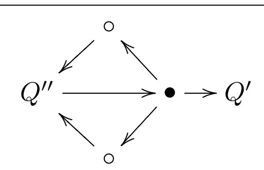
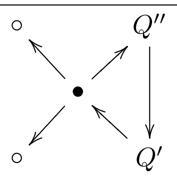
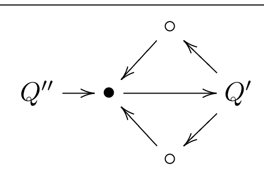
I.1		$\mu_\bullet^+$		$\mu_\bullet^-$	good
I.2		$\mu_\bullet^-$		$\mu_\bullet^+$	good
I.3a		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
I.3b		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
I.4a		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
I.4b		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
I.4c		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
I.5a		$\mu_\bullet^-$		$\mu_\bullet^+$	good
I.5b		$\mu_\bullet^+$		$\mu_\bullet^-$	good

Table 4.2. Mutations involving type I quivers.

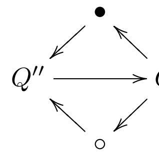
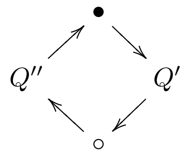
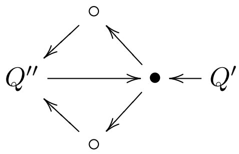
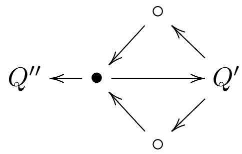
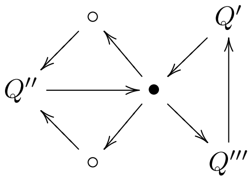
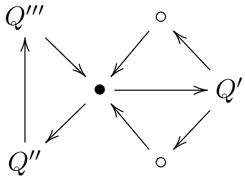
II.1		$\mu_\bullet^-, \mu_\bullet^+$		none	bad
II.2		$\mu_\bullet^-$		$\mu_\bullet^+$	good
II.3		$\mu_\bullet^-, \mu_\bullet^+$		$\mu_\bullet^-, \mu_\bullet^+$	good

Table 4.3. Mutations involving type II quivers.

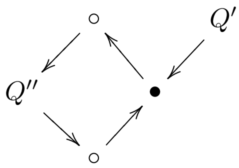
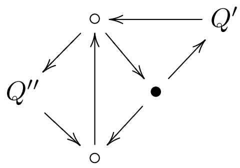
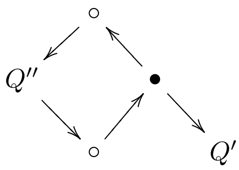
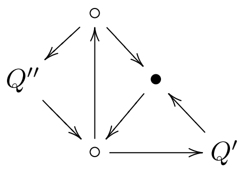
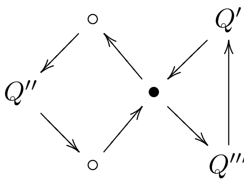
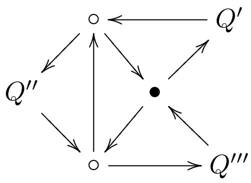
III.1		$\mu_\bullet^-$		$\mu_\bullet^+$	good
III.2		$\mu_\bullet^+$		$\mu_\bullet^-$	good
III.3		$\mu_\bullet^-, \mu_\bullet^+$		none	bad

Table 4.4. Mutations involving type III quivers.

The two algebra mutations $\mu_0^-(A')$ and $\mu_0^+(A')$ are not defined since the two compositions of the arrows $1 \to 0$ and $2 \to 0$, respectively, with the arrow $0 \to 3$ vanish in A'.

□

Lemma 4.2.20 (I.4a - I.4c). (a) *Consider the two cluster-tilted algebras A and A' with the following quivers*

$$\begin{tikzcd}[column sep=small] \bullet_1 \arrow[dr] & & \\ & \bullet_0 \arrow[r] \arrow[dl] & \bullet_3 \\ \bullet_2 & & \end{tikzcd} \qquad\qquad \begin{tikzcd}[column sep=small] & \bullet_3 \arrow[dr] & \\ \bullet_1 \arrow[ur] \arrow[dr] & & \bullet_0 \arrow[ll] \\ & \bullet_2 \arrow[ur] & \end{tikzcd} .$$

Then the mutations $\mu_0^-(A)$ and $\mu_0^+(A)$ are defined, whereas $\mu_0^-(A')$ and $\mu_0^+(A')$ are not defined.

(b) *Consider the two cluster-tilted algebras A and A' with the following quivers*

$$\begin{tikzcd}
\bullet_1 \arrow[dr] & & \\
 & \bullet_0 \arrow[dl] & \bullet_3 \arrow[l] \\
\bullet_2 & &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \bullet_3 \arrow[dr] & \\
\bullet_0 \arrow[ur] \arrow[dr] & & \bullet_2 \arrow[ll] \\
 & \bullet_1 \arrow[ur] &
\end{tikzcd} .$$

Then the mutations $\mu_0^-(A)$ and $\mu_0^+(A)$ are defined, whereas $\mu_0^-(A')$ and $\mu_0^+(A')$ are not defined.

(c) *Consider the two cluster-tilted algebras A and A' with the following quivers*

$$\begin{tikzcd}
\bullet_1 \arrow[dr] & & \bullet_4 \arrow[dd] \\
 & \bullet_0 \arrow[ur] \arrow[dl] & \\
\bullet_2 & & \bullet_3 \arrow[ul]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet_1 \arrow[rr] \arrow[dd] & & \bullet_4 \arrow[dl] \\
 & \bullet_0 \arrow[ul] \arrow[dr] & \\
\bullet_2 \arrow[ur] & & \bullet_3 \arrow[ll]
\end{tikzcd} .$$

Then the mutations $\mu_0^-(A)$ and $\mu_0^+(A)$ are defined, whereas $\mu_0^-(A')$ and $\mu_0^+(A')$ are not defined.

Proof. We use the criterion of Proposition 4.2.3.

(a) Since the arrow $1 \to 0$ does not appear in any relation in A, its composition with any non-zero path starting at 0 is non-zero. Thus, the negative mutation $\mu_0^-(A)$ is defined. Similarly, since the arrow $0 \to 2$ (or $0 \to 3$) does not appear in any relation of A, its composition with the arrow $1 \to 0$ is non-zero, and the positive mutation $\mu_0^+(A)$ is also defined.

The two algebra mutations $\mu_0^-(A')$ and $\mu_0^+(A')$ are not defined since the two compositions of the arrows $2 \to 0$ and $3 \to 0$, respectively, with the arrow $0 \to 1$ vanish in A'.

(b) Since the arrow $1 \to 0$ (or $3 \to 0$) does not appear in any relation in A, the negative mutation $\mu_0^-(A)$ is defined. Similarly, the positive mutation $\mu_0^+(A)$ is defined since the arrow $0 \to 2$ does not appear in any relation.

The two algebra mutations $\mu_0^-(A')$ and $\mu_0^+(A')$ are not defined since the two compositions of the arrow $2 \to 0$ with the arrows $0 \to 1$ and $0 \to 3$, respectively, vanish in A'.

(c) Since the arrow $1 \to 0$ does not appear in any relation of A, the negative mutation $\mu_0^-(A)$ is defined. Similarly, since $0 \to 2$ does not appear in any relation of A, the positive mutation $\mu_0^+(A)$ is also defined.

Now consider A'. The path $0, 1, 2$ is non-zero, as it equals $0, 3, 2$, but both compositions $2, 0, 1, 2$ and $4, 0, 1, 2$ vanish because of the zero-relations $2, 0, 1$ and $4, 0, 1$, hence $\mu_0^-(A')$ is not defined. Similarly, the path $1, 2, 0$ is non-zero, as it equals $1, 4, 0$, but both compositions $1, 2, 0, 1$ and $1, 2, 0, 3$ vanish because of the zero-relations $2, 0, 1$ and $2, 0, 3$, showing that $\mu_0^+(A')$ is not defined.

□

Lemma 4.2.21 (I.5a, I.5b). (a) *Consider the two cluster-tilted algebras A and A' with the following quivers*

$$\begin{tikzcd}
\bullet_1 \arrow[dr] & & \bullet_4 \arrow[dd] \\
 & \bullet_0 \arrow[ur] & \\
\bullet_2 \arrow[ur] & & \bullet_3 \arrow[ul]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \bullet_1 \arrow[dl] & & \\
\bullet_4 \arrow[rr] & & \bullet_0 \arrow[ul] \arrow[dl] \arrow[r] & \bullet_3 \\
 & \bullet_2 \arrow[ul] & &
\end{tikzcd} .$$

Then the mutations $\mu_0^-(A)$ and $\mu_0^+(A')$ are defined, whereas $\mu_0^+(A)$ and $\mu_0^-(A')$ are not defined.

(b) *Consider the two cluster-tilted algebras A and A' with the following quivers*

$$
\begin{tikzcd}
\bullet_1 & & \bullet_4 \arrow[dd] \\
& \bullet_0 \arrow[ul] \arrow[ur] \arrow[dl] & \\
\bullet_2 & & \bullet_3 \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \bullet_1 \arrow[dl] & \\
\bullet_4 \arrow[r] & \bullet_0 \arrow[rr] & & \bullet_3 \arrow[ul] \arrow[dl] \\
& & \bullet_2 \arrow[ul] &
\end{tikzcd}
.
$$

Then the mutations $\mu_0^+(A)$ and $\mu_0^-(A')$ are defined, whereas $\mu_0^-(A)$ and $\mu_0^+(A')$ are not defined.

Proof. (a) Since the arrow $1 \to 0$ (or $2 \to 0$) does not appear in any relation of A, the negative mutation $\mu_0^-(A)$ is defined. But $\mu_0^+(A)$ is not defined since the composition of the arrow $3 \to 0$ with $0 \to 4$ vanishes. Similarly for A': the positive mutation $\mu_0^+(A')$ is defined since the arrow $0 \to 3$ does not appear in any relation, and $\mu_0^-(A')$ is not defined since the composition of the arrow $0 \to 1$ with the arrow $4 \to 0$ vanishes.

(b) The negative mutation $\mu_0^-(A)$ is not defined since the composition of the arrow $0 \to 4$ with $3 \to 0$ vanishes in A. The positive mutation $\mu_0^+(A)$ is defined since the arrow $0 \to 1$ (or $0 \to 2$) does not appear in any relation of A.

$\mu_0^-(A')$ is defined since the arrow $4 \to 0$ does not appear in any relation in A', and the positive mutation $\mu_0^+(A')$ is not defined since the composition of the arrow $1 \to 0$ with the arrow $0 \to 3$ vanishes in A'.

□

Lemma 4.2.22 (II.1, II.2). (a) *Consider the two cluster-tilted algebras A and A' with the following quivers*

$$
\begin{tikzcd}
& \bullet_0 \arrow[dl] & \\
\bullet_1 \arrow[rr] & & \bullet_2 \arrow[ul] \arrow[dl] \\
& \bullet_3 \arrow[ul] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \bullet_0 \arrow[dr] & \\
\bullet_1 \arrow[ur] & & \bullet_2 \arrow[dl] \\
& \bullet_3 \arrow[ul] &
\end{tikzcd}
.
$$

Then the mutations $\mu_0^-(A)$ and $\mu_0^+(A)$ are defined, whereas $\mu_0^-(A')$ and $\mu_0^+(A')$ are not defined.

(b) *Consider the two cluster-tilted algebras A and A' with the following quivers*

$$
\begin{tikzcd}
& \bullet_3 \arrow[dl] & & \\
\bullet_2 \arrow[rr] & & \bullet_0 \arrow[ul] \arrow[dl] & \bullet_1 \arrow[l] \\
& \bullet_4 \arrow[ul] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \bullet_3 \arrow[dl] & \\
\bullet_2 & \bullet_0 \arrow[l] \arrow[rr] & & \bullet_1 \arrow[ul] \arrow[dl] \\
& & \bullet_4 \arrow[ul] &
\end{tikzcd}
.
$$

Then the mutations $\mu_0^-(A)$ and $\mu_0^+(A')$ are defined, whereas $\mu_0^+(A)$ and $\mu_0^-(A')$ are not defined.

Proof. (a) The two algebra mutations $\mu_0^-(A)$ and $\mu_0^+(A)$ are defined since the composition of the arrow $2 \to 0$ with $0 \to 1$ is not zero in A (as it equals the path $2, 3, 1$).

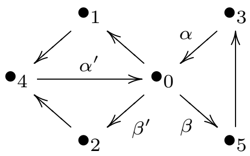
Now consider A'. The path $0, 2, 3$ is non-zero, but the composition $1, 0, 2, 3$ vanishes in A', hence $\mu_0^-(A')$ is not defined. Similarly, the path $3, 1, 0$ is non-zero, but the composition $3, 1, 0, 2$ vanishes in A', showing that $\mu_0^+(A')$ is not defined.

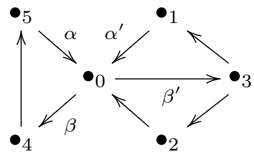
(b) Since the arrow $1 \to 0$ does not appear in any relation of A, the negative mutation $\mu_0^-(A)$ is defined. But $\mu_0^+(A)$ is not defined since the two compositions of the arrow $2 \to 0$ with the arrows $0 \to 3$ and $0 \to 4$, respectively, vanish in A.

The positive mutation $\mu_0^+(A')$ is defined since the arrow $0 \to 2$ does not appear in any relation, and $\mu_0^-(A')$ is not defined since the two compositions of the arrow $0 \to 1$ with the arrows $3 \to 0$ and $4 \to 0$, respectively, vanish in A'.

□

Lemma 4.2.23 (II.3). *Let A be one of the cluster-tilted algebras with the quivers given below*





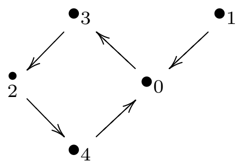
.

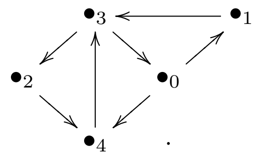
Then both algebra mutations $\mu_0^-(A)$ and $\mu_0^+(A)$ are defined.

Proof. Indeed, let $p = \gamma_2\gamma_1$ be a non-zero path starting at vertex 0 written as a concatenation of arrows. Note that p could also be just a single arrow. If $p \neq \beta$, then the composition $p \cdot \alpha$ is not zero, whereas otherwise the composition $\beta \cdot \alpha'$ is not zero, hence $\mu_0^-(A)$ is defined.

Similarly, if $p = \gamma_2\gamma_1$ is a non-zero path ending at 0, then the composition $\beta \cdot p$ is not zero if $p \neq \alpha$, and otherwise $\beta' \cdot \alpha$ is not zero, hence $\mu_0^+(A)$ is defined as well. □

Lemma 4.2.24 (III.1, III.2). (a) *Consider the two cluster-tilted algebras A and A' with the following quivers*

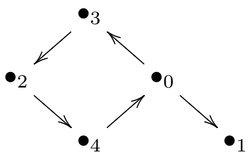


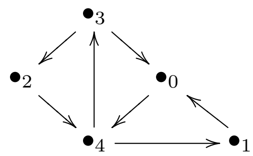


.

Then the mutations $\mu_0^-(A)$ and $\mu_0^+(A')$ are defined, whereas $\mu_0^+(A)$ and $\mu_0^-(A')$ are not defined.

(b) *Consider the two cluster-tilted algebras A and A' with the following quivers*





.

Then the mutations $\mu_0^+(A)$ and $\mu_0^-(A')$ are defined, whereas $\mu_0^-(A)$ and $\mu_0^+(A')$ are not defined.

Proof. We use the criterion of Proposition 4.2.3.

(a) The negative mutation $\mu_0^-(A)$ is defined since the arrow $1 \to 0$ does not appear in any relation. The path $2, 4, 0$ is non-zero in A, but the composition $2, 4, 0, 3$ vanishes, hence $\mu_0^+(A)$ is not defined.

The negative mutation $\mu_0^-(A')$ is not defined since the composition of the arrow $0 \to 1$ with $3 \to 0$ vanishes in A'. The positive mutation $\mu_0^+(A')$ is defined since the composition of the arrow $3 \to 0$ with $0 \to 4$ is non-zero in A, as it equals the path $3, 2, 4$.

(b) The path $0, 3, 2$ is non-zero in A, but the composition $4, 0, 3, 2$ vanishes, hence $\mu_0^-(A)$ is not defined. The positive mutation $\mu_0^+(A)$ is defined since the arrow $0 \to 1$ does not appear in any relation of A.

The negative mutation $\mu_0^-(A')$ is defined since the composition of the arrow $0 \to 4$ with $3 \to 0$ is non-zero in A, as it equals the path $3, 2, 4$. The positive mutation $\mu_0^+(A')$ is not defined since the composition of the arrow $1 \to 0$ with $0 \to 4$ vanishes in A'.

□

Lemma 4.2.25 (III.3). *Consider the two cluster-tilted algebras A and A' with the following quivers*

$$
\begin{tikzcd}
& \bullet_4 \arrow[dl] & & \bullet_1 \arrow[dl, "\alpha"'] \\
\bullet_2 \arrow[dr] & & \bullet_0 \arrow[ul, "\beta'"'] \arrow[dr, "\beta"] & \\
& \bullet_5 \arrow[ur, "\alpha'"] & & \bullet_3 \arrow[uu]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \bullet_4 \arrow[dl] \arrow[dr] & & \bullet_1 \arrow[ll] \\
\bullet_2 \arrow[dr] & & \bullet_0 \arrow[ur] \arrow[dl] & \\
& \bullet_5 \arrow[uu] \arrow[rr] & & \bullet_3 \arrow[ul]
\end{tikzcd}
.
$$

Then the mutations $\mu_0^-(A)$ and $\mu_0^+(A)$ are defined, whereas $\mu_0^-(A')$ and $\mu_0^+(A')$ are not defined.

Proof. Let $p = \gamma_2\gamma_1$ be a non-zero path in A starting at vertex 0 (written as a concatenation of at most two arrows). If $p \neq \beta$, then the composition $p \cdot \alpha$ is not zero. If $p = \beta$, then $\beta \cdot \alpha'$ is not zero, hence $\mu_0^-(A)$ is defined. Similarly, if $p = \gamma_2\gamma_1$ is a non-zero path in A ending at vertex 0, then the composition $\beta \cdot p$ is not zero if $p \neq \alpha$ and otherwise $\beta' \cdot \alpha$ is not zero. It follows that $\mu_0^+(A)$ is defined as well.

Now consider A': the path $0, 5, 4$ is non-zero in A', as it equals $0, 1, 4$, but both compositions $3, 0, 5, 4$ and $4, 0, 5, 4 = 4, 0, 1, 4$ vanish because of the zero-relations $3, 0, 5$ and $4, 0, 1$. Thus, $\mu_0^-(A')$ is not defined. Similarly, the path $5, 4, 0$ is non-zero in A', as it equals $5, 3, 0$, but both compositions $5, 4, 0, 1$ and $5, 4, 0, 5 = 5, 3, 0, 5$ vanish because of the zero-relations $4, 0, 1$ and $3, 0, 5$. It follows that $\mu_0^+(A')$ is not defined. □

Now we consider quivers of type IV. The good mutations involving these quivers are given in Table 4.5. In this table, the dotted lines indicate the central cycle, and the two vertices at the sides may be identified. The proof that all the mutations listed in Table 4.5 are good relies on the Lemmas 4.2.26 - 4.2.34 below.

Mutations at vertices on the central cycle are discussed in Lemmas 4.2.26, 4.2.30 and 4.2.34, whereas mutations at the spikes are discussed in Lemmas 4.2.27 and 4.2.32. The moves IV.1a and IV.1b in Table 4.5 follow from Corollary 4.2.28. The moves IV.2a and IV.2b follow from Corollary 4.2.33. Lemma 4.2.34 implies that there are no additional good mutations involving type IV quivers.

Lemma 4.2.26. *Let $m \geq 2$ and consider a cluster-tilted algebra A of type IV with the quiver*

(4.2.2)
$$
\begin{tikzcd}
& & \bullet_0 \arrow[dl] & & \\
& \bullet_1 \arrow[dd] & & \bullet_m \arrow[ul] \arrow[dr, dashed] & \\
Q_+ \arrow[ur, dashed] & & & & Q_- \arrow[dl, dashed] \\
& \bullet_2 \arrow[ul, dashed] \arrow[rr, dotted, no head, bend right] & & \bullet_{m-1} \arrow[uu] &
\end{tikzcd}
$$

having a central cycle $0, 1, \ldots, m$ and optional spikes Q_- and Q_+ (which coincide when $m = 2$). Then:

(a) *$\mu_0^-(A)$ is defined if and only if the spike Q_- is present.*
(b) *$\mu_0^+(A)$ is defined if and only if the spike Q_+ is present.*

Proof. We use the criterion of Proposition 4.2.3.

(a) The negative mutation $\mu_0^-(A)$ is defined if and only if the composition of the arrow $m \to 0$ with any non-zero path starting at 0 is not zero. This holds for all such paths of length smaller than $m-1$, so we only need to consider the path $0, 1, \ldots, m-1$. Now, the composition $m, 0, 1, \ldots, m-1$ vanishes if Q_- is not present, and otherwise equals the (non-zero) path $m, v_-, m-1$ where v_- denotes the root of Q_-.
(b) The positive mutation $\mu_0^+(A)$ is defined if and only if the composition of the arrow $0 \to 1$ with any non-zero path ending at 0 is not zero. This holds for all such paths of length smaller than $m-1$, so we only need to consider the path $2, \ldots, m, 0$. Now, the composition $2, \ldots, m, 0, 1$ vanishes if Q_+ is not present, and otherwise equals the (non-zero) path $2, v_+, 1$ where v_+ denotes the root of Q_+.

□

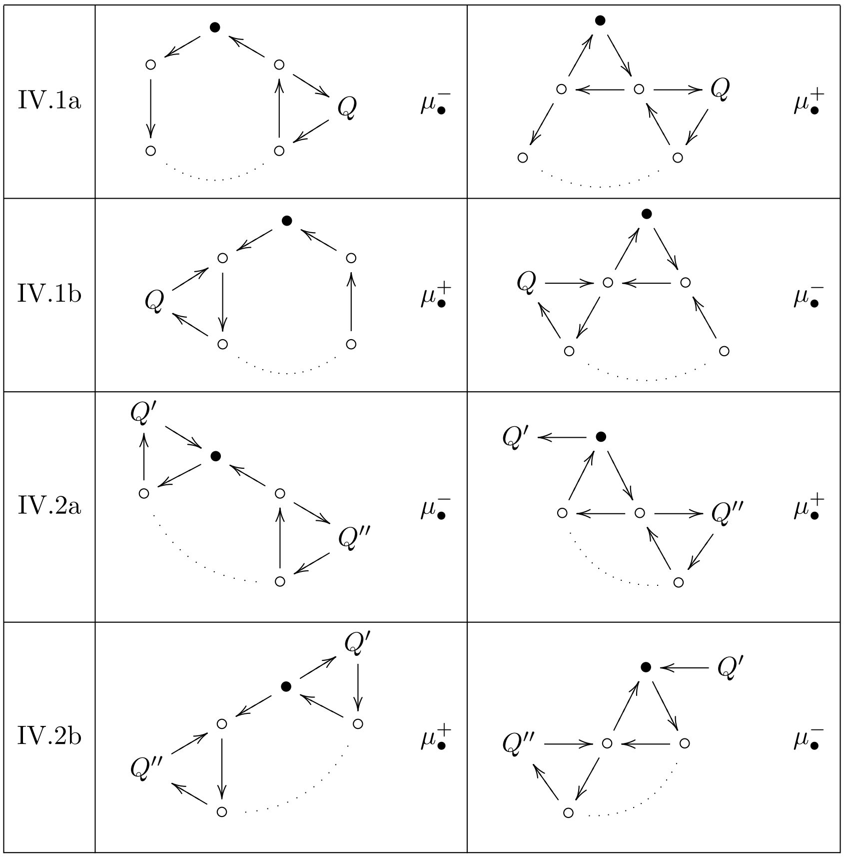
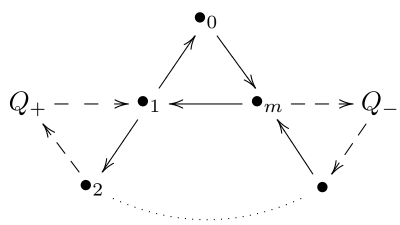


Table 4.5. Good mutations involving type IV quivers.

Lemma 4.2.27. *Let $m \geq 3$ and consider a cluster-tilted algebra A' of type IV with the quiver*

(4.2.3)



having a central cycle $1, 2, \ldots, m$ and optional spikes Q_- and Q_+. Then:

(a) $\mu_0^-(A')$ *is defined if and only if the spike Q_- is not present.*
(b) $\mu_0^+(A')$ *is defined if and only if the spike Q_+ is not present.*

Proof. We use the criterion of Proposition 4.2.3.

(a) The negative mutation $\mu_0^-(A')$ is defined if and only if the composition of the arrow $1 \to 0$ with any non-zero path starting at 0 is not zero. For the path $0, m$, the composition $1, 0, m$ equals the path $1, 2, \ldots, m$ and hence it is non-zero. This shows that $\mu_0^-(A')$ is defined when Q_- is not present. When Q_- is present, the path $0, m, v_-$ to the root v_- of Q_- is non-zero, but the composition $1, 0, m, v_-$ equals the path $1, 2, \ldots, m, v_-$ which is zero since the path $m-1, m, v_-$ vanishes.
(b) The positive mutation $\mu_0^+(A')$ is defined if and only if the composition of the arrow $0 \to m$ with any non-zero path ending at 0 is not zero. For the path $1, 0$, the composition $1, 0, m$ equals the path $1, 2, \ldots, m$ and hence it is non-zero. This shows that $\mu_0^+(A')$ is defined

when Q_+ is not present. When Q_+ is present, the path $v_+, 1, 0$ from the root v_+ of Q_+ is non-zero, but the composition $v_+, 1, 0, m$ equals the path $v_+, 1, 2, \ldots, m$ which is zero since the path $v_+, 1, 2$ vanishes.

□

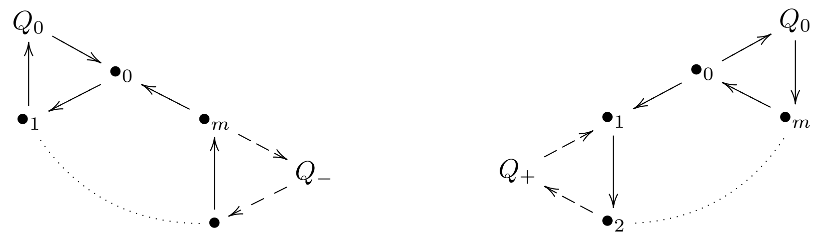
Corollary 4.2.28. *Let A be a cluster-tilted algebra corresponding to a quiver as in* (4.2.2) *with $m \geq 3$ and let A' be the one corresponding to its mutation at* 0*, as in* (4.2.3)*. The following table lists which of the algebra mutations at* 0 *are defined for A and A' depending on whether the optional spikes Q_- or Q_+ are present ("yes") or not ("no").*

Q_-	Q_+	A	A'	
yes	*yes*	μ_0^-, μ_0^+	*none*	*bad*
yes	*no*	μ_0^-	μ_0^+	*good*
no	*yes*	μ_0^+	μ_0^-	*good*
no	*no*	*none*	μ_0^-, μ_0^+	*bad*

Remark 4.2.29. Note that if $m = 2$ in (4.2.2) of Lemma 4.2.26 (and Q_- is present), then mutation at vertex 0 leads to a quiver of type III (and the quiver mutation is bad as in Table 4.3 II.1). This quiver of type III is indeed a degenerate version of (4.2.3) for $m = 2$ (see Remark 4.1.5).

Lemma 4.2.30. *Let $m \geq 2$ and consider cluster-tilted algebras A_- and A_+ of type IV with the following quivers*

(4.2.4)



having a central cycle $0, 1, \ldots, m$ and optional spikes Q_- and Q_+, respectively. Then:

(a) $\mu_0^+(A_-)$ *is never defined;*
(b) $\mu_0^-(A_-)$ *is defined if and only if the spike Q_- is present;*
(c) $\mu_0^-(A_+)$ *is never defined;*
(d) $\mu_0^+(A_+)$ *is defined if and only if the spike Q_+ is present.*

PROOF.

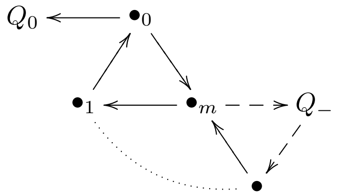
(a) Let v_0 denote the root of Q_0. Then the path $v_0, 0$ is non-zero whereas $v_0, 0, 1$ is zero.
(b) Since the path $v_0, 0, 1$ is zero, the composition of the arrow $v_0 \to 0$ with any non-trivial path starting at 0 is zero. Therefore, the negative mutation at 0 is defined if and only if the composition of the arrow $m \to 0$ with any non-zero path starting at 0 is not zero, and the proof goes in the same manner as in Lemma 4.2.26.
(c) Let v_0 denote the root of Q_0. Then the path $0, v_0$ is non-zero whereas $m, 0, v_0$ is zero.
(d) Since the path $m, 0, v_0$ is zero, the composition of the arrow $0 \to v_0$ with any non-trivial path ending at 0 is zero. Therefore, the positive mutation at 0 is defined if and only if the composition of the arrow $0 \to 1$ with any non-zero path ending at 0 is not zero, and the proof goes in the same manner as in Lemma 4.2.26.

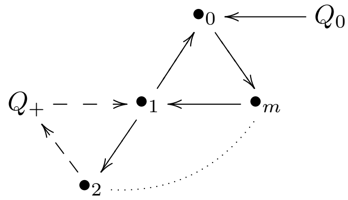
□

Remark 4.2.31. Note that if $m = 2$ in (4.2.4) of Lemma 4.2.30, then the two cluster-tilted algebras A_- and A_+ with optional spikes Q_- and Q_+ present are the same as the cluster-tilted algebras appearing at the right hand side of III.2 and III.1 in Table 4.4, respectively.

Lemma 4.2.32. *Let $m \geq 3$ and consider cluster-tilted algebras A'_- and A'_+ of type IV with the following quivers*

(4.2.5)





having a central cycle $1, \dots, m$ and optional spikes Q_- and Q_+, respectively. Then:

(a) *$\mu_0^+(A'_-)$ is always defined;*
(b) *$\mu_0^-(A'_-)$ is defined if and only if the spike Q_- is not present;*
(c) *$\mu_0^-(A'_+)$ is always defined;*
(d) *$\mu_0^+(A'_+)$ is defined if and only if the spike Q_+ is not present.*

Proof.

(a) Let v_0 denote the root of Q_0. Then the composition of any non-zero path ending at 0 with the arrow $0 \to v_0$ is not zero.
(b) Since the composition of the arrow $1 \to 0$ with any non-zero path whose first arrow is $0 \to v_0$ is not zero, we only need to consider paths whose first arrow is $0 \to m$. The proof is then the same as in Lemma 4.2.27.
(c) Let v_0 denote the root of Q_0. Then the composition of any non-zero path starting at 0 with the arrow $v_0 \to 0$ is not zero.
(d) Since the composition of the arrow $0 \to m$ with any non-zero path whose last arrow is $v_0 \to 0$ is not zero, we only need to consider paths whose last arrow is $1 \to 0$. The proof is then the same as in Lemma 4.2.27.

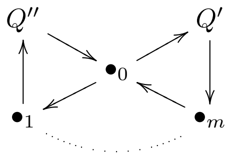
□

Corollary 4.2.33. *Let A_- and A_+ be cluster-tilted algebras corresponding to quivers as in* (4.2.4) *with $m \geq 3$ and let A'_- and A'_+ be the ones corresponding to their mutations at 0, as in* (4.2.5)*. The following tables list which of the algebra mutations at 0 are defined for A_-, A'_-, A_+ and A'_+ depending on whether the optional spikes Q_- or Q_+ are present ("yes") or not ("no").*

Q_-	A_-	A'_-	
yes	μ_0^-	μ_0^+	*good*
no	*none*	μ_0^-, μ_0^+	*bad*

Q_+	A_+	A'_+	
yes	μ_0^+	μ_0^-	*good*
no	*none*	μ_0^-, μ_0^+	*bad*

Lemma 4.2.34. *Let $m \geq 2$ and consider a cluster-tilted algebra A of type IV with the following quiver*



having a central cycle $0, 1, \dots, m$. Then the algebra mutations $\mu_0^-(A)$ and $\mu_0^+(A)$ are never defined.

Proof. Denote by v', v'' the roots of Q' and Q'', respectively.

Consider the path $0, 1, \dots, m$. It is non-zero, since it equals the path $0, v', m$. However, its composition with the arrow $v'' \to 0$ is zero since the path $v'', 0, 1$ vanishes, and its composition with the arrow $m \to 0$ is zero as well, since it equals $m, 0, v', m$ and the path $m, 0, v'$ vanishes. By Proposition 4.2.3, the mutation $\mu_0^-(A)$ is not defined.

Now, consider the path $1, \dots, m, 0$. It is non-zero, since it equals the path $1, v'', 0$. However, its composition with the arrow $0 \to v'$ is zero since the path $m, 0, v'$ vanishes, and its composition with the arrow $0 \to 1$ is zero as well, since it equals $1, v'', 0, 1$ and the path $v'', 0, 1$ vanishes. By Proposition 4.2.3, the mutation $\mu_0^+(A)$ is also not defined. □

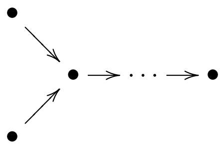
4.2.4. Main theorem for good mutation equivalences. In this section we state the results concerning the good mutation equivalence classification in type D_n.

Proposition 4.2.35. *Given two quivers which are mutation equivalent to D_n in parametric notation (i.e. specified by their type I,II,III,IV and the parameters), there is an algorithm which decides whether or not the corresponding cluster-tilted algebras are good mutation equivalent.*

We also provide a list of 'canonical forms' for good mutation equivalence.

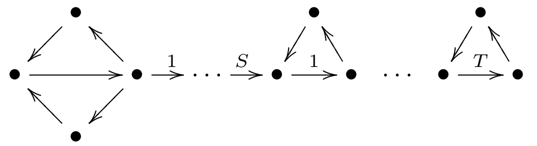
Theorem 4.2.36. *A cluster-tilted algebra of type D_n is good mutation equivalent to one (and only one) of the cluster-tilted algebras with the following quivers:*

(a) *D_n (i.e. type I with a linearly oriented A_{n-2} quiver attached):*



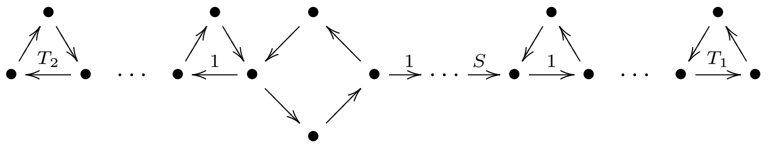
;

(b) *Type II as in the following figure, where $S, T \geq 0$ and $S + 2T = n - 4$:*



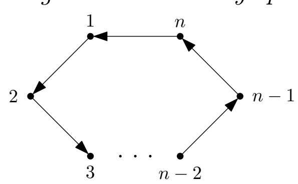
;

(c) *Type III with parameters $(S, T_1, 0, T_2)$ as in the following figure, with $S \geq 0$, the non-negative integers T_1, T_2 are considered up to rotation of the sequence (T_1, T_2), i.e. up to interchanging T_1 and T_2, and $S + 2(T_1 + T_2) = n - 4$:*



;

(d_1) *Type IV with a central cycle of length n without any spikes:*

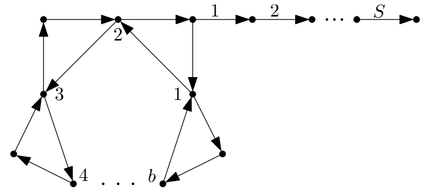


;

($d_{2,1}$) *Type IV with parameter sequence (as defined in Definition 4.1.4)*

$$\big((1, S, 0), (1, 0, 0), \dots, (1, 0, 0)\big)$$

for some $S \geq 0$, where the number of the triples is $b \geq 3$ such that $n = 2b + S$ and the attached rooted quiver of type A is linearly oriented of type A_{S+1}:



;

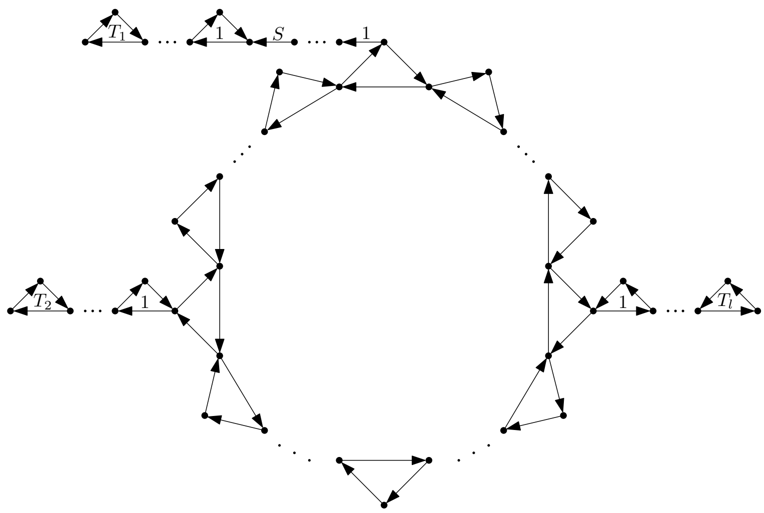
($d_{2,2}$) *Type IV with parameter sequence*

$$\Big(\underbrace{(1, S, T_1), (1, 0, 0), \dots, (1, 0, 0)}_{b_1}, \underbrace{(1, 0, T_2), (1, 0, 0), \dots, (1, 0, 0)}_{b_2}, \dots, \underbrace{(1, 0, T_l), (1, 0, 0), \dots, (1, 0, 0)}_{b_l}\Big)$$

which is a concatenation of $l \geq 1$ sequences with positive numbers of triples $b_1, b_2, \dots, b_l$ whose sum is not smaller than 3, with $S \geq 0$ and $T_1, \dots, T_l > 0$ considered up to rotation of the sequence

$$\big((b_1, T_1), (b_2, T_2), \dots, (b_l, T_l)\big),$$

$n = 2(b_1 + \dots + b_l + T_1 + \dots + T_l) + S$ and the attached rooted quivers of type A are in standard form:

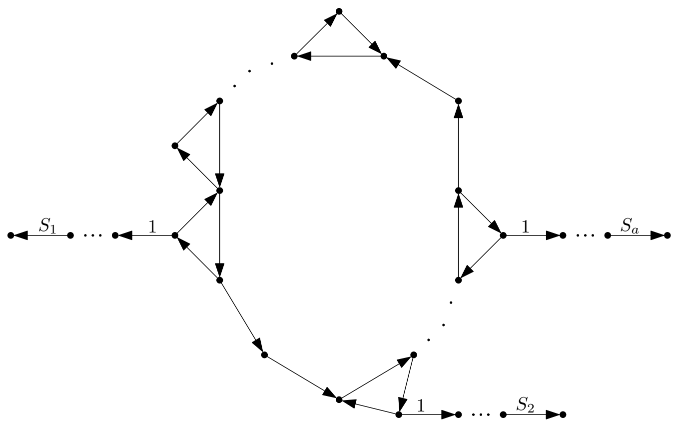


;

(d$_{3,1}$) *Type IV with parameter sequence*

$$\big((1,0,0), \dots, (1,0,0), (3, S_1, 0), \dots, (3, S_a, 0)\big)$$

for some $a > 0$, where the number of the triples $(1,0,0)$ is $b \geq 0$, the sequence of non-negative integers $(S_1, \dots, S_a)$ is considered up to a cyclic permutation, $n = 4a + 2b + S_1 + \dots + S_a$ and the attached rooted quivers of type A are in standard form (i.e. linearly oriented $A_{S_1+1}, \dots, A_{S_a+1}$):



;

(d$_{3,2}$) *Type IV with parameter sequence which is a concatenation of $l \geq 1$ sequences of the form*

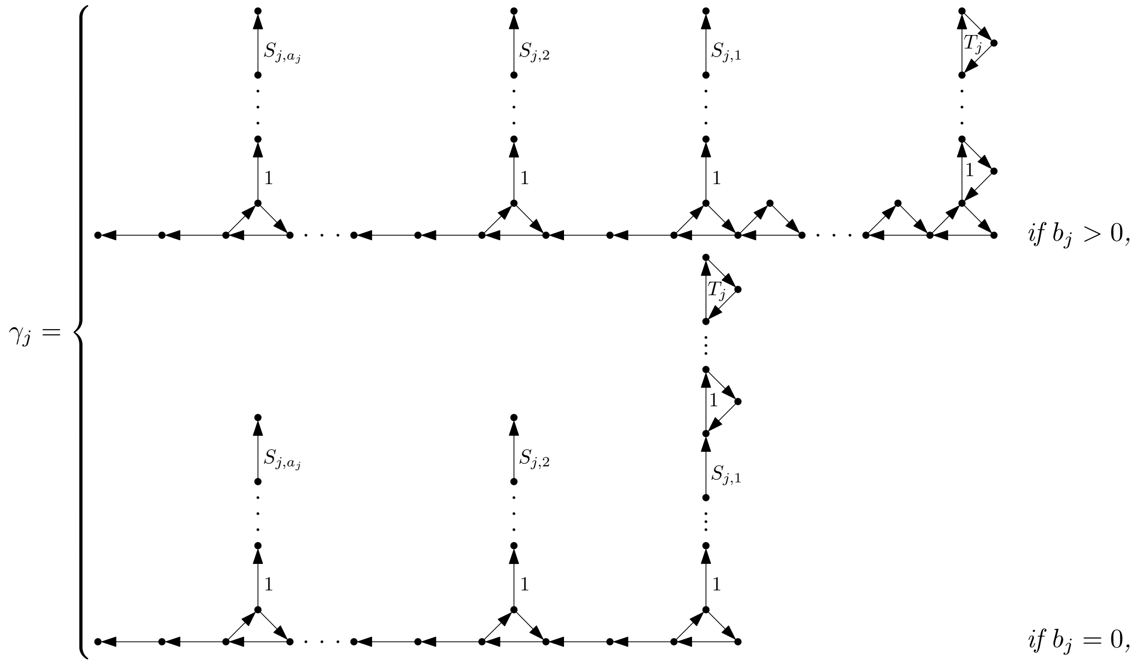
$$\gamma_j = \begin{cases} \big(\underbrace{(1,0,T_j), (1,0,0), \dots, (1,0,0)}_{b_j}, (3, S_{j,1}, 0), (3, S_{j,2}, 0), \dots, (3, S_{j,a_j}, 0)\big) & \text{if } b_j > 0, \\ \big((3, S_{j,1}, T_j), (3, S_{j,2}, 0), \dots, (3, S_{j,a_j}, 0)\big) & \text{otherwise,} \end{cases}$$

where each sequence γ_j for $1 \leq j \leq l$ is defined by non-negative integers a_j and b_j not both zero. The integer b_j is the number of triples with distance 1. For each j there is a sequence of a_j non-negative integers $S_{j,1}, \dots, S_{j,a_j}$ and a positive integer T_j. Note that we require that not all the a_j are zero. All these numbers are considered up to rotation of the l-term *sequence*

$$\Big(\big(b_1,(S_{1,1},\dots,S_{1,a_1}),T_1\big),\big(b_2,(S_{2,1},\dots,S_{2,a_2}),T_2\big),\dots,\big(b_l,(S_{l,1},\dots,S_{l,a_l}),T_l\big)\Big),$$

they satisfy $n=\sum_{j=1}^{l}(4a_j+2b_j+S_{j,1}+\cdots+S_{j,a_j}+2T_j)$*, and the attached rooted quivers of type* A *are in standard form.*

That is, the quiver is a concatenation of $l\geq 1$ *quivers* γ_j *of the form*



where the last vertex of γ_l *is glued to the first vertex of* γ_1.

For the proofs we start by describing all the good mutations determined in the previous subsection using the parametric notation of Section 4.1 which will be useful later. Note that by Proposition 4.2.15, any two rooted quivers of type A with the same parameters $s(Q)$ and $t(Q)$ are good mutation equivalent. Thus, two quivers which are mutation equivalent to D_n with the same type and parameters are indeed equivalent by good mutations, so the parametric notation makes sense.

Each row of Table 4.6 describes a good mutation between two quivers of cluster-tilted algebras of type D_n given in parametric form. All entries in that table follow immediately from the Tables 4.2 - 4.5. The numbers s',s'',s''',t',t'',t''' are arbitrary, non-negative integers, and correspond to the parameters of the rooted quivers Q', Q'', Q''' of type A appearing in the corresponding pictures of Tables 4.2, 4.3 and 4.4 (indicated by the column “Move”). Additionally, the numbers s_1,s_2,t_1,t_2 are arbitrary, non-negative integers, and correspond to the parameters of the rooted quivers of type A appearing in Table 4.5; and the numbers d_1,d_2 are positive integers according to the distances between the spikes appearing in the corresponding pictures of Table 4.5.

Remark 4.2.37. A careful look at Table 4.6 shows that one can regard type III quivers with parameters (s',t',s'',t'') as ‘formal’ type IV quivers with parameters $\big((1,s',t'),(1,s'',t'')\big)$ (see also Remark 4.1.5). Indeed, the good mutation moves III.1 and III.2 in Table 4.6 then become just specific cases of moves IV.2b and IV.2a, respectively.

By looking at the first six rows of the table we can immediately draw the following conclusions:

Lemma 4.2.38. *Consider quivers of type I or II.*

(a) *The set consisting of the quivers of type I or II is closed under good mutations.*
(b) *The set consisting of all the orientations of a* D_n *diagram is closed under good mutations.*
(c) *A quiver of type I with parameters* $(s,t+1)$ *for some* $s,t\geq 0$ *is equivalent by good mutations to one of type II with parameters* $(s+1,t,0,0)$.

Move	Type	Parameters	Type	Parameters	Remarks
I.1	I	(s', t')	I	(s', t')	
I.2	I	$(s'+1, t')$	I	$(s'+1, t')$	
I.5a	I	$(s'+s'', t'+t''+1)$	II	$(s'+1, t', s'', t'')$	
I.5b	I	$(s'+s'', t'+t''+1)$	II	$(s', t', s''+1, t'')$	
II.2	II	$(s'+1, t', s'', t'')$	II	$(s', t', s''+1, t'')$	
II.3	II	$(s'+s''', t'+t'''+1, s'', t'')$	II	$(s', t', s''+s''', t''+t'''+1)$	
III.1	III	$(s'+1, t', s'', t'')$	IV	$\big((2, s', t'), (1, s'', t'')\big)$	
III.2	III	$(s'+1, t', s'', t'')$	IV	$\big((1, s', t'), (2, s'', t'')\big)$	
IV.1a	IV	$\big((d_1, s_1, t_1), (d_2, s_2, t_2), \dots\big)$	IV	$\big((1, s_1, t_1), (d_1-2, 0, 0), (d_2, s_2, t_2), \dots\big)$	$d_1 \geq 4$
IV.1b	IV	$\big((d_1, s_1, t_1), (d_2, s_2, t_2), \dots\big)$	IV	$\big((d_1-2, s_1, t_1), (1, 0, 0), (d_2, s_2, t_2), \dots\big)$	$d_1 \geq 4$
IV.2a	IV	$\big((2, s_1, t_1), (d_2, s_2, t_2), \dots\big)$	IV	$\big((1, s_1, t_1), (d_2, s_2+1, t_2), \dots\big)$	
IV.2b	IV	$\big((2, s_1, t_1), (d_2, s_2, t_2), \dots\big)$	IV	$\big((1, s_1+1, t_1), (d_2, s_2, t_2), \dots\big)$	

Table 4.6. All good mutations in parametric form.

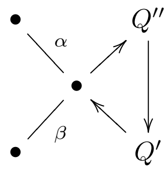
(d) *A quiver of type II with parameters (s', t', s'', t'') is equivalent by good mutations to one of type II with parameters $(s'+s'', t'+t'', 0, 0)$.*

(e) *Two quivers of type II with parameters $(s_1, t_1, 0, 0)$ and $(s_2, t_2, 0, 0)$ are equivalent by good mutations if and only if $s_1 = s_2$ and $t_1 = t_2$.*

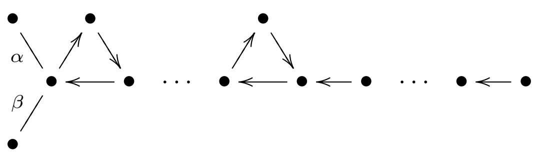
Proof. (a) All good mutation moves I.1 - II.3 in Table 4.6 involving quivers of type I or type II yield a quiver of type I and type II, respectively.

(b) Any orientation of a D_n diagram is a quiver of type I with parameters $(s, 0)$. Hence, the only possible good mutations for such a quiver are the ones appearing in I.1 and I.2, respectively. Then the claim follows since the number of oriented 3-cycles remains unchanged in both moves.

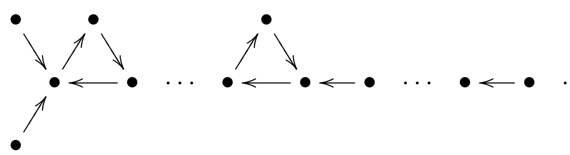
(c) Let Q be a quiver of type I with parameters $(s, t+1)$. By applying the good mutation moves 1, 2a, 3 and 4 of Table 4.1, we can assume that Q has the following form



where the orientations of α and β can be chosen arbitrarily. Using again the good mutation moves 1, 2a, 3 and 4 of Table 4.1, we can transform the rooted quiver of type A to the following form (similar to the proof of Proposition 4.2.15)

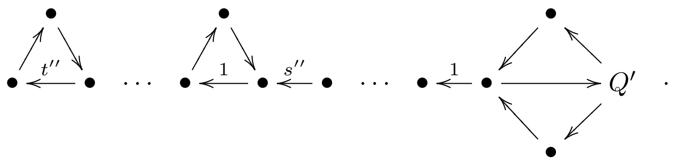


and this quiver of type I has still the parameters $(s, t+1)$. Using move I.1 of Table 4.2 we can assume that α and β are oriented as follows:

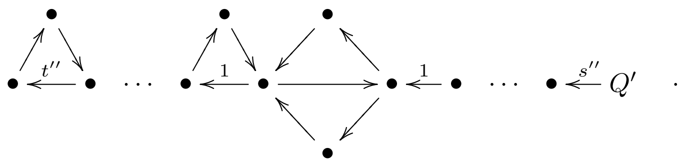


Applying I.5a of Table 4.2 yields a quiver of type II with parameters $(s+1, t, 0, 0)$.

(d) Let Q be a quiver of type II with parameters (s', t', s'', t''). By applying the good mutation moves 1, 2a, 3 and 4 of Table 4.1, we can assume that the second rooted quiver Q'' of type A is in standard form as in (4.2.1):



If $s'' \neq 0$, we can apply move II.2 s''-times and get a quiver of type II with parameters $(s' + s'', t', 0, t'')$:



If $t'' \neq 0$, we can apply move II.3 t''-times to get a quiver of type II with parameters $(s' + s'', t' + t'', 0, 0)$.

(e) Follows from Proposition 4.2.15 and Theorem 4.2.18.

□

Lemma 4.2.39. *Consider quivers of type III.*

(a) *A quiver of type III with parameters $(s' + 1, t', s'', t'')$ is good mutation equivalent to one of type III with parameters $(s', t', s'' + 1, t'')$.*

(b) *A quiver of type III with parameters (s', t', s'', t'') is good mutation equivalent to one of type III with parameters $(s' + s'', t', 0, t'')$.*

PROOF. (a) We have

$$\begin{aligned} \mathrm{III}(s'+1, t', s'', t'') &\xrightarrow{\mathrm{III.1}} \mathrm{IV}\big((2, s', t'), (1, s'', t'')\big) \simeq \mathrm{IV}\big((1, s'', t''), (2, s', t')\big) \\ &\xrightarrow{\mathrm{III.2}} \mathrm{III}(s''+1, t'', s', t') \simeq \mathrm{III}(s', t', s''+1, t'') \end{aligned}$$

where the isomorphisms follow from rotational symmetries.

(b) Follows from the first part.

□

Remark 4.2.40. The previous lemma shows that in type III, it is possible to move linear parts in the rooted quivers of type A from side to side by using good mutations. It is not possible, however, to move oriented 3-cycles by good mutations (see III.3 in Table 4.4 and Example 4.3.5 below).

For the next two lemmas we need the following terminology for spikes in quivers of type IV. Spikes are *consecutive* if the distance (d_i) between them is 1. A spike is *free* if the attached rooted quiver of type A consists of just a single vertex.

Lemma 4.2.41. *Consider quivers of type IV. A free spike at the end of a group of at least two consecutive spikes can be moved by good mutations to the next group of consecutive spikes. In other words, the two quivers with parameters $\big((1, s, t), (d, 0, 0), \dots\big)$ and $\big((d, s, t), (1, 0, 0), \dots\big)$ are connected by good mutations.*

PROOF. Since the free spike is at the end of a group of at least two consecutive spikes, the distance d is at least two. Then

$$\big((1, s, t), (d, 0, 0), \dots\big) \xrightarrow{\mathrm{IV.1a}} \big((d+2, s, t), \dots\big) \xrightarrow{\mathrm{IV.1b}} \big((d, s, t), (1, 0, 0), \dots\big).$$

□

Lemma 4.2.42. *Arrows in a rooted quiver of type A attached to a spike in a group of consecutive spikes in a quiver of type IV can be moved by good mutations to a rooted quiver attached to any spike in that group. Note that by 'arrows' we mean arrows which are not part of an oriented* 3*-cycle.*

PROOF. It suffices to show that the two quivers with parameters $\big((1,s_1,t_1),(d_2,s_2+1,t_2),\dots\big)$ and $\big((1,s_1+1,t_1),(d_2,s_2,t_2),\dots\big)$ are good mutation equivalent. Indeed,

$$\big((1,s_1,t_1),(d_2,s_2+1,t_2),\dots\big)\xrightarrow{\text{IV.2a}}\big((2,s_1,t_1),(d_2,s_2,t_2),\dots\big)\xrightarrow{\text{IV.2b}}\big((1,s_1+1,t_1),(d_2,s_2,t_2),\dots\big).$$

□

Now we prove Proposition 4.2.35 and Theorem 4.2.36. We first observe that, by Lemma 4.2.38, the set of quivers of types I or II is closed under good mutations, and moreover the lemma completely characterises good mutation equivalence among these quivers in terms of their parameters, leading to the classes (a) and (b) in Theorem 4.2.36. We also observe that the cyclic quiver of type IV without any spikes does not admit any good mutations, thus it falls into a separate equivalence class (d_1). Therefore we are left to deal with only quivers of types III and IV (with spikes). Before describing the algorithm, we introduce some notation.

Notation. Given $r\geq 1$ and a non-empty subset $\mathcal{I}\subseteq\{1,2,\dots,r\}$, we define the following two partitions of the set $\{1,2,\dots,r\}$. Write the elements of $\mathcal{I}$ in increasing order $1\leq i_1<i_2<\dots<i_l\leq r$ for $l=|\mathcal{I}|$, and define the intervals

$$\begin{aligned}
i_1^+ &= \{i_1,i_1+1,\dots,i_2-1\} & i_1^- &= \{i_l+1,\dots,r,1,\dots,i_1\}\\
i_2^+ &= \{i_2,i_2+1,\dots,i_3-1\} & i_2^- &= \{i_1+1,\dots,i_2-1,i_2\}\\
&\vdots & &\vdots\\
i_l^+ &= \{i_l,\dots,r,1,\dots,i_1-1\} & i_l^- &= \{i_{l-1}+1,\dots,i_l-1,i_l\}
\end{aligned}$$

We call the partition $i_1^+\cup i_2^+\cup\dots\cup i_l^+$ the *positive* partition defined by $\mathcal{I}$. Similarly, we call $i_1^-\cup i_2^-\cup\dots\cup i_l^-$ the *negative* partition defined by $\mathcal{I}$.

Notation. We partition the set of positive integers as $N_1\cup N_2\cup N_3$, where

$$N_1=\{1\},\qquad N_2=\{n\geq 2\,:\,n\text{ is even}\},\qquad N_3=\{n\geq 3\,:\,n\text{ is odd}\}.$$

Notation. Given a sequence $\big((d_1,s_1,t_1),(d_2,s_2,t_2),\dots,(d_r,s_r,t_r)\big)$ of triples of non-negative integers and a subset $\mathcal{I}$ of $\{1,\dots,r\}$, we define the quantities

$$\begin{aligned}
a(\mathcal{I}) &= |\{i\in\mathcal{I}\,:\,d_i\in N_3\}|,\\
b(\mathcal{I}) &= |\{i\in\mathcal{I}\,:\,d_i\in N_1\}|+\sum_{i\in\mathcal{I}\,:\,d_i\in N_2}\frac{d_i}{2}+\sum_{i\in\mathcal{I}\,:\,d_i\in N_3}\frac{d_i-3}{2},\\
s(\mathcal{I}) &= |\{i\in\mathcal{I}\,:\,d_i\in N_2\}|+\sum_{i\in\mathcal{I}}s_i.
\end{aligned}$$

Notation. We call two sequences $(v_0,v_1,\dots,v_{m-1})$ and $(w_0,w_1,\dots,w_{m-1})$ *cyclic equivalent* if there is some $0\leq j\leq m$ such that $w_i=v_{(i+j)\bmod m}$ for all $0\leq i<m$.

Definition 4.2.43. We define the set $\mathcal{S}$ of *good mutation parameters* as a disjoint union of the following five sets. We also define an equivalence relation $\sim$ on $\mathcal{S}$ inside each set, and agree that elements from different sets are inequivalent.

(c) Triples (T_1,T_2,S) of non-negative integers. $(T_1,T_2,S)\sim(T_1',T_2',S')$ if and only if $S=S'$ and (T_1,T_2), (T_1',T_2') are cyclic equivalent, i.e. $(T_1',T_2')=(T_1,T_2)$ or $(T_1',T_2')=(T_2,T_1)$.

($\mathrm{d}_{2,1}$) Pairs (b,S) with $b\geq 3$ and $S\geq 0$. $(b,S)\sim(b',S')$ if and only if $b=b'$ and $S=S'$.

($\mathrm{d}_{2,2}$) Pairs

$$\Big(\big((b_1,T_1),(b_2,T_2),\dots,(b_l,T_l)\big),S\Big)$$

for some $l \geq 1$, where the numbers b_j, T_j are positive, $b_1 + \cdots + b_l \geq 3$ and $S \geq 0$. Two such pairs are equivalent if and only if $S = S'$ and $\big((b_1, T_1), (b_2, T_2), \ldots, (b_l, T_l)\big)$, $\big((b'_1, T'_1), (b'_2, T'_2), \ldots, (b'_l, T'_l)\big)$ are cyclic equivalent.

($\mathrm{d}_{3,1}$) Pairs $\big(b, (S_1, \ldots, S_a)\big)$ where $b \geq 0$ and $(S_1, \ldots, S_a)$ is a sequence of $a > 0$ non-negative integers. Two such pairs are equivalent if and only if $b = b'$ and $(S_1, \ldots, S_a)$, $(S'_1, \ldots, S'_a)$ are cyclic equivalent.

($\mathrm{d}_{3,2}$) Sequences

$$\Big(\big(b_1, (S_{1,1}, \ldots, S_{1,a_1}), T_1\big), \big(b_2, (S_{2,1}, \ldots, S_{2,a_2}), T_2\big), \ldots, \big(b_l, (S_{l,1}, \ldots, S_{l,a_l}), T_l\big)\Big),$$

of any length $l \geq 1$, where for any $1 \leq j \leq l$ the numbers a_j, b_j are non-negative integers not both zero, $(S_{j,1}, \ldots, S_{j,a_j})$ is a (possibly empty) sequence of a_j non-negative integers and T_j is a positive integer. The relation $\sim$ is just cyclic equivalence.

Remark 4.2.44. It is easy to decide whether or not two good mutation parameters are equivalent, because this only involves checking for cyclic equivalence.

Now we are able to describe the algorithm.

Algorithm 4.2.45 (Good mutation class). Let

$$\big((d_1, s_1, t_1), (d_2, s_2, t_2), \ldots, (d_r, s_r, t_r)\big)$$

be a non-empty sequence of triples of non-negative integers such that

- $d_i \geq 1$ and $s_i, t_i \geq 0$ for all $1 \leq i \leq r$,
- $d_1 + d_2 + \cdots + d_r \geq 2$ and $(d_1, \ldots, d_r) \neq (2)$.

This sequence parametrises a quiver of type III (see Remark 4.1.5) or a quiver of type IV with spikes. The algorithm then outputs the class (c), ($\mathrm{d}_{2,1}$), ($\mathrm{d}_{2,2}$), ($\mathrm{d}_{3,1}$) or ($\mathrm{d}_{3,2}$) and the corresponding good mutation parameters of that class as specified in Definition 4.2.43. The algorithm is as follows:

1. Compute the subsets
$$\mathcal{I}_D = \{1 \leq i \leq r \,:\, d_i \in N_3\}, \qquad\qquad \mathcal{I}_T = \{1 \leq i \leq r \,:\, t_i > 0\}.$$
2. If $\mathcal{I}_D = \emptyset$ and $\mathcal{I}_T = \emptyset$, set b, S as
$$b = b(\{1, 2, \ldots, r\}), \qquad\qquad S = s(\{1, 2, \ldots, r\}).$$
If $b \geq 3$, we are in class ($\mathrm{d}_{2,1}$) with good mutation parameters (b, S). Otherwise we are in class (c) with good mutation parameters $(0, 0, S)$.
3. If $\mathcal{I}_D = \emptyset$ and $\mathcal{I}_T \neq \emptyset$, enumerate the elements of $\mathcal{I}_T$ in increasing order as $\mathcal{I}_T = \{i_1 < i_2 < \cdots < i_l\}$ with $l = |\mathcal{I}_T|$, and set b_j, T_j for $1 \leq j \leq l$ and S as
$$b_j = b(i_j^+), \qquad\qquad T_j = t_{i_j}, \qquad\qquad S = s(\{1, 2, \ldots, r\}).$$
If $b_1 + \cdots + b_l \geq 3$, we are in class ($\mathrm{d}_{2,2}$) with good mutation parameters $\big(((b_1, T_1), (b_2, T_2), \ldots, (b_l, T_l)), S\big)$. Otherwise, we are in class (c) with good mutation parameters $(T_1, 0, S)$ if $l = 1$ or (T_1, T_2, S) if $l = 2$.
4. If $\mathcal{I}_D \neq \emptyset$ and $\mathcal{I}_T = \emptyset$, enumerate the elements of $\mathcal{I}_D$ in increasing order as $\mathcal{I}_D = \{i_{1,1} < i_{1,2} < \cdots < i_{1,a}\}$ and set a, b and $S_1, \ldots, S_a$ as
$$a = a(\{1, 2, \ldots, r\}) = |\mathcal{I}_D|, \qquad\qquad b = b(\{1, 2, \ldots, r\}), \qquad\qquad S_j = s(i_{1,j}^-).$$
We are in class ($\mathrm{d}_{3,1}$) with good mutation parameters $\big(b, (S_1, \ldots, S_a)\big)$.
5. If $\mathcal{I}_D \neq \emptyset$ and $\mathcal{I}_T \neq \emptyset$, enumerate the elements of $\mathcal{I}_T$ in increasing order as $\mathcal{I}_T = \{i_1 < i_2 < \cdots < i_l\}$ with $l = |\mathcal{I}_T|$. For any $1 \leq j \leq l$,
    - Enumerate the elements of $i_j^+ \cap \mathcal{I}_D$ (where the positive partition is taken with respect to the subset $\mathcal{I}_T$) in the order they appear within the interval i_j^+ as $i_{j,1} < i_{j,2} < \cdots < i_{j,a(i_j^+)}$.

- Set a_j, b_j, T_j and $S_{j,1}, \dots, S_{j,a_j}$ as
$a_j = a(i_j^+), \quad b_j = b(i_j^+), \quad T_j = t_{i_j}, \quad (S_{j,1}, \dots, S_{j,a_j}) = \big(s(i_{j,1}^-), s(i_{j,2}^-), \dots, s(i^-_{j,a(i_j^+)})\big)$
with the positive partition taken with respect to $\mathcal{I}_T$ and the negative one with respect to $\mathcal{I}_D$.

We are in class $(\mathrm{d}_{3,2})$ with good mutation parameters

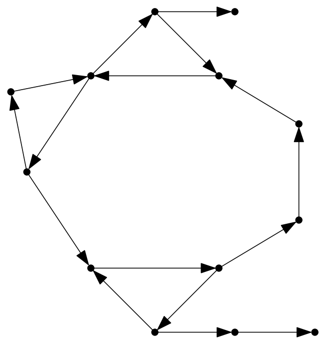
$$\Big(\big(b_1, (S_{1,1}, \dots, S_{1,a_1}), T_1\big), \big(b_2, (S_{2,1}, \dots, S_{2,a_2}), T_2\big), \dots, \big(b_l, (S_{l,1}, \dots, S_{l,a_l}), T_l\big)\Big).$$

Algorithm 4.2.45 computes a map $\Sigma : \mathcal{Q} \to \mathcal{S}$ from the set $\mathcal{Q}$ of all quivers of types III or IV (with spikes) to the set $\mathcal{S}$ of good mutation parameters. On the other hand, the canonical forms stated in Theorem 4.2.36 can be viewed as a map $Q : \mathcal{S} \to \mathcal{Q}$. In particular, the numerical data in Definition 4.2.43 and Theorem 4.2.36 are the same. We also have two natural equivalence relations on the sets $\mathcal{S}$ and $\mathcal{Q}$: the equivalence relation $\sim$ defined on $\mathcal{S}$ via cyclic equivalence, and the good mutation equivalence on $\mathcal{Q}$, which we also denote by $\sim$.

Now we present examples for Algorithm 4.2.45.

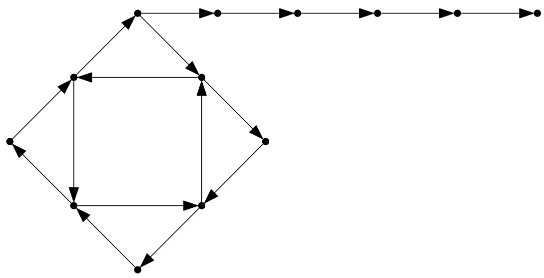
Example 4.2.46. The numbering of the examples refers to the steps in Algorithm 4.2.45.

2a) Let $q = ((1,1,0),(2,0,0),(4,2,0))$ be a sequence of triples corresponding to the following quiver (with the rooted quivers of type A in standard form):



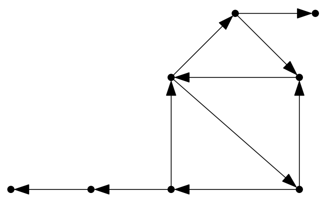
.

In the first step of Algorithm 4.2.45 we have to compute the sets $\mathcal{I}_D$ and $\mathcal{I}_T$, but both sets are empty. Thus, we are in Step 2 of the algorithm. Then $b = b(\{1,2,3\}) = 4 \geq 3$ and $S = s(\{1,2,3\}) = 5$ and thus, we are in class $(\mathrm{d}_{2,1})$ with good mutation parameters $(b, S) = (4, 5)$. According to Theorem 4.2.36 these correspond to a quiver in canonical form with parameter sequence $((1,5,0),(1,0,0),(1,0,0),(1,0,0))$:



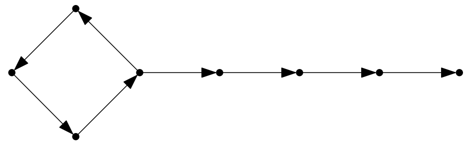
.

2b) Let $q = ((1,1,0),(2,2,0))$ be a sequence corresponding to the following quiver (with the rooted quivers of type A in standard form):



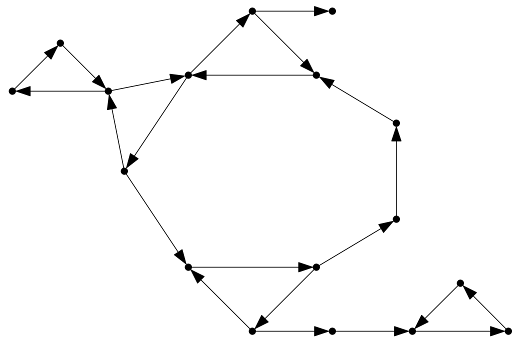
.

Since both sets $\mathcal{I}_D$ and $\mathcal{I}_T$ are empty, we are in Step 2 of the algorithm. Then $b = 2 < 3$ and $S = 4$ and thus, we are in class (c) with good mutation parameters $(0, 0, S) = (0, 0, 4)$. Then the canonical form of Theorem 4.2.36 has parameters $(4, 0, 0, 0)$ and quiver



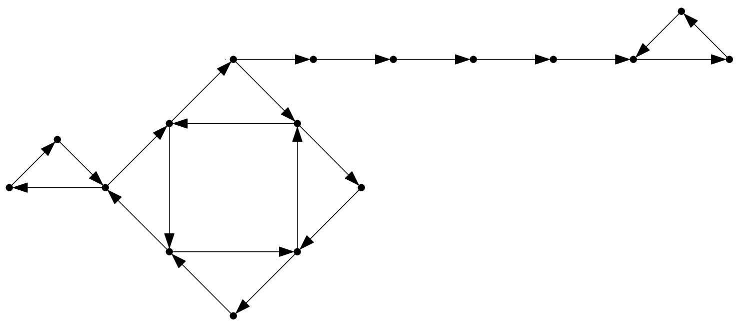
.

3a) Let $q = ((1, 1, 0), (2, 0, 1), (4, 2, 1))$ be a sequence corresponding to the following quiver (with the rooted quivers of type A in standard form):



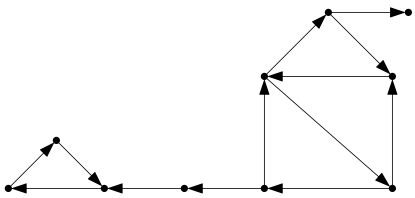
.

The set $\mathcal{I}_D$ is empty and $\mathcal{I}_T = \{2, 3\}$. Thus, we are in Step 3 of the algorithm. We compute $i_1^+ = 2^+ = \{2\}$ and $i_2^+ = 3^+ = \{3, 1\}$. Then we get $b_1 = b(i_1^+) = 1$ and $b_2 = b(i_2^+) = 3$, i.e. $b_1 + b_2 = 4 \geq 3$. Additionally, we have $S = s(\{1, 2, 3\}) = 5$ and $T_1 = t_2 = 1$, $T_2 = t_3 = 1$. We are in class $(\mathrm{d}_{2,2})$ with good mutation parameters $\big(((b_1, T_1), (b_2, T_2)), S\big) = \big(((1, 1), (3, 1)), 5\big)$, and the corresponding canonical form has parameter sequence $((1, 5, 1), (1, 0, 1), (1, 0, 0), (1, 0, 0))$:



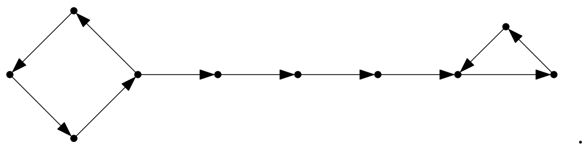
.

3b) Let $q = ((1, 1, 0), (2, 2, 1))$ be a sequence corresponding to the following quiver (with the rooted quivers of type A in standard form):

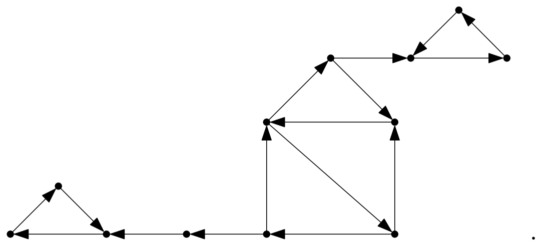


.

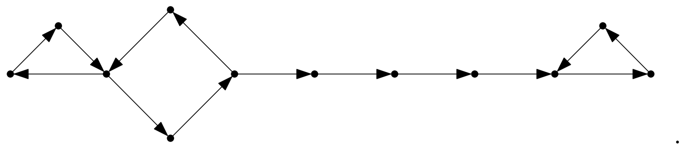
The set $\mathcal{I}_D$ is empty and $\mathcal{I}_T = \{2\}$. Hence, we are in Step 3 of the algorithm. We compute $i_1^+ = 2^+ = \{2, 1\}$ and get $b_1 = b(\{2, 1\}) = 2 < 3$. Additionally, we have $S = s(\{1, 2\}) = 4$ and $T_1 = t_2 = 1$. Hence, we are in class (c) with good mutation parameters $(T_1, 0, S) = (1, 0, 4)$. The corresponding canonical form has parameters $(4, 1, 0, 0)$:



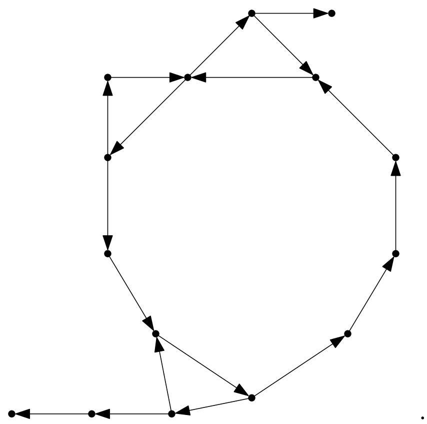
3c) Let $q = ((1,1,1),(2,2,1))$ be a sequence corresponding to the following quiver (with the rooted quivers of type A in standard form):



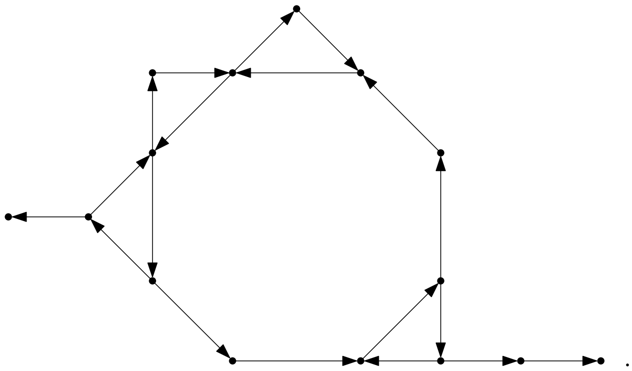
The set $\mathcal{I}_D$ is empty and $\mathcal{I}_T = \{1,2\}$. Hence, we are in Step 3 of the algorithm. We compute $i_1^+ = 1^+ = \{1\}$, $i_2^+ = 2^+ = \{2\}$ and get $b_1 = 1$, $b_2 = 1$, i.e. $b_1 + b_2 = 2 < 3$. We also have $S = 4$ and $T_1 = t_1 = 1$, $T_2 = t_2 = 1$. We are in class (c) with good mutation parameters $(T_1, T_2, S) = (1,1,4)$. The corresponding canonical form has parameters $(4,1,0,1)$



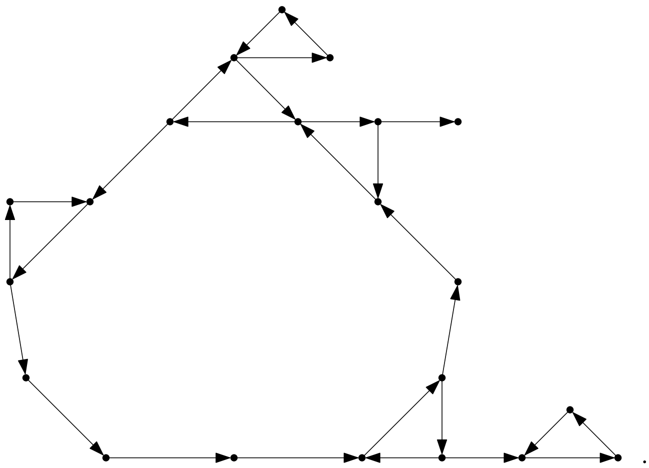
4) Let $q = ((1,1,0),(3,0,0),(5,2,0))$ be a sequence corresponding to the following quiver (with the rooted quivers of type A in standard form):



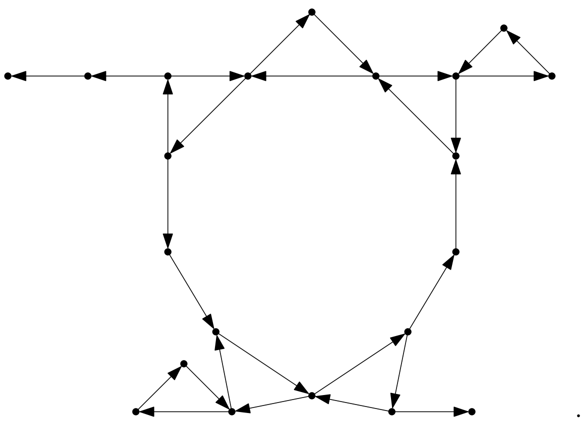
The set $\mathcal{I}_T$ is empty and $\mathcal{I}_D = \{2,3\}$. Thus, we are in Step 4 of the algorithm. We compute $i_{1,1}^- = 2^- = \{1,2\}$ and $i_{1,2}^- = 3^- = \{3\}$ and get $S_1 = s(i_{1,1}^-) = 1$, $S_2 = s(i_{1,2}^-) = 2$. We also have $a = a(\{1,2,3\}) = |\mathcal{I}_D| = 2$ and $b = b(\{1,2,3\}) = 2$. We are in class $(\mathrm{d}_{3,1})$ with good mutation parameters $(b,(S_1,S_2)) = (2,(1,2))$. The corresponding canonical form has parameter sequence $((1,0,0),(1,0,0),(3,1,0),(3,2,0))$



5) Let $q = ((1,1,0),(2,0,1),(5,0,0),(3,1,1))$ be a sequence corresponding to the following quiver (with the rooted quivers of type A in standard form):



We compute the sets $\mathcal{I}_T = \{2,4\}$ and $\mathcal{I}_D = \{3,4\}$. Thus, we are in Step 5 of the algorithm. Additionally, we compute $i_1^+ = 2^+ = \{2,3\}$ and $i_2^+ = 4^+ = \{4,1\}$. We also have $a_1 = a(\{2,3\}) = 1$, $a_2 = a(\{4,1\}) = 1$ and $b_1 = b(\{2,3\}) = 2$, $b_2 = b(\{4,1\}) = 1$. Then we get $i_1^+ \cap \mathcal{I}_D = \{3\}$, $i_2^+ \cap \mathcal{I}_D = \{4\}$ and hence, $i_{1,1}^- = 3^- = \{1,2,3\}$ and $i_{2,1}^- = 4^- = \{4\}$. Finally, we have $T_1 = t_2 = 1$, $T_2 = t_4 = 1$ and $(S_{1,1}) = (s(i_{1,1}^-)) = (2)$, $(S_{2,1}) = (s(i_{2,1}^-)) = (1)$. We are in class $(\mathrm{d}_{3,2})$ with good mutation parameters $((b_1,(S_{1,1}),T_1),(b_2,(S_{2,1}),T_2)) = ((2,(2),1),(1,(1),1))$. According to Theorem 4.2.36 the corresponding canonical form has parameter sequence $(\underbrace{(1,0,1),(1,0,0),(3,2,0)}_{\gamma_1},\underbrace{(1,0,1),(3,1,0)}_{\gamma_2})$



The correctness of the output of Algorithm 4.2.45 is guaranteed by the following proposition.

Proposition 4.2.47. *Let $q, q' \in \mathcal{Q}$ and $\sigma, \sigma' \in \mathcal{S}$.*

1. *If $q \sim q'$ then $\Sigma(q) \sim \Sigma(q')$.*
2. *If $\sigma \sim \sigma'$ then $Q(\sigma) \sim Q(\sigma')$.*
3. *If $\sigma \in \mathcal{S}$ then $\Sigma(Q(\sigma)) = \sigma$. In other words, applying Algorithm 4.2.45 to a canonical form as in Theorem 4.2.36 recovers the parameters of that form.*
4. *If $q \in \mathcal{Q}$ then there exists $\sigma \in \mathcal{S}$ such that $q \sim Q(\sigma)$. In other words, a quiver can be transformed by good mutations to a quiver in canonical form as in Theorem 4.2.36.*

Proof. Let $q = ((d_1, s_1, t_1), (d_2, s_2, t_2), \dots, (d_r, s_r, t_r)) \in \mathcal{Q}$ and let $\mathcal{Q}_2$ and $\mathcal{Q}_{\geq 4}$ be the subsets of $\mathcal{Q}$ consisting of the elements q with $d_1 = 2$ and $d_1 \geq 4$, respectively. According to Table 4.6 (moves III.1 - IV.2b) and Lemmas 4.2.39, 4.2.41 and 4.2.42 we can define five functions,

$$\mu_0 : \mathcal{Q} \to \mathcal{Q}, \qquad \mu_{1a}, \mu_{1b} : \mathcal{Q}_{\geq 4} \to \mathcal{Q}, \qquad \mu_{2a}, \mu_{2b} : \mathcal{Q}_2 \to \mathcal{Q},$$

corresponding to the rotation and good mutation moves as follows:

$$\mu_0(q) = \big((d_2, s_2, t_2), \dots, (d_r, s_r, t_r), (d_1, s_1, t_1)\big),$$

and if $d_1 \geq 4$,

$$\mu_{1a}(q) = \big((1, s_1, t_1), (d_1 - 2, 0, 0), (d_2, s_2, t_2), \dots, (d_r, s_r, t_r)\big),$$
$$\mu_{1b}(q) = \big((d_1 - 2, s_1, t_1), (1, 0, 0), (d_2, s_2, t_2), \dots, (d_r, s_r, t_r)\big).$$

Finally, if $d_1 = 2$, that is, $q = \big((2, s_1, t_1), (d_2, s_2, t_2), \dots, (d_r, s_r, t_r)\big)$, we set

$$\mu_{2a}(q) = \big((1, s_1, t_1), (d_2, s_2 + 1, t_2), \dots, (d_r, s_r, t_r)\big),$$
$$\mu_{2b}(q) = \big((1, s_1 + 1, t_1), (d_2, s_2, t_2), \dots, (d_r, s_r, t_r)\big).$$

Note that the use of this notation for good mutations is different from the use of notation for mutations elsewhere in this thesis. In particular, the subscript refers to the type of good mutations in Table 4.6 rather than the vertex at which the mutation is performed. This will not cause any ambiguity because it is used only in the proof of this proposition.

(1) We have to check that $\Sigma(\mu_{(-)}(q)) \sim \Sigma(q)$ for $(-) \in \{0, 1a, 1b, 2a, 2b\}$. Note that since the parameter sequence $((d_1, s_1, t_1), (d_2, s_2, t_2), \dots, (d_r, s_r, t_r))$ is defined up to rotation, $\Sigma(\mu_0(q)) \sim \Sigma(q)$ ensures that the output of Algorithm 4.2.45 is well-defined.

First we look at $\mu_0(q)$. We indicate the parameters of $\Sigma(\mu_0(q))$ by a zero. We observe that $|\mathcal{I}_D| = |\mathcal{I}_D^0|$ and $|\mathcal{I}_T| = |\mathcal{I}_T^0|$.

- If $\mathcal{I}_D = \mathcal{I}_T = \emptyset$, then $b^0 = b(\{1, \dots, r\}) = b$, $S^0 = s(\{1, \dots, r\}) = S$, and thus, $\Sigma(q) = \Sigma(\mu_0(q))$.
- Suppose $\mathcal{I}_D \neq \emptyset$ and $\mathcal{I}_T = \emptyset$, i.e. we are in Step 4 of Algorithm 4.2.45. Then $a^0 = |\mathcal{I}_D^0| = |\mathcal{I}_D| = a$ and $b^0 = b(\{1, \dots, r\}) = b$.
  If $i_{1,1} = 1$, i.e. $d_1 \in N_3$, then
  $$i^0_{1,j} = \begin{cases} i_{1,j+1} - 1, & \text{for } 1 \leq j < a \\ i_{1,1} - 1 \equiv r, & \text{for } j = a \end{cases}.$$
  Thus, $S^0_j = s((i^0_{1,j})^-) = S_{j+1}$ for $1 \leq j < a$ and $S^0_a = s((i^0_{1,a})^-) = S_1$.
  Then $\Sigma(q) = (b, (S_1, \dots, S_a))$ and $\Sigma(\mu_0(q)) = (b, (S_2, \dots, S_a, S_1))$. Hence, $\Sigma(q)$ and $\Sigma(\mu_0(q))$ are cyclic equivalent.
  If $i_{1,1} \geq 2$, i.e. $d_1 \notin N_3$, then $i^0_{1,j} = i_{1,j} - 1$ for $1 \leq j \leq a$. Thus, $S^0_j = S_j$ for all j and $\Sigma(q) = \Sigma(\mu_0(q))$.
- Suppose $\mathcal{I}_D = \emptyset$ and $\mathcal{I}_T \neq \emptyset$, i.e. we are in Step 3 of Algorithm 4.2.45. Then $S^0 = s(\{1, \dots, r\}) = S$.
  If $i_1 = 1$, i.e. $t_1 > 0$, then
  $$i^0_j = \begin{cases} i_{j+1} - 1, & \text{for } 1 \leq j < l \\ i_1 - 1 \equiv r, & \text{for } j = l \end{cases}.$$

Thus, $b_j^0 = b((i_j^0)^+) = b_{j+1}$ for $1 \leq j < l$ and $b_l^0 = b((i_l^0)^+) = b_1$. Additionally, $T_j^0 = t_{i_j}^0 = T_{j+1}$ for $1 \leq j < l$ and $T_l^0 = t_{i_l}^0 = t_{i_1} = T_1$.
If $b_1 + \cdots + b_l \geq 3$, then $\Sigma(q) = (((b_1, T_1), \ldots, (b_l, T_l)), S)$ and $\Sigma(\mu_0(q)) = (((b_2, T_2), \ldots, (b_l, T_l), (b_1, T_1)), S)$ are cyclic equivalent.
If $b_1 + \cdots + b_l < 3$ and $l = 1$, then $\Sigma(q) = (T_1, 0, S) = \Sigma(\mu_0(q))$.
If $b_1 + \cdots + b_l < 3$ and $l = 2$, then $\Sigma(q) = (T_1, T_2, S) \sim (T_2, T_1, S) = \Sigma(\mu_0(q))$.

If $i_1 \geq 2$, i.e. $t_1 = 0$, then $i_j^0 = i_j - 1$ for all $1 \leq j \leq l$. Thus, $b_j^0 = b_j$, $T_j^0 = T_j$ for $1 \leq j \leq l$ and $\Sigma(q) = \Sigma(\mu_0(q))$.

- Finally, suppose $\mathcal{I}_D \neq \emptyset$ and $\mathcal{I}_T \neq \emptyset$, i.e. we are in Step 5 of Algorithm 4.2.45.
As above, we get

$$i_j^0 = \begin{cases} i_{j+1} - 1, & \text{for } 1 \leq j < l \text{ and } i_1 = 1 \\ i_1 - 1 \equiv r, & \text{for } j = l \text{ and } i_1 = 1 \\ i_j - 1, & \text{for } i_1 > 1 \end{cases}.$$

Now we look at $(i_j^0)^+ \cap \mathcal{I}_D^0$.
If $i_1 = 1$, then

$$i_{j,p}^0 = \begin{cases} i_{1,p}, & \text{for } j = l \\ i_{j+1,p} - 1, & \text{for } 1 \leq j < l \end{cases},$$

for $1 \leq p \leq a(i_1^+)$ and $1 \leq p \leq a(i_{j+1}^+)$, respectively.
Then $a_j^0 = a((i_j^0)^+) = a_{j+1}$, $a_l^0 = a_1$, $b_j^0 = b_{j+1}$, $b_l^0 = b_1$, $T_j^0 = T_{j+1}$ and $T_l^0 = T_1$ for all $1 \leq j < l$. Continuing, we get $(S_{j,1}^0, \ldots, S_{j,a_j}^0) = (S_{j+1,1}, \ldots, S_{j+1,a_{j+1}})$ for $j < l$ and $(S_{l,1}^0, \ldots, S_{l,a_l}^0) = (S_{1,1}, \ldots, S_{1,a_1})$.
Hence, $\Sigma(q) = ((b_1, (S_{1,1}, \ldots, S_{1,a_1}), T_1), \ldots, (b_l, (S_{l,1}, \ldots, S_{l,a_l}), T_l))$ is equivalent to $\Sigma(\mu_0(q)) = ((b_2, (S_{2,1}, \ldots, S_{2,a_2}), T_2), \ldots, (b_l, (S_{l,1}, \ldots, S_{l,a_l}), T_l), (b_1, (S_{1,1}, \ldots, S_{1,a_1}), T_1))$.

If $i_1 \geq 2$, then $i_{j,p}^0 = i_{j,p} - 1 \mod r$, for $1 \leq j \leq l$ and $1 \leq p \leq a(i_j^+)$. Thus, $a_j^0 = a_j$, $b_j^0 = b_j$, $T_j^0 = T_j$, $(S_{j,1}^0, \ldots, S_{j,a_j}^0) = (S_{j,1}, \ldots, S_{j,a_j})$ for $1 \leq j \leq l$ and $\Sigma(q) = \Sigma(\mu_0(q))$.

Now, we consider $\mu_{1a/1b}(q)$. We indicate the parameters of $\Sigma(\mu_{1a/1b}(q))$ by a prime. Observe that $|\mathcal{I}_D| = |\mathcal{I}_D'|$ and $|\mathcal{I}_T| = |\mathcal{I}_T'|$ since for $d_1 \geq 4$ we have $d_1 - 2 \in \begin{cases} N_3 & \text{if } d_1 \in N_3 \\ N_2 & \text{if } d_1 \in N_2 \end{cases}$.

- If $\mathcal{I}_D = \mathcal{I}_T = \emptyset$, then $b' = b$ since the distance 1 in $(1, s_1, t_1)$ and $(1, 0, 0)$, respectively, counts one in b' and the distance $d_1 - 2$ in $(d_1 - 2, 0, 0)$, resp. $(d_1 - 2, s_1, t_1)$, counts one less in b' than d_1 in b. In addition, $S' = S$ since $d_1 - 2 \in N_2$ if $d_1 \in N_2$. Hence, $\Sigma(q) = \Sigma(\mu_{1a/1b}(q))$.
- Suppose $\mathcal{I}_D \neq \emptyset$ and $\mathcal{I}_T = \emptyset$, i.e. we are in Step 4 of Algorithm 4.2.45. Then $a' = |\mathcal{I}_D'| = |\mathcal{I}_D| = a$ and $b' = b$ with the same arguments as above.

If $1 \in \mathcal{I}_D$, i.e. $d_1 \geq 4$ is odd, then also $d_1 - 2 \geq 2$ is odd and $2 \in \mathcal{I}_D'$ for $\mu_{1a}(q)$ or $1 \in \mathcal{I}_D'$ for $\mu_{1b}(q)$, respectively. Then

$$i_{1,1}' = \begin{cases} 2, & \text{for } \mu_{1a}(q) \\ 1, & \text{for } \mu_{1b}(q) \end{cases},$$

and $i_{1,j}' = i_{1,j} + 1$ for $j > 1$. Thus, $S_j' = s((i_{1,j}')^-) = S_j$ for all $1 \leq j \leq a$ and $\Sigma(q) = \Sigma(\mu_{1a/1b}(q))$.

If $1 \notin \mathcal{I}_D$, i.e. $d_1 \geq 4$ is even, then $d_1 - 2 \geq 2$ is also even. Moreover, $1 \notin \mathcal{I}_D'$ for $\mu_{1a}(q)$ and $2 \notin \mathcal{I}_D'$ for $\mu_{1b}(q)$, respectively, since the distances are one. Then $i_{1,j}' = i_{1,j} + 1$, $i_{1,1}' \geq 3$ and $1, 2 \in (i_{1,1}')^- = \{i_{1,a}' + 1, \ldots, r, 1, \ldots, i_{1,1}'\}$. Hence, $S_j' = s((i_{1,j}')^-) = S_j$ for all j and $\Sigma(q) = \Sigma(\mu_{1a/1b}(q))$.

- Suppose $\mathcal{I}_D = \emptyset$ and $\mathcal{I}_T \neq \emptyset$, i.e. we are in Step 3 of Algorithm 4.2.45.
  If $i_1 = 1$, i.e. $t_1 > 0$, then also $i'_1 = 1$ for $\mu_{1a/b}(q)$ and $2 \in (i'_1)^+ = \{i'_1, i'_1+1, \ldots, i'_2-1\}$ since $i'_2 \geq 3$. Hence, $S' = S$ and $T'_j = T_j$ for all $1 \leq j \leq l$. The only possible difference between $\Sigma(q)$ and $\Sigma(\mu_{1a/b}(q))$ may occur for the parameters b_1 and b'_1, respectively. However, $b'_1 = b_1$ since the distance 1 in $(1, s_1, t_1)$ (resp., $(1,0,0)$) counts one in b'_1 and the distance $d_1 - 2$ in $(d_1-2,0,0)$ (resp., (d_1-2, s_1, t_1)) counts one less in b'_1 than d_1 in b. Thus, $\Sigma(q) = \Sigma(\mu_{1a/1b}(q))$.
  If $i_1 \geq 2$, i.e. $t_1 = 0$, then $i'_1 \geq 3$ for $\mu_{1a/b}(q)$ and $1, 2 \in (i'_l)^+ = \{i'_l, \ldots, r, 1, \ldots, i'_1-1\}$. Hence, $S' = S$ and $T'_j = T_j$ for all $1 \leq j \leq l$. Using the same arguments as above we get $b'_l = b_l$. Thus, $\Sigma(q) = \Sigma(\mu_{1a/1b}(q))$.
- Finally, suppose $\mathcal{I}_D \neq \emptyset$ and $\mathcal{I}_T \neq \emptyset$, i.e. we are in Step 5 of Algorithm 4.2.45. As above we get
  $$i'_j = \begin{cases} 1, & \text{if } i_1 = 1 \text{ and } j = 1 \\ i_j + 1, & \text{if } (i_1 = 1 \text{ and } j \geq 2) \text{ or } (i_1 \geq 2) \end{cases}.$$
  Then
  $$(i'_j)^+ = \begin{cases} i_1^+ \cup \{i_2\}, & \text{if } i_1 = 1 \text{ and } j = 1 \\ ((i_l^+) + 1) \cup \{1\}, & \text{if } i_1 \geq 2 \text{ and } j = l \\ (i_j^+) + 1, & \text{otherwise} \end{cases},$$
  where $(i_j^+) + 1$ means that each element of i_j^+ has increased by one.
  Now we look at $(i'_j)^+ \cap \mathcal{I}'_D$. For $\mu_{1a}(q)$ we get $i'_{j,p} = i_{j,p} + 1$ for all $1 \leq j \leq l$, $1 \leq p \leq a((i'_j)^+)$ and thus, $a((i'_j)^+) = a(i_j^+) = a_j$ for all j.
  For $\mu_{1b}(q)$ we have to consider two cases. If $d_1 \notin N_3$, i.e. $1 \notin \mathcal{I}_D$, then $i'_{j,p} = i_{j,p} + 1$ for all j, p and thus $a((i'_j)^+) = a(i_j^+) = a_j$ for all j. If $d_1 \in N_3$, i.e. $1 \in \mathcal{I}_D$, then
  $$i'_{j,p} = \begin{cases} 1, & \text{if } (i_1 = 1 \text{ and } j = p = 1) \text{ or } (i_1 \geq 2 \text{ and } j = l, p = a((i'_l)^+)) \\ i_{j,p} + 1, & \text{otherwise} \end{cases}.$$
  However, $a((i'_j)^+) = a(i_j^+) = a_j$ since if $1 \in \mathcal{I}_D$, then $1 \in \mathcal{I}'_D$ and $2 \notin \mathcal{I}'_D$.
  Using the same arguments as the case $\mathcal{I}_D = \emptyset$, $\mathcal{I}_T \neq \emptyset$, we get $b((i'_j)^+) = b(i_j^+) = b_j$ for all j. We also get $S'_{j,p} = s((i'_{j,p})^-) = S_{j,p}$ and thus, $\Sigma(q) = \Sigma(\mu_{1a/1b}(q))$.

At the end, we look at $\mu_{2a/2b}(q)$. We indicate the parameters of $\Sigma(\mu_{2a/2b}(q))$ by a prime. We observe that $\mathcal{I}_D = \mathcal{I}'_D$ and $\mathcal{I}_T = \mathcal{I}'_T$ since we only change $d_1 = 2$ to $d'_1 = 1$ and s_1, s_2.

- If $\mathcal{I}_D = \mathcal{I}_T = \emptyset$, then $b' = b$ since $d_1 = 2$ is replaced by $d'_1 = 1$ and both count one in b and b', respectively. Additionally, $S' = S$ since there is only one more arrow in $\mu_{2a/2b}(q)$ than in q, and there is also one distance less in N_2 in $\mu_{2a/2b}(q)$ (namely, d'_1). Thus, $\Sigma(q) = \Sigma(\mu_{2a/2b}(q))$.
- Let $\mathcal{I}_D \neq \emptyset$ and $\mathcal{I}_T = \emptyset$, i.e. we are in Step 4 of Algorithm 4.2.45. Then $a' = |\mathcal{I}_D| = a$ and $b' = b$ using the same arguments as above.
  $i'_{1,j} = i_{1,j}$ for all $1 \leq j \leq a' = a$ and $S'_j = s((i'_{1,j})^-) = S_j$ for all $2 \leq j \leq a$.
  Moreover, since $d_1 = 2$ and $d'_1 = 1$ we have $1 \notin \mathcal{I}_D$ and $i'_{1,1} = i_{1,1} \geq 2$. Thus, $1, 2 \in i_{1,1}^- = \{i_{1,a}+1, \ldots, r, 1, \ldots, i_{1,1}\}$. We get the following table for the indices 1 and 2 in $s(i_{1,1}^-)$:

	q	$\mu_{2a}(q)$	$\mu_{2b}(q)$
distances	$d_1 = 2, d_2$	$1, d_2$	$1, d_2$
arrows	s_1, s_2	$s_1, s_2 + 1$	$s_1 + 1, s_2$

  Thus, $S'_1 = s(i_{1,1}^-) = S_1$ is the same in all three cases and $\Sigma(q) = \Sigma(\mu_{2a/2b}(q))$.
- Let $\mathcal{I}_D = \emptyset$ and $\mathcal{I}_T \neq \emptyset$, i.e. we are in Step 3 of Algorithm 4.2.45.
  We have $i'_j = i_j$ for all $1 \leq j \leq l$ and thus, $T'_j = T_j$ for all j. $S' = S$ with the same arguments as the case $\mathcal{I}_D = \mathcal{I}_T = \emptyset$. In addition, $b'_j = b_j$ for all $2 \leq j \leq l$.

The only possible differences may occur for the parameter b'_1. However, $b'_1 = b(i_1^+) = b(\{i_1, i_1+1, \ldots, i_2-1\}) = b_1$ since the distance 1 of $(1, s_1, t_1)$ (resp., $(1, s_1+1, t_1)$) counts one in b'_1 and this is the same as the distance $d_1 = 2$ counts in b_1. Hence, $\Sigma(q) = \Sigma(\mu_{2a/2b}(q))$.

- Finally, let $\mathcal{I}_D \neq \emptyset$ and $\mathcal{I}_T \neq \emptyset$, i.e. we are in Step 5 of Algorithm 4.2.45.
  We have $i'_j = i_j$ for all $1 \leq j \leq l$ and thus, $i'_{j,p} = i_{j,p}$ for all $1 \leq j \leq l$ and $1 \leq p \leq a(i_j^+)$, $a'_j = a_j$ and $T'_j = T_j$ for all j. In addition, $b'_j = b_j$ for all $1 \leq j \leq l$ with the same arguments as the case $\mathcal{I}_D = \emptyset$, $\mathcal{I}_T \neq \emptyset$.
  The only possible differences may occur for the parameter $S'_{1,1}$. However, $S'_{1,1} = s(i_{1,1}^-) = S_{1,1}$ with the same arguments as the case $\mathcal{I}_D \neq \emptyset$, $\mathcal{I}_T = \emptyset$ and thus, $S'_{j,p} = S_{j,p}$ for all j and for all p. Hence, $\Sigma(q) = \Sigma(\mu_{2a/2b}(q))$.

We have shown that $\Sigma(\mu_{(-)}(q)) \sim \Sigma(q)$ for $(-) \in \{0, 1a, 1b, 2a, 2b\}$.

(2) Let $\sigma, \sigma' \in \mathcal{S}$ with $\sigma \sim \sigma'$. Since elements from different sets of $\mathcal{S}$ are inequivalent (see Definition 4.2.43) σ and σ' have to be in the same set. Thus, we have to consider five different cases.

(c) Let $\sigma = (T_1, T_2, S)$ and $\sigma' = (T'_1, T'_2, S')$. Since $\sigma \sim \sigma'$ we get $S = S'$ and (T_1, T_2) is cyclic equivalent to (T'_1, T'_2), i.e. $\sigma' = \sigma$ or $\sigma' = (T_2, T_1, S)$. In the first case it is clear that $Q(\sigma) = Q(\sigma')$, so suppose that $\sigma' = (T_2, T_1, S)$. Then $Q(\sigma) = ((1, S, T_1)(1, 0, T_2))$ and $Q(\sigma') = ((1, S, T_2)(1, 0, T_1))$. Now, we apply $\mu_{2a}(\mu_{2b}^{-1})$ to $Q(\sigma)$ S-times and get $Q(\sigma) \sim ((1, 0, T_1), (1, S, T_2))$. Then $Q(\sigma') = \mu_0((1, 0, T_1), (1, S, T_2))$ and thus, $Q(\sigma) \sim Q(\sigma')$.

($\mathrm{d}_{2,1}$) Let $\sigma = (b, S)$ and $\sigma' = (b', S')$. Then $(b, S) \sim (b', S')$ if and only if $b = b'$ and $S = S'$. Thus, $\sigma = \sigma'$ and $Q(\sigma) = Q(\sigma')$.

($\mathrm{d}_{2,2}$) Let $\sigma = \big(((b_1, T_1), (b_2, T_2), \ldots, (b_l, T_l)), S\big)$ and $\sigma' = \big(((b'_1, T'_1), (b'_2, T'_2), \ldots, (b'_l, T'_l)), S'\big)$. Then $S = S'$ and $\big(((b_1, T_1), (b_2, T_2), \ldots, (b_l, T_l)), S\big)$ is cyclic equivalent to $\big(((b'_1, T'_1), (b'_2, T'_2), \ldots, (b'_l, T'_l)), S'\big)$. Hence, it suffices to show the claim for $\sigma' = \big(((b_2, T_2), \ldots, (b_l, T_l), (b_1, T_1)), S\big)$.
We get

$$\begin{aligned} Q(\sigma) = \big(&\underbrace{(1,S,T_1),(1,0,0),\ldots,(1,0,0)}_{b_1},\underbrace{(1,0,T_2),(1,0,0),\ldots,(1,0,0)}_{b_2},\ldots,\\ &\underbrace{(1,0,T_l),(1,0,0),\ldots,(1,0,0)}_{b_l}\big),\\ Q(\sigma') = \big(&\underbrace{(1,S,T_2),(1,0,0),\ldots,(1,0,0)}_{b_2},\underbrace{(1,0,T_3),(1,0,0),\ldots,(1,0,0)}_{b_3},\ldots,\\ &\underbrace{(1,0,T_l),(1,0,0),\ldots,(1,0,0)}_{b_l},\underbrace{(1,0,T_1),(1,0,0),\ldots,(1,0,0)}_{b_1}\big). \end{aligned}$$

By applying $\mu_{2a}(\mu_{2b}^{-1})$ to $Q(\sigma)$ S-times we can shift the S arrows to the second triple to get

$$\begin{aligned} Q(\sigma) \sim \big(&\underbrace{(1,0,T_1),(1,S,0),(1,0,0),\ldots,(1,0,0)}_{b_1},\underbrace{(1,0,T_2),(1,0,0),\ldots,(1,0,0)}_{b_2},\ldots,\\ &\underbrace{(1,0,T_l),(1,0,0),\ldots,(1,0,0)}_{b_l}\big). \end{aligned}$$

Then using μ_0 we can shift the triple $(1, 0, T_1)$ to the end of the sequence. Iteratively repeating this procedure, we obtain the desired order of $Q(\sigma')$.

($\mathrm{d}_{3,1}$) Let $\sigma = (b, (S_1, \ldots, S_a))$ and $\sigma' = (b', (S'_1, \ldots, S'_a))$. Then $b = b'$ and $(S_1, \ldots, S_a)$ is cyclic equivalent to $(S'_1, \ldots, S'_a)$. It suffices to show that $Q(\sigma) \sim Q(\sigma')$ for $\sigma' = (b, (S_2, \ldots, S_a, S_1))$.

Then

$$Q(\sigma) = (\underbrace{(1,0,0),\dots,(1,0,0)}_{b},(3,S_1,0),(3,S_2,0),\dots,(3,S_a,0)),$$
$$Q(\sigma') = (\underbrace{(1,0,0),\dots,(1,0,0)}_{b},(3,S_2,0),\dots,(3,S_a,0),(3,S_1,0)).$$

If $b=0$, then $Q(\sigma') = \mu_0(Q(\sigma))$.
If $b=1$, then we can apply $\mu_{2b}(\mu_{2a}^{-1})$ to $Q(\sigma)$ S_1-times to shift the S_1 arrows to the first triple to get

$$Q(\sigma) \sim ((1,S_1,0),(3,0,0),(3,S_2,0),\dots,(3,S_a,0)).$$

Applying $\mu_{1b}(\mu_{1a}^{-1})$ leads to

$$Q(\sigma) \sim ((3,S_1,0),(1,0,0),(3,S_2,0),\dots,(3,S_a,0)).$$

Using μ_0 we then get that $Q(\sigma) \sim Q(\sigma')$.
Now suppose that $b \geq 2$. First, we explain the operations we need to obtain the quiver $Q(\sigma')$ from the quiver $Q(\sigma)$. Afterwards, we explain what happens with these quivers in each step:

	Step	Result
i)	μ_0^{b-1}	$\mu_0^{b-1}(Q(\sigma)) = ((1,0,0),(3,S_1,0),(3,S_2,0),\dots,(3,S_a,0),\underbrace{(1,0,0),\dots,(1,0,0)}_{(b-1)})$
ii)	$(\mu_{2b}(\mu_{2a}^{-1}))^{S_1}$	$= ((1,S_1,0),(3,0,0),(3,S_2,0)\dots,(3,S_a,0),\underbrace{(1,0,0),\dots,(1,0,0)}_{(b-1)})$
iii)	$((\mu_{2b}(\mu_{2a}^{-1}))^{S_1} \circ \mu_0^{-1})^{b-1}$	$= ((1,S_1,0),\underbrace{(1,0,0),\dots,(1,0,0)}_{(b-1)},(3,0,0),(3,S_2,0)\dots,(3,S_a,0))$
iv)	μ_0^{b-1}	$= ((1,0,0),(3,0,0),(3,S_2,0),\dots,(3,S_a,0),(1,S_1,0),\underbrace{(1,0,0),\dots,(1,0,0)}_{(b-2)})$
v)	$(\mu_0^{-1}(\mu_{1a}^{-1}))^{b-1}$	$= ((1,S_1,0),(3+2(b-1),0,0),(3,S_2,0),\dots,(3,S_a,0))$
vi)	μ_{1a}^{-1}	$= ((3+2b,S_1,0),(3,S_2,0),\dots,(3,S_a,0))$
vii)	μ_{1b}^{b}	$= ((3,S_1,0),\underbrace{(1,0,0),\dots,(1,0,0)}_{b},(3,S_2,0)\dots,(3,S_a,0))$
viii)	μ_0	$= (\underbrace{(1,0,0),\dots,(1,0,0)}_{b},(3,S_2,0)\dots,(3,S_a,0),(3,S_1,0))$
		$= Q(\sigma')$

Steps	Explanation
i)	rotation of the quiver to be able to apply μ_{2a}^{-1}
ii) - iii)	arrows attached to the last spike of the big group of consecutive spikes will be moved to the first spike of this group
iv)	rotation of the quiver to be able to apply μ_{1a}^{-1}
v) - vi)	all free spikes of the big group of consecutive spikes will be destroyed
vii)	the same number b of free spikes will be built up (in the next group)
viii)	rotation to get $Q(\sigma')$

($\mathrm{d}_{3,2}$) Let

$$\sigma = \big((b_1,(S_{1,1},\dots,S_{1,a_1}),T_1),(b_2,(S_{2,1},\dots,S_{2,a_2}),T_2),\dots,(b_l,(S_{l,1},\dots,S_{l,a_l}),T_l)\big),$$
$$\sigma' = \big((b_1',(S_{1,1}',\dots,S_{1,a_1}'),T_1'),(b_2',(S_{2,1}',\dots,S_{2,a_2}'),T_2'),\dots,(b_l',(S_{l,1}',\dots,S_{l,a_l}'),T_l')\big).$$

Then by Definition 4.2.43 σ and σ' are cyclic equivalent. Without loss of generality let

$$\sigma' = \big((b_2, (S_{2,1}, \dots, S_{2,a_2}), T_2), \dots, (b_l, (S_{l,1}, \dots, S_{l,a_l}), T_l), (b_1, (S_{1,1}, \dots, S_{1,a_1}), T_1)\big).$$

By Theorem 4.2.36 part ($\mathrm{d}_{3,2}$), $Q(\sigma)$ is a concatenation of $l \geq 1$ sequences γ_j and $Q(\sigma')$ is a concatenation of $l \geq 1$ sequences γ'_j, where $\gamma'_j = \gamma_{(j+1) \bmod l}$ for all $1 \leq j \leq l$. Since the last vertex of γ_l (resp., γ'_l) is glued to the first vertex of γ_1 (resp., γ'_1), $Q(\sigma)$ and $Q(\sigma')$ are the same quivers (up to rotation).

Thus, we have shown that if $\sigma \sim \sigma'$ then $Q(\sigma) \sim Q(\sigma')$.

(3) Let $\sigma \in \mathcal{S}$. We indicate the parameters of $\Sigma(Q(\sigma))$ by a prime.

(c) $\sigma = (T_1, T_2, S)$, then $Q(\sigma) = ((1, S, T_1), (1, 0, T_2))$. Now we apply Algorithm 4.2.45 and we have to consider three cases:

If $T_1 = T_2 = 0$, then $\mathcal{I}_D = \mathcal{I}_T = \emptyset$. We are in Step 2 of the algorithm and obtain $b = b(\{1, 2\}) = 2$. We get as good mutation parameters $(0, 0, S)$ and hence, $\Sigma(Q(\sigma)) = \sigma$.

If $T_1, T_2 > 0$, then $\mathcal{I}_D = \emptyset$ and $\mathcal{I}_T = \{1, 2\} \neq \emptyset$. We are in Step 3 of the algorithm. We get $i_1^+ = \{1\}$, $i_2^+ = \{2\}$, $b_1 = b(\{1\}) = 1$ and $b_2 = b(\{2\}) = 1$. Since $b_1 + b_2 < 3$ and $l = 2$, we obtain as good mutation parameters (T_1, T_2, S) and hence, $\Sigma(Q(\sigma)) = \sigma$.

If $T_1 > 0$ and $T_2 = 0$, then $\mathcal{I}_D = \emptyset$ and $\mathcal{I}_T = \{1\} \neq \emptyset$. Note that the case $T_1 = 0$ and $T_2 > 0$ is similar since (T_1, T_2), (T_2, T_1) are cyclic equivalent. We are in Step 3 of the algorithm and get $i_1^+ = \{1, 2\}$, $b_1 = 2 < 3$. Since $l = 1$, we obtain as good mutation parameters $(T_1, 0, S)$ and hence, $\Sigma(Q(\sigma)) = \sigma$.

($\mathrm{d}_{2,1}$) $\sigma = (b, S)$, $b \geq 3$, $S \geq 0$ and $Q(\sigma) = ((1, S, 0), (1, 0, 0), \dots, (1, 0, 0))$ of length b. Now we apply Algorithm 4.2.45. We have $\mathcal{I}_D = \mathcal{I}_T = \emptyset$. Thus, we are in Step 2 of the algorithm and get $b' = b$, $S' = S$. Hence, $\Sigma(Q(\sigma)) = \sigma$.

($\mathrm{d}_{2,2}$) $\sigma = \big(((b_1, T_1), (b_2, T_2), \dots, (b_l, T_l)), S\big)$ with $l \geq 1$, $b_j, T_j > 0$, $b_1 + \dots + b_l \geq 3$ and $S \geq 0$. Then

$$Q(\sigma) = \big(\underbrace{(1, S, T_1), (1, 0, 0), \dots, (1, 0, 0)}_{b_1}, \underbrace{(1, 0, T_2), (1, 0, 0), \dots, (1, 0, 0)}_{b_2}, \dots,$$
$$\underbrace{(1, 0, T_l), (1, 0, 0), \dots, (1, 0, 0)}_{b_l}\big)$$

is a concatenation of $l \geq 1$ sequences of positive lengths $b_1, b_2, \dots, b_l$.

We apply the algorithm and get $\mathcal{I}_D = \emptyset$, $\mathcal{I}_T = \{1, b_1 + 1, b_1 + b_2 + 1, \dots, \Big(\sum_{i=1}^{l-1} b_i\Big) + 1\}$.

Thus, we are in Step 3 with $i_j = \Big(\sum_{i=1}^{j-1} b_i\Big) + 1$ and $i_j^+ = \Big\{\Big(\sum_{i=1}^{j-1} b_i\Big) + 1, \dots, \sum_{i=1}^{j} b_i\Big\}$, $1 \leq j \leq l$. Additionally, we get $S' = S$, $b'_j = b(i_j^+) = |\{i \in i_j^+ : d_i = 1\}| = b_j$ and $T'_j = t_{i_j} = T_j$, $1 \leq j \leq l$. Hence, $\Sigma(Q(\sigma)) = \sigma$.

($\mathrm{d}_{3,1}$) $\sigma = \big(b, (S_1, \dots, S_a)\big)$ where $b \geq 0$ and $(S_1, \dots, S_a)$ is a sequence of $a > 0$ non-negative integers. Then $Q(\sigma) = \big((1, 0, 0), \dots, (1, 0, 0), (3, S_1, 0), \dots, (3, S_a, 0)\big)$, where the number of the triples $(1, 0, 0)$ is b. By applying Algorithm 4.2.45 we get $\mathcal{I}_D = \{b + 1, \dots, b + a\}$ and $\mathcal{I}_T = \emptyset$. Thus, we are in Step 4 and get $i_{1,j} = b + j$, $1 \leq j \leq a$,

$$i_{1,j}^- = \begin{cases} \{1, \dots, b + 1\}, & j = 1 \\ \{b + j\}, & 1 < j \leq a \end{cases}.$$

Additionally, we get $a' = |\mathcal{I}_D| = a$, $b' = b$ and $S'_j = s(i_{1,j}^-) = S_j$ for all $1 \leq j \leq a$. Hence, $\Sigma(Q(\sigma)) = \sigma$.

($\mathrm{d}_{3,2}$) $\sigma = \Big(\big(b_1, (S_{1,1}, \dots, S_{1,a_1}), T_1\big), \big(b_2, (S_{2,1}, \dots, S_{2,a_2}), T_2\big), \dots, \big(b_l, (S_{l,1}, \dots, S_{l,a_l}), T_l\big)\Big)$ with $l \geq 1$. For any $1 \leq j \leq l$ the numbers a_j, b_j are non-negative integers not both zero, $(S_{j,1}, \dots, S_{j,a_j})$ is a (possibly empty) sequence of a_j non-negative integers and

$T_j > 0$. Then $Q(\sigma)$ is a concatenation of $l \geq 1$ sequences of the form

$$\gamma_j = \begin{cases} \big(\underbrace{(1,0,T_j),(1,0,0),\dots,(1,0,0)}_{b_j},(3,S_{j,1},0),(3,S_{j,2},0),\dots,(3,S_{j,a_j},0)\big) & \text{if } b_j > 0, \\ ((3,S_{j,1},T_j),(3,S_{j,2},0),\dots,(3,S_{j,a_j},0)) & \text{otherwise.} \end{cases}$$

We can consider $b_j \geq 0$ for all $1 \leq j \leq l$, since the computations for both cases $b_j > 0$ and $b_j = 0$ are the same. We get

$$\mathcal{I}_T = \{1, b_1+a_1+1, (b_1+a_1)+(b_2+a_2)+1, \dots, \Big(\sum_{i=1}^{l-1}(b_i+a_i)\Big)+1\} \text{ and}$$

$$\mathcal{I}_D = \Big\{b_1+1,\dots,b_1+a_1,(b_1+a_1)+(b_2+1),\dots,(b_1+a_1)+(b_2+a_2),\dots, \Big(\sum_{i=1}^{l-1}(b_i+a_i)\Big)+(b_l+1),\dots,\sum_{i=1}^{l}(b_i+a_i)\Big\}.$$

Thus, we are in Step 5 of the algorithm and have

$$i_j^+ = \Big\{\Big(\sum_{i=1}^{j-1}(b_i+a_i)\Big)+1,\dots,\sum_{i=1}^{j}(b_i+a_i)\Big\} \text{ and}$$

$$i_j^+ \cap \mathcal{I}_D = \begin{cases} \Big\{\Big(\sum_{i=1}^{j-1}(b_i+a_i)\Big)+(b_j+1),\dots,\sum_{i=1}^{j}(b_i+a_i)\Big\}, & \text{if } a_j > 0 \\ \emptyset, & \text{if } a_j = 0 \end{cases}.$$

Additionally, we get $a'_j = a(i_j^+) = a_j$, $b'_j = b(i_j^+) = |\{i \in i_j^+ : d_i = 1\}| = b_j$ and

$$i_{1,k}^- = (b_1+k)^- = \begin{cases} \{1,\dots,b_1+1\}, & \text{if } k = 1 \\ \{b_1+k\}, & \text{if } 2 \leq k \leq a_1 \end{cases},$$

$$\begin{aligned} i_{j,k}^- &= \Big(\big(\sum_{i=1}^{j-1}(b_i+a_i)\big)+(b_j+k)\Big)^- \\ &= \begin{cases} \Big\{\Big(\sum_{i=1}^{j-1}(b_i+a_i)\Big)+1,\dots,\Big(\sum_{i=1}^{j-1}(b_i+a_i)\Big)+(b_j+1)\Big\}, & \text{if } k = 1 \\ \Big\{\Big(\sum_{i=1}^{j-1}(b_i+a_i)\Big)+(b_j+k)\Big\}, & \text{if } 2 \leq k \leq a_j \end{cases} \end{aligned}$$

for $2 \leq j \leq l$.
Thus, we have

$$S'_{1,k} = s(i_{1,k}^-) = \begin{cases} \sum_{i=1}^{b_1+1} s_i = S_{1,1}, & \text{if } k = 1 \\ s(\{b_1+k\}) = S_{1,k}, & \text{if } 2 \leq k \leq a_1 \end{cases} \text{ and}$$

$$S'_{j,k} = s(i_{j,k}^-) = \begin{cases} S_{j,1}, & \text{if } k = 1 \\ S_{j,k}, & \text{if } 2 \leq k \leq a_j \end{cases} \text{ for } 2 \leq j \leq l.$$

Hence, $\Sigma(Q(\sigma)) = \sigma$.

We have thus shown that applying Algorithm 4.2.45 to a canonical form as in Theorem 4.2.36 recovers the parameters of that form.

(4) Let $q = ((d_1,s_1,t_1),(d_2,s_2,t_2),\dots,(d_r,s_r,t_r))$.

If q is a quiver of type III, i.e. $q = ((1,s_1,t_1),(1,s_2,t_2))$, then by Lemma 4.2.39 the corresponding cluster-tilted algebra is good mutation equivalent to one in class (c) of Theorem 4.2.36. In particular, there exists $\sigma = (t_1,t_2,s_1+s_2) \in \mathcal{S}$ with $Q(\sigma) \sim q$.

Now suppose that q is a quiver of type IV with some spikes.

- If all the distances $d_i \in N_1 \cup N_2$, then we proceed as follows: by iteratively applying the good mutation move IV.1b of Table 4.6 we can repeatedly shorten all the distances $d_i \geq 4$ by 2 until they become 2 (this corresponds to μ_{1b}). By applying the good mutation move IV.2b of Table 4.6 we can further shorten any distance 2 to a distance of 1 (this corresponds to μ_{2b}). Thus, we get a parameter sequence where all the new distances are 1:

$$q \sim ((1, \tilde{s}_1, t_1), \underbrace{(1,0,0), \dots, (1,0,0)}_{b_1}, (1, \tilde{s}_2, t_2), \underbrace{(1,0,0), \dots, (1,0,0)}_{b_2}, \dots,$$
$$(1, \tilde{s}_r, t_r), \underbrace{(1,0,0), \dots, (1,0,0)}_{b_r}),$$

with

$$b_i = \begin{cases} \frac{d_i}{2} - 1, & \text{if } d_i \in N_2 \\ 0, & \text{if } d_i = 1 \end{cases} \quad \text{and } \tilde{s}_i = \begin{cases} s_i + 1, & \text{if } d_i \in N_2 \\ s_i, & \text{if } d_i = 1 \end{cases}.$$

If these new distances sum up to 2, then by Lemma 4.2.39 the corresponding cluster-tilted algebra is good mutation equivalent to one in class (c) of Theorem 4.2.36. In particular, q could be either $((2, s_1, t_1), (1, s_2, t_2)) \sim ((1, s_1+1, t_1), (1, s_2, t_2))$, $((2, s_1, t_1), (2, s_2, t_2)) \sim ((1, s_1+1, t_1), (1, s_2+1, t_2))$ or $((4, s_1, t_1)) \sim ((1, s_1+1, t_1), (1,0,0))$. Then σ can be chosen to be either (t_1, t_2, s_1+s_2+1), (t_1, t_2, s_1+s_2+2) or $(t_1, 0, s_1+1)$ with $Q(\sigma) = ((1, s_1+s_2+1, t_1), (1, 0, t_2))$, $Q(\sigma) = ((1, s_1+s_2+2, t_1), (1, 0, t_2))$ or $Q(\sigma) = ((1, s_1+1, t_1), (1,0,0))$.

Otherwise, if these new distances sum up to at least 3, we will get one of the classes $(\mathrm{d}_{2,1})$ or $(\mathrm{d}_{2,2})$ of Theorem 4.2.36. In particular, by Lemma 4.2.42, we can successively move all the arrows of the attached rooted quivers of type A and concentrate them on the first spike, i.e. on the first triple in the sequence. Note that by 'arrows' we mean arrows which are not part of an oriented 3-cycle. This can be done using the operations $\mu_{2b}(\mu_{2a}^{-1})$ and μ_0.

This yields the canonical form of $(\mathrm{d}_{2,1})$, if there are no oriented 3-cycles in the rooted quivers of type A. That is,

$$q \sim ((1, \sum_{i=1}^{r} \tilde{s}_i, 0), \underbrace{(1,0,0), \dots, (1,0,0)}_{\left(\sum_{i=1}^{r} b_i\right) + (r-1)}),$$

with $\sum_{i=1}^{r} \tilde{s}_i = \sum_{i=1}^{r} s_i + |\{i \in \{1, \dots, r\} : d_i \in N_2\}| = s(\{1, \dots, r\})$. The length of the sequence is $\left(\sum_{i=1}^{r} b_i\right) + r = \left(\sum_{d_i \in N_2} \frac{d_i}{2} - 1\right) + r = \left(\sum_{d_i \in N_2} \frac{d_i}{2}\right) + |\{i \in \{1, \dots, r\} : d_i = 1\}| = b(\{1, \dots, r\})$. Then the corresponding σ is given by $\sigma = (b(\{1, \dots, r\}), s(\{1, \dots, r\}))$.

If there is at least one oriented 3-cycle in a rooted quiver of type A we get the canonical form of $(\mathrm{d}_{2,2})$ as follows:

$$q \sim ((1, \sum_{i=1}^{r} \tilde{s}_i, t_1), \underbrace{(1,0,0), \dots, (1,0,0)}_{b_1}, (1, 0, t_2), \underbrace{(1,0,0), \dots, (1,0,0)}_{b_2}, \dots,$$
$$(1, 0, t_r), \underbrace{(1,0,0), \dots, (1,0,0)}_{b_r}),$$

with $\sum_{i=1}^{r} \tilde{s}_i = \sum_{i=1}^{r} s_i + |\{i \in \{1, \dots, r\} : d_i \in N_2\}| = s(\{1, \dots, r\})$ and b_i as above.

Now, let $|\{1 \le i \le r : t_i > 0\}| = l$ and denote the elements of this set by $i_1 < i_2 < \cdots < i_l$. If necessary, i.e. if $t_1 = 0$, we shift all the arrows of the rooted quivers of type A to the triple which contains t_{i_1}. That is,

$$q \sim (\underbrace{(1, \sum_{i=1}^{r} \tilde{s}_i, t_{i_1}), (1,0,0), \dots, (1,0,0)}_{b_{i_1}}, \underbrace{(1,0,t_{i_2}), (1,0,0), \dots, (1,0,0)}_{b_{i_2}}, \dots,$$
$$\underbrace{(1,0,t_{i_l}), (1,0,0), \dots, (1,0,0)}_{b_{i_l}}),$$

with

$$\begin{aligned} b_{i_j} &= \Big(\sum_{i_j \le i \le i_{j+1}-1,\ d_i \in N_2} \frac{d_i}{2} - 1 \Big) + (i_{j+1} - i_j) \\ &= \Big(\sum_{i_j \le i \le i_{j+1}-1,\ d_i \in N_2} \frac{d_i}{2} \Big) + |\{ i_j \le i \le i_{j+1} - 1 : d_i = 1 \}| \\ &= b(\{i_j, \dots, i_{j+1} - 1\}) = b(i_j^+),\ 1 \le j < l, \end{aligned}$$

$$\begin{aligned} b_{i_l} &= \Big(\sum_{i=i_l,\ d_i \in N_2}^{r} \frac{d_i}{2} - 1 \Big) + \Big(\sum_{i=1,\ d_i \in N_2}^{i_1 - 1} \frac{d_i}{2} - 1 \Big) + (r - i_l + i_1) \\ &= \Big(\sum_{i \in i_l^+,\ d_i \in N_2} \frac{d_i}{2} \Big) + |\{ i \in i_l^+ : d_i = 1 \}| \\ &= b(i_l^+). \end{aligned}$$

Thus, the corresponding σ is given by $\sigma = (((b(i_1^+), t_{i_1}),\ (b(i_2^+), t_{i_2}),\ \dots,\ (b(i_l^+), t_{i_l})),\ s(\{1, \dots, r\}))$.

- Otherwise, when there is at least one distance $d_i \in N_3$, we proceed as follows: by iteratively applying the good mutation move IV.1a of Table 4.6 we can repeatedly shorten all the distances $d_i \ge 4$ by 2 until they become 2 or 3 (this corresponds to μ_{1a}). By applying the good mutation move IV.2b of Table 4.6 we can further shorten any distance 2 to a distance of 1 (this corresponds to μ_{2b}). Thus, we get a parameter sequence where all the new distances are either 1 or 3. In particular, for each triple (d_i, s_i, t_i) we get

$$(d_i, s_i, t_i) \rightsquigarrow \begin{cases} (1, s_i, t_i), & \text{if } d_i = 1 \\ (1, s_i + 1, t_i), & \text{if } d_i = 2 \\ (3, s_i, t_i), & \text{if } d_i = 3 \\ (1, s_i, t_i), \underbrace{(1,0,0), \dots, (1,0,0)}_{b_i}, (1,1,0), & \text{if } d_i \in N_2 \backslash \{2\} \\ (1, s_i, t_i), \underbrace{(1,0,0), \dots, (1,0,0)}_{b_i}, (3,0,0), & \text{if } d_i \in N_3 \backslash \{3\} \end{cases}$$

with

$$b_i = \begin{cases} \frac{d_i - 2}{2} - 1, & \text{if } d_i \in N_2 \backslash \{2\} \\ \frac{d_i - 3}{2} - 1, & \text{if } d_i \in N_3 \backslash \{3\} \end{cases}.$$

If there are no oriented 3-cycles in the attached rooted quivers of type A, that is, $t_i = 0$ for all $1 \le i \le r$, we observe the following: consider a group of consecutive spikes, by Lemma 4.2.42, inside this group we can concentrate the attached rooted quivers of type A at the last spike in that group, thus creating free spikes at the beginning of the group. Note that by the 'last' spike we mean the last spike in the direction of the orientation of the central cycle. This can be done by using the operations $\mu_{2a}(\mu_{2b}^{-1})$

and μ_0. Then, by Lemma 4.2.41, the free spikes at the beginning of this group can be moved to the previous group, i.e. the group of (consecutive) spikes immediately preceding, with respect to the orientation of the central cycle, the group which we started with. This can be done by applying the operations $\mu_{1a}(\mu_{1b}^{-1})$ and μ_0. In this way, we can move all spikes of this group of consecutive spikes except one to the previous group, thus creating a single spike with some linear rooted quiver of type A attached. Iteratively repeating this procedure for all the previous groups of consecutive spikes, we can eventually merge all groups of at least two consecutive spikes into one large group, with all the other spikes being single spikes. In other words, the sequence of distances will look like $(1, 1, \dots, 1, 3, 3, \dots, 3)$. In this large group of consecutive spikes, we concentrate the linear rooted quivers of type A at the last spike, yielding exactly the canonical form appearing in $(\mathrm{d}_{3,1})$. That is,

$$\begin{aligned} q &\sim (\underbrace{(1,0,0),\dots,(1,0,0)}_{b}, (3, s_{i_j}, 0), \dots, (3, s_{i_a}, 0), (3, s_{i_1}, 0), \dots, (3, s_{i_{j-1}}, 0)) \\ &\overset{(2)}{\sim} (\underbrace{(1,0,0),\dots,(1,0,0)}_{b}, (3, s_{i_1}, 0), (3, s_{i_2}, 0), \dots, (3, s_{i_a}, 0)), \end{aligned}$$

where $i_1 < i_2 < \dots < i_a$ are the elements of the set $\{1 \le i \le r : d_i \in N_3\}$,

$$s_{i_j} = \Big(\sum_{i \in i_j^-} s_i\Big) + |\{i \in i_j^- : d_i \in N_2\}| = s(i_j^-)$$

for $1 \le j \le a$, and

$$b = \sum_{1 \le i \le r,\ d_i \in N_2} \frac{d_i}{2} + \sum_{1 \le i \le r,\ d_i \in N_3} \frac{d_i - 3}{2} + |\{1 \le i \le r : d_i = 1\}| = b(\{1, \dots, r\}).$$

The corresponding σ is then given by $\sigma = ((b(\{1, \dots, r\}), (s(i_1^-), \dots, s(i_a^-)))$.

If there is at least one oriented 3-cycle in the attached rooted quivers of type A, we proceed in a similar way as above: consider a group of consecutive spikes. By Lemma 4.2.42, inside this group we can concentrate all arrows of the attached rooted quivers of type A at the last spike in that group. Note that by 'arrows' we mean arrows which are not part of an oriented 3-cycle. This can be done using the operations $\mu_{2a}(\mu_{2b}^{-1})$ and μ_0. Then, by Lemma 4.2.41, we can move all the possible free spikes at the beginning of this group to the previous group (by applying the operations $\mu_{1a}(\mu_{1b}^{-1})$ and μ_0). Then we iteratively repeat this procedure for all the previous groups of consecutive spikes. Having done this procedure for all of these groups, we consider the group with which we started again, and possibly move all arrows of the attached rooted quivers of type A to the last spike in that group.

Thus, for each rooted quiver of type A with at least one oriented 3-cycle, we get a partial quiver with parameter sequence γ_j as in the canonical form appearing in $(\mathrm{d}_{3,2})$. That is, the cluster-tilted algebra corresponding to q is good mutation equivalent to a cluster-tilted algebra whose quiver is a concatenation of $l := |\{1 \le i \le r : t_i > 0\}| \ge 1$ sequences

$$\gamma_{i_j} = \begin{cases} \big(\underbrace{(1, 0, t_{i_j}), (1,0,0), \dots, (1,0,0)}_{b_{i_j}}, (3, S_{j,1}, 0), \dots, (3, S_{j,a_{i_j}}, 0)\big) & \text{if } b_{i_j} > 0, \\ \big((3, S_{j,1}, t_{i_j}), (3, S_{j,2}, 0), \dots, (3, S_{j,a_{i_j}}, 0) & \text{otherwise}, \end{cases}$$

where the elements of the set $|\{1 \le i \le r : t_i > 0\}|$ are denoted by $i_1 < i_2 < \dots < i_l$.

$$b_{i_j} = \sum_{i \in i_j^+,\ d_i \in N_2} \frac{d_i}{2} + \sum_{i \in i_j^+,\ d_i \in N_3} \frac{d_i - 3}{2} + |\{i \in i_j^+ : d_i = 1\}| = b(i_j^+)$$

is the number of triples with distance 1 and

$$a_{i_j} = |\{i \in i_j^+ : d_i \in N_3\}| = a(i_j^+)$$

is the number of triples with distance 3 in γ_{i_j}, $1 \leq j \leq l$. In addition, denote the elements of $\{i \in i_j^+ : d_i \in N_3\}$ in the order they appear within the interval as $i_{j,1}, \dots, i_{j,a_{i_j}}$. Then

$$S_{j,k} = \sum_{i \in i_{j,k}^-} s_i + \sum_{i \in i_{j,k}^-,\ d_i \in N_2} 1 = s(i_{j,k}^-),$$

for all $1 \leq j \leq l$ and $1 \leq k \leq a_{i_j}$. The corresponding σ is then given by

$$\begin{aligned}\sigma &= ((b_{i_1}, (S_{1,1}, \dots, S_{1,a_{i_1}}), t_{i_1}), \dots, (b_{i_l}, (S_{l,1}, \dots, S_{l,a_{i_l}}), t_{i_l}))\\ &= ((b(i_1^+), (s(i_{1,1}^-), \dots, s(i_{1,a(i_1^+)}^-)), t_{i_1}), \dots, (b(i_l^+), (s(i_{l,1}^-), \dots, s(i_{l,a(i_l^+)}^-)), t_{i_l})).\end{aligned}$$

Hence, a quiver can be transformed by good mutations to a quiver in canonical form as in Theorem 4.2.36.

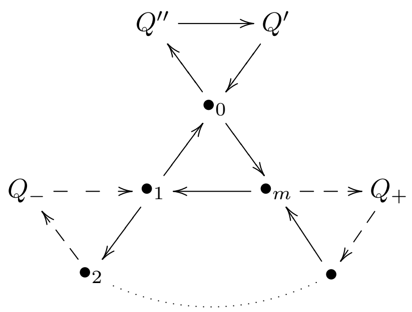
□

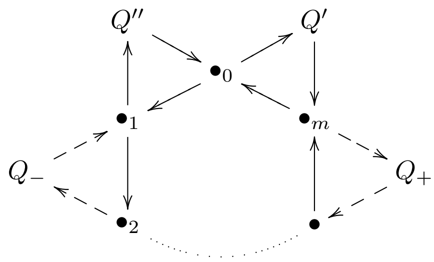
This completes the proof of Proposition 4.2.35 and Theorem 4.2.36.

4.3. Derived equivalences for cluster-tilted algebras of type D_n

4.3.1. Good double mutations in types III and IV. The good double mutations we consider in this section consist of two algebra mutations. The first takes a cluster-tilted algebra A to a derived equivalent algebra which is not cluster-tilted, whereas the second takes that algebra to another cluster-tilted algebra A', thus obtaining a derived equivalence of A and A'.

Lemma 4.3.1. *Let $m \geq 3$ and consider a cluster-tilted algebra $A = A_{\widetilde{Q}}$ of type IV with the quiver $\widetilde{Q}$ as in the left picture*





having a central cycle $1, \dots, m$ and optional spikes Q_- and Q_+. Let $\mu_0(\widetilde{Q})$ denote the mutation of $\widetilde{Q}$ at the vertex 0, as in the right picture. Then:

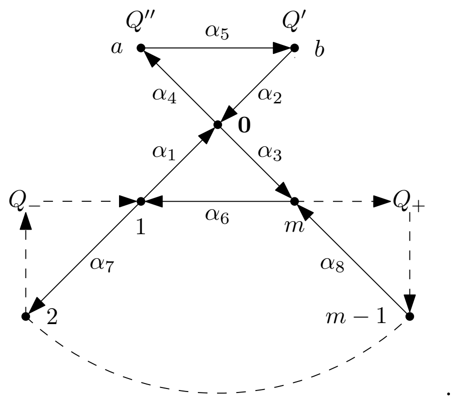
(a) *$\mu_0^-(A)$ is always defined and is isomorphic to the quotient of the cluster-tilted algebra $A_{\mu_0(\widetilde{Q})}$ by the ideal generated by the path p given by*

$$p = \begin{cases} 1, 2, \dots, m, 0 & \textit{if the spike } Q_- \textit{ is present,} \\ 2, \dots, m, 0 & \textit{otherwise.} \end{cases}$$

(b) *$\mu_0^+(A)$ is always defined and is isomorphic to the quotient of the cluster-tilted algebra $A_{\mu_0(\widetilde{Q})}$ by the ideal generated by the path p given by*

$$p = \begin{cases} 0, 1, \dots, m & \textit{if the spike } Q_+ \textit{ is present,} \\ 0, 1, \dots, m-1 & \textit{otherwise.} \end{cases}$$

Proof. Let $A = A_{\widetilde{Q}}$ be the cluster-tilted algebra corresponding to the quiver $\widetilde{Q}$ depicted as



It is easily seen using Proposition 4.2.3 that the negative mutation $\mu_0^-(A)$ and the positive mutation $\mu_0^+(A)$ are defined (since $\alpha_4\alpha_1 \neq 0$ and $\alpha_3\alpha_2 \neq 0$). In order to describe them explicitly, we recall that $\mu_0^-(A) = \mathrm{End}_{D^b(A)}(T_0^-(A))$ and $\mu_0^+(A) = \mathrm{End}_{D^b(A)}(T_0^+(A))$, where

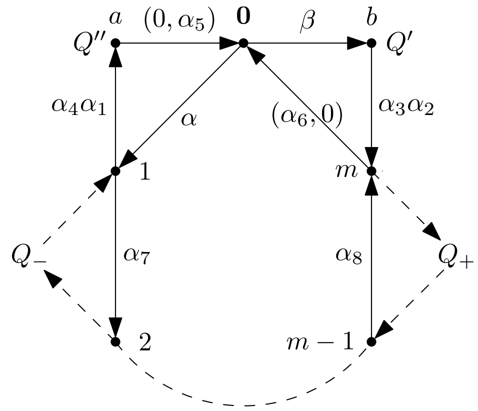
$$\begin{aligned} T_0^-(A) &= \big(P_0 \xrightarrow{(\alpha_1,\alpha_2)} (P_1 \oplus P_b)\big) \oplus \big(\bigoplus_{i\neq 0} P_i\big) =: L_0 \oplus \big(\bigoplus_{i\neq 0} P_i\big), \\ T_0^+(A) &= \big((P_m \oplus P_a) \xrightarrow{(\alpha_3,\alpha_4)} P_0\big) \oplus \big(\bigoplus_{i\neq 0} P_i\big) =: R_0 \oplus \big(\bigoplus_{i\neq 0} P_i\big). \end{aligned}$$

- First we compute the Cartan matrix of $\mu_0^-(A)$ from the Cartan matrix of A by only changing the row and column of vertex 0 according to Happel's alternating sum formula of Proposition 2.3.2. Hence, it is given by

$$C_{\mu_0^-(A)} = \begin{array}{c|ccccc:ccc} & 0 & 1 & m & a & b & 2 & (m-1) & \cdots \\ \hline 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & \cdots \\ 1 & \boxed{0} & 1 & 1 & 1 & 0 & 1 & 1 & \cdots \\ m & 1 & 1 & 1 & 0 & 0 & 1 & ? & \cdots \\ a & 1 & 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ b & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\ \hdashline 2 & 1/\boxed{0} & 1/0 & 1 & 0 & 0 & 1 & 1 & \cdots \\ (m-1) & 1 & 1 & 1 & 0 & 0 & ? & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

where the entries 1/0 depend on whether or not Q_- is present (i.e. 1 if Q_- is present and 0 if Q_- is not present). One of the question marks depends on whether or not Q_+ is present and the other on the (undrawn) rest of the non-oriented cycle. The boxes around 0 indicate the path p of Lemma 4.3.1 (a), that is, $C_{\mu_0^-(A)}$ is just the Cartan matrix of the cluster-tilted algebra corresponding to the right hand quiver in Lemma 4.3.1 except for these two boxes.

Corresponding to each arrow of the following quiver we define a homomorphism of complexes between the summands of $T_0^-(A)$ (in the opposite direction).



First we have the embeddings $\alpha := (\mathrm{id}, 0) : P_1 \to L_0$ and $\beta := (0, \mathrm{id}) : P_b \to L_0$ (in degree zero). Moreover, we have the homomorphisms $\alpha_4\alpha_1 : P_a \to P_1$, $\alpha_3\alpha_2 : P_m \to P_b$, $(\alpha_6, 0) : L_0 \to P_m$ and $(0, \alpha_5) : L_0 \to P_a$. All the other homomorphisms are as before.

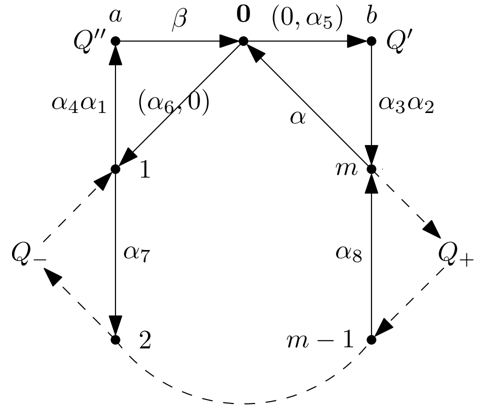
Now we have to show that these homomorphisms satisfy the defining relations of the algebra $A_{\mu_0(\tilde{Q})}/I(p)$, up to homotopy, where $I(p)$ is the ideal generated by the path p stated in Lemma 4.3.1 (a). Clearly, the composition of $(0, \alpha_5)$ and α and the composition of $(\alpha_6, 0)$ and β are zero-relations. The composition of $\alpha_3\alpha_2$ and $(\alpha_6, 0)$ is zero as before. There is one commutativity relation between vertex 0 and vertex m. This is given by the two homomorphisms from $P_m[0]$ to the first and second summand of L_0. These are indeed the same since $(0, \alpha_3\alpha_2)$ is homotopic to $(\alpha_8 \dots \alpha_7, 0)$ (and $\alpha_8 \dots \alpha_7 = \alpha_3\alpha_1 \neq 0$ in A). The path from vertex 0 to vertex a is zero since $\alpha_4\alpha_2 = 0$ and thus, $(\alpha_4\alpha_1, 0)$ is homotopic to zero. There is no non-zero path from vertex 1 to vertex 0 since $(0, \alpha_5\alpha_4\alpha_1) = 0 = (\alpha_6\alpha_8 \dots \alpha_7, 0)$. This corresponds to the path p in the case that Q_- is present and is marked in the Cartan matrix by a box. If Q_- is not present, then the path from vertex 2 to vertex 0 is already zero since $(\alpha_6\alpha_8 \dots, 0) = 0$ which is also marked in the Cartan matrix above. Thus, $\mu_0^-(A)$ is isomorphic to the quotient of the cluster-tilted algebra $A_{\mu_0(\tilde{Q})}$ by the ideal generated by the path p.

- Now we compute the Cartan matrix of $\mu_0^+(A)$ from the Cartan matrix of A by only changing the row and column of vertex 0 according to Happel's alternating sum formula of Proposition 2.3.2. Thus, it is given by

$$C_{\mu_0^+(A)} = \begin{array}{c|ccccc:ccc}
 & 0 & 1 & m & a & b & 2 & (m-1) & \cdots \\
\hline
0 & 1 & 1 & \boxed{0} & 0 & 1 & 1 & 1/\boxed{0} & \cdots \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & \cdots \\
m & 1 & 1 & 1 & 0 & 0 & 1 & 1/0 & \cdots \\
a & 1 & 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\
b & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\
\hdashline
2 & 1 & ? & 1 & 0 & 0 & 1 & 1 & \cdots \\
(m-1) & 1 & 1 & 1 & 0 & 0 & ? & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

where the entries $1/0$ depend on whether or not Q_+ is present (i.e. 1 if Q_+ is present and 0 if Q_+ is not present). One of the question marks depends on whether or not Q_- is present and the other on the (undrawn) rest of the non-oriented cycle. The boxes around 0 indicate the path p of Lemma 4.3.1 (b), that is, $C_{\mu_0^+(A)}$ is just the Cartan matrix of the cluster-tilted algebra corresponding to the right hand quiver in Lemma 4.3.1 except for these two boxes.

Now corresponding to each arrow of the following quiver we define a homomorphism of complexes between the summands of $T_0^+(A)$.

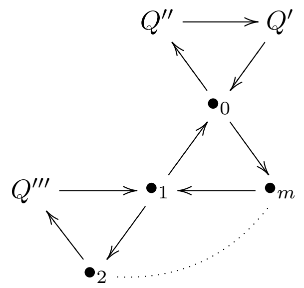


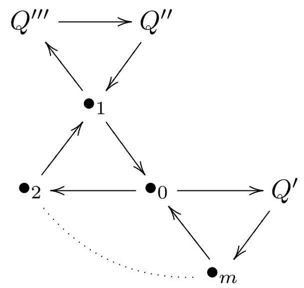
First we have the embeddings $\alpha := (\mathrm{id}, 0) : R_0 \to P_m$ and $\beta := (0, \mathrm{id}) : R_0 \to P_a$ (in degree zero). Moreover, we have the homomorphisms $\alpha_4\alpha_1 : P_a \to P_1$, $\alpha_3\alpha_2 : P_m \to P_b$, $(\alpha_6, 0) : P_1 \to R_0$, and $(0, \alpha_5) : P_b \to R_0$. All the other homomorphisms are as before.

We leave it to the reader to verify the defining relations of the algebra $A_{\mu_0(\widetilde{Q})}/I(p)$ other than the path p. There is no non-zero path from vertex 0 to vertex m since $(0, \alpha_3\alpha_2\alpha_5) = (\alpha_8 \dots \alpha_7\alpha_6, 0)$ is zero. This corresponds to the path p in Lemma 4.3.1 (b) in the case that Q_+ is present and is marked in the Cartan matrix by a box. If Q_+ is not present then the path from vertex 0 to vertex $m-1$ is already zero since $(\dots \alpha_7\alpha_6, 0) = 0$ which is also marked in the Cartan matrix above. Hence, $\mu_0^+(A)$ is isomorphic to the algebra $A_{\mu_0(\widetilde{Q})}/I(p)$, where p is the path stated in Lemma 4.3.1 (b).

□

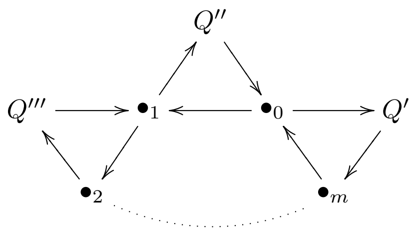
Corollary 4.3.2. *The two cluster-tilted algebras of type IV with quivers*





(where Q', Q'' and Q''' are rooted quivers of type A) are related by a good double mutation (at the vertex 0 *and then at* 1*).*

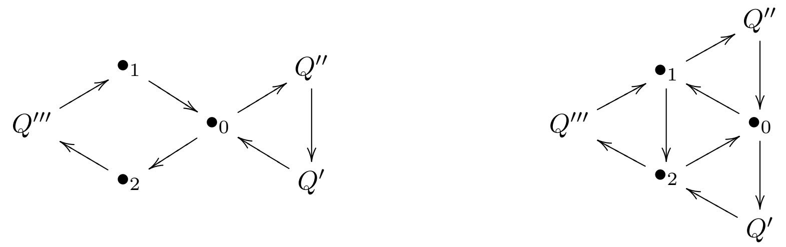
PROOF. Denoting the algebra defined as the cluster-tilted algebra of the quiver on the left as A_L, and that on the right as A_R, we see that $\mu_0^-(A_L) \cong \mu_1^+(A_R)$, since by Lemma 4.3.1 these algebra mutations are isomorphic to the quotient of the cluster-tilted algebra of the quiver



by the ideal generated by the path $1, 2, \dots, m, 0$. □

There is an analogue of Lemma 4.3.1 for cluster-tilted algebras of type III:

Lemma 4.3.3. *Consider the cluster-tilted algebra $A = A_{\widetilde{Q}}$ of type III whose quiver $\widetilde{Q}$ is shown in the picture on the left, where Q', Q'' and Q''' are rooted quivers of type A.*

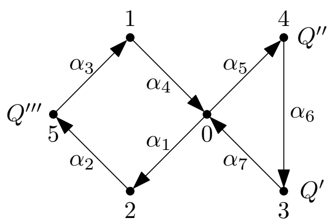


Let $\mu_0(\widetilde{Q})$ denote the mutation of $\widetilde{Q}$ at the vertex 0, as in the picture on the right. Then:

(a) *$\mu_0^-(A)$ is always defined and is isomorphic to the quotient of the cluster-tilted algebra $A_{\mu_0(\widetilde{Q})}$ of type IV by the ideal generated by the path 1, 2, 0.*

(b) *$\mu_0^+(A)$ is always defined and is isomorphic to the quotient of the cluster-tilted algebra $A_{\mu_0(\widetilde{Q})}$ of type IV by the ideal generated by the path 0, 1, 2.*

Proof. Let $A = A_{\widetilde{Q}}$ be the cluster-tilted algebra corresponding to the quiver $\widetilde{Q}$ depicted as



.

It is easily seen using Proposition 4.2.3 that the negative mutation $\mu_0^-(A)$ and the positive mutation $\mu_0^+(A)$ are defined (since $\alpha_5\alpha_4 \neq 0$ and $\alpha_1\alpha_7 \neq 0$).

We recall that $\mu_0^-(A) = \mathrm{End}_{D^b(A)}(T_0^-(A))$ and $\mu_0^+(A) = \mathrm{End}_{D^b(A)}(T_0^+(A))$, where

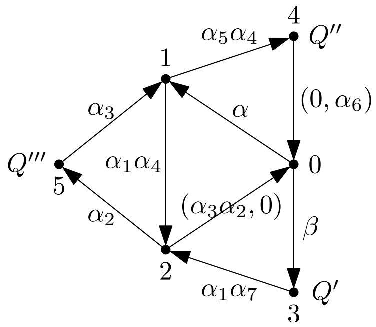
$$
\begin{aligned}
T_0^-(A) &= \big(P_0 \xrightarrow{(\alpha_4,\alpha_7)} (P_1 \oplus P_3)\big) \oplus \big(\bigoplus_{i\neq 0} P_i\big) =: L_0 \oplus \big(\bigoplus_{i\neq 0} P_i\big), \\
T_0^+(A) &= \big((P_2 \oplus P_4) \xrightarrow{(\alpha_1,\alpha_5)} P_0\big) \oplus \big(\bigoplus_{i\neq 0} P_i\big) =: R_0 \oplus \big(\bigoplus_{i\neq 0} P_i\big).
\end{aligned}
$$

- First we compute the Cartan matrix of $\mu_0^-(A)$ from the Cartan matrix of A by only changing the row and column of vertex 0 according to Happel's alternating sum formula of Proposition 2.3.2. Hence, it is given by

$$
C_{\mu_0^-(A)} = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & \cdots \\
\boxed{0} & 1 & 1 & 0 & 1 & 0 & \cdots \\
1 & 1 & 1 & 0 & 0 & 1 & \cdots \\
0 & 0 & 1 & 1 & 0 & 1 & \cdots \\
1 & 0 & 0 & 1 & 1 & 0 & \cdots \\
0 & 1 & 0 & 0 & 1 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix}
$$

where the box around 0 indicates the path 1, 2, 0 of Lemma 4.3.3 (a), that is, $C_{\mu_0^-(A)}$ is just the Cartan matrix of the cluster-tilted algebra corresponding to the right hand quiver in Lemma 4.3.3 except for this box.

Corresponding to each arrow of the following quiver we define a homomorphism of complexes between the summands of $T_0^-(A)$ (in the opposite direction).



First we have the embeddings $\alpha := (\mathrm{id}, 0) : P_1 \to L_0$ and $\beta := (0, \mathrm{id}) : P_3 \to L_0$ (in degree zero). Moreover, we have the homomorphisms $\alpha_1\alpha_4 : P_2 \to P_1$, $\alpha_5\alpha_4 : P_4 \to P_1$, $\alpha_1\alpha_7 : P_2 \to P_3$, $(0, \alpha_6) : L_0 \to P_4$ and $(\alpha_3\alpha_2, 0) : L_0 \to P_2$. All the other homomorphisms are as before.

Now we have to show that these homomorphisms satisfy the defining relations of the algebra $A_{\mu_0(\widetilde{Q})}/I(1,2,0)$, up to homotopy, where $I(1,2,0)$ is the ideal generated by the path $1,2,0$ stated in Lemma 4.3.3 (a).

There are six zero-relations in $\mu_0^-(A)$. The path $5,1,2$ is zero since $\alpha_1\alpha_4\alpha_3 = 0$, the path $1,2,5$ is zero since $\alpha_2\alpha_1\alpha_4 = 0$, the path $3,2,0$ is zero since $(\alpha_3\alpha_2\alpha_1\alpha_7, 0) = 0$, the path $2,0,3$ is zero since the composition of $(\alpha_3\alpha_2, 0)$ and β is zero, the path $4,0,1$ is zero since the composition of $(0, \alpha_6)$ and α is zero and the path $0,1,4$ is zero since $\alpha_5\alpha_7 = 0$ and thus, $(\alpha_5\alpha_4, 0)$ is homotopic to zero. Additionally, there are two commutativity relations in $\mu_0^-(A)$. There is one between vertex 0 and vertex 2, and this is given by the two homomorphisms from $P_2[0]$ to the first and second summand of L_0. These are indeed the same since $(\alpha_1\alpha_4, 0)$ is homotopic to $(0, \alpha_1\alpha_7)$. It is easily seen that the second commutativity relation is between vertex 2 and vertex 1. There is no non-zero path from vertex 1 to vertex 0 since $(0, \alpha_6\alpha_5\alpha_4) = 0 = (\alpha_3\alpha_2\alpha_1\alpha_4, 0)$. This corresponds to the path $1,2,0$ and is marked in the Cartan matrix by a box.

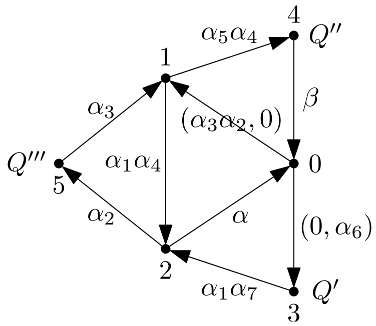
Thus, $\mu_0^-(A)$ is isomorphic to the quotient of the cluster-tilted algebra $A_{\mu_0(\widetilde{Q})}$ by the ideal generated by the path $1,2,0$.

- Now we compute the Cartan matrix of $\mu_0^+(A)$ from the Cartan matrix of A by only changing the row and column of vertex 0 according to Happel's alternating sum formula of Proposition 2.3.2. Hence, it is given by

$$C_{\mu_0^+(A)} = \begin{pmatrix} 1 & 1 & \boxed{0} & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

where the box around 0 indicates the path $0,1,2$ of Lemma 4.3.3 (b), that is, $C_{\mu_0^+(A)}$ is just the Cartan matrix of the cluster-tilted algebra corresponding to the right hand quiver in Lemma 4.3.3 except for this box.

Now corresponding to each arrow of the following quiver we define a homomorphism of complexes between the summands of $T_0^+(A)$.



First we have the embeddings $\alpha := (\mathrm{id}, 0) : R_0 \to P_2$ and $\beta := (0, \mathrm{id}) : R_0 \to P_4$ (in degree zero). Moreover, we have the homomorphisms $\alpha_1\alpha_4 : P_2 \to P_1$, $\alpha_5\alpha_4 : P_4 \to P_1$, $\alpha_1\alpha_7 : P_2 \to P_3$, $(0, \alpha_6) : P_3 \to R_0$ and $(\alpha_3\alpha_2, 0) : P_1 \to R_0$. All the other homomorphisms are as before.

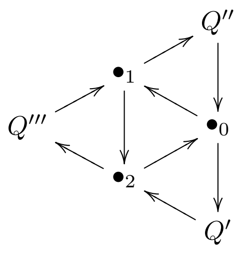
We leave it to the reader to verify the defining relations of $A_{\mu_0(\tilde{Q})}/I(0,1,2)$ other than the path $0,1,2$. There is no non-zero path from vertex 0 to vertex 2 since $(0, \alpha_1\alpha_7\alpha_6) = 0 = (\alpha_1\alpha_4\alpha_3\alpha_2, 0)$. This corresponds to the path $0,1,2$ and is marked in the Cartan matrix by a box. Thus, $\mu_0^+(A)$ is isomorphic to the quotient of the cluster-tilted algebra $A_{\mu_0(\tilde{Q})}$ by the ideal generated by the path $0,1,2$.

$\square$

Corollary 4.3.4. *The cluster-tilted algebras of type III with quivers*

(where Q', Q'' and Q''' are rooted quivers of type A) are related by a good double mutation (at 0 *and then at* 1*).*

Proof. Denoting the algebra defined as the cluster-tilted algebra of the quiver on the left as A_L, and that on the right as A_R, we see that $\mu_0^-(A_L) \cong \mu_1^+(A_R)$, since by Lemma 4.3.3 these algebra mutations are isomorphic to the quotient of the cluster-tilted algebra of the quiver



by the ideal generated by the path $1,2,0$. $\square$

The good double mutations considered in this section consist of two algebra mutations. The first takes a cluster-tilted algebra A to a derived equivalent algebra which is not cluster-tilted. The second then takes that algebra to another cluster-tilted algebra A', thus obtaining a derived equivalence of A and A'. The following example shows that these derived equivalences cannot, in general, be obtained by performing a sequence of only good mutations. In other words, any sequence of algebra mutations connecting A and A' must pass through an algebra which is not cluster-tilted. Thus, the situation in Dynkin type D is somewhat more complicated than that in types A and E, where any two derived equivalent cluster-tilted algebras can be connected by a sequence of good mutations (see Section 4.2.2 for type A and [**12**, Theorem 1.1] for type E).

Example 4.3.5. The two cluster-tilted algebras of type D_8 with quivers,

of type III are derived equivalent but cannot be connected by a sequence of good mutations. Indeed, the mutations at all the vertices of the quiver on the right hand side are bad (see cases 2b, II.1 and III.3 in Tables 4.1, 4.3 and 4.4).

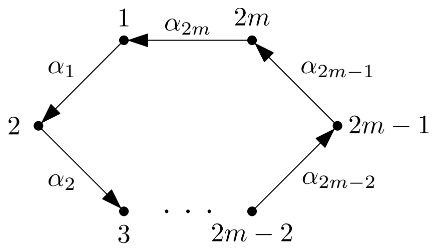
4.3.2. Main theorem for derived equivalences. First we recall a result about derived equivalences for self-injective cluster-tilted algebras.

The self-injective cluster-tilted algebras were determined by Ringel in [**51**]. They are all of Dynkin type D_n, $n \geq 3$. Fixing the number n of vertices, there are one or two such algebras depending on whether n is odd or even. Namely, there is the algebra corresponding to the cycle of length n without spikes, and when $n = 2m$ is even, there is also the one of type IV with parameter sequence $\big((1,0,0),(1,0,0),\dots,(1,0,0)\big)$ of length m.

The following lemma shows that these two algebras are in fact derived equivalent.

Lemma 4.3.6 (Asashiba [2]). *Let $m \geq 3$. Then the cluster-tilted algebra of type IV with a central cycle of length $2m$ without any spike is derived equivalent to that of type IV with parameter sequence $\big((1,0,0),(1,0,0),\dots,\ (1,0,0)\big)$ of length m.*

Remark 4.3.7. We denote the vertices and the arrows of the cycle of length $2m$ as in the picture below.



Then the result of Lemma 4.3.6 can also be proved using the tilting complex

$$T = \Big(\bigoplus_{i=1}^{m}(P_{2i} \xrightarrow{\alpha_{2i-1}} P_{2i-1})\Big) \oplus \big(\bigoplus_{i=1}^{m} P_{2i-1}\big)$$

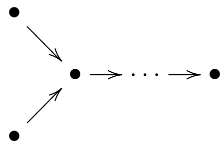
where the terms P_{2i-1} are always in degree 0.

Remark 4.3.8. There is no sequence of good mutations connecting the two self-injective algebras above. Indeed, none of the algebra mutations at any vertex is defined (see Lemma 4.2.26 for the algebra corresponding to the oriented n-cycle, and Lemmas 4.2.27 and 4.2.34 for the algebra with parameter sequence $\big((1,0,0),(1,0,0),\dots,(1,0,0)\big)$). The smallest such pair occurs in type D_6; the corresponding quivers are shown below.

Now we will provide 'standard forms' for derived equivalence for cluster-tilted algebras of type D_n.

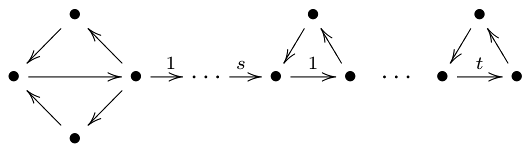
Theorem 4.3.9. *A cluster-tilted algebra of type D_n is derived equivalent to one of the cluster-tilted algebras with the following quivers, which we call 'standard forms' for derived equivalence:*

(a) D_n *(i.e. type I with a linearly oriented* A_{n-2} *quiver attached):*



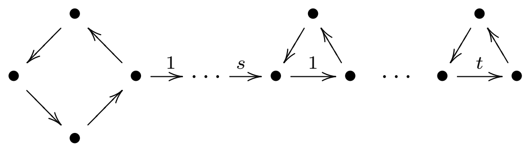
;

(b) *Type II as in the following figure, where* $s, t \geq 0$ *and* $s + 2t = n - 4$*:*



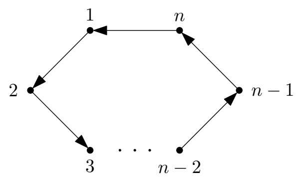
;

(c) *Type III as in the following figure, where* $s, t \geq 0$ *and* $s + 2t = n - 4$*:*



;

(d$_1$) *(only when* n *is odd) Type IV with a central cycle of length* n *without spikes, as in the following picture:*

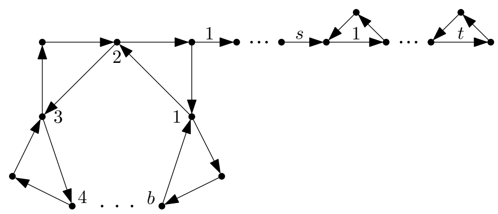


;

(d$_2$) *Type IV with parameter sequence*

$$\big((1, s, t), (1, 0, 0), \dots, (1, 0, 0)\big)$$

of length $b \geq 3$*, with* $s, t \geq 0$ *such that* $n = 2b + s + 2t$*, and the attached rooted quiver of type A is in standard form:*



;

(d$_3$) *Type IV with parameter sequence*

$$\big((1, 0, 0), (1, 0, 0), \dots, (1, 0, 0), (3, s_1, t_1), (3, s_2, t_2), \dots, (3, s_k, t_k)\big)$$

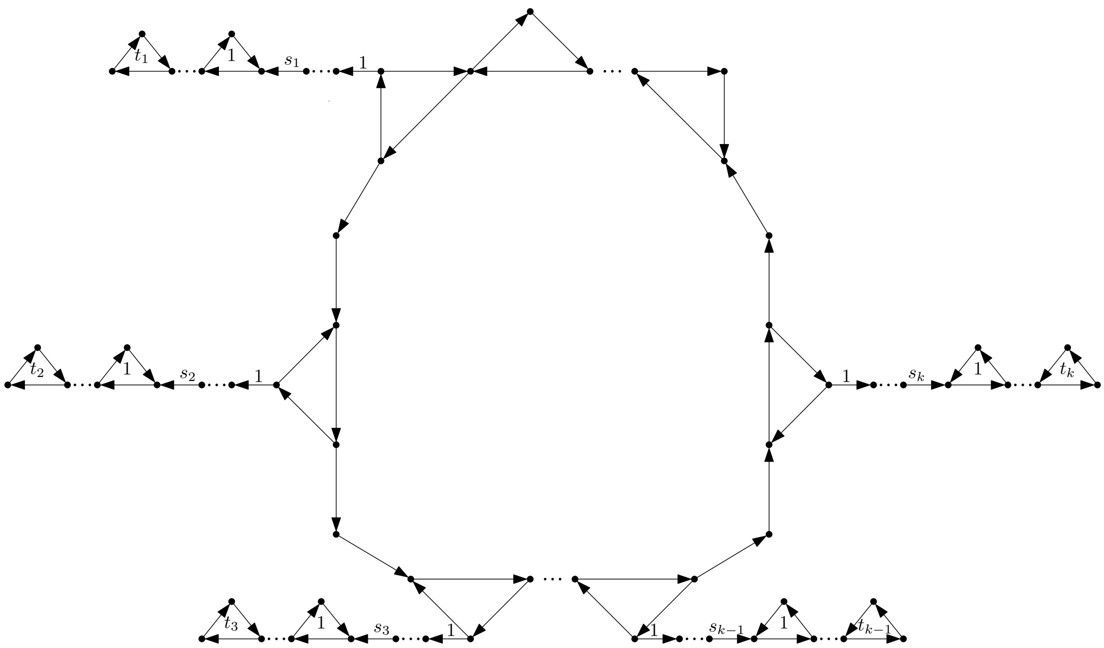
for some $k > 0$*, where the number of triples* $(1, 0, 0)$ *is* $b \geq 0$*, the non-negative integers* $s_1, t_1, \dots, s_k, t_k$ *are considered up to rotation of the sequence*

$$\big((s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)\big),$$

$n = 4k + 2b + s_1 + 2t_1 + \dots + s_k + 2t_k > 4$ *and the attached rooted quivers of type A are in standard form:*

Type	Parameters	Type	Parameters
III	$(s'+s'', t'+t''+1, s''', t''')$	III	$(s', t', s''+s''', t''+t'''+1)$
IV	$\big((1, s'+s'', t'+t''+1), (d_2, s''', t'''), \dots\big)$	IV	$\big((1, s', t'), (d_2, s''+s''', t''+t'''+1), \dots\big)$

Table 4.8. Good double mutations in parametric form.



To prove Theorem 4.3.9, we now turn to good double mutations as determined in Section 4.3.1. They are presented in parametric form in Table 4.8, based on Corollaries 4.3.2 and 4.3.4. Using them, we can obtain further transformations of quivers of types III and IV described in the next lemmas.

Lemma 4.3.10. *Consider quivers of type III.*

(a) *A quiver of type III with parameters $(s', t'+1, s'', t'')$ is equivalent by good mutations and good double mutations to one of type III with parameters $(s', t', s'', t''+1)$.*
(b) *A quiver of type III with parameters (s', t', s'', t'') can be transformed using good mutations and good double mutations to one of type III with parameters $(s'+s'', t'+t'', 0, 0)$.*

PROOF. (a) There is at least one oriented 3-cycle in Q'. By applying the good mutation moves 1, 2a, 3 and 4 of Table 4.1 we can assume that this oriented 3-cycle is directly attached to the oriented 4-cycle, that is, there are no intermediate arrows between this 3-cycle and the oriented 4-cycle. Then the statement follows from the first row of Table 4.8.
(b) Follows from the first part together with Lemma 4.2.39.

□

Lemma 4.3.11. *Oriented 3-cycles in a rooted quiver of type A attached to a spike in a group of consecutive spikes in a quiver of type IV can be moved by good mutations and good double mutations to a rooted quiver attached to any spike in that group.*

PROOF. It suffices to show that the two quivers with parameters $\big((1, s_1, t_1), (d_2, s_2, t_2+1), \dots\big)$ and $\big((1, s_1, t_1+1), (d_2, s_2, t_2), \dots\big)$ are equivalent by good mutations and good double mutations. By applying the good mutation moves 1, 2a, 3 and 4 of Table 4.1 we can assume that there is at least one oriented 3-cycle in $Q^{(2)}$ of the first quiver which is directly attached to the spike, that is, there are no intermediate arrows between this 3-cycle and the spike. Then setting $(s'', t'') = (0, 0)$ in the second row of Table 4.8 shows this equivalence. □

Remark 4.3.12. By regarding type III quivers as 'formal' type IV quivers (as in Remarks 4.1.5 and 4.2.37), the first row of Table 4.8 becomes a specific instance of the second.

Proof of Theorem 4.3.9. Given a quiver Q of a cluster-tilted algebra of Dynkin type D_n, we show how to find a quiver in one of the standard forms of Theorem 4.3.9 whose cluster-tilted algebra is derived equivalent to that of Q.

First we note that by applying the algorithm in the proof of Proposition 4.2.15 we can transform any rooted quiver of type A to a standard form as in Definition 4.2.14.

Let Q be a quiver of type I. If Q is any orientation of a D_n diagram, then by Lemma 4.2.38 (b) we can transform it by good mutations to a quiver in the class (a) in Theorem 4.3.9, thus proving the derived equivalence for this case. If Q contains at least one oriented 3-cycle, then by Lemma 4.2.38 (c) we can transform it by good mutations to a quiver of type II.

Now suppose that Q is a quiver of type II, then by Lemma 4.2.38 (d) we can transform it by good mutations to a quiver in the class (b) in Theorem 4.3.9.

Similarly, if Q is of type III, then by Lemma 4.3.10 (b) the corresponding cluster-tilted algebra is derived equivalent to one in class (c) in the theorem.

Let Q be a quiver of type IV. If it is a cycle without any spikes, we distinguish two cases. If the number of vertices is even, then by Lemma 4.3.6 the corresponding cluster-tilted algebra is derived equivalent to another one in type IV with spikes. Otherwise, it gives rise to the standard form (d_1).

If Q is of type IV with some spikes, let $\big((d_1, s_1, t_1), (d_2, s_2, t_2), \dots, (d_r, s_r, t_r)\big)$ be its parameters. By iteratively applying the good mutations IV.1a or IV.1b (see Table 4.6 for parametric notation) we can repeatedly shorten all the distances $d_i \geq 4$ by 2 until they become 2 or 3. By applying the good mutations IV.2a or IV.2b (see Table 4.6 for parametric notation) we can shorten further any distance 2 to a distance of 1. Thus we get a parameter sequence where all distances are either 1 or 3.

If all the distances are 1, we distinguish two cases. If all the distances sum up to at least 3, then by Lemma 4.2.42 and Lemma 4.3.11 we can successively move all the arrows and oriented 3-cycles of the attached rooted quivers of type A and concentrate them on a single spike, yielding the standard form of (d_2). If the distances sum up to 2, then by Remark 4.1.5 this is a quiver of type III (and Lemma 4.3.10 (b) yields the standard form (c)).

Otherwise, when there is at least one distance of 3, we observe the following. If all the distances are 3, we are in class (d_3). Therefore, we may assume that there is at least one distance of 1, i.e. we can consider a group of consecutive spikes. By Lemma 4.2.42 and Lemma 4.3.11, inside such a group we can always concentrate the attached rooted quivers of type A at one of the spikes in the group, thus creating a free spike at the end of the group. By Lemma 4.2.41 this free spike can then be moved to the beginning of the next group. In this way, we can move all spikes of the group except one to the next group, thus creating a single spike with some rooted quiver of type A attached.

Continuing in this way, we can eventually merge all groups of at least two consecutive spikes into one large group, with all the other spikes being single spikes. In other words, the sequence of distances will look like $(1, 1, \dots, 1, 3, 3, \dots, 3)$. In this large group of consecutive spikes, we can concentrate the rooted quivers of type A at the last spike, yielding exactly the standard form appearing in (d_3). $\square$

CHAPTER 5

Invariants of derived equivalence

Section 5.1 is a brief review of background material. The results of Section 5.2 are a joint work with Thorsten Holm and Sefi Ladkani, and appeared in [**13**]. The results of Section 5.3 are new in this thesis.

5.1. Asymmetry matrices

Let A be a finite-dimensional K-algebra and let $P_1, \dots, P_n$ be a complete collection of pairwise non-isomorphic indecomposable projective A-modules. The Cartan matrix of A is then the $n \times n$ matrix C_A defined by $(C_A)_{ij} = \dim_K \operatorname{Hom}_A(P_j, P_i)$ (see Section 2.3). An important invariant of derived equivalence is given by the following well-known proposition. For a proof see the proof of Proposition 1.5 in [**14**], and also [**12**, Prop. 2.6].

Proposition 5.1.1. *Let A and B be two finite-dimensional, derived equivalent algebras. Then the matrices C_A and C_B represent equivalent bilinear forms over $\mathbb{Z}$, that is, there exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $PC_AP^T = C_B$, where n denotes the number of indecomposable projective modules of A and B (up to isomorphism).*

In general, to decide whether two integral bilinear forms are equivalent is computationally intensive. Therefore, it is useful to introduce somewhat weaker invariants that are computationally easier to handle. In order to do this, assume further that C_A is invertible over $\mathbb{Q}$. In this case one can consider the rational matrix $S_A = C_A C_A^{-T}$ (here C_A^{-T} denotes the inverse of the transpose of C_A), known in the theory of non-symmetric bilinear forms as the *asymmetry* of C_A.

Proposition 5.1.2. *Let A and B be two finite-dimensional, derived equivalent algebras with invertible (over $\mathbb{Q}$) Cartan matrices. Then we have the following assertions, each implied by the preceding one:*

(a) *There exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $PC_AP^T = C_B$.*
(b) *There exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $PS_AP^{-1} = S_B$.*
(c) *There exists $P \in \mathrm{GL}_n(\mathbb{Q})$ such that $PS_AP^{-1} = S_B$.*
(d) *The matrices S_A and S_B have the same characteristic polynomial.*

Proof. If A and B are derived equivalent algebras then condition (a) follows from [**42**, Corollary 3.13]. In particular, this corollary follows from a sequence of technical lemmas regarding the Euler form and the Möbius function of a poset (see [**42**, Section 3]). We only check that each assertion is implied by the preceding one:

(a) $\Rightarrow$ (b) If $C_B = PC_AP^T$ for some $P \in \mathrm{GL}_n(\mathbb{Z})$, then

$$\begin{aligned} S_B &= C_B C_B^{-T} = (PC_AP^T)(P^{-T}C_A^{-T}P^{-1}) \\ &= P(C_AC_A^{-T})P^{-1} = PS_AP^{-1}. \end{aligned}$$

(b) $\Rightarrow$ (c) This implication is clear.

(c) $\Rightarrow$ (d) If $PS_AP^{-1} = S_B$ for some $P \in \mathrm{GL}_n(\mathbb{Q})$, then S_A and S_B are similar over $\mathbb{Q}$ and thus, S_A and S_B have the same characteristic polynomial. That is, since

$$P(xE - S_A)P^{-1} = xE - PS_AP^{-1} = xE - S_B,$$

we get

$$\det(xE - S_B) = \det(P)\det(xE - S_A)\det(P^{-1}) = \det(xE - S_A).$$

□

Since the determinant of the Cartan matrix is also invariant under derived equivalence (see [**14**, Proposition 1.5]), we obtain the following discrete invariant of derived equivalence.

Definition 5.1.3. For an algebra A with invertible Cartan matrix C_A over $\mathbb{Q}$, we define its *associated polynomial* as $(\det C_A) \cdot \chi_{S_A}(x)$, where $\chi_{S_A}(x)$ is the characteristic polynomial of the asymmetry matrix $S_A = C_A C_A^{-T}$.

Remark 5.1.4. The matrix S_A (or better, minus its inverse $-C_A^T C_A^{-1}$) is related to the *Coxeter transformation* which has been widely studied in the case when A has finite global dimension (so that C_A is invertible over $\mathbb{Z}$), see [**44**]. The characteristic polynomial is then known as the *Coxeter polynomial* of the algebra.

5.2. Cartan determinants of cluster-tilted algebras of type D_n

The main tool for distinguishing the various standard forms appearing in Theorem 4.3.9 is the computation of numerical invariants of derived equivalence. We start by computing the formulae for the determinants of the Cartan matrices of all cluster-tilted algebras of type D_n.

Notation. Throughout this section, for a quiver Q which is mutation equivalent to a Dynkin quiver, we denote by A_Q the corresponding cluster-tilted algebra and its Cartan matrix by C_Q.

The following theorem was obtained independently by Vatne in [**52**].

Theorem 5.2.1. *Let Q be a quiver which is mutation equivalent to D_n for $n \geq 4$. Using the notation from Section 4.1 we have the following formulae for the determinants of the Cartan matrices of the corresponding cluster-tilted algebras.*

(I) *If Q is of type I, then* $\det C_Q = 2^{t(Q')} = \det C_{Q'}$.

(II) *If Q is of type II, then* $\det C_Q = 2 \cdot 2^{t(Q')+t(Q'')} = 2 \cdot \det C_{Q'} \cdot \det C_{Q''}$.

(III) *If Q is of type III, then* $\det C_Q = 3 \cdot 2^{t(Q')+t(Q'')} = 3 \cdot \det C_{Q'} \cdot \det C_{Q''}$.

(IV) *For a quiver Q of type IV with central cycle of length $m \geq 3$, let $Q^{(1)}, \dots, Q^{(r)}$ be the rooted quivers of type A glued to the spikes and let $c(Q)$ be the number of vertices on the central cycle which are part of two (consecutive) spikes, i.e. $c(Q) = |\{1 \leq j \leq r \,:\, d_j = 1\}|$, cf.* (4.1.1). *Then*

$$\det C_Q = (m + c(Q) - 1) \cdot \prod_{j=1}^{r} 2^{t(Q^{(j)})} = (m + c(Q) - 1) \cdot \prod_{j=1}^{r} \det C_{Q^{(j)}}.$$

By using that the Cartan determinant of a cluster-tilted algebra of type A_n is a power of 2 (see Proposition 5.2.6 below), we immediately obtain the following.

Corollary 5.2.2.

(a) *A cluster-tilted algebra of type II is not derived equivalent to any cluster-tilted algebra of type III.*

(b) *A cluster-tilted algebra of type II is not derived equivalent to any cluster-tilted algebra of type IV whose Cartan determinant is not a power of* 2.

Note that the determinant alone is not enough to distinguish types II and IV, an example already occurs in type D_5.

Example 5.2.3. The Cartan matrices of the cluster-tilted algebra of type II with parameters $(1, 0, 0, 0)$ and the one of type IV with parameters $\big((3, 1, 0)\big)$ whose quivers are given by

have both determinant 2. But the characteristic polynomials of their asymmetries differ, namely $x^5 - x^3 + x^2 - 1$ and $x^5 - 2x^3 + 2x^2 - 1$, respectively, as we will show in Propositions 5.3.11 and 5.3.12 (c).

Corollary 5.2.4. *Any two distinct standard forms of Theorem 4.3.9 which are not of the class* (d_3) *are not derived equivalent.*

PROOF. We show how to distinguish between standard forms which are not of class (d_3). First, observe that when the number, n, of vertices is odd, the standard form in class (d_1) corresponds to the unique self-injective cluster-tilted algebra with n vertices (see Section 4.3.2) hence it is distinguished by the fact that self-injectivity is invariant under derived equivalence (see [**1**, Theorem 3]).

The standard forms in all other classes (except (d_3)) can be distinguished by the determinants of their Cartan matrices. These are given in the list below:

$$\begin{array}{cccc} 1 & 2^{t+1} & 3\cdot 2^t & (2b-1)\cdot 2^t \\ (a) & (b) & (c) & (d_2)\ , \end{array}$$

from which it is clear how to distinguish between the standard forms (since $b \geq 3$ in (d_2)). □

Remark 5.2.5. There is no known example of two distinct standard forms of Theorem 4.3.9 which are derived equivalent.

Now, we want to prove Theorem 5.2.1. As the quivers of cluster-tilted algebras of type D_n are defined by gluing of rooted quivers of type A, it is also useful to have formulae for cluster-tilted algebras of Dynkin type A_n. Since cluster-tilted algebras of type A_n are gentle, the Cartan determinants can be obtained as a special case of [**36**, Theorem 1] where formulae for the Cartan determinants of arbitrary gentle algebras are given; for a simplified proof for the special case of cluster-tilted algebras of type A_n see also [**23**, Proposition 4.1].

Proposition 5.2.6. *Let Q be a quiver mutation equivalent to a Dynkin quiver of type A_n. Then the Cartan matrix of the cluster-tilted algebra corresponding to Q has determinant* $\det C_Q = 2^{t(Q)}$.

To prove Theorem 5.2.1, we shall first show a useful reduction lemma. We need the following notation: if Q is a quiver and V a set of vertices in Q, then $Q \setminus V$ is the quiver obtained from Q by removing all vertices in V from Q and all arrows incident to them.

Lemma 5.2.7. *Let Q be a quiver in the mutation class of a quiver of Dynkin type D_n, i.e. Q is of one of the types I,II,III,IV given in Section 4.1.*

(i) *Suppose Q contains a vertex a of valency 1. Then* $\det C_Q = \det C_{Q\setminus\{a\}}$.
(ii) *Suppose that Q contains an oriented 3-cycle with vertices a, b, c (in this order, i.e. there is an arrow from b to c etc.) where a and b have valency 2 in Q and where the quiver $Q' = Q \setminus \{a, b\}$ is mutation equivalent to a quiver of Dynkin type A or D. Then* $\det C_Q = 2 \cdot \det C_{Q\setminus\{a,b\}} = 2 \cdot \det C_{Q'}$.

PROOF. (i) Since taking transposes does not change the determinant we can assume that a is a sink. Then the Cartan matrix of the cluster-tilted algebra corresponding to Q has the form

$$C_Q = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & C_{Q\setminus\{a\}} & \\ * & & & \end{array}\right)$$

from which the desired formula follows directly by Laplace expansion.

(ii) In the cluster-tilted algebra corresponding to Q, every product of two consecutive arrows in the oriented 3-cycle a, b, c is zero. Moreover, in the quiver $Q \setminus \{a\}$ there is a one-to-one correspondence between non-zero paths starting in c and non-zero paths starting in b by extending any of the former paths by the arrow from b to c (in fact, by the unique relations in the cluster-tilted algebra all these extensions remain non-zero). Therefore, the Cartan matrix of the cluster-tilted algebra corresponding to Q has the form

$$C_Q = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & *_1 & \cdots & \cdots & *_n \\ 1 & 0 & 1 & *_1 & \cdots & \cdots & *_n \\ * & 0 & * & & & & \\ \vdots & \vdots & \vdots & & C_{Q\setminus\{a,b,c\}} & & \\ \vdots & \vdots & \vdots & & & & \\ * & 0 & * & & & & \end{pmatrix}$$

where the first three rows are labelled by a, b and c, respectively. The entries marked by $*_i$ are really the same in the rows for b and c because of the one-to-one correspondence just mentioned. Note that in the row for a (the first row) and in the column for b (the second column) we have 0s except the two 1s indicated because of the zero-relations in the oriented 3-cycle with vertices a, b, c.

Denote by r_v the row in the above matrix corresponding to the vertex v. We now perform an elementary row operation, namely replace the first row r_a by $r_a - r_b + r_c$. Then we get

$$\det C_Q = \det \begin{pmatrix} 2 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & *_1 & \cdots & \cdots & *_n \\ 1 & 0 & 1 & *_1 & \cdots & \cdots & *_n \\ * & 0 & * & & & & \\ \vdots & \vdots & \vdots & & C_{Q\setminus\{a,b,c\}} & & \\ \vdots & \vdots & \vdots & & & & \\ * & 0 & * & & & & \end{pmatrix} = 2 \cdot \det C_{Q\setminus\{a,b\}}$$

where the last equality follows directly by Laplace expansion (with respect to the row of a and then the column of b).

□

We will also need three additional lemmas dealing with skeleta of type IV, i.e. the rooted quivers of type A consist of just one vertex.

Lemma 5.2.8. *Let Q be a quiver of type IV which contains no spikes at all, i.e. it is just an oriented cycle of length $m \geq 3$. Then $\det C_Q = m - 1$.*

Proof. We have

$$\det C_Q = \det \begin{pmatrix} 1 & \cdots & \cdots & 1 & 0 \\ 0 & 1 & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & 1 \end{pmatrix} = (-1)^{m-1} \det \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} = m - 1,$$

where, for the last equality, we have used the following formula whose verification is a standard exercise in linear algebra: for all $a, b \in K$, K any field, we have

$$\det \begin{pmatrix} b & a & \cdots & a \\ a & b & & \vdots \\ \vdots & & \ddots & a \\ a & \cdots & a & b \end{pmatrix} = (b-a)^{m-1}(b + (m-1)a).$$

□

Lemma 5.2.9. *Let Q be a quiver of type IV with parameter sequence $\big((1,0,0),(1,0,0),\dots,(1,0,0)\big)$ of length $m \geq 3$, in other words, it is an oriented cycle of length m with all spikes present. Then $\det C_Q = 2m - 1$.*

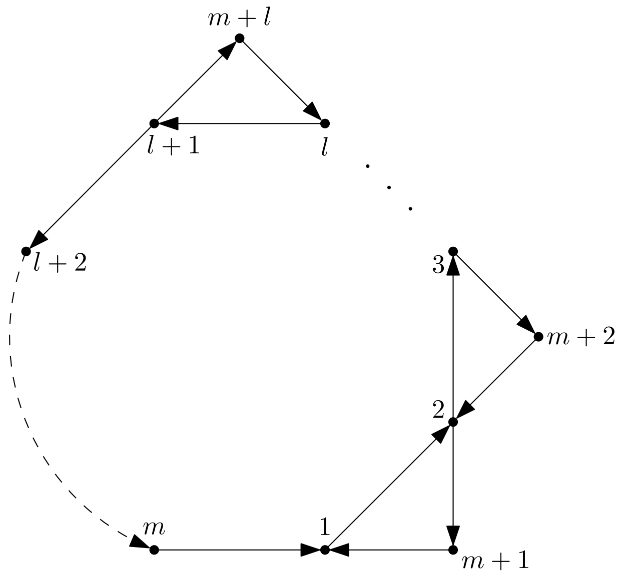


Figure 5.1

Proof. By Lemma 4.3.6, the cluster-tilted algebra A_Q is derived equivalent to the one corresponding to the cycle of length $2m$. Since the determinant of the Cartan matrix is invariant under derived equivalence, the result now follows from Lemma 5.2.8. $\square$

Lemma 5.2.10. *Let Q be a skeleton of type IV with central cycle of length $m \geq 3$ on which not all spikes are present. Let $c(Q)$ be the number of vertices on the central cycle which are part of two (consecutive) spikes. Then* $\det C_Q = m + c(Q) - 1$.

Proof. We shall closely look at one group of $l \geq 1$ consecutive spikes in Q and label the vertices as in Figure 5.1. Then the Cartan matrix has the following shape:

$$C_Q = \left(\begin{array}{cccc|cccc||cccc|cccc}
1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots & \ddots & \ddots & & \vdots & & & \vdots \\
1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\
\hline
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\hline
1 & 1 & \cdots & 1 & & & & & 0 & 0 & \cdots & 0 & & & & \\
1 & 1 & \cdots & 1 & & ? & ? & & 0 & 0 & \cdots & 0 & & ? & ? & \\
\vdots & \vdots & & \vdots & & ? & ? & & \vdots & \vdots & & \vdots & & ? & ? & \\
1 & 1 & \cdots & 1 & & & & & 0 & 0 & \cdots & 0 & & & & \\
\hline\hline
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & & \vdots & \vdots & & & \vdots & 1 & \ddots & & & \vdots & & & \vdots \\
\vdots & & \ddots & 0 & \vdots & & & \vdots & 0 & \ddots & \ddots & 0 & \vdots & & & \vdots \\
\hline
0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & \cdots & 0 & & & & & 0 & \cdots & \cdots & 0 & & & & \\
\vdots & & & \vdots & & ? & ? & & \vdots & & & \vdots & & ? & ? & \\
0 & \cdots & \cdots & 0 & & & & & 0 & \cdots & \cdots & 0 & & & &
\end{array}\right).$$

The two highlighted rows correspond to the vertices $l+1$ and $m+l$, respectively. For each vertex j let r_j be the row of C_Q corresponding to j.

Note that the column $m+l$ of C_Q has only two non-zero entries, namely in rows $l+1$ and $m+l$. We first replace row r_{l+1} by $r_{l+1} - r_{m+l} + r_l - r_1$ (in case $l = 1$ this indeed means just $r_{l+1} - r_{m+l}$).

Then column $m+l$ has only one non-zero entry, that on the diagonal; Laplace expansion along this column yields a new matrix $\tilde{C}$.

We consider the $(l+1)^{\text{th}}$ row in this new matrix which has the form

$$\left(\begin{array}{ccccc|ccccc||cccc|cccc} 1 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & 1 & N & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \end{array}\right)$$

where the number N at position $(l+1,m)$ is equal to 0 if $l=1$ and equal to 2 if $l>1$.

In case $l=1$ we see that $\tilde{C}$ is equal to the Cartan matrix of the cluster-tilted algebra corresponding to the quiver $Q \setminus \{m+l\}$. This means that when computing the Cartan determinant we can remove isolated spikes, i.e. spikes which are not neighbouring any other spike.

In this case $l=1$ the statement of the lemma follows immediately by induction on the number of spikes (with the case of no spikes treated earlier in Lemma 5.2.8 as the base of the induction). In fact, removing the isolated spike does not change the determinant (as we have just seen), and also the formula given in the lemma is not affected by removing an isolated spike.

Let us turn to the more complicated case $l>1$ (which we shall also show by induction). If $l>1$, the matrix $\tilde{C}$ is equal to the Cartan matrix of $Q \setminus \{m+l\}$, except for the $(l+1,m)$-entry which is 2 in $\tilde{C}$, but 1 in $C_{Q\setminus\{m+l\}}$.

To compare the determinants in this case we use the following easy observation. Let $C=(c_{ij})$ and $\tilde{C}=(\tilde{c}_{ij})$ be two matrices which only differ at the (k,h)-entry. Then

$$\det \tilde{C} - \det C = (-1)^{k+h}(\tilde{c}_{kh} - c_{kh})C_{kh}$$

where C_{kh} is the matrix obtained from C (or $\tilde{C}$) by removing row k and column h.

Applied to our situation we get

$$\det \tilde{C} - \det C_{Q\setminus\{m+l\}} = (-1)^{l+1+m}(2-1)\det C_{l+1,m} = (-1)^{l+1+m}\det C_{l+1,m}.$$

Since $\det \tilde{C} = \det C_Q$ we can rewrite this to get

$$\det C_Q = \det C_{Q\setminus\{m+l\}} + (-1)^{l+1+m}\det C_{l+1,m}.$$

By induction on the number of spikes of Q (with the case of no spikes treated earlier as the base of the induction) we can deduce that $\det C_{Q\setminus\{m+l\}} = m + c(Q) - 2$ and hence

$$\det C_Q = m + c(Q) - 2 + (-1)^{l+1+m}\det C_{l+1,m}. \tag{5.2.1}$$

To prove the assertion of the lemma we therefore have to show that $(-1)^{l+1+m}\det C_{l+1,m} = 1$.

We keep the labelling of the rows and columns also for $C_{l+1,m}$ (i.e. there is no row with label $l+1$ or $m+l$, and no column with label m or $m+l$). For convenience, the vertices $1,\ldots,m$ on the cycle will be called *cycle vertices* and the remaining vertices will be called *outer vertices*.

Note that in $C_{l+1,m}$ we have 0s on the diagonal in all rows indexed by cycle vertices which have no spike attached. More precisely, the rows corresponding to cycle vertices are of the form

$$\left(\begin{array}{ccc|ccc||ccccccc|ccc} 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right)$$

if the vertex has a spike attached, and

$$\left(\begin{array}{cccc|ccc||ccc|ccc} 1 & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right)$$

if there is no spike attached to the vertex.

For each cycle vertex $j \neq 1$ with no spike attached we perform the elementary row operation replacing r_j by $r_j - r_1$; this gives a unit vector with -1 on the diagonal. Laplace expansion along all these rows removes from $C_{l+1,m}$ all rows and columns corresponding to cycle vertices (not equal to vertex 1) with no spikes attached; for the determinant we thus get a sign $(-1)^{m-p(Q)-1}$ where $p(Q)$ is the total number of spikes of Q.

The matrix obtained after this removal process has rows indexed by vertex 1, the cycle vertices with spikes attached except vertex $l+1$, and the outer vertices except vertex $m+l$. It has the form

$$\left(\begin{array}{ccccccc|cccc}
1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & 0 & \cdots & \cdots & 0 \\
1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & & & & \vdots & \vdots & 0 & 1 & \ddots & \vdots \\
1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & 0 \\
1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & 0 & \cdots & 0 & 1 \\
\hline
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & ? & ? & \\
 & & & 0 & 1 & & & & ? & ? & \\
 & & & & \ddots & \ddots & & & & & \\
 & & & & & 0 & 1 & & & &
\end{array}\right)$$

where in the lower left block the crucial 0 on the main diagonal occurs in the column labelled by vertex l (because the row indexed by $m+l$ has been removed).

Now for each cycle vertex j with a spike attached we replace the row r_j by $r_j - r_1$, giving a unit vector. Consecutive Laplace expansion along these rows removes all columns corresponding to outer vertices (and all rows corresponding to cycle vertices with spikes attached). For each of these Laplace expansions we get a sign $(-1)^{p(Q)+1}$ and there are $p(Q)-1$ such expansions in total, giving an overall sign of $(-1)^{p(Q)^2-1} = (-1)^{p(Q)-1}$.

We are left with a $p(Q) \times p(Q)$-matrix of the form

$$\begin{pmatrix}
1 & 1 & \cdots & \cdots & \cdots & 1 & 1 \\
1 & & & & & & \\
 & \ddots & & & & & \\
 & & 1 & & & & \\
 & & & 0 & 1 & & \\
 & & & & \ddots & \ddots & \\
 & & & & & 0 & 1
\end{pmatrix}.$$

We expand consecutively along the last rows until we get a $l \times l$-matrix of the form

$$\begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & & & \\
 & \ddots & & \\
 & & 1 & 0
\end{pmatrix}.$$

Laplace expansion along the last column leads to the determinant $(-1)^{l+1}$.

Summarising our arguments, we get

$$\det C_{l+1,m} = (-1)^{m-p(Q)-1} \cdot (-1)^{p(Q)-1} \cdot (-1)^{l+1} = (-1)^{m+l-1}.$$

Substituting this into equation (5.2.1) we get the following expression for the Cartan determinant of Q

$$\begin{aligned}
\det C_Q &= m + c(Q) - 2 + (-1)^{l+1+m} \det C_{l+1,m} \\
&= m + c(Q) - 2 + (-1)^{l+1+m}(-1)^{m+l-1} = m + c(Q) - 1
\end{aligned}$$

which is exactly the formula claimed in Lemma 5.2.10. □

PROOF OF THEOREM 5.2.1. (I) Applying part (i) of Lemma 5.2.7 twice gives $\det C_Q = \det C_{Q\setminus\{a,b\}}$. By definition $Q' = Q\setminus\{a,b\}$ is a quiver of Dynkin type A, thus $\det C_{Q'} = 2^{t(Q')}$ by Proposition 5.2.6. Clearly, $t(Q) = t(Q')$ for quivers of type I and hence

$$\det C_Q = \det C_{Q\setminus\{a,b\}} = \det C_{Q'} = 2^{t(Q')} = 2^{t(Q)}.$$

(II) Let Q be a quiver of type II. By applying Lemma 5.2.7 inductively we can shrink each of the quivers Q' and Q'' (which are of Dynkin type A) to one vertex where, for the Cartan determinant of the corresponding cluster-tilted algebra, we get a factor 2 for each oriented 3-cycle we remove (see part (i) and (ii) of Lemma 5.2.7). Thus, we get

$$\det C_Q = 2^{t(Q')} \cdot 2^{t(Q'')} \cdot \det C_{\tilde{Q}} \tag{5.2.2}$$

where $\tilde{Q}$ is the skeleton with vertices a, b, c', c'' obtained after shrinking both Q' and Q'' to one vertex. Labelling the rows and columns in the order a, b, c', c'' the cluster-tilted algebra corresponding to $\tilde{Q}$ has Cartan matrix $C_{\tilde{Q}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ which has determinant 2. This gives the desired formula as

$$\det C_Q = 2^{t(Q')} \cdot 2^{t(Q'')} \cdot \det C_{\tilde{Q}} = 2^{t(Q')+t(Q'')+1} = 2 \cdot \det C_{Q'} \det C_{Q''}.$$

(III) Completely analogous to the previous argument in type II we can shrink the subquivers Q' and Q'' of any quiver of type III to one vertex, ending up with an oriented 4-cycle $\tilde{Q}$. Labelling the rows and columns in the order a, b, c', c'' the cluster-tilted algebra corresponding to this 4-cycle has Cartan matrix $C_{\tilde{Q}} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ which has determinant 3. As above we then get

$$\det C_Q = 2^{t(Q')} \cdot 2^{t(Q'')} \cdot \det C_{\tilde{Q}} = 3 \cdot 2^{t(Q')+t(Q'')} = 3 \cdot \det C_{Q'} \det C_{Q''}.$$

(IV) If there are no spikes at all, the result follows from Lemma 5.2.8. Otherwise, by Lemma 5.2.7 we can again assume that all the rooted quivers $Q^{(1)}, \dots, Q^{(r)}$ of type A attached to the spikes have been shrunk to one vertex, yielding a factor

$$\prod_{j=1}^{r} 2^{t(Q^{(j)})} = \prod_{j=1}^{r} \det C_{Q^{(j)}}$$

for the Cartan determinant $\det C_Q$. The result then follows from Lemmas 5.2.9 and 5.2.10. □

Now we list further properties of the Cartan determinant for derived equivalences.

Proposition 5.2.11. *Let Q be a quiver which is mutation equivalent to D_n. Then*

$$\det C_Q \leq \frac{3}{4} \cdot 2^{n/2},$$

with equality if and only if Q is of type III with the attached rooted quivers of type A consisting only of oriented 3-cycles.

For the proof, we need the following lemmas.

Lemma 5.2.12. *Let Q be a quiver mutation equivalent to A_{n+1}. Then $\det C_Q \leq 2^{n/2}$, with equality if and only if there are only oriented 3-cycles in Q.*

Proof. We denote by s the number of arrows in Q which are not part of an oriented 3-cycle, and by t the number of oriented 3-cycles in Q. Then $n = s+2t$ and by Proposition 5.2.6, $\det C_Q = 2^t$, and the result follows. □

Lemma 5.2.13. *Let Q be a skeleton of type IV. Then $\det C_Q \leq \frac{3}{4} \cdot 2^{n/2}$, where n is the number of vertices of Q, with equality if and only if Q is a cycle of length 4.*

Proof. Let $m \geq 3$ be the length of the central cycle and $0 \leq r \leq m$ the number of spikes on that cycle. Then $n = m + r \geq 3$ and by Theorem 5.2.1,

$$\det C_Q \leq (m-1) + r = n - 1.$$

Now, one can easily show that $a_n := \frac{3}{4} \cdot 2^{n/2} - n + 1 > 0$ and (a_n) is monotonically increasing for any $n \geq 5$. Hence, the result follows since $n - 1 < \frac{3}{4} \cdot 2^{n/2}$ for any $n \geq 3$ except $n = 4$. For $n = 4$ we have that either $m = 3$ and $r = 1$, so that $\det C_Q = 2 < 3$, or that $m = 4$ and $r = 0$, and then $\det C_Q = 3$, yielding the only case where equality holds. □

Proof of Proposition 5.2.11. We use Theorem 5.2.1 and proceed according to the type of Q.

(I) If Q is of type I and Q' is the attached rooted quiver of type A with $n' + 1$ vertices, then $n' = n - 3$ and by Lemma 5.2.12,

$$\det C_Q = \det C_{Q'} \leq 2^{(n-3)/2} = \frac{\sqrt{2}}{4} \cdot 2^{n/2} < \frac{3}{4} \cdot 2^{n/2}.$$

(II) If Q is of type II and Q', Q'' are the attached rooted quivers of type A with $n' + 1$ and $n'' + 1$ vertices, then $n' + n'' = n - 4$ and by Lemma 5.2.12,

$$\det C_Q = 2 \cdot \det C_{Q'} \cdot \det C_{Q''} \leq 2 \cdot 2^{(n-4)/2} = \frac{2}{4} \cdot 2^{n/2} < \frac{3}{4} \cdot 2^{n/2}.$$

(III) If Q is of type III and Q', Q'' are the attached rooted quivers of type A with $n' + 1$ and $n'' + 1$ vertices, then $n' + n'' = n - 4$ and by Lemma 5.2.12,

$$\det C_Q = 3 \cdot \det C_{Q'} \cdot \det C_{Q''} \leq 3 \cdot 2^{(n-4)/2} = \frac{3}{4} \cdot 2^{n/2}$$

with equality if and only if Q' and Q'' consist only of oriented 3-cycles.

(IV) Finally, if Q is of type IV, we may assume that it is not a cycle of length 4 (as this case falls into type III). Let m be the length of the central cycle, r be the number of spikes and $Q^{(1)}, \dots, Q^{(r)}$ be the corresponding rooted quivers of type A attached, with $n_1+1, \dots, n_r+1$ vertices, respectively.

Then $n = m + r + n_1 + \cdots + n_r$, and by Theorem 5.2.1

$$\det C_Q = (m + c(Q) - 1) \cdot \prod_{i=1}^{r} \det C_{Q^{(i)}}$$

where $(m + c(Q) - 1)$ is the determinant of the skeleton of type IV with $m + r$ vertices (see Lemmas 5.2.8, 5.2.9 and 5.2.10).

Hence by Lemmas 5.2.13 and 5.2.12,

$$\begin{aligned}\det C_Q &< \frac{3}{4} \cdot 2^{(m+r)/2} \cdot \det C_{Q^{(1)}} \cdot \det C_{Q^{(2)}} \cdot \ldots \cdot \det C_{Q^{(r)}} \\ &\leq \frac{3}{4} \cdot 2^{(m+r)/2} \cdot 2^{n_1/2} \cdot 2^{n_2/2} \cdot \ldots \cdot 2^{n_r/2} = \frac{3}{4} \cdot 2^{n/2}\end{aligned}$$

so the assertion holds in this case as well.

□

Corollary 5.2.14. *Let Q and Q' be quivers of derived equivalent cluster-tilted algebras of type D_n. If Q is of type III with its rooted quivers of type A consisting only of oriented* 3*-cycles, then so is Q'.*

Proposition 5.2.15. *Two standard forms of skeleta of type IV appearing in Theorem 4.3.9 with the same number of vertices and the same Cartan determinant are isomorphic.*

Proof. First observe that when the number of vertices is odd, the standard form in class (d_1) is distinguished from (d_2) and (d_3) by the fact that self-injectivity is invariant under derived equivalence (see [**1**, Theorem 3]). Hence, we suppose that Q is a standard form of a skeleton of type IV with some spikes. Recall that according to Theorem 4.3.9, the corresponding parameters are

$$\big((1,0,0), \dots, (1,0,0), (3,0,0), \dots, (3,0,0)\big)$$

with the numbers of $(1,0,0)$ and $(3,0,0)$ entries being $b \geq 0$ and $k \geq 0$, respectively, so that Q is controlled by the two numbers b and k.

Then the number of vertices of Q is $4k+2b$, the length of the central cycle is $m = 3k+b$, and the Cartan determinant is thus given by Theorem 5.2.1(IV) as $\det C_Q = (m-1)+b = 3k+2b-1$.

Now if Q' is another standard form corresponding to the numbers b' and k' having the same number of vertices and the same Cartan determinant as Q, then

$$\begin{aligned} 4k+2b &= 4k'+2b', \\ 3k+2b-1 &= 3k'+2b'-1, \end{aligned}$$

hence $b' = b$ and $k' = k$. $\square$

Corollary 5.2.16. *Two standard forms of skeleta of type IV are derived equivalent if and only if they are isomorphic.*

5.3. Characteristic polynomials of the asymmetry matrices

Since the determinants of the Cartan matrices of all cluster-tilted algebras of types A_n and D_n do not vanish, we can consider their asymmetry matrices and the corresponding characteristic polynomials.

5.3.1. Methods to determine characteristic polynomials of the asymmetry matrices. In this section we provide methods to determine the characteristic polynomials of the asymmetry matrices for cluster-tilted algebras of type D_n. We also compute these polynomials for cluster-tilted algebras of type A_n.

First we provide a method to determine the characteristic polynomials of the asymmetry matrices. We reduce the determination of the characteristic polynomials to the computation of certain 'smaller' polynomials. For this, we recall some background from [**15**].

Let Q be a quiver, v a vertex of Q and p a path in Q. We say that p *properly passes through* v if p can be written in the form $p = p_2 e_v p_1$ with paths p_1, p_2 in Q of length ≥ 1 and e_v the trivial path corresponding to v. Moreover, an admissible ideal I of relations in KQ is called *v-separated* when I can be generated as an ideal by a set R of relations such that for every $\sum_i \lambda_i p_i \in R$ with $\lambda_i \in K\backslash\{0\}$ and distinct paths p_i in Q, none of the p_i properly passes through v. We denote by $Q\backslash\{v\}$ the quiver obtained from Q by deleting the vertex v and all incident arrows. The *union* of quivers is given by the union of the vertex sets Q_0 and the disjoint union of the arrow sets Q_1. In [**15**, Theorem 2.2] there is a formula for determining the characteristic polynomial of the matrix $\Phi_Q := -C^T C^{-1}$, where $C := C_{KQ/I}$, given as follows:

Theorem 5.3.1 (Boldt [15]). *Let Γ_1, Γ_2 be two quivers with $(\Gamma_1)_0 \cap (\Gamma_2)_0 = \{v\}$ and let Q be the union of Γ_1 and Γ_2. Suppose that $I \subset KQ$ is an v-separated ideal of relations such that $A_Q := KQ/I$ is finite dimensional. Let $\Gamma_1' = \Gamma_1\backslash\{v\}$ and $\Gamma_2' = \Gamma_2\backslash\{v\}$, and define the algebras A_{Γ_1}, $A_{\Gamma_1'}$, A_{Γ_2}, $A_{\Gamma_2'}$ canonically:*

$$A_{\Gamma_i} = K\Gamma_i/(I \cap K\Gamma_i) \text{ and } A_{\Gamma_i'} = K\Gamma_i'/(I \cap K\Gamma_i') \text{ for } i = 1,2.$$

If the determinant $\det(C_{A_Q}) = \det(C_{A_{\Gamma_1}})\det(C_{A_{\Gamma_2}})$ is non-zero, the characteristic polynomial of Φ_Q is

$$\chi_{\Phi_Q}(x) = \chi_{\Phi_{\Gamma_1}}(x) \cdot \chi_{\Phi_{\Gamma_2'}}(x) + \chi_{\Phi_{\Gamma_1'}}(x) \cdot \chi_{\Phi_{\Gamma_2}}(x) - (x+1) \cdot \chi_{\Phi_{\Gamma_1'}}(x) \cdot \chi_{\Phi_{\Gamma_2'}}(x).$$

Example 5.3.2. Let $A = KQ$ be the path algebra of the following quiver of type A_3.

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$$

We can choose $v = 2$ since there are no relations in A. Thus Γ_1 and Γ_2 are both quivers of type A_2 and the corresponding Cartan matrices of A_{Γ_1} and A_{Γ_2}, respectively, have the same shape $C_{A_{\Gamma_1}} = C_{A_{\Gamma_2}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then Γ_1' and Γ_2' are both quivers consisting of just a single vertex, hence, the Cartan matrices of $A_{\Gamma_1'}$ and $A_{\Gamma_2'}$ are just $C_{A_{\Gamma_1'}} = C_{A_{\Gamma_2'}} = 1$.

Next, we compute the matrices Φ_{Γ_1}, Φ_{Γ_2}, $\Phi_{\Gamma_1'}$ and $\Phi_{\Gamma_2'}$ and the corresponding characteristic polynomials to be

$$\begin{aligned}
\Phi_{\Gamma_1} = \Phi_{\Gamma_2} = -C_{A_{\Gamma_1}}^T C_{A_{\Gamma_1}}^{-1} &= \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \\
\Phi_{\Gamma_1'} = \Phi_{\Gamma_2'} = -C_{A_{\Gamma_1'}}^T C_{A_{\Gamma_1'}}^{-1} &= -1 \cdot 1 = -1, \\
\chi_{\Phi_{\Gamma_1}}(x) = \chi_{\Phi_{\Gamma_2}}(x) &= \begin{vmatrix} x+1 & -1 \\ 1 & x \end{vmatrix} = x^2 + x + 1, \\
\chi_{\Phi_{\Gamma_1'}}(x) = \chi_{\Phi_{\Gamma_2'}}(x) &= x + 1.
\end{aligned}$$

Using Theorem 5.3.1 we then get the following expression for the characteristic polynomial of Φ_Q:

$$\begin{aligned}
\chi_{\Phi_Q}(x) &= \chi_{\Phi_{\Gamma_1}}(x) \cdot \chi_{\Phi_{\Gamma_2'}}(x) + \chi_{\Phi_{\Gamma_1'}}(x) \cdot \chi_{\Phi_{\Gamma_2}}(x) - (x+1) \cdot \chi_{\Phi_{\Gamma_1'}}(x) \cdot \chi_{\Phi_{\Gamma_2'}}(x) \\
&= 2(x^2 + x + 1)(x+1) - (x+1)^3 \\
&= (x+1)(2x^2 + 2x + 2 - x^2 - 2x - 1) \\
&= (x+1)(x^2+1).
\end{aligned}$$

With the help of Theorem 5.3.1, we get a formula for the characteristic polynomial of the asymmetry matrix as follows:

Proposition 5.3.3. *Let Γ_1, Γ_2, Γ_1', Γ_2' and A_Q be as in Theorem 5.3.1. Then the characteristic polynomial of the asymmetry matrix $S_Q = C_{A_Q} C_{A_Q}^{-T}$ is given by*

$$\chi_{S_Q}(x) = \chi_{S_{\Gamma_1}}(x) \cdot \chi_{S_{\Gamma_2'}}(x) + \chi_{S_{\Gamma_1'}}(x) \cdot \chi_{S_{\Gamma_2}}(x) - (x-1) \cdot \chi_{S_{\Gamma_1'}}(x) \cdot \chi_{S_{\Gamma_2'}}(x).$$

Proof. Let $C := C_{A_Q}$ and let n be the number of vertices of Q. Since $C^{-1}C^T = C^{-1}(C^T C^{-1})C$, the matrices $C^{-1}C^T$ and $C^T C^{-1}$ have the same characteristic polynomial. Thus, $\Phi_Q = -C^T C^{-1}$ and $-CC^{-T} = (-C^{-1}C^T)^T$ have the same characteristic polynomial as well. Then, we get

$$\begin{aligned}
\chi_{\Phi_Q}(x) = \chi_{-CC^{-T}}(x) &= \det(xE + CC^{-T}) \\
&= (-1)^n \det(-xE - CC^{-T}) \\
&= (-1)^n \chi_{S_Q}(-x). \qquad (5.3.1)
\end{aligned}$$

Now, suppose that the number of vertices of Γ_1 is k and thus, the number of vertices of Γ_2 is $n-k+1$. Hence, the number of vertices of Γ_1' is $k-1$ and the number of vertices of Γ_2' is $n-k$. Using (5.3.1) we can rewrite the formula in Theorem 5.3.1 as follows:

$$\begin{aligned}
(-1)^n \chi_{S_Q}(-x) &= (-1)^k \chi_{S_{\Gamma_1}}(-x) \cdot (-1)^{n-k} \chi_{S_{\Gamma_2'}}(-x) + (-1)^{k-1} \chi_{S_{\Gamma_1'}}(-x) \cdot (-1)^{n-k+1} \chi_{S_{\Gamma_2}}(-x) \\
&\quad -(x+1) \cdot (-1)^{k-1} \chi_{S_{\Gamma_1'}}(-x) \cdot (-1)^{n-k} \chi_{S_{\Gamma_2'}}(-x) \\
&= (-1)^n \chi_{S_{\Gamma_1}}(-x) \cdot \chi_{S_{\Gamma_2'}}(-x) + (-1)^n \chi_{S_{\Gamma_1'}}(-x) \cdot \chi_{S_{\Gamma_2}}(-x) \\
&\quad -(x+1) \cdot (-1)^{n-1} \chi_{S_{\Gamma_1'}}(-x) \cdot \chi_{S_{\Gamma_2'}}(-x).
\end{aligned}$$

Now let $y = -x$. We get

$$\begin{aligned}
\chi_{S_Q}(y) &= \chi_{S_{\Gamma_1}}(y) \cdot \chi_{S_{\Gamma_2'}}(y) + \chi_{S_{\Gamma_1'}}(y) \cdot \chi_{S_{\Gamma_2}}(y) - (-y+1) \cdot (-1)^{2n-1} \chi_{S_{\Gamma_1'}}(y) \cdot \chi_{S_{\Gamma_2'}}(y) \\
&= \chi_{S_{\Gamma_1}}(y) \cdot \chi_{S_{\Gamma_2'}}(y) + \chi_{S_{\Gamma_1'}}(y) \cdot \chi_{S_{\Gamma_2}}(y) - (y-1) \cdot \chi_{S_{\Gamma_1'}}(y) \cdot \chi_{S_{\Gamma_2'}}(y).
\end{aligned}$$

□

We will also need the following well-known lemma.

Lemma 5.3.4. *Let M be a circulant $n \times n$-matrix, i.e. a matrix of the form*

$$M = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

The eigenvectors of such a circulant matrix are given by

$$y_j = (1, \omega_j, \omega_j^2, \dots, \omega_j^{n-1})^T, \text{ with } \omega_j = (e^{\frac{2\pi i}{n}})^j, \; j = 0, 1, \dots, n-1,$$

and the corresponding eigenvalues are then given by

$$\lambda_j = \sum_{k=0}^{n-1} c_k \omega_j^k, \; j = 0, 1, \dots, n-1.$$

As a consequence, the determinant of such a circulant matrix can be computed as:

$$\det(M) = \prod_{j=0}^{n-1} \lambda_j = \prod_{j=0}^{n-1} (c_0 + c_1\omega_j + c_2\omega_j^2 + \cdots + c_{n-1}\omega_j^{n-1}), \tag{5.3.2}$$

or equivalently (when taking the transpose):

$$\det(M) = \prod_{j=0}^{n-1} (c_0 + c_{n-1}\omega_j + c_{n-2}\omega_j^2 + \cdots + c_1\omega_j^{n-1}). \tag{5.3.3}$$

Example 5.3.5. Let M be a circulant $n \times n$-matrix with entries $c_0 = x$, $c_1 = -1$ and $c_l = 0$ for all $l \in \{2, \dots, n-1\}$, i.e. M has the following shape

$$M = \begin{pmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ 0 & x & -1 & \cdots & 0 & 0 \\ 0 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & -1 \\ -1 & 0 & 0 & \cdots & 0 & x \end{pmatrix}.$$

Then by (5.3.2) we get

$$\begin{aligned} \det(M) &= \prod_{j=0}^{n-1} (c_0 + c_1\omega_j + c_2\omega_j^2 + \cdots + c_{n-1}\omega_j^{n-1}) \\ &= \prod_{j=0}^{n-1} (x - \omega_j) \\ &= \prod_{j=0}^{n-1} (x - \omega^j), \text{ for } \omega := \omega_1 = e^{\frac{2\pi i}{n}} \\ &= (x-1)(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1}) \\ &= (x^n - 1). \end{aligned}$$

With the help of these results, we can compute the characteristic polynomials of the asymmetry matrices for cluster-tilted algebras of type A_n.

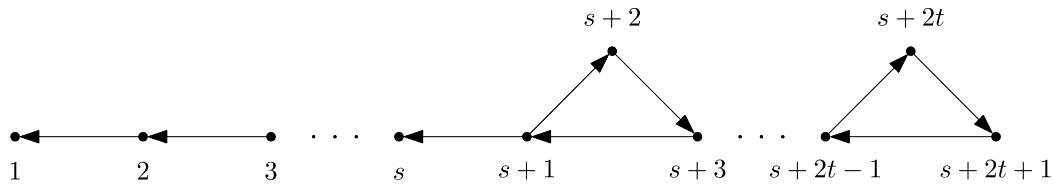
Notation. For a quiver Q mutation equivalent to a Dynkin quiver, we denote by $\chi_{S_Q}(x)$ the characteristic polynomial of the asymmetry matrix of the Cartan matrix C_Q of the cluster-tilted algebra corresponding to Q.

Proposition 5.3.6. *Let Q be the quiver of a cluster-tilted algebra of type A_n. Then*

$$\chi_{S_Q}(x) = (x+1)^{t-1}\left(x^{s+t+2} + (-1)^{s+1}\right)$$

where $t = t(Q)$ is the number of oriented 3-cycles in Q and $s = s(Q)$ is the number of arrows in Q which are not part of any oriented 3-cycle.

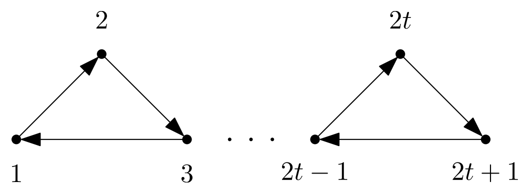
Proof. Let Q be a quiver which is mutation equivalent to A_n. By using the derived equivalence classification of cluster-tilted algebras of type A_n (see [**23**, Theorem 5.1] or Theorem 4.2.18), we can assume that Q has the following shape



since by Proposition 5.1.2 the characteristic polynomial of the asymmetry matrix S_Q is invariant under derived equivalence. Here, s is the number of arrows which are not part of an oriented 3-cycle and t is the number of oriented 3-cycles in Q. Note that this is not the standard form as in Definition 4.2.14. However, Q can be easily transformed to this shape using similar methods as in the proof of Proposition 4.2.15.

Now we want to determine the characteristic polynomial of the asymmetry matrix S_Q. For this, we consider the cases $s = 0$, $s = 1$ and $s \geq 2$.

Case 1) Let $s = 0$, i.e. Q has the following shape (up to derived equivalence)



By using the elementary matrices E_{ij}, with entries $(E_{ij})_{kl} = \begin{cases} 1 & \text{if } k = i, l = j \\ 0 & \text{otherwise} \end{cases}$, we can write the Cartan matrix of the corresponding cluster-tilted algebra as follows:

$$C = E + \sum_{i=1}^{2t} E_{i,i+1} + \sum_{i=1}^{t-1} E_{2i,2i+2} + \sum_{i=1}^{t} \sum_{j=1}^{i} E_{2i+1,2i+1-2j},$$

that is, C has the following form

$$C = \left(\begin{array}{cccccccccccc} 1 & 1 & 0 & & & & & & & & & \\ 0 & 1 & 1 & 1 & & & & & & & & \\ 1 & 0 & 1 & 1 & 0 & & & & & & & \\ 0 & 0 & 0 & 1 & 1 & 1 & & & & & & \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & & & & \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 \end{array} \right). \tag{5.3.4}$$

Then we can compute the inverse of the Cartan matrix and the asymmetry matrix $S_Q = C \cdot C^{-T}$ to be

$$C^{-1} = \frac{1}{2}\big(E - \sum_{k=1}^{t-1} E_{2k+1,2k+1} - \sum_{k=1}^{2t} E_{k,k+1} + \sum_{k=2}^{2t+1} E_{k,k-1} + \sum_{k=1}^{t} E_{2k-1,2k+1} - \sum_{k=1}^{t} E_{2k+1,2k-1}\big),$$

$$S_Q = \sum_{l=1}^{t} E_{2l+1,1} + \sum_{l=1}^{t} E_{2l-1,2l} + \sum_{l=1}^{t-1} E_{2l,2l+2} - \sum_{l=1}^{t-1} E_{2l+1,2l+1} + E_{2t,2t+1}.$$

That is, the matrices C^{-1} and S_Q have the following shapes

$$C^{-1} = \frac{1}{2} \cdot \left(\begin{array}{ccccccccccc}
1 & -1 & 1 & & & & & & & & \\
1 & 1 & -1 & & & & & & & & \\
-1 & 1 & 0 & -1 & 1 & & & & & & \\
 & & 1 & 1 & -1 & & & & & & \\
 & & -1 & 1 & 0 & -1 & 1 & & & & \\
 & & & & 1 & 1 & -1 & & & & \\
 & & & & -1 & 1 & 0 & -1 & 1 & & \\
 & & & & & & & \ddots & & & \\
 & & & & & & -1 & 1 & 0 & -1 & 1 \\
 & & & & & & & & 1 & 1 & -1 \\
 & & & & & & & & -1 & 1 & 1
\end{array}\right), \tag{5.3.5}$$

$$S_Q = \left(\begin{array}{cccccccccccc}
0 & 1 & 0 & & & & & & & & & \\
0 & 0 & 0 & 1 & & & & & & & & \\
1 & 0 & -1 & 1 & 0 & & & & & & & \\
0 & & 0 & 0 & 0 & 1 & & & & & & \\
1 & & 0 & 0 & -1 & 1 & 0 & & & & & \\
0 & & & & 0 & 0 & 0 & 1 & & & & \\
1 & & & & 0 & 0 & -1 & 1 & 0 & & & \\
0 & & & & & & 0 & 0 & 0 & 1 & & \\
\vdots & & & & & & & & \ddots & \ddots & \ddots & \\
1 & & & & & & & & & -1 & 1 & 0 \\
0 & & & & & & & & & 0 & 0 & 1 \\
1 & & & & & & & & & 0 & 0 & 0
\end{array}\right). \tag{5.3.6}$$

First we prove the formula for C^{-1}:

$$\begin{aligned}
C \cdot C^{-1} = \frac{1}{2}\Big(& E - \sum_{k=1}^{t-1} E_{2k+1,2k+1} - \sum_{k=1}^{2t} E_{k,k+1} + \sum_{k=2}^{2t+1} E_{k,k-1} + \sum_{k=1}^{t} E_{2k-1,2k+1} - \sum_{k=1}^{t} E_{2k+1,2k-1} \\
&+ \sum_{k=1}^{2t} E_{k,k+1} - \sum_{k=1}^{t-1} E_{2k,2k+1} - \sum_{k=1}^{2t-1} E_{k,k+2} + \sum_{k=1}^{2t} E_{k,k} + \sum_{k=1}^{t-1} E_{2k,2k+3} - \sum_{k=1}^{t} E_{2k,2k-1} \\
&+ \sum_{k=1}^{t-1} E_{2k,2k+2} - \sum_{k=1}^{t-1} E_{2k,2k+3} + \sum_{k=1}^{t-1} E_{2k,2k+1} + \sum_{k=1}^{t} \sum_{l=1}^{k} E_{2k+1,2k+1-2l} \\
&- \sum_{k=2}^{t} \sum_{l=1}^{k-1} E_{2k+1,2k+1-2l} - \sum_{k=1}^{t} \sum_{l=1}^{k} E_{2k+1,2k+2-2l} + \sum_{k=2}^{t} \sum_{l=1}^{k-1} E_{2k+1,2k-2l} \\
&+ \sum_{k=1}^{t} \sum_{l=1}^{k} E_{2k+1,2k+3-2l} - \sum_{k=2}^{t} \sum_{l=1}^{k-1} E_{2k+1,2k-1-2l}\Big)
\end{aligned}$$

The six double sums at the end reduce to

$$\sum_{k=1}^{t} E_{2k+1,2k-1} - \sum_{k=1}^{t} E_{2k+1,2k} + \sum_{k=1}^{t} E_{2k+1,2k+1}.$$

Simplifying the first part of the expression for $C \cdot C^{-1}$ and putting this together with the reduced expression for the six double sums gives

$$\begin{aligned} C \cdot C^{-1} &= \frac{1}{2}\Big(E - \sum_{k=1}^{t-1} E_{2k+1,2k+1} + \sum_{k=2}^{2t+1} E_{k,k-1} - \sum_{k=1}^{t} E_{2k+1,2k-1} + \sum_{k=1}^{2t} E_{k,k} \\ &\quad - \sum_{k=1}^{t} E_{2k,2k-1} + \sum_{k=1}^{t} E_{2k+1,2k-1} - \sum_{k=1}^{t} E_{2k+1,2k} + \sum_{k=1}^{t} E_{2k+1,2k+1}\Big) \\ &= \frac{1}{2}\Big(E + \sum_{k=1}^{2t+1} E_{k,k}\Big) \\ &= E \ . \end{aligned}$$

Now we verify the formula for the asymmetry matrix S_Q:

$$\begin{aligned} C \cdot C^{-T} &= \Big(E + \sum_{i=1}^{2t} E_{i,i+1} + \sum_{i=1}^{t-1} E_{2i,2i+2} + \sum_{i=1}^{t}\sum_{j=1}^{i} E_{2i+1,2i+1-2j}\Big) \\ &\quad \cdot \frac{1}{2}\Big(E - \sum_{k=1}^{t-1} E_{2k+1,2k+1} - \sum_{k=1}^{2t} E_{k+1,k} + \sum_{k=2}^{2t+1} E_{k-1,k} + \sum_{k=1}^{t} E_{2k+1,2k-1} - \sum_{k=1}^{t} E_{2k-1,2k+1}\Big) \\ &= \frac{1}{2}\Big(E - \sum_{k=1}^{t-1} E_{2k+1,2k+1} - \sum_{k=1}^{2t} E_{k+1,k} + \sum_{k=2}^{2t+1} E_{k-1,k} + \sum_{k=1}^{t} E_{2k+1,2k-1} - \sum_{k=1}^{t} E_{2k-1,2k+1} \\ &\quad + \sum_{k=1}^{2t} E_{k,k+1} - \sum_{k=1}^{t-1} E_{2k,2k+1} - \sum_{k=1}^{2t} E_{k,k} + \sum_{k=1}^{2t-1} E_{k,k+2} + \sum_{k=1}^{t} E_{2k,2k-1} - \sum_{k=1}^{t-1} E_{2k,2k+3} \\ &\quad + \sum_{k=1}^{t-1} E_{2k,2k+2} - \sum_{k=1}^{t-1} E_{2k,2k+1} + \sum_{k=1}^{t-1} E_{2k,2k+3} + \sum_{k=1}^{t}\sum_{l=1}^{k} E_{2k+1,2k+1-2l} \\ &\quad - \sum_{k=2}^{t}\sum_{l=1}^{k-1} E_{2k+1,2k+1-2l} - \sum_{k=2}^{t}\sum_{l=1}^{k-1} E_{2k+1,2k-2l} + \sum_{k=1}^{t}\sum_{l=1}^{k} E_{2k+1,2k+2-2l} \\ &\quad + \sum_{k=2}^{t}\sum_{l=1}^{k-1} E_{2k+1,2k-1-2l} - \big(\sum_{k=1}^{t} E_{2k+1,2k+1} + \sum_{k=2}^{t}\sum_{l=1}^{k-1} E_{2k+1,2k+1-2l}\big)\Big) \end{aligned}$$

The six double sums reduce to

$$\sum_{k=1}^{t} E_{2k+1,1} + \sum_{k=1}^{t} E_{2k+1,2k} + \sum_{k=2}^{t} E_{2k+1,1} - \sum_{k=2}^{t} E_{2k+1,2k-1}.$$

Similarly to the computation of $C \cdot C^{-1}$ the expression for $C \cdot C^{-T}$ simplifies to

$$\begin{aligned} C \cdot C^{-T} &= \frac{1}{2}\Big(E_{2t+1,2t+1} - \sum_{k=1}^{t-1} E_{2k+1,2k+1} - \sum_{k=1}^{2t} E_{k+1,k} + 2\sum_{k=1}^{2t} E_{k,k+1} + \sum_{k=1}^{t} E_{2k+1,2k-1} \\ &\quad - \sum_{k=1}^{t} E_{2k-1,2k+1} - \sum_{k=1}^{t-1} E_{2k,2k+1} + \sum_{k=1}^{2t-1} E_{k,k+2} + \sum_{k=1}^{t} E_{2k,2k-1} + \sum_{k=1}^{t-1} E_{2k,2k+2} \\ &\quad - \sum_{k=1}^{t-1} E_{2k,2k+1} + \sum_{k=1}^{t} E_{2k+1,1} + \sum_{k=1}^{t} E_{2k+1,2k} + \sum_{k=2}^{t} E_{2k+1,1} - \sum_{k=2}^{t} E_{2k+1,2k-1} \\ &\quad - \sum_{k=1}^{t} E_{2k+1,2k+1}\Big) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(-2\sum_{k=1}^{t-1} E_{2k+1,2k+1} + 2\sum_{k=1}^{2t} E_{k,k+1} + E_{31} - \sum_{k=1}^{t} E_{2k-1,2k+1} - 2\sum_{k=1}^{t-1} E_{2k,2k+1} \\
&\quad + 2\sum_{k=1}^{t-1} E_{2k,2k+2} + \sum_{k=1}^{t} E_{2k-1,2k+1} + 2\sum_{k=2}^{t} E_{2k+1,1} + E_{31}\Big) \\
&= \frac{1}{2}\Big(-2\sum_{k=1}^{t-1} E_{2k+1,2k+1} + 2\sum_{k=1}^{t} E_{2k-1,2k} + 2E_{2t,2t+1} + 2\sum_{k=1}^{t} E_{2k+1,1} + 2\sum_{k=1}^{t-1} E_{2k,2k+2}\Big) \\
&= S_Q.
\end{aligned}
$$

Next we compute the characteristic polynomial $\chi_{S_Q}(x)$:

(5.3.7)

$$
\chi_{S_Q}(x) = \det(xE - S_Q) = \left|\begin{array}{cccccccccccc}
x & -1 & 0 & & & & & & & & & \\
0 & x & 0 & -1 & & & & & & & & \\
-1 & 0 & (x+1) & -1 & 0 & & & & & & & \\
0 & & 0 & x & 0 & -1 & & & & & & \\
-1 & & 0 & 0 & (x+1) & -1 & 0 & & & & & \\
0 & & & & 0 & x & 0 & -1 & & & & \\
-1 & & & & 0 & 0 & (x+1) & -1 & 0 & & & \\
0 & & & & & & 0 & x & 0 & -1 & & \\
\vdots & & & & & & & & \ddots & \ddots & \ddots & \\
-1 & & & & & & & & & (x+1) & -1 & 0 \\
0 & & & & & & & & & 0 & x & -1 \\
-1 & & & & & & & & & 0 & 0 & x
\end{array}\right|.
$$

We expand the determinant along columns using Laplace's formula. Laplace expansion along every odd column of the original matrix, except for the first and the last one (i.e. along the 3^{rd}, 5^{th}, $\ldots$, $(2t-1)^{\text{th}}$ column), leads to a determinant of a $(t+2)\times(t+2)$-matrix

$$
\chi_{S_Q}(x) = (x+1)^{t-1}\cdot \left|\begin{array}{ccccccc}
x & -1 & & & & & \\
0 & x & -1 & & & & \\
0 & & x & -1 & & & \\
0 & & & x & -1 & & \\
\vdots & & & & \ddots & \ddots & \\
0 & & & & & x & -1 \\
-1 & & & & & & x
\end{array}\right|.
$$

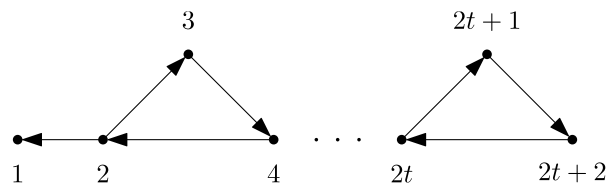
This matrix is a circulant matrix as in Example 5.3.5 with entries $c_0 = x$, $c_1 = -1$, $c_l = 0$ for all $l \neq 0, 1$, and determinant $x^{t+2} - 1$.

Hence, we get the characteristic polynomial $\chi_{S_Q}(x)$ as

$$\chi_{S_Q}(x) = (x+1)^{t-1}(x^{t+2} - 1),$$

which is exactly the formula given in Proposition 5.3.6 in case $s = 0$.

Case 2) Let $s = 1$, i.e. Q has the following shape (up to derived equivalence)



.

By using the elementary matrices E_{ij}, we can write the Cartan matrix of the corresponding cluster-tilted algebra as follows:

$$C = E + \sum_{i=1}^{t+1} E_{2i,1} + \sum_{i=2}^{2t+1} E_{i,i+1} + \sum_{i=2}^{t} E_{2i-1,2i+1} + \sum_{i=2}^{t+1}\sum_{j=1}^{i-1} E_{2i,2i-2j},$$

that is, C has the following form

$$
C=\left(\begin{array}{c|cccccccccccc}
1 & 0 & 0 & \cdots & & & & & & & & & 0 \\ \hline
1 & 1 & 1 & 0 & & & & & & & & & \\
0 & 0 & 1 & 1 & 1 & & & & & & & & \\
1 & 1 & 0 & 1 & 1 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & & & & & & \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & & & & \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1
\end{array}\right).
$$

Then we can compute the inverse of the Cartan matrix and the asymmetry matrix $S_Q = C\cdot C^{-T}$ to be

$$
\begin{aligned}
C^{-1} = \frac{1}{2}\big(& E + E_{11} - 2E_{21} - \sum_{k=1}^{t-1} E_{2k+2,2k+2} - \sum_{k=2}^{2t+1} E_{k,k+1} + \sum_{k=2}^{2t+1} E_{k+1,k} \\
&+ \sum_{k=1}^{t} E_{2k,2k+2} - \sum_{k=1}^{t} E_{2k+2,2k}\big), \\
S_Q = E_{11} &- E_{12} - \sum_{l=1}^{t} E_{2l,2l} + \sum_{l=1}^{t} E_{2l,2l+1} + \sum_{l=2}^{t} E_{2l-1,2l+1} + \sum_{l=1}^{t+1} E_{2l,1} + E_{2t+1,2t+2},
\end{aligned}
$$

that is, the matrices C^{-1} and S_Q have the following shapes

$$
C^{-1}=\frac{1}{2}\cdot\left(\begin{array}{c|ccccccccccc}
2 & & & & & & & & & & & \\ \hline
-2 & 1 & -1 & 1 & & & & & & & & \\
 & 1 & 1 & -1 & & & & & & & & \\
 & -1 & 1 & 0 & -1 & 1 & & & & & & \\
 & & & 1 & 1 & -1 & & & & & & \\
 & & & -1 & 1 & 0 & -1 & 1 & & & & \\
 & & & & & 1 & 1 & -1 & & & & \\
 & & & & & -1 & 1 & 0 & -1 & 1 & & \\
 & & & & & & & & \ddots & & & \\
 & & & & & & & -1 & 1 & 0 & -1 & 1 \\
 & & & & & & & & & 1 & 1 & -1 \\
 & & & & & & & & & -1 & 1 & 1
\end{array}\right),
$$

$$
S_Q=\left(\begin{array}{c|ccccccccccc}
1 & -1 & & & & & & & & & & \\ \hline
1 & -1 & 1 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 1 & & & & & & & \\
1 & 0 & 0 & -1 & 1 & 0 & & & & & & \\
0 & & & 0 & 0 & 0 & 1 & & & & & \\
1 & & & 0 & 0 & -1 & 1 & 0 & & & & \\
0 & & & & & 0 & 0 & 0 & 1 & & & \\
1 & & & & & 0 & 0 & -1 & 1 & 0 & & \\
\vdots & & & & & & & & \ddots & & & \\
1 & & & & & & & & & -1 & 1 & 0 \\
0 & & & & & & & & & 0 & 0 & 1 \\
1 & & & & & & & & & 0 & 0 & 0
\end{array}\right).
$$

First we verify the formula for C^{-1}:

$$\begin{aligned}
C\cdot C^{-1} = \frac{1}{2}\Big(& E + E_{11} - 2E_{21} - \sum_{k=1}^{t-1} E_{2k+2,2k+2} - \sum_{k=2}^{2t+1} E_{k,k+1} + \sum_{k=2}^{2t+1} E_{k+1,k} + \sum_{k=1}^{t} E_{2k,2k+2}\\
& - \sum_{k=1}^{t} E_{2k+2,2k} + 2\sum_{k=1}^{t+1} E_{2k,1} + \sum_{k=2}^{2t+1} E_{k,k+1} - \sum_{k=1}^{t-1} E_{2k+1,2k+2} - \sum_{k=2}^{2t} E_{k,k+2} + \sum_{k=2}^{2t+1} E_{kk}\\
& + \sum_{k=2}^{t} E_{2k-1,2k+2} - \sum_{k=1}^{t} E_{2k+1,2k} + \sum_{k=2}^{t} E_{2k-1,2k+1} - \sum_{k=2}^{t} E_{2k-1,2k+2} + \sum_{k=2}^{t} E_{2k-1,2k}\\
& + \sum_{k=2}^{t+1}\sum_{l=1}^{k-1} E_{2k,2k-2l} - 2\sum_{k=2}^{t+1} E_{2k,1} - \sum_{k=3}^{t+1}\sum_{l=1}^{k-2} E_{2k,2k-2l} - \sum_{k=2}^{t+1}\sum_{l=1}^{k-1} E_{2k,2k+1-2l}\\
& + \sum_{k=3}^{t+1}\sum_{l=1}^{k-2} E_{2k,2k-1-2l} + \big(\sum_{k=2}^{t+1} E_{2k,2k} + \sum_{k=3}^{t+1}\sum_{l=1}^{k-2} E_{2k,2k-2l}\big) - \sum_{k=3}^{t+1}\sum_{l=1}^{k-2} E_{2k,2k-2-2l}\Big)
\end{aligned}$$

The six double sums reduce to

$$\sum_{k=1}^{t} E_{2k+2,2k} - \sum_{k=2}^{t+1} E_{2k,2k-1}.$$

Simplifying again gives

$$\begin{aligned}
C\cdot C^{-1} &= \frac{1}{2}\Big(2E + \sum_{k=2}^{2t+1} E_{k+1,k} - \sum_{k=1}^{t} E_{2k+2,2k} - \sum_{k=1}^{t} E_{2k+1,2k} + \sum_{k=1}^{t} E_{2k+2,2k} - \sum_{k=2}^{t+1} E_{2k,2k-1}\Big)\\
&= E\ .
\end{aligned}$$

Now we prove the formula for S_Q:

$$\begin{aligned}
C\cdot C^{-T} = \Big(& E + \sum_{i=1}^{t+1} E_{2i,1} + \sum_{i=2}^{2t+1} E_{i,i+1} + \sum_{i=2}^{t} E_{2i-1,2i+1} + \sum_{i=2}^{t+1}\sum_{j=1}^{i-1} E_{2i,2i-2j}\Big)\cdot\frac{1}{2}\Big(E + E_{11} - 2E_{12}\\
& - \sum_{k=1}^{t-1} E_{2k+2,2k+2} - \sum_{k=2}^{2t+1} E_{k+1,k} + \sum_{k=2}^{2t+1} E_{k,k+1} + \sum_{k=1}^{t} E_{2k+2,2k} - \sum_{k=1}^{t} E_{2k,2k+2}\Big)\\
= \frac{1}{2}\Big(& E + E_{11} - 2E_{12} - \sum_{k=1}^{t-1} E_{2k+2,2k+2} - \sum_{k=2}^{2t+1} E_{k+1,k} + \sum_{k=2}^{2t+1} E_{k,k+1} + \sum_{k=1}^{t} E_{2k+2,2k}\\
& - \sum_{k=1}^{t} E_{2k,2k+2} + 2\sum_{k=1}^{t+1} E_{2k,1} - 2\sum_{k=1}^{t+1} E_{2k,2} + \sum_{k=2}^{2t+1} E_{k,k+1} - \sum_{k=1}^{t-1} E_{2k+1,2k+2} - \sum_{k=2}^{2t+1} E_{kk}\\
& + \sum_{k=2}^{2t} E_{k,k+2} + \sum_{k=1}^{t} E_{2k+1,2k} - \sum_{k=2}^{t} E_{2k-1,2k+2} + \sum_{k=2}^{t} E_{2k-1,2k+1} - \sum_{k=2}^{t} E_{2k-1,2k}\\
& + \sum_{k=2}^{t} E_{2k-1,2k+2} + \sum_{k=2}^{t+1}\sum_{l=1}^{k-1} E_{2k,2k-2l} - \sum_{k=3}^{t+1}\sum_{l=1}^{k-2} E_{2k,2k-2l} - \sum_{k=3}^{t+1}\sum_{l=1}^{k-2} E_{2k,2k-1-2l}\\
& + \sum_{k=2}^{t+1}\sum_{l=1}^{k-1} E_{2k,2k+1-2l} + \sum_{k=3}^{t+1}\sum_{l=1}^{k-2} E_{2k,2k-2-2l} - \big(\sum_{k=2}^{t+1} E_{2k,2k} + \sum_{k=3}^{t+1}\sum_{l=1}^{k-2} E_{2k,2k-2l}\big)\Big)
\end{aligned}$$

The six double sums reduce to

$$2\sum_{k=2}^{t+1} E_{2k,2} - E_{42} - \sum_{k=2}^{t} E_{2k+2,2k} + \sum_{k=2}^{t+1} E_{2k,2k-1}.$$

Simplifying again gives

$$\begin{aligned}
C\cdot C^{-T} &= \frac{1}{2}\Big(E_{11}+E_{2t+2,2t+2}+E_{11}-2E_{12}-2\sum_{k=2}^{t}E_{2k,2k}-E_{2t+2,2t+2}-\sum_{k=2}^{t+1}E_{2k,2k-1}\\
&\quad+2\sum_{k=2}^{2t+1}E_{k,k+1}+\sum_{k=1}^{t}E_{2k+2,2k}+2\sum_{k=2}^{t}E_{2k-1,2k+1}+2\sum_{k=1}^{t+1}E_{2k,1}-2\sum_{k=1}^{t+1}E_{2k,2}\\
&\quad-2\sum_{k=2}^{t}E_{2k-1,2k}+2\sum_{k=2}^{t+1}E_{2k,2}-\sum_{k=2}^{t}E_{2k+2,2k}+\sum_{k=2}^{t+1}E_{2k,2k-1}-E_{42}\Big)\\
&=\frac{1}{2}\Big(2E_{11}-2E_{12}-2\sum_{k=2}^{t}E_{2k,2k}+2\sum_{k=1}^{t}E_{2k,2k+1}+2E_{2t+1,2t+2}+2\sum_{k=2}^{t}E_{2k-1,2k+1}\\
&\quad+2\sum_{k=1}^{t+1}E_{2k,1}-2E_{22}\Big)\\
&=S_Q.
\end{aligned}$$

Finally we compute the characteristic polynomial $\chi_{S_Q}(x)$:

$$\chi_{S_Q}(x)=\det(xE-S_Q)=\left|\begin{array}{ccccccccccccc}
(x-1) & 1 & & & & & & & & & & & \\
-1 & (x+1) & -1 & 0 & & & & & & & & & \\
0 & 0 & x & 0 & -1 & & & & & & & & \\
-1 & 0 & 0 & (x+1) & -1 & 0 & & & & & & & \\
0 & & & 0 & x & 0 & -1 & & & & & & \\
-1 & & & 0 & 0 & (x+1) & -1 & 0 & & & & & \\
0 & & & & & 0 & x & 0 & -1 & & & & \\
-1 & & & & & 0 & 0 & (x+1) & -1 & 0 & & & \\
0 & & & & & & & 0 & x & 0 & -1 & & \\
\vdots & & & & & & & & & \ddots & \ddots & \ddots & \\
-1 & & & & & & & & & & (x+1) & -1 & 0\\
0 & & & & & & & & & & 0 & x & -1\\
-1 & & & & & & & & & & 0 & 0 & x
\end{array}\right|.$$

Laplace expansion along every even column of the original matrix, except for the second and the last one (i.e. along the 4^{th}, 6^{th}, …, $2t^{\text{th}}$ column), leads to a determinant of a $(t+3)\times(t+3)$-matrix

$$\chi_{S_Q}(x)=(x+1)^{t-1}\cdot\left|\begin{array}{ccccccccc}
(x-1) & 1 & & & & & & & \\
-1 & (x+1) & -1 & & & & & & \\
0 & & x & -1 & & & & & \\
0 & & & x & -1 & & & & \\
0 & & & & x & -1 & & & \\
0 & & & & & x & -1 & & \\
\vdots & & & & & & \ddots & \ddots & \\
0 & & & & & & & x & -1\\
-1 & & & & & & & & x
\end{array}\right|.$$

Now we expand along the first column to get

$$\chi_{S_Q}(x) = (x+1)^{t-1} \cdot (x-1) \cdot \begin{vmatrix} (x+1) & -1 & & & & & \\ & x & -1 & & & & \\ & & x & -1 & & & \\ & & & x & -1 & & \\ & & & & \ddots & \ddots & \\ & & & & & x & -1 \\ & & & & & & x \end{vmatrix} + (x+1)^{t-1} \cdot \begin{vmatrix} 1 & & & & & & \\ & x & -1 & & & & \\ & & x & -1 & & & \\ & & & x & -1 & & \\ & & & & \ddots & \ddots & \\ & & & & & x & -1 \\ & & & & & & x \end{vmatrix}$$

$$+ (x+1)^{t-1} \cdot (-1)^{t+5} \cdot \begin{vmatrix} 1 & & & & & & \\ (x+1) & -1 & & & & & \\ & x & -1 & & & & \\ & & x & -1 & & & \\ & & & \ddots & \ddots & & \\ & & & & x & -1 & \\ & & & & & x & -1 \end{vmatrix}.$$

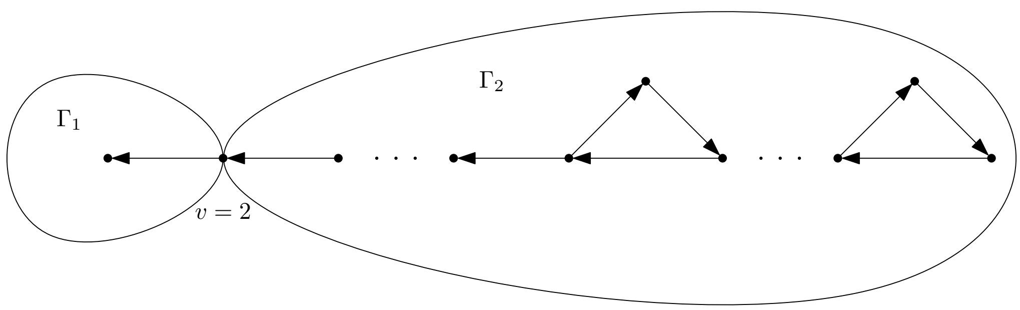
Thus, we have

$$\begin{aligned} \chi_{S_Q}(x) &= (x+1)^{t-1} \cdot \big((x-1)(x+1)x^{t+1} + x^{t+1} + (-1)^{t+5} \cdot (-1)^{t+1}\big) \\ &= (x+1)^{t-1} \cdot \big((x^2-1)x^{t+1} + x^{t+1} + (-1)^{2t+6}\big) \\ &= (x+1)^{t-1} \cdot \big(x^{t+1} \cdot (x^2-1+1) + 1\big) \\ &= (x+1)^{t-1} \cdot \big(x^{t+3} + 1\big), \end{aligned}$$

which is exactly the formula given in Proposition 5.3.6 in case $s = 1$.

Case 3) Finally, we compute the formula of $\chi_{S_Q}(x)$ for $s \geq 2$ using induction on s (with the two cases $s = 0$ and $s = 1$ as the base of the induction).

To apply Proposition 5.3.3, we need the following notations for the quiver Q



where Γ_1 is a quiver of type A_2 and Γ_2 is a rooted quiver of type A with parameters $s-1$ and t. Additionally, $\Gamma_1' = \Gamma_1 \backslash \{2\}$ is just a single vertex and $\Gamma_2' = \Gamma_2 \backslash \{2\}$ is a rooted quiver of type A with parameters $s-2$ and t.

We compute the Cartan matrices, the asymmetry matrices and their characteristic polynomials of the cluster-tilted algebras corresponding to the quivers Γ_1 and Γ_1' as

$$\begin{aligned}
C_{A_{\Gamma_1}} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\
S_{\Gamma_1} &= C_{A_{\Gamma_1}} C_{A_{\Gamma_1}}^{-T} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \\
\chi_{S_{\Gamma_1}}(x) &= \begin{vmatrix} x-1 & 1 \\ -1 & x \end{vmatrix} = x^2 - x + 1, \\
C_{A_{\Gamma_1'}} &= 1, \\
S_{\Gamma_1'} &= C_{A_{\Gamma_1'}} C_{A_{\Gamma_1}'}^{-T} = 1, \\
\chi_{S_{\Gamma_1}'}(x) &= x - 1.
\end{aligned}$$

Hence, we get by Proposition 5.3.3

$$\begin{aligned}
\chi_{S_Q}(x) &= \chi_{S_{\Gamma_1}}(x) \cdot \chi_{S_{\Gamma_2'}}(x) + \chi_{S_{\Gamma_1'}}(x) \cdot \chi_{S_{\Gamma_2}}(x) - (x-1) \cdot \chi_{S_{\Gamma_1'}}(x) \cdot \chi_{S_{\Gamma_2'}}(x) \\
&= (x^2 - x + 1) \cdot \chi_{S_{\Gamma_2'}}(x) + (x-1) \cdot \chi_{S_{\Gamma_2}}(x) - (x-1)^2 \chi_{S_{\Gamma_2'}}(x).
\end{aligned}$$

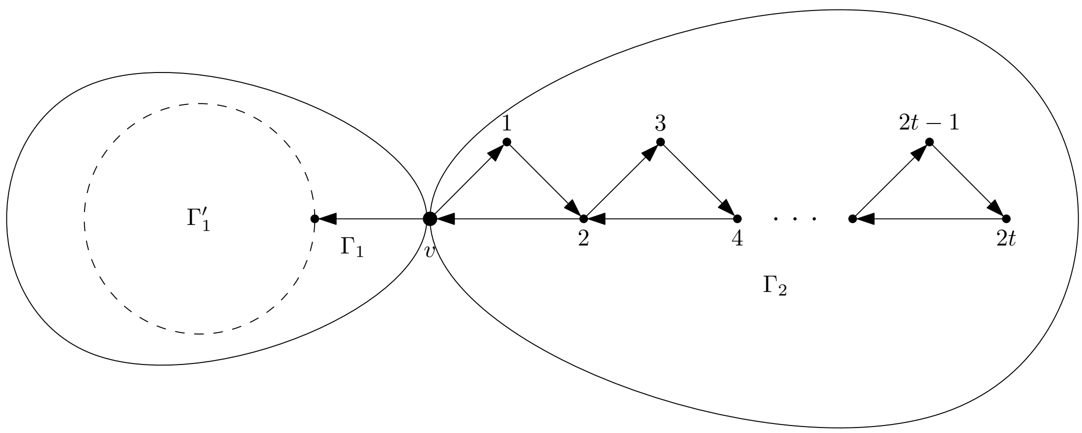
Using the induction hypothesis for Γ_2 and Γ_2', we have

$$\begin{aligned}
\chi_{S_Q}(x) &= (x^2 - x + 1)(x+1)^{t-1}(x^{s+t} + (-1)^{s-1}) + (x-1)(x+1)^{t-1}(x^{s+t+1} + (-1)^s) \\
&\quad - (x-1)^2(x+1)^{t-1}(x^{s+t} + (-1)^{s-1}) \\
&= (x+1)^{t-1}\big((x^2 - x + 1 - (x-1)^2)(x^{s+t} + (-1)^{s-1}) + (x-1)(x^{s+t+1} + (-1)^s)\big) \\
&= (x+1)^{t-1}\big(x(x^{s+t} + (-1)^{s-1}) + x^{s+t+2} - x^{s+t+1} + (-1)^s x + (-1)^{s+1}\big) \\
&= (x+1)^{t-1}(x^{s+t+2} + (-1)^{s+1}),
\end{aligned}$$

and this proves Proposition 5.3.6 in case $s \geq 2$. □

Now we derive another method to determine the characteristic polynomials of the asymmetry matrices for certain cluster-tilted algebras of type D_n.

Lemma 5.3.7. *Let $A := A_Q$ be the cluster-tilted algebra of type D_n corresponding to the following quiver Q*



where Γ_1' is the skeleton of type I,II,III or IV and $\Gamma_2' = \Gamma_2 \backslash \{v\}$ is the quiver consisting of the vertices $1, 2, \ldots, 2t$. Furthermore, let $t \geq 1$ be the number of oriented 3-cycles in Γ_2. Then the characteristic polynomial of $\Phi_Q = -C^T C^{-1}$ is given by

$$\chi_{\Phi_Q}(x) = (x-1)^{t-1}(x^{t+1} + (-1)^{t+1}) \cdot \big(\chi_{\Phi_{\Gamma_1}}(x) - (x+1)\chi_{\Phi_{\Gamma_1'}}(x)\big) + \chi_{\Phi_{\Gamma_1'}}(x) \cdot \chi_{\Phi_{\Gamma_2}}(x).$$

PROOF. We shall use the same methods as Boldt used in the proof of Theorem 5.3.1. We also need the following notation. For every $e \in (\Gamma'_1)_0$, let $a_e := \dim_K eAv = \sharp\{\text{paths from vertex } v \text{ to vertex } e\}$. Note that $\dim_K vAe = 0$ for all $e \in (\Gamma'_1)_0$. Similarly, for every $e \in (\Gamma'_2)_0$, let $b_e := \dim_K eAv$ and $\tilde{b}_e := \dim_K vAe$. We consider a, b and $\tilde{b}$ as column vectors with entries a_e, b_e and $\tilde{b}_e$, respectively, and write C, C_1, C_2, C'_1 and C'_2 instead of C_A, $C_{A_{\Gamma_1}}$, $C_{A_{\Gamma_2}}$, $C_{A_{\Gamma'_1}}$ and $C_{A_{\Gamma'_2}}$.

Then the Cartan matrices of A_{Γ_1} and A_{Γ_2} are given by

$$
\begin{aligned}
C_1 &= \left(\begin{array}{c|c} C'_1 & 0 \\ \hline a^T & 1 \end{array}\right), \\
C_2 &= \left(\begin{array}{c|c} 1 & b^T \\ \hline \tilde{b} & C'_2 \end{array}\right),
\end{aligned}
$$

where the last row and column of C_1 and the first row and column of C_2 belong to the vertex v. Moreover, $b^T = (1, 0, \dots, 0)$ and $\tilde{b}^T = (0, 1, 0, 1, \dots, 0, 1)$. We then get the Cartan matrix of A as

$$
C = \left(\begin{array}{c|c|c} C'_1 & 0 & \mathbf{0} \\ \hline a^T & 1 & b^T \\ \hline \tilde{b}a^T & \tilde{b} & C'_2 \end{array}\right).
$$

Note that all paths between Γ'_1 and Γ'_2 traverse the vertex v. Hence, the lower left block of C is $\tilde{b}a^T$ and the upper right block is $\mathbf{0}$ (since there are no non-zero paths from Γ'_1 to Γ'_2).

Let M and N be invertible matrices such that $N = SMS^T$ for some invertible matrix S. Then we will write $M \sim N$. Note that in this case $S(-M^T M^{-1})S^{-1} = -N^T N^{-1}$, and therefore, $-M^T M^{-1}$ and $-N^T N^{-1}$ have the same characteristic polynomial.

Thus, we have

$$
\begin{aligned}
C_1 &\sim \left(\begin{array}{c|c} E & -a \\ \hline 0^T & 1 \end{array}\right) \left(\begin{array}{c|c} C'_1 & 0 \\ \hline a^T & 1 \end{array}\right) \left(\begin{array}{c|c} E & 0 \\ \hline -a^T & 1 \end{array}\right) = \left(\begin{array}{c|c} C'_1 & -a \\ \hline 0^T & 1 \end{array}\right) =: F_1, \\
C_2 &\sim \left(\begin{array}{c|c} 1 & 0^T \\ \hline -b & E \end{array}\right) \left(\begin{array}{c|c} 1 & b^T \\ \hline \tilde{b} & C'_2 \end{array}\right) \left(\begin{array}{c|c} 1 & -b^T \\ \hline 0 & E \end{array}\right) = \left(\begin{array}{c|c} 1 & 0^T \\ \hline \tilde{b} - b & C'_2 - \tilde{b}b^T \end{array}\right) =: F_2, \\
C &\sim \left(\begin{array}{c|c|c} E & -a & \mathbf{0} \\ \hline 0^T & 1 & 0^T \\ \hline \mathbf{0} & -b & E \end{array}\right) \left(\begin{array}{c|c|c} C'_1 & 0 & \mathbf{0} \\ \hline a^T & 1 & b^T \\ \hline \tilde{b}a^T & \tilde{b} & C'_2 \end{array}\right) \left(\begin{array}{c|c|c} E & 0 & \mathbf{0} \\ \hline -a^T & 1 & -b^T \\ \hline \mathbf{0} & 0 & E \end{array}\right) \\
&= \left(\begin{array}{c|c|c} C'_1 & -a & \mathbf{0} \\ \hline 0^T & 1 & 0^T \\ \hline \mathbf{0} & \tilde{b} - b & C'_2 - \tilde{b}b^T \end{array}\right) =: F.
\end{aligned}
$$

Let $G := C'_2 - \tilde{b}b^T$ with $\tilde{b}b^T = \begin{pmatrix} \begin{smallmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{smallmatrix} & \mathbf{0} \end{pmatrix}$. We will compute the inverse matrices of F_1, F_2 and F. For this, we need to know that C'_1 and G are invertible. Since C'_1 is the Cartan matrix of a cluster-tilted algebra corresponding to the skeleton Γ'_1 of type I, II, III or IV, we have $\det(C'_1) \neq 0$ by Theorem 5.2.1. We see that the matrix G is invertible by direct calculation, the inverse is given in (5.3.8) below.

Hence, we can compute F_1^{-1}, F_2^{-1} and F^{-1} to be

$$F_1^{-1} = \left(\begin{array}{c|c} C_1'^{-1} & C_1'^{-1}a \\ \hline 0^T & 1 \end{array}\right),$$

$$F_2^{-1} = \left(\begin{array}{c|c} 1 & 0^T \\ \hline G^{-1}(b-\tilde{b}) & G^{-1} \end{array}\right),$$

$$F^{-1} = \left(\begin{array}{c|c|c} C_1'^{-1} & C_1'^{-1}a & \mathbf{0} \\ \hline 0^T & 1 & 0^T \\ \hline \mathbf{0} & G^{-1}(b-\tilde{b}) & G^{-1} \end{array}\right).$$

Then $\Phi_1 := \Phi_{\Gamma_1} = -C_1^T C_1^{-1}$, $\Phi_2 := \Phi_{\Gamma_2} = -C_2^T C_2^{-1}$ and $\Phi_Q = -C^T C^{-1}$ have the same characteristic polynomials as

$$-F_1^T F_1^{-1} = \left(\begin{array}{c|c} \Phi_1' & \Phi_1' a \\ \hline a^T C_1'^{-1} & a^T C_1'^{-1} a - 1 \end{array}\right),$$

$$-F_2^T F_2^{-1} = \left(\begin{array}{c|c} (b-\tilde{b})^T G^{-1}(b-\tilde{b}) - 1 & (b-\tilde{b})^T G^{-1} \\ \hline -G^T G^{-1}(b-\tilde{b}) & -G^T G^{-1} \end{array}\right),$$

$$-F^T F^{-1} = \left(\begin{array}{c|c|c} \Phi_1' & \Phi_1' a & \mathbf{0} \\ \hline a^T C_1'^{-1} & (*) & (b-\tilde{b})^T G^{-1} \\ \hline \mathbf{0} & -G^T G^{-1}(b-\tilde{b}) & -G^T G^{-1} \end{array}\right),$$

where $\Phi_1' := \Phi_{\Gamma_1'}$, $\Phi_2' := \Phi_{\Gamma_2'}$ and $(*) = a^T C_1'^{-1} a - 1 + (b-\tilde{b})^T G^{-1}(b-\tilde{b})$.

Now, let

$$\begin{aligned} \alpha &:= -(x+1), \\ \alpha_1 &:= (x+1) - a^T C_1'^{-1} a, \\ \alpha_2 &:= (x+1) - (b-\tilde{b})^T G^{-1}(b-\tilde{b}), \end{aligned}$$

such that $\alpha_1 + \alpha + \alpha_2 = x - (*)$. Together with Lemma 2.3 in [**15**] (also stated below as Lemma 5.3.8) we get

$$\begin{aligned} \chi_{-F^T F^{-1}}(x) &= \left|\begin{array}{c|c} xE - \Phi_1' & -\Phi_1' a \\ \hline -a^T C_1'^{-1} & \alpha_1 \end{array}\right| \cdot \left|xE + G^T G^{-1}\right| \\ &\quad + \chi_{\Phi_1'}(x) \cdot \left|\begin{array}{c|c} \alpha_2 & -(b-\tilde{b})^T G^{-1} \\ \hline G^T G^{-1}(b-\tilde{b}) & xE + G^T G^{-1} \end{array}\right| \\ &\quad + \alpha \cdot \chi_{\Phi_1'}(x) \cdot \left|xE + G^T G^{-1}\right| \\ &= \chi_{\Phi_1}(x) \cdot \chi_{-G^T G^{-1}}(x) + \chi_{\Phi_1'}(x) \cdot \chi_{\Phi_2}(x) - (x+1) \cdot \chi_{\Phi_1'}(x) \cdot \chi_{-G^T G^{-1}}(x). \end{aligned}$$

Hence, we need the characteristic polynomial of the matrix $-G^T G^{-1}$ to complete the proof.

The matrices G, G^{-1} and $-G^T G^{-1}$ are $2t \times 2t$-matrices and given as follows (compare with the matrices in Case 2) of the proof of Proposition 5.3.6):

$$\begin{aligned} G &= E - \sum_{i=1}^{t} E_{2i,1} + \sum_{i=2}^{t}\sum_{j=1}^{i-1} E_{2i,2j} + \sum_{i=1}^{2t-1} E_{i,i+1} + \sum_{i=1}^{t-1} E_{2i-1,2i+1}, \\ G^{-1} &= \frac{1}{2}\left(\sum_{i=1}^{2t-1} E_{i+1,i} + \sum_{i=1}^{t} E_{2i-1,2i-1} + \sum_{i=1}^{t-1} E_{2i,2i+2} + E_{2t,2t} - \sum_{i=1}^{2t-1} E_{i,i+1} - \sum_{i=1}^{t-1} E_{2i+2,2i}\right), \\ -G^T G^{-1} &= E_{1,2t} - \sum_{i=1}^{t-1} E_{2i,2t} + \sum_{i=1}^{t-1} E_{2i,2i} - \sum_{i=1}^{t} E_{2i,2i-1} - \sum_{i=1}^{t-1} E_{2i+1,2i-1}. \end{aligned}$$

For $t=1$, we get $G=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $G^{-1}=\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $-G^TG^{-1}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In this case we compute the characteristic polynomial to be $\chi_{-G^TG^{-1}}(x)=x^2+1$ which is exactly $(x-1)^{t-1}(x^{t+1}+(-1)^{t+1})$ for $t=1$.

Now let $t>1$. Then the matrices G, G^{-1} and $-G^TG^{-1}$ have the following shapes:

$$G=\left(\begin{array}{r|ccccccccccccc}
1 & 1 & 1 & 0 & \cdots & & & & & & & & & 0 \\ \hline
-1 & 1 & 1 & 0 & & & & & & & & & & \\
0 & 0 & 1 & 1 & 1 & & & & & & & & & \\
-1 & 1 & 0 & 1 & 1 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & & & & & & & \\
-1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & & & & & \\
-1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & & & & \\
\vdots & \vdots & & & & & & & \ddots & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & & \\
-1 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & \\
-1 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1 &
\end{array}\right),$$

$$G^{-1}=\frac{1}{2}\cdot\left(\begin{array}{c|ccccccccccc}
1 & -1 & & & & & & & & & & \\ \hline
1 & 0 & -1 & 1 & & & & & & & & \\
0 & 1 & 1 & -1 & & & & & & & & \\
 & -1 & 1 & 0 & -1 & 1 & & & & & & \\
 & & & 1 & 1 & -1 & & & & & & \\
 & & & -1 & 1 & 0 & -1 & 1 & & & & \\
 & & & & & 1 & 1 & -1 & & & & \\
 & & & & & -1 & 1 & 0 & -1 & 1 & & \\
\vdots & & & & & & & & \ddots & & & \\
 & & & & & & & -1 & 1 & 0 & -1 & 1 \\
 & & & & & & & & & 1 & 1 & -1 \\
0 & & & & & & & & & -1 & 1 & 1
\end{array}\right), \tag{5.3.8}$$

$$-G^TG^{-1}=\begin{pmatrix}
0 & 0 & & & & & & & & & 1 \\
-1 & 1 & & & & & & & & & -1 \\
-1 & 0 & & & & & & & & & 0 \\
 & & -1 & 1 & & & & & & & -1 \\
 & & -1 & 0 & & & & & & & 0 \\
 & & & & -1 & 1 & & & & & -1 \\
 & & & & -1 & 0 & & & & & 0 \\
 & & & & & & \ddots & & & & \vdots \\
 & & & & & & & -1 & 1 & & -1 \\
 & & & & & & & -1 & 0 & & 0 \\
 & & & & & & & & & -1 & 0
\end{pmatrix}.$$

We compute the characteristic polynomial of $-G^TG^{-1}$ to be

$$\chi_{-G^TG^{-1}}(x) = \left|xE + G^TG^{-1}\right| = \begin{vmatrix} x & 0 & & & & & & & & & -1 \\ 1 & x-1 & & & & & & & & & 1 \\ 1 & 0 & x & & & & & & & & 0 \\ & & 1 & x-1 & & & & & & & 1 \\ & & 1 & 0 & x & & & & & & 0 \\ & & & & 1 & x-1 & & & & & 1 \\ & & & & 1 & 0 & x & & & & 0 \\ & & & & & & \ddots & \ddots & & & \vdots \\ & & & & & & & 1 & x-1 & & 1 \\ & & & & & & & 1 & 0 & x & 0 \\ & & & & & & & & & 1 & x \end{vmatrix}.$$

We expand along every even column of the original matrix, except for the last one. Thus, we expand $(t-1)$-times and get

$$\chi_{-G^TG^{-1}}(x) = (x-1)^{t-1} \begin{vmatrix} x & & & & & & -1 \\ 1 & x & & & & & 0 \\ & 1 & x & & & & 0 \\ & & 1 & x & & & 0 \\ & & & \ddots & \ddots & & \vdots \\ & & & & 1 & x & 0 \\ & & & & & 1 & x \end{vmatrix}.$$

Now, we expand along the first row and get the following two $t \times t$-matrices:

$$\begin{aligned} \chi_{-G^TG^{-1}}(x) &= (x-1)^{t-1}\left(x \cdot \begin{vmatrix} x & & & & & \\ 1 & x & & & & \\ & 1 & x & & & \\ & & \ddots & \ddots & & \\ & & & 1 & x & \\ & & & & 1 & x \end{vmatrix} + (-1)^{t+3} \cdot \begin{vmatrix} 1 & x & & & & \\ & 1 & x & & & \\ & & 1 & x & & \\ & & & \ddots & \ddots & \\ & & & & 1 & x \\ & & & & & 1 \end{vmatrix}\right) \\ &= (x-1)^{t-1}(x^{t+1} + (-1)^{t+1}). \end{aligned}$$

Finally, we obtain that the characteristic polynomial of the asymmetry matrix is

$$\begin{aligned} \chi_{-F^TF^{-1}}(x) &= \chi_{\Phi_1}(x) \cdot (x-1)^{t-1}(x^{t+1} + (-1)^{t+1}) + \chi_{\Phi_1'}(x) \cdot \chi_{\Phi_2}(x) \\ &\quad -(x+1) \cdot \chi_{\Phi_1'}(x) \cdot (x-1)^{t-1}(x^{t+1} + (-1)^{t+1}) \\ &= (x-1)^{t-1}(x^{t+1} + (-1)^{t+1}) \cdot \big(\chi_{\Phi_1}(x) - (x+1)\chi_{\Phi_1'}(x)\big) + \chi_{\Phi_1'}(x) \cdot \chi_{\Phi_2}(x), \end{aligned}$$

which is exactly the formula claimed in Lemma 5.3.7. □

Lemma 5.3.8 (Boldt [15]). *Let R be a commutative ring, and $F \in M_n(R)$ a matrix of the following form*

$$F = \left(\begin{array}{c|c|c} F_1 & f_1 & \mathbf{0} \\ \hline g_1^T & \alpha_1 + \alpha + \alpha_2 & g_2^T \\ \hline \mathbf{0} & f_2 & F_2 \end{array}\right),$$

where $F_1 \in M_{n_1}(R)$, $F_2 \in M_{n_2}(R)$, $n_1 + n_2 + 1 = n$, α, α_1, $\alpha_2 \in R$, f_1, $g_1 \in R^{n_1}$ and f_2, $g_2 \in R^{n_2}$. Then

$$|F| = \left|\begin{array}{c|c} F_1 & f_1 \\ \hline g_1^T & \alpha_1 \end{array}\right| |F_2| + |F_1| \left|\begin{array}{c|c} \alpha_2 & g_2^T \\ \hline f_2 & F_2 \end{array}\right| + \alpha |F_1||F_2|.$$

Proof. Expand the determinant of F along the $(n_1+1)^{\text{th}}$ column. Then expand $\left|\begin{array}{c|c} F_1 & f_1 \\ \hline g_1^T & \alpha_1 \end{array}\right|$ along the last column and $\left|\begin{array}{c|c} \alpha_2 & g_2^T \\ \hline f_2 & F_2 \end{array}\right|$ along the first column. □

Corollary 5.3.9. *Let $A := A_Q$ be the same cluster-tilted algebra of type D_n as in Lemma 5.3.7 and $t \geq 1$. Then the characteristic polynomial of the asymmetry matrix S_Q is given by*

$$\chi_{S_Q}(x) = (x+1)^{t-1}(x^{t+1}+1) \cdot \big(\chi_{S_{\Gamma_1}}(x) - (x-1) \cdot \chi_{S_{\Gamma'_1}}(x)\big) + \chi_{S_{\Gamma'_1}}(x) \cdot \chi_{S_{\Gamma_2}}(x).$$

Proof. Recall that by (5.3.1) in the proof of Proposition 5.3.3 we have

$$\chi_{\Phi_Q}(x) = (-1)^n \chi_{S_Q}(-x),$$

or equivalently,

$$\chi_{S_Q}(x) = (-1)^n \chi_{\Phi_Q}(-x),$$

where n is the number of vertices of Q.

Now, suppose that the number of vertices of Γ_1 is k and thus, the number of vertices of Γ_2 is $n-k+1 = 2t+1$. Hence, the number of vertices of Γ'_1 is $k-1$ and the number of vertices of Γ'_2 is $n-k = 2t$. Note that $n = 2t+k$. We can rewrite the formula in Lemma 5.3.7 as follows:

$$\begin{aligned}
\chi_{S_Q}(x) &= (-1)^n \chi_{\Phi_Q}(-x) \\
&= (-1)^n \Big((-x-1)^{t-1}((-x)^{t+1} + (-1)^{t+1}) \cdot \big(\chi_{\Phi_{\Gamma_1}}(-x) - (-x+1)\chi_{\Phi_{\Gamma'_1}}(-x)\big) \\
&\quad + \chi_{\Phi_{\Gamma'_1}}(-x)\chi_{\Phi_{\Gamma_2}}(-x)\Big) \\
&= (-1)^n \Big((-1)^{t-1}(x+1)^{t-1}(-1)^{t+1}(x^{t+1}+1)\big((-1)^k \chi_{S_{\Gamma_1}}(x) + (x-1)(-1)^{k-1}\chi_{S_{\Gamma'_1}}(x)\big) \\
&\quad + (-1)^{k-1}\chi_{S_{\Gamma'_1}}(x)(-1)^{n-k+1}\chi_{S_{\Gamma_2}}(x)\Big) \\
&= (-1)^n \Big((x+1)^{t-1}(x^{t+1}+1)(-1)^k \big(\chi_{S_{\Gamma_1}}(x) - (x-1)\chi_{S_{\Gamma'_1}}(x)\big) + (-1)^n \chi_{S_{\Gamma'_1}}(x)\chi_{S_{\Gamma_2}}(x)\Big) \\
&= (-1)^{n+k}(x+1)^{t-1}(x^{t+1}+1)\big(\chi_{S_{\Gamma_1}}(x) - (x-1)\chi_{S_{\Gamma'_1}}(x)\big) + \chi_{S_{\Gamma'_1}}(x)\chi_{S_{\Gamma_2}}(x).
\end{aligned}$$

Since $n+k = 2t+2k$, we get the desired formula

$$\chi_{S_Q}(x) = (x+1)^{t-1}(x^{t+1}+1)\big(\chi_{S_{\Gamma_1}}(x) - (x-1)\chi_{S_{\Gamma'_1}}(x)\big) + \chi_{S_{\Gamma'_1}}(x)\chi_{S_{\Gamma_2}}(x).$$

□

5.3.2. Polynomials for type D_n. We derive the characteristic polynomials of the asymmetry matrices for cluster-tilted algebras of types I, II and III of Dynkin type D_n and for certain cases of type IV. Combining this with Theorem 5.2.1, we get the corresponding associated polynomials. Using these it is possible to distinguish several further standard forms of Theorem 4.3.9 up to derived equivalence.

Note that these associated polynomials are not a complete list of all the associated polynomials of type D_n. However, up to D_{14}, the the full list of associated polynomials can be obtained by direct computation. This, together with Theorem 4.3.9, gives a complete derived equivalence classification up to D_{14} (see [**13**, Section 2.3]).

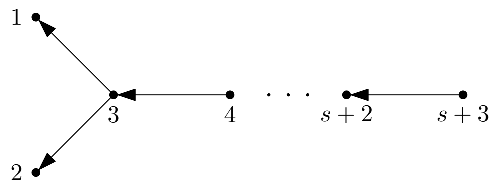
Proposition 5.3.10. *Consider a cluster-tilted algebra of type D_n with quiver Q of type I and parameters as defined in Section 4.1. Then*

$$\chi_{S_Q}(x) = (x+1)^t(x-1)\Big(x^{s+t+2} + (-1)^s\Big)$$

where $s = s(Q')$ and $t = t(Q')$.

Proof. We only have to compute the polynomial for $t = 0$ and $s \geq 0$ since a quiver of type I with parameters $(s, t+1)$ for some $s, t \geq 0$ is equivalent by good mutations to one of type II with parameters $(s+1, t, 0, 0)$ (see Lemma 4.2.38 (c)). Thus, for $t \geq 1$, we can consider a quiver of type II.

If $s = 0$, we get a quiver of type A_3. Thus, let $s \geq 1$. The quiver of a cluster-tilted algebra of type I with $t = 0$ looks as follows (up to sink/source equivalence)



We can use the formula given in Proposition 5.3.3 for $v = 3$ since there are no relations in this cluster-tilted algebra. Let Γ_1 be the quiver consisting of the vertex set $\{1, 2, 3\}$ and the corresponding incident arrows. Thus, $\Gamma'_1 = \Gamma_1 \backslash \{3\}$ consists of just two single vertices. Let Γ_2 be the linearly oriented type A quiver consisting of the vertex set $\{3, \ldots, s+3\}$ and let $\Gamma'_2 = \Gamma_2 \backslash \{3\}$. Thus, using Proposition 5.3.6, we get

$$\begin{aligned}
\chi_{S_{\Gamma_1}}(x) &= \frac{x^4 - 1}{x+1}, \\
\chi_{S_{\Gamma_2}}(x) &= \frac{x^{s+2} + (-1)^{s+1}}{x+1}, \\
\chi_{S_{\Gamma'_2}}(x) &= \frac{x^{s+1} + (-1)^{s}}{x+1}.
\end{aligned}$$

We also have

$$\chi_{S_{\Gamma'_1}}(x) = (x-1)^2.$$

Hence, using Proposition 5.3.3, the polynomial can be computed as

$$\begin{aligned}
\chi_{S_Q}(x) &= \chi_{S_{\Gamma_1}}(x) \cdot \chi_{S_{\Gamma'_2}}(x) + \chi_{S_{\Gamma'_1}}(x) \cdot \chi_{S_{\Gamma_2}}(x) - (x-1) \cdot \chi_{S_{\Gamma'_1}}(x) \cdot \chi_{S_{\Gamma'_2}}(x) \\
&= \frac{x^4 - 1}{x+1} \cdot \frac{x^{s+1} + (-1)^s}{x+1} + (x-1)^2 \cdot \frac{x^{s+2} + (-1)^{s+1}}{x+1} \\
&\quad -(x-1) \cdot (x-1)^2 \cdot \frac{x^{s+1} + (-1)^s}{x+1} \\
&= \frac{x^{s+1} + (-1)^s}{x+1} \big(x^3 - x^2 + x - 1 - (x-1)^3\big) + (x-1)^2 \cdot \frac{x^{s+2} + (-1)^{s+1}}{x+1} \\
&= \frac{x^{s+1} + (-1)^s}{x+1} \big(2x(x-1)\big) + (x-1)^2 \cdot \frac{x^{s+2} + (-1)^{s+1}}{x+1} \\
&= \frac{x-1}{x+1} \big(2x(x^{s+1} + (-1)^s) + (x-1)(x^{s+2} + (-1)^{s+1})\big) \\
&= \frac{x-1}{x+1} \big(x^{s+3} + x^{s+2} + (-1)^s x + (-1)^s\big) \\
&= (x-1)(x^{s+2} + (-1)^s),
\end{aligned}$$

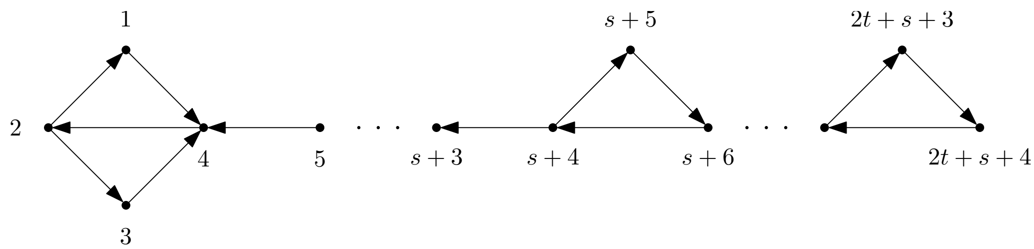
which is exactly the formula given in Proposition 5.3.10 in case $t = 0$. □

Proposition 5.3.11. *Consider a cluster-tilted algebra of type D_n with quiver Q of type II or type III, and parameters as defined in Section 4.1. Then*

$$\chi_{S_Q}(x) = (x+1)^{t+1}(x-1)\Big(x^{s+t+2} + (-1)^{s+1}\Big)$$

where $s = s(Q') + s(Q'')$ and $t = t(Q') + t(Q'')$.

Proof.
Type II: Let Q be a quiver of type II. Using the derived equivalence classification of cluster-tilted algebras of type A (see [**23**, Theorem 5.1] or Theorem 4.2.18) and the fact that we can move the rooted quivers of type A in type II from side to side (see II.2 and II.3 in Tables 4.3 and 4.6), we can assume that Q has the following shape



since by Proposition 5.1.2 the characteristic polynomial of the asymmetry matrix S_Q is invariant under derived equivalence.

Case 1) Let $s > 1$. We can use the formula given in Proposition 5.3.3 since, with vertex $v = 5$, the ideal I is v-separated. Let Γ_1 be the quiver consisting of the vertex set $\{1, \dots, 5\}$ and the corresponding incident arrows. Thus, $\Gamma_1' = \Gamma_1 \backslash \{5\}$ is the quiver of the skeleton of type II. Let Γ_2 be the rooted quiver of type A consisting of the vertex set $\{5, \dots, 2t+s+4\}$ and thus, $\Gamma_2' = \Gamma_2 \backslash \{5\}$ is a rooted quiver of type A with parameters $s-2$ and t. Thus, with Proposition 5.3.6, we get

$$\begin{aligned}\chi_{S_{\Gamma_2}}(x) &= (x+1)^{t-1}(x^{s+t+1} + (-1)^s),\\ \chi_{S_{\Gamma_2'}}(x) &= (x+1)^{t-1}(x^{s+t} + (-1)^{s-1}).\end{aligned}$$

Since the quivers of Γ_1 and Γ_1' are small, we can compute their asymmetry matrices and the corresponding characteristic polynomials by hand to get

$$\begin{aligned}\chi_{S_{\Gamma_1'}}(x) &= (x+1)(x-1)(x^2-1),\\ \chi_{S_{\Gamma_1}}(x) &= (x+1)(x-1)(x^3+1).\end{aligned}$$

Using Proposition 5.3.3, we obtain the following expression for the polynomial:

$$\begin{aligned}\chi_{S_Q}(x) &= \chi_{S_{\Gamma_1}}(x) \cdot \chi_{S_{\Gamma_2'}}(x) + \chi_{S_{\Gamma_1'}}(x) \cdot \chi_{S_{\Gamma_2}}(x) - (x-1) \cdot \chi_{S_{\Gamma_1'}}(x) \cdot \chi_{S_{\Gamma_2'}}(x)\\ &= (x+1)(x-1)(x^3+1)(x+1)^{t-1}(x^{s+t} + (-1)^{s-1})\\ &\quad + (x+1)(x-1)(x^2-1)(x+1)^{t-1}(x^{s+t+1} + (-1)^s)\\ &\quad - (x-1)(x+1)(x-1)(x^2-1)(x+1)^{t-1}(x^{s+t} + (-1)^{s-1})\\ &= (x+1)^t(x-1)\Big(2x^{s+t+3} + (-1)^{s-1}x^3 + x^{s+t} + 2(-1)^{s-1} + (-1)^s x^2 - x^{s+t+1}\\ &\quad - (x+1)(x^{s+t+2} + (-1)^{s-1}x^2 - x^{s+t} + (-1)^s)\Big)\\ &= (x+1)^t(x-1)\Big(x^{s+t+3} + x^{s+t+2} + (-1)^{s+1}x + (-1)^{s+1}\Big)\\ &= (x+1)^{t+1}(x-1)(x^{s+t+2} + (-1)^{s+1}).\end{aligned}$$

This is exactly the formula given in Proposition 5.3.11 in case $s > 1$ (i.e. Proposition 5.3.11 is proven for type II in this case).

Case 2) Let $s = 1$. If $t = 0$, we can compute the characteristic polynomial $\chi_{S_Q}(x)$ to be $\chi_{S_Q}(x) = (x+1)(x-1)(x^3+1)$ which is exactly the formula given in Proposition 5.3.11 in case $s = 1$, $t = 0$. Thus, let $t > 0$. We can use the formula given in Corollary 5.3.9. Let Γ_1 be the quiver consisting of the vertex set $\{1, \dots, 5\}$ and the corresponding incident arrows. Thus, $\Gamma_1' = \Gamma_1 \backslash \{5\}$ is the quiver of the skeleton of type II. Let Γ_2 be the rooted quiver of type A consisting of only t oriented 3-cycles. As in Case 1) we have

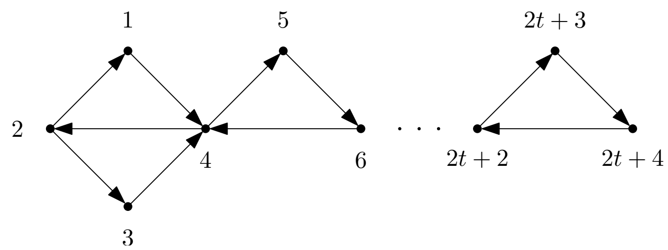
$$\begin{aligned}\chi_{S_{\Gamma_1}}(x) &= (x+1)(x-1)(x^3+1),\\ \chi_{S_{\Gamma_1'}}(x) &= (x+1)(x-1)(x^2-1),\\ \chi_{S_{\Gamma_2}}(x) &= (x+1)^{t-1}(x^{t+2}-1).\end{aligned}$$

Using Corollary 5.3.9, the polynomial can be computed to be

$$\begin{aligned}
\chi_{S_Q}(x) &= (x+1)^{t-1}(x^{t+1}+1)\cdot\big(\chi_{S_{\Gamma_1}}(x)-(x-1)\cdot\chi_{S_{\Gamma_1'}}(x)\big)+\chi_{S_{\Gamma_1'}}(x)\cdot\chi_{S_{\Gamma_2}}(x)\\
&= (x+1)^{t-1}(x^{t+1}+1)\Big((x+1)(x-1)(x^3+1)-(x-1)(x+1)(x-1)(x^2-1)\Big)\\
&\quad+(x+1)(x-1)(x^2-1)(x+1)^{t-1}(x^{t+2}-1)\\
&= (x+1)^t(x-1)\Big((x^{t+1}+1)\big((x^3+1)-(x-1)(x^2-1)\big)+(x^2-1)(x^{t+2}-1)\Big)\\
&= (x+1)^t(x-1)\Big((x^{t+1}+1)(x^2+x)+x^{t+4}-x^{t+2}-x^2+1\Big)\\
&= (x+1)^t(x-1)\Big(x^{t+4}+x^{t+3}+x+1\Big)\\
&= (x+1)^{t+1}(x-1)(x^{t+3}+1),
\end{aligned}$$

which is exactly the formula given in Proposition 5.3.11 in case $s=1$.

Case 3) Now we assume that $s=0$. Up to derived equivalence, the quiver of the corresponding cluster-tilted algebra looks as follows



.

We denote the rooted quiver of type A which begins with vertex 4 and ends with vertex $2t+4$ by $A := A_{2t+1}$. We compute the Cartan matrix, its inverse and the corresponding asymmetry matrix of the cluster-tilted algebra corresponding to Q to be the following:

$$C=\left(\begin{array}{ccc|ccc}
1 & 0 & 0 & 1 & 1 & \\
1 & 1 & 1 & 1 & 1 & \mathbf{0}\\
0 & 0 & 1 & 1 & 1 & \\
\hline
0 & 1 & 0 & & & \\
0 & 0 & 0 & & & \\
0 & 1 & 0 & & & \\
\vdots & \vdots & \vdots & & \mathbf{C_A} & \\
0 & 0 & 0 & & & \\
0 & 1 & 0 & & &
\end{array}\right),$$

$$C^{-1}=\frac{1}{2}\cdot\left(\begin{array}{ccc|cc}
1 & 1 & -1 & -1 & \\
-1 & 1 & -1 & 1 & \mathbf{0}\\
-1 & 1 & 1 & -1 & \\
\hline
1 & -1 & 1 & & \\
0 & 0 & 0 & & \\
\vdots & \vdots & \vdots & & 2\mathbf{C_A^{-1}}\\
0 & 0 & 0 & &
\end{array}\right)-\frac{1}{2}E_{44},$$

$$S_Q=\left(\begin{array}{ccc|ccc}
0 & 0 & -1 & 0 & 1 & \\
0 & 0 & 0 & 0 & 1 & \mathbf{0}\\
-1 & 0 & 0 & 0 & 1 & \\
\hline
0 & 1 & 0 & & & \\
0 & 0 & 0 & & & \\
0 & 1 & 0 & & & \\
\vdots & \vdots & \vdots & & (*) & \\
0 & 0 & 0 & & & \\
0 & 1 & 0 & & &
\end{array}\right),\ (*)=\mathbf{S_A}-\sum_{i=1}^{t+1}E_{2i-1,1}=\mathbf{S_A}-\begin{pmatrix}
1 & & \\
0 & & \\
1 & & \\
\vdots & & \mathbf{0}\\
0 & & \\
1 & &
\end{pmatrix},$$

where $\mathbf{C_A}$ is the Cartan matrix of A as in (5.3.4), $\mathbf{C_A^{-1}}$ is the inverse as in (5.3.5) and $\mathbf{S_A}$ is the asymmetry matrix as in (5.3.6) (see Case 1) in the proof of Proposition 5.3.6).

Hence, S_Q is a $(2t{+}4)\times(2t{+}4)$-matrix and the characteristic polynomial $\chi_{S_Q}(x)$ has the following form:

$$\chi_{S_Q}(x) = \left|\begin{array}{ccc|ccc}
x & 0 & 1 & 0 & -1 & \\
0 & x & 0 & 0 & -1 & \mathbf{0} \\
1 & 0 & x & 0 & -1 & \\
\hline
0 & -1 & 0 & & & \\
0 & 0 & 0 & & & \\
0 & -1 & 0 & & & \\
\vdots & \vdots & \vdots & \multicolumn{3}{c}{xE - S_A + \begin{pmatrix} 1 & \\ 0 & \\ 1 & \\ \vdots & \mathbf{0} \\ 0 & \\ 1 & \end{pmatrix}} \\
0 & 0 & 0 & & & \\
0 & -1 & 0 & & &
\end{array}\right|.$$

First, we exchange column 2 and column 4 and afterwards, we add the (new) second column to the (new) fourth column to get

$$\chi_{S_Q}(x) = -\left|\begin{array}{ccc|ccc}
x & 0 & 1 & 0 & -1 & \\
0 & 0 & 0 & x & -1 & \mathbf{0} \\
1 & 0 & x & 0 & -1 & \\
\hline
0 & (x+1) & 0 & & & \\
0 & 0 & 0 & & & \\
\vdots & \vdots & \vdots & \multicolumn{3}{c}{xE - S_A} \\
0 & 0 & 0 & & &
\end{array}\right|.$$

Laplace expansion along the second column leads to

$$\chi_{S_Q}(x) = -(x+1)\left|\begin{array}{cc|ccc}
x & 1 & 0 & -1 & \\
0 & 0 & x & -1 & \mathbf{0} \\
1 & x & 0 & -1 & \\
\hline
 & & \multicolumn{3}{c}{xE - S_A} \\
 & \mathbf{0} & \multicolumn{3}{c}{\text{without}} \\
 & & \multicolumn{3}{c}{1^{\text{st}} \text{ row}}
\end{array}\right|.$$

Now, we add the first column to the fourth column to get

$$\chi_{S_Q}(x) = -(x+1)\left|\begin{array}{cc|ccc}
x & 1 & 0 & (x-1) & \\
0 & 0 & x & -1 & \mathbf{0} \\
1 & x & 0 & 0 & \\
\hline
 & & \multicolumn{3}{c}{xE - S_A} \\
 & \mathbf{0} & \multicolumn{3}{c}{\text{without}} \\
 & & \multicolumn{3}{c}{1^{\text{st}} \text{ row}}
\end{array}\right|.$$

We expand along the third row to get

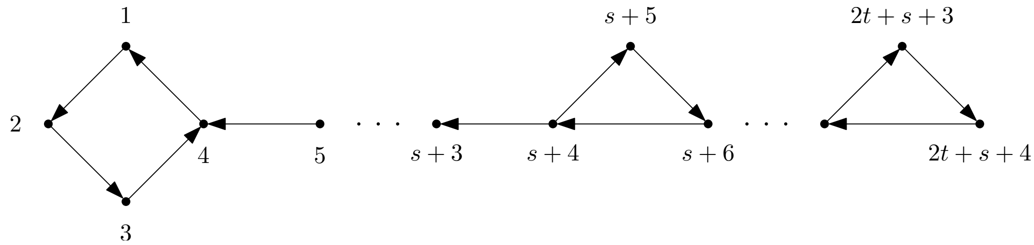
$$\chi_{S_Q}(x) = -(x+1)\left|\begin{array}{c|ccccc}
1 & 0 & (x-1) & 0 & \cdots & 0 \\
\hline
0 & x & -1 & 0 & \cdots & 0 \\
0 & & & \multicolumn{3}{c}{xE - S_A} \\
\vdots & & & \multicolumn{3}{c}{\text{without}} \\
0 & & & \multicolumn{3}{c}{1^{\text{st}} \text{ row}}
\end{array}\right| + x(x+1)\left|\begin{array}{c|ccccc}
x & 0 & (x-1) & 0 & \cdots & 0 \\
\hline
0 & x & -1 & 0 & \cdots & 0 \\
0 & & & \multicolumn{3}{c}{xE - S_A} \\
\vdots & & & \multicolumn{3}{c}{\text{without}} \\
0 & & & \multicolumn{3}{c}{1^{\text{st}} \text{ row}}
\end{array}\right|.$$

Since the lower right hand matrix is then $xE - S_A$ (see (5.3.7) in Case 1) of the proof of Proposition 5.3.6), we can compute the characteristic polynomial as follows:

$$\begin{aligned}
\chi_{S_Q}(x) &= -(x+1)\chi_{S_A}(x) + x^2(x+1)\chi_{S_A}(x) \\
&= (x+1)(x^2-1)\chi_{S_A}(x) \\
&= (x+1)^2(x-1)\cdot(x+1)^{t-1}(x^{t+2}-1) \\
&= (x+1)^{t+1}(x-1)(x^{t+2}-1).
\end{aligned}$$

This is exactly the formula given in Proposition 5.3.11 in case $s = 0$. Hence, Proposition 5.3.11 is proven for type II.

Type III: Let Q be a quiver of type III. Using the derived equivalence classification of cluster-tilted algebras of type A (see [**23**, Theorem 5.1] or Theorem 4.2.18) and Lemma 4.3.10 (b), we can assume that Q has the following shape



since by Proposition 5.1.2 the characteristic polynomial of the asymmetry matrix S_Q is invariant under derived equivalence.

Case 1) Let $s > 1$. We can use the formula given in Proposition 5.3.3 since, with vertex $v = 5$, the ideal I is v-separated. Let Γ_1 be the quiver consisting of the vertex set $\{1, \ldots, 5\}$ and the corresponding incident arrows. Thus, $\Gamma_1' = \Gamma_1 \backslash \{5\}$ is the quiver of the skeleton of type III. Let Γ_2 be the rooted quiver of type A consisting of the vertex set $\{5, \ldots, 2t + s + 4\}$ and thus, $\Gamma_2' = \Gamma_2 \backslash \{5\}$ is a rooted quiver of type A with parameters $s - 2$ and t. Thus, with Proposition 5.3.6, we get

$$\begin{aligned} \chi_{S_{\Gamma_2}}(x) &= (x+1)^{t-1}(x^{s+t+1} + (-1)^s), \\ \chi_{S_{\Gamma_2'}}(x) &= (x+1)^{t-1}(x^{s+t} + (-1)^{s-1}). \end{aligned}$$

Since the quivers of Γ_1 and Γ_1' are small, we can compute their asymmetry matrices and the corresponding characteristic polynomials by hand to get

$$\begin{aligned} \chi_{S_{\Gamma_1'}}(x) &= (x+1)(x-1)(x^2 - 1), \\ \chi_{S_{\Gamma_1}}(x) &= (x+1)(x-1)(x^3 + 1). \end{aligned}$$

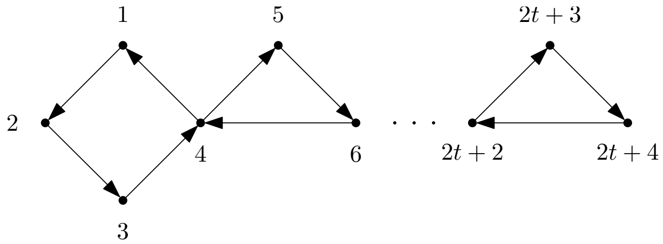
These are indeed the same polynomials as in Case 1) of type II. Hence, the computations are the same as the computations in type II.

Case 2) Let $s = 1$. If $t = 0$, we can compute the characteristic polynomial $\chi_{S_Q}(x)$ to be $\chi_{S_Q}(x) = (x+1)(x-1)(x^3+1)$ which is exactly the formula given in Proposition 5.3.11 in case $s = 1$, $t = 0$. Thus, let $t > 0$. We can use the formula given in Corollary 5.3.9. Let Γ_1 be the quiver consisting of the vertex set $\{1, \ldots, 5\}$ and the corresponding incident arrows. Thus, $\Gamma_1' = \Gamma_1 \backslash \{5\}$ is the quiver of the skeleton of type III. Let Γ_2 be the rooted quiver of type A consisting of only t oriented 3-cycles. As in Case 1) we have

$$\begin{aligned} \chi_{S_{\Gamma_1}}(x) &= (x+1)(x-1)(x^3 + 1), \\ \chi_{S_{\Gamma_1'}}(x) &= (x+1)(x-1)(x^2 - 1), \\ \chi_{S_{\Gamma_2}}(x) &= (x+1)^{t-1}(x^{t+2} - 1), \end{aligned}$$

and hence, the computations are the same as in type II.

Case 3) Now we assume that $s = 0$. Up to derived equivalence, the quiver of the corresponding cluster-tilted algebra looks as follows



.

We denote the rooted quiver of type A which begins with vertex 4 and ends with vertex $2t+4$ by $A := A_{2t+1}$. We compute the Cartan matrix, its inverse and the corresponding asymmetry matrix of the cluster-tilted algebra corresponding to Q to be the following:

$$C = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 1 & 1 & \mathbf{0} \\ 1 & 0 & 1 & 1 & 1 & \\ \hline 1 & 1 & 0 & & & \\ 0 & 0 & 0 & & & \\ 1 & 1 & 0 & & & \\ \vdots & \vdots & \vdots & & \mathbf{C_A} & \\ 0 & 0 & 0 & & & \\ 1 & 1 & 0 & & & \end{array}\right),$$

$$C^{-1} = \frac{1}{3}\cdot \left(\begin{array}{ccc|cc} 1 & -2 & 1 & 1 & \\ 1 & 1 & -2 & 1 & \mathbf{0} \\ 1 & 1 & 1 & -2 & \\ \hline -2 & 1 & 1 & & \\ 0 & 0 & 0 & & \\ \vdots & \vdots & \vdots & & 3\mathbf{C_A^{-1}} \\ 0 & 0 & 0 & & \end{array}\right) - \frac{2}{3}E_{44},$$

$$S_Q = \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \mathbf{0} \\ 1 & 0 & 0 & -1 & 1 & \\ \hline 0 & 1 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ \vdots & \vdots & \vdots & & & (*) \\ 0 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \end{array}\right), \; (*) = \mathbf{S_A} - \sum_{i=1}^{t+1} E_{2i-1,1} = \mathbf{S_A} - \left(\begin{array}{cc} 1 & \\ 0 & \\ 1 & \\ \vdots & \mathbf{0} \\ 0 & \\ 1 & \end{array}\right),$$

where $\mathbf{C_A}$ is the Cartan matrix of A as in (5.3.4), $\mathbf{C_A^{-1}}$ is the inverse as in (5.3.5) and $\mathbf{S_A}$ is the asymmetry matrix as in (5.3.6) (see Case 1) in the proof of Proposition 5.3.6).

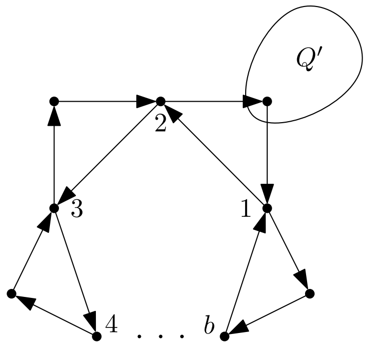
We leave it to the reader to compute the characteristic polynomial of S_Q since this computation is similar to the computation in Case 3) of type II. $\square$

Proposition 5.3.12. *Consider a cluster-tilted algebra of type D_n with quiver Q of type IV and parameters as defined in Section 4.1.*

(a) *If Q is an oriented cycle of length n without spikes then*

$$\chi_{S_Q}(x) = \begin{cases} x^n - 1, & \textit{if } n \textit{ is odd,} \\ \left(x^{\frac{n}{2}} - 1\right)^2, & \textit{if } n \textit{ is even.} \end{cases}$$

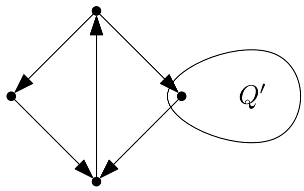
(b) *If Q has parameter sequence $\big((1,s,t),(1,0,0),\dots,(1,0,0)\big)$ of length $b \geq 3$ as in the picture below*



then

$$\chi_{S_Q}(x) = (x+1)^t(x^b-1)\Big(x^{s+t+b} + (-1)^{s+1}\Big).$$

(c) *If Q has parameters $\big((3,s,t)\big)$ as in the picture below*



then

$$\chi_{S_Q}(x) = (x+1)^{t-1}(x-1)\Big(x^{s+t+4} + 2\cdot x^{s+t+3} + (-1)^{s-1}\cdot 2x + (-1)^{s-1}\Big).$$

By applying the Propositions 5.1.2, 5.3.11 and 5.3.12 (c) we immediately obtain the following.

Corollary 5.3.13. *A cluster-tilted algebra of type II is not derived equivalent to a cluster-tilted algebra of type IV with parameters $\big((3,s,t)\big)$, and s, t not both zero.*

Proof of Proposition 5.3.12. (a) Let Q be an oriented n-cycle without spikes. We number the vertices in consecutive order along the cycle. Then we compute the Cartan matrix, its inverse and the corresponding asymmetry matrix of the cluster-tilted algebra corresponding to Q as follows:

$$\begin{aligned}
C &= \sum_{i=1}^{n}\sum_{j=1}^{n} E_{ij} - \sum_{i=2}^{n} E_{i,i-1} - E_{1,n},\\
C^{-1} &= \frac{1}{n-1}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} E_{ij} - (n-1)\sum_{i=1}^{n-1} E_{i,i+1} - (n-1)E_{n,1}\Big),\\
S_Q &= \sum_{i=3}^{n} E_{i,i-2} + E_{1,n-1} + E_{2,n},
\end{aligned}$$

that is,

$$C = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0\\ 0 & 1 & 1 & \cdots & 1 & 1 & 1\\ 1 & 0 & 1 & \cdots & 1 & 1 & 1\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ 1 & 1 & 1 & \cdots & 0 & 1 & 1\\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 \end{pmatrix},$$

$$C^{-1} = \frac{1}{n-1}\begin{pmatrix} 1 & -(n-2) & 1 & 1 & \cdots & 1 & 1\\ 1 & 1 & -(n-2) & 1 & \cdots & 1 & 1\\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ 1 & 1 & 1 & 1 & \cdots & 1 & -(n-2)\\ -(n-2) & 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix},$$

$$S_Q = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix}.$$

Hence, S_Q is a $n \times n$-matrix and the characteristic polynomial $\chi_{S_Q}(x)$ has the following shape:

$$\chi_{S_Q}(x) = \begin{vmatrix} x & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \\ 0 & x & 0 & \cdots & 0 & 0 & 0 & -1 \\ -1 & 0 & x & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & x & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & x \end{vmatrix}.$$

The matrix $xE - S_Q$ is a circulant matrix as in Lemma 5.3.4 with entries $c_0 = x$, $c_{n-2} = -1$ and $c_l = 0$ for all $l \neq 0, n-2$.

Using Equation (5.3.3) of Lemma 5.3.4 we get the characteristic polynomial $\chi_{S_Q}(x)$ as follows:

$$\begin{aligned} \chi_{S_Q}(x) &= \prod_{j=0}^{n-1}(x - \omega_j^2) \\ &= \prod_{j=0}^{n-1}(x - \omega^{2j}) \\ &= (x-1)(x-\omega^2)(x-\omega^4)\cdots(x-\omega^{2(n-1)}), \end{aligned}$$

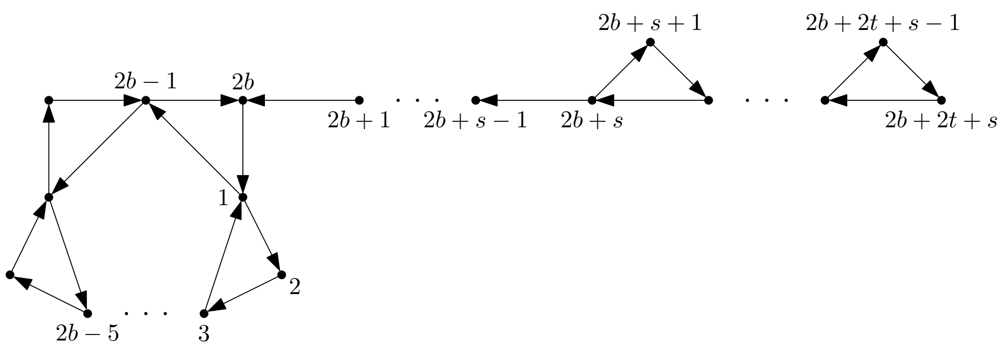
where $\omega := \omega_1 = e^{\frac{2\pi i}{n}}$. Since

$$\operatorname{ord}(\omega^2) = \frac{\operatorname{ord}(\omega)}{\gcd(2, \operatorname{ord}(\omega))} = \begin{cases} n, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases},$$

we get the desired polynomials

$$\chi_{S_Q}(x) = \begin{cases} x^n - 1, & \text{if } n \text{ is odd} \\ (x^{\frac{n}{2}} - 1)^2, & \text{if } n \text{ is even} \end{cases}.$$

(b) Let Q be a quiver of type IV with parameter sequence $\big((1,s,t),(1,0,0),\ldots,(1,0,0)\big)$ of length $b \geq 3$. Using the derived equivalence classification of cluster-tilted algebras of type A (see [**23**, Theorem 5.1] or Theorem 4.2.18), we can assume that Q has the following shape



since by Proposition 5.1.2 the characteristic polynomial of the asymmetry matrix S_Q is invariant under derived equivalence.

We denote the quiver of the corresponding skeleton consisting of the vertices $1, 2, \ldots, 2b$ by the symbol $\lozenge$. To compute the polynomial of the skeleton we use that the corresponding cluster-tilted algebra is self-injective. Thus, it is derived equivalent to an oriented $2b$-cycle (see Lemma 4.3.6) and by Proposition 5.3.12 (a), the polynomial is given by

$$\chi_{S_\lozenge} = (x^b - 1)^2.$$

Indeed, this is exactly the formula given in Proposition 5.3.12 (b) in case $s = t = 0$, so this case is proven.

However, we will need the matrices of the skeleton in the other computations for this type. Thus, the Cartan matrix $C_\lozenge$, its inverse and the corresponding asymmetry matrix are given as follows:

$$\begin{aligned}
C_\lozenge &= \sum_{i=1}^{b}\sum_{j=1}^{b} E_{2i-1,2j-1} + \sum_{i=1}^{b} E_{2i-1,2i} + \sum_{i=1}^{b-1}(E_{2i,2i} - E_{2i,2i+1} + E_{2i,2i+1}) \\
&\quad + E_{2b,1} + E_{2b,2} + E_{2b,2b}, \\
C_\lozenge^{-1} &= \frac{1}{2b-1}\Big(\sum_{i=1}^{b}\sum_{j=1}^{b} E_{2i-1,2j-1} + \sum_{i=1}^{b}\sum_{j=1}^{b} E_{2i,2j} - 2\sum_{i=1}^{b}\sum_{j=1}^{b}(E_{2i-1,2j} + E_{2i,2j-1}) \\
&\quad + (2b-1)\big(-\sum_{i=1}^{b} E_{2i-1,2i-1} + \sum_{i=1}^{2b-1} E_{i+1,i} + E_{1,2b} - \sum_{i=1}^{b-1} E_{2i+1,2i+1} - E_{1,2b-1}\big)\Big), \\
S_\lozenge &= \sum_{i=1}^{2b-2} E_{i,i+2} + E_{2b-1,1} + E_{2b,2},
\end{aligned}$$

that is,

$$C_\lozenge = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{pmatrix}, \tag{5.3.9}$$

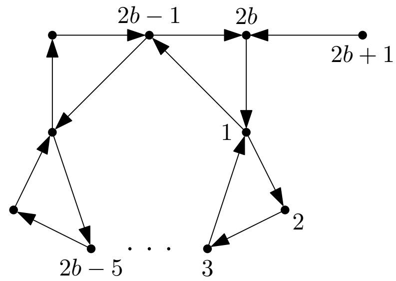
$$C_\lozenge^{-1} = \frac{1}{2b-1}\cdot\begin{pmatrix}
-2b+5 & -2 & 4 & -2 & 4 & -2 & 4 & \cdots & 4 & -2 & -2b+5 & 2b-3 \\
2b-3 & 1 & -2 & 1 & -2 & 1 & -2 & \cdots & -2 & 1 & -2 & 1 \\
-2b+5 & 2b-3 & -2b+5 & -2 & 4 & -2 & 4 & \cdots & 4 & -2 & 4 & -2 \\
-2 & 1 & 2b-3 & 1 & -2 & 1 & -2 & \cdots & -2 & 1 & -2 & 1 \\
4 & -2 & -2b+5 & 2b-3 & -2b+5 & -2 & 4 & \cdots & 4 & -2 & 4 & -2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
-2 & 1 & -2 & 1 & -2 & 1 & -2 & \cdots & 2b-3 & 1 & -2 & 1 \\
4 & -2 & 4 & -2 & 4 & -2 & 4 & \cdots & -2b+5 & 2b-3 & -2b+5 & -2 \\
-2 & 1 & -2 & 1 & -2 & 1 & -2 & \cdots & -2 & 1 & 2b-3 & 1
\end{pmatrix}, \tag{5.3.10}$$

$$S_\lozenge = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}. \tag{5.3.11}$$

We leave it to the reader to verify the expressions for the matrices $C_\Diamond^{-1}$ and $S_\Diamond$ since the computations are very long, but straightforward.

To compute the polynomials in the cases $s > 1$ and $s = 1$, we need the polynomial of the cluster-tilted algebra corresponding to the quiver in the following case.

Case 1) Let $s = 1$ and $t = 0$, that is, Q is a quiver of type IV with parameter sequence $((1,1,0),(1,0,0),\ldots,(1,0,0))$,



.

The Cartan matrix, its inverse and the corresponding asymmetry matrix are given as follows:

$$C = \left(\begin{array}{cccccc|c} & & & & & & 0 \\ & & \mathbf{C}_\Diamond & & & & \vdots \\ & & & & & & 0 \\ \hline 1 & 1 & 0 & \cdots & 0 & 1 & 1 \end{array}\right), \quad C^{-1} = \left(\begin{array}{cccccc|c} & & & & & & 0 \\ & & \mathbf{C}_\Diamond^{-1} & & & & \vdots \\ & & & & & & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{array}\right),$$

$$S_Q = \left(\begin{array}{cccccc|c} & & & & & & 0 \\ & & & & & & \vdots \\ & & \mathbf{S}_\Diamond & & & & 0 \\ & & & & & & -1 \\ & & & & & & -1 \\ & & & & & & -1 \\ \hline 0 & 1 & 0 & \cdots & 0 & 0 & 0 \end{array}\right).$$

Note that the last row of C^{-1} is as shown because the last row of $\mathbf{C}_\Diamond$ is the same as the last row of C (except for the last column).

The asymmetry matrix S_Q is a $(2b+1)\times(2b+1)$-matrix and the characteristic polynomial $\chi_{S_Q}(x)$ has the following shape:

$$\chi_{S_Q}(x) = \left|\begin{array}{cccccc|c} & & & & & & 0 \\ & & & & & & \vdots \\ & & xE - S_\Diamond & & & & 0 \\ & & & & & & 1 \\ & & & & & & 1 \\ & & & & & & 1 \\ \hline 0 & -1 & 0 & \cdots & 0 & 0 & x \end{array}\right| = \left|\begin{array}{cccccccc|c} x & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & x & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & x & 0 & 1 \\ 0 & -1 & 0 & 0 & \cdots & 0 & 0 & x & 1 \\ \hline 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & x \end{array}\right|.$$

First, we expand along the last row to get

$$\chi_{S_Q}(x) = (-1)^{2b+4} \left|\begin{array}{ccccccccc} x & -1 & 0 & 0 & & & & & \\ 0 & 0 & -1 & 0 & & & & & \\ 0 & x & 0 & -1 & & & & & \\ 0 & 0 & x & 0 & & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & x & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & x & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x & 1 \end{array}\right| + x \cdot \chi_{S_\Diamond}(x).$$

Then we successively expand along every even row of this matrix, except for the $(2b-2)^{\text{th}}$ and the $2b^{\text{th}}$ row (i.e. along the 2^{nd}, 4^{th}, $6^{\text{th}}, \dots, (2b-4)^{\text{th}}$ row). Hence, we expand $(b-2)$-times. This yields a $(b+2)\times(b+2)$-matrix

$$\chi_{S_Q}(x) = \begin{vmatrix} x & -1 & 0 & 0 & & & & & \\ 0 & x & -1 & 0 & & & & & \\ 0 & 0 & x & -1 & & & & & \\ 0 & 0 & 0 & x & & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & x & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & x & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x & 1 \end{vmatrix} + x \cdot \chi_{S_\Diamond}(x).$$

Laplace expansion along the b^{th} row then yields the following two $(b+1)\times(b+1)$-matrices:

$$\chi_{S_Q}(x) = (-1)^{2b+2} \cdot \begin{vmatrix} x & -1 & 0 & 0 & & & & \\ 0 & x & -1 & 0 & & & & \\ 0 & 0 & x & -1 & & & & \\ 0 & 0 & 0 & x & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & x & -1 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & x & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{vmatrix}$$
$$+ (-1)^{2b+2} \cdot \begin{vmatrix} x & -1 & 0 & 0 & & & & \\ 0 & x & -1 & 0 & & & & \\ 0 & 0 & x & -1 & & & & \\ 0 & 0 & 0 & x & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & x & -1 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & x & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x \end{vmatrix} + x \cdot \chi_{S_\Diamond}(x).$$

Finally, we expand along the last rows of these two matrices to get two circulant $b\times b$-matrices with determinant (x^b-1) (see Example 5.3.5). Thus,

$$\begin{aligned} \chi_{S_Q}(x) &= (x^b-1) + x(x^b-1) + x(x^b-1)^2 \\ &= (x^b-1)(x^{b+1}+1). \end{aligned}$$

This is exactly the formula given in Proposition 5.3.12 (b) in case $t=0$, $s=1$ (i.e. Proposition 5.3.12 (b) is proven in this case).

Case 2) If $s>1$ we can use the formula given in Proposition 5.3.3 since, with vertex $v=2b+1$, the ideal I is v-separated. Let Γ_1 be the quiver consisting of the vertex set $\{1,\dots,2b+1\}$, i.e. Γ_1 is a quiver with parameter sequence $((1,1,0),(1,0,0),\dots,(1,0,0))$. Thus, $\Gamma_1' = \Gamma_1\backslash\{2b+1\}$ is the quiver of the skeleton $\Diamond$. Let Γ_2 be the quiver consisting of the vertex set $\{2b+1,\dots,2b+2t+s\}$ and thus, $\Gamma_2' = \Gamma_2\backslash\{2b+1\}$. With Proposition 5.3.6 and the computations above we get

$$\begin{aligned} \chi_{S_{\Gamma_1}}(x) &= (x^b-1)(x^{b+1}+1), \\ \chi_{S_{\Gamma_1'}}(x) &= (x^b-1)^2, \\ \chi_{S_{\Gamma_2}}(x) &= (x+1)^{t-1}(x^{s+t+1}+(-1)^s), \\ \chi_{S_{\Gamma_2'}}(x) &= (x+1)^{t-1}(x^{s+t}+(-1)^{s-1}). \end{aligned}$$

Hence, using Proposition 5.3.3, the polynomial can be computed as

$$\begin{aligned}
\chi_{S_Q}(x) &= \chi_{S_{\Gamma_1}}(x)\cdot\chi_{S_{\Gamma_2'}}(x)+\chi_{S_{\Gamma_1'}}(x)\cdot\chi_{S_{\Gamma_2}}(x)-(x-1)\cdot\chi_{S_{\Gamma_1'}}(x)\cdot\chi_{S_{\Gamma_2'}}(x)\\
&= (x^b-1)(x^{b+1}+1)(x+1)^{t-1}(x^{s+t}+(-1)^{s-1})\\
&\quad +(x^b-1)^2(x+1)^{t-1}(x^{s+t+1}+(-1)^s)\\
&\quad -(x-1)(x^b-1)^2(x+1)^{t-1}(x^{s+t}+(-1)^{s-1})\\
&= (x+1)^{t-1}(x^b-1)\Big(x^{s+t+b+1}+x^{s+t}+(-1)^{s-1}x^{b+1}+(-1)^{s-1}\Big)\\
&\quad +(x+1)^{t-1}(x^b-1)\Big(x^{s+t+b+1}-x^{s+t+1}+(-1)^s x^b+(-1)^{s+1}\Big)\\
&\quad +(x+1)^{t-1}(x^b-1)\Big(-x^{s+t+b+1}+x^{s+t+b}+x^{s+t+1}-x^{s+t}\\
&\quad +(-1)^s x^{b+1}+(-1)^{s-1}x^b+(-1)^{s+1}x+(-1)^s\Big)\\
&= (x+1)^{t-1}(x^b-1)(x^{s+t+b+1}+x^{s+t+b}+(-1)^{s+1}x+(-1)^{s+1})\\
&= (x+1)^t(x^b-1)(x^{s+t+b}+(-1)^{s+1}),
\end{aligned}$$

which is exactly the formula given in Proposition 5.3.12 (b) in case $s>1$.

Case 3) Let $s=1$. We can assume that $t>0$ since the polynomial for the case $t=0$ is $\chi_{S_Q}(x)=(x^b-1)(x^{b+1}+1)$ as computed in Case 1).

We can use the formula given in Corollary 5.3.9. Let Γ_1 be the quiver consisting of the vertex set $\{1,\dots,2b+1\}$. Thus, $\Gamma_1'=\Gamma_1\backslash\{2b+1\}$ is the quiver of the skeleton $\Diamond$. Let Γ_2 be the quiver consisting of the vertex set $\{2b+1,\dots,2b+2t+1\}$. With Proposition 5.3.6 and the computations at the beginning of this part of the proof we get

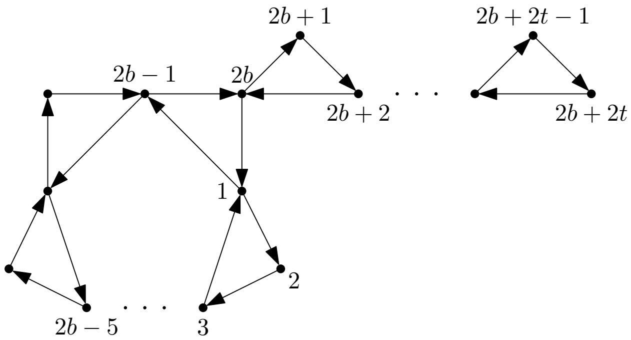
$$\begin{aligned}
\chi_{S_{\Gamma_1}}(x) &= (x^b-1)(x^{b+1}+1),\\
\chi_{S_{\Gamma_1'}}(x) &= (x^b-1)^2,\\
\chi_{S_{\Gamma_2}}(x) &= (x+1)^{t-1}(x^{t+2}-1).
\end{aligned}$$

Thus, using Corollary 5.3.9, the polynomial can be computed to be

$$\begin{aligned}
\chi_{S_Q}(x) &= (x+1)^{t-1}(x^{t+1}+1)\cdot\big(\chi_{S_{\Gamma_1}}(x)-(x-1)\cdot\chi_{S_{\Gamma_1'}}(x)\big)+\chi_{S_{\Gamma_1'}}(x)\cdot\chi_{S_{\Gamma_2}}(x)\\
&= (x+1)^{t-1}(x^{t+1}+1)\Big((x^b-1)(x^{b+1}+1)-(x-1)(x^b-1)^2\Big)\\
&\quad +(x^b-1)^2(x+1)^{t-1}(x^{t+2}-1)\\
&= (x+1)^{t-1}(x^b-1)\Big((x^{t+1}+1)\big((x^{b+1}+1)-(x-1)(x^b-1)\big)+(x^b-1)(x^{t+2}-1)\Big)\\
&= (x+1)^{t-1}(x^b-1)\Big((x^{t+1}+1)(x^b+x)+x^{t+b+2}-x^b-x^{t+2}+1\Big)\\
&= (x+1)^{t-1}(x^b-1)(x^{t+b+2}+x^{t+b+1}+x+1)\\
&= (x+1)^t(x^b-1)(x^{t+b+1}+1),
\end{aligned}$$

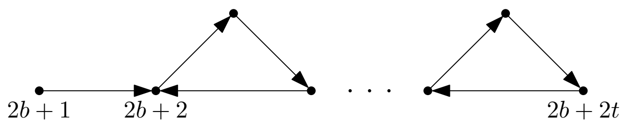
which is exactly the formula given in Proposition 5.3.12 (b) in case $s=1$.

Case 4) Now we assume that $s=0$. Up to derived equivalence, the quiver of the corresponding cluster-tilted algebra has the form



.

Let A be the quiver with vertex set $\{2b+1, 2b+2, \dots, 2b+2t\}$, i.e.



,

and corresponding matrices

$$C_A = \begin{pmatrix}
1 & 1 & 1 & 0 & \cdots & & & & & & & & 0 \\
0 & 1 & 1 & 0 & & & & & & & & & \\
0 & 0 & 1 & 1 & 1 & & & & & & & & \\
0 & 1 & 0 & 1 & 1 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & & & & & & \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & & & & \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 & 0 & 1
\end{pmatrix},$$

$$C_A^{-1} = \frac{1}{2} \cdot \begin{pmatrix}
2 & -2 & & & & & & & & & & \\
& 1 & -1 & 1 & & & & & & & & \\
& 1 & 1 & -1 & & & & & & & & \\
& -1 & 1 & 0 & -1 & 1 & & & & & & \\
& & & 1 & 1 & -1 & & & & & & \\
& & & -1 & 1 & 0 & -1 & 1 & & & & \\
& & & & & 1 & 1 & -1 & & & & \\
& & & & & -1 & 1 & 0 & -1 & 1 & & \\
& & & & & & & & \ddots & & & \\
& & & & & & & -1 & 1 & 0 & -1 & 1 \\
& & & & & & & & & 1 & 1 & -1 \\
& & & & & & & & & -1 & 1 & 1
\end{pmatrix},$$

$$S_A = \left(\begin{array}{cccccccccccc}
0 & 0 & 1 & & & & & & & & & \\
-1 & 0 & 1 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 1 & & & & & & & \\
-1 & 1 & 0 & -1 & 1 & 0 & & & & & & \\
0 & 0 & & 0 & 0 & 0 & 1 & & & & & \\
-1 & 1 & & 0 & 0 & -1 & 1 & 0 & & & & \\
0 & 0 & & & & 0 & 0 & 0 & 1 & & & \\
-1 & 1 & & & & 0 & 0 & -1 & 1 & 0 & & \\
\vdots & \vdots & & & & & & & \ddots & & & \\
-1 & 1 & & & & & & & & -1 & 1 & 0 \\
0 & 0 & & & & & & & & 0 & 0 & 1 \\
-1 & 1 & & & & & & & & 0 & 0 & 0
\end{array}\right).$$

Now we compute the Cartan matrix, its inverse and the corresponding asymmetry matrix of the cluster-tilted algebra corresponding to Q as the following block matrices:

$$C = \left(\begin{array}{cccccc|c}
 & & & & & & 0 \\
 & & & & & & \vdots \\
 & & & \mathbf{C}_{\Diamond} & & & 0 \\
 & & & & & & 1 \\
 & & & & & & 1 \\
 & & & & & & 1 \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & \\
1 & 1 & 0 & \cdots & 0 & 1 & \\
0 & 0 & 0 & \cdots & 0 & 0 & \\
1 & 1 & 0 & \cdots & 0 & 1 & \mathbf{C_A} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \\
0 & 0 & 0 & \cdots & 0 & 0 & \\
1 & 1 & 0 & \cdots & 0 & 1 &
\end{array}\right),$$

$$C^{-1} = \left(\begin{array}{cc|ccc}
\mathbf{C}_{\Diamond}^{-1} & & 0 & 0 & \\
 & & \vdots & \vdots & \mathbf{0} \\
 & & 0 & 0 & \\
 & & -\frac{1}{2} & \frac{1}{2} & \\ \hline
 & \frac{1}{2} & & & \\
 & -\frac{1}{2} & & & \\
 & 0 & & \mathbf{C}_{\mathbf{A}}^{-1} & \\
\mathbf{0} & \vdots & & & \\
 & 0 & & &
\end{array}\right) \quad \begin{array}{l} +\dfrac{1}{2}\cdot(-E_{2b,2b} - E_{2b+1,2b+1} + E_{2b+1,2b+2} \\ \qquad +E_{2b+2,2b+1} - E_{2b+2,2b+2}), \end{array}$$

$$S_Q = \left(\begin{array}{ccccc|c}
 & & & & & 0 \\
 & & & & & \vdots \\
 & & \mathbf{S}_{\Diamond} & & & 0 \\
 & & & & & 1 \\
 & & & & & 1 \\
 & & & & & 1 \\ \hline
0 & 0 & 0 & \cdots & 0 & \\
0 & 1 & 0 & \cdots & 0 & \\
0 & 0 & 0 & \cdots & 0 & \\
0 & 1 & 0 & \cdots & 0 & (*) \\
\vdots & \vdots & \vdots & & \vdots & \\
0 & 0 & 0 & \cdots & 0 & \\
0 & 1 & 0 & \cdots & 0 &
\end{array}\right) -(E_{2b-2,2b} + E_{2b-1,2b} + E_{2b,2b}),$$

$$(*) = \mathbf{S_A} + \sum_{i=1}^{t+1} E_{2i,1} - \sum_{i=1}^{t+1} E_{2i,2} = \mathbf{S_A} + \left(\begin{array}{ccc} 0 & 0 & \\ 1 & -1 & \\ 0 & 0 & \\ 1 & -1 & \\ \vdots & \vdots & \mathbf{0} \\ 0 & 0 & \\ 1 & -1 & \end{array}\right),$$

where $\mathbf{C}_\Diamond$ is the Cartan matrix of the skeleton as in (5.3.9), $\mathbf{C}_\Diamond^{-1}$ is the inverse as in (5.3.10) and $\mathbf{S}_\Diamond$ is the asymmetry matrix as in (5.3.11).

Hence, S_Q is a $(2b+2t) \times (2b+2t)$-matrix and the characteristic polynomial $\chi_{S_Q}(x)$ has the following shape:

$$\left|\begin{array}{cccccccccc|ccccccccc}
x & 0 & -1 & 0 & 0 & & & & & & 0 & & & & & & & & \\
0 & x & 0 & -1 & 0 & & & & & & 0 & & & & & & & & \\
0 & 0 & x & 0 & -1 & & & & & & 0 & & & & & & & & \\
0 & 0 & 0 & x & 0 & & & & & & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & x & & & & & & 0 & & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & & & \vdots & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & x & 0 & -1 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & x & 0 & 0 & -1 & & & & & & & & \\
-1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x & 1 & -1 & & & & & & & & \\
0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & (x+1) & -1 & & & & & & & & \\
\hline
 & 0 & & & & & & & & & x & 0 & -1 & 0 & 0 & & & & \\
 & -1 & & & & & & & & & 0 & (x+1) & -1 & 0 & 0 & & & & \\
 & 0 & & & & & & & & & 0 & 0 & x & 0 & -1 & & & & \\
 & -1 & & & & & & & & & 0 & 0 & 0 & (x+1) & -1 & & & & \\
 & 0 & & & & & & & & & 0 & 0 & 0 & 0 & x & & & & \\
 & -1 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & & & & \\
 & \vdots & & & & & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\
 & 0 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\
 & -1 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & \cdots & (x+1) & -1 & 0 \\
 & 0 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & x & -1 \\
 & -1 & & & & & & & & & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x
\end{array}\right|$$

We expand along every even column in the matrix $S_A + \sum_{i=1}^{t+1} E_{2i,1} - \sum_{i=1}^{t+1} E_{2i,2}$, i.e. in the lower right hand matrix, except for the last one, to get:

$$\chi_{S_Q}(x) = (x+1)^{t-1} \cdot \left|\begin{array}{cccccccccc|cccccccc}
x & 0 & -1 & 0 & 0 & & & & & & 0 & & & & & & & \\
0 & x & 0 & -1 & 0 & & & & & & 0 & & & & & & & \\
0 & 0 & x & 0 & -1 & & & & & & 0 & & & & & & & \\
0 & 0 & 0 & x & 0 & & & & & & 0 & & & & & & & \\
0 & 0 & 0 & 0 & x & & & & & & 0 & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & & & \vdots & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & x & 0 & -1 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & x & 0 & 0 & -1 & & & & & & & \\
-1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x & 1 & -1 & & & & & & & \\
0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & (x+1) & -1 & & & & & & & \\
\hline
 & 0 & & & & & & & & & x & -1 & 0 & 0 & & & & \\
 & 0 & & & & & & & & & 0 & x & -1 & 0 & & & & \\
 & 0 & & & & & & & & & 0 & 0 & x & -1 & & & & \\
 & 0 & & & & & & & & & 0 & 0 & 0 & x & & & & \\
 & \vdots & & & & & & & & & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\
 & 0 & & & & & & & & & 0 & 0 & 0 & 0 & \cdots & x & -1 & 0 \\
 & 0 & & & & & & & & & 0 & 0 & 0 & 0 & \cdots & 0 & x & -1 \\
 & -1 & & & & & & & & & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x
\end{array}\right|.$$

Now, we expand along the last row to get the following two $(2b+t) \times (2b+t)$-matrices:

$$
\chi_{S_Q}(x) = (x+1)^{t-1}(-1)^{2b+t+4} \cdot \left|\begin{array}{cccccccccc|ccccccc}
x & -1 & 0 & 0 & 0 & & & & & 0 & & & & & & & \\
0 & 0 & -1 & 0 & 0 & & & & & 0 & & & & & & & \\
0 & x & 0 & -1 & 0 & & & & & 0 & & & & & & & \\
0 & 0 & x & 0 & -1 & & & & & 0 & & & & & & & \\
0 & 0 & 0 & x & 0 & & & & & 0 & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & & \vdots & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & x & 0 & 0 & -1 & & & & & & & \\
-1 & 0 & 0 & 0 & 0 & \cdots & 0 & x & 1 & -1 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & (x+1) & -1 & & & & & & & \\ \hline
& & & & & & & & & x & -1 & 0 & 0 & 0 & & & \\
& & & & & & & & & 0 & x & -1 & 0 & 0 & & & \\
& & & & & & & & & 0 & 0 & x & -1 & 0 & & & \\
& & & & & & & & & 0 & 0 & 0 & x & -1 & & & \\
& & & & & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\
& & & & & & & & & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\
& & & & & & & & & 0 & 0 & 0 & 0 & 0 & \cdots & x & -1
\end{array}\right|
$$

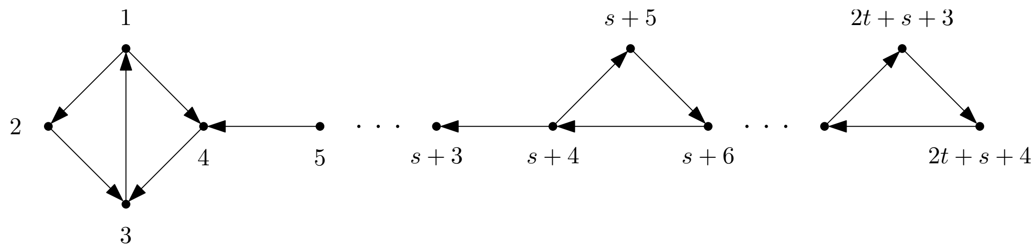
$$
+ (x+1)^{t-1} x \cdot \left|\begin{array}{cccccccccc|ccccccc}
x & 0 & -1 & 0 & 0 & & & & & & 0 & & & & & & \\
0 & x & 0 & -1 & 0 & & & & & & 0 & & & & & & \\
0 & 0 & x & 0 & -1 & & & & & & 0 & & & & & & \\
0 & 0 & 0 & x & 0 & & & & & & 0 & & & & & & \\
0 & 0 & 0 & 0 & x & & & & & & 0 & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & & & \vdots & & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & x & 0 & -1 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & x & 0 & 0 & -1 & & & & & & \\
-1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x & 1 & -1 & & & & & & \\
0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & (x+1) & -1 & & & & & & \\ \hline
& & & & & & & & & & x & -1 & 0 & 0 & & & \\
& & & & & & & & & & 0 & x & -1 & 0 & & & \\
& & & & & & & & & & 0 & 0 & x & -1 & & & \\
& & & & & & & & & & 0 & 0 & 0 & x & & & \\
& & & & & & & & & & \vdots & \vdots & \vdots & \vdots & \ddots & & \\
& & & & & & & & & & 0 & 0 & 0 & 0 & \cdots & x & -1 \\
& & & & & & & & & & 0 & 0 & 0 & 0 & \cdots & 0 & x
\end{array}\right| .
$$

Finally, we use Lemma 5.3.14 below to get

$$
\begin{aligned}
\chi_{S_Q}(x) &= (x+1)^{t-1} \cdot \Big((-1)^t \cdot (x+1)(-x^b+1) \cdot (-1)^t + x \cdot (x+1)(x^{2b-1} - x^{b-1}) \cdot x^t \Big) \\
&= (x+1)^t \cdot \Big(-x^b + 1 + x^{t+1}(x^{2b-1} - x^{b-1}) \Big) \\
&= (x+1)^t (x^b - 1)(x^{t+b} - 1),
\end{aligned}
$$

which is exactly the formula given in Proposition 5.3.12 (b) in case $s = 0$. Hence, Proposition 5.3.12 (b) is proven.

(c) Let Q be a quiver of type IV with parameters $((3, s, t))$. Using the derived equivalence classification of cluster-tilted algebras of type A (see [**23**, Theorem 5.1] or Theorem 4.2.18), we can assume that Q has the following shape



since by Proposition 5.1.2 the characteristic polynomial of the asymmetry matrix S_Q is invariant under derived equivalence.

Case 1) If $s > 1$ we can use the formula given in Proposition 5.3.3 since, with vertex $v = 5$, the ideal I is v-separated. Let Γ_1 be the quiver consisting of the vertex set $\{1, \dots, 5\}$ and the corresponding incident arrows. Thus, $\Gamma_1' = \Gamma_1 \backslash \{5\}$ is the quiver of the skeleton $((3, 0, 0))$. Let Γ_2 be the quiver consisting of the vertex set $\{5, \dots, 2t + s + 4\}$ and thus, $\Gamma_2' = \Gamma_2 \backslash \{5\}$ is a rooted quiver of type A with parameters $s - 2$ and t. Thus, using Proposition 5.3.6, we get

$$\begin{aligned} \chi_{S_{\Gamma_2}}(x) &= (x+1)^{t-1}(x^{s+t+1} + (-1)^s), \\ \chi_{S_{\Gamma_2'}}(x) &= (x+1)^{t-1}(x^{s+t} + (-1)^{s-1}). \end{aligned}$$

Since the quivers of Γ_1 and Γ_1' are small, we can compute their asymmetry matrices and the corresponding characteristic polynomials by hand to get

$$\begin{aligned} \chi_{S_{\Gamma_1}}(x) &= (x-1)(x^4 + x^3 - x^2 + x + 1), \\ \chi_{S_{\Gamma_1'}}(x) &= (x+1)(x-1)(x^2 - 1). \end{aligned}$$

Hence, a straightforward computation using Proposition 5.3.3 yields the polynomial,

$$\begin{aligned} \chi_{S_Q}(x) &= \chi_{S_{\Gamma_1}}(x) \cdot \chi_{S_{\Gamma_2'}}(x) + \chi_{S_{\Gamma_1'}}(x) \cdot \chi_{S_{\Gamma_2}}(x) - (x-1) \cdot \chi_{S_{\Gamma_1'}}(x) \cdot \chi_{S_{\Gamma_2'}}(x) \\ &= (x-1)(x^4 + x^3 - x^2 + x + 1)(x+1)^{t-1}(x^{s+t} + (-1)^{s-1}) \\ &\quad + (x+1)(x-1)(x^2-1)(x+1)^{t-1}(x^{s+t+1} + (-1)^s) \\ &\quad - (x-1)(x+1)(x-1)(x^2-1)(x+1)^{t-1}(x^{s+t} + (-1)^{s-1}) \\ &= (x-1)(x+1)^{t-1}\Big((x^4 + x^3 - x^2 + x + 1)(x^{s+t} + (-1)^{s-1}) \\ &\quad + (x+1)(x^2-1)(x^{s+t+1} + (-1)^s) - (x^2-1)^2(x^{s+t} + (-1)^{s-1})\Big) \\ &= (x-1)(x+1)^{t-1}\Big((x^3 + x^2 + x)(x^{s+t} + (-1)^{s-1}) + (x^3 + x^2 - x - 1)(x^{s+t+1} + (-1)^s)\Big) \\ &= (x-1)(x+1)^{t-1}(x^{s+t+4} + 2x^{s+t+3} + (-1)^{s-1}2x + (-1)^{s-1}). \end{aligned}$$

This is exactly the formula given in Proposition 5.3.12 (c) in case $s > 1$ (i.e. Proposition 5.3.12 (c) is proven in this case).

Case 2) Let $s = 1$. If $t = 0$, the characteristic polynomial is $\chi_{S_Q}(x) = (x-1)(x^4+x^3-x^2+x+1) = (x+1)^{-1}(x-1)(x^5+2x^4+2x+1)$. Thus, let $t > 0$. We can use the formula given in Corollary 5.3.9. Let Γ_1 be the quiver consisting of the vertex set $\{1, \dots, 5\}$ and the corresponding incident arrows. Thus, $\Gamma_1' = \Gamma_1 \backslash \{5\}$ is the quiver of the skeleton $((3, 0, 0))$. Let Γ_2 be the rooted quiver of type A consisting only of t oriented 3-cycles. As in Case 1) we have

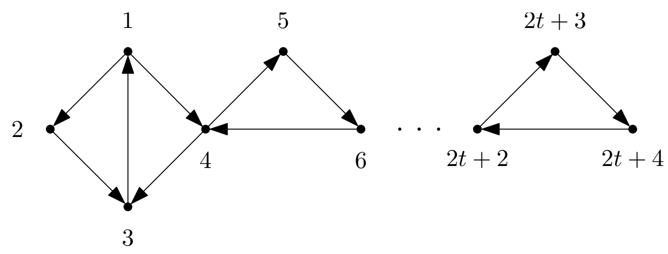
$$\begin{aligned} \chi_{S_{\Gamma_1}}(x) &= (x-1)(x^4 + x^3 - x^2 + x + 1), \\ \chi_{S_{\Gamma_1'}}(x) &= (x+1)(x-1)(x^2 - 1), \\ \chi_{S_{\Gamma_2}}(x) &= (x+1)^{t-1}(x^{t+2} - 1). \end{aligned}$$

Hence, using Corollary 5.3.9, we obtain the following expression for the polynomial:

$$\begin{aligned}
\chi_{S_Q}(x) &= (x+1)^{t-1}(x^{t+1}+1)\cdot\big(\chi_{S_{\Gamma_1}}(x)-(x-1)\cdot\chi_{S_{\Gamma_1'}}(x)\big)+\chi_{S_{\Gamma_1'}}(x)\cdot\chi_{S_{\Gamma_2}}(x)\\
&= (x+1)^{t-1}(x^{t+1}+1)\Big((x-1)(x^4+x^3-x^2+x+1)-(x-1)(x+1)(x-1)(x^2-1)\Big)\\
&\quad+(x+1)(x-1)(x^2-1)(x+1)^{t-1}(x^{t+2}-1)\\
&= (x+1)^{t-1}(x-1)\Big((x^{t+1}+1)\big((x^4+x^3-x^2+x+1)-(x^2-1)^2\big)\\
&\quad+(x+1)(x^2-1)(x^{t+2}-1)\Big)\\
&= (x+1)^{t-1}(x-1)\Big((x^{t+1}+1)(x^3+x^2+x)+(x^3+x^2-x-1)(x^{t+2}-1)\Big)\\
&= (x+1)^{t-1}(x-1)(x^{t+5}+2x^{t+4}+2x+1),
\end{aligned}$$

which is exactly the formula given in Proposition 5.3.12 (c) in case $s=1$.

Case 3) Now we assume that $s=0$. Up to derived equivalence, the quiver of the corresponding cluster-tilted algebra looks as follows



.

We denote the rooted quiver of type A which begins with vertex 4 and ends with vertex $2t+4$ by $A := A_{2t+1}$. Computing the Cartan matrix, its inverse and the corresponding asymmetry matrix of the cluster-tilted algebra corresponding to Q gives

$$C = \left(\begin{array}{ccc|cc}
1 & 1 & 1 & 1 & 1 \quad \mathbf{0} \\
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\ \hline
0 & 0 & 1 & & \\
0 & 0 & 0 & & \\
0 & 0 & 1 & & \\
\vdots & \vdots & \vdots & \multicolumn{2}{c}{\mathbf{C_A}} \\
0 & 0 & 0 & & \\
0 & 0 & 1 & &
\end{array}\right),$$

$$C^{-1} = \frac{1}{2}\cdot\left(\begin{array}{ccc|c}
1 & -1 & 1 & -1 \quad \mathbf{0} \\
1 & 1 & -1 & -1 \\
-1 & 1 & 1 & 1 \\ \hline
1 & -1 & -1 & \\
0 & 0 & 0 & \\
\vdots & \vdots & \vdots & 2\mathbf{C_A^{-1}} \\
0 & 0 & 0 &
\end{array}\right) - \frac{1}{2}E_{44},$$

$$S_Q = \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & -1 & 1 & \\ 0 & 0 & 1 & -1 & 0 & \mathbf{0} \\ 1 & 0 & 0 & 0 & 0 & \\ \hline 0 & -1 & 1 & & & \\ 0 & 0 & 0 & & & \\ 0 & -1 & 1 & & & \\ \vdots & \vdots & \vdots & & (*) & \\ 0 & 0 & 0 & & & \\ 0 & -1 & 1 & & & \end{array}\right), \ (*) = \mathbf{S_A} - \sum_{i=1}^{t+1} E_{2i-1,1} = \mathbf{S_A} - \left(\begin{array}{cc} 1 & \\ 0 & \\ 1 & \\ \vdots & \mathbf{0} \\ 0 & \\ 1 & \end{array}\right),$$

where $\mathbf{C_A}$ is the Cartan matrix of A as in (5.3.4), $\mathbf{C_A^{-1}}$ is the inverse as in (5.3.5) and $\mathbf{S_A}$ is the asymmetry matrix as in (5.3.6) (see Case 1) in the proof of Proposition 5.3.6).

Hence, S_Q is a $(2t+4)\times(2t+4)$-matrix and the characteristic polynomial $\chi_{S_Q}(x)$ has the following form:

$$\chi_{S_Q}(x) = \left|\begin{array}{ccc|ccc} x & 0 & -1 & 1 & -1 & \\ 0 & x & -1 & 1 & 0 & \mathbf{0} \\ -1 & 0 & x & 0 & 0 & \\ \hline 0 & 1 & -1 & & & \\ 0 & 0 & 0 & & & \\ 0 & 1 & -1 & & & \\ \vdots & \vdots & \vdots & & xE - S_A + \left(\begin{smallmatrix} 1 & \\ 0 & \\ 1 & \\ \vdots & \mathbf{0} \\ 0 & \\ 1 & \end{smallmatrix}\right) & \\ 0 & 0 & 0 & & & \\ 0 & 1 & -1 & & & \end{array}\right|.$$

First, we add the third column to the second column. Then we exchange column 3 and column 4 and afterwards, we add the (new) third column to the (new) fourth column to get

$$\chi_{S_Q}(x) = -\left|\begin{array}{ccc|ccc} x & -1 & 1 & 0 & -1 & \\ 0 & (x-1) & 1 & 0 & 0 & \mathbf{0} \\ -1 & x & 0 & x & 0 & \\ \hline 0 & 0 & (x+1) & & & \\ 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & & xE - S_A & \\ 0 & 0 & 0 & & & \end{array}\right|.$$

Laplace expansion along the third column leads to

$$\begin{aligned} \chi_{S_Q}(x) &= -\left|\begin{array}{cc|ccc} 0 & (x-1) & 0 & 0 & \mathbf{0} \\ -1 & x & x & 0 & \\ \hline & \mathbf{0} & & xE - S_A & \end{array}\right| + \left|\begin{array}{cc|ccc} x & -1 & 0 & -1 & \mathbf{0} \\ -1 & x & x & 0 & \\ \hline & \mathbf{0} & & xE - S_A & \end{array}\right| \\ &\quad +(x+1)\left|\begin{array}{cc|ccc} x & -1 & 0 & -1 & \\ 0 & (x-1) & 0 & 0 & \mathbf{0} \\ -1 & x & x & 0 & \\ \hline & \mathbf{0} & & \begin{array}{c} xE - S_A \\ \text{without} \\ 1^{\text{st}} \text{ row} \end{array} & \end{array}\right| \\ &= -(x-1)\chi_{S_A}(x) + (x^2-1)\chi_{S_A}(x) + (x+1)\left|\begin{array}{cc|ccc} x & -1 & 0 & -1 & \\ 0 & (x-1) & 0 & 0 & \mathbf{0} \\ -1 & x & x & 0 & \\ \hline & \mathbf{0} & & \begin{array}{c} xE - S_A \\ \text{without} \\ 1^{\text{st}} \text{ row} \end{array} & \end{array}\right|. \end{aligned}$$

Now, we add the first column to the fourth column to obtain

$$\chi_{S_Q}(x)=\chi_{S_A}(x)(-x+1+x^2-1)+(x+1)\left|\begin{array}{cc|ccc} x & -1 & 0 & (x-1) & \\ 0 & (x-1) & 0 & 0 & \mathbf{0} \\ -1 & x & x & -1 & \\ \hline & & & & xE-S_A \\ & \mathbf{0} & & & \text{without} \\ & & & & 1^{\text{st}}\text{ row} \end{array}\right|.$$

We expand along the second row to get

$$\chi_{S_Q}(x)=x(x-1)\chi_{S_A}(x)+(x+1)(x-1)\left|\begin{array}{c|ccccc} x & 0 & (x-1) & 0 & \cdots & 0 \\ \hline -1 & x & -1 & 0 & \cdots & 0 \\ 0 & & & xE-S_A & & \\ \vdots & & & \text{without} & & \\ 0 & & & 1^{\text{st}}\text{ row} & & \end{array}\right|.$$

Next, we expand along the first column. Note that the lower right hand matrix is $xE-S_A$ again (see (5.3.7) in Case 1) of the proof of Proposition 5.3.6).

$$\begin{aligned} \chi_{S_Q}(x) &= x(x-1)\chi_{S_A}(x)+(x+1)(x-1)\Bigg(x\cdot\chi_{S_A}(x)+\left|\begin{array}{ccccc} 0 & (x-1) & 0 & \cdots & 0 \\ & xE-S_A & & & \\ & \text{without} & & & \\ & 1^{\text{st}}\text{ row} & & & \end{array}\right|\Bigg) \\ &= \chi_{S_A}(x)\cdot x(x-1)(x+2)+(x+1)(x-1)\left|\begin{array}{ccccc} 0 & (x-1) & 0 & \cdots & 0 \\ & xE-S_A & & & \\ & \text{without} & & & \\ & 1^{\text{st}}\text{ row} & & & \end{array}\right| \end{aligned}$$

We compute the determinant of the last $(2t+1)\times(2t+1)$-matrix as follows (compare with the matrix (5.3.7) in Case 1) of the proof of Proposition 5.3.6):

$$\left|\begin{array}{cccccccccccc} 0 & (x-1) & 0 & & & & & & & & & \\ 0 & x & 0 & -1 & & & & & & & & \\ -1 & 0 & (x+1) & -1 & 0 & & & & & & & \\ 0 & & 0 & x & 0 & -1 & & & & & & \\ -1 & & 0 & 0 & (x+1) & -1 & 0 & & & & & \\ 0 & & & & 0 & x & 0 & -1 & & & & \\ -1 & & & & 0 & 0 & (x+1) & -1 & 0 & & & \\ 0 & & & & & & 0 & x & 0 & -1 & & \\ \vdots & & & & & & & & \ddots & \ddots & \ddots & \\ -1 & & & & & & & & & (x+1) & -1 & 0 \\ 0 & & & & & & & & & 0 & x & -1 \\ -1 & & & & & & & & & 0 & 0 & x \end{array}\right|$$

Laplace expansion along every odd column of this matrix, except for the first and the last one (i.e. along the 3^{rd}, 5^{th}, $\dots$, $(2t-1)^{\text{th}}$ column), leads to a determinant of a $(t+2)\times(t+2)$-matrix

$$(x+1)^{t-1}\cdot\left|\begin{array}{ccccccc} 0 & (x-1) & & & & & \\ 0 & x & -1 & & & & \\ 0 & & x & -1 & & & \\ 0 & & & x & -1 & & \\ \vdots & & & & \ddots & \ddots & \\ 0 & & & & & x & -1 \\ -1 & & & & & & x \end{array}\right|.$$

Now, we expand along the first column to get

$$\begin{array}{rcl} & & (x+1)^{t-1}(-1)^{t+4}\cdot \left|\begin{array}{cccccc} (x-1) & & & & & \\ x & -1 & & & & \\ & x & -1 & & & \\ & & x & -1 & & \\ & & & \ddots & \ddots & \\ & & & & x & -1 \end{array}\right| \\ & = & (x+1)^{t-1}(-1)^{t+4}(-1)^{t}(x-1) \\ & = & (x+1)^{t-1}(x-1). \end{array}$$

Altogether we get

$$\begin{array}{rcl} \chi_{S_Q}(x) & = & \chi_{S_A}(x)\cdot x(x-1)(x+2)+(x+1)(x-1)(x+1)^{t-1}(x-1) \\ & = & (x^{t+2}-1)(x+1)^{t-1}(x-1)(x^2+2x)+(x^2-1)(x+1)^{t-1}(x-1) \\ & = & (x+1)^{t-1}(x-1)(x^{t+4}+2x^{t+3}-x^2-2x+x^2-1) \\ & = & (x+1)^{t-1}(x-1)(x^{t+4}+2x^{t+3}-2x-1), \end{array}$$

which is exactly the formula given in Proposition 5.3.12 (c) in case $s=0$. Thus, Proposition 5.3.12 (c) is proven. ☐

Lemma 5.3.14. *Let $b\geq 3$. The determinants of the following $(2b)\times(2b)$-matrices are given by*

$$\text{(a)}\quad \left|\begin{array}{cccccccccc} x & -1 & 0 & 0 & 0 & & & & & \\ 0 & 0 & -1 & 0 & 0 & & & & & \\ 0 & x & 0 & -1 & 0 & & & & & \\ 0 & 0 & x & 0 & -1 & & & & & \\ 0 & 0 & 0 & x & 0 & \ddots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & x & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 0 & x & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & (x+1) & -1 \end{array}\right| = (x+1)(-x^b+1),$$

$$\text{(b)}\quad \left|\begin{array}{cccccccccc} x & 0 & -1 & 0 & 0 & & & & & \\ 0 & x & 0 & -1 & 0 & & & & & \\ 0 & 0 & x & 0 & -1 & & & & & \\ 0 & 0 & 0 & x & 0 & & & & & \\ 0 & 0 & 0 & 0 & x & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & x & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & x & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x & 1 \\ 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & (x+1) \end{array}\right| = (x+1)(x^{2b-1}-x^{b-1}).$$

PROOF. (a) First, let $\chi(x) = \left|\begin{array}{cccccccccc} x & -1 & 0 & 0 & 0 & & & & & \\ 0 & 0 & -1 & 0 & 0 & & & & & \\ 0 & x & 0 & -1 & 0 & & & & & \\ 0 & 0 & x & 0 & -1 & & & & & \\ 0 & 0 & 0 & x & 0 & \ddots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & x & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 0 & x & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & (x+1) & -1 \end{array}\right|$.

We successively expand along every even row of the original matrix, except for the last one (i.e. along the 2^{nd}, 4^{th}, …, $(2b-2)^{\text{th}}$ row). Hence, we expand $(b-1)$-times. In the first $b-2$ expansions, each non-zero entry is -1 and these -1s occur in the secondary diagonal. Hence, in the first $b-2$ expansions we get a sign $+1$. In the last expansion, the entry -1 occurs as the $(b, b+2)$-entry, giving a sign -1. Thus, we get the following $(b+1)\times(b+1)$-matrix:

$$\chi(x) = -\begin{vmatrix} x & -1 & 0 & 0 & 0 & & & & \\ 0 & x & -1 & 0 & 0 & & & & \\ 0 & 0 & x & -1 & 0 & & & & \\ 0 & 0 & 0 & x & -1 & & & & \\ 0 & 0 & 0 & 0 & x & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & x & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 0 & x & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & (x+1) \end{vmatrix}.$$

Laplace expansion along the last row yields a circulant $b\times b$-matrix with determinant (x^b-1) (see Example 5.3.5). Hence,

$$\chi(x) = -(x+1)(x^b-1) = (x+1)(-x^b+1).$$

(b) Now, let $\chi(x) = \begin{vmatrix} x & 0 & -1 & 0 & 0 & & & & & \\ 0 & x & 0 & -1 & 0 & & & & & \\ 0 & 0 & x & 0 & -1 & & & & & \\ 0 & 0 & 0 & x & 0 & & & & & \\ 0 & 0 & 0 & 0 & x & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & x & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & x & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x & 1 \\ 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & (x+1) \end{vmatrix}$.

We successively expand along the $(2b-2)^{\text{th}}$, $(2b-4)^{\text{th}}$, $\ldots$, 4^{th}, 2^{nd} row of the original matrix, i.e. we expand $(b-1)$-times, and get the $(b+1)\times(b+1)$-matrix

$$\chi(x) = x^{b-1}\cdot\begin{vmatrix} x & -1 & 0 & 0 & & & & \\ 0 & x & -1 & 0 & & & & \\ 0 & 0 & x & -1 & & & & \\ 0 & 0 & 0 & x & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & x & -1 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & x & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & (x+1) \end{vmatrix}.$$

Laplace expansion along the last row yields a circulant $b\times b$-matrix with determinant (x^b-1) (see Example 5.3.5), that is,

$$\chi(x) = x^{b-1}(x+1)(x^b-1) = (x+1)(x^{2b-1}-x^{b-1}).$$

□

Bibliography

[1] S. Al-Nofayee, *Equivalences of derived categories for selfinjective algebras*, J. Algebra 313 (2007), no. 2, 897-904

[2] H. Asashiba, *The derived equivalence classification of representation-finite selfinjective algebras*, J. Algebra 214 (1999), no. 1, 182-221

[3] I. Assem, T. Brüstle, G. Charbonneau-Jodoin, P. G. Plamondon, *Gentle algebras arising from surface triangulations*, Algebra Number Theory 4 (2010), no. 2, 201-229

[4] I. Assem, T. Brüstle, R. Schiffler, *Cluster-tilted algebras and slices*, J. Algebra 319 (2008), no. 8, 3464-3479

[5] I. Assem, T. Brüstle, R. Schiffler, *Cluster-tilted algebras as trivial extensions*, Bull. London Math. Soc. 40 (2008), no. 1, 151-162

[6] I. Assem, M. J. Redondo, *The first Hochschild cohomology group of a schurian cluster-tilted algebra*, Manuscripta Math. 128 (2009), no. 3, 373-388

[7] I. Assem, D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras, vol. 1, Techniques of representation theory*, London Math. Soc. Student Texts 65 (2006), Cambridge University Press, Cambridge

[8] M. Auslander, M. I. Platzeck, I. Reiten, *Coxeter functors without diagrams*, Trans. Amer. Math. Soc. 250 (1979), 1-46

[9] M. Auslander, I. Reiten, S. O. Smalø, *Representation theory of Artin algebras*, Cambridge Stud. Adv. Math. 36 (1997), Cambridge University Press, Cambridge

[10] D. Avella-Alaminos, C. Geiß, *Combinatorial derived invariants for gentle algebras*, J. Pure Appl. Algebra 212 (2008), no. 1, 228-243

[11] J. Bastian, *Mutation classes of $\tilde{A}_n$-quivers and derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$*, to appear in Algebra Number Theory, Preprint available at `arXiv:0901.1515`

[12] J. Bastian, T. Holm, S. Ladkani, *Derived equivalence classification of cluster-tilted algebras of Dynkin type E*, Preprint available at `arXiv:0906.3422`

[13] J. Bastian, T. Holm, S. Ladkani, *Derived equivalences for cluster-tilted algebras of Dynkin type D*, Preprint available at `arXiv:1012.4661`

[14] R. Bocian, A. Skowroński, *Weakly symmetric algebras of Euclidean type*, J. Reine Angew. Math. 580 (2005), 157-200

[15] A. Boldt, *Methods to determine Coxeter polynomials*, Linear Algebra Appl. 230 (1995), 151-164

[16] A. B. Buan, P. Caldero, B. Keller, R. J. Marsh, I. Reiten, G. Todorov, *Appendix to Clusters and seeds in acyclic cluster algebras*, Proc. Amer. Math. Soc. 135 (2007), no. 10, 3049-3060

[17] A. B. Buan, O. Iyama, I. Reiten, D. Smith, *Mutation of cluster-tilting objects and potentials*, Amer. J. Math 133 (2011), 835-887

[18] A. B. Buan, R. J. Marsh, M. Reineke, I. Reiten, G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. 204 (2006), no. 2, 572-618

[19] A. B. Buan, R. J. Marsh, I. Reiten, *Cluster mutation via quiver representations*, Coment. Math. Helvetici 83 (2008), no. 1, 143-177

[20] A. B. Buan, R. J. Marsh, I. Reiten, *Cluster-tilted algebras*, Trans. Amer. Math. Soc. 359 (2007), no. 1, 323-332

[21] A. B. Buan, R. J. Marsh, I. Reiten, *Cluster-tilted algebras of finite representation type*, J. Algebra 306 (2006), no. 2, 412-431

[22] A. B. Buan, I. Reiten, *Acyclic quivers of finite mutation type*, IMRN Art. ID 12804 (2006), no. 2, 1-10

[23] A. B. Buan, D. F. Vatne, *Derived equivalence classification for cluster-tilted algebras of type* A_n, J. Algebra 319 (2008), no. 7, 2723-2738

[24] M. C. R. Butler, C. M. Ringel, *Auslander-Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra 15 (1987), no. 1-2, 145-179

[25] P. Caldero, F. Chapoton, R. Schiffler, *Quivers with relations arising from clusters* (A_n *case)*, Trans. Amer. Math. Soc. 358 (2006), no. 3, 1347-1364

[26] P. Caldero, F. Chapoton, R. Schiffler, *Quivers with relations and cluster tilted algebras*, Algebr. Represent. Theory 9 (2006), no. 4, 359-376

[27] P. Caldero, B. Keller, *From triangulated categories to cluster algebras II*, Ann. Sci. Ecole Norm. Sup 39 (2006), 4eme serie, 983-1009

[28] H. Derksen, J. Weyman, A. Zelevinsky, *Quivers with potentials and their representations I: Mutations*, Selecta Math. (N.S.) 14 (2008), no. 1, 59-119

[29] A. Felikson, M. Shapiro, P. Tumarkin, *Skew-symmetric cluster algebras of finite mutation type*, Preprint available at `arXiv:0811.1703`

[30] S. Fomin, M. Shapiro, D. Thurston, *Cluster algebras and triangulated surfaces Part I: Cluster complexes*, Acta Math. 201 (2008), no. 1, 83-146

[31] S. Fomin, A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. 15 (2002), no. 2, 497-529

[32] S. Fomin, A. Zelevinsky, *Cluster algebras II: Finite type classification*, Invent. Math. 154 (2003), no. 1, 63-121

[33] P. Gabriel, *Unzerlegbare Darstellungen I*, Manuscripta Math. 6 (1972), no. 1, 71-103

[34] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Math. Soc. Lecture Note Series 119 (1988), Cambridge University Press, Cambridge

[35] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics 20 (1966), Springer-Verlag, Berlin - New York

[36] T. Holm, *Cartan determinants for gentle algebras*, Arch. Math. (Basel) 85 (2005), no. 3, 233-239

[37] T. Holm, *Derived categories, derived equivalences and representation theory*, Proceedings of the Summer School on "Representation theory of algebras, finite and reductive groups" , "Babes Bolyai" University Cluj (1998), 33-66

[38] T. Holm, P. Jørgensen, *Triangulated categories: definitions, properties, and examples*, Triangulated categories, 1-51, London Math. Soc. Lecture Note Series 375 (2010), Cambridge University Press, Cambridge

[39] B. Keller, *On triangulated orbit categories*, Doc. Math. 10 (2005), 551-581

[40] H. Krause, *Derived categories, resolutions, and Brown representability*, Proceedings of the Summer School on "Interactions between homotopy theory and algebra", Contemp. Math. 436 (2007), Amer. Math. Soc., Providence, RI, 101-139

[41] D. Labardini-Fragoso, *Quivers with potentials associated to triangulated surfaces*, Proc. Lond. Math. Soc. (3) 98 (2009), no. 3, 797-839

[42] S. Ladkani, *On derived equivalences of categories of sheaves over finite posets*, J. Pure Appl. Algebra 212 (2008), 435-451

[43] S. Ladkani, *Perverse equivalences, BB-tilting, mutations and applications*, Preprint available at `arXiv:1001.4765`

[44] H. Lenzing, *Coxeter transformations associated with finite-dimensional algebras*, Computational methods for representations of groups and algebras (Essen, 1997), Progr. Math. 173 (1999), Birkhäuser, Basel, 287-308

[45] G. Murphy, *Derived equivalence classification of m-cluster tilted algebras of type A*, J. Algebra 323 (2010), no. 4, 920-965

[46] A. Neeman, *Triangulated categories*, Ann. of Math. Stud. 148 (2001), Princeton University Press, Princeton, NJ

[47] J. Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra 61 (1989), 303-317

[48] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. 43 (1991), 37-48

[49] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. 39 (1989), no. 3, 436-456

[50] C. Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comment. Math. Helv. 55 (1980), no. 2, 199-224
[51] C. M. Ringel, *The self-injective cluster-tilted algebras*, Arch. Math. 91 (2008), no. 3, 218-225
[52] D. F. Vatne, *Finite type cluster-tilted algebras and cluster structures in 2-CY categories*, PhD thesis (2009), NTNU Trondheim
[53] D. F. Vatne, *The mutation class of* D_n *quivers*, Comm. Algebra 38 (2010), no. 3, 1137-1146
[54] J. L. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque 239 (1996)
[55] C. A. Weibel, *An introduction to homological algebra*, Cambridge Stud. Adv. Math. 38 (1994), Cambridge University Press, Cambridge

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J. Bastian, T. Prellberg, M. Rubey, C. Stump, *Counting the number of elements in the mutation classes of $\tilde{A}_n$-quivers*, Electronic Journal of Combinatorics 18 (2011), no. 1, P98

J. Bastian, *Mutation classes of $\tilde{A}_n$-quivers and derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$*, erscheint in: Algebra and Number Theory, arXiv:0901.1515

Preprints

J. Bastian, T. Holm, S. Ladkani, *Derived equivalences for cluster-tilted algebras of Dynkin type D*, Preprint (Dezember 2010), arXiv:1012.4661

J. Bastian, T. Holm, S. Ladkani, *Derived equivalence classification of cluster-tilted algebras of Dynkin type E*, Preprint (Juni 2009, letzte Version im Dezember 2009), arXiv:0906.3422