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Coulomb branches for rank 2 gauge groups in $3d$ $\mathcal{N}=4$ gauge theories

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Abstract: The Coulomb branch of 3-dimensional $\mathcal{N}=4$ gauge theories is the space of bare and dressed BPS monopole operators. We utilise the conformal dimension to define a fan which, upon intersection with the weight lattice of a GNO-dual group, gives rise to a collection of semi-groups. It turns out that the unique Hilbert bases of these semi-groups are a sufficient, finite set of monopole operators which generate the entire chiral ring. Moreover, the knowledge of the properties of the minimal generators is enough to compute the Hilbert series explicitly. The techniques of this paper allow an efficient evaluation of the Hilbert series for general rank gauge groups. As an application, we provide various examples for all rank two gauge groups to demonstrate the novel interpretation.

Keywords: Field Theories in Lower Dimensions, Solitons Monopoles and Instantons, Supersymmetric gauge theory

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1 Introduction

The moduli spaces of supersymmetric gauge theories with 8 supercharges have generically two branches: the Higgs and the Coulomb branch. In this paper we focus on 3-dimensional $\mathcal{N} = 4$ gauge theories, for which both branches are hyper-Kähler spaces. Despite this fact, the branches are fundamentally different.

The Higgs branch $\mathcal{M}_H$ is understood as hyper-Kähler quotient

$$\mathcal{M}_H = \mathbb{R}^{4N} /\!/\!/ \, \mathrm{G} \ , \tag{1.1}$$

in which the vanishing locus of the $\mathcal{N} = 4$ F-terms is quotient by the complexified gauge group. The F-term equations play the role of complex hyper-Kähler moment maps, while the transition to the complexified gauge group eliminates the necessity to impose the D-term constraints. Moreover, this classical description is sufficient as the Higgs branch is protected from quantum corrections. The explicit quotient construction can be supplemented by the study of the Hilbert series, which allows to gain further understanding of $\mathcal{M}_H$ as a complex space.

Classically, the Coulomb branch $\mathcal{M}_C$ is the hyper-Kähler space

$$\mathcal{M}_C \approx (\mathbb{R}^3 \times S^1)^{\mathrm{rk(G)}} / \mathcal{W}_{\mathrm{G}} \ , \tag{1.2}$$

where $\mathcal{W}_{\mathrm{G}}$ is the Weyl group of G and rk(G) denotes the rank of G. However, the geometry and topology of $\mathcal{M}_C$ are affected by quantum corrections. Recently, the understanding of the Coulomb branch has been subject of active research from various viewpoints: the authors of [1] aim to provide a description for the quantum-corrected Coulomb branch of any $3d$ $\mathcal{N} = 4$ gauge theory, with particular emphasis on the full Poisson algebra of the chiral ring $\mathbb{C}[\mathcal{M}_C]$. In contrast, a rigorous mathematical definition of the Coulomb branch itself lies at the heart of the attempts presented in [2–4]. In this paper, we take the perspective centred around the *monopole formula* proposed in [5]; that is, the computation of the Hilbert series for the Coulomb branch allows to gain information on $\mathcal{M}_C$ as a complex space.

Let us briefly recall the set-up. Select an $\mathcal{N}=2$ subalgebra in the $\mathcal{N}=4$ algebra, which implies a decomposition of the $\mathcal{N}=4$ vector multiplet into an $\mathcal{N}=2$ vector multiplet (containing a gauge field A and a real adjoint scalar σ) and an $\mathcal{N}=2$ chiral multiplet (containing a complex adjoint scalar Φ) which transforms in the adjoint representation of the gauge group G. In addition, the selection of an $\mathcal{N}=2$ subalgebra is equivalent to the choice of a complex structure on $\mathcal{M}_C$ and $\mathcal{M}_H$, which is the reason why one studies the branches only as complex and not as hyper-Kähler spaces.

The description of the Coulomb branch relies on 't Hooft monopole operators [6], which are local disorder operators [7] defined by specifying a Dirac monopole singularity

$$A_\pm \sim \frac{m}{2}\left(\pm 1 - \cos\theta\right)\mathrm{d}\varphi \tag{1.3}$$

for the gauge field, where $m \in \mathfrak{g} = \mathrm{Lie(G)}$ and (θ,φ) are coordinates on the 2-sphere around the insertion point. An important consequence is that the *generalised Dirac quantisation condition* [8]

$$\exp\left(2\pi\mathrm{i}m\right) = \mathbb{1}_{\mathrm{G}} \tag{1.4}$$

has to hold. As proven in [9], the set of solutions to (1.4) equals the weight lattice $\Lambda_w(\widehat{\mathrm{G}})$ of the GNO (or Langlands) dual group $\widehat{\mathrm{G}}$, which is uniquely associated to the gauge group G.

For Coulomb branches of supersymmetric gauge theories, the monopole operators need to be supersymmetric as well, see for instance [10]. In a pure $\mathcal{N}=2$ theory, the supersymmetry condition amounts to the singular boundary condition

$$\sigma \sim \frac{m}{2r} \qquad \text{for} \quad r \to \infty\ , \tag{1.5}$$

for the real adjoint scalar in the $\mathcal{N}=2$ vector multiplet. Moreover, an $\mathcal{N}=4$ theory also allows for a non-vanishing vacuum expectation value of the complex adjoint scalar Φ of the adjoint-valued chiral multiplet. Compatibility with supersymmetry requires Φ to take values in the stabiliser H_m of the "magnetic weight" m in G. This phenomenon gives rise to dressed monopole operators.

Dressed monopole operators and G-invariant functions of Φ are believed to generate the entire chiral ring $\mathbb{C}[\mathcal{M}_C]$. The corresponding Hilbert series allows for two points of view: seen via the *monopole formula*, each operator is precisely counted once in the Hilbert series — no over-counting appears. Evaluating the Hilbert series as rational function, however, provides an over-complete set of generators that, in general, satisfies relations. In order to count polynomials in the chiral ring, a notion of degree or dimension is required. Fortunately, in a CFT one employs the conformal dimension Δ, which for BPS states agrees with the $\mathrm{SU}(2)_R$ highest weight. Following [10–13], the conformal dimension of a BPS bare

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monopole operator of GNO-charge m is given by

$$\Delta(m) = \frac{1}{2}\sum_{i=1}^{n}\sum_{\rho\in\mathcal{R}_i}|\rho(m)| - \sum_{\alpha\in\Phi_+}|\alpha(m)| \ , \tag{1.6}$$

where $\mathcal{R}_i$ denotes the set of all weights ρ of the G-representation in which the i-th flavour of $\mathcal{N}=4$ hypermultiplets transform. Moreover, Φ_+ denotes the set of positive roots α of the Lie algebra $\mathfrak{g}$ and provides the contribution of the $\mathcal{N}=4$ vector multiplet. Bearing in mind the proposed classification of 3d $\mathcal{N}=4$ theories by [11], we restrict ourselves to "good" theories (i.e. $\Delta > \frac{1}{2}$ for all BPS monopoles).

If the centre $\mathcal{Z}(\widehat{\mathrm{G}})$ is non-trivial, then the monopole operators can be charged under this topological symmetry group and one can refine the counting on the chiral ring.

Putting all the pieces together, the by now well-established *monopole formula* of [5] reads

$$\mathrm{HS}_{\mathrm{G}}(t,z) = \sum_{m\in\Lambda_w(\widehat{\mathrm{G}})/\mathcal{W}_{\widehat{\mathrm{G}}}} z^{J(m)} t^{\Delta(m)} P_{\mathrm{G}}(t,m) \ . \tag{1.7}$$

Here, the fugacity t counts the $\mathrm{SU}(2)_R$-spin, while the (multi-)fugacity z counts the quantum numbers $J(m)$ of the topological symmetry $\mathcal{Z}(\widehat{\mathrm{G}})$.

This paper serves three purposes: firstly, we provide a geometric derivation of a sufficient set of monopole operators, called the *Hilbert basis*, that generates the entire chiral ring. Secondly, employing the Hilbert basis allows an explicit summation of (1.7), which we demonstrate for $\mathrm{rk}(\mathrm{G})=2$ explicitly. Thirdly, we provide various examples for all rank two gauge groups and display how the knowledge of the Hilbert basis completely determines the Hilbert series.

The remainder of this paper is organised as follows: section 2 is devoted to the exposition of our main points: after recapitulating basics on (root and weight) lattices and rational polyhedral cones in subsection 2.1, we explain in subsection 2.2 how the conformal dimension decomposes the Weyl chamber of $\widehat{\mathrm{G}}$ into a fan. Intersecting the fan with the weight lattice $\Lambda_w(\widehat{\mathrm{G}})$ introduces affine semi-groups, which are finitely generated by a unique set of irreducible elements — called the Hilbert basis. Moving on to subsection 2.3, we collect mathematical results that interpret the dressing factors $P_{\mathrm{G}}(t,m)$ as Poincaré series for the set of H_m-invariant polynomials on the Lie algebra $\mathfrak{h}_m$. Finally, we explicitly sum the unrefined Hilbert series in subsection 2.4 and the refined Hilbert series in 2.5 utilising the knowledge about the Hilbert basis. After establishing the generic results, we provide a comprehensive collection of examples for all rank two gauge groups in section 3–8. Lastly, section 9 concludes.

Before proceeding to the details, we present our main result (2.35) already at this stage: the refined Hilbert series for any rank two gauge group G.

$$\begin{aligned}\mathrm{HS}_{\mathrm{G}}(t,z) = \frac{P_{\mathrm{G}}(t,0)}{\prod_{p=0}^{L}\left(1-z^{J(x_p)}t^{\Delta(x_p)}\right)}\Bigg\{&\prod_{q=0}^{L}\left(1-z^{J(x_q)}t^{\Delta(x_q)}\right) \\ &+\sum_{q=0}^{L}\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)} z^{J(x_q)}t^{\Delta(x_q)}\prod_{\substack{r=0\\ r\neq q}}^{L}\left(1-z^{J(x_r)}t^{\Delta(x_r)}\right)\end{aligned} \tag{1.8}$$

$$+\sum_{q=1}^{L} \frac{P_{\mathrm{G}}(t,C_q^{(2)})}{P_{\mathrm{G}}(t,0)} \Bigg[z^{J(x_{q-1})+J(x_q)} t^{\Delta(x_{q-1})+\Delta(x_q)} + \sum_{s\in \mathrm{Int}(\mathcal{P}(C_q^{(2)}))} z^{J(s)} t^{\Delta(s)} \Bigg] \prod_{\substack{r=0\\ r\neq q-1,q}}^{L} \left(1 - z^{J(x_r)} t^{\Delta(x_r)}\right) \Bigg\} ,$$

where the ingredients can be summarised as follows:

- A fan $F_\Delta = \{C_p^{(2)}, p = 1, \dots, L\}$, and each 2-dimensional cone satisfies $\partial C_p^{(2)} = C_{p-1}^{(1)} \cup C_p^{(1)}$ and $C_{p-1}^{(1)} \cap C_p^{(1)} = \{0\}$.
- The Hilbert basis for $C_p^{(2)}$ comprises the ray generators x_{p-1}, x_p as well as other minimal generators $\{u_\kappa^p\}$.
- The x_{p-1}, x_p generate a fundamental parallelotope $\mathcal{P}(C_p^{(2)})$, where the discriminant counts the number of lattice points in the interior $\mathrm{Int}(\mathcal{P}(C_p^{(2)}))$ via $d(C_p^{(2)}) - 1 = \#\mathrm{pts.}\left(\mathrm{Int}(\mathcal{P}(C_p^{(2)}))\right)$.

The form of (1.8) is chosen to emphasis that the terms within the curly bracket represent the numerator of the Hilbert series as rational function, i.e. the curly bracket is a proper polynomial in t without poles. On the other hand, the first fraction represents the denominator of the rational function, which is again a proper polynomial by construction.

2 Hilbert basis for monopole operators

2.1 Preliminaries

Let us recall some basic properties of Lie algebras, cf. [14], and combine them with the description of *strongly convex rational polyhedral cones* and *affine semi-groups*, cf. [15]. Moreover, we recapitulate the definition and properties of the GNO-dual group, which can be found in [9, 16].

Root and weight lattices of $\mathfrak{g}$. Let G be a Lie group with semi-simple Lie algebra $\mathfrak{g}$ and $\mathrm{rk(G)} = r$. Moreover, $\widetilde{\mathrm{G}}$ is the universal covering group of G, i.e. the unique simply connected Lie group with Lie algebra $\mathfrak{g}$. Choose a maximal torus $\mathrm{T} \subset \mathrm{G}$ and the corresponding Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. Denote by $\mathbf{\Phi}$ the set of all roots $\alpha \in \mathfrak{t}^*$. By the choice of a hyperplane, one divides the root space into positive $\mathbf{\Phi}_+$ and negative roots $\mathbf{\Phi}_-$. In the half-space of positive roots one introduces the simple positive roots as irreducible basis elements and denotes their set by $\mathbf{\Phi}_s$. The roots span a lattice $\Lambda_r(\mathfrak{g}) \subset \mathfrak{t}^*$, the *root lattice*, with basis $\mathbf{\Phi}_s$.

Besides roots, one can always choose a basis in the complexified Lie algebra that gives rise to the notion of coroots $\alpha^\vee \in \mathfrak{t}$ which satisfy $\alpha(\beta^\vee) \in \mathbb{Z}$ for any $\alpha, \beta \in \mathbf{\Phi}$. Define $\alpha^\vee$ to be a simple coroot if and only if α is a simple root. Then the coroots span a lattice $\Lambda_r^\vee(\mathfrak{g})$ in $\mathfrak{t}$ — called the *coroot lattice* of $\mathfrak{g}$.

The dual lattice $\Lambda_w(\mathfrak{g})$ of the coroot lattice is the set of points $\mu \in \mathfrak{t}^*$ for which $\mu(\alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in \mathbf{\Phi}$. This lattice is called *weight lattice* of $\mathfrak{g}$. Choosing a basis $\boldsymbol{B}$ of simple coroots

$$\boldsymbol{B} := \left\{\alpha^\vee \,,\ \alpha \in \mathbf{\Phi}_s\right\} \subset \mathfrak{t} \,, \tag{2.1}$$

one readily defines a basis for the dual space via

$$\boldsymbol{B}^* := \{\lambda_\alpha \,,\ \alpha \in \mathbf{\Phi}_s\} \subset \mathfrak{t}^* \qquad \text{for} \quad \lambda_\alpha\left(\beta^\vee\right) = \delta_{\alpha,\beta} \,,\ \forall \alpha, \beta \in \mathbf{\Phi}_s \,. \tag{2.2}$$

The basis elements λ_α are precisely the fundamental weights of $\mathfrak{g}$ (or $\widetilde{\mathrm{G}}$) and they are a basis for the weight lattice.

Analogous, the dual lattice $\Lambda_{mw}(\mathfrak{g}) \subset \mathfrak{t}$ of the root lattice is the set of points $m \in \mathfrak{t}$ such that $\alpha(m) \in \mathbb{Z}$ for all $\alpha \in \mathbf{\Phi}$. In particular, the coroot lattice is a sublattice of $\Lambda_{mw}(\mathfrak{g})$.

As a remark, the lattices defined so far solely depend on the Lie algebra $\mathfrak{g}$, or equivalently on $\widetilde{G}$, but not on G. Because any group defined via $\widetilde{\mathrm{G}}/\Gamma$ for $\Gamma \subset \mathcal{Z}(\mathrm{G})$ has the same Lie algebra.

Weight and coweight lattice of G. The weight lattice of the group G is the lattice of the infinitesimal characters, i.e. a character $\chi : \mathrm{T} \to \mathrm{U}(1)$ is a homomorphism, which is then uniquely determined by the derivative at the identity. Let $X \in \mathfrak{t}$ then $\chi(\exp(X)) = \exp(i\mu(X))$, wherein $\mu \in \mathfrak{t}^*$ is an *infinitesimal character* or weight of G. The weights form then a lattice $\Lambda_w(\mathrm{G}) \subset \mathfrak{t}^*$, because the exponential map translates the multiplicative structure of the character group into an additive structure. Most importantly, the following inclusion of lattices holds:

$$\Lambda_r(\mathfrak{g}) \subset \Lambda_w(\mathrm{G}) \subset \Lambda_w(\mathfrak{g}) \,. \tag{2.3}$$

Note that the weight lattice Λ_w of $\mathfrak{g}$ equals the weight lattice of the universal cover $\widetilde{\mathrm{G}}$.

As before, the dual lattice for $\Lambda_w(\mathrm{G})$ in $\mathfrak{t}$ is readily defined

$$\Lambda_w^*(\mathrm{G}) := \mathrm{Hom}\left(\Lambda_w(\mathrm{G}), \mathbb{Z}\right) = \ker \left\{ \begin{matrix} \mathfrak{t} & \to & \mathrm{T} \\ X & \mapsto & \exp(2\pi \mathrm{i} X) \end{matrix} \right\} \,. \tag{2.4}$$

As we see, the *coweight lattice* $\Lambda_w^*(\mathrm{G})$ is precisely the set of solutions to the generalised Dirac quantisation condition (1.4) for G. In addition, an inclusion of lattices holds

$$\Lambda_r^\vee(\mathfrak{g}) \subset \Lambda_w^*(\mathrm{G}) \subset \Lambda_{mw}(\mathfrak{g}) \,, \tag{2.5}$$

which follows from dualising (2.3).

GNO-dual group and algebra. Following [9, 16], a Lie algebra $\widehat{\mathfrak{g}}$ is the *magnetic dual* of $\mathfrak{g}$ if its roots coincide with the coroots of $\mathfrak{g}$. Hence, the Weyl groups of $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ agree. The *magnetic dual group* $\widehat{\mathrm{G}}$ is, by definition, the unique Lie group with Lie algebra $\widehat{\mathfrak{g}}$ and weight lattice $\Lambda_w(\widehat{\mathrm{G}})$ equal to $\Lambda_w^*(\mathrm{G})$. In physics, $\widehat{\mathrm{G}}$ is called the GNO-dual group; while in mathematics, it is known under Langlands dual group.

Polyhedral cones. A *rational convex polyhedral cone* in $\mathfrak{t}$ is a set σ_B of the form

$$\sigma_B \equiv \mathrm{Cone}(\boldsymbol{B}) = \left\{ \sum_{\alpha^\vee \in \boldsymbol{B}} f_{\alpha^\vee} \, \alpha^\vee \mid f_{\alpha^\vee} \geq 0 \right\} \subseteq \mathfrak{t} \tag{2.6}$$

where $\boldsymbol{B} \subseteq \Lambda_r^\vee$, the basis of simple coroots, is finite. Moreover, we note that σ_B is a *strongly convex* cone, i.e. $\{0\}$ is a face of the cone, and of *maximal dimension*, i.e. $\dim(\sigma_B) = r$. Following [15], such cones σ_B are generated by the *ray generators* of their edges, where the ray generators in this case are precisely the simple coroots of $\mathfrak{g}$.

For a polyhedral cone $\sigma_B \subseteq \mathfrak{t}$ one naturally defines the *dual cone*

$$\sigma_B^\vee = \{ m \in \mathfrak{t}^* \mid m(u) \geq 0 \text{ for all } u \in \sigma_B \} \subseteq \mathfrak{t}^* \, . \tag{2.7}$$

One can prove that $\sigma_B^\vee$ equals the rational convex polyhedral cone generated by $\boldsymbol{B^*}$, i.e.

$$\sigma_B^\vee = \sigma_{B^*} = \mathrm{Cone}(\boldsymbol{B^*}) = \left\{ \sum_{\lambda \in \boldsymbol{B^*}} g_\lambda \, \lambda \mid g_\lambda \geq 0 \right\} \subseteq \mathfrak{t}^* \, , \tag{2.8}$$

which is well-known under the name *(closed) principal Weyl chamber*. By the very same arguments as above, the cone σ_{B^*} is generated by its ray generators, which are the fundamental weights of $\mathfrak{g}$.

For any $m \in \mathfrak{t}$ and $d \geq 0$, let us define an *affine hyperplane* $H_{m,d}$ and *closed linear half-spaces* $H_{m,d}^\pm$ in $\mathfrak{t}^*$ via

$$H_{m,d} \coloneqq \{ \mu \in \mathfrak{t}^* \mid \mu(m) = d \} \subseteq \mathfrak{t}^* \, , \tag{2.9a}$$

$$H_{m,d}^\pm \coloneqq \{ \mu \in \mathfrak{t}^* \mid \mu(m) \geq \pm d \} \subseteq \mathfrak{t}^* \, . \tag{2.9b}$$

If $d = 0$ then $H_{m,0}$ is hyperplane through the origin, sometimes denoted as *central* affine hyperplane. A theorem [17] then states: a cone $\sigma \subset \mathbb{R}^n$ is finitely generated if and only if it is the finite intersection of closed linear half spaces.

This result allows to make contact with the usual definition of the Weyl chamber. Since we know that σ_{B^*} is finitely generated by the fundamental weights $\{\lambda_\alpha\}$ and the dual basis is $\{\alpha^\vee\}$, one arrives at $\sigma_{B^*} = \cap_{\alpha \in \boldsymbol{\Phi}_s} H_{\alpha^\vee,0}^+$; thus, the dominant Weyl chamber is obtained by cutting the root space along the hyperplanes orthogonal to some root and selecting the cone which has only positive entries.

Remark. Consider the group SU(2), then the fundamental weight is simply $\frac{1}{2}$ such that $\Lambda_w^{\mathrm{SU}(2)} = \mathrm{Span}_{\mathbb{Z}}(\frac{1}{2}) = \mathbb{Z} \cup \{\mathbb{Z} + \frac{1}{2}\}$. Moreover, the corresponding cone (Weyl chamber) will be denoted by $\sigma_{\boldsymbol{B^*}}^{\mathrm{SU}(2)} = \mathrm{Cone}(\frac{1}{2})$.

2.2 Effect of conformal dimension

Next, while considering the conformal dimension $\Delta(m)$ as map between two Weyl chambers we will stumble across the notion of *affine semi-groups*, which are known to constitute the combinatorial background for toric varieties [15].

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Conformal dimensions — revisited. Recalling the conformal dimension Δ to be interpreted as the highest weight under $\mathrm{SU}(2)_R$, it can be understood as the following map

$$\Delta : \begin{array}{ccc} \sigma_{B^*}^{\widehat{\mathrm{G}}} \cap \Lambda_w(\widehat{\mathrm{G}}) & \to & \sigma_{\boldsymbol{B^*}}^{\mathrm{SU}(2)} \cap \Lambda_w(\mathrm{SU}(2)) \\ m & \mapsto & \Delta(m) \end{array} . \tag{2.10}$$

Where $\sigma_{B^*}^{\widehat{\mathrm{G}}}$ is the cone spanned by the fundamental weights of $\widehat{\mathfrak{g}}$, i.e. the dual basis of the simple roots $\boldsymbol{\Phi}_s$ of $\mathfrak{g}$. Likewise, $\sigma_{\boldsymbol{B^*}}^{\mathrm{SU}(2)}$ is the Weyl chamber for $\mathrm{SU}(2)_R$. Upon continuation, Δ becomes a map between the dominant Weyl chamber of $\widehat{\mathrm{G}}$ and $\mathrm{SU}(2)_R$

$$\Delta : \begin{array}{ccc} \sigma_{B^*}^{\widehat{\mathrm{G}}} & \to & \sigma_{\boldsymbol{B^*}}^{\mathrm{SU}(2)} \\ m & \mapsto & \Delta(m) \end{array} . \tag{2.11}$$

By definition, the conformal dimension (1.6) has two types of contributions: firstly, a positive contribution $|\rho(m)|$ for a weight $\rho \in \Lambda_w(\mathrm{G}) \subset \mathfrak{t}^*$ and a magnetic weight $m \in \Lambda_w(\widehat{\mathrm{G}}) \subset \widehat{\mathfrak{t}}^*$. By definition $\Lambda_w(\widehat{\mathrm{G}}) = \Lambda_w^*(\mathrm{G})$; thus, m is a coweight of G and $\rho(m)$ is the duality paring. Secondly, a negative contribution $-|\alpha(m)|$ for a positive root $\alpha \in \boldsymbol{\Phi}_+$ of $\mathfrak{g}$. By the same arguments, $\alpha(m)$ is the duality pairing of weights and coweights. The paring is also well-defined on the entire the cone.

Fan generated by conformal dimension. The individual absolute values in Δ allow for another interpretation; we use them to associate a collection of affine central hyperplanes and closed linear half-spaces

$$H_{\mu,0}^{\pm} = \left\{ m \in \mathfrak{t} \;\middle|\; \pm \mu(m) \geq 0 \right\} \subset \mathfrak{t} \quad \text{and} \quad H_{\mu,0} = \left\{ m \in \mathfrak{t} \;\middle|\; \mu(m) = 0 \right\} \subset \mathfrak{t} \,. \tag{2.12}$$

Here, μ ranges over all weights ρ and all positive roots α appearing in the theory. If two weights μ_1, μ_2 are (integer) multiples of each other, then $H_{\mu_1,0} = H_{\mu_2,0}$ and we can reduce the number of relevant weights. From now on, denote by Γ the set of weights ρ and positive roots α which are not multiples of one another. Then the conformal dimension contains $Q := |\Gamma| \in \mathbb{N}$ distinct hyperplanes such that there exist 2^Q different finitely generates cones

$$\sigma_{\epsilon_1,\epsilon_2,\dots,\epsilon_Q} := H_{\mu_1,0}^{\epsilon_1} \cap H_{\mu_2,0}^{\epsilon_2} \cap \dots \cap H_{\mu_Q,0}^{\epsilon_Q} \subset \mathfrak{t} \qquad \text{with} \quad \epsilon_i = \pm \qquad \text{for} \quad i = 1, \dots, Q \,. \tag{2.13}$$

By construction, each cone $\sigma_{\epsilon_1,\epsilon_2,\dots,\epsilon_Q}$ is a strongly convex rational polyhedral cone of dimension r, for non-trivial cones, or 0, for trivial intersections. Consequently, each cone is generated by its ray generators and these can be chosen to be lattice points of $\Lambda_w(\widehat{\mathrm{G}})$. Moreover, the restriction of Δ to any $\sigma_{\epsilon_1,\epsilon_2,\dots,\epsilon_Q}$ yields a linear function, because we effectively resolved the absolute values by defining these cones.

It is, however, sufficient to restrict the considerations to the Weyl chamber of $\widehat{\mathrm{G}}$; hence, we simply intersect the cones with the hyperplanes defining $\sigma_{B^*}^{\widehat{\mathrm{G}}}$, i.e.

$$C_p \equiv C_{\epsilon_1,\epsilon_2,\dots,\epsilon_Q} := \sigma_{\epsilon_1,\epsilon_2,\dots,\epsilon_Q} \cap \sigma_{B^*}^{\widehat{\mathrm{G}}} \qquad \text{with} \quad p = (\epsilon_1, \epsilon_2, \dots, \epsilon_Q) \,. \tag{2.14}$$

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Naturally, we would like to know for which $\mu \in \Lambda_w(\mathrm{G})$ the hyperplane $H_{\mu,0}$ intersects the Weyl chamber $\sigma_{B^*}^{\widehat{\mathrm{G}}}$ non-trivially, i.e. not only in the origin. Let us emphasis the differences of the Weyl chamber (and their dual cones) of G and $\widehat{\mathrm{G}}$:

$$\sigma_{B^*}^{\mathrm{G}} = \mathrm{Cone}\left(\lambda_\alpha \mid \lambda_\alpha(\beta^\vee) = \delta_{\alpha,\beta}\,, \forall \alpha,\beta \in \mathbf{\Phi}_s\right) \subset \mathfrak{t}^* \overset{*}{\longleftrightarrow} \sigma_B^{\mathrm{G}} = \mathrm{Cone}\left(\alpha^\vee \mid \forall \alpha \in \mathbf{\Phi}_s\right) \subset \mathfrak{t}\,, \tag{2.15a}$$

$$\sigma_{B^*}^{\widehat{\mathrm{G}}} = \mathrm{Cone}\left(m_\alpha \mid \beta(m_\alpha) = \delta_{\alpha,\beta}\,, \forall \alpha,\beta \in \mathbf{\Phi}_s\right) \subset \mathfrak{t} \quad \overset{*}{\longleftrightarrow} \sigma_B^{\widehat{\mathrm{G}}} = \mathrm{Cone}\left(\alpha \mid \forall \alpha \in \mathbf{\Phi}_s\right) \subset \mathfrak{t}^*\,. \tag{2.15b}$$

It is possible to prove the following statements:

1. If $\mu \in \mathrm{Int}\left(\sigma_B^{\widehat{\mathrm{G}}} \cup (-\sigma_B^{\widehat{\mathrm{G}}})\right)$, i.e. $\mu = \sum_{\alpha\in\mathbf{\Phi}_s} g_\alpha \alpha$ where either all $g_\alpha > 0$ *or* all $g_\alpha < 0$, then $H_{\mu,0} \cap \sigma_{B^*}^{\widehat{\mathrm{G}}} = \{0\}$.
2. If $\mu \in \partial\left(\sigma_B^{\widehat{\mathrm{G}}} \cup (-\sigma_B^{\widehat{\mathrm{G}}})\right)$ and $\mu \neq 0$, i.e. $\mu = \sum_{\alpha\in\mathbf{\Phi}_s} g_\alpha \alpha$ where at least one $g_\alpha = 0$, then $H_{\mu,0}$ intersects $\sigma_{B^*}^{\widehat{\mathrm{G}}}$ at one of its boundary faces.
3. If $\mu \notin \sigma_B^{\widehat{\mathrm{G}}} \cup (-\sigma_B^{\widehat{\mathrm{G}}})$, i.e. $\mu = \sum_{\alpha\in\mathbf{\Phi}_s} g_\alpha \alpha$ with at least one $g_\alpha > 0$ *and* at least one $g_\beta < 0$, then $\left(H_{\mu,0} \cap \sigma_{B^*}^{\widehat{\mathrm{G}}}\right) \setminus \{0\} \neq \emptyset$.

> *Consequently, a weight $\mu \in \Lambda_w(\mathrm{G})$ appearing in Δ leads to a hyperplane intersecting the Weyl chamber of $\widehat{\mathrm{G}}$ non-trivially if and only if neither μ nor $-\mu$ lies in the rational cone spanned by the simple roots $\mathbf{\Phi}_s$ of* G.

Therefore, the contributions $-|\alpha(m)|$, for $\alpha \in \mathbf{\Phi}_+$, of the vector multiplet never yield a relevant hyperplane. From now on, assume that trivial cones C_p are omitted in the index set I for p. The appropriate geometric object to consider is then the *fan* $F_\Delta \subset \mathfrak{t}$ defined by the family $F_\Delta = \{C_p\,,\ p \in I\}$ in $\mathfrak{t}$. A fan F is a family of non-empty polyhedral cones such that (i) every non-empty face of a cone in F is a cone in F and (ii) the intersection of any two cones in F is a face of both. In addition, the fan F_Δ defined above is a *pointed fan*, because $\{0\}$ is a cone in F_Δ (called the trivial cone).

Semi-groups. Although we already know the cone generators for the fan F_Δ, we have to distinguish them from the generators of $F_\Delta \cap \Lambda_w(\widehat{\mathrm{G}})$, i.e. we need to restrict to the weight lattice of $\widehat{\mathrm{G}}$. The first observation is that

$$S_p \coloneqq C_p \cap \Lambda_w(\widehat{\mathrm{G}}) \qquad \text{for} \quad p \in I \tag{2.16}$$

are semi-groups, i.e. sets with an associative binary operation. This is because the addition of elements is commutative, but there is no inverse defined as "subtraction" would lead out of the cone. Moreover, the S_p satisfy further properties, which we now simply collect, see for instance [17]. Firstly, the S_p are affine semi-groups, which are semi-groups that can be embedded in $\mathbb{Z}^n$ for some n. Secondly, every S_p possesses an identity element, here $m = 0$, and such semi-groups are called *monoids*. Thirdly, the S_p are *positive* because the only invertible element is $m = 0$.

Now, according to *Gordan's Lemma* [15, 17], we know that every S_p is finitely generated, because all C_p's are finitely generated, rational polyhedral cones. Even more is true, since the division into the C_p is realised via affine hyperplanes $H_{\mu_i,0}$ passing through the origin, the C_p are strongly convex rational cones of maximal dimension. Then [15, Prop. 1.2.22.] holds and we know that there exist a unique minimal generating set for S_p, which is called *Hilbert basis*.

The Hilbert basis $\mathcal{H}(S_p)$ is defined via

$$\mathcal{H}(S_p) \coloneqq \{m \in S_p \mid m \text{ is irreducible}\} \ , \tag{2.17}$$

where an element is called *irreducible* if and only if $m = x + y$ for $x, y \in S_p$ implies $x = 0$ or $y = 0$. The importance of the Hilbert basis is that it is a unique, finite, minimal set of irreducible elements that generate S_p. Moreover, $\mathcal{H}(S_p)$ always contains the ray generators of the edges of C_p. The elements of $\mathcal{H}(S_p)$ are sometimes called *minimal generators*.

As a remark, there exist various algorithms for computing the Hilbert basis, which are, for example, discussed in [18, 19]. For the computations presented in this paper, we used the `Sage` module *Toric varieties* programmed by A. Novoseltsev and V. Braun as well as the `Macaulay2` package *Polyhedra* written by René Birkner.

After the exposition of the idea to employ the conformal dimension to define a fan in the Weyl chamber of $\widehat{\mathrm{G}}$, for which the intersection with the weight lattice leads to affine semi-groups, we now state the main consequence:

> *The collection $\{\mathcal{H}(S_p)\ , p \in I\}$ of all Hilbert bases is the set of necessary (bare) monopole operators for a theory with conformal dimension Δ.*

At this stage we did not include the Casimir invariance described by the dressing factors $P_G(t, m)$. For a generic situation, the bare and dressed monopole operators for a GNO-charge $m \in \mathcal{H}(S_p)$ for some p are all necessary generators for the chiral ring $\mathbb{C}[\mathcal{M}_C]$. However, there will be scenarios for which there exists a further reduction of the number of generators. For those cases, we will comment and explain the cancellations.

2.3 Dressing of monopole operators

One crucial ingredient of the monopole formula of [5] are the dressing factors $P_{\mathrm{G}}(t, m)$ and this section provides an algebraic understanding. We refer to [14, 20, 21] for the exposition of the mathematical details used here.

It is known that in $\mathcal{N} = 4$ the $\mathcal{N} = 2$ BPS-monopole operator V_m is compatible with a constant background of the $\mathcal{N} = 2$ adjoint complex scalar Φ, provided Φ takes values on the Lie algebra $\mathfrak{h}_m$ of the residual gauge group $\mathrm{H}_m \subset \mathrm{G}$, i.e. the stabiliser of m in G. Consequently, each bare monopole operator V_m is compatible with *any* H_m-invariant polynomial on $\mathfrak{h}_m$. We will now argue that the dressing factors $P_{\mathrm{G}}(t, m)$ are to be understood as Hilbert (or Poincaré) series for this so-called *Casimir-invariance*.

Chevalley-Restriction Theorem. Let G be a Lie group of rank l with a semi-simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and G acts via the adjoint representation on $\mathfrak{g}$. Denote by $\mathfrak{P}(\mathfrak{g})$ the algebra of all polynomial functions on $\mathfrak{g}$. The action of G extends to $\mathfrak{P}(\mathfrak{g})$ and $\mathfrak{I}(\mathfrak{g})^{\mathrm{G}}$ denotes the set of G-invariant polynomials in $\mathfrak{P}(\mathfrak{g})$. In addition, denote by $\mathfrak{P}(\mathfrak{h})$ the algebra of all polynomial functions on $\mathfrak{h}$. The Weyl group $\mathcal{W}_{\mathrm{G}}$, which acts naturally on $\mathfrak{h}$, acts also on $\mathfrak{P}(\mathfrak{h})$ and $\mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}}$ denotes the Weyl-invariant polynomials on $\mathfrak{h}$. The *Chevalley-Restriction Theorem* now states

$$\mathfrak{I}(\mathfrak{g})^{\mathrm{G}} \cong \mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}} \,, \tag{2.18}$$

where the isomorphism is given by the restriction map $p \mapsto p|_{\mathfrak{h}}$ for $p \in \mathfrak{I}(\mathfrak{g})^{\mathrm{G}}$.

Therefore, the study of H_m-invariant polynomials on $\mathfrak{h}_m$ is reduced to $\mathcal{W}_{\mathrm{H}_m}$-invariant polynomials on a Cartan subalgebra $\mathfrak{t}_m \subset \mathfrak{h}_m$.

Finite reflection groups. It is due to a theorem by Chevalley [22], in the context of *finite reflection groups*, that there exist l algebraically independent homogeneous elements $p_1, \dots, p_l$ of positive degrees d_i, for $i = 1, \dots, l$, such that

$$\mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}} = \mathbb{C}\left[p_1, \dots, p_l\right] \,. \tag{2.19}$$

In addition, the degrees d_i satisfy

$$|\mathcal{W}_{\mathrm{G}}| = \prod_{i=1}^{l} d_i \quad \text{and} \quad \sum_{i=1}^{d} (d_i - 1) = \text{number of reflections in } \mathcal{W}_{\mathrm{G}} \,. \tag{2.20}$$

The degrees d_i are unique [21] and tabulated for all Weyl groups, see for instance [21, section 3.7]. However, the generators p_i are themselves not uniquely determined.

Poincaré or Molien series. On the one hand, the Poincaré series for the $\mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}}$ is simply given by

$$P_{\mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}}}(t) = \prod_{i=1}^{l} \frac{1}{1 - t^{d_i}} \,. \tag{2.21}$$

On the other hand, since $\mathfrak{h}$ is a l-dimensional complex vector space and $\mathcal{W}_{\mathrm{G}}$ a finite group, the generating function for the invariant polynomials is known as Molien series [23]

$$P_{\mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}}}(t) = \frac{1}{|\mathcal{W}_{\mathrm{G}}|} \sum_{g \in \mathcal{W}_{\mathrm{G}}} \frac{1}{\det\left(\mathbb{1} - t\ g\right)} \,. \tag{2.22}$$

Therefore, the dressing factors $P_{\mathrm{G}}(t,m)$ in the Hilbert series (1.7) for the Coulomb branch are the Poincaré series for graded algebra of H_m-invariant polynomials on $\mathfrak{h}_m$.

Harish-Chandra isomorphism. In [5], the construction of the $P_{\mathrm{G}}(t,m)$ is based on Casimir invariants of G and H_m; hence, we need to make contact with that idea. Casimir invariants live in the centre $\mathcal{Z}(\mathfrak{U}(\mathfrak{g}))$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. Fortunately, the Harish-Candra isomorphism [24] provides us with

$$\mathcal{Z}(\mathfrak{U}(\mathfrak{g})) \cong \mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}} \,. \tag{2.23}$$

Consequently, $\mathcal{Z}(\mathfrak{U}(\mathfrak{g}))$ is a polynomial algebra with l algebraically independent homogeneous elements that have the same positive degrees d_i as the generators of $\mathfrak{I}(\mathfrak{h})^{\mathcal{W}_{\mathrm{G}}}$. It is known that for semi-simple groups G these generators can be chosen to be the rk(G) Casimir invariants; i.e. the space of Casimir-invariants is freely generated by l generators (together with the unity).

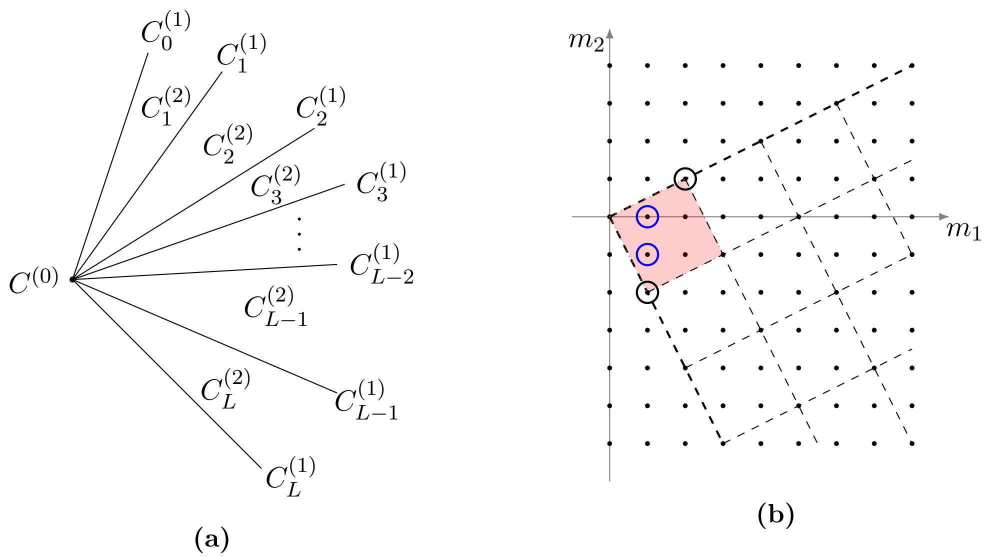

Figure 1. A representative fan, which is spanned by the 2-dim. cones $C_p^{(2)}$ for $p = 1, \ldots, L$, is displayed in 1a. In addition, 1b contains a 2-dim. cone with a Hilbert basis of the two ray generators (black) and two additional minimal generators (blue). The ray generators span the fundamental parallelotope (red region).

Conclusions. So far, G (and H_m) had been restricted to be semi-simple. However, in most cases H_m is a direct product group of semi-simple Lie groups and U(1)-factors. We proceed in two steps: firstly, U(1) acts trivially on its Lie-algebra $\cong \mathbb{R}$, thus all polynomials are invariant and we obtain

$$\mathfrak{I}(\mathbb{R})^{\mathrm{U}(1)} = \mathbb{R}[x] \quad \text{and} \quad P_{\mathrm{U}(1)}(t) = \frac{1}{1-t} \,. \tag{2.24}$$

Secondly, each factor G_i of a direct product $\mathrm{G}_1 \times \cdots \times \mathrm{G}_M$ acts via the adjoint representation on on its own Lie algebra $\mathfrak{g}_i$ and trivially on all other $\mathfrak{g}_j$ for $j \neq i$. Hence, the space of $\mathrm{G}_1 \times \cdots \times \mathrm{G}_M$-invariant polynomials on $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_M$ factorises into the product of the $\mathfrak{I}(\mathfrak{g}_i)^{\mathrm{G}_i}$ such that

$$\mathfrak{I}(\oplus_i \mathfrak{g}_i)^{\prod_i \mathrm{G}_i} = \prod_i \mathfrak{I}(\mathfrak{g}_i)^{\mathrm{G}_i} \quad \text{and} \quad P_{\mathfrak{I}(\oplus_i \mathfrak{g}_i)^{\prod_i \mathrm{G}_i}}(t) = \prod_i P_{\mathfrak{I}(\mathfrak{g}_i)^{\mathrm{G}_i}}(t) \,. \tag{2.25}$$

For abelian groups G, the Hilbert series for the Coulomb branch factorises in the Poincaré series G-invariant polynomials on $\mathfrak{g}$ times the contribution of the (bare) monopole operators. In contrast, the Hilbert series does not factorise for non-abelian groups G as the stabiliser $\mathrm{H}_m \subset \mathrm{G}$ depends on m.

2.4 Consequences for unrefined Hilbert series

The aforementioned dissection of the Weyl chamber $\sigma_{B*}^{\widehat{\mathrm{G}}}$ into a fan, induced by the conformal dimension Δ, and the subsequent collection of semi-groups in $\Lambda_w(\widehat{\mathrm{G}})/\mathcal{W}_{\widehat{\mathrm{G}}}$ provides an immediate consequence for the unrefined Hilbert series. For simplicity, we illustrate the consequences for a rank two example. Assume that the Weyl chamber is divided into a fan generated the 2-dimensional cones $C_p^{(2)}$ for $p = 1, \ldots, L$, as sketched in figure 1b. For each cone, one has two 1-dimensional cones $C^{(1)}_{p-1}$, $C^{(1)}_p$ and the trivial cone $C^{(0)} = \{0\}$ as boundary, i.e. $\partial C^{(2)}_p = C^{(1)}_{p-1} \cup C^{(1)}_p$, where $C^{(1)}_{p-1} \cap C^{(1)}_p = C^{(0)}$.

The Hilbert basis $\mathcal{H}(S^{(2)}_p)$ for $S^{(2)}_p \coloneqq C^{(2)}_p \cap \Lambda^{\widehat{\mathrm{G}}}_w$ contains the ray generators $\{x_{p-1}, x_p\}$, such that $\mathcal{H}(S^{(1)}_p) = \{x_p\}$, and potentially other minimal generators u^p_κ for κ in some *finite* index set. Although any element $s \in S^{(2)}_p$ can be generated by $\{x_{p-1}, x_p, \{u^p_\kappa\}_\kappa\}$, the representation $s = a_0 x_{p-1} + a_1 x_p + \sum_\kappa b_\kappa u^p_\kappa$ is *not* unique. Therefore, great care needs to be taken if one would like to sum over all elements in $S^{(2)}_p$. A possible realisation employs the *fundamental parallelotope*

$$\mathcal{P}(C^{(2)}_p) \coloneqq \{a_0 x_{p-1} + a_1 x_p \mid 0 \geq a_0, a_1 \geq 1\} \,, \tag{2.26}$$

see also figure 1b. The number of points contained in $\mathcal{P}(C^{(2)}_p)$ is computed by the discriminant

$$d(C^{(2)}_p) \coloneqq |\det(x_{p-1}, x_p)| \,. \tag{2.27}$$

However, as known from solid state physics, the discriminant counts each of the four boundary lattice points by $\frac{1}{4}$; thus, there are $d(C^{(2)}_p) - 1$ points in the interior. Remarkably, each point $s \in \mathrm{Int}(\mathcal{P}(C^{(2)}_p))$ is given by positive integer combinations of the $\{u^p_\kappa\}_\kappa$ alone. A translation of $\mathcal{P}(C^{(2)}_p)$ by non-negative integer combinations of the ray-generators $\{x_{p-1}, x_p\}$ fills the entire semi-group $S^{(2)}_p$ and each point is only realised once.

Now, we employ this fact to evaluate the un-refined Hilbert series explicitly.

$$\begin{aligned}
\mathrm{HS}_{\mathrm{G}}(t) &= \sum_{m \in \Lambda_w(\widehat{\mathrm{G}})/\mathcal{W}_{\widehat{\mathrm{G}}}} t^{\Delta(m)} P_{\mathrm{G}}(t,m) \\
&= P_{\mathrm{G}}(t,0) + \sum_{p=0}^{L} P_{\mathrm{G}}(t,x_p) \sum_{n_p>0} t^{n_p \Delta(x_p)} \\
&\quad + \sum_{p=1}^{L} \sum_{n_{p-1},n_p>0} P_{\mathrm{G}}(t, x_{p-1}+x_p) t^{\Delta(n_{p-1}x_{p-1}+n_p x_p)} \\
&\quad + \sum_{p=1}^{L} \sum_{s \in \mathrm{Int}(\mathcal{P}(C^{(2)}_p))} \sum_{n_{p-1},n_p \geq 0} P_{\mathrm{G}}(t,s) t^{\Delta(s+n_{p-1}x_{p-1}+n_p x_p)} \\
&= P_{\mathrm{G}}(t,0) + \sum_{p=0}^{L} P_{\mathrm{G}}(t,x_p) \frac{t^{\Delta(x_p)}}{1-t^{\Delta(x_p)}} + \sum_{p=1}^{L} \frac{P_{\mathrm{G}}(t,x_{p-1}+x_p)\; t^{\Delta(x_{p-1})+\Delta(x_p)}}{\left(1-t^{\Delta(x_{p-1})}\right)\left(1-t^{\Delta(x_p)}\right)} \\
&\quad + \sum_{p=1}^{L} \sum_{s \in \mathrm{Int}(\mathcal{P}(C^{(2)}_p))} \frac{P_{\mathrm{G}}(t,s)\; t^{\Delta(s)}}{\left(1-t^{\Delta(x_{p-1})}\right)\left(1-t^{\Delta(x_p)}\right)} \\
&= \frac{P_{\mathrm{G}}(t,0)}{\prod_{p=0}^{L}\left(1-t^{\Delta(x_p)}\right)} \left\{ \prod_{q=0}^{L} \left(1-t^{\Delta(x_q)}\right) + \sum_{q=0}^{L} \frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)} t^{\Delta(x_q)} \prod_{\substack{r=0 \\ r\neq q}}^{L} \left(1-t^{\Delta(x_r)}\right) \right. \\
&\quad \left. + \sum_{q=1}^{L} \frac{P_{\mathrm{G}}(t,C^{(2)}_q)}{P_{\mathrm{G}}(t,0)} \left[t^{\Delta(x_{q-1})+\Delta(x_q)} + \sum_{s \in \mathrm{Int}(\mathcal{P}(C^{(2)}_q))} t^{\Delta(s)} \right] \prod_{\substack{r=0 \\ r\neq q-1,q}}^{L} \left(1-t^{\Delta(x_r)}\right) \right\} .
\end{aligned} \tag{2.28}$$

Next, we utilise that the classical dressing factors, for rank two examples, only have three different values: in the (2-dim.) interior of the Weyl chamber W, the residual gauge group is the maximal torus T and $P_{\mathrm{G}}(t, \mathrm{Int}W) \equiv P_2(t) = \prod_{i=1}^{2} \frac{1}{(1-t)}$. Along the 1-dimensional boundaries, the residual gauge group is a non-abelian subgroup H such that $\mathrm{T} \subset \mathrm{H} \subset \mathrm{G}$ and the $P_{\mathrm{G}}(t, \partial W \setminus \{0\}) \equiv P_1(t) = \prod_{i=1}^{2} \frac{1}{(1-t^{b_i})}$, for the two degree b_i Casimir invariants of H. At the (0-dim.) boundary of the boundary, the group is unbroken and $P_{\mathrm{G}}(t,0) \equiv P_0(t) = \prod_{i=1}^{2} \frac{1}{(1-t^{d_i})}$ contains the Casimir invariants of G of degree d_i. Thus, there are a few observations to be addressed.

1. The numerator of (2.28), which is everything in the curly brackets $\{\dots\}$, starts with a one and is a polynomial with integer coefficients, which is required for consistency.

2. The denominator of (2.28) is given by $P_{\mathrm{G}}(t,0)/\prod_{p=0}^{L}(1-t^{\Delta(x_p)})$ and describes the poles due to the Casimir invariants of G and the bare monopole $(x_p, \Delta(x_p))$ which originate from ray generators x_p.

3. The numerator has contributions $\sim t^{\Delta(x_p)}$ for the ray generators with pre-factors $\frac{P_1(t)}{P_0(t)} - 1$ for the two outermost rays $p=0$, $p=L$ and pre-factors $\frac{P_2(t)}{P_0(t)} - 1$ for the remaining ray generators. None of the two pre-factors has a constant term as $P_i(t \to 0) = 1$ for each $i = 0, 1, 2$. Also $\deg(1/P_0(t)) \geq \deg(1/P_1(t)) \geq \deg(1/P_2(t)) = 2$ and
$$\frac{P_2(t)}{P_0(t)} = \frac{(1-t^{d_1})(1-t^{d_2})}{(1-t)(1-t)} = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} t^{i+j} \tag{2.29}$$
is a polynomial for any rank two group. For the examples considered here, we also obtain
$$\frac{P_1(t)}{P_0(t)} = \frac{(1-t^{d_1})(1-t^{d_2})}{(1-t^{b_1})(1-t^{b_2})} = \frac{(1-t^{k_1 b_1})(1-t^{k_2 b_2})}{(1-t^{b_1})(1-t^{b_2})} = \sum_{i=0}^{b_1-1} \sum_{j=0}^{b_2-1} t^{i \cdot k_1 + j \cdot k_2} \tag{2.30}$$
for some $k_1, k_2 \in \mathbb{N}$. In summary, $(\frac{P_{\mathrm{G}}(t,x_p)}{P_{\mathrm{G}}(t,0)} - 1) t^{\Delta(x_p)}$ describes the dressed monopole operators corresponding to the ray generators x_p.

4. The finite sums $\sum_{s \in \mathrm{Int}(\mathcal{P}(C_p^{(2)}))} t^{\Delta(s)}$ are entirely determined by the conformal dimensions of the minimal generators u_κ^p.

5. The first contributions for the minimal generators u_κ^p are of the form
$$\frac{P_2(t)}{P_0(t)} t^{\Delta(u_\kappa^p)} = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_2-1} t^{i+j+\Delta(u_\kappa^p)} \,, \tag{2.31}$$
which then comprise the bare and the dressed monopole operators simultaneously.

6. If $C_p^{(2)}$ is simplicial, i.e. $\mathcal{H}(S_p^{(2)}) = \{x_{p-1}, x_p\}$, then the sum over $s \in \mathrm{Int}(\mathcal{P}(C_p^{(2)}))$ in (2.28) is zero, as the interior is empty. Also indicated by $d(C_p^{(2)}) = 1$.

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In conclusion, the Hilbert series (2.28) suggests that *ray generators* are to be expected in the denominator, while other minimal generators are manifest in the numerator. Moreover, the entire Hilbert series is determined by a *finite* set of numbers: the conformal dimensions of the minimal generators $\{\Delta(x_p) \mid p=0,1,\dots,L\}$ and $\{\{\Delta(u_\kappa^{(p)}) \mid \kappa=1,\dots,d(C_p^{(2)})-1\} \mid p=1,\dots,L\}$ as well as the classical dressing factors.

Moreover, the dressing behaviour, i.e. number and degree, of a minimal generator m is described by the quotient $P_{\mathrm{G}}(t,m)/P_{\mathrm{G}}(t,0)$. Consolidating evidence for this statement comes from the analysis of the plethystic logarithm, which we present in appendix A. Together, the Hilbert series and the plethystic logarithm allow a better understanding of the chiral ring.

We illustrate the formula (2.28) for the two simplest cases in order to hint on the differences that arise if $d(C_p^{(2)})>1$ for cones within the fan.

Example: one simplicial cone Adapting the result (2.28) to one cone $C_1^{(2)}$ with cone/Hilbert basis $\{x_0,x_1\}$, we find

$$\mathrm{HS} = \frac{1+\left(\frac{P_1(t)}{P_0(t)}-1\right)\left(t^{\Delta(x_0)}+t^{\Delta(x_1)}\right)+\left(1-2\frac{P_1(t)}{P_0(t)}+\frac{P_2(t)}{P_0(t)}\right)t^{\Delta(x_0)+\Delta(x_1)}}{\prod_{i=1}^{2}\left(1-t^{d_i}\right)\prod_{p=0}^{1}\left(1-t^{\Delta(x_p)}\right)}\,. \tag{2.32}$$

Examples treated in this paper are as follows: firstly, the representation $[2,0]$ for the quotients $\mathrm{Spin}(4)$, $\mathrm{SO}(3)\times\mathrm{SU}(2)$, $\mathrm{SU}(2)\times\mathrm{SO}(3)$, $\mathrm{PSO}(4)$ of section 5.2; secondly, $\mathrm{USp}(4)$ for the case $N_3=0$ of section 6.5; thirdly, G_2 in the representations $[1,0]$, $[0,1]$ and $[2,0]$ of section 7.2. The corresponding expression for the plethystic logarithm is provided in (A.14).

Example: one non-simplicial cone Adapting the result (2.28) to one cone $C_1^{(2)}$ with Hilbert basis $\{x_0,x_1,\{u_\kappa\}\}$, fundamental parallelotope $\mathcal{P}$, and discriminant $d>1$, we find

$$\mathrm{HS} = \frac{1+\left(\frac{P_1(t)}{P_0(t)}-1\right)\left(t^{\Delta(x_0)}+t^{\Delta(x_1)}\right)+\left(1-2\frac{P_1(t)}{P_0(t)}+\frac{P_2(t)}{P_0(t)}\right)t^{\Delta(x_0)+\Delta(x_1)}+\frac{P_2(t)}{P_0(t)}\sum_{s\in\mathrm{Int}(\mathcal{P})}t^{\Delta(s)}}{\prod_{i=1}^{2}\left(1-t^{d_i}\right)\prod_{p=0}^{1}\left(1-t^{\Delta(x_p)}\right)}\,. \tag{2.33}$$

An example for this case is $\mathrm{SO}(4)$ with representation $[2,0]$ treated in section 5.2. For the plethystic logarithm we refer to (A.15).

The difference between (2.32) and (2.33) lies in the finite sum added in the numerator which accounts for the minimal generators that are not ray generators.

2.5 Consequences for refined Hilbert series

If the centre $\mathcal{Z}(\widehat{\mathrm{G}})$ of the GNO-dual group $\widehat{\mathrm{G}}$ is a non-trivial Lie-group of rank $\mathrm{rk}(\mathcal{Z}(\widehat{\mathrm{G}}))=\rho$, one introduces additional fugacities $\vec{z}\equiv(z_i)$ for $i=1,\dots,\rho$ such that the Hilbert series counts operators according to $\mathrm{SU}(2)_R$-spin $\Delta(m)$ and topological charges $\vec{J}(m)\equiv(J_i(m))$ for $i=1,\dots,\rho$. Let us introduce the notation

$$\vec{z}^{\vec{J}(m)} := \prod_{i=1}^{\rho} z_i^{J_i(m)} \quad \text{such that} \quad \vec{z}^{\vec{J}(m_1+m_2)} = \vec{z}^{\vec{J}(m_1)+\vec{J}(m_2)} = \vec{z}^{\vec{J}(m_1)}\cdot\vec{z}^{\vec{J}(m_2)}\,, \tag{2.34}$$

where we *assumed* each component $J_i(m)$ to be a linear function in m. By the very same arguments as in (2.28), one can evaluate the refined Hilbert series explicitly and obtains

$$
\begin{aligned}
\mathrm{HS}_{\mathrm{G}}(t,\vec{z}) &= \sum_{m\in\Lambda_w^{\widehat{\mathrm{G}}}/\mathcal{W}_{\widehat{\mathrm{G}}}} \vec{z}^{\vec{J}(m)} t^{\Delta(m)} P_{\mathrm{G}}(t,m) \\
&= \frac{P_{\mathrm{G}}(t,0)}{\prod_{p=0}^{L}\left(1-\vec{z}^{\vec{J}(x_p)}t^{\Delta(x_p)}\right)} \left\{ \prod_{q=0}^{L}\left(1-\vec{z}^{\vec{J}(x_q)}t^{\Delta(x_q)}\right)\right. \\
&\quad + \sum_{q=0}^{L} \frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)} \vec{z}^{\vec{J}(x_q)}t^{\Delta(x_q)} \prod_{\substack{r=0\\ r\neq q}}^{L}\left(1-\vec{z}^{\vec{J}(x_r)}t^{\Delta(x_r)}\right) \\
&\quad + \sum_{q=1}^{L} \frac{P_{\mathrm{G}}(t,C_q^{(2)})}{P_{\mathrm{G}}(t,0)} \left[\vec{z}^{\vec{J}(x_{q-1})+\vec{J}(x_q)} t^{\Delta(x_{q-1})+\Delta(x_q)} \right. \\
&\qquad\qquad \left.\left. + \sum_{s\in\mathrm{Int}(\mathcal{P}(C_q^{(2)}))} \vec{z}^{\vec{J}(s)}t^{\Delta(s)} \right] \prod_{\substack{r=0\\ r\neq q-1,q}}^{L}\left(1-\vec{z}^{\vec{J}(x_r)}t^{\Delta(x_r)}\right) \right\} .
\end{aligned}
\tag{2.35}
$$

The interpretation of the refined Hilbert series (2.35) remains the same as before: the minimal generators, i.e. their GNO-charge, $\mathrm{SU}(2)_R$-spin, topological charges $\vec{J}$, and their dressing factors, completely determine the Hilbert series. In principle, this data makes the (sometimes cumbersome) explicit summation of (1.7) obsolete.

3 Case: $\mathbf{U(1)\times U(1)}$

In this section we analyse the abelian product $\mathrm{U}(1)\times\mathrm{U}(1)$. By construction, the Hilbert series simplifies as the dressing factors are constant throughout the lattice of magnetic weights. Consequently, abelian theories do not exhibit dressed monopole operators.

3.1 Set-up

The weight lattice of the GNO-dual of $\mathrm{U}(1)$ is simply $\mathbb{Z}$ and no Weyl-group exists due the abelian character; thus, $\Lambda_w(\widehat{\mathrm{U}(1)\times\mathrm{U}(1)})=\mathbb{Z}^2$. Moreover, since $\mathrm{U}(1)\times\mathrm{U}(1)$ is abelian the classical dressing factors are the same for any magnetic weight (m_1,m_2), i.e.

$$
P_{\mathrm{U}(1)\times\mathrm{U}(1)}(t,m_1,m_2) = \frac{1}{(1-t)^2} \,, \tag{3.1}
$$

which reflects the two degree one Casimir invariants.

3.2 Two types of hypermultiplets

Set-up. To consider a rank 2 abelian gauge group of the form $\mathrm{U}(1)\times\mathrm{U}(1)$ requires a delicate choice of matter content. If one considers N_1 hypermultiplets with charges $(a_1,b_1)\in\mathbb{N}^2$ under $\mathrm{U}(1)\times\mathrm{U}(1)$, then the conformal dimension reads

$$
\Delta_{\text{1h-plet}}(m_1,m_2) = \frac{N_1}{2}\,|a_1m_1+b_1m_2| \qquad \text{for} \quad (m_1,m_2)\in\mathbb{Z}^2 \,. \tag{3.2a}
$$

However, there exists an infinite number of points $\{m_1 = b_1 k, m_2 = -a_1 k, k \in \mathbb{Z}\}$ with zero conformal dimension, i.e. the Hilbert series does not converge due to a decoupled U(1). Fixing this symmetry would reduce the rank to one.

Fortunately, we can circumvent this problem by introducing a second set of N_2 hypermultiplets with charges $(a_2, b_2) \in \mathbb{N}^2$, such that the matrix

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \tag{3.2b}$$

has maximal rank. The relevant conformal dimension then reads

$$\Delta_{\text{2h-plet}}(m_1, m_2) = \sum_{j=1}^{2} \frac{N_j}{2} |a_j m_1 + b_j m_2| \qquad \text{for} \quad (m_1, m_2) \in \mathbb{Z}^2 \,. \tag{3.2c}$$

Nevertheless, this set-up would introduce four charges and the summation of the Hilbert series becomes tricky. We evade the difficulties by the *choice* $a_2 = b_1$ and $b_2 = -a_1$. Dealing with such a scenario leads to summation bounds such as

$$a\, m_1 \geq b\, m_2 \Leftrightarrow m_1 \geq \frac{b}{a} m_2 \Leftrightarrow m_1 \geq \left\lceil \frac{b}{a} m_2 \right\rceil , \tag{3.2d}$$

$$a\, m_1 < b\, m_2 \Leftrightarrow m_1 < \frac{b}{a} m_2 \Leftrightarrow m_1 < \left\lceil \frac{b}{a} m_2 \right\rceil - 1 \,. \tag{3.2e}$$

Having the summation variable within a floor or ceiling function seems to be an elaborate task with `Mathematica`. Therefore, we simplify the setting by *assuming* $\exists\, k \in \mathbb{N}$ such that $b_1 = k a_1$. Then we arrive at

$$\Delta_{\text{2h-plet}}(m_1, m_2) = \frac{a_1}{2} \left(N_1 |m_1 + k m_2| + N_2 |k m_1 - m_2| \right) \qquad \text{for} \quad (m_1, m_2) \in \mathbb{Z}^2 \,. \tag{3.2f}$$

For this conformal dimension, there exists exactly one point (m_1, m_2) with zero conformal dimension — the trivial solution. Further, by a redefinition of N_1 and N_2 we can consider $a_1 = 1$.

Hilbert basis. Consider the conformal dimension (3.2f) for $a_1 = 1$. By resolving the absolute values, we divide $\mathbb{Z}^2$ into four semi-groups

$$S_1^{(2)} = \left\{ (m_1, m_2) \in \mathbb{Z}^2 |\, (k m_1 \geq m_2) \wedge (m_1 \geq -k m_2) \right\} \,, \tag{3.3a}$$

$$S_2^{(2)} = \left\{ (m_1, m_2) \in \mathbb{Z}^2 |\, (k m_1 \geq m_2) \wedge (m_1 \leq -k m_2) \right\} \,, \tag{3.3b}$$

$$S_3^{(2)} = \left\{ (m_1, m_2) \in \mathbb{Z}^2 |\, (k m_1 \leq m_2) \wedge (m_1 \geq -k m_2) \right\} \,, \tag{3.3c}$$

$$S_4^{(2)} = \left\{ (m_1, m_2) \in \mathbb{Z}^2 |\, (k m_1 \leq m_2) \wedge (m_1 \leq -k m_2) \right\} \,, \tag{3.3d}$$

which all descend from 2-dimensional rational polyhedral cones. The situation is depicted in figure 2. Next, one needs to compute the Hilbert basis $\mathcal{H}(S)$ for each semi-group S. In this example, it follows from the drawing that

$$\mathcal{H}(S_1^{(2)}) = \Big\{ (k, -1), \big\{ (1, l) \;\big|\; l = 0, 1, \ldots, k \big\} \Big\} \,, \tag{3.4a}$$

$$\mathcal{H}(S_2^{(2)}) = \Big\{ (-1, -k), \big\{ (l, -1) \;\big|\; l = 0, 1, \ldots, k \big\} \Big\} \,, \tag{3.4b}$$

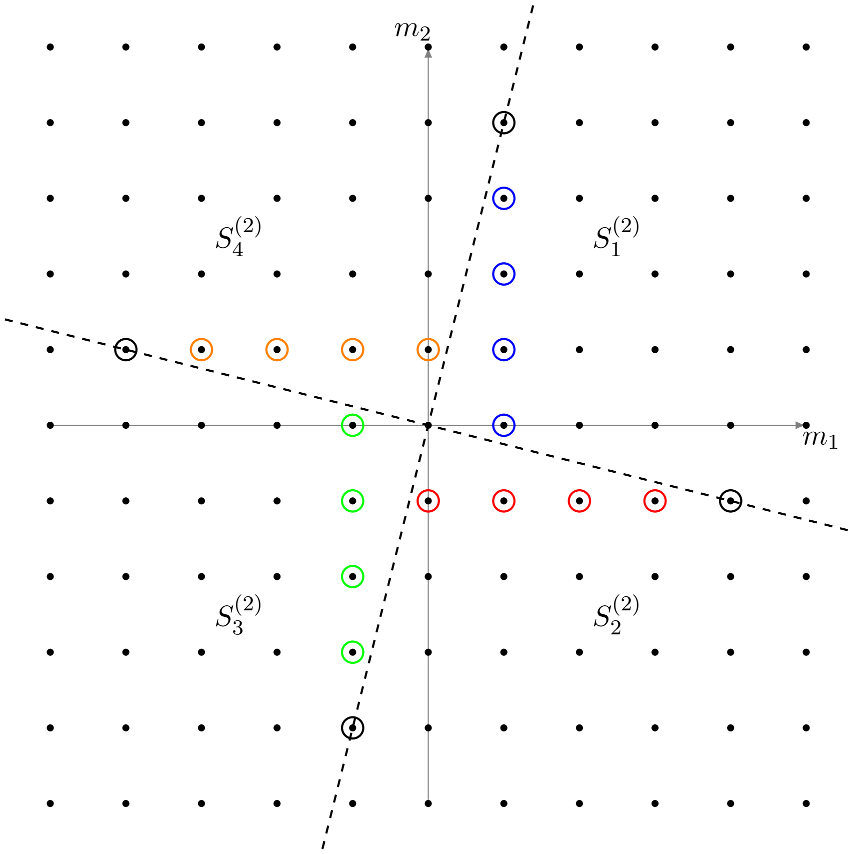

Figure 2. The dashed lines correspond the $k\,m_1 = m_2$ and $m_1 = -k\,m_2$ and divide the lattice $\mathbb{Z}^2$ into four semi-groups $S_j^{(2)}$ for $j = 1, 2, 3, 4$. The black circles denote the ray generators, while the blue circles complete the Hilbert basis for $S_1^{(2)}$, red circled points complete the basis for $S_2^{(2)}$. Green circles correspond to the remaining minimal generators of $S_3^{(2)}$ and orange circled points are the analogue for $S_4^{(2)}$. (Here, the example is $k = 4$.)

$$\mathcal{H}(S_3^{(2)}) = \Big\{(-k,1), \{(-1,-l) \mid l = 0,1,\dots,k\}\Big\} \,, \tag{3.4c}$$

$$\mathcal{H}(S_4^{(2)}) = \Big\{(1,k), \{(-l,1) \mid l = 0,1,\dots,k\}\Big\} \,. \tag{3.4d}$$

For a fixed $k \geq 1$ we obtain $4(k+1)$ basis elements.

Hilbert series. We then compute the following Hilbert series

$$\mathrm{HS}^k_{\mathrm{U}(1)\times\mathrm{U}(1)}(t,z_1,z_2) = \frac{1}{(1-t)^2} \sum_{m_1,m_2\in\mathbb{Z}} z_1^{m_1} z_2^{m_2} t^{\Delta_{\text{2h-plet}}(m_1,m_2)} \,, \tag{3.5}$$

for which we obtain

$$\mathrm{HS}^k_{\mathrm{U}(1)\times\mathrm{U}(1)}(t,z_1,z_2) = \frac{R(t,z_1,z_2)}{P(t,z_1,z_2)} \,, \tag{3.6a}$$

(m_1,m_2)	$\Delta(m_1,m_2)$	(m_1,m_2)	$\Delta(m_1,m_2)$
$(1,0)$, $(-1,0)$	$\frac{1}{2}(N_1+kN_2)$	$(0,1)$, $(0,-1)$	$\frac{1}{2}(kN_1+N_2)$
$(1,k)$, $(-1,-k)$	$\frac{1}{2}\left(1+k^2\right)N_1$	$(-k,1)$, $(k,-1)$	$\frac{1}{2}\left(1+k^2\right)N_2$

(a) The minimal generators which are ray generators or poles of the Hilbert series.

(m_1,m_2)	$\Delta(m_1,m_2)$	(m_1,m_2)	$\Delta(m_1,m_2)$
$(1,l)$, $(-1,-l)$	$\frac{1}{2}N_1(kl+1)+\frac{1}{2}N_2(k-l)$	$(-l,1)$, $(l,-1)$	$\frac{1}{2}N_1(k-l)+\frac{1}{2}N_2(kl+1)$

(b) The minimal generators, labelled by $l=1,2,\ldots,k-1$, which are not ray generators.

Table 1. The set of bare monopole operators for a $\mathrm{U}(1)\times\mathrm{U}(1)$ theory with conformal dimension (3.2f).

with denominator

$$\begin{aligned}P(t,z_1,z_2)=(1-t)^2&\left(1-\frac{1}{z_1}t^{\frac{kN_2-N_1}{2}}\right)\left(1-z_1t^{\frac{kN_2-N_1}{2}}\right)\left(1-\frac{1}{z_2}t^{\frac{kN_1-N_2}{2}}\right)\left(1-z_2t^{\frac{kN_1-N_2}{2}}\right)\\ \times&\left(1-\frac{1}{z_1}t^{\frac{kN_2+N_1}{2}}\right)\left(1-z_1t^{\frac{kN_2+N_1}{2}}\right)\left(1-\frac{1}{z_2}t^{\frac{kN_1+N_2}{2}}\right)\left(1-z_2t^{\frac{kN_1+N_2}{2}}\right)\\ \times&\left(1-\frac{1}{z_1z_2^k}t^{\frac{1}{2}(k^2+1)N_1}\right)\left(1-z_1z_2^kt^{\frac{1}{2}(k^2+1)N_1}\right)\\ \times&\left(1-\frac{z_1^k}{z_2}t^{\frac{1}{2}(k^2+1)N_2}\right)\left(1-\frac{z_2}{z_1^k}t^{\frac{1}{2}(k^2+1)N_2}\right)\,,\end{aligned}\tag{3.6b}$$

while the numerator $R(t,z_1,z_2)$ is too long to be displayed, as it contains 1936 monomials. Nonetheless, one can explicitly verify a few properties of the Hilbert series. For example, the Hilbert series (3.6) has a pole of order 4 at $t\to1$, because $R(1,z_1,z_2)=0$ and the derivatives $\frac{\mathrm{d}^n}{\mathrm{d}t^n}R(t,z_1,z_2)|_{t=1}=0$ for $n=1,2,\ldots 9$ (at least for $z_1=z_2=1$). Moreover, the degrees of numerator and denominator depend on the relations between N_1, N_2, and k; however, one can show that the difference in degrees is precisely 2, i.e. it matches the quaternionic dimension of the moduli space.

Discussion. Analysing the plethystic logarithm and the Hilbert series, the monopole operators corresponding to the Hilbert basis can be identified as follows: Eight poles of the Hilbert series (3.6) can be identified with monopole generators as shown in table 1a. Studying the plethystic logarithm clearly displays the remaining set, which is displayed in table 1b.

Remark. A rather special case of (3.2c) is $a_2=0=b_1$, for which the theory becomes the product of two U(1)-theories with N_1 or N_2 electrons of charge a or b, respectively. In detail, the conformal dimension is simply

$$\Delta_{\text{2h-plet}}(m_1,m_2)\overset{a_2=0=b_1}{=}\frac{N_1}{2}\left|a\,m_1\right|+\frac{N_2}{2}\left|b\,m_2\right|\qquad\text{for}\quad(m_1,m_2)\in\mathbb{Z}^2\,,\tag{3.7}$$

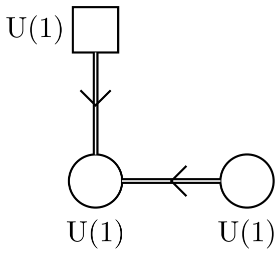

Figure 3. Quiver gauge theory whose Coulomb branch is the reduced moduli space of one SO(5)-instanton.

such that the Hilbert series becomes

$$\begin{aligned}\mathrm{HS}^{a,b}_{\mathrm{U}(1)^2}(t,z_1,z_2) &= \frac{1-t^{aN_1}}{(1-t)\left(1-z_1 t^{\frac{aN_1}{2}}\right)\left(1-\frac{1}{z_1}t^{\frac{aN_1}{2}}\right)} \\ &\quad\times \frac{1-t^{bN_2}}{(1-t)\left(1-z_2 t^{\frac{bN_2}{2}}\right)\left(1-\frac{1}{z_2}t^{\frac{bN_2}{2}}\right)} \\ &= \mathrm{HS}^{a}_{\mathrm{U}(1)}(t,z_1,N_1)\times \mathrm{HS}^{b}_{\mathrm{U}(1)}(t,z_2,N_2)\ . \end{aligned} \tag{3.8}$$

For the unrefined Hilbert series, that is $z_1 = 1 = z_2$, the rational function $\mathrm{HS}^{a}_{\mathrm{U}(1)}(t,N)$ equals the Hilbert series of the (abelian) ADE-orbifold $\mathbb{C}^2/\mathbb{Z}_{a\cdot N}$, see for instance [25]. Thus, the $\mathrm{U}(1)\times\mathrm{U}(1)$ Coulomb branch is the product of two A-type singularities.

Quite intuitively, taking the corresponding limit $k\to 0$ in (3.6) yields the product

$$\lim_{k\to 0}\mathrm{HS}^{k}_{\mathrm{U}(1)\times\mathrm{U}(1)}(t,z_1,z_2) = \mathrm{HS}_{\mathrm{U}(1)}(t,z_1,N_1)\times \mathrm{HS}_{\mathrm{U}(1)}(t,z_2,N_2)\ , \tag{3.9}$$

which are U(1) theories with N_1 and N_2 electrons of unit charge. The unrefined rational functions are the Hilbert series of $\mathbb{Z}_{N_1}$ and $\mathbb{Z}_{N_2}$ singularities in the ADE-classification. From figure 2 one observes that in the limit $k\to 0$ the relevant rational cones coincide with the four quadrants of $\mathbb{R}^2$ and the Hilbert basis reduces to the cone generators.

3.3 Reduced moduli space of one SO(5)-instanton

Consider the Coulomb branch of the quiver gauge theory depicted in figure 3 with conformal dimension given by

$$\Delta(m_1,m_2) = \frac{1}{2}\left(|m_1| + |m_1 - 2m_2|\right)\ . \tag{3.10}$$

Instead of associating (3.10) with the quiver of figure 3, one could equally well understand it as a special case of a $\mathrm{U}(1)^2$ theory with two different hypermultiplets (3.2c).

Hilbert basis. Similar to the previous case, the conformal dimensions induces a fan which, in this case, is generated by four 2-dimensional cones

$$C^{(2)}_1 = \mathrm{Cone}\left((2,1),(0,1)\right)\ , \qquad C^{(2)}_2 = \mathrm{Cone}\left((2,1),(0,-1)\right)\ , \tag{3.11a}$$

$$C^{(2)}_3 = \mathrm{Cone}\left((-2,-1),(0,-1)\right)\ , \qquad C^{(2)}_4 = \mathrm{Cone}\left((-2,-1),(0,1)\right)\ . \tag{3.11b}$$

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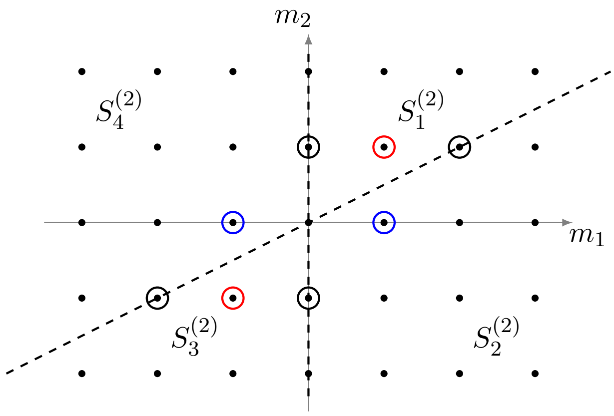

Figure 4. The dashed lines correspond the $m_1 = 2m_2$ and $m_1 = 0$ and divide the lattice $\mathbb{Z}^2$ into four semi-groups $S_j^{(2)}$ for $j = 1, 2, 3, 4$. The black circles denote the ray generators, while the red circles complete the Hilbert bases for $S_1^{(2)}$ and $S_3^{(2)}$. Blue circled lattice points complete the bases for $S_2^{(2)}$ and $S_4^{(2)}$.

The intersection with the $\mathbb{Z}^2$ lattice defines the semi-groups $S_p^{(2)} := C_p^{(2)} \cap \mathbb{Z}^2$ for which we need to compute the Hilbert bases. Figure 4 illustrates the situation and we obtain

$$\mathcal{H}(S_1^{(2)}) = \{(2,1),(1,1),(0,1)\} \ , \qquad \mathcal{H}(S_2^{(2)}) = \{(2,1),(1,0),(0,-1)\} \ , \tag{3.12a}$$

$$\mathcal{H}(S_3^{(2)}) = \{(-2,-1),(-1,-1),(0,-1)\} \ , \quad \mathcal{H}(S_4^{(2)}) = \{(-2,-1),(-1,0),(0,1)\} \ . \tag{3.12b}$$

Hilbert series. The Hilbert series is evaluated to

$$\mathrm{HS}_{\mathrm{U}(1)^2}^{\mathrm{SO}(5)}(t, z_1, z_2) = \frac{R(t, z_1, z_2)}{(1-t)^2 \left(1 - \frac{t}{z_2}\right)(1 - z_2 t)\left(1 - \frac{t}{z_1^2 z_2}\right)(1 - z_1^2 z_2 t)} \ , \tag{3.13a}$$

$$\begin{aligned} R(t, z_1, z_2) = 1 &+ t\left(z_1 + \frac{1}{z_1} + z_1 z_2 + \frac{1}{z_1 z_2}\right) \\ &- 2t^2\left(1 + z_1 + \frac{1}{z_1} + z_1 z_2 + \frac{1}{z_1 z_2}\right) \\ &+ t^3\left(z_1 + \frac{1}{z_1} + z_1 z_2 + \frac{1}{z_1 z_2}\right) + t^4 \ . \end{aligned} \tag{3.13b}$$

The Hilbert series (3.13) has a pole of order 4 at $t = 1$, because one can explicitly verify that $R(t = 1, z_1, z_2) = 0$, $\frac{\mathrm{d}}{\mathrm{d}t} R(t, z_1, z_2)|_{t=1} = 0$, but $\frac{\mathrm{d}^2}{\mathrm{d}t^2} R(t, z_1, z_2)|_{t=1} \neq 0$. Thus, the complex dimension of the moduli space is 4. Moreover, the difference in degrees of numerator and denominator is 2, which equals the quaternionic dimension of the Coulomb branch.

Plethystic logarithm. The plethystic logarithm for this scenario reads

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}_{\mathrm{U}(1)^2}^{\mathrm{SO}(5)}) = &\left(2 + z_1^2 z_2 + \frac{1}{z_1^2 z_2} + z_1 z_2 + \frac{1}{z_1 z_2} + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2}\right) t \\ &- \Bigg(4 + z_1^2 + \frac{1}{z_1^2} + z_2 + \frac{1}{z_2} + z_1^2 z_2^2 + \frac{1}{z_1^2 z_2^2} + z_1^2 z_2 + \frac{1}{z_1^2 z_2} \\ &\quad + 2z_1 + \frac{2}{z_1} + 2 z_1 z_2 + \frac{2}{z_1 z_2}\Bigg) t^2 + \mathcal{O}(t^3) \ . \end{aligned} \tag{3.14}$$

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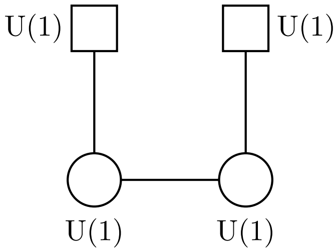

Figure 5. Quiver gauge theory whose Coulomb branch is the reduced moduli space of one SU(3)-instanton.

Symmetry enhancement. The information conveyed by the Hilbert basis (3.12), the Hilbert series (3.13), and the plethystic logarithm (3.14) is that there are eight minimal generators of conformal dimension one which, together with the two Casimir invariants, span the adjoint representation of SO(5). It is known [25, 26] that (3.13) is the Hilbert series for the reduced moduli space of one SO(5)-instanton over $\mathbb{C}^2$.

3.4 Reduced moduli space of one SU(3)-instanton

The quiver gauge theories associated to the affine Dynkin diagram $\hat{A}_n$ have been studied in [5]. Here, we consider the Coulomb branch of the $\hat{A}_2$ quiver gauge theory as depicted in figure (5) and with conformal dimension given by

$$\Delta(m_1, m_2) = \frac{1}{2}\left(|m_1| + |m_2| + |m_1 - m_2|\right) \ . \tag{3.15}$$

Hilbert basis. Similar to the previous case, the conformal dimensions induces a fan which, in this case, is generated by six 2-dimensional cones

$$C_1^{(2)} = \mathrm{Cone}\left((0,1),(1,1)\right) \ , \qquad C_2^{(2)} = \mathrm{Cone}\left((1,1),(1,0)\right) \ , \tag{3.16a}$$

$$C_3^{(2)} = \mathrm{Cone}\left((1,0),(0,-1)\right) \ , \qquad C_4^{(2)} = \mathrm{Cone}\left((0,-1),(-1,-1)\right) \ , \tag{3.16b}$$

$$C_5^{(2)} = \mathrm{Cone}\left((-1,-1),(-1,0)\right) \ , \qquad C_6^{(2)} = \mathrm{Cone}\left((-1,0),(0,1)\right) \ . \tag{3.16c}$$

The intersection with the $\mathbb{Z}^2$ lattice defines the semi-groups $S_p^{(2)} := C_p^{(2)} \cap \mathbb{Z}^2$ for which we need to compute the Hilbert bases. Figure 6 illustrates the situation. We compute the Hilbert bases to read

$$\mathcal{H}(S_1^{(2)}) = \{(0,1),(1,1)\} \qquad \mathcal{H}(S_2^{(2)}) = \{(1,1),(1,0))\} \ , \tag{3.17a}$$

$$\mathcal{H}(S_3^{(2)}) = \{(1,0),(0,-1)\} \qquad \mathcal{H}(S_4^{(2)}) = \{(0,-1),(-1,-1)\} \ , \tag{3.17b}$$

$$\mathcal{H}(S_5^{(2)}) = \{(-1,-1),(-1,0)\} \qquad \mathcal{H}(S_6^{(2)}) = \{(-1,0),(0,1)\} \ . \tag{3.17c}$$

Hilbert series.

$$\mathrm{HS}_{\mathrm{U}(1)^2}^{\mathrm{SU}(3)}(t,z_1,z_2) = \frac{R(t,z_1,z_2)}{(1-t)^2\left(1-\frac{t}{z_1}\right)(1-z_1t)\left(1-\frac{t}{z_2}\right)(1-z_2t)\left(1-\frac{t}{z_1z_2}\right)(1-z_1z_2t)} \tag{3.18a}$$

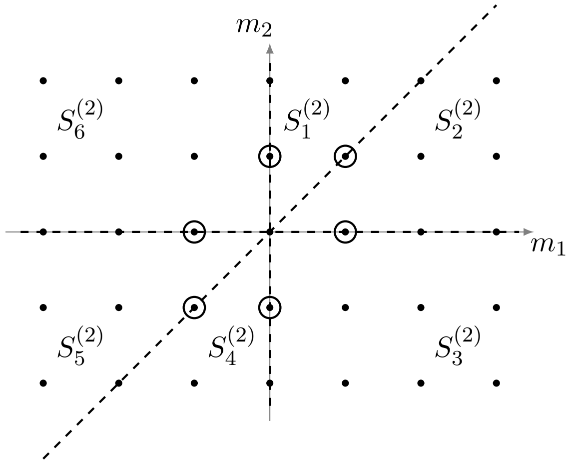

Figure 6. The dashed lines correspond the $m_1 = m_2$, $m_1 = 0$, and $m_2 = 0$ and divide the lattice $\mathbb{Z}^2$ into six semi-groups $S_j^{(2)}$ for $j = 1, \ldots, 6$. The black circled points denote the ray generators, which coincide with the minimal generators.

$$
\begin{aligned}
R(t, z_1, z_2) = 1 &- \left(3 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) t^2 \\
&+ 2\left(2 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) t^3 \\
&- \left(3 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) t^4 + t^6
\end{aligned}
\tag{3.18b}
$$

The Hilbert series (3.17) has a pole of order 4 as $t \to 1$, because $R(t = 1, z_1, z_2) = 0$ and $\frac{\mathrm{d}^n}{\mathrm{d}t^n} R(t, z_1, z_2)|_{t=1, z_1=z_2=1} = 0$ for $n = 1, 2, 3$. Thus, the Coulomb branch is of complex dimension 4. In addition, the difference in degrees of numerator and denominator is 2, which equals the quaternionic dimension.

Plethystic logarithm.

$$
\begin{aligned}
\mathrm{PL}(\mathrm{HS}_{\mathrm{U}(1)^2}^{\mathrm{SU}(3)}) = &\left(2 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) t \\
&- \left(3 + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} + z_1 z_2 + \frac{1}{z_1 z_2}\right) t^2 + \mathcal{O}(t^3)
\end{aligned}
\tag{3.19}
$$

Symmetry enhancement. The information conveyed by the Hilbert basis (3.17), the Hilbert series (3.18), and the plethystic logarithm (3.19) is that there are six minimal generators of conformal dimension one which, together with the two Casimir invariants, span the adjoint representation of SU(3). As proved in [5], the Hilbert series (3.18) can be resumed as

$$
\mathrm{HS}_{\mathrm{U}(1)^2}^{\mathrm{SU}(3)}(t, z_1, z_2) = \sum_{k=0}^{\infty} \chi_{[k,k]} t^k \tag{3.20}
$$

with $\chi_{[k,k]}$ being the character of the SU(3)-representation $[k, k]$. Therefore, this theory has an explicit SU(3)-enhancement in the Coulomb branch. It is known [27] that (3.20) is the reduced instanton moduli space of one SU(3)-instanton over $\mathbb{C}^2$.

4 Case: U(2)

In this section we aim to consider two classes of U(2) gauge theories wherein $\mathrm{U}(2) \cong \mathrm{SU}(2)\times\mathrm{U}(1)$, i.e. this is effectively an SU(2) theory with varying U(1)-charge. As a unitary group, U(2) is self-dual under GNO-duality.

4.1 Set-up

To start with, let consider the two view points and elucidate the relation between them.

U(2) view point. The GNO-dual of U(2) is U(2) itself; hence, the weight lattice is $\Lambda_w(\mathrm{U}(2)) \cong \mathbb{Z}^2$. Moreover, the Weyl-group is S_2 and acts via permuting the two Cartan generators; consequently, $\Lambda_w(\mathrm{U}(2))/S_2 = \{(m_1,m_2) \in \mathbb{Z}^2 : m_1 \geq m_2\}$.

U(1) × SU(2) view point. Considering $\mathrm{U}(1)\times\mathrm{SU}(2)$, we need to find the weight lattice of the GNO-dual, i.e. find all solutions to the Dirac quantisation condition, see for instance [9]. Since we consider the product, the exponential in (1.4) factorises in $\exp(2\pi\mathrm{i}\; n\; T_{\mathrm{U}(1)})$ and $\exp(2\pi\mathrm{i}\; m\; T_{\mathrm{SU}(2)})$, where the T's are the Cartan generators. Besides the solution

$$(n,m) \in H_0 \coloneqq \mathbb{Z}^2 = \mathbb{Z}\times\Lambda_w(\mathrm{SO}(3)) = \mathbb{Z}\times\Lambda_r(\mathrm{SU}(2)) \tag{4.1a}$$

corresponding to the weight lattice of $\mathrm{U}(1)\times\mathrm{SO}(3)$, there exists also the solution

$$(n,m) \in H_1 \coloneqq \mathbb{Z}^2 + (\tfrac{1}{2},\tfrac{1}{2}) = \left(\mathbb{Z}+\tfrac{1}{2}\right)\times\left(\Lambda_w(\mathrm{SU}(2))\setminus\Lambda_r(\mathrm{SU}(2))\right) \;, \tag{4.1b}$$

for which both factors are equal to -1. The action of the Weyl-group S_2 restricts then to non-negative m i.e. $H_0^+ = H_0 \cap \{m \geq 0\}$ and $H_1^+ = H_1 \cap \{m \geq 0\}$.

Relation between both. To identify both views with one another, we select the U(1) as diagonally embedded, i.e. identify the charges as follows:

$$\left.\begin{aligned} n &\coloneqq \tfrac{m_1+m_2}{2} \\ m &\coloneqq \tfrac{m_1-m_2}{2} \end{aligned}\right\} \qquad \Leftrightarrow \qquad \left\{\begin{aligned} m_1 &= n+m \\ m_2 &= n-m \end{aligned}\right. . \tag{4.2}$$

The two classes of U(2)-representations under consideration in this section are

$$[1,a] \qquad \text{with} \quad \chi_{[1,a]}^{\mathrm{U}(2)} = y_1^{a+1} y_2^a + y_1^a y_2^{a+1} \;, \tag{4.3a}$$

$$[2,a] \qquad \text{with} \quad \chi_{[2,a]}^{\mathrm{U}(2)} = y_1^{a+2} y_2^a + y_1^{a+1} y_2^{a+1} + y_1^a y_2^{a+2} \;, \tag{4.3b}$$

for $a \in \mathbb{N}_0$. Following (4.2), we define the fugacities

$$q \coloneqq \sqrt{y_1\, y_2} \qquad \text{for} \quad \mathrm{U}(1) \quad \text{and} \quad x \coloneqq \sqrt{\frac{y_1}{y_2}} \qquad \text{for} \quad \mathrm{SU}(2), \tag{4.4}$$

and consequently observe

$$\chi_{[1,a]}^{\mathrm{U}(2)} = q^{2a+1}\left(x+\frac{1}{x}\right) = \chi_{2a+1}^{\mathrm{U}(1)}\cdot\chi_{[1]}^{\mathrm{SU}(2)} \;, \tag{4.5a}$$

$$\chi_{[2,a]}^{\mathrm{U}(2)} = q^{2a+2}\left(x^2+1+\frac{1}{x^2}\right) = \chi_{2a+2}^{\mathrm{U}(1)}\cdot\chi_{[2]}^{\mathrm{SU}(2)} \;, \tag{4.5b}$$

where the SU(2)-characters are defined via

$$\chi_{[L]}^{\mathrm{SU}(2)} = \sum_{r=-\frac{L}{2}}^{\frac{L}{2}} x^{2r} \,. \tag{4.5c}$$

Therefore, the family $[1,a]$ corresponds to the fundamental representation of SU(2) with *odd* U(1)-charge $2a+1$; while the family $[2,a]$ represents the adjoint representation of SU(2) with *even* U(1)-charge $2a+2$.

Dressing factors. Lastly, the calculation employs the classical dressing function

$$P_{\mathrm{U}(2)}(t^2,m) := \begin{cases} \dfrac{1}{(1-t^2)^2} & ,m \neq 0 \\ \dfrac{1}{(1-t^2)(1-t^4)} & ,m=0 \end{cases} \,, \tag{4.6}$$

as presented in [5]. (Note that we rescaled t to be t^2 for later convenience.) Following the discussion of appendix A, monopoles with $m \neq 0$ have precisely one dressing by a U(1) Casimir invariant due to $P_{\mathrm{U}(2)}(t^2,m)/P_{\mathrm{U}(2)}(t^2,0) = 1+t^2$. In contrast, there are no dressed monopole operators for $m=0$.

4.2 N hypermultiplets in the fundamental representation of SU(2)

The conformal dimension for a U(2) theory with N hypermultiplets transforming in $[1,a]$ is given as

$$\Delta(n,m) = \frac{N}{2}\big(\,|(2a+1)\cdot n + m| + |(2a+1)\cdot n - m|\,\big) - 2|m| \tag{4.7}$$

such that the Hilbert series is computed via

$$\mathrm{HS}_{\mathrm{U}(2)}^{[1,a]}(t,z) = \sum_{n,m} P_{\mathrm{U}(2)}(t^2,m)\; t^{2\Delta(n,m)} z^{2n} \,, \tag{4.8}$$

where the ranges of n,m have been specified above. Here we use the fugacity t^2 instead of t to avoid half-integer powers.

Hilbert basis. The conformal dimension (4.7) divides $\Lambda_w(\mathrm{U}(2))/S_2$ into semi-groups via the absolute values $|m|$, $|(2a+1)n+m|$, and $|(2a+1)n-m|$. Thus, there are three semi-groups

$$S_{+}^{(2)} = \Big\{(m,n) \in \Lambda_w^{\mathrm{U}(2)}/S_2 \mid (n \geq 0) \,\wedge\, (0 \leq m \leq (2a+1)n)\Big\} \,, \tag{4.9a}$$

$$S_{0}^{(2)} = \Big\{(m,n) \in \Lambda_w^{\mathrm{U}(2)}/S_2 \mid -(2a+1)n \leq m \leq (2a+1)n\Big\} \,, \tag{4.9b}$$

$$S_{-}^{(2)} = \Big\{(m,n) \in \Lambda_w^{\mathrm{U}(2)}/S_2 \mid (n \leq 0) \,\wedge\, (0 \leq m \leq -(2a+1)n)\Big\} \tag{4.9c}$$

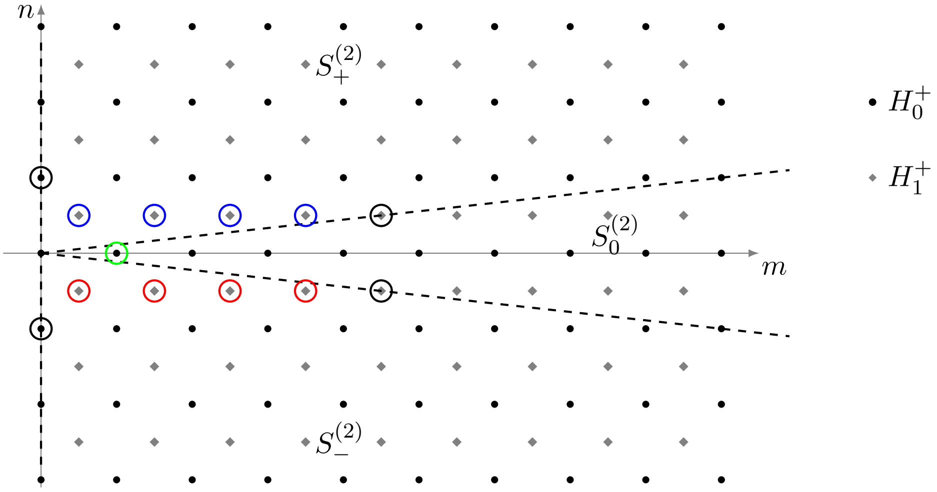

Figure 7. The Weyl-chamber for the example $a = 4$. The black circled lattice points are the ray generators. The blue circled lattice points complete the Hilbert basis (together with two ray generators) for $S_+^{(2)}$; while the red circled points analogously complete the Hilbert basis for $S_-^{(2)}$. The green circled point represents the missing minimal generator for $S_0^{(2)}$.

originating from 2-dimensional cones, see figure 7. Since all these semi-groups $S_\pm^{(2)}$, $S_0^{(2)}$ are finitely generated, one can compute the Hilbert basis $\mathcal{H}(S_p)$ for each p and obtains

$$\mathcal{H}(S_\pm^{(2)}) = \Big\{(0,\pm 1), \big\{\big(l+\tfrac{1}{2},\pm\tfrac{1}{2}\big) \mid l=0,1,\ldots,a\big\}\Big\}\,, \tag{4.10a}$$

$$\mathcal{H}(S_0^{(2)}) = \Big\{\big(a+\tfrac{1}{2},\tfrac{1}{2}\big),(1,0),\big(a+\tfrac{1}{2},-\tfrac{1}{2}\big)\Big\}\,. \tag{4.10b}$$

Hilbert series. Computing the Hilbert series yields

$$\mathrm{HS}_{\mathrm{U}(2)}^{[1,a]}(t,z,N) = \frac{R(t,z)}{P(t,z)}\,, \tag{4.11a}$$

$$\begin{aligned} P(t,z) = {}& \left(1-t^2\right)^2\left(1-t^4\right)\left(1-t^{2N-4}\right)\left(1-\tfrac{1}{z^2}t^{(4a+2)N}\right)\left(1-z^2t^{(4a+2)N}\right) \\ &\times\left(1-\tfrac{1}{z}t^{(2a+1)(N-2)}\right)\left(1-zt^{(2a+1)(N-2)}\right)\,, \end{aligned} \tag{4.11b}$$

$$\begin{aligned} R(t,z) = {}& 1-t^2+t^{2N-2}-t^{2N}+2t^{4aN-4a+2N}-t^{4aN-8a+2N-4}-t^{4aN-8a+2N-2} \\ &-2t^{4aN-4a+4N-4}+t^{4aN-8a+4N-6}+t^{4aN-8a+4N-4}+t^{8aN+4N}+t^{8aN+4N+2} \\ &-2t^{8aN-4a+4N}-t^{8aN+6N-2}-t^{8aN+6N}+2t^{8aN-4a+6N-4}-t^{12aN-8a+6N-4} \\ &+t^{12aN-8a+6N-2}-t^{12aN-8a+8N-6}+t^{12aN-8a+8N-4} \\ &+\left(z+\tfrac{1}{z}\right)\Big(t^{2aN-4a+N}-t^{2aN+N+2}+t^{2aN+3N-2}-t^{2aN-4a+3N-4}+t^{6aN+3N+2} \\ &-t^{6aN-8a+3N-2}-t^{6aN+5N-2}+t^{6aN-8a+5N-6}-t^{10aN-4a+5N}+t^{10aN-8a+5N-2} \\ &+t^{10aN-4a+7N-4}-t^{10aN-8a+7N-6}\Big) \end{aligned}$$

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(m,n)	(m_1,m_2)	$2\Delta(m,n)$	$H_{(m,n)}$	dressings
$(1,0)$	$(1,-1)$	$2N-4$	$\mathrm{U}(1)^2$	1 by U(1)
$(l+\frac{1}{2},\frac{1}{2})$, for $l=0,1,\ldots,a$	$(l+1,-l)$	$(2a+1)N-2(2l+1)$	$\mathrm{U}(1)^2$	1 by U(1)
$(l+\frac{1}{2},-\frac{1}{2})$, for $l=0,1,\ldots,a$	$(l,-(l+1))$	$(2a+1)N-2(2l+1)$	$\mathrm{U}(1)^2$	1 by U(1)
$(0,\pm1)$	$\pm(1,1)$	$(4a+2)N$	U(2)	none

Table 2. Bare and dressed monopole operators for the family $[1,a]$ of U(2)-representations.

$$+\left(z^2+\tfrac{1}{z^2}\right)\Big(t^{4aN-4a+2N}-t^{4aN+2N}+t^{4aN+4N}-t^{4aN-4a+4N-4}-t^{8aN-4a+4N}$$
$$+t^{8aN-8a+4N-4}+t^{8aN-4a+6N-4}-t^{8aN-8a+6N-4}\Big)\,. \tag{4.11c}$$

The Hilbert series (4.11) has a pole of order 4 at $t\to1$, because $R(t=1,z)=0$ and $\frac{\mathrm{d}^n}{\mathrm{d}t^n}R(t,z)|_{t=1}=0$ for $n=1,2,3$. Hence, the moduli space is of (complex) dimension 4. As a comment, the additional $(1-t^2)$-term in the denominator can be cancelled with a corresponding term in the numerator either explicitly for each $a=$ fixed or for any a, but the resulting expressions are not particularly insightful.

Discussion. The four poles of the Hilbert series (4.11), which are graded as $z^{\pm2}$ and $z^{\pm1}$, can be identified with the four ray generators $(0,\pm1)$ and $(a+\frac{1}{2},\pm\frac{1}{2})$, i.e. they correspond to bare monopole operators. In addition, the bare monopole operator for the minimal generator $(1,0)$ is present in the denominator (4.11b), too.

In contrast, the family of monopoles $\{(l+\frac{1}{2},\pm\frac{1}{2}),l=0,1,\ldots,a-1\}$ is not directly visible in the Hilbert series, but can be deduced unambiguously from the plethystic logarithm. These monopole operators correspond the *minimal generators* of $S_{\pm}^{(2)}$ which are not ray generators. Table 2 provides as summary of the monopole generators and their properties. As a remark, the family of monopole operators $(l+\frac{1}{2},\pm\frac{1}{2})$ is not always completely present in the plethystic logarithm. We observe that l-th bare operator is a generator if $N\geq2(a-l+1)$, while the dressing of the l-th object is a generator if $N>2(a-l+1)$. The reason for the disappearance lies in a relation at degree $\Delta(1,0)+\Delta(a+\frac{1}{2},\pm\frac{1}{2})+2$, which coincides with $\Delta(l+\frac{1}{2},\pm\frac{1}{2})$ for $N-1=2(a-l+1)$, such that the terms cancel in the PL. (See also appendix A.) Thus, for large N all above listed objects are generators.

4.2.1 Case: $a=0$, complete intersection

For the choice $a=1$, we obtain the Hilbert series for the 2-dimensional fundamental representation $[1,0]$ of U(2) as

$$\mathrm{HS}_{\mathrm{U}(2)}^{[1,0]}(t,z,N)=\frac{\left(1-t^{2N}\right)\left(1-t^{2N-2}\right)}{(1-t^2)\,(1-t^4)\left(1-\frac{1}{z}t^N\right)(1-zt^N)\left(1-\frac{1}{z}t^{N-2}\right)(1-zt^{N-2})} \tag{4.12}$$

which agrees with the results of [5].

Let us comment on the reduction of generators compared to the Hilbert basis (4.10). The minimal generators have conformal dimensions $2\Delta(\frac{1}{2},\pm\frac{1}{2})=N-2$, $2\Delta(1,0)=2N-4$, and $2\Delta(0,\pm1)=2N$. Thus, $(1,0)$ is generated by $(\frac{1}{2},\pm\frac{1}{2})$ and $(0,\pm1)$ are generated by utilising the dressed monopoles of $(\frac{1}{2},\pm\frac{1}{2})$ and suitable elements in their Weyl-orbits.

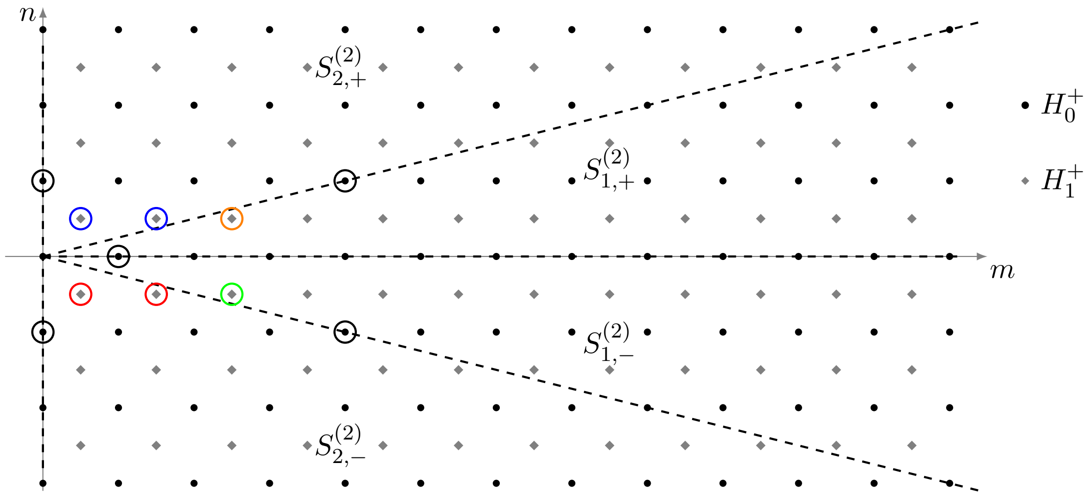

Figure 8. The Weyl-chamber for odd a, here with the example $a = 3$. The black circled lattice points correspond to the ray generators originating from the fan. The blue/red circled points are the remaining minimal generators for $S^{(2)}_{2,\pm}$, respectively. Similarly, the orange/green circled point are the generators that complete the Hilbert basis for $S^{(2)}_{1,\pm}$.

4.3 N hypermultiplets in the adjoint representation of SU(2)

The conformal dimension for a U(2)-theory with N hypermultiplets transforming in the adjoint representation of SU(2) and arbitrary even U(1)-charge is given by

$$\Delta(n,m) = \frac{N}{2}\big(|(2a+2)n+2m| + |(2a+2)n| + |(2a+2)n-2m|\big) - 2|m| \,. \tag{4.13}$$

Already at this stage, one can define the four semi-groups induced by the conformal dimension, which originate from 2-dimensional cones

$$S^{(2)}_{2,\pm} = \Big\{(m,n) \in \Lambda_w^{\mathrm{U}(2)}/S_2 \mid (m \geq 0) \wedge (m \leq \pm(a+1)n) \wedge (\pm n \geq 0)\Big\}\,, \tag{4.14a}$$
$$S^{(2)}_{1,\pm} = \Big\{(m,n) \in \Lambda_w^{\mathrm{U}(2)}/S_2 \mid (m \geq 0) \wedge (m \geq \pm(a+1)n) \wedge (\pm n \geq 0)\Big\}\,. \tag{4.14b}$$

It turns out that the precise form of the Hilbert basis depends on the divisibility of a by 2; thus, we split the considerations in two cases: $a = 2k-1$ and $a = 2k$.

4.3.1 Case: $a = 1 \mod 2$

Hilbert basis. The collection of semi-groups (4.14) is depicted in figure 8. As before, we compute the Hilbert basis $\mathcal{H}$ for each semi-group of the minimal generators.

$$\mathcal{H}(S^{(2)}_{2,\pm}) = \Big\{(0,\pm1),(2k,\pm1),\big\{\big(j+\tfrac{1}{2},\pm\tfrac{1}{2}\big) \mid j=0,\dots,k-1\big\}\Big\}\,, \tag{4.15a}$$
$$\mathcal{H}(S^{(2)}_{1,\pm}) = \Big\{(2k,\pm1),\big(k+\tfrac{1}{2},\pm\tfrac{1}{2}\big),(1,0)\Big\}\,. \tag{4.15b}$$

Hilbert series. The computation of the Hilbert series yields

$$\mathrm{HS}_{\mathrm{U}(2)}^{[2,2k-1]}(t,z,N) = \frac{R(t,z,N)}{P(t,z,N)} \,, \tag{4.16a}$$

$$\begin{aligned}
P(t,z,N) =\; & \left(1-t^2\right)^2 \left(1-t^4\right) \left(1-t^{4N-4}\right) \left(1-\tfrac{1}{z^2}t^{12kN}\right) \left(1-z^2 t^{12kN}\right) \\
& \times \left(1-\tfrac{1}{z^2}t^{12kN-8k}\right) \left(1-z^2 t^{12kN-8k}\right) \,, \\
R(t,z,N) =\; & 1 - t^2 + t^{4N-2} - t^{4N} t^{24kN} + t^{24kN+2} - t^{24kN-16k} - t^{24kN-16k+2} \\
& - t^{24kN+4N-2} - t^{24kN+4N} + t^{24kN-16k+4N} + t^{24kN-16k+4N-2} \\
& - t^{48kN-16k} + t^{48kN-16k+2} + t^{48kN-16k+4N} - t^{48kN-16k+4N-2} \\
& + \left(z+\tfrac{1}{z}\right) \Big(- t^{6kN+2} + t^{6kN-4k+2} + t^{6kN-4k+2N-2} - t^{6kN-4k+2N+2} \\
& + t^{6kN+4N-2} - t^{6kN-4k+4N-2} + t^{18kN+2} - t^{18kN-4k+2} + t^{18kN-8k+2} \\
& - t^{18kN-12k+2} - t^{18kN-4k+2N-2} + t^{18kN-4k+2N+2} - t^{18kN-12k+2N-2} \\
& + t^{18kN-12k+2N+2} - t^{18kN+4N-2} + t^{18kN-4k+4N-2} - t^{18kN-8k+4N-2} \\
& + t^{18kN-12k+4N-2} + t^{30kN-4k+2} - t^{30kN-8k+2} + t^{30kN-12k+2} \\
& - t^{30kN-16k+2} + t^{30kN-4k+2N-2} - t^{30kN-4k+2N+2} + t^{30kN-12k+2N-2} \\
& - t^{30kN-12k+2N+2} - t^{30kN-4k+4N-2} + t^{30kN-8k+4N-2} - t^{30kN-12k+4N-2} \\
& + t^{30kN-16k+4N-2} - t^{42kN-12k+2} + t^{42kN-16k+2} - t^{42kN-12k+2N-2} \\
& + t^{42kN-12k+2N+2} + t^{42kN-12k+4N-2} - t^{42kN-16k+4N-2} \Big) \\
& + \left(z^2+\tfrac{1}{z^2}\right) \Big(- t^{12kN} + t^{12kN-8k+2} + t^{12kN+4N} - t^{12kN-8k+4N-2} \\
& + t^{36kN-16k} - t^{36kN-8k+2} - t^{36kN-16k+4N} + t^{36kN-8k+4N-2} \Big) \\
& + \left(z^3+\tfrac{1}{z^3}\right) \Big(- t^{18kN-4k+2} + t^{18kN-8k+2} - t^{18kN-4k+2N-2} + t^{18kN-4k+2N+2} \\
& + t^{18kN-4k+4N-2} - t^{18kN-8k+4N-2} - t^{30kN-8k+2} + t^{30kN-12k+2} \\
& + t^{30kN-12k+2N-2} - t^{30kN-12k+2N+2} + t^{30kN-8k+4N-2} - t^{30kN-12k+4N-2} \Big) \,.
\end{aligned} \tag{4.16b}$$

Inspection of the Hilbert series (4.16) reveals that it has a pole of order 4 as $t \to 1$ because one explicitly verifies $R(t=1,z,N)=0$, $\frac{\mathrm{d}}{\mathrm{d}t}R(t,z,N)|_{t=1}=0$, and $\frac{\mathrm{d}^n}{\mathrm{d}t^n}R(t,z,N)|_{t=1,z=1}=0$ for $n=2,3$.

Discussion. The denominator of the Hilbert series (4.16) displays poles for the five bare monopole operators $(0,\pm1)$, $(2k,\pm1)$, and $(1,0)$, which are ray generators and charged under $\mathrm{U}(1)_J$ as ±2, ±2, and 0, respectively. The remaining operators, corresponding to the minimal generators which are not ray generators, are apparent in the analysis of the plethystic logarithm. The relevant bare and dressed monopole operators are summarised in table 3.

The plethystic logarithm, moreover, displays that not always all monopoles of the family $(j+\frac{1}{2},\pm\frac{1}{2})$ are generators (in the sense of the PL). The observation is: if $k-j<N$

(m,n)	(m_1,m_2)	$2\Delta(m,n)$	$H_{(m,n)}$	dressings
$(1,0)$	$(1,-1)$	$4N-4$	$\mathrm{U}(1)^2$	1 by U(1)
$(j+\frac{1}{2},\frac{1}{2})$, for $j=0,\dots,k-1$	$(j+1,-j)$	$6kN-4j-2$	$\mathrm{U}(1)^2$	1 by U(1)
$(j+\frac{1}{2},-\frac{1}{2})$, for $j=0,\dots,k-1$	$(j,-(j+1))$	$6kN-4j-2$	$\mathrm{U}(1)^2$	1 by U(1)
$(k+\frac{1}{2},\frac{1}{2})$	$(k+1,-k)$	$6kN+2N-4k-2$	$\mathrm{U}(1)^2$	1 by U(1)
$(k+\frac{1}{2},-\frac{1}{2})$	$(k,-(k+1))$	$6kN+2N-4k-2$	$\mathrm{U}(1)^2$	1 by U(1)
$(0,\pm1)$	$\pm(1,1)$	$12kN$	$\mathrm{U}(2)$	none
$(2k,1)$	$(2k+1,1-2k)$	$12kN-8k$	$\mathrm{U}(1)^2$	1 by U(1)
$(2k,-1)$	$(2k-1,-(2k+1))$	$12kN-8k$	$\mathrm{U}(1)^2$	1 by U(1)

Table 3. Summary of the monopole operators for odd a.

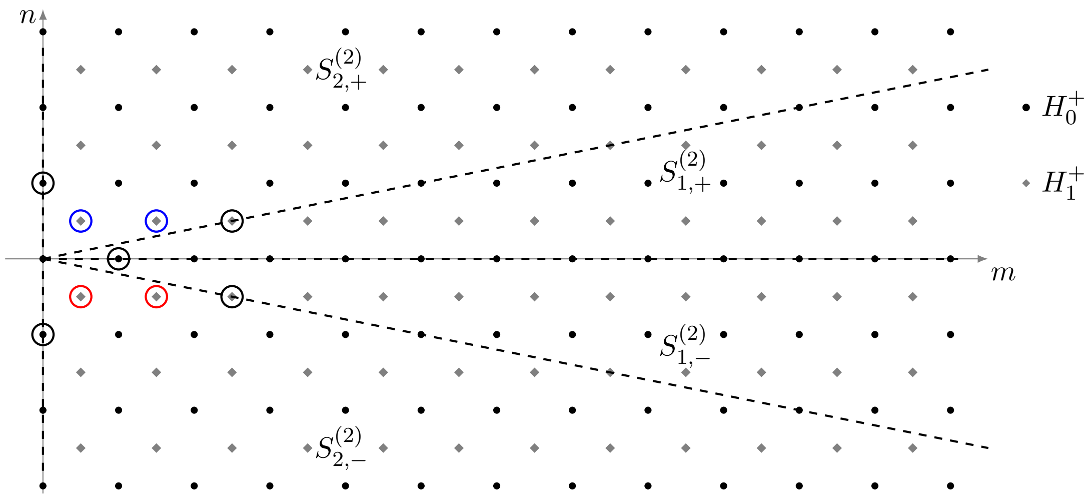

Figure 9. The Weyl-chamber for $a = 0 \mod 2$, here with the example $a = 4$. The black circled lattice points correspond to the ray generators originating from the fan. The blue/red circled points are the remaining minimal generators for $S^{(2)}_{2,\pm}$, respectively.

then the j-th operator (bare as well as dressed) is truely a generator in the PL. The reason behind lies in a relation at degree $\Delta(k-\frac{1}{2},\pm\frac{1}{2})+\Delta(1,0)$, which coincides with $\Delta(j+\frac{1}{2},\pm\frac{1}{2})$ for $k-j=N$. (See also appendix A.) Hence, for large enough N all above listed operators are generators.

4.3.2 Case: $a = 0 \mod 2$

Hilbert basis. The diagram for the minimal generators is provided in figure 9. Again, the appearing (bare) monopoles correspond to the Hilbert basis of the semi-groups.

$$\mathcal{H}(S^{(2)}_{2,\pm}) = \left\{(0,\pm1), \left\{ \left(j+\tfrac{1}{2},\pm\tfrac{1}{2}\right) \ , \ j=0,1,\dots,k \right\}\right\} , \tag{4.17a}$$

$$\mathcal{H}(S^{(2)}_{1,\pm}) = \left\{ \left(k+\tfrac{1}{2},\pm\tfrac{1}{2}\right), (1,0) \right\} . \tag{4.17b}$$

(m,n)	(m_1,m_2)	$2\Delta(m,n)$	$H_{(m,n)}$	dressings
$(1,0)$	$(1,-1)$	$4N-4$	$\mathrm{U}(1)^2$	1 by U(1)
$(j+\frac{1}{2},\frac{1}{2})$, for $j=0,1,\dots,k$	$(j+1,-j)$	$6kN+3N-4j-2$	$\mathrm{U}(1)^2$	1 by U(1)
$(j+\frac{1}{2},-\frac{1}{2})$, for $j=0,1,\dots,k$	$(j,-(j+1))$	$6kN+3N-4j-2$	$\mathrm{U}(1)^2$	1 by U(1)
$(0,\pm1)$	$\pm(1,1)$	$12kN+6N$	U(2)	none

Table 4. Summary of the monopole operators for even a.

Hilbert series. The computation of the Hilbert series for this case yields

$$\mathrm{HS}_{\mathrm{U}(2)}^{[2,2k]}(t,z,N)=\frac{R(t,z,N)}{P(t,z,N)}\,, \tag{4.18a}$$

$$\begin{aligned}P(t,z,N)=&\left(1-t^2\right)^2\left(1-t^4\right)\left(1-t^{4N-4}\right)\left(1-\tfrac{1}{z}t^{6kN-4k+3N-2}\right)\left(1-zt^{6kN-4k+3N-2}\right)\\&\times\left(1-\tfrac{1}{z^2}t^{12kN+6N}\right)\left(1-z^2t^{12kN+6N}\right)\,,\end{aligned} \tag{4.18b}$$

$$\begin{aligned}R(t,z,N)=1&-t^2+t^{4N-2}-t^{4N}+2t^{12kN-4k+6N}-t^{12kN-8k+6N-4}-t^{12kN-8k+6N-2}\\&-2t^{12kN-4k+10N-4}+t^{12kN-8k+10N-6}+t^{12kN-8k+10N-4}+t^{24kN+12N}\\&+t^{24kN+12N+2}-2t^{24kN-4k+12N}-t^{24kN+16N-2}-t^{24kN+16N}\\&+2t^{24kN-4k+16N-4}-t^{36kN-8k+18N-4}+t^{36kN-8k+18N-2}\\&-t^{36kN-8k+22N-6}+t^{36kN-8k+22N-4}\\&+\left(z+\tfrac{1}{z}\right)\Big(-t^{6kN+3N+2}+t^{6kN-4k+3N}+t^{6kN+7N-2}-t^{6kN-4k+7N-4}\\&\quad+t^{18kN+9N+2}-t^{18kN-8k+9N-2}-t^{18kN+13N-2}+t^{18kN-8k+13N-6}\\&\quad-t^{30kN-4k+15N}+t^{30kN-8k+15N-2}+t^{30kN-4k+19N-4}-t^{30kN-8k+19N-6}\Big)\\&+\left(z^2+\tfrac{1}{z^2}\right)\Big(-t^{12kN+6N}+t^{12kN-4k+6N}+t^{12kN+10N}-t^{12kN-4k+10N-4}\\&\quad-t^{24kN-4k+12N}+t^{24kN-8k+12N-4}+t^{24kN-4k+16N-4}-t^{24kN-8k+16N-4}\Big)\,.\end{aligned} \tag{4.18c}$$

The Hilbert series (4.18) has a pole of order 4 as $t\to1$ because one can explicitly verify that $R(t=1,z,N)=0$, $\frac{\mathrm{d}}{\mathrm{d}t}R(t,z,N)|_{t=1}=0$, and $\frac{\mathrm{d}^n}{\mathrm{d}t^n}R(t,z,N)|_{t=1,z=1}=0$ for $n=2,3$.

Discussion. The five monopoles corresponding to the ray generators, i.e. $(0,\pm1)$, $(k+\frac{1}{2},\pm\frac{1}{2})$, and $(1,0)$, appear as poles in the Hilbert series (4.18) and are charged under $\mathrm{U}(1)_J$ as ±2, ±1, and 0, respectively. The remaining minimal generator can be deduced by inspecting the plethystic logarithm. We summarise the monopole generators in table 4. Similarly to the case of odd a, the plethystic logarithm displays that not always all monopoles of the family $(j+\frac{1}{2},\pm\frac{1}{2})$ are generators. The observation is: if $k-j+1\geq N$ then the j-th bare operator is a generator in the PL, while for $k-j+2\geq N$ then also the dressing of the j-th monopole is a generator. The reason behind lies, again, in a relation at degree $\Delta(k-\frac{1}{2},\pm\frac{1}{2})+\Delta(1,0)+2$, which coincides with $\Delta(j+\frac{1}{2},\pm\frac{1}{2})$ for $k-j=N$. (See also appendix A.) Hence, for large enough N all above listed operators are generators.

4.4 Direct product of SU(2) and U(1)

A rather simple example is obtained by considering the non-interacting product of an SU(2) and a U(1) theory. Nonetheless, it illustrates how the rank two Coulomb branches contain the product of rank one Coulomb branches as subclasses.

As first example, take N_1 fundamentals of SU(2) and N_2 hypermultiplets charged under U(1) with charges $a \in \mathbb{N}$. The conformal dimension is given by

$$\Delta(m,n) = (N_1 - 2)|m| + \frac{N_2 \cdot a}{2}|n| \qquad \text{for} \quad m \in \mathbb{N} \quad \text{and} \quad n \in \mathbb{Z} \tag{4.19}$$

and the dressing factor splits as

$$P_{\mathrm{SU}(2)}(t,m,n) = P_{\mathrm{SU}(2)}(t,m) \times P_{\mathrm{U}(1)}(t,n) \; , \tag{4.20}$$

such that the Hilbert series factorises

$$\mathrm{HS}^{[1],a}_{\mathrm{SU}(2)\times\mathrm{U}(1)}(t,N_1,N_2) = \mathrm{HS}^{[1]}_{\mathrm{SU}(2)}(t,N_1) \times \mathrm{HS}^{a}_{\mathrm{U}(1)}(t,N_2) \; . \tag{4.21}$$

The rank one Hilbert series have been presented in [5]. Moreover, $\mathrm{HS}^{a}_{\mathrm{U}(1)}(t,N_2)$ equals the $A_{a\cdot N_2 -1}$ singularity $\mathbb{C}^2/\mathbb{Z}_{a\cdot N_2}$; whereas $\mathrm{HS}^{[1]}_{\mathrm{SU}(2)}(t,N_1)$ is precisely the D_{N_1} singularity.

The second, follow-up example is simply a theory comprise of N_1 hypermultiplets in the adjoint representation of SU(2) and N_2 hypermultiplets charged under U(1) as above. The conformal dimension is modified to

$$\Delta(m,n) = 2(N_1 - 1)|m| + \frac{N_2 \cdot a}{2}|n| \qquad \text{for} \quad m \in \mathbb{N} \quad \text{and} \quad n \in \mathbb{Z} \tag{4.22}$$

and Hilbert series is obtained as

$$\mathrm{HS}^{[2],a}_{\mathrm{SU}(2)\times\mathrm{U}(1)}(t,N_1,N_2) = \mathrm{HS}^{[2]}_{\mathrm{SU}(2)}(t,N_1) \times \mathrm{HS}^{a}_{\mathrm{U}(1)}(t,N_2) \; . \tag{4.23}$$

Applying the results of [5], $\mathrm{HS}^{[2]}_{\mathrm{SU}(2)}(t,N_1)$ is the Hilbert series of the D_{2N_1}-singularity on $\mathbb{C}^2$.

Summarising, the direct product of these SU(2)-theories with U(1)-theories results in moduli spaces that are products of A and D type singularities, which are complete intersections. Moreover, any non-trivial interactions between these two gauge groups, as discussed in subsection 4.2 and 4.3, leads to a very elaborate expression for the Hilbert series as rational functions. Also, the Hilbert basis becomes an important concept for understanding the moduli space.

5 Case: $A_1 \times A_1$

This section concerns all Lie groups with Lie algebra D_2, which allows to study products of the rank one gauge groups SO(3) and SU(2), but also the proper rank two group SO(4).

5.1 Set-up

Let us consider the Lie algebra $D_2 \cong A_1 \times A_1$. Following [9], there are five different groups with this Lie algebra. The reason is that the universal covering group $\widetilde{\mathrm{SO}}(4)$ of SO(4) has a non-trivial centre $\mathcal{Z}(\widetilde{\mathrm{SO}}(4)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ of order 4. The quotient of $\widetilde{\mathrm{SO}}(4)$ by any of the five

Quotient	isomorphic group G	GNO-dual $\widehat{\mathrm{G}}$	$\mathcal{Z}(\widehat{\mathrm{G}})$	GNO-charges (m_1, m_2)
$\frac{\widetilde{\mathrm{SO}}(4)}{\{1\}}$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$	$\mathrm{SO}(3) \times \mathrm{SO}(3)$	$\{1\}$	$K^{[0]}$
$\frac{\widetilde{\mathrm{SO}}(4)}{\mathbb{Z}_2 \times \{1\}}$	$\mathrm{SO}(3) \times \mathrm{SU}(2)$	$\mathrm{SU}(2) \times \mathrm{SO}(3)$	$\mathbb{Z}_2 \times \{1\}$	$K^{[0]} \cup K^{[1]}$
$\frac{\widetilde{\mathrm{SO}}(4)}{\mathrm{diag}(\mathbb{Z}_2)}$	$\mathrm{SO}(4)$	$\mathrm{SO}(4)$	$\mathbb{Z}_2$	$K^{[0]} \cup K^{[2]}$
$\frac{\widetilde{\mathrm{SO}}(4)}{\{1\} \times \mathbb{Z}_2}$	$\mathrm{SU}(2) \times \mathrm{SO}(3)$	$\mathrm{SO}(3) \times \mathrm{SU}(2)$	$\{1\} \times \mathbb{Z}_2$	$K^{[0]} \cup K^{[3]}$
$\frac{\widetilde{\mathrm{SO}}(4)}{\mathbb{Z}_2 \times \mathbb{Z}_2}$	$\mathrm{SO}(3) \times \mathrm{SO}(3)$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$K^{[0]} \cup K^{[1]} \cup K^{[2]} \cup K^{[3]}$

Table 5. All the Lie groups that arise taking the quotient of $\widetilde{\mathrm{SO}}(4)$ by a subgroup of its centre; hence, their Lie algebra is D_2.

different subgroups $\mathcal{Z}(\widetilde{\mathrm{SO}}(4))$ yields a Lie group with the same Lie algebra. Fortunately, working with SO(4) allows to use the isomorphism $\widetilde{\mathrm{SO}}(4) = \mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. We can summarise the setting as displayed in table 5. Here, we employed $\widehat{\mathrm{SU}(2)} = \mathrm{SO}(3)$ and that for semi-simple groups G_1, G_2

$$\widehat{\mathrm{G}_1 \times \mathrm{G}_2} = \widehat{\mathrm{G}}_1 \times \widehat{\mathrm{G}}_2 \tag{5.1}$$

holds [9]. Moreover, the GNO-charges are defined via the following sublattices of the weight lattice of Spin(4) (see also figure 10)

$$K^{[0]} = \Big\{ (m_1, m_2) \mid m_i = p_i \in \mathbb{Z} \,,\ p_1 + p_2 = \text{even} \Big\} \,, \tag{5.2a}$$

$$K^{[1]} = \Big\{ (m_1, m_2) \mid m_i = p_i + \tfrac{1}{2} \,,\ p_i \in \mathbb{Z} \,,\ p_1 + p_2 = \text{even} \Big\} \,, \tag{5.2b}$$

$$K^{[2]} = \Big\{ (m_1, m_2) \mid m_i = p_i \in \mathbb{Z} \,,\ p_1 + p_2 = \text{odd} \Big\} \,, \tag{5.2c}$$

$$K^{[3]} = \Big\{ (m_1, m_2) \mid m_i = p_i + \tfrac{1}{2} \,,\ p_i \in \mathbb{Z} \,,\ p_1 + p_2 = \text{odd} \Big\} \,. \tag{5.2d}$$

The important consequence of this set-up is that the fan defined by the conformal dimension will be the same for a given representation in each of the five quotients, but the semi-groups will differ due to the different lattices $\Lambda_w(\widehat{\mathrm{G}})$ used in the intersection. Hence, we will find different Hilbert basis in each quotient group. Nevertheless, we are forced to consider representations on the root lattice as we otherwise cannot compare all quotients.

Dressings. In addition, we have chosen to parametrise the principal Weyl chamber via $m_1 \geq |m_2|$ such that the classical dressing factors are given by [5]

$$P_{A_1 \times A_1}(t, m_1, m_2) = \begin{cases} \dfrac{1}{(1-t^2)^2} \,, & \text{for} \quad m_1 = m_2 = 0 \,, \\ \dfrac{1}{(1-t)(1-t^2)} \,, & \text{for} \quad m_1 = |m_2| > 0 \,, \\ \dfrac{1}{(1-t)^2} \,, & \text{for} \quad m_1 > |m_2| \geq 0 \,. \end{cases} \tag{5.3}$$

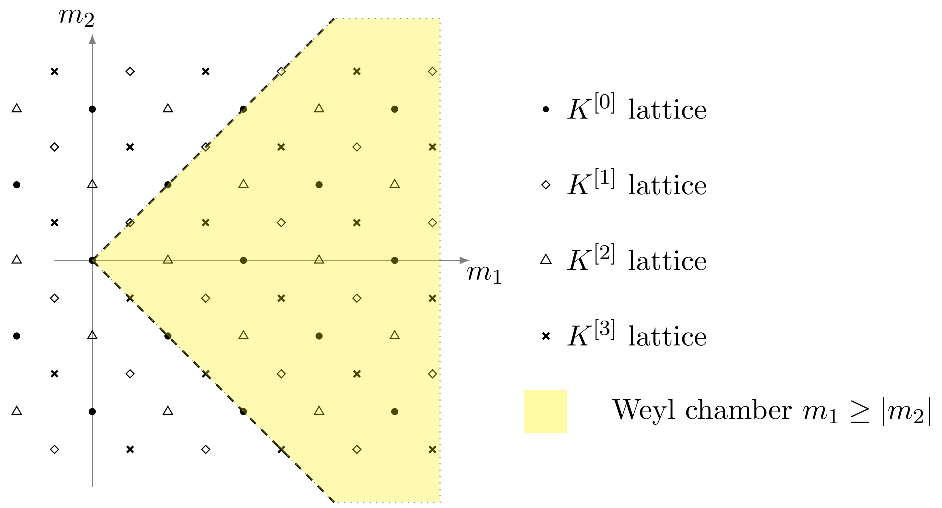

Figure 10. The four different sublattices of the covering group of SO(4). One recognises the root lattice $\Lambda_r^{\widetilde{\mathrm{SO}}(4)} = K^{[0]}$ and the weight lattice $\Lambda_w^{\widetilde{\mathrm{SO}}(4)} = K^{[0]} \cup K^{[1]} \cup K^{[2]} \cup K^{[3]}$.

Regardless of the quotient $\widetilde{\mathrm{SO}(4)}/\Gamma$, the space of Casimir invariance is 2-dimensional. We choose a basis such that the two degree 2 Casimir invariants stem either from SU(2) or SO(3), i.e.[1]

$$\mathrm{diag}(\Phi) = (\phi_1, \phi_2) \quad \longrightarrow \quad \mathcal{C}_2^{(i)} = (\phi_i)^2 \,. \tag{5.4}$$

Next, we can clarify all relevant bare and dressed monopole operators for an (m_1, m_2) that is a minimal generator. There are two cases: on the one hand, for $m_2 = \pm m_1$, i.e. at the boundary of the Weyl chamber, the residual gauge group is either $\mathrm{U}(1)_i \times \mathrm{SU}(2)_j$ or $\mathrm{U}(1)_i \times \mathrm{SO}(3)_j$ (for $i, j = 1, 2$ and $i \neq j$), depending on the quotient under consideration. Thus, only the degree 1 Casimir invariant of the $\mathrm{U}(1)_i$ can be employed for a dressing, as the Casimir invariant of $\mathrm{SU}(2)_j$ or $\mathrm{SO}(3)_j$ belongs to the quotient $\widetilde{\mathrm{SO}(4)}/\Gamma$ itself. Hence, we get

$$V^{\mathrm{dress},0}_{(m_1,\pm m_1)} = (m_1, \pm m_1) \quad \text{and} \quad V^{\mathrm{dress},1}_{(m_1,\pm m_1)} = \phi_i \ (m_1, \pm m_1) \,. \tag{5.5a}$$

Alternatively, we can apply the results of appendix A and deduce the dressing behaviour at the boundary of the Weyl chamber to be $P_{A_1 \times A_1}(t, m_1, \pm m_1)/P_{A_1 \times A_1}(t, 0, 0) = 1 + t$, i.e. only one dressed monopole arises.

On the other hand, for $m_1 > |m_2| \geq 0$, i.e. in the interior of the Weyl chamber, the residual gauge group is $\mathrm{U}(1)^2$. From the resulting two degree 1 Casimir invariants one constructs the following monopole operators:

$$V^{\mathrm{dress},0}_{(m_1,m_2)} = (m_1, m_2) \quad \longrightarrow \quad \begin{cases} V^{\mathrm{dress},1,i}_{(m_1,m_2)} = \phi_i \ (m_1, m_2) \,, & \text{for} \quad i = 1, 2 \\ V^{\mathrm{dress},2}_{(m_1,m_2)} = \phi_1 \phi_2 \ (m_1, m_2) \,. & \end{cases} \tag{5.5b}$$

[1] In a different basis, the Casimir invariants for SO(4) are the quadratic Casimir and the Pfaffian.

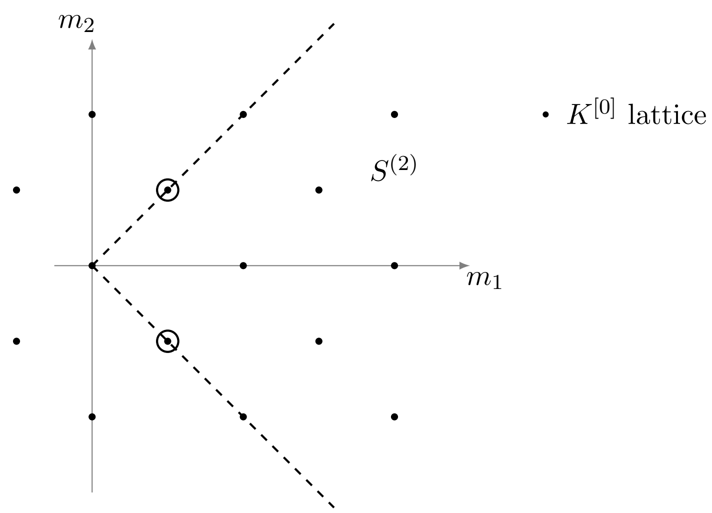

Figure 11. The semi-group $S^{(2)}$ and its ray-generators (black circled points) for the quotient Spin(4) and the representation $[2, 0]$.

Using appendix A, we obtain that monopole operator with GNO-charge in the interior of the Weyl chamber exhibit the following dressings $P_{A_1\times A_1}(t, m_1, m_2)/P_{A_1\times A_1}(t, 0, 0) = 1 + 2t + t^2$, which agrees with our discussion above.

5.2 Representation $[2, 0]$

The conformal dimension for this case reads

$$\Delta(m_1, m_2) = (N - 1) \left(|m_1 + m_2| + |m_1 - m_2|\right) \ . \tag{5.6}$$

Following the ideas outlined earlier, the conformal dimension (5.6) defines a fan in the dominant Weyl chamber. In this example, Δ is already a linear function on the entire dominant Weyl chamber; thus, we generate a fan which just consists of one 2-dimensional rational cone

$$C^{(2)} = \Big\{ (m_1 \geq m_2) \wedge (m_1 \geq -m_2) \Big\} \ . \tag{5.7}$$

5.2.1 Quotient Spin(4)

Hilbert basis. Starting from the fan (5.7) with the cone $C^{(2)}$, the Hilbert basis for the semi-group $S^{(2)} := C^{(2)} \cap K^{[0]}$ is simply given by the ray generators

$$\mathcal{H}(S^{(2)}) = \Big\{ (1, 1), (1, -1) \Big\} \ , \tag{5.8}$$

see for instance figure 11. Both minimal generators exhibit a bare monopole operator and one dressed operators, as explained in (5.5).

Hilbert series. We compute the Hilbert series to

$$\mathrm{HS}^{[2,0]}_{\mathrm{Spin}(4)}(t, N) = \frac{\left(1 - t^{4N-2}\right)^2}{\left(1 - t^2\right)^2 \left(1 - t^{2N-2}\right)^2 \left(1 - t^{2N-1}\right)^2} \ , \tag{5.9}$$

which is a complete intersection with 6 generators and 2 relations. The generators are given in table 6.

object	(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(1,\pm 1)$	$K^{[0]}$	$2N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by U(1)

Table 6. Bare and dressed monopole generators for a Spin(4) gauge theory with matter transforming in $[2,0]$.

Remark. The Hilbert series (5.9) can be compared to the case of SU(2) with n fundamentals and n_a adjoints such that $2N = n + 2n_a$, cf. [5]. One derives at

$$\mathrm{HS}_{\mathrm{Spin}(4)}^{[2,0]}(t,N) = \mathrm{HS}_{\mathrm{SU}(2)}^{[1]+[2]}(t,n,n_a) \times \mathrm{HS}_{\mathrm{SU}(2)}^{[1]+[2]}(t,n,n_a)\ , \tag{5.10}$$

which equals the product of two D_{2N} singularities. As a consequence, the minimal generator $(1,1)$ belongs to one SU(2) Hilbert series with adjoint matter content, while $(1,-1)$ belongs to the other.

5.2.2 Quotient SO(4)

The centre of the GNO-dual SO(4) is a $\mathbb{Z}_2$, which we choose to count if (m_1,m_2) belongs to $K^{[0]}$ or $K^{[2]}$. A realisation is given by

$$z^{m_1+m_2} = \begin{cases} z^{\mathrm{even}} = 1 & \text{for} \quad (m_1,m_2) \in K^{[0]}\ , \\ z^{\mathrm{odd}} = z & \text{for} \quad (m_1,m_2) \in K^{[2]}\ . \end{cases} \tag{5.11}$$

In other words, z is a $\mathbb{Z}_2$-fugacity.

Hilbert basis. The semi-group $S^{(2)} := C^{(2)} \cap \left(K^{[0]} \cup K^{[2]\cdot}\right)$ has a Hilbert basis as displayed in figure 12 or explicitly

$$\mathcal{H}(S^{(2)}) = \Big\{(1,\pm1),(1,0)\Big\}\ . \tag{5.12}$$

Hilbert series. The Hilbert series for SO(4) is given by

$$\mathrm{HS}_{\mathrm{SO}(4)}^{[2,0]}(t,z,N) = \frac{1+t^{2N-2}+2t^{2N-1}+zt^{2N}+2zt^{2N-1}+zt^{4N-2}}{(1-t^2)^2\,(1-t^{2N-2})\,(1-zt^{2N-2})}\ , \tag{5.13}$$

which is a rational function with a palindromic polynomial of degree $4N-2$ as numerator, while the denominator is of degree $4N$. Hence, the difference in degrees is 2, i.e. the quaternionic dimension of the moduli space. In addition, the denominator (5.13) has a pole of order 4 at $t \to 1$, which equals the complex dimension of the moduli space.

Plethystic logarithm. Analysing the PL yields for $N \geq 3$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}_{\mathrm{SO}(4)}^{[2,0]}) = 2t^2 &+ zt^{\Delta(1,0)}(1+2t^2+t^2) + 2t^{\Delta(1,\pm1)}(1+t) \\ &- t^{2\Delta(1,0)}(1+2(1+z)t+(6+4z)t^2+2(1+z)t^3+t^4) + \ldots \end{aligned} \tag{5.14}$$

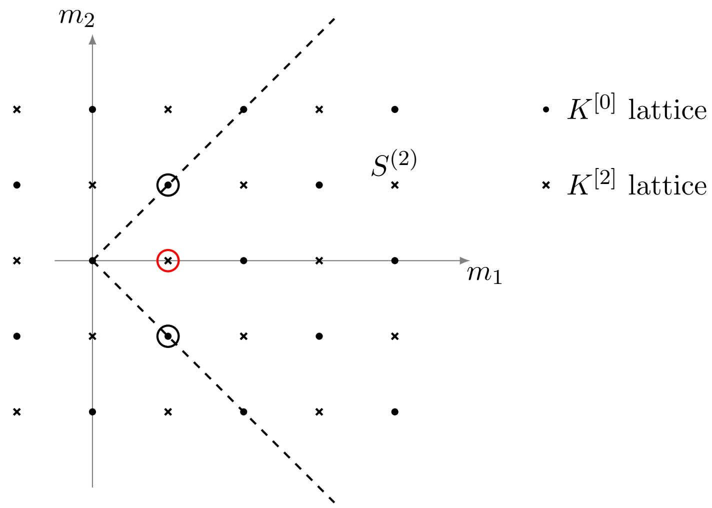

Figure 12. The semi-group $S^{(2)}$ and its ray-generators (black circled points) for the quotient SO(4) and the representation $[2,0]$. The red circled lattice point completes the Hilbert basis for $S^{(2)}$.

object	(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(1,0)$	$K^{[2]}$	$2N-2$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
monopole	$(1,\pm1)$	$K^{[0]}$	$2N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$

Table 7. Bare and dressed monopole generators for a SO(4) gauge theory with matter transforming in $[2,0]$.

and for $N=2$

$$\mathrm{PL}(\mathrm{HS}^{[2,0]}_{\mathrm{SO}(4)}) = 2t^2 + zt^2(1+2t+t^2) + 2t^2(1+t) - t^4(1+2(1+z)t+(6+4z)t^2) + \dots \tag{5.15}$$

such that we have generators as summarised in table 7.

Gauging a $\mathbb{Z}_2$. Although the Hilbert series (5.13) is not a complete intersection, the gauging of the topological $\mathbb{Z}_2$ reproduces the Spin(4) result (5.9), that is

$$\mathrm{HS}^{[2,0]}_{\mathrm{Spin}(4)}(t,N) = \frac{1}{2}\left(\mathrm{HS}^{[2,0]}_{\mathrm{SO}(4)}(t,z{=}1,N) + \mathrm{HS}^{[2,0]}_{\mathrm{SO}(4)}(t,z{=}-1,N)\right) . \tag{5.16}$$

5.2.3 Quotient SO(3) × SU(2)

The dual group is $\mathrm{SU}(2)\times\mathrm{SO}(3)$ and the summation extends over $(m_1,m_2)\in K^{[0]}\cup K^{[1]}$. The non-trivial centre $\mathbb{Z}_2\times\{1\}$ gives rise to a $\mathbb{Z}_2$-action, which we choose to distinguish the two lattices $K^{[0]}$ and $K^{[1]}$ as follows:

$$z_1^{m_1+m_2} = \begin{cases} z_1^{p_1+p_2} = z_1^{\mathrm{even}} = 1 & \text{for} \quad (m_1,m_2)\in K^{[0]} \, , \\ z_1^{p_1+\frac{1}{2}+p_2+\frac{1}{2}} = z_1^{\mathrm{even}+1} = z_1 & \text{for} \quad (m_1,m_2)\in K^{[1]} \, . \end{cases} \tag{5.17}$$

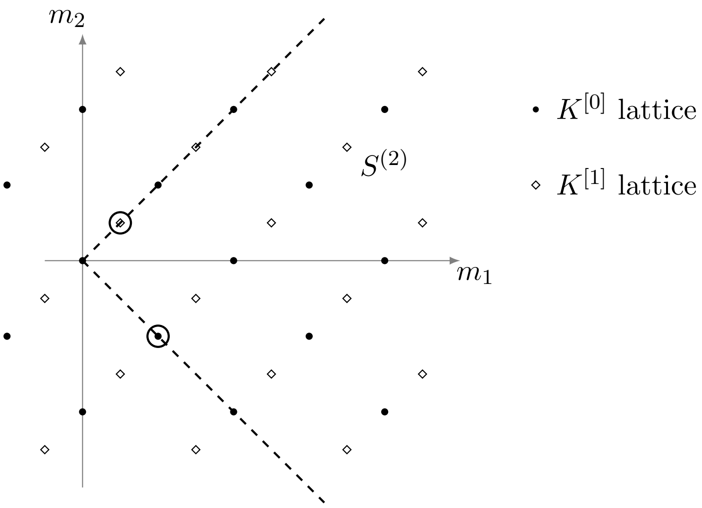

Figure 13. The semi-group $S^{(2)}$ for the quotient $\mathrm{SO}(3) \times \mathrm{SU}(2)$ and the representation $[2,0]$. The black circled points are the ray generators.

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(\frac{1}{2}, \frac{1}{2})$	$K^{[1]}$	$N-1$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(1, -1)$	$K^{[0]}$	$2N-2$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$

Table 8. Bare and dressed monopole generators for a $\mathrm{SO}(3) \times \mathrm{SU}(2)$ gauge theory with matter transforming in $[2,0]$.

Hilbert basis. The semi-group $S^{(2)} := C^{(2)} \cap \left(K^{[0]} \cup K^{[1]}\right)$ has a Hilbert basis comprised of the ray generators. We refer to figure 13 and provide the minimal generators for completeness:

$$\mathcal{H}(S^{(2)}) = \left\{ (\tfrac{1}{2}, \tfrac{1}{2}), (1, -1) \right\} . \tag{5.18}$$

Hilbert series. Computing the Hilbert series and using explicitly the $\mathbb{Z}_2$-properties of z_1 yields

$$\mathrm{HS}^{[2,0]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t, z_1, N) = \frac{\left(1-t^{2N}\right)\left(1-t^{4N-2}\right)}{\left(1-t^2\right)^2\left(1-t^{2N-2}\right)\left(1-t^{2N-1}\right)\left(1-z_1 t^{N-1}\right)\left(1-z_1 t^N\right)} , \tag{5.19}$$

which is a complete intersection with 6 generators and 2 relations. The generators are displayed in table 8.

Remark. Comparing to the case of $\mathrm{SU}(2)$ with n_a adjoints and $\mathrm{SO}(3)$ with n fundamentals presented in [5], we can re-express the Hilbert series (5.19) as the product

$$\mathrm{HS}^{[2,0]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t, z_1, N) = \mathrm{HS}^{[1]}_{\mathrm{SO}(3)}(t, z_1, n=N) \times \mathrm{HS}^{[2]}_{\mathrm{SU}(2)}(t, n_a=N) , \tag{5.20}$$

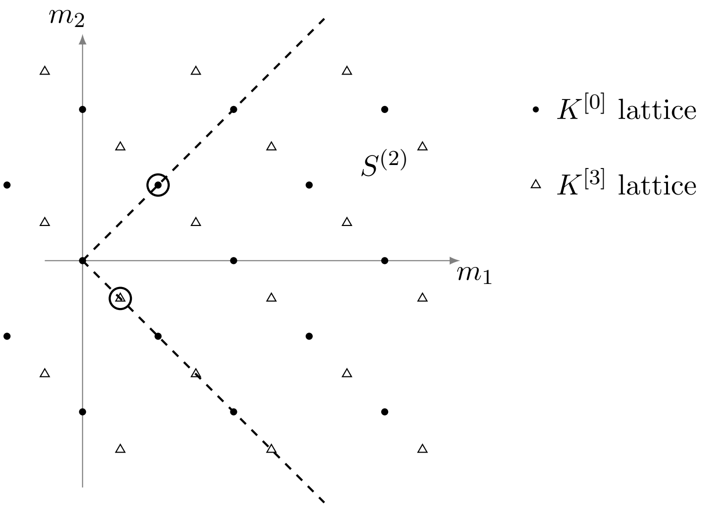

Figure 14. The semi-group $S^{(2)}$ for the quotient $\mathrm{SU}(2) \times \mathrm{SO}(3)$ and the representation $[2, 2]$. The black circled points are the ray generators.

where the z_1-grading belongs to SO(3) with N fundamentals. The minimal generator $(\frac{1}{2}, \frac{1}{2})$ is the minimal generator for SO(3) with N fundamentals, while $(1, -1)$ is the minimal generator for SU(2) with N adjoints.

5.2.4 Quotient SU(2) × SO(3)

The dual group is $\mathrm{SO}(3) \times \mathrm{SU}(2)$ and the summation extends over $(m_1, m_2) \in K^{[0]} \cup K^{[3]}$. The non-trivial centre $\{1\} \times \mathbb{Z}_2$ gives rise to a $\mathbb{Z}_2$-action, which we choose to distinguish the two lattices $K^{[0]}$ and $K^{[3]}$ as follows:

$$z_2^{p_1+p_2} = \begin{cases} z_2^{\mathrm{even}} = 1 & \text{for} \quad (m_1, m_2) \in K^{[0]} , \\ z_2^{\mathrm{odd}} = z_2 & \text{for} \quad (m_1, m_2) \in K^{[3]} . \end{cases} \tag{5.21}$$

Hilbert basis. The semi-group $S^{(2)} := C^{(2)} \cap \left(K^{[0]} \cup K^{[3]}\right)$ has as Hilbert basis the set of ray generators

$$\mathcal{H}(S^{(2)}) = \left\{(1, 1), (\tfrac{1}{2}, -\tfrac{1}{2})\right\} . \tag{5.22}$$

Figure 14 depicts the situation. We observe that bases (5.18) and (5.22) are related by reflection along the $m_2 = 0$ axis, which in turn corresponds to the interchange of $K^{[1]}$ and $K^{[3]}$.

Hilbert series. Similar to the previous case, employing the $\mathbb{Z}_2$-properties of z_2 we obtain the following Hilbert series:

$$\mathrm{HS}^{[2,0]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t, z_2, N) = \frac{\left(1 - t^{2N}\right)\left(1 - t^{4N-2}\right)}{\left(1 - t^2\right)^2\left(1 - t^{2N-2}\right)\left(1 - t^{2N-1}\right)\left(1 - z_2 t^{N-1}\right)\left(1 - z_2 t^N\right)} , \tag{5.23}$$

which is a complete intersection with 6 generators and 2 relations. We summarise the generators in table 9.

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(\frac{1}{2}, -\frac{1}{2})$	$K^{[3]}$	$N-1$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(1,1)$	$K^{[0]}$	$2N-2$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$

Table 9. Bare and dressed monopole generators for a $\mathrm{SU}(2) \times \mathrm{SO}(3)$ gauge theory with matter transforming in $[2, 0]$.

lattice	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\widetilde{\mathbb{Z}}_2 \times \widetilde{\mathbb{Z}}_2$
$K^{[0]}$	$(z_1)^0, (z_2)^0$	$(w_1)^0, (w_2)^0$
$K^{[1]}$	$(z_1)^1, (z_2)^0$	$(w_1)^1, (w_2)^1$
$K^{[2]}$	$(z_1)^0, (z_2)^1$	$(w_1)^0, (w_2)^1$
$K^{[3]}$	$(z_1)^1, (z_2)^1$	$(w_1)^1, (w_2)^0$

Table 10. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ distinguishes the four different lattice $K^{[j]}$, $j = 0, 1, 2, 3$. The choice of fugacities z_1, z_2 is used in the computation, while the second choice w_1, w_2 is convenient for gauging $\mathrm{PSO}(4)$ to $\mathrm{SU}(2) \times \mathrm{SO}(3)$.

Remark. Also, the equivalence

$$\mathrm{HS}^{[2,0]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t, z_1, N) \xleftrightarrow{z_1 \leftrightarrow z_2} \mathrm{HS}^{[2,0]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t, z_2, N) \tag{5.24}$$

holds, which then also implies

$$\mathrm{HS}^{[2,0]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t, z_2, N) = \mathrm{HS}^{[1]}_{\mathrm{SO}(3)}(t, z_2, n = N) \times \mathrm{HS}^{[2]}_{\mathrm{SU}(2)}(t, n_a = N) \, . \tag{5.25}$$

Thus, the moduli space is a product of two complete intersections.

5.2.5 Quotient PSO(4)

Taking the quotient with respect to the entire centre of $\widetilde{\mathrm{SO}(4)}$ yields the projective group $\mathrm{PSO}(4)$, which has as GNO-dual $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. Consequently, the summation extends over the whole weight lattice $K^{[0]} \cup K^{[1]} \cup K^{[2]} \cup K^{[3]}$ and there is an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on this lattice, which is chosen as displayed in table 10.

Hilbert basis. The semi-group $S^{(2)} := C^{(2)} \cap \left(K^{[0]} \cup K^{[1]} \cup K^{[2]} \cup K^{[3]}\right)$ has a Hilbert basis that is determined by the ray generators. Figure 15 depicts the situation and the Hilbert basis reads

$$\mathcal{H}(S^{(2)}) = \left\{ (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, -\tfrac{1}{2}) \right\} . \tag{5.26}$$

Hilbert series. An evaluation of the Hilbert series yields

$$\mathrm{HS}^{[2,0]}_{\mathrm{PSO}(4)}(t, z_1, z_2, N) = \frac{\left(1 - t^{2N}\right)^2}{(1-t^2)^2 \left(1 - z_1 t^{N-1}\right)\left(1 - z_1 t^{N}\right)\left(1 - z_1 z_2 t^{N-1}\right)\left(1 - z_1 z_2 t^{N}\right)} \, , \tag{5.27}$$

which is a complete intersection with 6 generators and 2 relations. Table 11 summarises the generators with their properties.

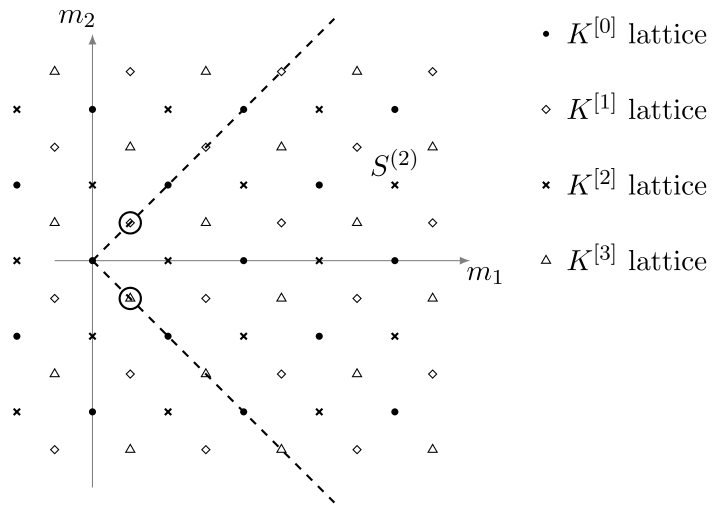

Figure 15. The semi-group $S^{(2)}$ and its ray-generators (black circled points) for the quotient PSO(4) and the representation $[2,0]$.

object	(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(\frac{1}{2},\frac{1}{2})$	$K^{[1]}$	$N-1$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by U(1)
monopole	$(\frac{1}{2},-\frac{1}{2})$	$K^{[3]}$	$N-1$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by U(1)

Table 11. Bare and dressed monopole generators for a PSO(4) gauge theory with matter transforming in $[2,0]$.

Gauging a $\mathbb{Z}_2$. Now, we utilise the $\mathbb{Z}_2\times\mathbb{Z}_2$ global symmetry to recover the Hilbert series for *all* five quotients solely from the PSO(4) result. Firstly, to obtain the SO(4) result, we need to average out the contributions of $K^{[1]}$ and $K^{[3]}$, which is achieved for $z_1\to\pm1$ (we also relabel z_2 for consistence), see also table 10. This yields

$$\mathrm{HS}_{\mathrm{SO}(4)}^{[2,0]}(t,z,N)=\frac{1}{2}\left(\mathrm{HS}_{\mathrm{PSO}(4)}^{[2,0]}(t,z_1{=}1,z_2{=}z,N)+\mathrm{HS}_{\mathrm{PSO}(4)}^{[2,0]}(t,z_1{=}-1,z_2{=}z,N)\right)\,. \tag{5.28a}$$

Secondly, a subsequent gauging leads to the Spin(4) result as demonstrated in (5.16), because one averages the $K^{[2]}$ contributions out. Thirdly, one can gauge the other $\mathbb{Z}_2$-factor corresponding to $z_2\to\pm1$, which then eliminates the contributions of $K^{[2]}$ and $K^{[3]}$ due to the choices of table 10. The result then reads

$$\mathrm{HS}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}^{[2,0]}(t,z_1,N)=\frac{1}{2}\left(\mathrm{HS}_{\mathrm{PSO}(4)}^{[2,0]}(t,z_1,z_2{=}1,N)+\mathrm{HS}_{\mathrm{PSO}(4)}^{[2,0]}(t,z_1,z_2{=}-1,N)\right)\,. \tag{5.28b}$$

Lastly, for obtaining the $\mathrm{SU}(2)\times\mathrm{SO}(3)$ Hilbert series one needs to eliminate the $K^{[1]}$ and $K^{[2]}$ contributions. For that, we have to redefine the $\mathbb{Z}_2$-fugacities conveniently. One choice is

$$z_1\cdot z_2\mapsto w_1\,,\qquad z_1\mapsto w_1\cdot w_2\,,\quad\text{and}\quad z_2\mapsto w_2\,, \tag{5.28c}$$

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which is consistent in $\mathbb{Z}_2 \times \mathbb{Z}_2$. The effect on the lattices is summarised in table 10. Hence, $w_2 \to \pm 1$ has the desired effect and leads to

$$\mathrm{HS}^{[2,0]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t, z_2{=}w_1, N) = \frac{1}{2}\left(\mathrm{HS}^{[2,0]}_{\mathrm{PSO}(4)}(t, w_1, w_2{=}1, N) + \mathrm{HS}^{[2,0]}_{\mathrm{PSO}(4)}(t, w_1, w_2{=}-1, N)\right) . \tag{5.28d}$$

Consequently, the Hilbert series for *all* five quotients can be computed from the PSO(4)-result by gauging $\mathbb{Z}_2$-factors.

Remark. As for most of the cases in this section, the Hilbert series (5.27) can be written as a product of two complete intersections. Employing the results of [5] for SO(3) with n fundamentals, we obtain

$$\mathrm{HS}^{[2,0]}_{\mathrm{PSO}(4)}(t, z_1, z_2, N) = \mathrm{HS}^{[1]}_{\mathrm{SO}(3)}(t, z_1, n = N) \times \mathrm{HS}^{[1]}_{\mathrm{SO}(3)}(t, z_1 z_2, n = N) . \tag{5.29}$$

5.3 Representation [2, 2]

Let us use the representation $[2, 2]$ to further compare the results for the five different quotient groups. The conformal dimension reads

$$\Delta(m_1, m_2) = N(|m_1 - m_2| + |m_1 + m_2| + 2\,|m_1| + 2\,|m_2|) - |m_1 - m_2| - |m_1 + m_2| . \tag{5.30}$$

As described in the introduction, the conformal dimension (5.30) defines a fan in the dominant Weyl chamber, which is spanned by two 2-dimensional rational cones

$$C^{(2)}_{\pm} = \left\{(m_1 \geq \pm m_2) \wedge (m_2 \geq \pm 0)\right\} . \tag{5.31}$$

5.3.1 Quotient Spin(4)

Hilbert basis. Starting from the fan (5.31) with cones $C^{(2)}_{\pm}$, the Hilbert bases for the semi-groups $S^{(2)}_{\pm} := C^{(2)}_{\pm} \cap K^{[0]}$ are simply given by the ray generators, see for instance figure 16.

$$\mathcal{H}(S^{(2)}_{\pm}) = \left\{(1, \pm 1), (2, 0)\right\} . \tag{5.32}$$

Hilbert series. The GNO-dual $\mathrm{SO}(3) \times \mathrm{SO}(3)$ has a trivial centre and the Hilbert series reads

$$\mathrm{HS}^{[2,2]}_{\mathrm{Spin}(4)}(t, N) = \frac{1 + t^{6N-2} + 2t^{6N-1} + 2t^{8N-3} + t^{8N-2} + t^{14N-4}}{(1-t^2)^2\,(1-t^{6N-2})\,(1-t^{8N-4})} . \tag{5.33}$$

The numerator of (5.33) is a palindromic polynomial of degree $14N - 4$; while the denominator is a polynomial of degree $14N - 2$. Hence, the difference in degree is two, which equals the quaternionic dimension of the moduli space. In addition, denominator of (5.33) has a pole of order four at $t = 1$, which equals the complex dimension of the moduli space.

Plethystic logarithm. The plethystic logarithm takes the form

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[2,2]}_{\mathrm{Spin}(4)}) = {} & 2t^2 + 2t^{\Delta(1,\pm1)}(1+t) + t^{\Delta(2,0)}(1 + 2t + t^2) \\ & - t^{2\Delta(1,\pm1)}(1 + 2t + 3t^2 + 2t^3 + 4t^4 + 2t^5 + 3t^6 + 2t^7 + t^8) + \ldots\end{aligned} \tag{5.34}$$

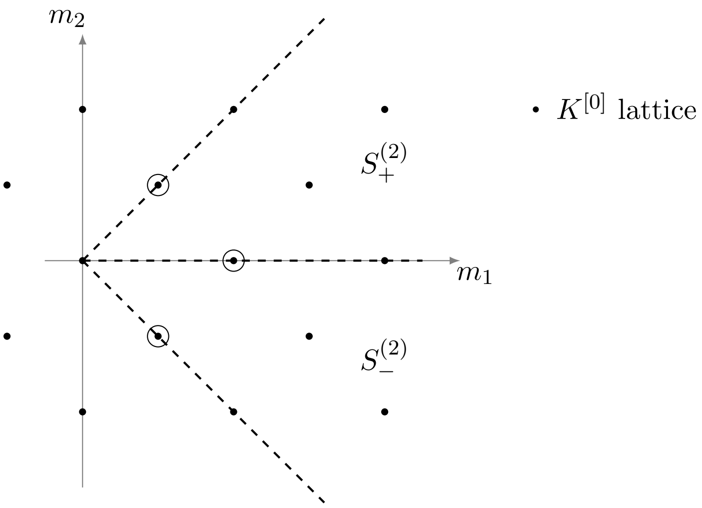

Figure 16. The semi-groups and their ray-generators (black circled points) for the quotient Spin(4) and the representation [2, 2].

The appearing terms agree with the minimal generators of the Hilbert bases (16). One has two independent degree two Casimir invariants. Further, there are monopole operators of GNO-charge (1, 1) and (1, −1) at conformal dimension $6N-2$ with an independent dressed monopole generator of conformal dimension $6N-1$ for both charges. Moreover, there is a monopole operator of GNO-charge (2, 0) at dimension $8N-4$ with two dressing operators at dimension $8N-3$ and one at $8N-2$.

5.3.2 Quotient SO(4)

Hilbert basis. The semi-groups $S_{\pm}^{(2)} \coloneqq C_{\pm}^{(2)} \cap \left(K^{[0]} \cup K^{[2]}\right)$ have Hilbert bases which again equal (the now different) ray generators. The situation is depicted in figure 17 and the Hilbert bases are as follows:

$$\mathcal{H}(S_{\pm}^{(2)}) = \Big\{(1, \pm 1), (1, 0)\Big\} \,. \tag{5.35}$$

Hilbert series. The Hilbert series reads

$$\mathrm{HS}_{\mathrm{SO}(4)}^{[2,2]}(t, z, N) = \frac{1 + zt^{4N} + 2zt^{4N-1} + t^{6N-2} + 2t^{6N-1} + zt^{10N-2}}{(1-t^2)^2\,(1-zt^{4N-2})\,(1-t^{6N-2})} \,. \tag{5.36}$$

The numerator of (5.36) is a palindromic polynomial of degree $10N-2$ (neglecting the dependence on z); while the denominator is a polynomial of degree $10N$. Hence, the difference in degree is two equals the quaternionic dimension of the moduli space. Moreover, the denominator has a pole of order four at $t = 1$, which equals the complex dimension of the moduli space.

Plethystic logarithm. Studying the PL, we observe

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}_{\mathrm{SO}(4)}^{[2,2]}) = {} & 2t^2 + zt^{\Delta(1,0)}(1 + 2t^2 + t) + 2t^{\Delta(1,\pm 1)}(1+t) \\ & - t^{2\Delta(1,0)+2}(3 + 2t^2 + t^2 + 2t^3 + 4t^4 + 2t^5 + t^6 + 2t^7 + 3t^8) + \ldots \end{aligned} \tag{5.37}$$

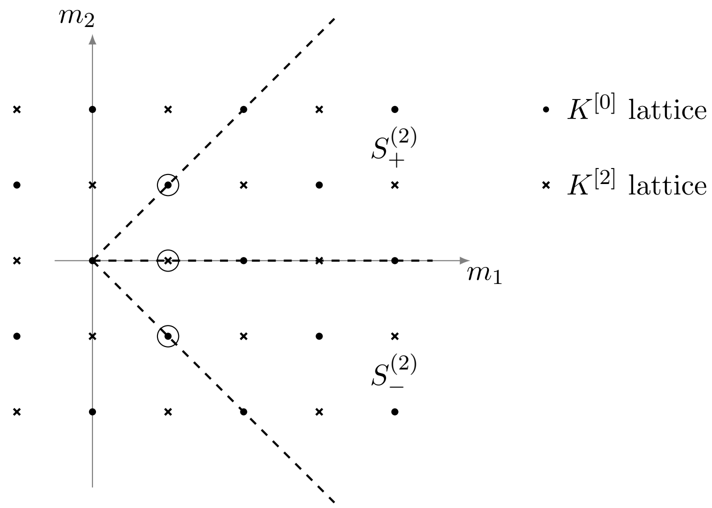

Figure 17. The semi-groups and their ray-generators (black circled points) for the quotient SO(4) and the representation $[2, 2]$.

such that we can associate the generators as follows: two degree two Casimir invariants of SO(4), i.e. the quadratic Casimir and the Pfaffian; a monopole of GNO-charge $(1, 0) \in K^{[2]}$ at conformal dimension $4N-2$ with two dressings at dimension $4N-1$ and another dressing at $4N$; and two monopole operators of GNO-charges $(1, 1)$, $(1, -1) \in K^{[0]}$ at dimension $6N-2$ one dressed monopoles at dimension $6N-1$ each.

Gauging the $\mathbb{Z}_2$. In addition, one can gauge the topological $\mathbb{Z}_2$ in (5.36) and obtains

$$\mathrm{HS}_{\mathrm{Spin}(4)}^{[2,2]}(t, N) = \frac{1}{2}\left(\mathrm{HS}_{\mathrm{SO}(4)}^{[2,2]}(t, z{=}1, N) + \mathrm{HS}_{\mathrm{SO}(4)}^{[2,2]}(t, z{=}-1, N)\right) . \tag{5.38}$$

5.3.3 Quotient $\mathrm{SO(3)} \times \mathrm{SU(2)}$

Hilbert basis. The semi-groups $S_\pm^{(2)} := C_\pm^{(2)} \cap \left(K^{[0]} \cup K^{[1]}\right)$ have Hilbert bases that go beyond the set of ray generators. We refer to figure 18 and the Hilbert bases are obtained as follows:

$$\mathcal{H}(S_+^{(2)}) = \left\{(\tfrac{1}{2}, \tfrac{1}{2}), (2, 0)\right\} \quad \text{and} \quad \mathcal{H}(S_-^{(2)}) = \left\{(1, -1), (\tfrac{3}{2}, -\tfrac{1}{2}), (2, 0)\right\} . \tag{5.39}$$

Hilbert series. The Hilbert series is computed to be

$$\mathrm{HS}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}^{[2,2]}(t, z_1, N) = \frac{R(t, z_1, N)}{(1-t^2)^2\,(1-t^{6N-2})\,(1-t^{8N-4})} , \tag{5.40a}$$

$$\begin{aligned} R(t, z_1, N) = 1 &+ z_1 t^{3N} + z_1 t^{3N-1} + t^{6N-2} + 2t^{6N-1} + z_1 t^{7N-3} \\ &+ 2z_1 t^{7N-2} + z_1 t^{7N-1} + 2t^{8N-3} + t^{8N-2} + z_1 t^{11N-4} \\ &+ z_1 t^{11N-3} + t^{14N-4} . \end{aligned} \tag{5.40b}$$

Again, the numerator of (5.40) is a palindromic polynomial of degree $14N-4$; while the denominator is a polynomial of degree $14N-2$. Hence, the difference in degree is two,

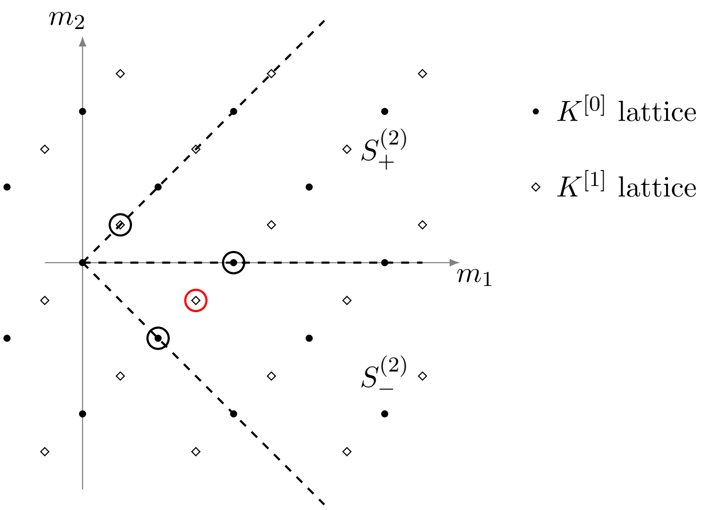

Figure 18. The semi-groups for the quotient $\mathrm{SO}(3) \times \mathrm{SU}(2)$ and the representation $[2, 2]$. The black circled points are the ray generators and the red circled point completes the Hilbert basis for $S_-^{(2)}$.

(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
$(\frac{1}{2}, \frac{1}{2})$	$K^{[1]}$	$3N-1$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
$(1, -1)$	$K^{[0]}$	$6N-2$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
$(\frac{3}{2}, -\frac{1}{2})$	$K^{[1]}$	$7N-3$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
$(2, 0)$	$K^{[0]}$	$8N-4$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 12. The generators for the chiral ring of a $\mathrm{SO}(3) \times \mathrm{SU}(2)$ gauge theory with matter in $[2, 2]$.

which matches the quaternionic dimension of the moduli space. Also, the denominator has a pole of order four at $t = 1$, which equals the complex dimension of the moduli space.

Plethystic logarithm. The inspection of the PL for $N \geq 2$ reveals

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}^{[2,2]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}) &= 2t^2 + z_1 t^{\Delta(\frac{1}{2},\frac{1}{2})}(1+t) + t^{\Delta(1,\pm1)}(1+t-t^2) \\ &\quad + z_1 t^{\Delta(1+\frac{1}{2},-1+\frac{1}{2})}(1+2t+t^2) + t^{\Delta(2,0)}(1+2t+t^2) \\ &\quad - z_1 t^{3\Delta(\frac{1}{2},\frac{1}{2})}(1+2t+t^2) + \dots \,. \end{aligned} \tag{5.41}$$

We summarise the generators in table 12.

5.3.4 Quotient $\mathrm{SU}(2) \times \mathrm{SO}(3)$

Hilbert basis. The semi-groups $S_\pm^{(2)} := C_\pm^{(2)} \cap \left(K^{[0]} \cup K^{[3]}\right)$ have Hilbert bases that go beyond the set of ray generators. Figure 19 depicts the situation and the Hilbert bases are computed to be

$$\mathcal{H}(S_+^{(2)}) = \left\{(1,1), (\tfrac{3}{2}, \tfrac{1}{2}), (2,0)\right\} \quad \text{and} \quad \mathcal{H}(S_-^{(2)}) = \left\{(\tfrac{1}{2}, -\tfrac{1}{2}), (2,0)\right\}. \tag{5.42}$$

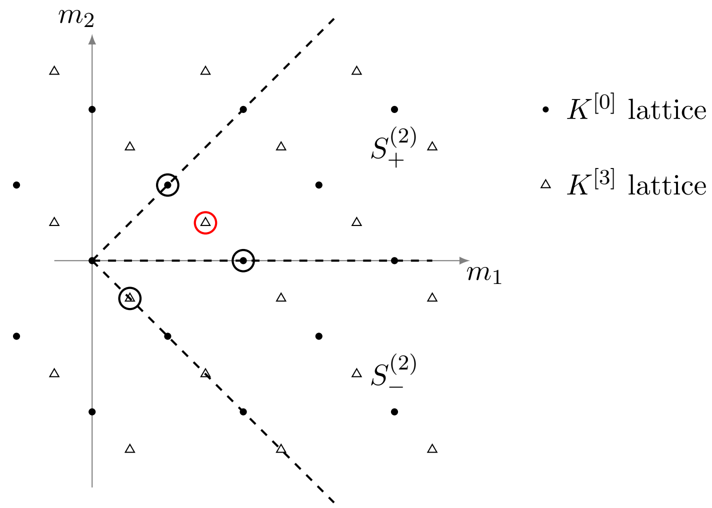

Figure 19. The semi-groups for the quotient SU(2) × SO(3) and the representation [2, 2]. The black circled points are the ray generators and the red circled point completes the Hilbert basis for $S_+^{(2)}$.

We observe that the bases (5.39) and (5.42) are related by reflection along the $m_2 = 0$ axis, which in turn corresponds to the interchange of $K^{[1]}$ and $K^{[3]}$.

Hilbert series. The Hilbert series reads

$$\mathrm{HS}^{[2,2]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t,z_2,N) = \frac{R(t,z_2,N)}{(1-t^2)^2\,(1-t^{6N-2})\,(1-t^{8N-4})}\,, \tag{5.43a}$$

$$\begin{aligned} R(t,z_2,N) &= 1 + z_2 t^{3N} + z_2 t^{3N-1} + t^{6N-2} + 2t^{6N-1} + z_2 t^{7N-3} \\ &\quad + 2z_2 t^{7N-2} + z_2 t^{7N-1} + 2t^{8N-3} + t^{8N-2} + z_2 t^{11N-4} \\ &\quad + z_2 t^{11N-3} + t^{14N-4}\,. \end{aligned} \tag{5.43b}$$

The numerator of (5.43) is palindromic polynomial of degree $14N-4$; while the denominator is a polynomial of degree $14N-2$. Hence, the difference in degree is two, which equals the quaternionic dimension of the moduli space. In addition, the denominator has a pole of order four at $t = 1$, which matches the complex dimension of the moduli space.

As before, comparing the quotients SO(3) × SU(2) and SU(2) × SO(3) as well as the symmetry of (5.30), it is natural to expect the relationship

$$\mathrm{HS}^{[2,2]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t,z_1,N) \xleftrightarrow{z_1\leftrightarrow z_2} \mathrm{HS}^{[2,2]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t,z_2,N)\,, \tag{5.44}$$

which is verified explicitly for (5.40) and (5.43).

Plethystic logarithm. The equivalence to SO(3) × SU(2) is further confirmed by the inspection of the PL for $N \geq 2$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}^{[2,2]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}) &= 2t^2 + z_2 t^{\Delta(\frac{1}{2},-\frac{1}{2})}(1+t) + t^{\Delta(1,1)}(1+t-t^2) \\ &\quad + z_2 t^{\Delta(\frac{3}{2},\frac{1}{2})}(1+2t+t^2) + t^{\Delta(2,0)}(1+2t+t^2) + \dots \end{aligned} \tag{5.45}$$

(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
$(\frac{1}{2}, -\frac{1}{2})$	$K^{[3]}$	$3N-1$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
$(1,1)$	$K^{[0]}$	$6N-2$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
$(\frac{3}{2}, \frac{1}{2})$	$K^{[3]}$	$7N-3$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
$(2,0)$	$K^{[0]}$	$8N-4$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 13. The generators for the chiral ring of a $\mathrm{SU}(2) \times \mathrm{SO}(3)$ gauge theory with matter in $[2,2]$.

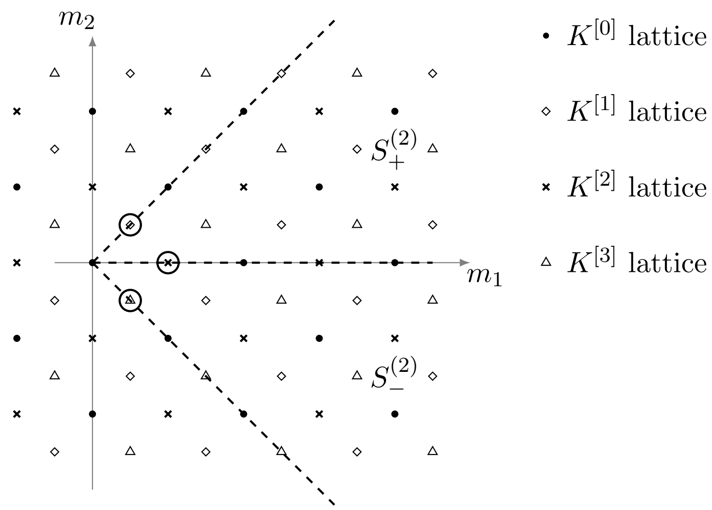

Figure 20. The semi-groups and their ray-generators (black circled points) for the quotient PSO(4) and the representation $[2,2]$.

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where we can summarise the monopole generators as in table 13. Note the change in GNO-charges in accordance with the use of $K^{[3]}$ instead of $K^{[1]}$.

5.3.5 Quotient PSO(4)

Hilbert basis. The semi-groups $S_\pm^{(2)} := C_\pm^{(2)} \cap \left(K^{[0]} \cup K^{[1]} \cup K^{[2]} \cup K^{[3]}\right)$ have Hilbert bases that are determined by the ray generators. Figure 20 depicts the situation and the Hilbert bases read

$$\mathcal{H}(S_\pm^{(2)}) = \left\{ (\tfrac{1}{2}, \pm\tfrac{1}{2}), (1,0) \right\} . \tag{5.46}$$

Hilbert series. The Hilbert series reads

$$\mathrm{HS}_{\mathrm{PSO}(4)}^{[2,2]}(t, z_1, z_2, N) = \frac{R(t, z_1, z_2, N)}{(1-t^2)^2 \, (1-t^{6N-2}) \, (1-z_2 t^{4N-2})} , \tag{5.47a}$$

$$\begin{aligned} R(t, z_1, z_2, N) = 1 &+ z_1 t^{3N} + z_1 t^{3N-1} + z_1 z_2 t^{3N} + z_1 z_2 t^{3N-1} + z_2 t^{4N} + 2 z_2 t^{4N-1} \\ &+ t^{6N-2} + 2 t^{6N-1} + z_1 z_2 t^{7N-2} + z_1 z_2 t^{7N-1} + z_1 t^{7N-2} \\ &+ z_1 t^{7N-1} + z_2 t^{10N-2} . \end{aligned} \tag{5.47b}$$

The numerator of (5.47) is palindromic polynomial of degree $10N-2$; while the denominator is a polynomial of degree $10N$. Hence, the difference in degree is two, which corresponds

(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
$(\frac{1}{2}, \frac{1}{2})$	$K^{[1]}$	$3N-1$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
$(\frac{1}{2}, -\frac{1}{2})$	$K^{[3]}$	$3N-1$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
$(1,0)$	$K^{[2]}$	$4N-2$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 14. The generators for the chiral ring of a PSO(4) gauge theory with matter in $[2,2]$.

to the quaternionic dimension of the moduli space. Similarly to the previous cases, the denominator of (5.47) has a pole of order four at $t=1$, which equals the complex dimension of the moduli space.

Gauging a $\mathbb{Z}_2$. As before, by gauging the $\mathbb{Z}_2$-factor corresponding to z_1 we recover the SO(4)-result

$$\mathrm{HS}_{\mathrm{SO}(4)}^{[2,2]}(t,z,N) = \frac{1}{2}\left(\mathrm{HS}_{\mathrm{PSO}(4)}^{[2,2]}(t,z_1{=}1,z_2{=}z,N) + \mathrm{HS}_{\mathrm{PSO}(4)}^{[2,2]}(t,z_1{=}-1,z_2{=}z,N)\right) , \tag{5.48a}$$

while gauging the $\mathbb{Z}_2$-factor with fugacity z_2 provides the $\mathrm{SO}(3)\times\mathrm{SU}(2)$-result

$$\mathrm{HS}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}^{[2,2]}(t,z_1,N) = \frac{1}{2}\left(\mathrm{HS}_{\mathrm{PSO}(4)}^{[2,2]}(t,z_1,z_2{=}1,N) + \mathrm{HS}_{\mathrm{PSO}(4)}^{[2,2]}(t,z_1,z_2{=}{-1},N)\right) . \tag{5.48b}$$

Furthermore, employing the redefined fugacities w_1, w_2 of (5.28c) one reproduces the $\mathrm{SU}(2)\times\mathrm{SO}(3)$ Hilbert series as follows:

$$\mathrm{HS}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}^{[2,2]}(t,z_2{=}w_1,N){=}\frac{1}{2}\left(\mathrm{HS}_{\mathrm{PSO}(4)}^{[2,2]}(t,w_1,w_2{=}1,N){+}\mathrm{HS}_{\mathrm{PSO}(4)}^{[2,2]}(t,w_1,w_2{=}{-1},N)\right) . \tag{5.48c}$$

Therefore, one can obtain the Hilbert series for *all* five quotients from the PSO(4)-result (5.47) by employing the $\mathbb{Z}_2$-gaugings (5.38) and (5.48).

Plethystic logarithm. Inspecting the PL leads to

$$\mathrm{PL}(\mathrm{HS}_{\mathrm{PSO}(4)}^{[2,2]}) = 2t^2 + z_1 t^{\Delta(\frac{1}{2},\frac{1}{2})}(1+t) + z_1 z_2 t^{\Delta(\frac{1}{2},-\frac{1}{2})}(1+t) + z_2 t^{\Delta(1,0)}(1+2t+t^2) + \dots \tag{5.49}$$

such that we can summarise the monopole generators as in table 14.

5.4 Representation [4, 2]

The conformal dimension for this case reads

$$\begin{aligned}\Delta(m_1,m_2) = N\big(&|3m_1-m_2| + |m_1-3m_2| + |m_1+m_2| + 3\,|m_1-m_2| + 2\,|m_1| + 2\,|m_2|\big) \\ &- |m_1+m_2| - |m_1-m_2| \, .\end{aligned} \tag{5.50}$$

The interesting feature of this representation is its asymmetric behaviour under exchange of m_1 and m_2.

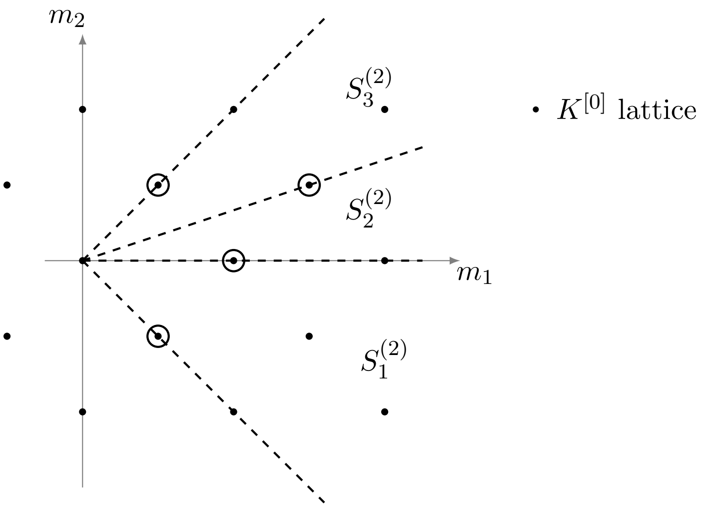

Figure 21. The semi-groups and their ray-generators (black circled points) for the quotient Spin(4) and the representation $[4, 2]$.

As before, the conformal dimension (5.50) defines a fan in the dominant Weyl chamber of, which is spanned by three 2-dimensional cones

$$C_1^{(2)} = \Big\{(m_1 \geq -m_2) \wedge (m_2 \leq 0)\Big\} , \tag{5.51a}$$
$$C_2^{(2)} = \Big\{(m_1 \geq 3m_2) \wedge (m_2 \geq 0)\Big\} , \tag{5.51b}$$
$$C_3^{(2)} = \Big\{(m_1 \geq m_2) \wedge (m_1 \leq 3m_2)\Big\} . \tag{5.51c}$$

5.4.1 Quotient Spin(4)

Hilbert basis. Starting from the fan (5.51) with cones $C_p^{(2)}$ ($p = 1, 2, 3$), the Hilbert bases for the semi-groups $S_p^{(2)} := C_p^{(2)} \cap K^{[0]}$ are simply given by the ray generators, see for instance figure 21.

$$\mathcal{H}(S_1^{(2)}) = \Big\{(2, 0), (1, -1)\Big\}, \quad \mathcal{H}(S_2^{(2)}) = \Big\{(3, 1), (2, 0)\Big\}, \quad \mathcal{H}(S_3^{(2)}) = \Big\{(1, 1), (3, 1)\Big\}. \tag{5.52}$$

Hilbert series. The Hilbert series reads

$$\mathrm{HS}^{[4,2]}_{\mathrm{Spin}(4)}(t, N) = \frac{R(t, N)}{(1 - t^2)^2 \left(1 - t^{18N-2}\right)\left(1 - t^{20N-4}\right)\left(1 - t^{26N-6}\right)} , \tag{5.53a}$$

$$\begin{aligned} R(t, N) = 1 &+ t^{10N-2}(1 + t) + t^{18N-1} + t^{20N-4}(1 + 3t + t^2) \\ &+ t^{26N-5}(2 + t) - t^{28N-4}(1 + t) + t^{36N-7}(1 + t) \\ &- t^{38N-6}(1 + 2t) - t^{44N-8}(1 + 3t + t^2) - t^{46N-9} \\ &- t^{54N-9}(1 + t) - t^{64N-10} . \end{aligned} \tag{5.53b}$$

The numerator of (5.53) is an anti-palindromic polynomial of degree $64N - 10$, while the denominator is of degree $64N - 8$. Consequently, the difference in degree is two. Moreover, the rational function (5.53) has a pole of order four as $t \to 1$ because $R(t{=}1, N) = 0$, but $\frac{\mathrm{d}}{\mathrm{d}t} R(t, N)|_{t=1} \neq 0$.

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object	(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	# dressings
Casimirs	—	—	2	—	—
monopole	$(1,1)$	$K^{[0]}$	$10N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(1,-1)$	$K^{[0]}$	$18N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(2,0)$	$K^{[0]}$	$20N-4$	$\mathrm{U}(1)\times\mathrm{U}(1)$	2 by $\mathrm{U}(1)^2$
monopole	$(3,1)$	$K^{[0]}$	$26N-6$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 15. The chiral ring generators for a Spin(4) gauge theory with matter transforming in $[4,2]$.

Plethystic logarithm. Inspecting the PL yields for $N\geq 3$

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[4,2]}_{\mathrm{Spin}(4)}) = 2t^2 &+ t^{\Delta(1,1)}(1+t) + t^{\Delta(1,-1)}(1+t) + t^{\Delta(2,0)}(1+2t)\\ &+ t^{\Delta(3,1)}(1+2t+t^2) - t^{\Delta(1,1)+\Delta(1,-1)}(1+2t+t^2)\\ &- t^{\Delta(1,1)+\Delta(2,0)}(1+3t+3t^2+t^3) + \dots \end{aligned} \tag{5.54}$$

leads to an identification of generators as in table 15. We observe that $(2,0)$ has only 2 dressings, although we would expect 3. We know from other examples that there should be a relation at $2\Delta(1,1)+2 = 20N-2$ which is precisely the dimension of the second dressing of $(2,0)$.

5.4.2 Quotient SO(4)

Hilbert basis. The semi-groups $S_p^{(2)} \coloneqq C_p^{(2)} \cap \left(K^{[0]}\cup K^{[2]}\right)$ have Hilbert bases as shown in figure 22 or explicitly:

$$\mathcal{H}(S_1^{(2)}) = \Big\{(1,0),(1,-1)\Big\}\,, \qquad \mathcal{H}(S_2^{(2)}) = \Big\{(3,1),(1,0)\Big\}\,, \tag{5.55a}$$

$$\mathcal{H}(S_3^{(2)}) = \Big\{(1,1),(2,1),(3,0)\Big\}\,. \tag{5.55b}$$

Hilbert series.

$$\mathrm{HS}^{[4,2]}_{\mathrm{SO}(4)}(t,z,N) = \frac{R(t,z,N)}{(1-t^2)^2\,(1-t^{10N-2})\,(1-t^{18N-2})\,(1-t^{26N-6})\,(1-zt^{10N-2})}\,, \tag{5.56a}$$

$$\begin{aligned}R(t,z,N) = 1 + t^{10N-1} &+ zt^{10N-1}(2+t) + zt^{18N-4}(1+2t+t^3) + t^{18N-1}\\ &- zt^{20N-4}(1+3t+t^2) + 2t^{26N-5}(2+t)\\ &- t^{28N-6}(1+2t+2t^2+2t^3) - zt^{28N-3}\\ &- t^{36N-7} - zt^{36N-7}(2+2t+2t^2+t^3) + zt^{38N-6}(1+2t)\\ &- t^{44N-8}(1+3t+t^2) + zt^{46N-9} + t^{46N-8}(1+2t+t^2)\\ &+ t^{54N-10}(1+2t) + zt^{54N-9} + zt^{64N-10}\,. \end{aligned} \tag{5.56b}$$

The numerator (5.56b) is a palindromic polynomial of degree $64N-10$, while the denominator is of degree $64N-8$. Consequently, the difference of the degree is two. Also, the Hilbert series (5.56) has a pole of order four as $t\to 1$, because $R(t{=}1,z,N) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}R(t,z,N)|_{t=1} = 0$, but $\frac{\mathrm{d}^2}{\mathrm{d}t^2}R(t,z,N)|_{t=1} \neq 0$.

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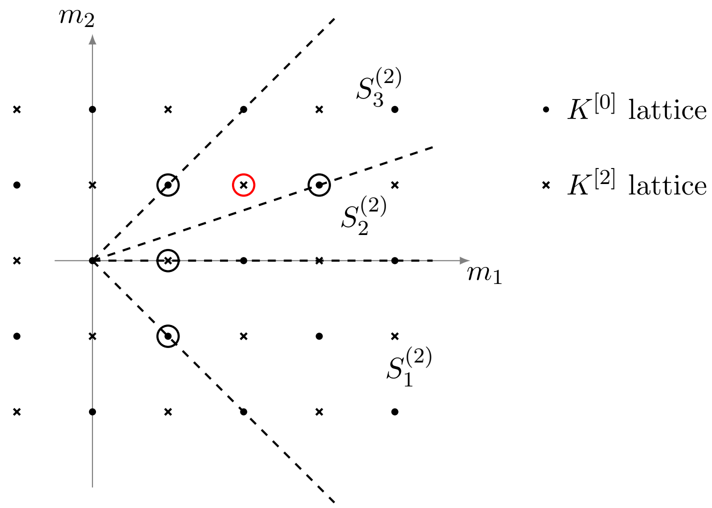

Figure 22. The semi-groups for the quotient SO(4) and the representation $[4,2]$. The black circled points are the ray generators and the red circled point completes the Hilbert basis for $S_3^{(2)}$.

object	(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(1,0)$	$K^{[2]}$	$10N-2$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
monopole	$(1,1)$	$K^{[0]}$	$10N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(2,1)$	$K^{[2]}$	$18N-4$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
monopole	$(1,-1)$	$K^{[0]}$	$18N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(3,1)$	$K^{[0]}$	$26N-6$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 16. The chiral ring generators for a SO(4) gauge theory with matter transforming in $[4,2]$.

Plethystic logarithm. Inspecting the PL reveals

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{SO}(4)}^{[4,2]}) = 2t^2 &+ zt^{\Delta(1,0)}(1+2t+t^2) + t^{\Delta(1,1)}(1+t) + zt^{\Delta(2,1)}(1+2t+t^2) \\ &+ t^{\Delta(1,-1)}(1+t) - zt^{2\Delta(1,0)}(1+3t+3t^2+t^3) - t^{2\Delta(1,1)+2}(4+2t+t^2) \\ &+ t^{\Delta(3,1)}(1+2t+t^2) + \dots \,,\end{aligned} \tag{5.57}$$

such that the monopole generators can be summarised as in table 16.

Gauging the $\mathbb{Z}_2$. Again, one can gauge the finite symmetry to recover the Spin(4) Hilbert series

$$\mathrm{HS}_{\mathrm{Spin}(4)}^{[4,2]}(t,N) = \frac{1}{2}\left(\mathrm{HS}_{\mathrm{SO}(4)}^{[4,2]}(t,z{=}1,N) + \mathrm{HS}_{\mathrm{SO}(4)}^{[4,2]}(t,z{=}-1,N)\right) \,. \tag{5.58}$$

5.4.3 Quotient SO(3) × SU(2)

Hilbert basis. The semi-groups $S_p^{(2)} := C_p^{(2)} \cap \left(K^{[0]} \cup K^{[1]}\right)$ $(p=1,2,3)$ have Hilbert bases that go beyond the set of ray generators. We refer to figure 23 and the Hilbert bases

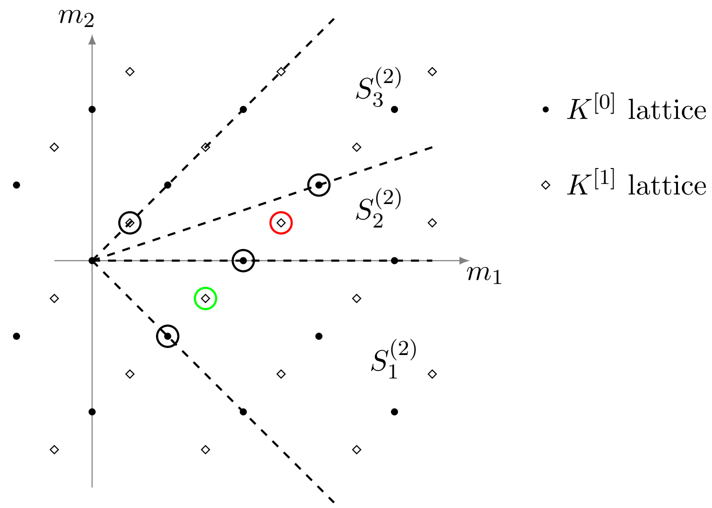

Figure 23. The semi-groups for the quotient $SO(3) \times SU(2)$ and the representation $[4, 2]$. The black circled points are the ray generators, the red circled point completes the Hilbert basis for $S_2^{(2)}$, while the green circled point completes the Hilbert basis of $S_1^{(2)}$.

are obtained as follows:

$$\mathcal{H}(S_1^{(2)}) = \left\{(2,1), (\tfrac{3}{2}, -\tfrac{1}{2}), (1,-1)\right\} , \qquad \mathcal{H}(S_2^{(2)}) = \left\{(3,1), (\tfrac{5}{2}, \tfrac{1}{2}), (2,0)\right\} , \tag{5.59a}$$

$$\mathcal{H}(S_3^{(2)}) = \left\{(1,1), (3,1)\right\} . \tag{5.59b}$$

Hilbert series. We compute the Hilbert series to

$$\mathrm{HS}^{[4,2]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t, z_1, N) = \frac{R(t, z_1, N)}{(1-t^2)^2\,(1-t^{18N-2})\,(1-t^{20N-4})\,(1-t^{26N-6})} , \tag{5.60a}$$

$$\begin{aligned} R(t,z_1,N) = 1 &+ z_1 t^{5N-1}(1+t) + t^{10N-2}(1+t) + z_1 t^{15N-3}(1+t) \\ &+ t^{18N-1} + z_1 t^{19N-3}(1+2t+t^3) \\ &+ t^{20N-4}(1+3t+t^2) + z_1 t^{23N-5}(1+2t-t^3) \\ &+ t^{26N-5}(2+t) - t^{28N-4}(1+t) + z_1 t^{31N-6}(1+t) \\ &- z_1 t^{33N-5}(1+t) + t^{36N-7}(1+t) - t^{38N-6}(1+2t) \\ &+ z_1 t^{41N-8}(1-2t^2-t^3) - t^{44N-8}(1+3t+t^2) \\ &- z_1 t^{45N-9}(1+2t+t^2) - t^{46N-9} - z_1 t^{49N-8}(1+t) \\ &- t^{54N-9}(1+t) - z_1 t^{59N-10}(1+t) - t^{64N-10} . \end{aligned} \tag{5.60b}$$

The numerator of (5.60) is an anti-palindromic polynomial of degree $64N - 10$, while the denominator is of degree $64N - 8$. Thus, the difference in degrees is again 2. In addition, the Hilbert series (5.60) has a pole of order 4 as $t \to 1$, because $R(t{=}1, z_1, N) = 0$, but $\frac{\mathrm{d}}{\mathrm{d}t} R(t, z_1, N)|_{t=1} \neq 0$.

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object	(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(\frac{1}{2},\frac{1}{2})$	$K^{[1]}$	$5N-1$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(1,-1)$	$K^{[0]}$	$18N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(\frac{3}{2},-\frac{1}{2})$	$K^{[1]}$	$19N-3$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
monopole	$(2,0)$	$K^{[0]}$	$20N-4$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)$
monopole	$(\frac{5}{2},\frac{1}{2})$	$K^{[1]}$	$23N-5$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3(2) by $\mathrm{U}(1)^2$
monopole	$(3,1)$	$K^{[0]}$	$26N-6$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 17. The chiral ring generators for a $\mathrm{SO}(3)\times\mathrm{SU}(2)$ gauge theory with matter transforming in $[4,2]$.

Plethystic logarithm. Analysing the PL yields

$$\begin{aligned} PL = {} & 2t^2 + z_1 t^{\Delta(\frac{1}{2},\frac{1}{2})}(1+t) - t^{\Delta(\frac{1}{2},\frac{1}{2})+2} + t^{\Delta(1,-1)}(1+t) \\ & + z_1 t^{\Delta(\frac{3}{2},-\frac{1}{2})}(1+2t+t^2) + t^{\Delta(2,0)}(1+2t+t^2) \\ & + z_1 t^{\Delta(\frac{5}{2},\frac{1}{2})}(1+2t+1) - z_1 t^{\Delta(\frac{1}{2},\frac{1}{2})+\Delta(1,-1)}(1+2t+t^2) \\ & - t^{\Delta(\frac{1}{2},\frac{1}{2})+\Delta(\frac{3}{2},-\frac{1}{2})}(1+3t+3t^2+t^3) \\ & - z_1 t^{\Delta(\frac{1}{2},\frac{1}{2})+\Delta(2,0)}(1+3t+3t^2+t^3) \\ & + t^{\Delta(3,1)}(1+2t+t^2) + \dots\,, \end{aligned} \tag{5.61}$$

verfies the set of generators as presented in table 17. The coloured term indicates that we suspect a cancellation between one dressing of $(\frac{5}{2},\frac{1}{2})$ and one relation because $\Delta(\frac{5}{2},\frac{5}{2})+2 = 23N-3 = \Delta(\frac{1}{2},\frac{1}{2})+\Delta(1,-1) = 5N-1+18N-2$.

5.4.4 Quotient $\mathrm{SU}(2)\times\mathrm{SO}(3)$

Hilbert basis. The semi-groups $S_p^{(2)} := C_p^{(2)} \cap \left(K^{[0]}\cup K^{[3]}\right)$ (for $p=1,2,3$) have Hilbert bases consist of the ray generators as shown in figure 24 and we obtain explicitly

$$\mathcal{H}(S_1^{(2)}) = \left\{(2,0),(\tfrac{1}{2},-\tfrac{1}{2})\right\}, \quad \mathcal{H}(S_2^{(2)}) = \left\{(\tfrac{3}{2},\tfrac{1}{2}),(2,0)\right\}, \quad \mathcal{H}(S_3^{(2)}) = \left\{(1,1),(\tfrac{3}{2},\tfrac{1}{2})\right\}. \tag{5.62}$$

Hilbert series. We compute the Hilbert series to

$$\mathrm{HS}^{[4,2]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t,z_2,N) = \frac{R(t,z_2,N)}{(1-t^2)^2\,(1-t^{18N-2})\,(1-t^{20N-4})\,(1-t^{26N-6})}\,, \tag{5.63a}$$

$$\begin{aligned} R(t,z_2,N) = {} & 1 + z_2 t^{9N-1}(1+t) + t^{10N-2}(1+t) + z_2 t^{13N-3}(1+2t+t^2) \\ & + t^{18N-1} + t^{20N-4}(1+3t+t^2) + z_2 t^{23N-5}(1+2t+t^2) \\ & + t^{26N-5}(2+t) - t^{28N-4}(1+t) + z_2 t^{29N-4}(1+t) \\ & - z_2 t^{31N-5}(1+2t+t^2) + z_2 t^{33N-7}(1+2t+t^2) \end{aligned} \tag{5.63b}$$

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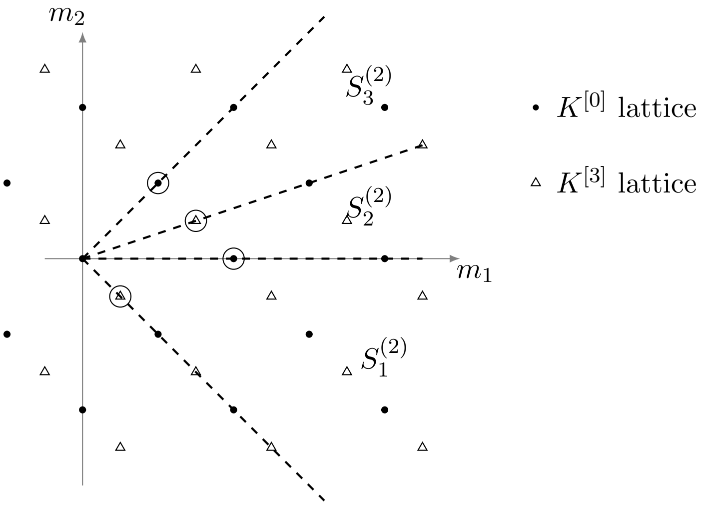

Figure 24. The semi-groups for the quotient $\mathrm{SU}(2) \times \mathrm{SO}(3)$ and the representation $[4, 2]$. The black circled points are the ray generators.

$$
\begin{aligned}
&- z_2 t^{35N-7}(1+t) + t^{36N-7}(1+t) - t^{38N-6}(1+2t) \\
&- z_2 t^{41N-7}(1+2t+t^2) - t^{44N-8}(1+3t+t^2) \\
&- t^{46N-9} - z_2 t^{51N-9}(1+2t+t^2) - t^{54N-9}(1+t) \\
&- z_2 t^{55N-10}(1+t) - t^{64N-10} \,.
\end{aligned}
$$

As before, we can try to compare the quotients $\mathrm{SO}(3)\times\mathrm{SU}(2)$ and $\mathrm{SU}(2)\times\mathrm{SO}(3)$. However, due to the asymmetry in m_1, m_2 or the asymmetry of the fan in the Weyl chamber, the Hilbert series for the two quotients are *not* related by an exchange of z_1 and z_2.

Plethystic logarithm. Upon analysing the PL we find

$$
\begin{aligned}
\mathrm{PL}(\mathrm{HS}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}^{[4,2]}) = 2t^2 &+ z_2 t^{\Delta(\frac{1}{2},-\frac{1}{2})}(1+t) + t^{\Delta(1,1)}(1+t) + z_2 t^{\Delta(\frac{3}{2},\frac{1}{2})}(1+2t+t^2) \\
&- t^{2\Delta(\frac{1}{2},-\frac{1}{2})+2} - z_2 t^{\Delta(\frac{1}{2},-\frac{1}{2})+\Delta(1,1)}(1+2t+t^2) \\
&+ t^{\Delta(2,0)}(1+2t) - t^{\Delta(\frac{1}{2},-\frac{1}{2})+\Delta(\frac{3}{2},\frac{1}{2})}(1+3t+3t^2+t^3) + \dots \,,
\end{aligned} \tag{5.64}
$$

through which one identifies the generators as given in table 18. The terms in the denominator of the Hilbert series can be seen to reproduce these generators

$$
(1-t^{18N-2}) = (1-z_2 t^{9N-1})(1+z_2 t^{9N-1}) \,, \tag{5.65a}
$$
$$
(1-t^{26N-6}) = (1-z_2 t^{13N-3})(1+z_2 t^{13N-3}) \,. \tag{5.65b}
$$

Unfortunately, we are unable to reduce the numerator accordingly.

5.4.5 Quotient PSO(4)

Hilbert basis. The semi-groups $S_p^{(2)} \coloneqq C_p^{(2)} \cap \left(K^{[0]} \cup K^{[1]} \cup K^{[2]} \cup K^{[3]}\right)$ (for $p = 1, 2, 3$) have Hilbert bases that are determined by the ray generators. Figure 25 depicts the situ-

object	(m_1,m_2)	lattice	$\Delta(m_1,m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(\frac{1}{2},-\frac{1}{2})$	$K^{[3]}$	$9N-1$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(1,1)$	$K^{[0]}$	$10N-2$	$\mathrm{U}(1)\times\mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(\frac{3}{2},\frac{1}{2})$	$K^{[3]}$	$13N-3$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
monopole	$(2,0)$	$K^{[0]}$	$20N-4$	$\mathrm{U}(1)\times\mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 18. The chiral ring generators for a $\mathrm{SU}(2)\times\mathrm{SO}(3)$ gauge theory with matter transforming in $[4,2]$.

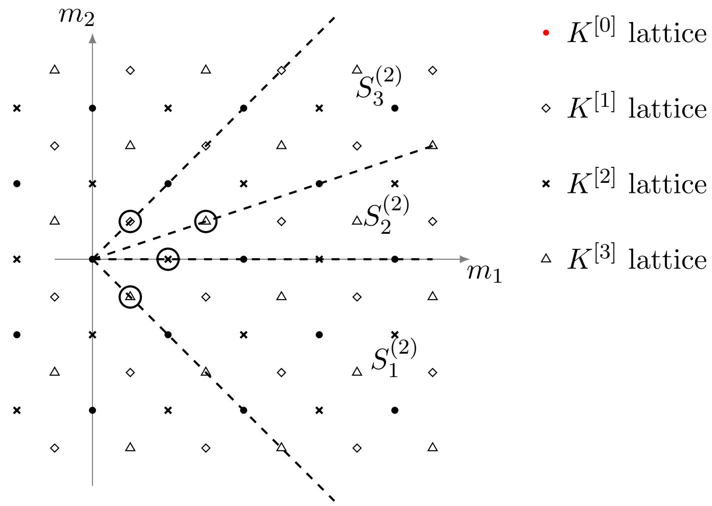

Figure 25. The semi-groups and their ray-generators (black circled points) for the quotient $\mathrm{PSO}(4)$ and the representation $[4,2]$.

ation and the Hilbert bases read:

$$\mathcal{H}(S_1^{(2)})=\Big\{(1,0),(\tfrac{1}{2},-\tfrac{1}{2})\Big\},\quad \mathcal{H}(S_2^{(2)})=\Big\{(\tfrac{3}{2},\tfrac{1}{2}),(1,0)\Big\},\quad \mathcal{H}(S_3^{(2)})=\Big\{(\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{3}{2},\tfrac{1}{2})\Big\}. \tag{5.66}$$

Hilbert series. We obtain the following Hilbert series

$$\mathrm{HS}_{\mathrm{PSO}(4)}^{[4,2]}(t,z_1,z_2,N)=\frac{R(t,z_1,z_2,N)}{(1-t^2)^2\,(1-t^{10N-2})\,(1-t^{18N-2})\,(1-t^{26N-6})\,(1-t^{10N-2}z_2)}\,, \tag{5.67a}$$

$$\begin{aligned}R(t,z_1,z_2,N)=1&+z_1t^{5N-1}(1+t)+z_1z_2t^{9N-1}(1+t)+z_1z_2t^{9N}+t^{10N-1}\\&+z_2t^{10N-1}(2+t)+z_1z_2t^{13N-3}(1+2t+t^2)-z_1z_2t^{15N-3}(1+t)\\&+z_2t^{18N-4}(1+2t+t^2)+t^{18N-1}-z_1z_2t^{19N-3}(1+t)\\&+z_1t^{19N-2}(1+t)-z_2t^{20N-4}(1+3t+t^2)-z_1t^{23N-3}(1+t)\\&+t^{26N-5}(2+t)-t^{28N-6}(1+2t+2t^2+2t^3)-z_2t^{28N-3}\end{aligned}$$

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2	—	—
monopole	$(\frac{1}{2}, \frac{1}{2})$	$K^{[1]}$	$5N-1$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(\frac{1}{2}, -\frac{1}{2})$	$K^{[3]}$	$9N-1$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	1 by $\mathrm{U}(1)$
monopole	$(1,0)$	$K^{[2]}$	$10N-2$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$
monopole	$(\frac{3}{2}, \frac{1}{2})$	$K^{[3]}$	$13N-3$	$\mathrm{U}(1) \times \mathrm{U}(1)$	3 by $\mathrm{U}(1)^2$

Table 19. The chiral ring generators for a PSO(4) gauge theory with matter transforming in $[4,2]$.

$$\begin{aligned}
&- z_1 t^{29N-4}(1+t) + z_1 t^{31N-6}(1+t) - z_1 z_2 t^{31N-5}(1+2t+t) \\
&- z_1 t^{33N-7}(1+2t+t^2) + z_1 z_2 t^{33N-5}(1+t) \\
&- z_1 z_2 t^{35N-7}(1+t) - z_2 t^{36N-7}(2+2t+2t^2+t^3) - t^{36N-7} \\
&+ z_2 t^{38N-6}(1+2t) - z_1 z_2 t^{41N-8}(1+t) - t^{44N-8}(1+3t+t^2) \\
&+ z_1 z_2 t^{45N-9}(1+t) - z_1 t^{45N-8}(1+t) \\
&+ z_2 t^{46N-9} + t^{46N-8}(1+2t+t^2) - z_1 t^{49N-8}(1+t) \\
&+ z_1 t^{51N-9}(1+2t+t^2) + t^{54N-10}(1+2t) + z_2 t^{54N-9} \\
&+ z_1 t^{55N-10}(1+t) + z_1 z_2 t^{59N-10}(1+t) + z_2 t^{64N-10} \,. \qquad (5.67\text{b})
\end{aligned}$$

The numerator of (5.67) is a palindromic polynomial of degree $64N-10$, while the denominator is of degree $64N-8$. Hence, the difference in degrees is again 2. Moreover, the Hilbert series (5.67) has a pole of order 4 as $t \to 1$ because $R(1, z_1, z_2, N) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t} R(t, z_1, z_2, N)|_{t\to 1} = 0$, while $\frac{\mathrm{d}^2}{\mathrm{d}t^2} R(t, z_1, z_2, N)|_{t\to 1} \neq 0$.

Plethystic logarithm. Working with the PL instead reveals further insights

$$\begin{aligned}
\mathrm{PL}(\mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}) = 2t^2 &+ z_1 t^{\Delta(\frac{1}{2},\frac{1}{2})}(1+t) + z_1 z_2 t^{\Delta(\frac{1}{2},-\frac{1}{2})}(1+t) + z_2 t^{\Delta(1,0)}(1+2t+t^2) \\
&- t^{2\Delta(\frac{1}{2},\frac{1}{2})+2} + z_1 z_2 t^{\Delta(\frac{3}{2},\frac{1}{2})}(1+2t+t^2) - z_2 t^{\Delta(\frac{1}{2},\frac{1}{2})+\Delta(\frac{1}{2},\frac{1}{2})}(1+2t+t^2) \\
&- z_1 z_2 t^{\Delta(\frac{1}{2},\frac{1}{2})+\Delta(1,0)}(1+3t+3t^2+t^3) + \ldots \,. \qquad (5.68)
\end{aligned}$$

The list of generators, together with their properties, is provided in table 19.

Gauging a $\mathbb{Z}_2$. The global $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry allows us to compute the Hilbert series for all five quotients from the PSO(4) result. We start by gauging the $\mathbb{Z}_2$-factor with fugacity z_1 (and relabel z_2 as z) and recover the SO(4)-result

$$\mathrm{HS}^{[4,2]}_{\mathrm{SO}(4)}(t,z,N) = \frac{1}{2}\left(\mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}(t, z_1{=}1, z_2{=}z, N) + \mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}(t, z_1{=}-1, z_2{=}z, N)\right). \qquad (5.69\text{a})$$

In contrast, gauging the other $\mathbb{Z}_2$-factor with fugacity z_1 provides the $\mathrm{SO}(3) \times \mathrm{SU}(2)$-result

$$\mathrm{HS}^{[4,2]}_{\mathrm{SO}(3)\times\mathrm{SU}(2)}(t, z_1, N) = \frac{1}{2}\left(\mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}(t, z_1, z_2{=}1, N) + \mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}(t, z_1, z_2{=}-1, N)\right). \qquad (5.69\text{b})$$

Lastly, switching to w_1, w_2 fugacities as in (5.28c) allows to recover the Hilbert series for $\mathrm{SU}(2) \times \mathrm{SO}(3)$ as follows:

$$\mathrm{HS}^{[4,2]}_{\mathrm{SU}(2)\times\mathrm{SO}(3)}(t, z_2{=}w_1, N) = \frac{1}{2}\left(\mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}(t, w_1, w_2{=}1, N) + \mathrm{HS}^{[4,2]}_{\mathrm{PSO}(4)}(t, w_1, w_2{=}-1, N)\right) . \tag{5.69c}$$

In conclusion, the PSO(4) result is sufficient to obtain the remaining four quotients by gauging of various $\mathbb{Z}_2$ global symmetries as in (5.69) and (5.58).

5.5 Comparison to O(4)

In this subsection we explore the orthogonal group O(4), related to SO(4) by $\mathbb{Z}_2$. To begin with, we summarise the set-up as presented in [28, appendix A]. The dressing factor $P_{\mathrm{O}(4)}(t)$ and the GNO lattice of O(4) equal those of SO(5). Moreover, the dominant Weyl chamber is parametrised by (m_1, m_2) subject to $m_1 \geq m_2 \geq 0$. Graphically, the Weyl chamber is the upper half of the yellow-shaded region in figure 10 with the lattices $K^{[0]} \cup K^{[2]}$ present. Consequently, the dressing function is given as

$$P_{\mathrm{O}(4)}(t, m_1, m_2) = \begin{cases} \dfrac{1}{(1-t^2)(1-t^4)} \, , & m_1 = m_2 = 0 \, , \\ \dfrac{1}{(1-t)(1-t^2)} \, , & m_1 = m_2 > 0 \, , \\ \dfrac{1}{(1-t)(1-t^2)} \, , & m_1 > 0, \ m_2 = 0 \, , \\ \dfrac{1}{(1-t)^2} \, , & m_1 > m_2 > 0 \, . \end{cases} \tag{5.70}$$

It is apparent that O(4) has a different Casimir invariant as SO(4), which comes about as the Levi-Civita tensor ε is not an invariant tensor under O(4). In other words, the Pfaffian of SO(4) is not an invariant of O(4).

Now, we provide the Hilbert series for the three different representations studied above.

5.5.1 Representation [2, 0]

The conformal dimension is the same as in (5.6) and the rational cone of the Weyl chamber is simply

$$C^{(2)} = \mathrm{Cone}\left((1,0),(1,1)\right) \, , \tag{5.71}$$

such that the cone generators and the Hilbert basis for $S^{(2)} := C^{(2)} \cap \left(K^{[0]} \cup K^{[2]}\right)$ coincide. The upper half-space of figure 12 depicts the situation.

The Hilbert series is then computed to read

$$\mathrm{HS}^{[2,0]}_{\mathrm{O}(4)}(t, N) = \frac{1 + 2t^{2N-1} + 2t^{2N} + 2t^{2N+1} + t^{4N}}{(1-t^2)(1-t^4)(1-t^{2N-2})^2} \, , \tag{5.72}$$

which clearly displays the palindromic numerator, the order four pole for $t \to 1$, and the order two pole for $t \to \infty$, i.e. the difference in degrees of denominator and numerator is two. By inspection of (5.72) and use of the plethystic logarithm

$$\mathrm{PL}(\mathrm{HS}^{[2,0]}_{\mathrm{O}(4)}) = t^2 + t^4 + t^{\Delta(1,0)}(1+t+t^2+t^3) + t^{\Delta(1,1)}(1+t+t^2+t^3) - \mathcal{O}(t^{2\Delta(1,0)+2}) \, , \tag{5.73}$$

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2, 4	—	—
monopole	$(1,0)$	$K^{[2]}$	$2N-2$	$\mathrm{U}(2)$	3
monopole	$(1,1)$	$K^{[0]}$	$2N-2$	$\mathrm{U}(1)\times\mathrm{O}(2)$	3

Table 20. Bare and dressed monopole generators for a $\mathrm{O}(4)$ gauge theory with matter transforming in $[2,0]$.

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2, 4	—	—
monopole	$(1,0)$	$K^{[2]}$	$4N-2$	$\mathrm{U}(2)$	3
monopole	$(1,1)$	$K^{[0]}$	$6N-2$	$\mathrm{U}(1)\times\mathrm{O}(2)$	3

Table 21. Bare and dressed monopole generators for a $\mathrm{O}(4)$ gauge theory with matter transforming in $[2,2]$.

for $N \geq 2$, we can summarise the generators as in table 20. The different dressing behaviour of the $\mathrm{O}(4)$ monopole generators $(1,0)$ and $(1,1)$ compared to their $\mathrm{SO}(4)$ counterparts can be deduced from dividing the relevant dressing factor by the trivial one. In detail

$$\frac{P_{\mathrm{O}(4)}(t,\{(1,0) \text{ or } (1,1)\})}{P_{\mathrm{O}(4)}(t,0,0)} = \frac{(1-t^2)(1-t^4)}{(1-t)(1-t^2)} = 1+t+t^2+t^3 \ . \tag{5.74}$$

5.5.2 Representation [2, 2]

The conformal dimension is the same as in (5.30) and the rational cone of the Weyl chamber is still

$$C^{(2)} = \mathrm{Cone}\left((1,0),(1,1)\right) \ , \tag{5.75}$$

such that the cone generators and the Hilbert basis for $S^{(2)} := C^{(2)} \cap \left(K^{[0]} \cup K^{[2]}\right)$ coincide. The upper half-space of figure 17 depicts the situation. We note that the Weyl chamber for $\mathrm{SO}(4)$ is already divided into a fan by two rational cones, while the Weyl chamber for $\mathrm{O}(4)$ is not.

The computation of the Hilbert series then yields

$$\mathrm{HS}_{\mathrm{O}(4)}^{[2,2]}(t,N) = \frac{1+t^{4N-1}+t^{4N}+t^{4N+1}+t^{6N-1}+t^{6N}+t^{6N+1}+t^{10N}}{(1-t^2)\left(1-t^4\right)\left(1-t^{4N-2}\right)\left(1-t^{6N-2}\right)} \ . \tag{5.76}$$

Again, the rational function clearly displays a palindromic numerator, an order four pole for $t \to 1$, and an order two pole for $t \to \infty$, i.e. the difference in degrees of denominator and numerator is two. By inspection of (5.76) and use of the plethystic logarithm

$$\mathrm{PL}(\mathrm{HS}_{\mathrm{O}(4)}^{[2,2]}) = t^2+t^4+t^{\Delta(1,0)}(1+t+t^2+t^3)+t^{\Delta(1,1)}(1+t+t^2+t^3)-\mathcal{O}(t^{2\Delta(1,0)+2}) \ , \tag{5.77}$$

for $N \geq 2$, we can summarise the generators as in table 21. The dressings behave as discussed earlier.

object	(m_1, m_2)	lattice	$\Delta(m_1, m_2)$	$\mathrm{H}_{(m_1,m_2)}$	dressings
Casimirs	—	—	2, 4	—	—
monopole	$(1, 0)$	$K^{[2]}$	$10N - 2$	$\mathrm{U}(2)$	3
monopole	$(1, 1)$	$K^{[0]}$	$10N - 2$	$\mathrm{U}(1) \times \mathrm{O}(2)$	3
monopole	$(2, 1)$	$K^{[2]}$	$18N - 4$	$\mathrm{U}(1)^2$	7
monopole	$(3, 1)$	$K^{[0]}$	$26N - 6$	$\mathrm{U}(1)^2$	7

Table 22. Bare and dressed monopole generators for a $\mathrm{O}(4)$ gauge theory with matter transforming in $[4, 2]$.

5.5.3 Representation $[4, 2]$

The conformal dimension is given in (5.50) and the Weyl chamber is split into a fan generated by two rational cones

$$C_2^{(2)} = \mathrm{Cone}\left((1, 0), (3, 1)\right) \quad \text{and} \quad C_3^{(2)} = \mathrm{Cone}\left((3, 1), (1, 1)\right) \,, \tag{5.78}$$

where we use the notation of the $\mathrm{SO}(4)$ setting, see the upper half plan of figure 22. The Hilbert bases for $S_p^{(2)} := C_p^{(2)} \cap \left(K^{[0]} \cup K^{[2]}\right)$ differ from the cone generators and are obtained as

$$\mathcal{H}(S_2^{(2)}) = \{(1, 0), (3, 1)\} \quad \text{and} \quad \mathcal{H}(S_3^{(2)}) = \{(3, 1), (2, 1), (1, 1)\} \,. \tag{5.79}$$

The computation of the Hilbert series then yields

$$\mathrm{HS}_{\mathrm{O}(4)}^{[4,2]}(t, N) = \frac{R(t, N)}{(1 - t^2)\,(1 - t^4)\,(1 - t^{10N-2})\,(1 - t^{26N-6})} \,, \tag{5.80a}$$

$$\begin{aligned} R(t, N) = 1 &+ t^{10N-2} + 2t^{10N-1} + 2t^{10N} + 2t^{10N+1} \\ &+ t^{18N-4} + 2t^{18N-3} + 2t^{18N-2} + 2t^{18N-1} + t^{18N} \\ &+ 2t^{26N-5} + 2t^{26N-4} + 2t^{26N-3} + t^{26N-2} + t^{36N-4} \end{aligned} \tag{5.80b}$$

As before, the rational function (5.80) clearly displays a palindromic numerator, an order four pole for $t \to 1$, and an order two pole for $t \to \infty$, i.e. the difference in degrees of denominator and numerator is two. By inspection of (5.80) and use of the plethystic logarithm

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}_{\mathrm{O}(4)}^{[4,2]}) = t^2 + t^4 &+ t^{\Delta(1,0)}(1 + t + t^2 + t^3) + t^{\Delta(1,1)}(1 + t + t^2 + t^3) \\ &+ t^{\Delta(2,1)}(1 + 2(t + t^2 + t^3) + t^4) \\ &- t^{\Delta(1,0)+\Delta(1,1)}(1 + 2t + 5t^2 + 6t^3 + 7t^4 + 4t^5 + 3t^6) \\ &+ t^{\Delta(3,1)}(1 + 2(t + t^2 + t^3) + t^4) - \mathcal{O}(t^{\Delta(1,0)+\Delta(2,1)}) \,, \end{aligned} \tag{5.81}$$

for $N \geq 2$, we can summarise the generators as in table 22. The dressing behaviour of $(1, 0)$, $(1, 1)$ is as discussed earlier; however, we need to describe the dressings of $(2, 1)$ and

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(3, 1) as it differs from the SO(4) counterparts. Again, we compute the quotient of the dressing factor of the maximal torus divided by the trivial one, i.e.

$$\frac{P_{\mathrm{O}(4)}(t, m_1 > m_2 > 0)}{P_{\mathrm{O}(4)}(t, 0, 0)} = \frac{(1-t^2)(1-t^4)}{(1-t)^2} = 1 + 2(t + t^2 + t^3) + t^4 \,. \tag{5.82}$$

Consequently, each bare monopole (2, 1), (3, 1) is accompanied by seven dressings, which is in agreement with (5.81).

6 Case: USp(4)

This section is devoted to the study of the compact symplectic group USp(4) with corresponding Lie algebra C_2. GNO-duality relates them with the special orthogonal group SO(5) and the Lie algebra B_2.

6.1 Set-up

For studying the non-abelian group USp(4), we start by providing the contributions of $N_{a,b}$ hypermultiplets in various representations $[a, b]$ of USp(4) to the conformal dimensions

$$\Delta^{[1,0]}_{\mathrm{h-plet}} = N_{1,0} \sum_i |m_i| \,, \tag{6.1a}$$

$$\Delta^{[0,1]}_{\mathrm{h-plet}} = N_{0,1} \left(\sum_{i<j} |m_i - m_j| + \sum_{i<j} |m_i + m_j| \right) , \tag{6.1b}$$

$$\Delta^{[2,0]}_{\mathrm{h-plet}} = 2N_{2,0} \sum_i |m_i| + N_{2,0} \left(\sum_{i<j} |m_i - m_j| + \sum_{i<j} |m_i + m_j| \right) , \tag{6.1c}$$

$$\Delta^{[0,2]}_{\mathrm{h-plet}} = 2N_{0,2} \sum_i |m_i| + 3N_{0,2} \left(\sum_{i<j} |m_i - m_j| + \sum_{i<j} |m_i + m_j| \right) , \tag{6.1d}$$

$$\begin{aligned} \Delta^{[1,1]}_{\mathrm{h-plet}} = 2N_{1,1} \sum_i |m_i| + N_{1,1} \Bigg(&\sum_{i<j} \left(|2m_i - m_j| + |m_i - 2m_j| \right) \\ &+ \sum_{i<j} \left(|2m_i + m_j| + |m_i + 2m_j| \right) \Bigg) , \end{aligned} \tag{6.1e}$$

$$\begin{aligned} \Delta^{[3,0]}_{\mathrm{h-plet}} = 5N_{3,0} \sum_i |m_i| + N_{3,0} \Bigg(&\sum_{i<j} \left(|2m_i - m_j| + |m_i - 2m_j| \right) \\ &+ \sum_{i<j} \left(|2m_i + m_j| + |m_i + 2m_j| \right) \Bigg) , \end{aligned} \tag{6.1f}$$

wherein $i, j = 1, 2$, and the contribution of the vector multiplet is given by

$$\Delta_{\mathrm{V-plet}} = -2 \sum_i |m_i| - \left(\sum_{i<j} |m_i - m_j| + \sum_{i<j} |m_i + m_j| \right) . \tag{6.1g}$$

Such that we will consider the following conformal dimension

$$\Delta(m_1, m_2) = (N_1 - 2)(|m_1| + |m_2|) + (N_2 - 1) \left(|m_1 - m_2| + |m_1 + m_2| \right) \tag{6.2a}$$

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$$+ N_3 \left(|2m_1 - m_2| + |m_1 - 2m_2| + |2m_1 + m_2| + |m_1 + 2m_2|\right)$$

and we can vary the representation content via

$$N_1 = N_{1,0} + 2N_{2,0} + 2N_{0,2} + 2N_{1,1} + 5N_{3,0} \ , \tag{6.2b}$$
$$N_2 = N_{0,1} + N_{2,0} + 3N_{0,2} \ , \tag{6.2c}$$
$$N_3 = N_{1,1} + N_{3,0} \ . \tag{6.2d}$$

The Hilbert series is computed as usual

$$\mathrm{HS}_{\mathrm{USp}(4)}(t,N) = \sum_{m_1 \geq m_2 \geq 0} t^{\Delta(m_1,m_2)} P_{\mathrm{USp}(4)}(t,m_1,m_2) \ , \tag{6.3}$$

where the summation for m_1, m_2 has been restricted to the principal Weyl chamber of the GNO-dual group SO(5), whose Weyl group is $S_2 \ltimes (\mathbb{Z}_2)^2$. Thus, we use the reflections to restrict to non-negative $m_i \geq 0$ and the permutations to restrict to a ordering $m_1 \geq m_2$. The classical dressing factor takes the following form [5]:

$$P_{\mathrm{USp}(4)}(t,m_1,m_2) = \begin{cases} \dfrac{1}{(1-t)^2} \ , & m_1 > m_2 > 0 \ , \\ \dfrac{1}{(1-t)(1-t^2)} \ , & (m_1 > m_2 = 0) \vee (m_1 = m_2 > 0) \ , \\ \dfrac{1}{(1-t^2)(1-t^4)} \ , & m_1 = m_2 = 0 \ . \end{cases} \tag{6.4}$$

6.2 Hilbert basis

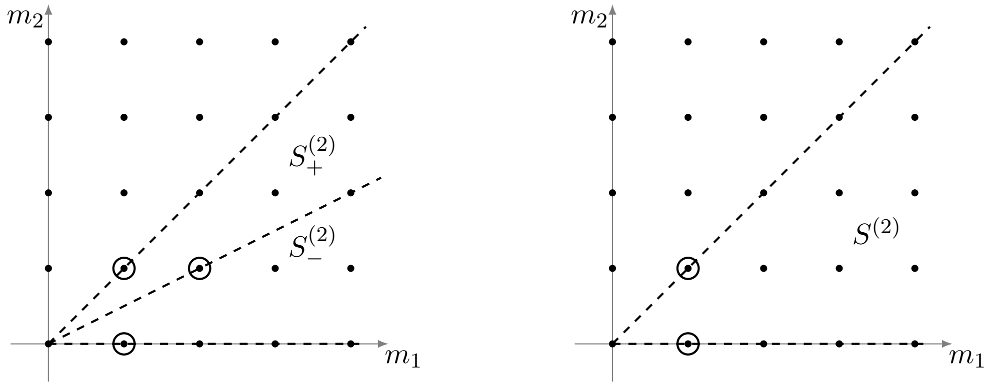
The conformal dimension (6.2a) divides the dominant Weyl chamber of SO(5) into a fan. The intersection with the corresponding weight lattice $\Lambda_w(\mathrm{SO}(5))$ introduces semi-groups S_p, which are sketched in figure 26. As displayed, the set of semi-groups (and rational cones that constitute the fan) differ if $N_3 \neq 0$. The Hilbert bases for both case are readily computed, because they coincide with the set of ray generators.

- For $N_3 \neq 0$, which is displayed in figure 26a, there exists one hyperplane $|m_1 - 2m_2| = 0$ which intersects the Weyl chamber non-trivially. Therefore, $\Lambda_w(\mathrm{SO}(5))/\mathcal{W}_{\mathrm{SO}(5)}$ becomes a fan generated by two 2-dimensional cones. The Hilbert bases of the corresponding semi-groups are computed to

$$\mathcal{H}(S_+^{(2)}) = \Big\{(1,1),(2,1)\Big\} \ , \qquad \mathcal{H}(S_-^{(2)}) = \Big\{(2,1),(1,0)\Big\} \ . \tag{6.5}$$

- For $N_3 = 0$, as shown in figure 26b, there exists no hyperplane that intersects the dominant Weyl chamber non-trivially. As a consequence, the $\Lambda_w(\mathrm{SO}(5))/\mathcal{W}_{\mathrm{SO}(5)}$ is described by one rational polyhedral cone of dimension 2. The Hilbert basis for the semi-group is given by

$$\mathcal{H}(S^{(2)}) = \Big\{(1,1),(1,0)\Big\} \ . \tag{6.6}$$



(a) $N_3 \neq 0$ **(b)** $N_3 = 0$

Figure 26. The various semi-groups for USp(4) depending on whether $N_3 \neq 0$ or $N_3 = 0$. For both cases the black circled points are the ray generators.

6.3 Dressings

Before evaluating the Hilbert series, let us analyse the classical dressing factors for the minimal generators (6.5) or (6.6). Firstly, the classical Lie group USp(4) has two Casimir invariants of degree 2 and 4 and can they can be written as $\mathrm{Tr}(\Phi^2) = \sum_{i=1}^{2}(\phi_i)^2$ and $\mathrm{Tr}(\Phi^4) = \sum_{i=1}^{2}(\phi_i)^4$, respectively. Again, we employ the diagonal form of the adjoint valued scalar field Φ.

Secondly, the bare monopole operator corresponding to GNO-charge $(1,0)$ has conformal dimension $N_1 + 2N_2 + 6N_3 - 4$ and the residual gauge group is $\mathrm{H}_{(1,0)} = \mathrm{U}(1) \times \mathrm{SU}(2)$, i.e. allowing for dressings by degree 1 and 2 Casimirs. The resulting set of bare and dressed monopoles is

$$V_{(1,0)}^{\mathrm{dress},0} = (1,0) + (-1,0) + (0,1) + (0,-1) \,, \tag{6.7a}$$
$$V_{(1,0)}^{\mathrm{dress},2} = ((1,0) + (-1,0))\,(\phi_2)^2 + ((0,1) + (0,-1))\,(\phi_1)^2 \,, \tag{6.7b}$$
$$V_{(1,0)}^{\mathrm{dress},1} = ((1,0) - (-1,0))\,\phi_1 + ((0,1) - (0,-1))\,\phi_2 \,, \tag{6.7c}$$
$$V_{(1,0)}^{\mathrm{dress},3} = ((1,0) - (-1,0))\,(\phi_1)^3 + ((0,1) - (0,-1))\,(\phi_2)^3 \,. \tag{6.7d}$$

Thirdly, the bare monopole operators of GNO-charge $(1,1)$ has conformal dimension $2N_1 + 2N_2 + 8N_3 - 6$ and residual gauge group $\mathrm{H}_{(1,1)} = \mathrm{U}(1) \times \mathrm{SU}(2)$. The bare and dressed monopole operators can be written as

$$V_{(1,1)}^{\mathrm{dress},0} = (1,1) + (1,-1) + (-1,1) + (-1,-1) \,, \tag{6.8a}$$
$$V_{(1,1)}^{\mathrm{dress},2} = ((1,1) + (-1,-1))((\phi_1)^2 + (\phi_2)^2) + (1,-1)(\phi_2)^2 + (-1,1)(\phi_1)^2 \,, \tag{6.8b}$$
$$V_{(1,1)}^{\mathrm{dress},1} = (1,1)(\phi_1 + \phi_2) + (-1,-1)(-\phi_1 - \phi_2) + (1,-1)(-\phi_2) + (-1,1)(-\phi_1) \,, \tag{6.8c}$$
$$\begin{aligned} V_{(1,1)}^{\mathrm{dress},3} &= (1,1)((\phi_1)^3 + (\phi_2)^3) + (-1,-1)(-(\phi_1)^3 - (\phi_2)^3) \\ &\quad + (1,-1)(-(\phi_2)^3) + (-1,1)(-(\phi_1)^3) \,. \end{aligned} \tag{6.8d}$$

The two magnetic weights $(1,0)$, $(1,1)$ lie at the boundary of the dominant Weyl chamber such that the dressing behaviour can be predicted by $P_{\mathrm{USp}(4)}(t,m_1,m_2)/P_{\mathrm{USp}(4)}(t,0,0) = 1{+}t{+}t^2{+}t^3$, following appendix A. The above description of the bare and dressed monopole operators is therefore a valid choice of generating elements for the chiral ring.

Lastly, the bare monopole for $(2,1)$ has conformal dimension $3N_1+4N_2+12N_3-10$ and residual gauge group $H_{(2,1)}=\mathrm{U}(1)^2$. Thus, the dressing proceeds by two independent degree 1 Casimir invariants.

$$
\begin{aligned}
V^{\text{dress},0}_{(2,1)} &= (2,1){+}(2,-1){+}(-2,1){+}(1,2){+}(1,-2){+}(-1,2){+}(-1,-2){+}(-2,-1)\\
&\equiv (2,1)+(2,-1)+(-2,1)+(-2,-1)+\text{permutations}\,, && \text{(6.9a)}\\
V^{\text{dress},2j-1,1}_{(2,1)} &= (2,1)(\phi_1)^{2j-1}+(2,-1)(\phi_1)^{2j-1}+(-2,1)(-\phi_1)^{2j-1} && \text{(6.9b)}\\
&\qquad +(-2,-1)(-\phi_1)^{2j-1}+\text{permutations} \qquad \text{for} \quad j=1,2\,,\\
V^{\text{dress},2j-1,2}_{(2,1)} &= (2,1)(\phi_2)^{2j-1}+(2,-1)(-\phi_2)^{2j-1}+(-2,1)(\phi_2)^{2j-1} && \text{(6.9c)}\\
&\qquad +(-2,-1)(-\phi_2)^{2j-1}+\text{permutations} \qquad \text{for} \quad j=1,2\,,\\
V^{\text{dress},2,1}_{(2,1)} &= (2,1)(\phi_1)^2+(2,-1)(-(\phi_1)^2)+(-2,1)(-(\phi_1)^2) && \text{(6.9d)}\\
&\qquad +(-2,-1)(\phi_1)^2+\text{permutations}\,,\\
V^{\text{dress},2,2}_{(2,1)} &= (2,1)(\phi_1\phi_2)+(2,-1)(-\phi_1\phi_2)+(-2,1)(-\phi_1\phi_2) && \text{(6.9e)}\\
&\qquad +(-2,-1)(\phi_1\phi_2)+\text{permutations}\,,\\
V^{\text{dress},4}_{(2,1)} &= (2,1)(\phi_1^3\phi_2)+(2,-1)(-(\phi_1)^3\phi_2)+(-2,1)(-(\phi_1)^3\phi_2) && \text{(6.9f)}\\
&\qquad +(-2,-1)((\phi_1)^3\phi_2)+\text{permutations}\,.
\end{aligned}
$$

The number and the degrees of dressed monopole operators of charge $(2,1)$ are consistent with the quotient $P_{\mathrm{USp}(4)}(t,m_1>m_2>0)/P_{\mathrm{USp}(4)}(t,0,0)=1+2t+2t^2+2t^3+t^4$ of the dressing factors.

For "generic" values of N_1, N_2 and N_3 the Coulomb branch will be generated by the two Casimir invariants together with the bare and dressed monopole operators corresponding to the minimal generators of the Hilbert bases. However, we will encounter choices of the three parameters such that the set of monopole generators can be further reduced; for example, in the case of complete intersections.

6.4 Generic case

The computation for arbitrary N_1, N_2, and N_3 yields

$$
\mathrm{HS}_{\mathrm{USp}(4)}(t,N_1,N_2,N_3)=\frac{R(t,N_1,N_2,N_3)}{P(t,N_1,N_2,N_3)}\,, \tag{6.10a}
$$

with

$$
\begin{aligned}
P(t,N_1,N_2,N_3) = {}&\left(1-t^2\right)\left(1-t^4\right)\left(1-t^{N_1+2N_2+6N_3-4}\right)\left(1-t^{2N_1+2N_2+8N_3-6}\right)\\
&\times\left(1-t^{3N_1+4N_2+12N_3-10}\right)\,,
\end{aligned} \tag{6.10b}
$$

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$$\begin{aligned}R(t,N_1,N_2,N_3) =& 1+t^{N_1+2N_2+6N_3-3}(1+t+t^2)+t^{2N_1+2N_2+8N_3-5}(1+t+t^2) \\ &+t^{3N_1+4N_2+12N_3-9}(2+2t+2t^2+t^3)-t^{3N_1+4N_2+14N_3-10}(1+2t+2t^2+2t^3) \\ &-t^{4N_1+6N_2+18N_3-13}(1+t+t^2)-t^{5N_1+6N_2+20N_3-15}(1+t+t^2) \\ &-t^{6N_1+8N_2+26N_3-16}\,.\end{aligned} \tag{6.10c}$$

The numerator (6.10c) is an anti-palindromic polynomial of degree $6N_1+8N_2+26N_3-16$; while the denominator is of degree $6N_1+8N_2+26N_3-14$. The difference in degrees is 2, which equals the quaternionic dimension of the moduli space. In addition, the pole of (6.10) at $t\to 1$ is of order 4, which matches the complex dimension of the moduli space. For that, one verifies explicitly $R(t=1,N_1,N_2,N_3)=0$, but $\frac{\mathrm{d}}{\mathrm{d}t}R(t,N_1,N_2,N_3)|_{t=1}\neq 0$.

Consequently, the above interpretation of bare and dressed monopoles from the Hilbert series (6.10) is correct for "generic" choices of N_1, N_2, and N_3. In particular, $N_3\neq 0$ for this arguments to hold. Moreover, we will now exemplify the effects of the Casimir invariance in various special case of (6.10) explicitly. There are cases for which the inclusion of the Casimir invariance, i.e. dressed monopole operators, leads to a reduction of basis of monopole generators.

6.5 Category $N_3=0$

6.5.1 Representation $[1,0]$

Hilbert series. This choice is realised for $N_1=N$, $N_2=N_3=0$ and the Hilbert series simplifies drastically to a complete intersection

$$\mathrm{HS}_{\mathrm{USp}(4)}^{[1,0]}(t,N)=\frac{(1-t^{2N-4})(1-t^{2N-2})}{(1-t^2)(1-t^4)(1-t^{N-4})(1-t^{N-3})(1-t^{N-2})(1-t^{N-1})}\,, \tag{6.11}$$

which was first obtained in [5]. Due to the complete intersection property, the plethystic logarithm terminates and for $N>4$ we obtain

$$\mathrm{PL}(\mathrm{HS}_{\mathrm{USp}(4)}^{[1,0]})=t^2+t^4+t^{N-4}(1+t+t^2+t^3)-t^{2N-4}-t^{2N-2}\,. \tag{6.12}$$

Hilbert basis. Naively, the Hilbert series (6.11) should be generated by the Hilbert basis (6.6) plus their dressings. However, due to the particular form (6.2a) in representation $[1,0]$ and the Casimir invariance, the bare monopole operator of GNO-charge $(1,1)$ can be generated by the dressings of $(1,0)$. To see this, consider the Weyl-orbit $\mathcal{O}_{\mathcal{W}}(1,0)=\big\{(1,0),(0,1),(-1,0),(0,-1)\big\}$ and note the conformal dimensions align suitably, i.e. $\Delta(V_{(1,0)}^{\mathrm{dress},1})=N-3$, while $\Delta(V_{(1,1)}^{\mathrm{dress},0})=2N-6$. Thus, we can symbolically write

$$V_{(1,1)}^{\mathrm{dress},0}=V_{(1,0)}^{\mathrm{dress},1}+V_{(0,1)}^{\mathrm{dress},1}\,. \tag{6.13}$$

The moduli space is then generated by the Casimir invariants and the bare and dressed monopole operators corresponding to $(1,0)$, but this is to be understood as a rather "non-generic" situation.

6.5.2 Representation [0, 1]

This choice is realised for $N_2 = N$, and $N_1 = N_3 = 0$ and the Hilbert series simplifies to

$$\mathrm{HS}_{\mathrm{USp}(4)}^{[0,1]}(t,N) = \frac{1 + t^{2N-5} + t^{2N-4} + 2t^{2N-3} + t^{2N-2} + t^{2N-1} + t^{4N-6}}{(1-t^2)\,(1-t^4)\,(1-t^{2N-6})\,(1-t^{2N-4})} \,. \tag{6.14}$$

The Hilbert series (6.14) has a pole of order 4 at $t = 1$ as well as a palindromic polynomial as numerator. Moreover, the result (6.14) reflects the expected basis of monopole operators as given in the Hilbert basis (6.6).

6.5.3 Representation [2, 0]

This choice is realised for $N_1 = 2N$, $N_2 = N$, and $N_3 = 0$ and the Hilbert series reduces to

$$\mathrm{HS}_{\mathrm{USp}(4)}^{[2,0]}(t,N) = \frac{1 + t^{4N-3} + t^{4N-2} + t^{4N-1} + t^{6N-5} + t^{6N-4} + t^{6N-3} + t^{10N-6}}{(1-t^2)\,(1-t^4)\,(1-t^{4N-4})\,(1-t^{6N-6})} \,. \tag{6.15}$$

Also, the rational function (6.15) has a pole of order 4 for $t \to 1$ and a palindromic numerator. Evaluating the plethystic logarithm yields for all $N > 1$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}_{\mathrm{USp}(4)}^{[2,0]}) &= t^2 + t^4 + t^{4N-4}(1+t+t^2+t^3) \\ &\quad + t^{6N-6}(1+t+t^2+t^3) - t^{8N-6} + \mathcal{O}(t^{8N-5}) \,. \end{aligned} \tag{6.16}$$

This proves that bare monopole operators, corresponding to the the minimal generators of (6.6), together with their dressing generate all other monopole operators.

6.5.4 Representation [0, 2]

For $N_1 = 2N$, $N_2 = 3N$, and $N_3 = 0$ and the Hilbert series is given by

$$\mathrm{HS}_{\mathrm{USp}(4)}^{[0,2]}(t,N) = \frac{1 + t^{8N-3} + t^{8N-2} + t^{8N-1} + t^{10N-5} + t^{10N-4} + t^{10N-3} + t^{18N-6}}{(1-t^2)\,(1-t^4)\,(1-t^{8N-4})\,(1-t^{10N-6})} \,. \tag{6.17}$$

Evaluating the plethystic logarithm yields for all $N > 1$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}_{\mathrm{USp}(4)}^{[0,2]}) &= t^2 + t^4 + t^{8N-4}(1+t+t^2+t^3) \\ &\quad + t^{10N-6}(1+t+t^2+t^3) - t^{16N-6} + \mathcal{O}(t^{16N-5}) \,, \end{aligned} \tag{6.18a}$$

and for $N = 1$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}_{\mathrm{USp}(4)}^{[0,2]}) &= t^2 + t^4 + t^4(1+t+t^2+t^3) \\ &\quad + t^4(1+t+t^2+t^3) - 3t^{10} + \mathcal{O}(t^{11}) \,. \end{aligned} \tag{6.18b}$$

The inspection of the Hilbert series (6.17), together with the PL, proves that Hilbert basis (6.6), alongside all their dressings, are a sufficient set for all monopole operators.

6.6 Category $N_3 \neq 0$

6.6.1 Representation $[1,1]$

This choice corresponds to $N_1 = 2N$, $N_2 = 0$, and $N_3 = N$ and we obtain the Hilbert series to be

$$\mathrm{HS}_{\mathrm{USp}(4)}^{[1,1]}(t,N) = \frac{R(t,N)}{(1-t^2)\,(1-t^4)\,(1-t^{8N-4})\,(1-t^{12N-6})\,(1-t^{18N-10})}\,, \tag{6.19a}$$

$$\begin{aligned} R(t,N) = 1 &+ t^{8N-3}(1+t+t^2) + t^{12N-5}(1+t+t^2) \\ &+ t^{18N-9}(2+2t+2t^2+t^3) - t^{20N-10}(1+2t+2t^2+2t^3) \\ &- t^{26N-13}(1+t+t^2) - t^{30N-15}(1+t+t^2) - t^{38N-16}\,. \end{aligned} \tag{6.19b}$$

Considering the plethystic logarithm, we observe the following behaviour:

- For $N \geq 5$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}_{\mathrm{USp}(4)}^{[1,1]}) = t^2 + t^4 &+ t^{8N-4}(1+t+t^2+t^3) + t^{12N-6}(1+t+t^2+t^3) \\ &- t^{2(8N-4)+2}(1+t+2t^2+t^3+t^4) \\ &+ t^{18N-10}(1+2t+2t^2+2t^3+t^4) \\ &- t^{20N-10}(1+2t+3t^2+4t^3+3t^4+2t^5+t^6) + \ldots \end{aligned} \tag{6.20a}$$

- For $N = 4$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}_{\mathrm{USp}(4)}^{[1,1]}) = t^2 + t^4 &+ t^{28}(1+t+t^2+t^3) + t^{42}(1+t+t^2+t^3) \\ &- t^{58}(1+t+2t^2+t^3+t^4) \\ &+ t^{62}(1+2t+2t^2+2t^3+t^4) \\ &- t^{70}(1+2t+3t^2+4t^3+3t^4+2t^5+t^6) + \ldots \end{aligned} \tag{6.20b}$$

  We see, employing the previous results for $N > 4$, that the bare monopole $(2,1)$ and the last relation at t^{62} coincide. Hence, the term $\sim t^{62}$ disappears from the PL.

- For $N = 3$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}_{\mathrm{USp}(4)}^{[1,1]}) = t^2 + t^4 &+ t^{20}(1+t+t^2+t^3) + t^{30}(1+t+t^2+t^3) \\ &- t^{42}(1+t+2t^2+t^3+t^4) \\ &+ t^{44}(1+2t+2t^2+2t^3+t^4) \\ &- t^{70}(1+2t+3t^2+4t^3+3t^4+2t^5+t^6) + \ldots \end{aligned} \tag{6.20c}$$

  We see, employing again the previous results for $N > 4$, that the some monopole contributions of $(2,1)$ and the some of the relations coincide, cf. the coloured terms. Hence, there are, presumably, cancellations between generators and relations.

- For $N = 2$

$$\begin{aligned} \mathrm{PL}(\mathrm{HS}_{\mathrm{USp}(4)}^{[1,1]}) = t^2 + t^4 &+ t^{12}(1+t+t^2+t^3) + t^{18}(1+t+t^2+t^3) \\ &- t^{26}(1+t+2t^2+t^3+t^4) \end{aligned} \tag{6.20d}$$

$$\begin{aligned}&+t^{26}(1+2t+2t^2+2t^3+t^4)\\&-t^{30}(1+2t+3t^2+4t^3+3t^4+2t^5+t^6)+\dots\\=t^2+t^4+t^{12}(1+t+t^2+t^3)+t^{18}(1+t+t^2+t^3)&\\&+t^{26}(0+t+0+t^3+0)\\&-t^{30}(1+2t+3t^2+4t^3+3t^4+2t^5+t^6)+\dots\end{aligned} \tag{6.20e}$$

- For $N=1$

$$\mathrm{PL}(\mathrm{HS}^{[1,1]}_{\mathrm{USp}(4)})=t^2+2t^4+t^5+2t^6+2t^7+2t^8+3t^9-t^{11}+\dots \tag{6.20f}$$

Summarising, the Hilbert series (6.19) and its plethystic logarithm display that the minimal generators of (6.5) are indeed the basis for the bare monopole operators, and the corresponding dressings generate the remaining operators.

6.6.2 Representation $[3,0]$

For the choice $N_1=5N$, $N_2=0$, and $N_3=N$ the Hilbert series is given by

$$\mathrm{HS}^{[3,0]}_{\mathrm{USp}(4)}(t,N)=\frac{R(t,N)}{(1-t^2)(1-t^4)(1-t^{11N-4})(1-t^{18N-6})(1-t^{27N-10})}\,, \tag{6.21a}$$

$$\begin{aligned}R(t,N)=1&+t^{11N-3}(1+t+t^2)+t^{18N-5}(1+t+t^2)\\&+t^{27N-9}(2+2t+2t^2+t^3)-t^{29N-10}(1+2t+2t^2+2t^3)\\&-t^{38N-13}(1+t+t^2)-t^{45N-15}(1+t+t^2)-t^{56N-16}\,.\end{aligned} \tag{6.21b}$$

The inspection of the plethystic logarithm provides further insights:

- For $N\geq 3$

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[3,0]}_{\mathrm{USp}(4)})=t^2+t^4&+t^{11N-4}(1+t+t^2+t^3)+t^{18N-6}(1+t+t^2+t^3)\\&-t^{2(11N-4)+2}(1+t+2t^2+t^3+t^4)\\&+t^{27N-10}(1+2t+2t^2+2t^3+t^4)\\&-t^{(11N-4)+(18N-6)}(1+2t+3t^2+4t^3+3t^4+2t^5+t^6)+\dots\end{aligned} \tag{6.22a}$$

- For $N=2$

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[3,0]}_{\mathrm{USp}(4)})=t^2+t^4&+t^{18}(1+t+t^2+t^3)+t^{30}(1+t+t^2+t^3)\\&-t^{38}(1+t+2t^2+t^3+t^4)\\&+t^{44}(1+2t+2t^2+2t^3+t^4)\\&-t^{48}(1+2t+3t^2+4t^3+3t^4+2t^5+t^6)+\dots\end{aligned} \tag{6.22b}$$

  We see that, presumably, one generator and one relation cancel at t^{48}.

- For $N=1$

$$\mathrm{PL}(\mathrm{HS}^{[3,0]}_{\mathrm{USp}(4)})=t^2+t^4+t^7(1+t+t^2+t^3)+t^{12}(1+t+t^2+t^3)-t^{16}-t^{20}+\dots \tag{6.22c}$$

Again, we confirm that the minimal generators of the Hilbert basis (6.5) are the relevant generators (together with their dressings) for the moduli space.

Dynkin label	[1, 0]	[0, 1]	[2, 0]	[1, 1]	[0, 2]	[3, 0]	[4, 0]	[2, 1]
Dim.	7	14	27	64	77	77	182	189
	category 1			category 2			category 3	

Table 23. An overview of the G_2-representations considered in this paper.

7 Case: $\mathbf{G_2}$

Here, we study the Coulomb branch for the only exceptional simple Lie group of rank two.

7.1 Set-up

The group G_2 has irreducible representations labelled by two Dynkin labels and the dimension formula reads

$$\dim[a,b] = \frac{1}{120}(a+1)(b+1)(a+b+2)(a+2b+3)(a+3b+4)(2a+3b+5)\,. \tag{7.1}$$

In the following, we study the representations given in table 23. The three categories defined are due to the similar form of the conformal dimensions.

The Weyl group of G_2 is D_6 and the GNO-dual group is another G_2. Any element in the Cartan subalgebra $\mathfrak{h} = \mathrm{span}(H_1, H_2)$ can be written as $H = n_1 H_1 + n_2 H_2$. Restriction to the principal Weyl chamber is realised via $n_1, n_2 \geq 0$.

The group G_2 has two Casimir invariants of degree 2 and 6. Therefore, the classical dressing function is [5]

$$P_{G_2}(t, n_1, n_2) = \begin{cases} \dfrac{1}{(1-t^2)(1-t^6)}\,, & n_1 = n_2 = 0\,, \\ \dfrac{1}{(1-t)(1-t^2)}\,, & n_1 > 0, n_2 = 0 \text{ or } n_1 = 0, n_2 > 0\,, \\ \dfrac{1}{(1-t)^2}\,, & n_1, n_2 > 0\,. \end{cases} \tag{7.2}$$

7.2 Category 1

Hilbert basis. The representations [1, 0], [0, 1], and [2, 0] have schematically conformal dimensions of the form

$$\Delta(n_1, n_2) = \sum_j A_j |a_j n_1 + b_j n_2| + B_1 |n_1| + B_2 |n_2| \tag{7.3}$$

for $a_j, b_j \in \mathbb{N}$ and $A_j, B_1, B_2 \in \mathbb{Z}$. As a consequence, the usual fan within the Weyl chamber is simply one 2-dimensional rational polyhedral cone

$$C^{(2)} = \mathrm{Cone}((1,0),(0,1))\,. \tag{7.4}$$

The intersection with the weight lattice $\Lambda_w(G_2)$ yields the relevant semi-group $S^{(2)}$, as depicted in figure 27. The Hilbert bases are trivially given by the ray generators

$$\mathcal{H}(S^{(2)}) = \Big\{(1,0),(0,1)\Big\}\,. \tag{7.5}$$

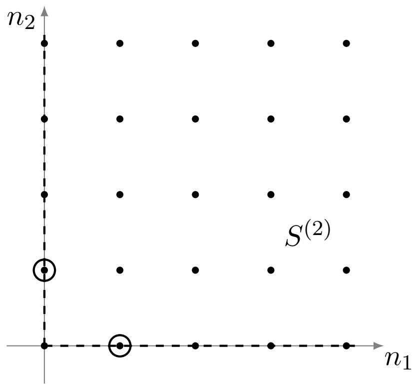

Figure 27. The semi-group $S^{(2)}$ for the representations $[1,0]$, $[0,1]$, and $[2,0]$ obtained from the G_2 Weyl chamber (considered as rational cone) and its ray generators (black circled points).

Dressings. The two minimal generators lie at the boundary of the Weyl chamber and, therefore, have residual gauge group $\mathrm{H}_{(1,0)} = \mathrm{H}_{(0,1)} = \mathrm{U}(2)$. Recalling that G_2 has two Casimir invariants $\mathcal{C}_2, \mathcal{C}_6$ at degree 2 and 6, one can analyse the dressed monopole operators associated to $(1,0)$ and $(0,1)$.

The residual gauge group $\mathrm{U}(2) \subset \mathrm{G}_2$ has a degree one Casimir $C_1 := \phi_1 + \phi_2$ and a degree two Casimir $C_2 := \phi_1^2 + \phi_2^2$. Again, we employed the diagonal form of the adjoint-valued scalar Φ. Consequently, the bare monopole $V_{(0,1)}^{\mathrm{dress},0}$ exhibits five dressed monopoles $V_{(0,1)}^{\mathrm{dress},i}$ ($i = 1, \ldots, 5$) of degrees $\Delta(0,1) + 1, \ldots, \Delta(0,1) + 5$. Since the highest degree Casimir invariant is of order 6 and the degree 2 Casimir invariant of G_2 differs from the pure sum of squares [29], one can build all dressings as follows:

$$C_1(0,1)\,, \quad C_2(0,1)\,, \quad C_1C_2(0,1)\,, \quad C_1^2C_2(0,1)\,, \quad (C_1C_2^2 + C_1^2C_2)(0,1)\,. \tag{7.6}$$

The very same arguments applies for the bare and dressed monopole generators associated to $(1,0)$. Thus, we expect six monopole operators: one bare $V_{(1,0)}^{\mathrm{dress},0}$ and five dressed $V_{(1,0)}^{\mathrm{dress},i}$ ($i = 1, \ldots, 5$).

Comparing with appendix A, we find that a magnetic weight at the boundary of the dominant Weyl chamber has dressings given by $P_{\mathrm{G}_2}(t, \{n_1 = 0 \text{ or } n_2 = 0\})/P_{\mathrm{G}_2}(t,0,0) = 1 + t + t^2 + t^3 + t^4 + t^5$, which is then consistent with the exposition above.

We will now exemplify the three different representations.

7.2.1 Representation $[1,0]$

The relevant computation has been presented in [5] and the conformal dimension reads

$$\begin{aligned} \Delta(n_1,n_2) =& N(|n_1+n_2| + |2n_1+n_2| + |n_1|) \\ &- (|n_1+n_2| + |2n_1+n_2| + |3n_1+n_2| + |3n_1+2n_2| + |n_1| + |n_2|)\,. \end{aligned} \tag{7.7}$$

Evaluating the Hilbert series for $N > 3$ yields

$$\mathrm{HS}_{\mathrm{G}_2}^{[1,0]}(t,N) = \frac{1 + t^{2N-4} + t^{2N-3} + t^{2N-2} + t^{2N-1} + t^{4N-5}}{(1-t^2)\,(1-t^6)\,(1-t^{2N-6})\,(1-t^{2N-5})}\,. \tag{7.8}$$

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V^{\mathrm{dress},0}_{(0,1)}$	$6(N-1)$	$\mathrm{U}(2)$
dressings $(i=1,\ldots,5)$	$V^{\mathrm{dress},i}_{(0,1)}$	$6(N-1)+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,0)}$	$10(N-1)$	$\mathrm{U}(2)$
dressings $(i=1,\ldots,5)$	$V^{\mathrm{dress},i}_{(1,0)}$	$10(N-1)+i$	—

Table 24. The chiral ring generators for a G_2 gauge theory and matter transforming in $[0,1]$.

We observe that the numerator of (7.8) is a palindromic polynomial of degree $4N-5$; while, the denominator has degree $4N-3$. Hence, the difference in degree between denominator and numerator is 2, which equals the quaternionic dimension of moduli space. In addition, the Hilbert series (7.8) has a pole of order 4 as $t \to 1$, which matches the complex dimension of the moduli space.

As discussed in [5], the plethystic logarithm has the following behaviour:

$$\mathrm{PL}(\mathrm{HS}^{[1,0]}_{\mathrm{G}_2}(t,N)) = t^2 + t^6 + t^{2N-6}(1+t+t^2+t^3+t^4+t^5) - t^{4N-8} + \dots \,. \tag{7.9}$$

Hilbert basis. According to [5], the monopole corresponding to GNO-charge $(1,0)$, which has $\Delta(1,0) = 4N-10$, can be generated. Again, this is due to the specific form (7.7).

7.2.2 Representation $[0,1]$

Hilbert series. For this representation, the conformal dimension is given as

$$\Delta(n_1,n_2) = (N-1)(|n_1+n_2| + |2n_1+n_2| + |3n_1+n_2| + |3n_1+2n_2| + |n_1| + |n_2|) \,, \tag{7.10}$$

and the computation of the Hilbert series for $N>1$ yields

$$\mathrm{HS}^{[0,1]}_{\mathrm{G}_2}(t,N) = \frac{1+t^{6N-5}(1+t+t^2+t^3+t^4)+t^{10N-9}(1+t+t^2+t^3+t^4)+t^{16N-10}}{(1-t^2)(1-t^6)\left(1-t^{6(N-1)}\right)\left(1-t^{10(N-1)}\right)} \,. \tag{7.11}$$

The numerator of (7.11) is a palindromic polynomial of degree $16N-10$; while, the denominator is of degree $16N-8$. Hence, the difference in degree between denominator and numerator is 2, which matches the quaternionic dimension of moduli space. Moreover, the Hilbert series has a pole of order 4 as $t \to 1$, i.e. it equals complex dimension of the moduli space. Employing the knowledge of the Hilbert basis (7.5), the appearing objects in (7.11) can be interpreted as in table 24.

Plethystic logarithm. For $N \geq 3$ the PL takes the form

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}^{[0,1]}_{\mathrm{G}_2}(t,N)) = t^2 + t^6 + t^{6(N-1)}(1+t+t^2+t^3+t^4+t^5) \\ + t^{10(N-1)}(1+t+t^2+t^3+t^4+t^5) - t^{12N-10} + \dots\end{aligned} \tag{7.12}$$

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V_{(0,1)}^{\mathrm{dress},0}$	$12N-6$	$\mathrm{U}(2)$
dressings ($i=1,\ldots,5$)	$V_{(0,1)}^{\mathrm{dress},i}$	$12N-6+i$	—
bare monopole	$V_{(1,0)}^{\mathrm{dress},0}$	$22N-10$	$\mathrm{U}(2)$
dressings ($i=1,\ldots,5$)	$V_{(1,0)}^{\mathrm{dress},i}$	$22N-10+i$	—

Table 25. The chiral ring generators for a G_2 gauge theory and matter transforming in $[2,0]$.

while for $N=2$ the PL is

$$\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[0,1]}(t,2)) = t^2 + t^6 + t^6(1+t+t^2+t^3+t^4+t^5) + t^{10}(1+t+t^2+t^3) - 2t^{16} + \ldots \tag{7.13}$$

In other words, the 4th and 5th dressing of $(1,0)$ are absent, because they can be generated.

7.2.3 Representation [2, 0]

Hilbert series. For this representation, the conformal dimensions is given by

$$\begin{aligned}\Delta(n_1,n_2) =& N\Big(2\left|n_1+n_2\right| + 2\left|2n_1+n_2\right| + \left|3n_1+n_2\right| + \left|2n_1+2n_2\right| + \left|3n_1+2n_2\right| \\ &+ \left|4n_1+2n_2\right| + 2\left|n_1\right| + \left|2n_1\right| + \left|n_2\right|\Big) \\ &- \left(\left|n_1+n_2\right| + \left|2n_1+n_2\right| + \left|3n_1+n_2\right| + \left|3n_1+2n_2\right| + \left|n_1\right| + \left|n_2\right|\right) .\end{aligned} \tag{7.14}$$

The calculation for the Hilbert series is analogous to the previous cases and we obtain

$$\mathrm{HS}_{\mathrm{G}_2}^{[2,0]}(t,N) = \frac{1 + t^{12N-5}(1+t+t^2+t^3+t^4) + t^{22N-9}(1+t+t^2+t^3+t^4) + t^{34N-10}}{(1-t^2)\,(1-t^6)\,(1-t^{12N-6})\,(1-t^{22N-10})} . \tag{7.15}$$

One readily observes, the numerator of (7.15) is a palindromic polynomial of degree $34N-10$ and the denominator is of degree $34N-8$. Hence, the difference in degree between denominator and numerator is 2, which is precisely the quaternionic dimension of moduli space. Also, the Hilbert series has a pole of order 4 as $t\to 1$, which equals the complex dimension of the moduli space. Having in mind the minimal generators (7.5), the appearing objects in (7.15) can be summarised as in table 25.

Plethystic logarithm.

- For $N\geq 3$ the PL takes the form

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[2,0]}(t,N)) = t^2 + t^6 + t^{12N-6}(1+t+t^2+t^3+t^4+t^5) \\ + t^{22N-10}(1+t+t^2+t^3+t^4+t^5) - t^{12N-10} + \ldots\end{aligned} \tag{7.16}$$

- While for $N=2$ the PL is

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[2,0]}(t,2)) = t^2 + t^6 + t^{18}(1+t+t^2+t^3+t^4+t^5) \\ + t^{34}(1+t+t^2+t^3) - 2t^{40} + \dots\end{aligned} \tag{7.17}$$

By the very same reasoning as before, $V_{(1,0)}^{\mathrm{dress},4}$ and $V_{(1,0)}^{\mathrm{dress},5}$ can be generated by monopoles associated to $(0,1)$.

- Moreover, for $N=1$ the PL looks as follows

$$\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[2,0]}(t,1)) = t^2 + t^6 + t^6(1+t+t^2+t^3+t^4+t^5) + t^{12}(1+t) - t^{16} + \dots \tag{7.18}$$

Looking at the conformal dimensions reveals that the missing dressed monopoles of GNO-charge $(1,0)$ can be generated.

7.3 Category 2

Hilbert basis. The representations $[1,1]$, $[0,2]$, and $[3,0]$ have schematically conformal dimensions of the form

$$\Delta(n_1,n_2) = \sum_j A_j |a_j n_1 + b_j n_2| + B_1|n_1| + B_2|n_2| + C|n_1 - n_2| \tag{7.19}$$

for $a_j, b_j \in \mathbb{N}$ and $A_j, B_1, B_2, C \in \mathbb{Z}$. The novelty of this conformal dimension, compared to (7.3), is the difference $|n_1 - n_2|$, i.e. a hyperplane that intersects the Weyl chamber non-trivially. As a consequence, there is a fan generated by two 2-dimensional rational polyhedral cones

$$C_1^{(2)} = \mathrm{Cone}((1,0),(1,1)) \quad \text{and} \quad C_2^{(2)} = \mathrm{Cone}((1,1),(0,1)) \ . \tag{7.20}$$

The intersection with the weight lattice $\Lambda_w(\mathrm{G}_2)$ yields the relevant semi-groups S_p ($p = 1,2$), as depicted in figure 28. The Hilbert bases are again given by the ray generators

$$\mathcal{H}(S_1^{(2)}) = \Big\{(1,0),(1,1)\Big\} \quad \text{and} \quad \mathcal{H}(S_2^{(2)}) = \Big\{(1,1),(0,1)\Big\} \ . \tag{7.21}$$

Dressings. The three minimal generators have different residual gauge groups, as two lie on the boundary and one in the interior of the Weyl chamber. The GNO-charges $(1,0)$ and $(0,1)$ are to be treated as in subsection 7.2.

The novelty is the magnetic weight $(1,1)$ with $\mathrm{H}_{(1,1)} = \mathrm{U}(1)^2$. Thus, the dressing can be constructed with two independent U(1)-Casimir invariants, proportional to ϕ_1 and ϕ_2. We choose a basis of dressed monopoles

$$V_{(1,1)}^{\mathrm{dress},j,\alpha} = (1,1)(\phi_\alpha)^j \ , \qquad \text{for} \quad j = 1,\dots 5 \ , \ \alpha = 1,2 \ , \tag{7.22a}$$

$$V_{(1,1)}^{\mathrm{dress},6} = (1,1)\left((\phi_1)^6 + (\phi_2)^6\right) \ . \tag{7.22b}$$

The reason behind the large number of dressings of the bare monopole $(1,1)$ lies in the delicate G_2 structure [29], i.e. the degree two Casimir $\mathcal{C}_2$ is not just the sum of the squares

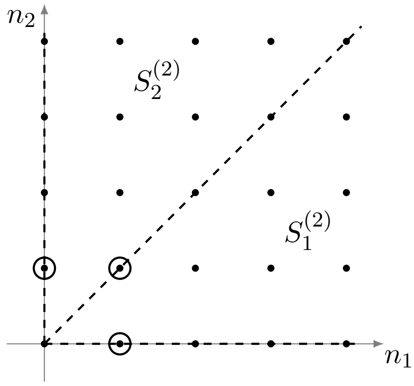

Figure 28. The semi-groups $S_p^{(2)}$ ($p = 1, 2$) for the representations $[1,1]$, $[0,2]$, and $[3,0]$ obtained from the G_2 Weyl chamber (considered as rational cone) and their ray generators (black circled points).

of ϕ_i and the next G_2-Casimir $\mathcal{C}_6$ is by four higher in degree and has a complicated structure as well.

The number and degrees of the dressed monopole operators associated to $(1,1)$ can be confirmed by $P_{G_2}(t, n_1 > 0, n_2 > 0)/P_{G_2}(t,0,0) = 1 + 2t + 2t^2 + 2t^3 + 2t^4 + 2t^5 + t^6$, following appendix A.

We will now exemplify the three different representations.

7.3.1 Representation $[1, 1]$

Hilbert series. The conformal dimension of the 64-dimensional representation is given by

$$\begin{aligned}\Delta(n_1,n_2) = N\Big(&|n_1-n_2| + 8|n_1+n_2| + 8|2n_1+n_2| + 2|3n_1+n_2| + |4n_1+n_2| \\ &+ |n_1+2n_2| + 2|3n_1+2n_2| + |5n_1+2n_2| + |4n_1+3n_2| + |5n_1+3n_2| \\ &+ 8|n_1| + 2|n_2| \Big) \\ &- \Big(|n_1+n_2| + |2n_1+n_2| + |3n_1+n_2| + |3n_1+2n_2| + |n_1| + |n_2| \Big) \,. \end{aligned} \tag{7.23}$$

Computing the Hilbert series provides the following expression

$$\mathrm{HS}_{G_2}^{[1,1]}(t,N) = \frac{R(t,N)}{(1-t^2)(1-t^6)(1-t^{36N-6})(1-t^{64N-10})(1-t^{98N-16})} \,, \tag{7.24a}$$

$$\begin{aligned} R(t,N) = 1 &+ t^{36N-5}(1+t+t^2+t^3+t^4) + t^{64N-9}(1+t+t^2+t^3+t^4) \\ &+ t^{98N-15}(2+2t+2t^2+2t^3+2t^4+t^5) \\ &- t^{100N-16}(1+2t+2t^2+2t^3+2t^4+2t^5) \\ &- t^{134N-21}(1+t+t^2+t^3+t^4) - t^{162N-25}(1+t+t^2+t^3+t^4) - t^{198N-26} \,. \end{aligned} \tag{7.24b}$$

The numerator (7.24b) is a anti-palindromic polynomial of degree $198N - 26$; whereas the denominator is of degree $198N - 24$. Hence, the difference in degree between denominator and numerator is 2, which coincides with the quaternionic dimension of moduli space.

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V_{(0,1)}^{\mathrm{dress},0}$	$134N-6$	U(2)
dressings $(i=1,\ldots,5)$	$V_{(0,1)}^{\mathrm{dress},i}$	$134N-6+i$	—
bare monopole	$V_{(1,0)}^{\mathrm{dress},0}$	$238N-10$	U(2)
dressings $(i=1,\ldots,5)$	$V_{(1,0)}^{\mathrm{dress},i}$	$238N-10+i$	—
bare monopole	$V_{(1,1)}^{\mathrm{dress},0}$	$364N-16$	$\mathrm{U}(1)\times\mathrm{U}(1)$
dressings $(i=1,\ldots,5;\alpha=1,2)$	$V_{(1,1)}^{\mathrm{dress},i,\alpha}$	$364N-16+i$	—
dressing	$V_{(1,1)}^{\mathrm{dress},6}$	$364N-16+6$	—

Table 26. The chiral ring generators for a G_2 gauge theory and matter transforming in $[1,1]$.

The Hilbert series (7.24) has a pole of order 4 as $t\to 1$, which agrees with the complex dimension of the moduli space. (One can explicitly show that $R(t=1,N)=0$, but $\frac{\mathrm{d}}{\mathrm{d}t}R(t,N)|_{t=1}\neq 0$.) The appearing operators agree with the general setting outlined above and we summarise them in table 26. The new monopole corresponds to GNO-charge $(1,1)$ and displays a different dressing behaviour than $(1,0)$ and $(0,1)$. The reason behind lies in the residual gauge group being $\mathrm{U}(1)^2$.

Plethystic logarithm. Although the bare monopole $V_{(1,1)}^{\mathrm{dress},0}$ is generically a necessary generator due to its origin as an ray generators of (7.21), not all dressings $V_{(1,1)}^{\mathrm{dress}}$ might be independent.

- For $N\geq 4$ the PL takes the form

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[1,1]}(t,N)) = t^2+t^6&+t^{36N-6}(1+t+t^2+t^3+t^4+t^5)\\&+t^{64N-10}(1+t+t^2+t^3+t^4+t^5)\\&-t^{2(36N-6)+2}(1+t+2t^2+2t^3+3t^4+2t^5+2t^6+t^7+t^8)\\&+t^{98N-16}(1+2t+2t^2+2t^3+2t^4+2t^5+t^6)-t^{100N-16}+\ldots\end{aligned} \tag{7.25}$$

- For $N=3$ the PL is

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[1,1]}(t,N=3)) = t^2+t^6&+t^{102}(1+t+t^2+t^3+t^4+t^5)\\&+t^{182}(1+t+t^2+t^3+t^4+t^5)\\&-t^{206}(1+t+2t^2+2t^3+3t^4+2t^5+2t^6+t^7+t^8)\\&+t^{278}(1+2t+2t^2+2t^3+2t^4+2t^5)-2t^{285}+\ldots\end{aligned} \tag{7.26}$$

Here, $\Delta(1,0)+\Delta(0,1)=284$ is precisely the conformal dimension of $V_{(1,1)}^{\mathrm{dress},6}$; i.e. it is generated and absent from the PL.

- For $N = 2$ the PL is

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[1,1]}(t, N = 2)) = t^2 + t^6 &+ t^{66}(1 + t + t^2 + t^3 + t^4 + t^5) \\ &+ t^{118}(1 + t + t^2 + t^3 + t^4 + t^5) \\ &- t^{134}(1 + t + 2t^2 + 2t^3 + 3t^4 + 2t^5 + 2t^6 + t^7 + t^8) \\ &+ t^{180}(1 + 2t + 2t^2 + 2t^3 + t^4) - 2t^{186} + \ldots\end{aligned} \tag{7.27}$$

Here, $\Delta(1, 0) + \Delta(0, 1) = 184$ is precisely the conformal dimension of $V_{(1,1)}^{\mathrm{dress},4,\alpha}$; i.e. only one of the dressings by the 4th power of U(1)-Casimir is a generator. Consequently, the other one is absent from the PL.

- For $N = 1$ the PL is

$$\begin{aligned}\mathrm{PL}(\mathrm{HS}_{\mathrm{G}_2}^{[1,1]}(t, N = 1)) = t^2 + t^6 &+ t^{30}(1 + t + t^2 + t^3 + t^4 + t^5) \\ &+ t^{54}(1 + t + t^2 + t^3 + t^4 + t^5) \\ &- t^{62}(1 + t + 2t^2 + 2t^3 + 3t^4 + 2t^5 + 2t^6 + t^7 + t^8) \\ &+ t^{82}(1 + 2t + t^2) - t^{62} + \ldots\end{aligned} \tag{7.28}$$

Here, $\Delta(1, 0)+\Delta(0, 1) = 64$ is precisely the conformal dimension of $V_{(1,1)}^{\mathrm{dress},2,\alpha}$; i.e. only one of the dressings by the 2th power of U(1)-Casimir is a generator. Consequently, the other one is absent from the PL.

7.3.2 Representation [3, 0]

Hilbert series. The conformal dimension in this representation is given by

$$\begin{aligned}\Delta(n_1, n_2) = N\Big(&|5n_1+3n_2|+|5n_1+2n_2|+|4n_1+3n_2|+|4n_1+n_2|+|n_1+2n_2|+|n_1-n_2| \\ &+ 10\big(|2n_1 + n_2|+|n_1 + n_2|+|n_1|\big)+3\big(|3n_1 + 2n_2|+|3n_1 + n_2|+|n_2|\big)\Big) \\ &- \Big(|n_1 + n_2| + |2n_1 + n_2| + |3n_1 + n_2| + |3n_1 + 2n_2| + |n_1| + |n_2|\Big) ,\end{aligned} \tag{7.29}$$

such that we obtain for the Hilbert series

$$\mathrm{HS}_{\mathrm{G}_2}[3, 0](t, N) = \frac{R(t, N)}{(1 - t^2)(1 - t^6)(1 - t^{46N-6})(1 - t^{82N-10})(1 - t^{126N-16})} , \tag{7.30a}$$

$$\begin{aligned}R(t, N) = 1 &+ t^{46N-5}(1 + t + t^2 + t^3 + t^4) + t^{82N-9}(1 + t + t^2 + t^3 + t^4) \\ &+ t^{126N-15}(2 + 2t + 2t^2 + 2t^3 + 2t^4 + t^5) \\ &- t^{128N-16}(1 + 2t + 2t^2 + 2t^3 + 2t^4 + 2t^5) \\ &- t^{172N-21}(1+t+t^2+t^3+t^4) - t^{208N-25}(1+t+t^2+t^3+t^4) - t^{254N-26} .\end{aligned} \tag{7.30b}$$

The numerator (7.30b) is a anti-palindromic polynomial of degree $254N - 26$; while the denominator is of degree $254N - 24$. Hence, the difference in degree between denominator and numerator is 2, which coincides with the quaternionic dimension of moduli space. The Hilbert series (7.30) has a pole of order 4 as $t \to 1$, which equals the complex dimension of the moduli space. (One can explicitly show that $R(t = 1, N) = 0$, but $\frac{\mathrm{d}}{\mathrm{d}t}R(t, N)|_{t=1} \neq 0$.)

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V^{\mathrm{dress},0}_{(0,1)}$	$46N-6$	U(2)
dressings ($i=1,\dots,5$)	$V^{\mathrm{dress},i}_{(0,1)}$	$46N-6+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,0)}$	$82N-10$	U(2)
dressings ($i=1,\dots,5$)	$V^{\mathrm{dress},i}_{(1,0)}$	$82N-10+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,1)}$	$126N-16$	$\mathrm{U}(1)\times\mathrm{U}(1)$
dressings ($i=1,\dots,5;\alpha=1,2$)	$V^{\mathrm{dress},i,\alpha}_{(1,1)}$	$126N-16+i$	—
dressing	$V^{\mathrm{dress},6}_{(1,1)}$	$126N-16+6$	—

Table 27. The chiral ring generators for a G_2 gauge theory and matter transforming in $[3,0]$.

Interpreting the appearing operators leads to a list of chiral ring generators as presented in table 27. The behaviour of the Hilbert series is absolutely identical to the case $[1,1]$, because the conformal dimension is structurally identical. Therefore, we do not provide further details.

7.3.3 Representation $[0,2]$

Hilbert series. The following conformal dimension reads

$$\begin{aligned}\Delta(n_1,n_2) = N\Big(&|5n_1+3n_2|+|5n_1+2n_2|+|4n_1+3n_2|+|4n_1+n_2|+|n_1+2n_2|+|n_1-n_2| \\ &+10\big(|2n_1+n_2|+|n_1+n_2|+|n_1|\big)+5\big(|3n_1+2n_2|+|3n_1+n_2|+|n_2|\big)\Big) \\ &-\Big(|n_1+n_2|+|2n_1+n_2|+|3n_1+n_2|+|3n_1+2n_2|+|n_1|+|n_2|\Big)\,. \end{aligned} \tag{7.31}$$

The computation of the Hilbert series results in

$$\mathrm{HS}^{[0,2]}_{\mathrm{G}_2}(t,N) = \frac{R(t,N)}{(1-t^2)\,(1-t^6)\,(1-t^{52N-6})\,(1-t^{90N-10})\,(1-t^{140N-16})}\,, \tag{7.32a}$$

$$\begin{aligned} R(t,N) = 1 &+ t^{52N-5}(1+t+t^2+t^3+t^4) + t^{90N-9}(1+t+t^2+t^3+t^4) \\ &+ t^{140N-15}(2+2t+2t^2+2t^3+2t^4+t^5) \\ &- t^{142N-16}(1+2t+2t^2+2t^3+2t^4+2t^5) \\ &- t^{192N-21}(1+t+t^2+t^3+t^4) - t^{230N-25}(1+t+t^2+t^3+t^4) - t^{282N-26}\,. \end{aligned} \tag{7.32b}$$

The numerator (7.32b) is a anti-palindromic polynomial of degree $282N-26$; while, the denominator is of degree $282N-24$. Hence, the difference in degree between denominator and numerator is 2, which agrees with the quaternionic dimension of moduli space. The Hilbert series (7.32) has a pole of order 4 as $t\to 1$, which equals complex dimension of

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V^{\mathrm{dress},0}_{(0,1)}$	$52N-6$	$\mathrm{U}(2)$
dressings $(i=1,\ldots,5)$	$V^{\mathrm{dress},i}_{(0,1)}$	$52N-6+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,0)}$	$90N-10$	$\mathrm{U}(2)$
dressings $(i=1,\ldots,5)$	$V^{\mathrm{dress},i}_{(1,0)}$	$90N-10+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,1)}$	$140N-16$	$\mathrm{U}(1)\times\mathrm{U}(1)$
dressings $(i=1,\ldots,5;\alpha=1,2)$	$V^{\mathrm{dress},i,\alpha}_{(1,1)}$	$140N-16+i$	—
dressing	$V^{\mathrm{dress},6}_{(1,1)}$	$140N-16+6$	—

Table 28. The chiral ring generators for a G_2 gauge theory and matter transforming in $[0,2]$.

the moduli space. (One can explicitly show that $R(t=1,N)=0$, but $\frac{\mathrm{d}}{\mathrm{d}t}R(t,N)|_{t=1}\neq 0$.) Table 28 summarises the appearing operators. The behaviour of the Hilbert series is identical to the cases $[1,1]$ and $[3,0]$, because the conformal dimension is structurally identical. Again, we do not provide further details.

7.4 Category 3

Hilbert basis. Investigating the representations $[2,1]$ and $[4,0]$, one recognises the common structural form of the conformal dimensions

$$\Delta(n_1,n_2)=\sum_j A_j|a_jn_1+b_jn_2|+B_1|n_1|+B_2|n_2|+C|n_1-n_2|+D|2n_1-n_2| \tag{7.33}$$

for $a_j,b_j\in\mathbb{N}$ and $A_j,B_1,B_2,C,D\in\mathbb{Z}$. The novelty of this conformal dimension, compared to (7.3) and (7.19), is the difference $|2n_1-n_2|$, i.e. a second hyperplane that intersects the Weyl chamber non-trivially. As a consequence, the Weyl chamber is decomposed into a fan generated by three rational polyhedral cones of dimension 2. These are

$$C_1^{(2)}=\mathrm{Cone}((1,0),(1,1))\,,\quad C_2^{(2)}=\mathrm{Cone}((1,1),(1,2))\quad\text{and}\quad C_3^{(2)}=\mathrm{Cone}((1,2),(0,1))\,. \tag{7.34}$$

The intersection with the weight lattice $\Lambda_w(\mathrm{G}_2)$ yields the relevant semi-groups S_p (for $p=1,2,3$), as depicted in figure 29. The Hilbert bases are again given by the ray generators

$$\mathcal{H}(S_1^{(2)})=\Big\{(1,0),(1,1)\Big\}\,,\quad \mathcal{H}(S_2^{(2)})=\Big\{(1,1),(1,2)\Big\}\quad\text{and}\quad \mathcal{H}(S_3^{(2)})=\Big\{(1,2),(0,1)\Big\}\,. \tag{7.35}$$

Dressings. Compared to subsection 7.2 and 7.3, the additional magnetic weight $(1,2)$ has the same dressing behaviour as $(1,1)$, because the residual gauge groups is $\mathrm{U}(1)^2$, too.

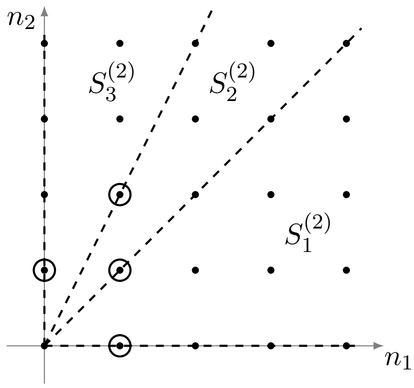

Figure 29. The semi-groups $S_p^{(2)}$ (p=1,2,3) for the representations $[2,1]$ and $[4,0]$ obtained from the G_2 Weyl chamber (considered as rational cone) and their ray generators (black circled points).

Thus, the additional necessary monopole operators are the bare operator $V_{(1,2)}^{\mathrm{dress},0}$ and the dressed monopoles $V_{(1,2)}^{\mathrm{dress},i,\alpha}$ for $i = 1, \ldots, 5$, $\alpha = 1, 2$ as well as $V_{(1,2)}^{\mathrm{dress},6}$.

We will now exemplify the three different representations.

7.4.1 Representation $[4, 0]$

Hilbert series. The conformal dimension reads

$$\begin{aligned}\Delta(n_1, n_2) = N\Big(&3\,|n_1 - n_2| + |2n_1 - n_2| + 27\,|n_1 + n_2| + 30\,|2n_1 + n_2| + 7\,|3n_1 + n_2| \\ &+ 3\,|4n_1{+}n_2| + |5n_1{+}n_2| + 3\,|n_1{+}2n_2| + 7\,|3n_1{+}2n_2| + 3\,|5n_1{+}2n_2| \\ &+ |2n_1{+}3n_2| + 3\,|4n_1{+}3n_2| + 3\,|5n_1{+}3n_2| + |7n_1{+}3n_2| + |5n_1{+}4n_2| \\ &+ |7n_1 + 4n_2| + 27\,|n_1| + 7\,|n_2|\Big) \\ -\Big(&|n_1 + n_2| + |2n_1 + n_2| + |3n_1 + n_2| + |3n_1 + 2n_2| + |n_1| + |n_2|\Big), \end{aligned} \tag{7.36}$$

from which we compute the Hilbert series to be

$$\mathrm{HS}_{\mathrm{G}_2}^{[4,0]}(t, N) = \frac{R(t, N)}{(1 - t^2)(1 - t^6)(1 - t^{134N-6})(1 - t^{238N-10})(1 - t^{364N-16})(1 - t^{496N-22})}, \tag{7.37a}$$

$$\begin{aligned} R(t, N) = 1 &+ t^{134N-5}(1 + t + t^2 + t^3 + t^4) + t^{238N-9}(1 + t + t^2 + t^3 + t^4) \\ &+ t^{364N-15}(2 + 2t + 2t^2 + 2t^3 + 2t^4 + t^5) \\ &- t^{372N-16}(1 + 2t + 2t^2 + 2t^3 + 2t^4 + 2t^5) \\ &+ t^{496N-21}(2 + 2t + 2t^2 + 2t^3 + 2t^4 + t^5) \\ &- t^{498N-22}(1 + 3t + 3t^2 + 3t^3 + 3t^4 + 3t^5 + t^6) \\ &- t^{602N-25}(1 + t + t^2 + t^3 + t^4) - t^{630N-27}(1 + t + t^2 + t^3 + t^4) \\ &- t^{734N-32}(1 + 3t + 3t^2 + 3t^3 + 3t^4 + 3t^5 + t^6) \\ &+ t^{736N-32}(1 + 2t + 2t^2 + 2t^3 + 2t^4 + 2t^5) \\ &- t^{860N-37}(2 + 2t + 2t^2 + 2t^3 + 2t^4 + t^5) \end{aligned} \tag{7.37b}$$

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V^{\mathrm{dress},0}_{(0,1)}$	$134N-6$	U(2)
dressings ($i=1,\ldots,5$)	$V^{\mathrm{dress},i}_{(0,1)}$	$134N-6+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,0)}$	$238N-10$	U(2)
dressings ($i=1,\ldots,5$)	$V^{\mathrm{dress},i}_{(1,0)}$	$238N-10+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,1)}$	$364N-16$	$\mathrm{U}(1)\times\mathrm{U}(1)$
dressings ($i=1,\ldots,5;\alpha=1,2$)	$V^{\mathrm{dress},i,\alpha}_{(1,1)}$	$364N-16+i$	—
dressing	$V^{\mathrm{dress},6}_{(1,1)}$	$364N-16+6$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,2)}$	$496N-22$	$\mathrm{U}(1)\times\mathrm{U}(1)$
dressings ($i=1,\ldots,5;\alpha=1,2$)	$V^{\mathrm{dress},i,\alpha}_{(1,2)}$	$496N-22+i$	—
dressing	$V^{\mathrm{dress},6}_{(1,2)}$	$496N-22+6$	—

Table 29. The chiral ring generators for a G_2 gauge theory and matter transforming in $[4,0]$.

$$+\,t^{868N-38}(1+2t+2t^2+2t^3+2t^4+2t^5)$$
$$+\,t^{994N-43}(1+t+t^2+t^3+t^4)+t^{1098N-47}(1+t+t^2+t^3+t^4)+t^{1232N-48}\,.$$

The numerator (7.37b) is a palindromic polynomial of degree $1232N-48$; while, the denominator is of degree $1232N-46$. Hence, the difference in degree between denominator and numerator is 2, which equals the quaternionic dimension of moduli space. The Hilbert series (7.37) has a pole of order 4 as $t\to 1$, which coincides with the complex dimension of the moduli space. (One can explicitly show that $R(t=1,N)=0$ and $\frac{\mathrm{d}}{\mathrm{d}t}R(t,N)|_{t=1}=0$, but $\frac{\mathrm{d}^2}{\mathrm{d}t^2}R(t,N)|_{t=1}\neq 0$.) The appearing operators can be summarised as in table 29. The new monopole corresponds to GNO-charge $(1,2)$ and displays the same dressing behaviour as $(1,1)$. Contrary to the cases $[1,1]$, $[3,0]$, and $[0,2]$, the bare and dressed monopoles of GNO-charge $(1,1)$ are always independent generators as

$$\Delta(1,1)=364N-16<372N-16=134N-6+238N-10=\Delta(0,1)+\Delta(1,0) \quad (7.38)$$

holds for all $N\geq 1$.

Plethystic logarithm. By means of the minimal generators (7.35), the bare monopole $V^{\mathrm{dress},0}_{(1,2)}$ is a necessary generator. Nevertheless, not all dressings $V^{\mathrm{dress}}_{(1,2)}$ need to be independent. For $N\geq 1$ the PL takes the form

$$\mathrm{PL}(\mathrm{HS}^{[0,2]}_{\mathrm{G}_2}(t,N))=t^2+t^6+t^{134N-6}(1+t+t^2+t^3+t^4+t^5) \quad (7.39)$$
$$+\,t^{238N-10}(1+t+t^2+t^3+t^4+t^5)$$

$$- t^{2(134N-6)+2}(1+t+2t^2+2t^3+3t^4+2t^5+2t^6+t^7+t^8) + t^{364N-16}(1+2t+2t^2+2t^3+2t^4+2t^5+t^6) - t^{372N-16} + \dots$$

Based purely in conformal dimension and GNO-charge, we can argue the following:

- For $N=3$, $\Delta(1,1)+\Delta(0,1)=1472$ is precisely the conformal dimension of $V_{(1,2)}^{\mathrm{dress},6}$, i.e. it is generated.
- For $N=2$, $\Delta(1,1)+\Delta(0,1)=974$ is precisely the conformal dimension of $V_{(1,2)}^{\mathrm{dress},4,\alpha}$, i.e. only one of the dressings by the 4th power of U(1)-Casimir is a generator.
- For $N=1$, $\Delta(1,1)+\Delta(0,1)=476$ is precisely the conformal dimension of $V_{(1,2)}^{\mathrm{dress},2,\alpha}$, i.e. only one of the dressings by the 2th power of U(1)-Casimir is a generator.

7.4.2 Representation $[2,1]$

Hilbert series. The conformal dimension reads

$$\begin{aligned}\Delta(n_1,n_2) = N\Big(&3\,|n_1-n_2| + |2n_1-n_2| + 24\,|n_1+n_2| + 24\,|2n_1+n_2| + 8\,|3n_1+n_2| \\ &+ 3\,|4n_1+n_2| + |5n_1+n_2| + 3\,|n_1+2n_2| + 8\,|3n_1+2n_2| + 3\,|5n_1+2n_2| \\ &+ |2n_1+3n_2| + 3\,|4n_1+3n_2| + 3\,|5n_1+3n_2| + |7n_1+3n_2| + |5n_1+4n_2| \\ &+ |7n_1+4n_2| + 24\,|n_1| + 8\,|n_2|\Big) \\ &- \Big(|n_1+n_2| + |2n_1+n_2| + |3n_1+n_2| + |3n_1+2n_2| + |n_1| + |n_2|\Big)\,,\end{aligned} \tag{7.40}$$

from which we compute the Hilbert series to be

$$\mathrm{HS}_{\mathrm{G}_2}^{[2,1]}(t,N) = \frac{R(t,N)}{(1-t^2)(1-t^6)(1-t^{132N-6})(1-t^{232N-10})(1-t^{356N-16})(1-t^{486N-22})}, \tag{7.41a}$$

$$\begin{aligned}R(t,N) = 1 &+ t^{132N-5}(1+t+t^2+t^3+t^4) + t^{232N-9}(1+t+t^2+t^3+t^4) \\ &+ t^{356N-15}(2+2t+2t^2+2t^3+2t^4+t^5) \\ &- t^{364N-16}(1+2t+2t^2+2t^3+2t^4+2t^5) \\ &+ t^{486N-21}(2+2t+2t^2+2t^3+2t^4+t^5) \\ &- t^{488N-22}(1+3t+3t^2+3t^3+3t^4+3t^5+t^6) \\ &- t^{588N-25}(1+t+t^2+t^3+t^4) - t^{618N-27}(1+t+t^2+t^3+t^4) \\ &- t^{718N-32}(1+3t+3t^2+3t^3+3t^4+3t^5+t^6) \\ &+ t^{720N-32}(1+2t+2t^2+2t^3+2t^4+2t^5) \\ &- t^{842N-37}(2+2t+2t^2+2t^3+2t^4+t^5) \\ &+ t^{850N-38}(1+2t+2t^2+2t^3+2t^4+2t^5) \\ &+ t^{974N-43}(1+t+t^2+t^3+t^4) + t^{1074N-47}(1+t+t^2+t^3+t^4) + t^{1206N-48}.\end{aligned} \tag{7.41b}$$

object		$\Delta(n_1,n_2)$	$\mathrm{H}_{(n_1,n_2)}$
Casimir	$\mathcal{C}_2$	2	—
	$\mathcal{C}_6$	6	—
bare monopole	$V^{\mathrm{dress},0}_{(0,1)}$	$132N-6$	U(2)
dressings ($i=1,\ldots,5$)	$V^{\mathrm{dress},i}_{(0,1)}$	$132N-6+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,0)}$	$232N-10$	U(2)
dressings ($i=1,\ldots,5$)	$V^{\mathrm{dress},i}_{(1,0)}$	$232N-10+i$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,1)}$	$356N-16$	$\mathrm{U}(1)\times\mathrm{U}(1)$
dressings ($i=1,\ldots,5;\alpha=1,2$)	$V^{\mathrm{dress},i,\alpha}_{(1,1)}$	$356N-16+i$	—
dressing	$V^{\mathrm{dress},6}_{(1,1)}$	$356N-16+6$	—
bare monopole	$V^{\mathrm{dress},0}_{(1,2)}$	$486N-22$	$\mathrm{U}(1)\times\mathrm{U}(1)$
dressings ($i=1,\ldots,5;\alpha=1,2$)	$V^{\mathrm{dress},i,\alpha}_{(1,2)}$	$486N-22+i$	—
dressing	$V^{\mathrm{dress},6}_{(1,2)}$	$486N-22+6$	—

Table 30. The chiral ring generators for a G_2 gauge theory and matter transforming in $[2,1]$.

The numerator (7.41b) is a palindromic polynomial of degree $1206N-48$; whereas, the denominator is of degree $1206N-46$. Hence, the difference in degree between denominator and numerator is 2, which agrees with the quaternionic dimension of moduli space. The Hilbert series (7.41) has a pole of order 4 as $t\to1$, which equals the complex dimension of the moduli space. (One can explicitly show that $R(t=1,N)=0$ and $\frac{\mathrm{d}}{\mathrm{d}t}R(t,N)|_{t=1}=0$, but $\frac{\mathrm{d}^2}{\mathrm{d}t^2}R(t,N)|_{t=1}\neq0$.) The list of appearing operators is presented in table 30. Due to the structure of the conformal dimension the behaviour of the $[2,1]$ representation is identical to that of $[4,0]$. Consequently, we do not discuss further details.

8 Case: SU(3)

The last rank two example we would like to cover is SU(3), for which the computation takes a detour over the corresponding U(3) theory, similar to [5]. The advantage is that we can simultaneously investigate the rank three example U(3) and demonstrate that the method of Hilbert bases for semi-groups works equally well in higher rank cases.

8.1 Set-up

In the following, we systematically study a number of SU(3) representation, where we understand a SU(3)-representation $[a,b]$ as an U(3)-representation with a fixed U(1)-charge.

Preliminaries for U(3). The GNO-dual group of U(3), which is again a U(3), has a weight lattice characterised by $m_1,m_2,m_3\in\mathbb{Z}$ and the dominant Weyl chamber is given by the restriction $m_1 \geq m_2 \geq m_3$, cf. [5]. The classical dressing factors associated to the interior and boundaries of the dominant Weyl chamber are the following:

$$P_{\mathrm{U}(3)}(t^2, m_1, m_2, m_3) = \begin{cases} \frac{1}{(1-t^2)^3} \ , & m_1 > m_2 > m_3 \ , \\ \dfrac{1}{(1-t^2)^2(1-t^4)} \ , & (m_1 = m_2 > m_3) \vee (m_1 > m_2 = m_3) \ , \\ \dfrac{1}{(1-t^2)(1-t^4)(1-t^6)} \ , & m_1 = m_2 = m_3 \ . \end{cases} \tag{8.1}$$

Note that we already introduced the fugacity t^2 instead of t. Moreover, the GNO-dual $\mathrm{U}(3)$ has a non-trivial centre, i.e. $\mathcal{Z}(\mathrm{U}(3)) = \mathrm{U}(1)_J$; thus, the topological symmetry is a $\mathrm{U}(1)_J$ counted by $z^{m_1+m_2+m_3}$.

The contributions of $N_{(a,b)}$ hypermultiplets transforming in $[a, b]$ to the conformal dimension are as follows:

$$\Delta_{\mathrm{h-plet}}^{[1,0]} = \frac{N_{(1,0)}}{2} \sum_i |m_i| \ , \tag{8.2a}$$

$$\Delta_{\mathrm{h-plet}}^{[2,0]} = \frac{3N_{(2,0)}}{2} \sum_i |m_i| \ , \tag{8.2b}$$

$$\Delta_{\mathrm{h-plet}}^{[1,1]} = N_{(1,1)} \sum_{i<j} |m_i - m_j| \ , \tag{8.2c}$$

$$\Delta_{\mathrm{h-plet}}^{[3,0]} = \frac{3N_{(3,0)}}{2} \sum_i |m_i| + N_{[3,0]} \sum_{i<j} |m_i - m_j| \ , \tag{8.2d}$$

$$\Delta_{\mathrm{h-plet}}^{[2,2]} = 3N_{(2,2)} \sum_i |m_i| + 4N_{(2,2)} \sum_{i<j} |m_i - m_j| \ , \tag{8.2e}$$

$$\Delta_{\mathrm{h-plet}}^{[2,1]} = 4N_{(2,1)} \sum_i |m_i| + \frac{N_{(2,1)}}{2} \sum_{i<j} \left(|2m_i - m_j| + |m_i - 2m_j|\right) \ , \tag{8.2f}$$

where $i, j = 1, 2, 3$. In addition, the contribution of the vector-multiplets reads as

$$\Delta_{\mathrm{v-plet}} = -\sum_{i<j} |m_i - m_j| \ . \tag{8.3}$$

Consequently, one can study a pretty wild matter content if one considers the conformal dimension to be of the form

$$\begin{aligned} \Delta(m_1, m_2, m_3) = \ & \frac{N_F}{2} \sum_i |m_i| + (N_A - 1) \sum_{i<j} |m_i - m_j| \\ & + \frac{N_R}{2} \sum_{i<j} \left(|2m_i - m_j| + |m_i - 2m_j|\right) \end{aligned} \tag{8.4}$$

and the relation to the various representations (8.2) is established via

$$N_F = N_{(1,0)} + 3N_{(2,0)} + 3N_{(3,0)} + 6N_{(2,2)} + 4N_{(2,1)} \ , \tag{8.5a}$$

$$N_A = N_{(1,1)} + N_{(3,0)} + 4N_{(2,2)} \ , \tag{8.5b}$$

$$N_R = N_{(2,1)} \ . \tag{8.5c}$$

Preliminaries for SU(3). As noted in [5], the reduction from U(3) to SU(3) (with the same matter content) is realised by averaging over $\mathrm{U}(1)_J$, for the purpose of setting $m_1+m_2+m_3=0$, and multiplying by $(1-t^2)$, such that $\mathrm{Tr}(\Phi)=0$ for the adjoint scalar Φ. In other words

$$\mathrm{HS}_{\mathrm{SU}(3)}^{[a,b]}(t^2) = (1-t^2)\oint_{|z|=1}\frac{\mathrm{d}z}{2\pi\mathrm{i}z}\mathrm{HS}_{\mathrm{U}(3)}^{[a,b]}(t^2,z)\,. \tag{8.6}$$

As a consequence, the conformal dimension for SU(3) itself is obtained from (8.4) via

$$\Delta(m_1,m_2) \coloneqq \Delta(m_1,m_2,m_3)\big|_{m_3=-m_1-m_2}\,. \tag{8.7}$$

The Weyl chamber is now characterised by $m_1\geq\max\{m_2,-2m_2\}$. Multiplying (8.1) by $(1-t^2)$ and employing $m_3=-m_1-m_2$ results in the classical dressing factors for SU(3)

$$P_{\mathrm{SU}(3)}(t^2,m_1,m_2) = \begin{cases} \dfrac{1}{(1-t^2)^2}\,, & m_1>\max\{m_2,-2m_2\}\,,\\ \dfrac{1}{(1-t^2)(1-t^4)}\,, & (m_1=m_2)\vee(m_1=-2m_2)\,,\\ \dfrac{1}{(1-t^4)(1-t^6)}\,, & m_1=m_2=0\,. \end{cases} \tag{8.8}$$

8.2 Hilbert basis

8.2.1 Fan and cones for U(3)

Following the ideas outline previously, $\Lambda_w(\widehat{\mathrm{U}(3)})/\mathcal{W}_{\mathrm{U}(3)}$ can be described as a collection of semi-groups that originate from a fan. Since this is our first 3-dimensional example, we provide a detail description on how to obtain the fan. Consider the absolute values $|am_1+bm_2+cm_2|$ in (8.7) as *Hesse normal form* for the hyperplanes

$$\vec{n}\cdot\vec{m} \equiv \begin{pmatrix} a\\ b\\ c\end{pmatrix}\cdot\begin{pmatrix} m_1\\ m_2\\ m_3\end{pmatrix} = 0 \tag{8.9}$$

which pass through the origin. Take all normal vectors $\vec{n}_j$, define the matrices $M_{i,j} = (\vec{n}_i,\vec{n}_j)^T$ (for $i<j$) and compute the null spaces (or kernel) $K_{i,j}\coloneqq\ker(M_{i,j})$. Linear algebra tell us that $\dim(K_{i,j})\geq 1$, but by the specific form[2] of Δ we have the stronger condition $\mathrm{rk}(M_{i,j})=2$ for all $i<j$; thus, we always have $\dim(K_{i,j})=1$. Next, we select a basis vector $e_{i,j}$ of $K_{i,j}$ and check if $e_{i,j}$ or $-e_{i,j}$ intersect the Weyl-chamber. If it does, then it is going to be an edge for the fan and, more importantly, will turn out to be a ray generator (provided one defines $e_{i,j}$ via the intersection with the corresponding weight lattice). Now, one has to define all 3-dimensional cones, merge them into a fan, and, lastly, compute the Hilbert bases. The programs `Macaulay2` and `Sage` are convenient tools for such tasks.

As two examples, we consider the conformal dimension (8.7) for $N_R=0$ and $N_R\neq 0$ and preform the entire procedure. That is: firstly, compute the edges of the fan; secondly, define the all 3-dimensional cones and; thirdly, compute the Hilbert bases.

[2] Δ is homogeneous and all hyperplanes pass through the origin; hence, no two hyperplanes can be parallel. This implies that no two normal vectors can be multiples of each other.

Case $N_R = 0$: in this circumstance, we deduce the following edges

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}0\\0\\-1\end{pmatrix}, \quad \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \quad \begin{pmatrix}-1\\-1\\-1\end{pmatrix}. \tag{8.10}$$

All these vectors are on the boundaries of the Weyl chamber. The set of 3-dimensional cones that generate the corresponding fan is given by

$$C_1^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}, \qquad C_2^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}, \tag{8.11a}$$

$$C_3^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}, C_4^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}. \tag{8.11b}$$

A computation shows that all four cones are strictly convex, smooth, and simplicial. The Hilbert bases for the resulting semi-groups comprise solely the ray generators

$$\mathcal{H}(S_1^{(3)}) = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}, \qquad \mathcal{H}(S_2^{(3)}) = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}, \tag{8.12a}$$

$$\mathcal{H}(S_3^{(3)}) = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}, \mathcal{H}(S_4^{(3)}) = \left\{\begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}. \tag{8.12b}$$

From the above, we expect 6 bare monopole operators plus their dressings for a generic theory with $N_R = 0$. Since all ray generators lie at the boundary of the Weyl chamber, the residual gauge groups are U(3) for $\pm(1,1,1)$ and $\mathrm{U}(2) \times \mathrm{U}(1)$ for the other four GNO-charges.

Case $N_R \neq 0$: here, we compute the following edges:

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\\0\end{pmatrix}, \quad \begin{pmatrix}2\\1\\1\end{pmatrix}, \quad \begin{pmatrix}2\\2\\1\end{pmatrix}, \quad \begin{pmatrix}4\\2\\1\end{pmatrix}, \tag{8.13a}$$

$$\begin{pmatrix}0\\0\\-1\end{pmatrix}, \quad \begin{pmatrix}0\\-1\\-1\end{pmatrix}, \quad \begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \quad \begin{pmatrix}0\\-1\\-2\end{pmatrix}, \quad \begin{pmatrix}-1\\-1\\-2\end{pmatrix}, \quad \begin{pmatrix}-1\\-2\\-2\end{pmatrix}, \quad \begin{pmatrix}-1\\-2\\-4\end{pmatrix}. \tag{8.13b}$$

Now, we need to proceed and define all 3-dimensional cones that constitute the fan and, in turn, will lead to the semi-groups that we wish to study.

$$C_1^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}4\\2\\1\end{pmatrix}\right\}, \qquad C_2^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}4\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}\right\}, \tag{8.14a}$$

$$C_3^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}\right\}, \qquad C_4^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}4\\2\\1\end{pmatrix}\right\}, \tag{8.14b}$$

$$C_5^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}4\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}\right\}, \qquad C_6^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}, \tag{8.14c}$$

$$C_7^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}\right\}, \qquad C_8^{(3)} = \mathrm{Cone}\left\{\begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}\right\}, \tag{8.14d}$$

$$C_9^{(3)} = \mathrm{Cone}\left\{ \begin{pmatrix} 0\\0\\-1 \end{pmatrix}, \begin{pmatrix} 0\\-1\\-2 \end{pmatrix}, \begin{pmatrix} 1\\0\\0 \end{pmatrix} \right\}, \quad C_{10}^{(3)} = \mathrm{Cone}\left\{ \begin{pmatrix} 0\\-1\\-2 \end{pmatrix}, \begin{pmatrix} 0\\-1\\-1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0 \end{pmatrix} \right\}, \tag{8.14e}$$

$$C_{11}^{(3)} = \mathrm{Cone}\left\{ \begin{pmatrix} 0\\0\\-1 \end{pmatrix}, \begin{pmatrix} 0\\-1\\-2 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-4 \end{pmatrix} \right\}, \quad C_{12}^{(3)} = \mathrm{Cone}\left\{ \begin{pmatrix} 0\\0\\-1 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-4 \end{pmatrix}, \begin{pmatrix} -1\\-1\\-2 \end{pmatrix} \right\}, \tag{8.14f}$$

$$C_{13}^{(3)} = \mathrm{Cone}\left\{ \begin{pmatrix} 0\\-1\\-2 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-4 \end{pmatrix}, \begin{pmatrix} 0\\-1\\-1 \end{pmatrix} \right\}, \quad C_{14}^{(3)} = \mathrm{Cone}\left\{ \begin{pmatrix} 0\\-1\\-1 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-4 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-2 \end{pmatrix} \right\}, \tag{8.14g}$$

$$C_{15}^{(3)} = \mathrm{Cone}\left\{ \begin{pmatrix} -1\\-2\\-4 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-2 \end{pmatrix}, \begin{pmatrix} -1\\-1\\-2 \end{pmatrix} \right\}, \quad C_{16}^{(3)} = \mathrm{Cone}\left\{ \begin{pmatrix} -1\\-2\\-2 \end{pmatrix}, \begin{pmatrix} -1\\-1\\-2 \end{pmatrix}, \begin{pmatrix} -1\\-1\\-1 \end{pmatrix} \right\}. \tag{8.14h}$$

All of the cones are strictly convex and simplicial, but only the cones C_p for $p = 1,2,3,6,\ldots,13,16$ are smooth. Now, we compute the Hilbert bases for semi-groups $S_p^{(3)}$ for $p = 1,2,\ldots,16$ and obtain

$$\mathcal{H}(S_1^{(3)}) = \left\{ \begin{pmatrix} 1\\0\\0 \end{pmatrix}, \begin{pmatrix} 2\\1\\0 \end{pmatrix}, \begin{pmatrix} 4\\2\\1 \end{pmatrix} \right\}, \quad \mathcal{H}(S_2^{(3)}) = \left\{ \begin{pmatrix} 4\\2\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0 \end{pmatrix}, \begin{pmatrix} 2\\1\\1 \end{pmatrix} \right\}, \tag{8.15a}$$

$$\mathcal{H}(S_3^{(3)}) = \left\{ \begin{pmatrix} 2\\2\\1 \end{pmatrix}, \begin{pmatrix} 1\\1\\0 \end{pmatrix}, \begin{pmatrix} 2\\1\\0 \end{pmatrix} \right\}, \quad \mathcal{H}(S_4^{(3)}) = \left\{ \begin{pmatrix} 2\\2\\1 \end{pmatrix}, \begin{pmatrix} 2\\1\\0 \end{pmatrix}, \begin{pmatrix} 4\\2\\1 \end{pmatrix}, \begin{pmatrix} 3\\2\\1 \end{pmatrix} \right\}, \tag{8.15b}$$

$$\mathcal{H}(S_5^{(3)}) = \left\{ \begin{pmatrix} 2\\2\\1 \end{pmatrix}, \begin{pmatrix} 4\\2\\1 \end{pmatrix}, \begin{pmatrix} 2\\1\\1 \end{pmatrix}, \begin{pmatrix} 3\\2\\1 \end{pmatrix} \right\}, \quad \mathcal{H}(S_6^{(3)}) = \left\{ \begin{pmatrix} 2\\2\\1 \end{pmatrix}, \begin{pmatrix} 2\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right\}, \tag{8.15c}$$

$$\mathcal{H}(S_7^{(3)}) = \left\{ \begin{pmatrix} 0\\0\\-1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0 \end{pmatrix}, \begin{pmatrix} 2\\1\\0 \end{pmatrix} \right\}, \quad \mathcal{H}(S_8^{(3)}) = \left\{ \begin{pmatrix} 0\\0\\-1 \end{pmatrix}, \begin{pmatrix} 1\\1\\0 \end{pmatrix}, \begin{pmatrix} 2\\1\\0 \end{pmatrix} \right\}, \tag{8.15d}$$

$$\mathcal{H}(S_9^{(3)}) = \left\{ \begin{pmatrix} 0\\0\\-1 \end{pmatrix}, \begin{pmatrix} 0\\-1\\-2 \end{pmatrix}, \begin{pmatrix} 1\\0\\0 \end{pmatrix} \right\}, \quad \mathcal{H}(S_{10}^{(3)}) = \left\{ \begin{pmatrix} 0\\-1\\-2 \end{pmatrix}, \begin{pmatrix} 0\\-1\\-1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0 \end{pmatrix} \right\}, \tag{8.15e}$$

$$\mathcal{H}(S_{11}^{(3)}) = \left\{ \begin{pmatrix} 0\\0\\-1 \end{pmatrix}, \begin{pmatrix} 0\\-1\\-2 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-4 \end{pmatrix} \right\}, \quad \mathcal{H}(S_{12}^{(3)}) = \left\{ \begin{pmatrix} 0\\0\\-1 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-4 \end{pmatrix}, \begin{pmatrix} -1\\-1\\-2 \end{pmatrix} \right\}, \tag{8.15f}$$

$$\mathcal{H}(S_{13}^{(3)}) = \left\{ \begin{pmatrix} 0\\-1\\-2 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-4 \end{pmatrix}, \begin{pmatrix} 0\\-1\\-1 \end{pmatrix} \right\}, \quad \mathcal{H}(S_{14}^{(3)}) = \left\{ \begin{pmatrix} 0\\-1\\-1 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-4 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-2 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-3 \end{pmatrix} \right\}, \tag{8.15g}$$

$$\mathcal{H}(S_{15}^{(3)}) = \left\{ \begin{pmatrix} -1\\-2\\-4 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-2 \end{pmatrix}, \begin{pmatrix} -1\\-1\\-2 \end{pmatrix}, \begin{pmatrix} -1\\-2\\-3 \end{pmatrix} \right\}, \quad \mathcal{H}(S_{16}^{(3)}) = \left\{ \begin{pmatrix} -1\\-2\\-2 \end{pmatrix}, \begin{pmatrix} -1\\-1\\-2 \end{pmatrix}, \begin{pmatrix} -1\\-1\\-1 \end{pmatrix} \right\}. \tag{8.15h}$$

We observe that there are four semi-groups S_p for $p = 4,5,14,15$ for which the Hilbert bases exceeds the set of ray generators by an additional element. Consequently, we expect 16 bare monopoles plus their dressings for a generic theory with $N_R \neq 0$. However, the dressings exhibit a much richer structure compared to $N_R = 0$, because some minimal generators lie in the interior of the Weyl chamber. The residual gauge groups are U(3) for $\pm(1,1,1)$; $\mathrm{U}(2) \times \mathrm{U}(1)$ for $(1,0,0)$, $(0,0,-1)$, $(1,1,0)$, $(0,-1,-1)$, $(2,1,1)$, $(-1,-1,-2)$, $(2,2,1)$, and $(-1,-2,-2)$; and $\mathrm{U}(1)^3$ for $(2,1,0)$, $(0,-1,-2)$,$(4,2,1)$, $(-1,-2,-4)$, $(3,2,1)$, and $(-1,-2,-3)$.

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8.2.2 Fan and cones for SU(3)

The conformal dimension (8.7) divides the Weyl chamber of the GNO-dual into two different fans, depending on $N_R = 0$ or $N_R \neq 0$.

Case $N_R = 0$: for this situation, which is depicted in figure 30a, there are three rays $\sim |m_1|, |m_1 - m_2|, |m_1 + 2m_2|$ present that intersect the Weyl chamber non-trivially. The corresponding fan is generated by two 2-dimensional cones

$$C_1^{(2)} = \mathrm{Cone}((2,-1),(1,0)) \quad \text{and} \quad C_2^{(2)} = \mathrm{Cone}((1,0),(1,1)) \ . \tag{8.16}$$

The Hilbert bases for the semi-groups, obtained by intersecting the cones with the weight lattice, are solely given by the ray generators, i.e.

$$\mathcal{H}(S_1^{(2)}) = \Big\{(2,-1),(1,0)\Big\} \quad \text{and} \quad \mathcal{H}(S_2^{(2)}) = \Big\{(1,0),(1,1)\Big\} \ . \tag{8.17}$$

As a consequence, we expect three bare monopole operators (plus dressings) for a generic $N_R = 0$ theory. The residual gauge group is $\mathrm{SU}(2) \times \mathrm{U}(1)$ for $(2,-1)$ and $(1,1)$, because these GNO-charges are at the boundary of the Weyl-chamber. In contrast, $(1,0)$ has residual gauge group $\mathrm{U}(1)^2$ as it lies in the interior of the dominant Weyl chamber.

Case $N_R \neq 0$: for this circumstance, which is depicted in figure 30b, there are two additional rays $\sim |m_1 - 2m_2|, |m_1 + 3m_2|$ present, compared to $N_R = 0$, that intersect the Weyl chamber non-trivially. The corresponding fan is now generated by four 2-dimensional cones

$$C_{1-}^{(2)} = \mathrm{Cone}((2,-1),(3,-1)) \ , \qquad C_{1+}^{(2)} = \mathrm{Cone}((3,-1),(1,0)) \ , \tag{8.18a}$$

$$C_{2-}^{(2)} = \mathrm{Cone}((1,0),(2,1)) \ , \qquad C_{2+}^{(2)} = \mathrm{Cone}((2,1),(1,1)) \ . \tag{8.18b}$$

The Hilbert bases for the resulting semi-groups are given by the ray generators, i.e.

$$\mathcal{H}(S_{1-}^{(2)}) = \Big\{(2,-1),(3,-1)\Big\} \ , \qquad \mathcal{H}(S_{1+}^{(2)}) = \Big\{(3,-1),(1,0)\Big\} \ , \tag{8.19a}$$

$$\mathcal{H}(S_{2-}^{(2)}) = \Big\{(1,0),(2,1)\Big\} \ , \qquad \mathcal{H}(S_{2+}^{(2)}) = \Big\{(2,1),(1,1)\Big\} \ . \tag{8.19b}$$

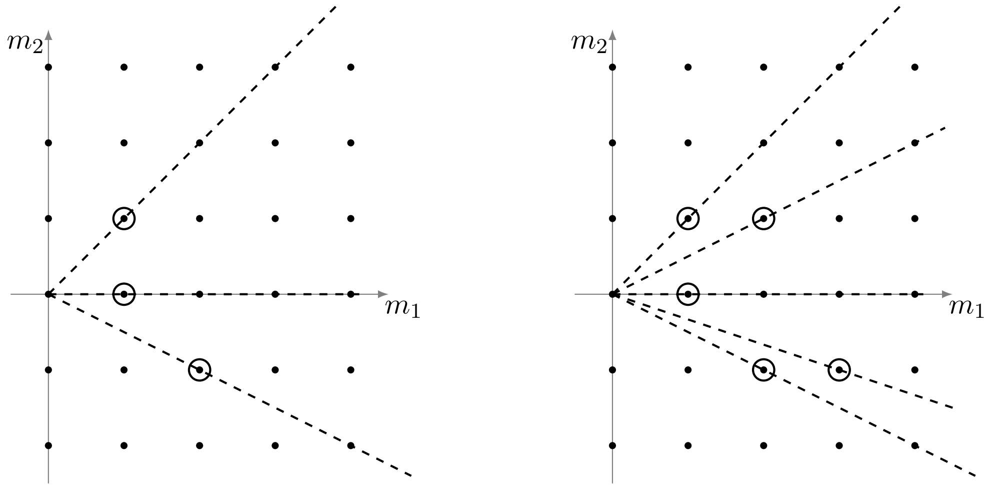
Judging from the Hilbert bases, there are five bare monopole operators present in the generic case. The residual gauge group for $(1,0)$, $(3,-1)$, and $(2,1)$ is $\mathrm{U}(1)^2$, as they lie in the interior. For $(1,1)$ and $(2,-1)$ the residual gauge group is $\mathrm{SU}(2) \times \mathrm{U}(1)$, because these points lie at the boundary of the Weyl chamber.

8.3 Casimir invariance

8.3.1 Dressings for U(3)

Following the description of dressed monopole operators as in [5], we diagonalise the adjoint-valued scalar Φ along the moduli space, i.e.

$$\mathrm{diag}\Phi = (\phi_1, \phi_2, \phi_3) \ . \tag{8.20}$$



(a) $N_R = 0$ **(b)** $N_R \neq 0$

Figure 30. The semi-groups for SU(3) and the corresponding ray generators (black circled points).

Moreover, the Casimir invariants of U(3) can then be written as $\mathcal{C}_j = \mathrm{Tr}(\Phi^j) = \sum_{l=1}^{3} (\phi_l)^j$ for $j = 1, 2, 3$. We will now elaborate on the possible dressed monopole operators by means of the insights gained in section 2.3 and appendix A.

To start with, for a monopole with GNO-charge such that $\mathrm{H}_{(m_1,m_2,m_3)} = \mathrm{U}(3)$ the dressings are described by

$$\frac{P_{\mathrm{U}(3)}(t, m_1, m_1, m_1)}{P_{\mathrm{U}(3)}(t, 0)} - 1 = 0 \,, \tag{8.21}$$

i.e. there are no dressings, because the Casimir invariants of the centraliser $\mathrm{H}_{(m_1,m_2,m_3)}$ are identical to those of G, since the groups coincide. Prominent examples are the (bare) monopoles of GNO-charge $\pm(1, 1, 1)$.

Next, a monopole of GNO-charge such that $\mathrm{H}_{(m_1,m_2,m_3)} = \mathrm{U}(1) \times \mathrm{U}(2)$ exhibit dressings governed by

$$\frac{P_{\mathrm{U}(3)}(t, m_1, m_2, m_3)}{P_{\mathrm{U}(3)}(t, 0)} - 1 = \frac{(1-t^2)(1-t^4)(1-t^6)}{(1-t^2)^2(1-t^4)} - 1 = t^2 + t^4 \,, \tag{8.22}$$

implying there to be exactly one dressing by a degree 2 Casimir and one dressing by a degree 4 Casimir. The two degree 2 Casimir invariants of $\mathrm{H}_{(m_1,m_2,m_3)}$, one by U(1) and one by U(2), are not both independent because there is the overall Casimir $\mathcal{C}_1$ of U(3). Therefore, only one of them leads to an independent dressed monopole generator. The second dressing is then due to the second Casimir of U(2). For example, the monopole of GNO-charge $(1, 1, 0)$, $(0, -1, -1)$, $(2, 1, 1)$, $(-1, -2, -2)$, $(2, 2, 1)$, and $(-1, -2, -2)$ exhibit these two dressings options.

Lastly, if the residual gauge group is $\mathrm{H}_{(m_1,m_2,m_3)} = \mathrm{U}(1)^3$ then the dressings are determined via

$$\frac{P_{\mathrm{U}(3)}(t, m_1, m_2, m_3)}{P_{\mathrm{U}(3)}(t, 0)} - 1 = \frac{(1-t^2)(1-t^4)(1-t^6)}{(1-t^2)^3} - 1 = 2t^2 + 2t^4 + t^6 \,. \tag{8.23}$$

Consequently, there are generically five dressings for each such bare monopole operator. Examples for this instance are $(2,1,0)$, $(0,-1,-2)$, $(3,2,1)$, $(-1,-2,-3)$, $(4,2,1)$, $(-1,-2,-4)$.

8.3.2 Dressings for SU(3)

To determine the dressings, we take the adjoint scalar Φ and diagonalise it, keeping in mind that it now belongs to SU(3), that is

$$\mathrm{diag}\Phi = (\phi_1, \phi_2, -(\phi_1+\phi_2)) \; . \tag{8.24}$$

While keeping in mind that each ϕ_i has dimension one, we can write down the dressings (in the dominant Weyl chamber): $(1,0)$ can be dressed by two independent U(1)-Casimir invariants, i.e. directly by ϕ_1 and ϕ_2

$$V^{\mathrm{dress},(0,0)}_{(1,0)} \equiv (1,0) \longrightarrow \begin{cases} V^{\mathrm{dress},(1,0)}_{(1,0)} \equiv \phi_1 \; (1,0) \; , \\ V^{\mathrm{dress},(0,1)}_{(1,0)} \equiv \phi_2 \; (1,0) \; , \end{cases} \tag{8.25}$$

such that the dressings have conformal dimension $\Delta(1,0)+1$. Next, out of the three degree 2 combinations of ϕ_i, only two of them are independent and we choose them to be

$$V^{\mathrm{dress},(0,0)}_{(1,0)} \equiv (1,0) \longrightarrow \begin{cases} V^{\mathrm{dress},(2,0)}_{(1,0)} \equiv \phi_1^2 \; (1,0) \; , \\ V^{\mathrm{dress},(0,2)}_{(1,0)} \equiv \phi_2^2 \; (1,0) \; , \end{cases} \tag{8.26}$$

and these second order dressings have conformal dimension $\Delta(1,0)+2$. Finally, one last dressing is possible

$$V^{\mathrm{dress},(0,0)}_{(1,0)} \equiv (1,0) \longrightarrow V^{\mathrm{dress},(3,0)+(0,3)}_{(1,0)} \equiv (\phi_1^3+\phi_2^3) \; (1,0) \; , \tag{8.27}$$

having dimension $\Delta(1,0)+3$. Alternatively, we utilise appendix A and compute the number and degrees of the dressed monopole operators of magnetic charge $(1,0)$ via the quotient $P_{\mathrm{SU}(3)}(t^2,1,0)/P_{\mathrm{SU}(3)}(t^2,0,0) = 1+2t^2+2t^4+t^6$.

For the two monopoles of GNO-charge $(1,1)$ and $(2,-1)$, the residual gauge group is $\mathrm{SU}(2)\times\mathrm{U}(1)$, i.e. the monopoles can be dressed by a degree one Casmir invariant of the U(1) and by a degree two Casimir invariant of the SU(2). These increase the dimensions by one and two, respectively. Consequently, we obtain

$$V^{\mathrm{dress},0}_{(1,1)} \equiv (1,1) \longrightarrow \begin{cases} V^{\mathrm{dress},\mathrm{U}(1)}_{(1,1)} \equiv (\phi_1+\phi_2) \; (1,1) \; , \\ V^{\mathrm{dress},\mathrm{SU}(2)}_{(1,1)} \equiv (\phi_1^2+\phi_2^2) \; (1,1) \; , \end{cases} \tag{8.28}$$

and similarly

$$V^{\mathrm{dress},0}_{(2,-1)} \equiv (2,-1) \longrightarrow \begin{cases} V^{\mathrm{dress},\mathrm{U}(1)}_{(2,-1)} \equiv (\phi_1+\phi_2) \; (2,-1) \; , \\ V^{\mathrm{dress},\mathrm{SU}(2)}_{(2,-1)} \equiv (\phi_1^2+\phi_2^2) \; (2,-1) \; . \end{cases} \tag{8.29}$$

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Since the magnetic weights $(1,1)$, $(2,-1)$ lie at the boundary of the dominant Weyl chamber, we can derive the dressing behaviour via $P_{\mathrm{SU}(3)}(t^2,(1,1) \text{ or } (2,-1))/P_{\mathrm{SU}(3)}(t^2,0,0) = 1+t^2+t^4$ and obtain agreement with our choice of generators.

The remaining monopoles of GNO-charge $(2,1)$ and $(3,-1)$ can be treated analogously to $(1,0)$ and we obtain

$$V^{\mathrm{dress},(0,0)}_{(2,1)} \equiv (2,1) \longrightarrow \begin{cases} V^{\mathrm{dress},(1,0)}_{(2,1)} \equiv \phi_1 \ (2,1) \ , \\ V^{\mathrm{dress},(0,1)}_{(2,1)} \equiv \phi_2 \ (2,1) \ , \\ V^{\mathrm{dress},(2,0)}_{(2,1)} \equiv \phi_1^2 \ (2,1) \ , \\ V^{\mathrm{dress},(0,2)}_{(2,1)} \equiv \phi_2^2 \ (2,1) \ , \\ V^{\mathrm{dress},(3,0)+(0,3)}_{(2,1)} \equiv (\phi_1^3+\phi_2^3) \ (2,1) \ , \end{cases} \tag{8.30}$$

$$V^{\mathrm{dress},(0,0)}_{(3,-1)} \equiv (3,-1) \longrightarrow \begin{cases} V^{\mathrm{dress},(1,0)}_{(3,-1)} \equiv \phi_1 \ (3,-1) \ , \\ V^{\mathrm{dress},(0,1)}_{(3,-1)} \equiv \phi_2 \ (3,-1) \ , \\ V^{\mathrm{dress},(2,0)}_{(3,-1)} \equiv \phi_1^2 \ (3,-1) \ , \\ V^{\mathrm{dress},(0,2)}_{(3,-1)} \equiv \phi_2^2 \ (3,-1) \ , \\ V^{\mathrm{dress},(3,0)+(0,3)}_{(3,-1)} \equiv (\phi_1^3+\phi_2^3) \ (3,-1) \ . \end{cases} \tag{8.31}$$

There can be circumstances in which not all dressings for the minimal generators determined by the Hilbert bases (8.19) are truly independent. However, this will only occur for special configurations of (N_F, N_A, F_R) and, therefore, is considered as "non-generic" case.

8.4 Category $N_R = 0$

8.4.1 N_F hypermultiplets in $[1,0]$ and N_A hypermultiplets in $[1,1]$

Intermediate step at U(3). The conformal dimension (8.4) reduces for $N_R = 0$ to the following:

$$\Delta(m_1,m_2,m_3) = \frac{N_F}{2}\sum_i |m_i| + (N_A-1)\sum_{i<j}|m_i-m_j| \ . \tag{8.32}$$

The Hilbert series is then readily computed

$$\mathrm{HS}^{[1,0]+[1,1]}_{\mathrm{U}(3)}(N_F,N_A,t,z) = \frac{R(N_F,N_A,t,z)}{P(N_F,N_A,t,z)} \ , \tag{8.33a}$$

$$\begin{aligned} P(N_F,N_A,t,z) = \prod_{j=1}^{3}\left(1-t^{2j}\right)&\left(1-\tfrac{1}{z}t^{4N_A+N_F-4}\right)\left(1-zt^{4N_A+N_F-4}\right) \\ &\times\left(1-\tfrac{1}{z^2}t^{4N_A+2N_F-4}\right)\left(1-z^2t^{4N_A+2N_F-4}\right)\left(1-\tfrac{1}{z^3}t^{3N_F}\right)\left(1-z^3t^{3N_F}\right) , \end{aligned} \tag{8.33b}$$

$$\begin{aligned} R(N_F,N_A,t,z) = 1 &+ t^{8N_A+2N_F-2} - t^{8N_A+4N_F-8}(1+2t^2+2t^4) + 2t^{8N_A+6N_F-8}(1-t^6) \\ &+ t^{8N_A+8N_F-6}(2+2t^2+t^4) - t^{8N_A+10N_F-8} + t^{16N_A+6N_F-10} \\ &- t^{16N_A+12N_F-10} - t^{6N_F} \end{aligned}$$

(m_1, m_2, m_3)		$2\Delta(m_1, m_2, m_3)$	$\mathrm{H}_{(m_1,m_2,m_3)}$
$(1,0,0)$	$(0,0,-1)$	$N_F + 4N_A - 4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(1,1,0)$	$(0,-1,-1)$	$2N_F + 4N_A - 4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(1,1,1)$	$(-1,-1,-1)$	$3N_F$	$\mathrm{U}(3)$

Table 31. The monopole generators for a U(3) gauge theory with $N_R = 0$ that together with the Casimir invariants generate the chiral ring.

$$\begin{aligned}
&+\left(z+\tfrac{1}{z}\right)\Big(t^{4N_A+N_F-2}(1+t^2)+t^{4N_A+7N_F-4}-t^{4N_A+5N_F-4}(1+t^2+t^4)\\
&\qquad -t^{8N_A+3N_F-6}(1+t^2)+t^{8N_A+9N_F-6}(1+t^2)-t^{12N_A+5N_F-6}\\
&\qquad +t^{12N_A+7N_F-10}(1+t^2+t^4)-t^{12N_A+11N_F-10}(1+t^2)\Big)\\
&+\left(z^2+\tfrac{1}{z^2}\right)\Big(t^{4N_A+2N_F-2}+t^{4N_A+2N_F}-t^{4N_A+4N_F-4}(1+t^2+t^4)\\
&\qquad +t^{4N_A+8N_F-4}-t^{12N_A+4N_F-6}+t^{12N_A+8N_F-10}(1+t^2+t^4)\\
&\qquad -t^{12N_A+10N_F-10}(1+t^2)\Big)\\
&+\left(z^3+\tfrac{1}{z^3}\right)\Big(t^{8N_A+3N_F-2}-t^{8N_A+5N_F-6}(1+t^2+t^4)\\
&\qquad +t^{8N_A+7N_F-8}(1+t^2+t^4)-t^{8N_A+9N_F-8}\Big)\,.
\end{aligned}\tag{8.33c}$$

One can check that $R(N_F, N_A, t=1, z) = 0$ and $\frac{\mathrm{d}^n}{\mathrm{d}t^n} R(N_F, N_A, t, z)|_{t=1,z=1} = 0$ for $n = 1, 2$. Thus, the Hilbert series (8.33) has a pole of order 6, which matches the dimension of the moduli space. Moreover, one computes the degree of the numerator (8.33c) to be $12N_F + 16N_A - 10$ and the degree of the denominator (8.33b) to be $12N_F + 16N_A - 4$, such that their difference equals the dimension of the moduli space. The interpretation follows the results (8.12) obtained from the Hilbert bases and we summarise the minimal generators in table 31.

Reduction to SU(3). Following the prescription (8.6), we derive the following Hilbert series:

$$\mathrm{HS}^{[1,0]+[1,1]}_{\mathrm{SU}(3)}(N_F, N_A, t) = \frac{R(N_F, N_A, t)}{(1-t^4)(1-t^6)(1-t^{8N_A+2N_F-8})(1-t^{12N_A+4N_F-12})}\,, \tag{8.34a}$$

$$\begin{aligned}
R(N_F, N_A, t) &= 1 + t^{8N_A+2N_F-6}(2+2t^2+t^4)\\
&\quad + t^{12N_A+4N_F-12}(1+2t^2+2t^4) + t^{20N_A+6N_F-14}\,.
\end{aligned}\tag{8.34b}$$

An inspection yields that the numerator (8.34b) is a palindromic polynomial of degree $20N_A + 6N_F - 14$; while the degree of the denominator is $20N_A + 6N_F - 10$. Thus, the difference in the degrees is 4, which equals the complex dimension of the moduli space. In addition, the Hilbert series (8.34) has a pole of order four at $t \to 1$, which agrees with the dimension of Coulomb branch as well.

The minimal generators of (8.17) are given by $V^{\mathrm{dress},(0,0)}_{(1,0)}$ with $2\Delta(1,0) = 8N_A + 2N_F - 8$, and $V^{\mathrm{dress},0}_{(1,1)}$ and $V^{\mathrm{dress},0}_{(2,-1)}$ with $2\Delta(2,-1) = 2\Delta(1,1) = 12N_A + 4N_F - 12$. The dressed monopole operators are as described in subsection 8.3.2.

8.4.2 N hypermultiplets in $[1,0]$ representation

Considering N hypermultiplets in the fundamental representation is on extreme case of (8.4), as $N_A = 0 = N_R$. We recall the results of [5] and discuss them in the context of Hilbert bases for semi-groups.

Intermediate step at U(3). The Hilbert series has been computed to read

$$\mathrm{HS}_{\mathrm{U}(3)}^{[1,0]}(N,t,z) = \prod_{j=1}^{3} \frac{1-t^{2N+2-2j}}{(1-t^{2j})(1-zt^{N+2-2j})(1-\frac{t^{N+2-2j}}{z})} \,. \tag{8.35}$$

Notably, it is a complete intersection in which the (bare and dressed) monopole operators of GNO-charge $(1,0,0)$ and $(0,0,-1)$ generate all other monopole operators. The to be expected minimal generators $(1,1,0)$, $(0,-1,-1)$, $(1,1,1)$, and $(-1,-1,-1)$ are now generated because

$$V_{(1,1,0)}^{\mathrm{dress},0} = V_{(1,0,0)}^{\mathrm{dress},1} + V_{(0,1,0)}^{\mathrm{dress},1} \,, \tag{8.36a}$$

$$V_{(1,1,0)}^{\mathrm{dress},0} = V_{(1,0,0)}^{\mathrm{dress},2} + V_{(0,1,0)}^{\mathrm{dress},2} + V_{(0,0,1)}^{\mathrm{dress},2} \,. \tag{8.36b}$$

Reduction to SU(3). The reduction leads to

$$\mathrm{HS}_{\mathrm{SU}(3)}^{[1,0]}(N,t) = \frac{1+t^{2N-6}+2t^{2N-4}+t^{2N-2}+t^{4N-8}}{(1-t^4)(1-t^6)(1-t^{2N-6})(1-t^{2N-8})} \,. \tag{8.37}$$

Although the form of the Hilbert series (8.37) is suggestive: it has a pole of order 4 for $t \to 1$ and the numerator is palindromic, there is one drawback: no monopole operator of conformal dimension $(2N-6)$ exists. Therefore, we provide a equivalent rational function to emphasis the minimal generators:

$$\mathrm{HS}_{\mathrm{SU}(3)}^{[1,0]}(N,t) = \frac{1+t^{2N-6}(2+2t^2+t^4)+t^{4N-12}(1+2t^2+2t^4)+t^{6N-14}}{(1-t^4)(1-t^6)(1-t^{2N-8})(1-t^{4N-12})} \,. \tag{8.38}$$

The equivalent form (8.38) still has a pole of order 4 and a palindromic numerator. Moreover, the monopole generators are clearly visible, as we know the set of minimal generators (8.17), and can be summarise for completeness: $2\Delta(1,0) = 2N-8$ and $2\Delta(1,1) = 2\Delta(2,-1) = 4N-12$.

8.4.3 N hypermultiplets in $[1,1]$ representation

Investigating N hypermultiplets in the adjoint representation is another extreme case of (8.4) as $N_F = 0 = N_R$. The conformal dimension in this circumstance reduces to

$$\Delta(m_1,m_2,m_3) = (N-1)\sum_{i<j} |m_i - m_j| \,, \tag{8.39}$$

and we notice that there is the shift symmetry $m_i \to m_i + a$ present. Due to this, the naive calculation of the U(3) Hilbert series is divergent, which we understand as follows: define overall U(1)-charge $M \coloneqq m_1+m_2+m_3$, then the Hilbert series becomes

$$\mathrm{HS}_{\mathrm{U}(3)}^{(1,1)} = \sum_{M\in\mathbb{Z}} \sum_{\substack{m_1,m_2 \\ m_1 \geq \max(m_2, M-2m_2)}} t^{2(N-1)(3m_1+3m_2-2M+|m_1-m_2|)}\, z^M\, P_{\mathrm{U}(3)}(t,m_1,m_2,m_3) \,. \tag{8.40}$$

Since we want to use the U(3)-calculation as an intermediate step to derive the SU(3)-case, the only meaningful choice to fix the shift-symmetry is $m_1 + m_2 + m_3 = 0$. But then

$$\mathrm{HS}^{(1,1)}_{\mathrm{U}(3),\text{fixed}} = \sum_{\substack{m_1,m_2 \\ m_1 \geq \max(m_2,-2m_2)}} t^{2(N-1)(3m_1+3m_2+|m_1-m_2|)} \; P_{\mathrm{U}(3)}(t,m_1,m_2,-m_1-m_2) \tag{8.41}$$

and the transition to SU(3) is simply

$$\begin{aligned}\mathrm{HS}^{(1,1)}_{\mathrm{SU}(3)} &= (1-t^2)\int_{|z|=1}\frac{\mathrm{d}z}{2\pi z} \\ &\qquad \times \sum_{\substack{m_1,m_2 \\ m_1 \geq \max(m_2,-2m_2)}} t^{2(N-1)(3m_1+3m_2+|m_1-m_2|)} \; P_{\mathrm{U}(3)}(t,m_1,m_2,-m_1-m_2) \\ &= \sum_{\substack{m_1,m_2 \\ m_1 \geq \max(m_2,-2m_2)}} t^{2(N-1)(3m_1+3m_2+|m_1-m_2|)} \; P_{\mathrm{SU}(3)}(t,m_1,m_2)\,. \end{aligned} \tag{8.42}$$

The computation then yields

$$\mathrm{HS}^{(1,1)}_{\mathrm{SU}(3)} = \frac{1+t^{8N-6}(2+2t^2+t^4)+t^{12N-12}(1+2t^2+2t^4)+t^{20N-14}}{(1-t^4)\,(1-t^6)\,(1-t^{8N-8})\,(1-t^{12N-12})}\,. \tag{8.43}$$

We see that numerator of (8.43) is a palindromic polynomial of degree $20N-14$; while the degree of the denominator is $20N-10$. Hence, the difference in the degrees is 4, which coincides with the complex dimension of the moduli space. The same holds for the order of the pole of (8.43) at $t \to 1$.

The interpretation of the appearing monopole operators, and their dressings, is completely analogous to (8.34) and reproduces the picture concluded from the Hilbert bases (8.12). To be specific, $2\Delta(1,0) = 8N-8$ and $2\Delta(1,1) = 2\Delta(2,-1) = 12N-12$.

8.4.4 N hypers in $[3,0]$ representation

Intermediate step at U(3). The conformal dimension reads

$$\Delta(m_1,m_2,m_3) = \frac{3}{2}N\sum_i |m_i| + (N-1)\sum_{i<j}|m_i - m_j|\,. \tag{8.44}$$

We then obtain for $N > 2$ the Hilbert series:

$$\mathrm{HS}^{[3,0]}_{\mathrm{U}(3)}(t,z) = \frac{R(N,t,z)}{P(N,t,z)}\,, \tag{8.45a}$$

$$\begin{aligned} P(N,t,z) = \prod_{j=1}^{3}&\left(1-t^{2j}\right)\left(1-\tfrac{1}{z}t^{7N-4}\right)\left(1-zt^{7N-4}\right)\left(1-\tfrac{1}{z^2}t^{10N-4}\right) \\ &\times\left(1-z^2t^{10N-4}\right)\left(1-\tfrac{1}{z^3}t^{9N}\right)\left(1-z^3t^{9N}\right)\,, \end{aligned} \tag{8.45b}$$

$$\begin{aligned} R(N,t,z) = 1 &+ t^{14N-2} - t^{18N} - t^{20N-8} - 2t^{20N-6} - 2t^{20N-4} + 2t^{26N-8} - 2t^{26N-2} \\ &+ 2t^{32N-6} + 2t^{32N-4} + t^{32N-2} + t^{34N-10} - t^{38N-8} - t^{52N-10} \end{aligned}$$

(m_1, m_2, m_3)		$2\Delta(m_1, m_2, m_3)$	$\mathrm{H}_{(m_1,m_2,m_3)}$
$(1,0,0)$	$(0,0,-1)$	$7N-4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(1,1,0)$	$(0,-1,-1)$	$10N-4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(1,1,1)$	$(-1,-1,-1)$	$9N$	$\mathrm{U}(3)$

Table 32. The monopole generators for a U(3) gauge theory with matter transforming in $[3,0]$ that together with the Casimir invariants generate the chiral ring.

$$\begin{aligned}
&+ (z+\tfrac{1}{z})\Big(t^{7N-2}+t^{7N}-t^{17N-6}-t^{17N-4}-t^{19N-4}-t^{19N-2}-t^{19N}\\
&\qquad +t^{25N-4}-t^{27N-6}+t^{33N-10}+t^{33N-8}+t^{33N-6}+t^{35N-6}\\
&\qquad +t^{35N-4}-t^{45N-10}-t^{45N-8}\Big)\\
&+ (z^2+\tfrac{1}{z^2})\Big(t^{10N-2}+t^{10N}-t^{16N-4}-t^{16N-2}-t^{16N}-t^{24N-6}\\
&\qquad +t^{28N-4}+t^{36N-10}+t^{36N-8}+t^{36N-6}-t^{42N-10}-t^{42N-8}\Big)\\
&+ (z^3+\tfrac{1}{z^3})\Big(t^{17N-2}-t^{23N-6}-t^{23N-4}-t^{23N-2}\\
&\qquad +t^{29N-8}+t^{29N-6}+t^{29N-4}-t^{35N-8}\Big)\,. \qquad (8.45c)
\end{aligned}$$

The Hilbert series (8.45) has a pole of order 6 as $t \to 1$, because $R(N, t = 1, z) = 0$ and $\frac{\mathrm{d}^n}{\mathrm{d}t^n} R(N, t, z)|_{t=1} = 0$ for $n = 1, 2$. Therefore, the moduli space is 6-dimensional. Also, the degree of (8.45c) is $52N - 10$, while the degree of (8.45b) us $52N - 4$; thus, the difference in degrees equals the dimension of the moduli space.

As this example is merely a special case of (8.33), we just summarise the minimal generators in table 32.

Reduction to SU(3). The Hilbert series reads

$$\mathrm{HS}^{[3,0]}_{\mathrm{SU}(3)}(t) = \frac{1+t^{14N-6}(2+2t^2+t^4)+t^{24N-12}(1+2t^2+2t^4)+t^{38N-14}}{(1-t^4)\,(1-t^6)\,(1-t^{14N-8})\,(1-t^{24N-12})}\,. \qquad (8.46)$$

It is apparent that the numerator of (8.46) is a palindromic polynomial of degree $38N-14$; while the degree of the denominator is $38N - 10$; hence, the difference in the degrees is 4, which equals the complex dimension of the moduli space.

The structure of (8.46) is merely a special case of (8.34), and the conformal dimensions of the minimal generators are $2\Delta(1, 0) = 14N - 8$ and $2\Delta(1, 1) = 2\Delta(2, -1) = 24N - 12$.

8.5 Category $N_R \neq 0$

8.5.1 N_F hypers in $[2, 1]$, N_A hypers in $[1, 1]$, N_R hypers in $[2, 1]$ representation

Intermediate step at U(3). The conformal dimension reads

$$\begin{aligned}
2\Delta(m_1,m_2,m_3) = (4N_R+N_A)\sum_{i=1}^{3}|m_i| + N_R\sum_{i<j}\left(|2m_i-m_j|+|m_i-2m_j|\right) \qquad (8.47)\\
+2(N_A-1)\sum_{i<j}|m_i-m_j|\ .
\end{aligned}$$

The Hilbert series reads

$$\mathrm{HS}_{\mathrm{U}(3)}^{[1,0]+[1,1]+[2,1]}(t,z) = \frac{R(N_F,N_A,N_R,t,z)}{P(N_F,N_A,N_R,t,z)} \,, \tag{8.48a}$$

with

$$\begin{aligned} P(N_F,N_A,N_R,t,z) = &\prod_{j=1}^{3}\left(1-t^{2j}\right)\left(1-\frac{t^{N_F+4N_A+10N_R-4}}{z}\right)\left(1-zt^{N_F+4N_A+10N_R-4}\right) \\ &\times\left(1-\frac{t^{2N_F+4N_A+16N_R-4}}{z^2}\right)\left(1-z^2t^{2N_F+4N_A+16N_R-4}\right) \\ &\times\left(1-\frac{t^{3N_F+18N_R}}{z^3}\right)\left(1-z^3t^{3N_F+18N_R}\right) \\ &\times\left(1-\frac{t^{3N_F+8N_A+24N_R-8}}{z^3}\right)\left(1-z^3t^{3N_F+8N_A+24N_R-8}\right) \\ &\times\left(1-\frac{t^{4N_F+4N_A+24N_R-4}}{z^4}\right)\left(1-z^4t^{4N_F+4N_A+24N_R-4}\right) \\ &\times\left(1-\frac{t^{5N_F+4N_A+30N_R-4}}{z^5}\right)\left(1-z^5t^{5N_F+4N_A+30N_R-4}\right) \\ &\times\left(1-\frac{t^{7N_F+12N_A+46N_R-12}}{z^7}\right)\left(1-z^7t^{7N_F+12N_A+46N_R-12}\right) \,, \end{aligned} \tag{8.48b}$$

and the numerator $R(N_F,N_A,N_R,t,z)$ is too long to be displayed, because it contains 28650 monomials. We checked explicitly that $R(N_F,N_A,N_R,t=1,z)=0$ and $\frac{\mathrm{d}^n}{\mathrm{d}t^n}R(N_F,N_A,N_R,t,z)|_{t=1,z=1}=0$ for all $n=1,2\dots,10$. Therefore, the Hilbert series (8.48) has a pole of order 6 at $t=1$, which equals the dimension of the moduli space. In addition, $R(N_F,N_A,N_R,t,z)$ is a polynomial of degree $50N_F+72N_A+336N_R-66$, while the denominator (8.48b) is of degree $50N_F+72N_A+336N_R-60$. The difference in degrees reflects the dimension of the moduli space as well.

Following the analysis of the Hilbert bases (8.19), we identify the bare monopole operators and provide their conformal dimensions in table 33. The result (8.48) has been tested against the independent calculations of the cases: N hypermultiplets in $[1,0]$; N_F hypermultiplets in $[1,0]$ together with N_A hypermultiplets in $[1,1]$; and N hypermultiplets in $[2,1]$. All the calculations agree.

Reduction to SU(3). The Hilbert series for the SU(3) theory reads

$$\mathrm{HS}_{\mathrm{SU}(3)}^{[1,0]+[1,1]+[2,1]}(N_F,N_A,N_R,t) = \frac{R(N_F,N_A,N_R,t)}{P(N_F,N_A,N_R,t)} \,, \tag{8.49a}$$

$$\begin{aligned} P(N_F,N_A,N_R,t) = &\left(1-t^4\right)\left(1-t^6\right)\left(1-t^{2N_F+8N_A+20N_R-8}\right) \\ &\times\left(1-t^{4N_F+12N_A+36N_R-12}\right)\left(1-t^{6N_F+20N_A+54N_R-20}\right) \,, \end{aligned} \tag{8.49b}$$

$$\begin{aligned} R(N_F,N_A,N_R,t) = &\,1+t^{2N_F+8N_A+20N_R-6}(2+2t^2+t^4) \\ &+t^{4N_F+12N_A+36N_R-12}(1+2t^2+2t^4) \end{aligned} \tag{8.49c}$$

(m_1, m_2, m_3)		$2\Delta(m_1, m_2, m_3)$	$\mathrm{H}_{(m_1,m_2,m_3)}$
$(1,0,0)$	$(0,0,-1)$	$N_F + 4N_A + 10N_R - 4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(1,1,0)$	$(0,-1,-1)$	$2N_F + 4N_A + 16N_R - 4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(1,1,1)$	$(-1,-1,-1)$	$3N_F + 18N_R$	$\mathrm{U}(3)$
$(2,1,0)$	$(0,-1,-2)$	$3N_F + 8N_A + 24N_R - 8$	$\mathrm{U}(1)^3$
$(2,1,1)$	$(-1,-1,-2)$	$4N_F + 4N_A + 24N_R - 4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(2,2,1)$	$(-1,-2,-2)$	$5N_F + 4N_A + 30N_R - 4$	$\mathrm{U}(1)\times\mathrm{U}(2)$
$(3,2,1)$	$(-1,-2,-3)$	$6N_F + 8N_A + 38N_R - 8$	$\mathrm{U}(1)^3$
$(4,2,1)$	$(-1,-2,-4)$	$7N_F + 12N_A + 46N_R - 12$	$\mathrm{U}(1)^3$

Table 33. The monopole generators for a U(3) gauge theory with a mixture of matter transforming in $[1,0]$, $[1,1]$, and $[2,1]$.

$$
\begin{aligned}
&+ t^{6N_F+20N_A+54N_R-20}(1+4t^2+4t^4+2t^6) \\
&- t^{6N_F+20N_A+56N_R-20}(2+4t^2+4t^4+t^6) \\
&- t^{8N_F+28N_A+74N_R-26}(2+2t^2+t^4) \\
&- t^{10N_F+32N_A+90N_R-32}(1+2t^2+2t^4) - t^{12N_F+40N_A+110N_R-34} \,.
\end{aligned}
$$

Again, the numerator (8.49c) is an anti-palindromic polynomial of degree $12N_F + 40N_A + 110N_R - 34$; while the denominator (8.49b) is of degree $12N_F + 40N_A + 110N_R - 30$, such that the difference is again 4.

The minimal generators from (8.19) are now realised with the following conformal dimensions: $2\Delta(1,0) = 2N_F + 8N_A + 20N_R - 8$, $2\Delta(1,1) = 2\Delta(2,-1) = 4N_F + 12N_A + 36N_R - 12$, and $2\Delta(2,1) = 2\Delta(3,-1) = 6N_F + 20N_A + 54N_R - 20$. Moreover, the appearing dressed monopoles are as described in subsection 8.3.2.

Remark. The SU(3) result (8.49) has been tested against the independent calculations of the cases: N hypermultiplets in $[1,0]$; N hypermultiplets in $[1,1]$; N_F hypermultiplets in $[1,0]$ together with N_A hypermultiplets in $[1,1]$; and N hypermultiplets in $[2,1]$. All the calculations agree.

Dressings of (2, 1) and (3, −1). From the generic analysis (8.19) the bare monopoles of GNO-charges $(3,-1)$ and $(2,1)$ are necessary generators. However, not all of their dressings need to be independent generators, cf. appendix A.

- $N_R = 0$: $(2,1)$ and $(3,-1)$ are generated by $(1,0)$, $(1,1)$, and $(2,-1)$, which is the generic result of (8.17).
- $N_R = 1$: here, $(2,1)$ and $(3,-1)$ are independent, but not all of their dressings, as we see

$$
(2,1) = (1,1) + (1,0) \quad \text{and} \quad \Delta(2,1) + 1 = \Delta(1,1) + \Delta(1,0) \,. \tag{8.50}
$$

Hence, *only one* of the degree one dressings $V^{\text{dress},(1,0)}_{(2,1)}$, $V^{\text{dress},(0,1)}_{(2,1)}$ is independent, while the other can be generated. (Same holds for $(3,-1)$.)

- $N_R = 2$: here, $(2,1)$ and $(3,-1)$ are independent, but not all of their dressings, as we see

$$(2,1) = (1,1) + (1,0) \quad \text{and} \quad \Delta(2,1) + 2 = \Delta(1,1) + \Delta(1,0) \; . \tag{8.51}$$

  Hence, *only one* of the degree two dressings $V^{\text{dress},(2,0)}_{(2,1)}$, $V^{\text{dress},(0,2)}_{(2,1)}$ is independent, while the other can be generated. However, *both* degree one dressings $V^{\text{dress},(1,0)}_{(2,1)}$, $V^{\text{dress},(0,1)}_{(2,1)}$ are independent. (Same holds for $(3,-1)$.)

- $N_R = 3$: here, $(2,1)$ and $(3,-1)$ are independent, but still not all of their dressings, as we see

$$(2,1) = (1,1) + (1,0) \quad \text{and} \quad \Delta(2,1) + 3 = \Delta(1,1) + \Delta(1,0) \; . \tag{8.52}$$

  Hence, the degree three dressing $V^{\text{dress},(3,0)+(0,3)}_{(2,1)}$ is not independent. However, *both* degree one dressings $V^{\text{dress},(1,0)}_{(2,1)}$, $V^{\text{dress},(0,1)}_{(2,1)}$ and *both* degree two dressings $V^{\text{dress},(2,0)}_{(2,1)}$, $V^{\text{dress},(0,2)}_{(2,1)}$ are independent. (Same holds for $(3,-1)$.)

- $N_R \geq 4$: the bare and the all dressed monopoles corresponding to $(2,1)$ and $(3,-1)$ are independent.

8.5.2 N hypers in $[2,1]$ representation

Intermediate step at U(3). The conformal dimension reads

$$2\Delta(m_1,m_2,m_3) = 4N \sum_{i=1}^{3} |m_i| + N \sum_{i<j} \left(|2m_i - m_j| + |m_i - 2m_j|\right) - 2\sum_{i<j} |m_i - m_j| \; . \tag{8.53}$$

From the calculations we obtain the Hilbert series

$$\mathrm{HS}^{[2,1]}_{\mathrm{U}(3)}(N,t,z) = \frac{R(N,t,z)}{P(N,t,z)} \; , \tag{8.54a}$$

$$\begin{aligned} P(N,t,z) = \prod_{j=1}^{3} \left(1-t^{2j}\right) &\left(1 - \frac{t^{10N-4}}{z}\right)\left(1 - z t^{10N-4}\right)\left(1 - \frac{t^{16N-4}}{z^2}\right)\left(1 - z^2 t^{16N-4}\right) \\ &\times \left(1 - \frac{t^{18N}}{z^3}\right)\left(1 - z^3 t^{18N}\right)\left(1 - \frac{t^{24N-8}}{z^3}\right)\left(1 - z^3 t^{24N-8}\right) \\ &\times \left(1 - \frac{t^{24N-4}}{z^4}\right)\left(1 - z^4 t^{24N-4}\right)\left(1 - \frac{t^{30N-4}}{z^5}\right)\left(1 - z^5 t^{30N-4}\right) \\ &\times \left(1 - \frac{t^{46N-12}}{z^7}\right)\left(1 - z^7 t^{46N-12}\right) \; , \end{aligned} \tag{8.54b}$$

(m_1, m_2, m_3)		$2\Delta(m_1, m_2, m_3)$	$\mathrm{H}_{(m_1,m_2,m_3)}$
$(1, 0, 0)$	$(0, 0, -1)$	$10N - 4$	$\mathrm{U}(1) \times \mathrm{U}(2)$
$(1, 1, 0)$	$(0, -1, -1)$	$16N - 4$	$\mathrm{U}(1) \times \mathrm{U}(2)$
$(1, 1, 1)$	$(-1, -1, -1)$	$18N$	$\mathrm{U}(3)$
$(2, 1, 0)$	$(0, -1, -2)$	$24N - 8$	$\mathrm{U}(1)^3$
$(2, 1, 1)$	$(-1, -1, -2)$	$24N - 4$	$\mathrm{U}(1) \times \mathrm{U}(2)$
$(2, 2, 1)$	$(-1, -2, -2)$	$30N - 4$	$\mathrm{U}(1) \times \mathrm{U}(2)$
$(3, 2, 1)$	$(-1, -2, -3)$	$38N - 8$	$\mathrm{U}(1)^3$
$(4, 2, 1)$	$(-1, -2, -4)$	$46N - 12$	$\mathrm{U}(1)^3$

Table 34. The monopole generators for a U(3) gauge theory with matter transforming in $[2, 1]$ that generate the chiral ring (together with the Casimir invariants).

and the numerator $R(N, t, z)$ is with 13492 monomials too long to be displayed. Nevertheless, we checked explicitly that $R(N, t = 1, z) = 0$ and $\frac{\mathrm{d}^n}{\mathrm{d}t^n} R(N, t, z)|_{t=1,z=1} = 0$ for all $n = 1, 2 \ldots, 10$. Therefore, the Hilbert series (8.54) has a pole of order 6 at $t = 1$, which equals the dimension of the moduli space. In addition, the degree of $R(N, t, z)$ is $296N - 62$, while the denominator (8.54b) is of degree $296N - 56$; therefore, the difference in degrees is again equal to the dimension of the moduli space.

The Hilbert series (8.54) appears as special case of (8.48) and as such the appearing monopole operators are the same. For completeness, we provide in table 34 the conformal dimensions of all minimal (bare) generators (8.15). The GNO-charge $(3, 2, 1)$ is not apparent in the Hilbert series, but we know it to be present due to the analysis of the Hilbert bases (8.15).

Reduction to SU(3). After reduction (8.6) of (8.54) to SU(3) we obtain the following Hilbert series:

$$HS_{\mathrm{SU}(3)}^{(2,1)} = \frac{R(N, t)}{(1 - t^4)\,(1 - t^6)\,(1 - t^{20N-8})\,(1 - t^{36N-12})\,(1 - t^{54N-20})}\,, \tag{8.55a}$$

$$\begin{aligned} R(N, t) = 1 &+ t^{20N-6}(2 + 2t^2 + t^4) + t^{36N-12}(1 + 2t^2 + 2t^4) \\ &+ t^{54N-20}(1 + 4t^2 + 4t^4 + 2t^6) - t^{56N-20}(2 + 4t^2 + 4t^4 + t^6) \\ &- t^{74N-26}(2 + 2t^2 + t^4) - t^{90N-32}(1 + 2t^2 + 2t^4) - t^{110N-34}\,. \end{aligned} \tag{8.55b}$$

The numerator of (8.55b) is an anti-palindromic polynomial of degree $110N - 34$; while the numerator is of degree $110N - 30$. Consequently, the difference in degree reflects the complex dimension of the moduli space.

The Hilbert series (8.55) is merely a special case of (8.49) and, thus, the appearing (bare and dressed) monopole operators are the same. For completeness we provide their conformal dimensions: $2\Delta(1, 0) = 20N - 8$, $2\Delta(1, 1) = 2\Delta(2, -1) = 36N - 12$, and $2\Delta(2, 1) = 2\Delta(3, -1) = 54N - 20$.

9 Conclusions

In this paper we introduced a geometric concept to identify and compute the set of bare and dressed monopole operators that are sufficient to describe the entire chiral ring $\mathbb{C}[\mathcal{M}_C]$ of any 3-dimensional $\mathcal{N}=4$ gauge theory. The methods can be summarised as follows:

1. The matter content together with the positive roots of the gauge group G define the conformal dimension, which in turn defines an arrangement of hyperplanes that divide the dominant Weyl chamber of $\widehat{\mathrm{G}}$ into a fan.

2. The intersection of the fan with the weight lattice of the GNO-dual group leads to a collection of affine semi-groups. All semi-groups are finitely generated and the unique, finite basis is called Hilbert basis.

3. The knowledge of the minimal generators, together with their properties $\mathrm{SU}(2)_R$-spin, residual gauge group H_m, and topological charges $J(m)$, is sufficient to explicitly sum and determine the Hilbert series as rational function.

Utilising the fan and the Hilbert bases for each semi-group also allows to deduce the dressing behaviour of monopole operators. The number of dressed operators is determined by a ratio of orders of Weyl groups, while the degrees are determined by the ratio of the dressing factors associated to the GNO-charge m divided by the dressing factor of the trivial monopole $m=0$.

Most importantly, the entire procedure works for any rank of the gauge group, as indicated in section 8 for U(3). For the main part of the paper, we, however, have chosen to provide a comprehensive collection of rank two examples.

Before closing, let us outline and comment on the approach to higher rank cases.

(a) The gauge group G determines the GNO-dual group $\widehat{\mathrm{G}}$ and the corresponding dominant Weyl chamber (or the product of several Weyl chambers). The Weyl chamber is understood as finite intersection of positive half-spaces $H_\alpha^+ \subset \mathfrak{t}$, where α ranges over all simple roots of G. (If G is a product, then the roots of one factor have to be embedded in a higher dimensional vector space.)

(b) The *relevant weights* μ_i, as identified in section 2.2, define a finite set of cones via the intersection of all possible upper and lower half-spaces with the Weyl chamber. This step can, for instance, be implemented by means of the package *Polyhedra* of `Macaulay2`.

(c) Having defined all cones in `Macaulay2`, one computes the dimension and the Hilbert basis for each cone. Identifying all cones $C_p^{(\mathrm{rk}(G))}$ of the maximal dimension rk(G) can typically reduce the number of cones one needs to consider.

(d) Define the fan $F=\{C_p^{(\mathrm{rk}(G))}|p=1,\dots,L\}$ generated by all top-dimensional cones in `Macaulay2`. This step is the computationally most demanding process so far.

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(e) Next, one employs the *inclusion-exclusion principle* for each cone in the fan: that is the number of points in the (relative) interior $\mathrm{Int}(S^{(p)}) \coloneqq \mathrm{Relint}(C^{(p)}) \cap \Lambda_w(\widehat{\mathrm{G}})$ is given by

$$\begin{aligned}
\#|\mathrm{Int}(S^{(p)})| = |S^{(p)}| - \Bigg(\sum_{j=1}^{\kappa_p} |S_j^{(p-1)}| - \sum_{1\leq i<j\leq\kappa_p} |S_i^{(p-1)} \cap S_j^{(p-1)}| \qquad\qquad (9.1a)\\
+ \sum_{1\leq i<j<k\leq\kappa_p} |S_i^{(p-1)} \cap S_j^{(p-1)} \cap S_k^{(p-1)}| - \dots \\
+ (-1)^{\kappa_p-1} \left| \bigcap_{i=1}^{\kappa_p} S_i^{(p-1)} \right| \Bigg) \\
\equiv |S^{(p)}| - |\partial S^{(p)}| \, , \qquad\qquad (9.1b)
\end{aligned}$$

where the $S_j^{(p-1)}$ for $j = 1, \dots, \kappa_p$ are the semi-groups resulting from the facets of $C^{(p)}$. Note that the last term $\bigcap_{i=1}^{\kappa_p} S_i^{(p-1)}$ equals the trivial semi-group, while the intermediate intersections give rise to all lower dimensional semi-groups contained in the boundary of $S^{(p)}$. Then, the contribution for $\mathrm{Int}(S^{(p)})$ to the monopole formula is computed as follows:

$$\mathrm{HS}(S^{(p)};t) \coloneqq P_{\mathrm{G}}(t;S^{(p)}) \cdot \left[\mathrm{H}_{S^{(p)}}(t) - \mathrm{H}_{\partial S^{(p)}}(t) \right] \, , \qquad (9.2a)$$

$$\mathrm{H}_{S^{(p)}}(t) \coloneqq \sum_{m \in S^{(p)}} z^{J(m)} \, t^{\Delta(m)} \, , \qquad (9.2b)$$

$$\begin{aligned}
\mathrm{H}_{\partial S^{(p)}}(t) \coloneqq \sum_{j=1}^{\kappa_p} \mathrm{H}_{S_j^{(p-1)}}(t) - \sum_{1\leq i<j\leq\kappa_p} \mathrm{H}_{S_i^{(p-1)} \cap S_j^{(p-1)}}(t) \qquad (9.2c)\\
+ \sum_{1\leq i<j<k\leq\kappa_p} \mathrm{H}_{S_i^{(p-1)} \cap S_j^{(p-1)} \cap S_k^{(p-1)}}(t) - \dots + (-1)^{\kappa_p-1} \mathrm{H}_{\bigcap_{i=1}^{\kappa_p} S_i^{(p-1)}}(t) \, .
\end{aligned}$$

Each contribution $\mathrm{H}_{S^{(p)}}(t)$ is evaluated as discussed in section 2.4 and 2.5. Although this step is algorithmically simple, it can be computationally demanding. It is, however, crucial that the fan F has been defined, in order to work with the correct faces of each cone and to sum over each cone in the fan only once.

(f) Finally, one has to add all contributions

$$\mathrm{HS}(F;t) = \sum_{C \in F} \mathrm{HS}(S) \, . \qquad (9.3)$$

This last step is a simple sum, but to obtain the Hilbert series as a rational function in a desirable form can be cumbersome.

Equipped with this procedure, we hope to report on Coulomb branches for higher rank gauge groups and quiver gauge theories in the future.

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A Plethystic logarithm

In this appendix we summarise the main properties of the plethystic logarithm. Starting with the definition, for a mulit-valued function $f(t_1, \ldots, t_m)$ with $f(0, \ldots, 0) = 1$, one defines

$$\mathrm{PL}[f] \coloneqq \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log \left(f(t_1^k, \ldots, t_m^k) \right) \,, \tag{A.1}$$

where $\mu(k)$ denote the Möbius function [30]. Some basic properties include

$$\mathrm{PL}[f \cdot g] = \mathrm{PL}[f] + \mathrm{PL}[g] \quad \text{and} \quad \mathrm{PL}\left[\frac{1}{\prod_n (1-t^n)^{a_n}} \right] = \sum_n a_n \, t^n \,. \tag{A.2}$$

Now, we wish to compute the plethystic logarithm. Given a Hilbert series as rational function, i.e. of the form (2.28) or (2.35), the denominator can be taken care of by means of (A.2), while the numerator is a polynomial with integer coefficients. In order to obtain an approximation of the PL, we employ the following two equivalent transformations for the numerator:

$$\begin{aligned} \mathrm{PL}\left[1 + a t^n + \mathcal{O}(t^{n+1})\right] &= \mathrm{PL}\left[\frac{(1-t^n)^a \ \left(1 + a t^n + \mathcal{O}(t^{n+1})\right)}{(1-t^n)^a} \right] \\ &= a t^n + \mathrm{PL}\left[1 + \mathcal{O}(t^{n+1})\right] \,, \end{aligned} \tag{A.3a}$$

$$\begin{aligned} \mathrm{PL}\left[1 - a t^n + \mathcal{O}(t^{n+1})\right] &= \mathrm{PL}\left[\frac{(1-t^n)^a \ (1+t^n)^a \ \left(1 - a t^n + \mathcal{O}(t^{n+1})\right)}{(1-t^{2n})^a} \right] \\ &= -a t^n + a t^{2n} + \mathrm{PL}\left[1 + \mathcal{O}(t^{n+1})\right] \,. \end{aligned} \tag{A.3b}$$

Now, we derive an approximation of the PL for a generic rank two gauge group in terms of t^{Δ}. More precisely, consider the Hilbert basis $\{X_i\}$ then we provide an approximation of the PL up to second order, i.e.

$$\mathrm{PL} = \text{Casimir inv.} + \left\{ t^{\Delta(X_i)}\text{-terms} \right\} + \left\{ t^{\Delta(X_i)+\Delta(X_j)}\text{-terms} \right\} + \mathcal{O}\left(t^{\Delta(X_i)+\Delta(X_j)+\Delta(X_k)} \right) \tag{A.4}$$

Considering (2.28), the numerator is denoted by $R(t)$, while the denominator $Q(t)$ is given by

$$Q(t) = \prod_{i=1}^{2} (1 - t^{d_i}) \ \prod_{p=0}^{L} \left(1 - t^{\Delta(x_p)} \right) \,, \tag{A.5}$$

with d_i the degrees of the Casimir invariants. Then expand the numerator as follows:

$$
\begin{aligned}
R(t) = 1 &+ \sum_{q=0}^{L}\left(\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)}-1\right)t^{\Delta(x_q)} + \sum_{q=0}^{L}\sum_{s\in\mathrm{Int}(\mathcal{P}(C_q^{(2)}))}\frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)}t^{\Delta(s)} \\
&- \sum_{\substack{q,p=0\\ q\neq p}}^{L}\left(\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)}-\frac{1}{2}\right)t^{\Delta(x_p)+\Delta(x_q)} + \sum_{q=1}^{L}\frac{P_{\mathrm{G}}(t,C_q^{(2)})}{P_{\mathrm{G}}(t,0)}t^{\Delta(x_{q-1})+\Delta(x_q)} \\
&- \sum_{q=1}^{L}\sum_{s\in\mathrm{Int}(\mathcal{P}(C_q^{(2)}))}\sum_{\substack{r=0\\ r\neq q-1,q}}^{L}\frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)}t^{\Delta(s)+\Delta(x_r)}\,.
\end{aligned}
\tag{A.6}
$$

Note that the appearing factor $\frac{1}{2}$ avoids double counting when changing summation $\sum_{q<p}$ to $\sum_{q\neq p}$. Still, the numerator is a polynomial with integer coefficients. The PL then reads

$$
\mathrm{PL}\left[\mathrm{HS}_{\mathrm{G}}(t)\right] = \sum_{i=1}^{2}t^{d_i} + \sum_{p=0}^{L}t^{\Delta(x_p)} + \mathrm{PL}\left[R(t)\right]\,. \tag{A.7}
$$

By step (A.3a) we factor out the order $t^{\Delta(x_q)}$ and $t^{\Delta(s)}$ terms. However, this introduces further terms at order $t^{\Delta(x_q)+\Delta(s)}$ and so forth, which are given by

$$
-\left(\sum_{q=0}^{L}\left(\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)}-1\right)t^{\Delta(x_q)} + \sum_{q=1}^{L}\sum_{s\in\mathrm{Int}(\mathcal{P}(C_q^{(2)}))}\frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)}t^{\Delta(s)}\right)^2\,. \tag{A.8}
$$

Subsequently factoring the terms of this order by means of (A.3b), one derives at the following expressing of the PL

$$
\begin{aligned}
\mathrm{PL}\left[\mathrm{HS}_{\mathrm{G}}(t)\right] = &\sum_{i=1}^{2}t^{d_i} + \sum_{q=0}^{L}\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)}\,t^{\Delta(x_q)} + \sum_{q=1}^{L}\sum_{s\in\mathrm{Int}(\mathcal{P}(C_q^{(2)}))}\frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)}\,t^{\Delta(s)} \\
&- \sum_{\substack{q,p=0\\ q\neq p}}^{L}\left(\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)}-\frac{1}{2}\right)t^{\Delta(x_p)+\Delta(x_q)} + \sum_{q=1}^{L}\frac{P_{\mathrm{G}}(t,C_q^{(2)})}{P_{\mathrm{G}}(t,0)}t^{\Delta(x_{q-1})+\Delta(x_q)} \\
&- \sum_{q=1}^{L}\sum_{s\in\mathrm{Int}(\mathcal{P}(C_q^{(2)}))}\sum_{\substack{r=0\\ r\neq q-1,q}}^{L}\frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)}t^{\Delta(s)+\Delta x_r} \\
&- \sum_{q,p=0}^{L}\left(\frac{P_{\mathrm{G}}(t,x_q)}{P_{\mathrm{G}}(t,0)}-1\right)\left(\frac{P_{\mathrm{G}}(t,x_p)}{P_{\mathrm{G}}(t,0)}-1\right)t^{\Delta(x_q)+\Delta(x_p)} \\
&- 2\sum_{p=0}^{L}\sum_{q=1}^{L}\left(\frac{P_{\mathrm{G}}(t,x_p)}{P_{\mathrm{G}}(t,0)}-1\right)\sum_{s\in\mathrm{Int}(\mathcal{P}(C_q^{(2)}))}\frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)}\,t^{\Delta(x_p)+\Delta(s)} \\
&- \sum_{q,p=1}^{L}\sum_{s\in\mathrm{Int}(\mathcal{P}(C_p^{(2)}))}\sum_{s'\in\mathrm{Int}(\mathcal{P}(C_q^{(2)}))}\frac{P_{\mathrm{G}}(t,s)}{P_{\mathrm{G}}(t,0)}\frac{P_{\mathrm{G}}(t,s')}{P_{\mathrm{G}}(t,0)}t^{\Delta(s)+\Delta(s')} \\
&+ \mathrm{PL}\left[1+\mathcal{O}\left(t^{\Delta(X_i)+\Delta(X_j)+\Delta(X_j)}\right)\right]\,.
\end{aligned}
\tag{A.9}
$$

Strictly speaking, the truncation (A.9) is only meaningful if

$$\begin{aligned}&\max\{\Delta(X)\} + \max\{d_i | i=1,2\} < \min\{\Delta(X)+\Delta(Y)\} = 2\cdot\min\{\Delta(X)\} \\ &\quad \text{for} \quad X,Y = x_q \text{ or } s\,, s\in \mathrm{Int}(\mathcal{P}(C_p^{(2)})), q=0,1,\ldots,l\end{aligned} \tag{A.10}$$

holds. Only in this case do the positive contributions, i.e. the generators, of the first line in (A.9) not mix with the negative contributions, i.e. first syzygies or relations, of the remaining lines. Moreover, the condition (A.10) ensures that the remained $\mathcal{O}\left(t^{\Delta(X_i)+\Delta(X_j)+\Delta(X_k)}\right)$ does not spoil the truncation.

From the examples of section 3–8, we see that (A.10) is at most satisfied for scenarios with just a few generators, but not for elaborate cases. Nevertheless, there are some observations we summarise as follows:

- The bare and dressed monopole operators associated to the GNO-charge m are described by $\frac{P_{\mathrm{G}}(t,m)}{P_{\mathrm{G}}(t,0)}t^{\Delta(m)}$. In particular, we emphasis that the quotient of dressing factors provides information on the number and degrees of the dressed monopole operators.

- The previous observation provides an upper bound on the number of dressed monopole operators associated to a magnetic weight m. In detail, the value of $\frac{P_{\mathrm{G}}(t,m)}{P_{\mathrm{G}}(t,0)}$ at $t=1$ equals the number of bare and dressed monopole operators associated to m. Let $\{d_i\}$ and $\{b_i\}$, for $i=1,\ldots,\mathrm{rk}(\mathrm{G})$ denote the degree of the Casimir invariants for G and H_m, respectively. Then

$$\begin{matrix}\#\text{ dressed monopoles}\\ +1\text{ bare monopole}\end{matrix} = \lim_{t\to1}\frac{P_{\mathrm{G}}(t,m)}{P_{\mathrm{G}}(t,0)} = \lim_{t\to1}\frac{\prod_{i=1}^{\mathrm{rk}(\mathrm{G})}\left(1-t^{d_i}\right)}{\prod_{j=1}^{\mathrm{rk}(\mathrm{G})}\left(1-t^{b_j}\right)} = \frac{\prod_{i=1}^{\mathrm{rk}(\mathrm{G})} d_i}{\prod_{j=1}^{\mathrm{rk}(\mathrm{G})} b_j} = \frac{|\mathcal{W}_{\mathrm{G}}|}{|\mathcal{W}_{\mathrm{H}_m}|}\,, \tag{A.11}$$

where the last equality holds because the order of the Weyl group equals the product of the degrees of the Casimir invariants. Since $\mathcal{W}_{\mathrm{H}_m}\subset\mathcal{W}_{\mathrm{G}}$ is a subgroup of the finite group $\mathcal{W}_{\mathrm{G}}$, Lagrange's theorem implies that $\frac{|\mathcal{W}_{\mathrm{G}}|}{|\mathcal{W}_{\mathrm{H}_m}|}\in\mathbb{N}$ holds.

The situation becomes obvious whenever m belongs to the interior of the Weyl chamber, because $\mathrm{H}_m = \mathrm{T}$ and thus

$$\left.\begin{matrix}\#\text{ dressed monopoles}\\ +1\text{ bare monopole}\end{matrix}\right|_{\substack{\text{interior of}\\ \text{Weyl chamber}}} = |\mathcal{W}_{\mathrm{G}}| \quad \text{and} \quad \frac{P_{\mathrm{G}}(t,m)}{P_{\mathrm{G}}(t,0)} = \prod_{i=1}^{\mathrm{rk}(\mathrm{G})}\sum_{l_i=0}^{d_i-1} t^{l_i}\,. \tag{A.12}$$

- The significance of the PL is limited, as, for instance, a positive contribution $\sim t^{\Delta(X_1)}$ can coincide with a negative contribution $\sim t^{\Delta(X_2)+\Delta(X_3)}$, but this does not necessarily imply that the object of degree $\Delta(X_1)$ can be generated by others. The situation becomes clearer if there exists an additional global symmetry $Z(\widehat{\mathrm{G}})$ on the moduli space. The truncated PL for (2.35) is obtained from (A.9) by the replacement

$$t^{\Delta(X)} \mapsto \vec{z}^{\vec{J}(X)}\ t^{\Delta(X)}\,. \tag{A.13}$$

Then the "syzygy" $\vec{z}^{\vec{J}(X_2+X_3)}t^{\Delta(X_2)+\Delta(X_3)}$ can cancel the "generator" $\vec{z}^{\vec{J}(X_1)}t^{\Delta(X_1)}$ only if the symmetry charges agree $\vec{z}^{\vec{J}(X_1)} = \vec{z}^{\vec{J}(X_2+X_3)}$, in addition to the $\mathrm{SU}(2)_R$ iso-spin.

Lastly, we illustrate the truncation with the two simplest examples:

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Example: one simplicial cone. For the Hilbert series (2.32) we obtain

$$\begin{aligned}\mathrm{PL} = \sum_{i=1}^{2} t^{d_i} + \frac{P_1(t)}{P_0(t)} \left(t^{\Delta(x_0)} + t^{\Delta(x_1)} \right) - \left(2\frac{P_1(t)}{P_0(t)} - 1 - \frac{P_2(t)}{P_0(t)} \right) t^{\Delta(x_0)+\Delta(x_1)} \\ - \left(\frac{P_1(t)}{P_0(t)} \right)^2 \left(t^{2\Delta(x_0)} + t^{2\Delta(x_1)} + 2t^{\Delta(x_0)+\Delta(x_1)} \right) + \dots \,.\end{aligned} \tag{A.14}$$

Example: one non-simplicial cone. In contrast, for the Hilbert series (2.33) we arrive at

$$\begin{aligned}\mathrm{PL} = \sum_{i=1}^{2} t^{d_i} &+ \frac{P_1(t)}{P_0(t)} \left(t^{\Delta(x_0)} + t^{\Delta(x_1)} \right) + \sum_{s \in \mathrm{Int}\mathcal{P}} \frac{P_2(t)}{P_0(t)} t^{\Delta(s)} \\ &- \left(2\frac{P_1(t)}{P_0(t)} - 1 - \frac{P_2(t)}{P_0(t)} \right) t^{\Delta(x_0)+\Delta(x_1)} \\ &- \left(\frac{P_1(t)}{P_0(t)} \right)^2 \left(t^{2\Delta(x_0)} + t^{2\Delta(x_1)} + 2t^{\Delta(x_0)+\Delta(x_1)} \right) \\ &- 2\left(\frac{P_1(t)}{P_0(t)} - 1 \right) \frac{P_2(t)}{P_0(t)} \sum_{s \in \mathrm{Int}\mathcal{P}} \left(t^{\Delta(s)+\Delta(x_0)} + t^{\Delta(s)+\Delta(x_1)} \right) \\ &- \sum_{s \in \mathrm{Int}\mathcal{P}} \sum_{s' \in \mathrm{Int}\mathcal{P}} \left(\frac{P_2(t)}{P_0(t)} \right)^2 t^{\Delta(s)+\Delta(s')} + \dots \,.\end{aligned} \tag{A.15}$$

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