

Superconformal $SU(1, 1|n)$ mechanics

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ABSTRACT: Recent years have seen an upsurge of interest in dynamical realizations of the superconformal group $SU(1, 1|2)$ in mechanics. Remarking that $SU(1, 1|2)$ is a particular member of a chain of supergroups $SU(1, 1|n)$ parametrized by an integer n , here we begin a systematic study of $SU(1, 1|n)$ multi-particle mechanics. A representation of the superconformal algebra $su(1, 1|n)$ is constructed on the phase space spanned by m copies of the $(1, 2n, 2n-1)$ supermultiplet. We show that the dynamics is governed by two prepotentials V and F , and the Witten-Dijkgraaf-Verlinde-Verlinde equation for F shows up as a consequence of a more general fourth-order equation. All solutions to the latter in terms of root systems reveal decoupled models only. An extension of the dynamical content of the $(1, 2n, 2n-1)$ supermultiplet by angular variables in a way similar to the $SU(1, 1|2)$ case is problematic.

KEYWORDS: Conformal and W Symmetry, Extended Supersymmetry

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1 Introduction

The recent increase of interest in dynamical realizations of the superconformal group $\text{SU}(1, 1|2)$ [1]–[14] and its $D(2, 1|\alpha)$ extension [15]–[23] was motivated by the proposal in [24, 25] that a study of superconformal mechanics may have applications to the quantum mechanics of black holes. In particular, according to [25] the large- m limit of the m -particle $\text{SU}(1, 1|2)$ superconformal Calogero model may provide a microscopic description of the extreme Reissner-Nordström black hole in the near-horizon limit.

The explicit construction of the m -particle $\text{SU}(1, 1|2)$ superconformal Calogero model reduces to solving a variant of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation [1, 4]. Although plenty of interesting solutions to the WDVV equation were found in terms of root systems and their deformations [1, 6, 7, 26–29], the construction of interacting models seems unfeasible beyond $m = 3$. Since, in the context of [25], it is the structure of the superconformal group which matters, any multi-particle $\text{SU}(1, 1|2)$ mechanics appears to be a good candidate. Yet, no attempt has been made to link the large- m limit of any known superconformal many-body quantum mechanics to the extreme Reissner-Nordström black hole in the near-horizon limit.

The studies in [1]–[23] proved useful for understanding the structure of interactions of various $\text{SU}(1, 1|2)$ and $D(2, 1|\alpha)$ supermultiplets. Supersymmetric couplings in $d=1$ are of interest on their own right because of novel features which are absent in higher dimensions.

The superconformal group $\text{SU}(1, 1|2)$ is a particular member of a chain of supergroups $\text{SU}(1, 1|n)$ parametrized by an integer n . The corresponding superconformal algebra $\text{su}(1, 1|n)$ involves n^2+3 bosonic and $4n$ fermionic generators. In particular, its bosonic sector includes $\text{so}(2, 1)$ and $\text{su}(n)$ subalgebras. The natural question arises whether the interesting features revealed for the $\text{SU}(1, 1|2)$ case persist for higher values of n . Interacting many-body $\text{SU}(1, 1|n)$ mechanics may also have applications to the quantum mechanics of higher-dimensional black holes. Specifically, the bosonic subgroup $\text{SO}(2, 1) \times \text{SU}(n)$ of

$SU(1, 1|n)$ coincides with the near-horizon symmetry group of the Myers-Perry black hole with all rotation parameters set equal (see, e.g., the discussion in [30, 31]).

In this work, we initiate a systematic study of $SU(1, 1|n)$ many-body mechanics. There are two competing approaches to analyzing superconformal mechanics, namely the direct construction of an $\mathfrak{su}(1, 1|n)$ representation within the Hamiltonian framework, and the superfield approach combined with the method of nonlinear realizations. In [32] for example, the second approach has been used to describe a single supermultiplet of type $(1, 2n, 2n-1)$. Although the superfield formulation is more powerful, the Hamiltonian approach yields on-shell components and allows one to comprehend the basic dynamical features and the structure of interactions in a simpler and more transparent way. In some instances it also offers notable technical simplifications in building interacting models [14]. In this work we thus adhere to the Hamiltonian formalism.

The paper is organized as follows. In section 2 we fix our notation and represent the structure relations of the superconformal algebra $\mathfrak{su}(1, 1|n)$ in a form analogous to the previously studied case of $\mathfrak{su}(1, 1|2)$. Section 3 is devoted to the construction of an $\mathfrak{su}(1, 1|n)$ representation on the phase space spanned by m copies of the $(1, 2n, 2n-1)$ supermultiplet. It is shown that similarly to the $SU(1, 1|2)$ case the dynamics is governed by two prepotentials V and F . However, the WDVV equation appears as a consequence of a more general fourth-order equation for F . The latter is absent in $SU(1, 1|2)$ mechanics because of a specific Fierz identity which exists for $SU(2)$ spinors only. In section 4 we consider prepotentials F constructed from root systems. It is demonstrated that the fourth-order structure equation characterizing $SU(1, 1|n)$ mechanics forces all root vectors to be mutually orthogonal. This implies decoupled dynamics. In section 5 we try to generalize also the analysis of [14] from $SU(1, 1|2)$ to $SU(1, 1|n)$. More specifically, we attempt to extend an arbitrary phase-space representation of $\mathfrak{su}(n)$ to an $\mathfrak{su}(1, 1|n)$ representation, with a negative result. The concluding section 6 contains a summary and an outlook. Throughout the paper summation over repeated indices is understood.

2 Superconformal algebra $\mathfrak{su}(1, 1|n)$

The superconformal algebra $\mathfrak{su}(1, 1|n)$ involves $n^2 + 3$ bosonic and $4n$ fermionic generators. Its even part is the direct sum $\mathfrak{so}(2, 1) \oplus \mathfrak{su}(n) \oplus \mathfrak{u}(1)$. The generators of $\mathfrak{so}(2, 1)$, which we designate as H, D, K , correspond to the time translation, dilatation and special conformal transformation, respectively. The R -symmetry subalgebra $\mathfrak{su}(n) \oplus \mathfrak{u}(1)$ is generated by J_a , with $a = 1, \dots, n^2 - 1$, and L . The odd part of the superalgebra includes the supersymmetry generators Q_α, \bar{Q}^α , where $\alpha = 1, \dots, n$, and their superconformal partners S_α, \bar{S}^α . It is assumed that the fermions are hermitian conjugates of each other

$$(Q_\alpha)^\dagger = \bar{Q}^\alpha, \quad (S_\alpha)^\dagger = \bar{S}^\alpha. \tag{2.1}$$

Q_α and S_α transform as $\mathfrak{su}(n)$ spinors. The structure relations of $\mathfrak{su}(1, 1|n)$ read

$$\begin{aligned} \{H, D\} &= H, & \{H, K\} &= 2D, \\ \{D, K\} &= K, & \{J_a, J_b\} &= f_{abc} J_c, \end{aligned}$$

$$\begin{aligned}
 \{Q_\alpha, \bar{Q}^\beta\} &= -2iH\delta_\alpha^\beta, & \{Q_\alpha, \bar{S}^\beta\} &= 2(\lambda_a)_\alpha{}^\beta J_a + \left(2iD - \frac{n-2}{n}L\right)\delta_\alpha^\beta, \\
 \{S_\alpha, \bar{S}^\beta\} &= -2iK\delta_\alpha^\beta, & \{\bar{Q}^\alpha, S_\beta\} &= -2(\lambda_a)_\beta{}^\alpha J_a + \left(2iD + \frac{n-2}{n}L\right)\delta_\beta^\alpha, \\
 \{D, Q_\alpha\} &= -\frac{1}{2}Q_\alpha, & \{D, S_\alpha\} &= \frac{1}{2}S_\alpha, \\
 \{K, Q_\alpha\} &= S_\alpha, & \{H, S_\alpha\} &= -Q_\alpha, \\
 \{J_a, Q_\alpha\} &= \frac{i}{2}(\lambda_a)_\alpha{}^\beta Q_\beta, & \{J_a, S_\alpha\} &= \frac{i}{2}(\lambda_a)_\alpha{}^\beta S_\beta, \\
 \{D, \bar{Q}^\alpha\} &= -\frac{1}{2}\bar{Q}^\alpha, & \{D, \bar{S}^\alpha\} &= \frac{1}{2}\bar{S}^\alpha, \\
 \{K, \bar{Q}^\alpha\} &= \bar{S}^\alpha, & \{H, \bar{S}^\alpha\} &= -\bar{Q}^\alpha, \\
 \{J_a, \bar{Q}^\alpha\} &= -\frac{i}{2}\bar{Q}^\beta(\lambda_a)_\beta{}^\alpha, & \{J_a, \bar{S}^\alpha\} &= -\frac{i}{2}\bar{S}^\beta(\lambda_a)_\beta{}^\alpha, \\
 \{L, Q_\alpha\} &= iQ_\alpha, & \{L, S_\alpha\} &= iS_\alpha, \\
 \{L, \bar{Q}^\alpha\} &= -i\bar{Q}^\alpha, & \{L, \bar{S}^\alpha\} &= -i\bar{S}^\alpha,
 \end{aligned} \tag{2.2}$$

where f_{abc} are the totally antisymmetric structure constants of $\mathfrak{su}(n)$ and λ_a are the hermitian and traceless $n \times n$ -matrices which obey the (anti)commutation relations

$$[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c, \quad \{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} + 2d_{abc}\lambda_c, \tag{2.3}$$

with the totally symmetric coefficients d_{abc} . In what follows the Fierz identity

$$\frac{1}{2}(\lambda_a)_\alpha{}^\beta(\lambda_a)_\gamma{}^\sigma = -\frac{1}{n}\delta_\alpha^\beta\delta_\gamma^\sigma + \delta_\gamma^\beta\delta_\alpha^\sigma \tag{2.4}$$

proves to be helpful.

3 Realization of $\mathfrak{su}(1, 1|n)$ in many-body mechanics

In order to realize the $\mathfrak{su}(1, 1|n)$ superconformal algebra in many-body mechanics, let us consider a phase space parametrized by m bosonic canonical pairs (x^i, p^i) , and m self-conjugate fermions $(\psi_\alpha^i)^\dagger = \bar{\psi}^{i\alpha}$, $i = 1, \dots, m$, $\alpha = 1, \dots, n$, which obey the conventional Poisson brackets

$$\{x^i, p^j\} = \delta^{ij}, \quad \{\psi_\alpha^i, \bar{\psi}^{j\beta}\} = -i\delta_\alpha^\beta\delta^{ij}. \tag{3.1}$$

It is assumed that each fermion belongs to the fundamental representation of $SU(n)$.

Guided by the previous studies of the $\mathfrak{su}(1, 1|2)$ -case [1, 4, 6], let us introduce two prepotentials $V(x^1, \dots, x^n)$, $F(x^1, \dots, x^n)$ and consider the following functions:

$$\begin{aligned}
 H &= \frac{1}{2}(p^i p^i + \partial^i V \partial^i V) + \partial^i \partial^j V (\bar{\psi}^i \psi^j) + \frac{1}{2} \partial^i \partial^j \partial^k \partial^l F (\bar{\psi}^i \psi^j) (\bar{\psi}^k \psi^l), & D &= tH - \frac{1}{2} x^i p^i, \\
 K &= t^2 H - t x^i p^i + \frac{1}{2} x^i x^i, & J_a &= \frac{1}{2} (\bar{\psi}^i \lambda_a \psi^i),
 \end{aligned}$$

$$\begin{aligned}
 Q_\alpha &= (p^i + i\partial^i V)\psi_\alpha^i + i\partial^i \partial^j \partial^k F \psi_\alpha^i (\bar{\psi}^j \psi^k), & S_\alpha &= x^i \psi_\alpha^i - tQ_\alpha, \\
 \bar{Q}^\alpha &= (p^i - i\partial^i V)\bar{\psi}^{i\alpha} - i\partial^i \partial^j \partial^k F \bar{\psi}^{i\alpha} (\bar{\psi}^j \psi^k), & \bar{S}^\alpha &= x^i \bar{\psi}^{i\alpha} - t\bar{Q}^\alpha, \\
 L &= \bar{\psi}^i \psi^i, & & (3.2)
 \end{aligned}$$

where $\bar{\psi}^i \psi^j = \bar{\psi}^{i\alpha} \psi_\alpha^j$, $\bar{\psi}^i \lambda_a \psi^i = \bar{\psi}^{i\alpha} (\lambda_a)_\alpha^\beta \psi_\beta^i$. It is straightforward to verify that these functions do obey the structure relations (2.2) under the Poisson bracket (3.1) provided the restrictions on the prepotentials

$$\begin{aligned}
 (\partial^i \partial^j \partial^k F)(\partial^k \partial^l \partial^m F) &= (\partial^m \partial^j \partial^k F)(\partial^k \partial^l \partial^i F), \quad x^i (\partial^i \partial^j \partial^k F) = -\delta^{jk}, \\
 x^i \partial^i V = C, \quad \partial^i \partial^j V &= (\partial^i \partial^j \partial^k F) \partial^k V, \quad \partial^i \partial^j \partial^k \partial^l F = (\partial^i \partial^j \partial^p F)(\partial^p \partial^k \partial^l F)
 \end{aligned} \tag{3.3}$$

hold, with C being an arbitrary constant.

Note that all the constraints in (3.3) coincide with those characterizing the $\text{su}(1,1|2)$ case, but for the rightmost equation entering the second line which is new. It arises when computing the bracket $\{Q_\alpha, \bar{Q}^\beta\}$ which explicitly involves the term

$$(\partial^i \partial^j \partial^k \partial^l F - (\partial^i \partial^j \partial^p F)(\partial^p \partial^k \partial^l F)) \psi_\alpha^i \bar{\psi}^{j\beta} (\bar{\psi}^k \psi^l). \tag{3.4}$$

For $n = 2$ the spinor index α takes only two values, and the spinors in the previous formula can be reordered so as to yield the piece proportional to $(\bar{\psi}^i \psi^j)(\bar{\psi}^k \psi^l) \delta_\alpha^\beta$, thus providing a contribution to the Hamiltonian which is quartic in fermions. For $n > 2$ such reordering is no longer possible, and one has to impose the extra condition

$$\partial^i \partial^j \partial^k \partial^l F = (\partial^i \partial^j \partial^p F)(\partial^p \partial^k \partial^l F), \tag{3.5}$$

which yields the main difference from the $\text{su}(1,1|2)$ case. Note that, by antisymmetrization of the indices i and l , (3.5) actually implies the WDVV equation visible in the first line in (3.3). Hence, the additional requirement as compared to the $n = 2$ case is the totally symmetric projection of (3.5),

$$\partial^i \partial^j \partial^k \partial^l F = (\partial^{(i} \partial^j \partial^p F)(\partial^p \partial^k \partial^l F), \tag{3.6}$$

where the symmetrisation (with weight $\frac{1}{4!}$) excludes the summation index p . Further differentiation of this relation, together with the WDVV equation, yields a hierarchy of equations,

$$\partial^{i_1} \partial^{i_2} \dots \partial^{i_{r+3}} F = r! (\partial^{i_1} \partial^{i_2} \partial^{k_1} F)(\partial^{k_1} \partial^{i_3} \partial^{k_2} F)(\partial^{k_2} \partial^{i_4} \partial^{k_3} F) \dots (\partial^{k_r} \partial^{i_{r+2}} \partial^{i_{r+3}} F) \tag{3.7}$$

together with $x^i (\partial^i \partial^{i_2} \dots \partial^{i_{r+3}} F) = -r \partial^{i_2} \dots \partial^{i_{r+3}} F$, for $r = 1, 2, \dots$

When computing the brackets $\{Q_\alpha, \bar{S}^\beta\}$ and $\{\bar{Q}^\alpha, S_\beta\}$, one has to use the Fierz identity (2.4). In particular, the constant C , which enters the homogeneity condition $x^i \partial^i V = C$, appears in the algebra as the central charge,

$$\begin{aligned}
 \{Q_\alpha, \bar{S}^\beta\} &= 2(\lambda_a)_\alpha^\beta J_a + \left(2iD - \frac{n-2}{n}L + C\right) \delta_\alpha^\beta, \\
 \{\bar{Q}^\alpha, S_\beta\} &= -2(\lambda_a)_\beta^\alpha J_a + \left(2iD + \frac{n-2}{n}L - C\right) \delta_\beta^\alpha.
 \end{aligned} \tag{3.8}$$

If desirable, C can be removed by redefining L . In the latter case the bosonic limit of L yields a constant rather than zero.

4 Prepotentials F related to root systems

The leftmost equation in the first line in (3.3) is a variant of the WDVV equation. With regard to the $SU(1,1|2)$ mechanics it has been extensively studied in [4, 6, 7, 27, 28]. In particular, each solution of the WDVV equation satisfying (3.6) will qualify to describe some $SU(1,1|n)$ superconformal mechanics. The known WDVV solutions are based on so-called \vee -systems [29], which are certain deformations of Coxeter root systems. For these, the prepotential F takes the form

$$F = -\frac{1}{4} \sum_{\alpha} h_{\alpha} (\alpha \cdot x)^2 \ln(\alpha \cdot x)^2, \tag{4.1}$$

where $\{\alpha\}$ is a set of positive m -dimensional root vectors, subject to the usual constraints for reflection groups or their \vee -system deformations, and h_{α} are real weights to be determined. Inserting (4.1) into (3.5), we obtain the condition

$$\sum_{\alpha} h_{\alpha} \frac{\alpha^i \alpha^j \alpha^k \alpha^l}{(\alpha \cdot x)^2} + \sum_{\alpha, \beta} h_{\alpha} h_{\beta} \frac{\alpha^i \alpha^j (\alpha \cdot \beta) \beta^k \beta^l}{(\alpha \cdot x)(\beta \cdot x)} = 0. \tag{4.2}$$

The diagonal terms in this double sum fix the weights,

$$(\alpha \cdot \alpha) h_{\alpha} = 1. \tag{4.3}$$

The projection antisymmetric in i and l ensures the WDVV equation; it is assumed to be fulfilled for our root systems. The symmetric projection gives further algebraic conditions: the vanishing of the double residues of the poles $(\alpha \cdot x)^{-1}(\beta \cdot x)^{-1}$ for any pair (α, β) yields

$$(\alpha \cdot \beta) (\alpha^i \alpha^j \beta^k \beta^l + \beta^i \beta^j \alpha^k \alpha^l) = 0. \tag{4.4}$$

Contracting this with $\alpha^i \beta^j \alpha^k \beta^l$ produces

$$(\alpha \cdot \alpha)(\beta \cdot \beta)(\alpha \cdot \beta)^2 = 0 \implies \alpha \cdot \beta = 0 \tag{4.5}$$

for any pair of distinct roots (α, β) . This admits only the direct sum of mutually orthogonal one-dimensional (i.e. rank-one) systems. By a rigid rotation of coordinates x^i , one can always bring it into the form

$$\{\alpha\} = \{(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, 0, \dots, 1)\}. \tag{4.6}$$

So we have arrived at a no-go theorem for interacting $SU(1,1|n)$ mechanics based on the on-shell supermultiplet of type $(1, 2n, 2n-1)$.

5 Angular variables

For $SU(1,1|2)$ mechanics one can extend the dynamical content of the simplest $(1, 4, 3)$ supermultiplet by introducing angular variables providing some realization of $\mathfrak{su}(2)$ in a

purely group-theoretic way [9, 14]. It suffices to consider a phase space parametrized by the canonical pairs $(\theta^A, p_{\theta A})$, $A = 1, \dots, n$, which obey the conventional Poisson brackets

$$\{\theta^A, p_{\theta B}\} = \delta^A_B \tag{5.1}$$

and realize on such a phase space the functions $J_a = J_a(\theta, p_\theta)$, $a = 1, 2, 3$, which obey the structure relations of the $\text{su}(2)$ R -symmetry subalgebra

$$\{J_a, J_b\} = \epsilon_{abc} J_c. \tag{5.2}$$

Then, the supersymmetry charges involve the angular variables only via the currents J_a ,

$$Q_\alpha = p \psi_\alpha + \frac{2i}{x} (\sigma_a \psi)_\alpha J_a - \frac{i}{x} \psi_\alpha (\bar{\psi} \psi), \tag{5.3}$$

where $(\sigma_a)_\alpha^\beta$ are the Pauli matrices [14].

Let us try to generalize this construction to the case of the superconformal algebra $\text{su}(1, 1|n)$. Introducing functions J_a of the angular variables subject to the $\text{su}(n)$ structure relations $\{J_a, J_b\} = f_{abc} J_c$, and employing the matrices $(\lambda_a)_\alpha^\beta$ from (2.3), it is straightforward to verify that the obvious candidate supersymmetry charge,

$$Q_\alpha = p \psi_\alpha + \frac{2i}{x} (\lambda_a \psi)_\alpha J_a - \frac{i}{x} \psi_\alpha (\bar{\psi} \psi), \tag{5.4}$$

is indeed nilpotent, i.e.

$$\{Q_\alpha, Q_\beta\} = 0. \tag{5.5}$$

However, in view of the properties of the λ -matrices in (2.3), the bracket of Q_α with \bar{Q}^β yields not just the Hamiltonian:

$$\{Q_\alpha, \bar{Q}^\beta\} = -2i H \delta_\alpha^\beta - \frac{4i}{x^2} (\lambda_a)_\alpha^\beta d_{abc} J_b J_c, \tag{5.6}$$

where d_{abc} are the symmetric structure coefficients appearing in (2.3). This means that the algebra does not close. One might try to modify the troublesome second term in Q_α . However, such a term seems indispensable for providing the structure relations (3.8).

We thus conclude that an extension of the $(1, 2n, 2n-1)$ supermultiplet by angular variables in a way similar to the $\text{su}(1, 1|2)$ case is problematic. Perhaps a more sophisticated construction involving extra auxiliary variables will help to circumvent the problem.

6 Discussion

To summarize, in this work we made the first step towards a systematic description of $\text{SU}(1, 1|n)$ multi-particle superconformal mechanics. Our consideration was primarily focused on the possibilities offered by the Hamiltonian formalism. The structure relations of the superconformal algebra $\text{su}(1, 1|n)$ were established in a form analogous to the previously studied $\text{su}(1, 1|2)$ case. A representation of $\text{su}(1, 1|n)$ on the phase space spanned by m copies of the $(1, 2n, 2n-1)$ supermultiplet was constructed. It was shown that the dynamics is governed by two prepotentials V and F , and that the WDVV equation for F

arises as a consequence of a more restrictive fourth-order equation. Solutions to the latter in terms of root systems allow decoupled models only. An attempt to extend the dynamical content of the $(1, 2n, 2n-1)$ supermultiplet by adding angular variables in a way similar to the $\text{su}(1, 1|2)$ case compromised the closure of the $\text{su}(1, 1|n)$ superconformal algebra. Hence, our results indicate that the construction of interacting $\text{SU}(1, 1|n)$ models with $n > 2$ appears to be a more difficult task than in the $\text{SU}(1, 1|2)$ case.

The Hamiltonian formulation adopted in this work automatically yields on-shell models. It is tempting to investigate $\text{SU}(1, 1|n)$ mechanics off-shell within the superfield approach combined with the method of nonlinear realizations, along the lines proposed in [32]. The key problem within the superfield method will be to guess the superfield constraints which will result in interacting dynamics. A possible link of $\text{SU}(1, 1|n)$ mechanics to the near-horizon Myers-Perry black hole with equal rotation parameters is worth studying as well. Finally, it might be rewarding to investigate the integrability of (3.5) on its own, which is more special than the WDVV equation.

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