Conformal Ghosts on the Sphere

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Zusammenfassung

In dieser Arbeit behandele ich die Verbindung von Geometrie und logarithmisch konformen Feldtheorien. Dabei betrachte ich zwei verschiedene geometrische Situationen: in Teil I das topologische A-Modell mit Einbettungsabbildung $x\colon \mathbb{R}\times S^1 \to \mathbb{CP}^1$ und in Teil II konforme, fermionische Geister auf dem Torus.

Das A-Modell lässt sich in eine Form bringen, in der das Pfadintegral eine δ-Distribution auf dem Modulraum der Instantonen ist. Integriert man die Abhängigkeit von S^1 heraus, erhält man eine Morsetheorie auf der universellen Überlagerung $\widetilde{L\mathbb{CP}^1}$ des Loop-Raumes. Deren Niedrigenergie-Zustandsräume lassen sich in Zellen dieser Mannigfaltigkeit störungstheoretisch bestimmen und durch Darstellungsräume des Chiralen de Rham-Komplexes beschreiben. Unter der Annahme, dass die Darstellungstheorien des A-Modelles und des Chiralen de Rham Komplexes übereinstimmen, betrachte ich im Folgenden den Chiralen de Rham-Komplex. Die Zustandsräume sind lokale, induzierte Darstellungen der Symmetrie, die durch das Gradientenfeld der Morsefunktion erzeugt wird. Entsprechend einer Hypothese von E. Frenkel, A. Losev und N. Nekrasov führt eine Verallgemeinerung dieser lokalen Darstellungen als Distributionen auf $\widetilde{L\mathbb{CP}^1}$ zu quantenexakten Zuständen der Theorie. Auf diesen Zuständen muss der Hamiltonoperator durch zusätzliche Terme korrigiert werden. Ich diskutiere die Darstellungstheorie der quantenexakten Zustände und bestimme die Deformationsterme des Hamiltonoperators. Ich zeige, dass diese eine geometrische Deutung als Kohomologieoperatoren in einem Komplex global erweiterter lokaler Darstellungsräume haben. Zuletzt zeige ich, dass den zusätzlichen Termen im Hamiltonoperator der Morsetheorie eine logarithmische Erweiterung des chiralen de Rham-Komplexes entspricht.

Die konformen, fermionischen Geister aus Teil II transformieren sich in irreduziblen Darstellungen der Monodromiegruppe $\mathbb{Z}_2$. Ich zeige, dass die durch sie beschriebene konforme Feldtheorie logarithmisch erweitert werden muss, sobald man zu den Darstellungen der Monodromiegruppe Felder assoziiert, die sich frei auf dem Parameterraum $\mathbb{CP}^1 \setminus \{0,1,\infty\}$ bewegen. Das Tripletmodell stellt eine minimale logarithmische Erweiterung dieser Theorie dar und bildet die Grundlage meines letzten Kapitels. Darin drücke ich die spektrale Kurve der $SU(2)$-Seiberg-Witten Theorie durch die Charaktere des Tripletmodelles aus, und führe ebenfalls das Präpotential auf dieses Modell zurück, indem ich es als Funktion des Modulus der spektralen Kurve gewinne.

Schlagworte: Nichtlineares Sigma Modell, Logarithmisch Konforme Geister, Seiberg Witten Theorie

Abstract

This thesis is about the relation of geometry and logarithmic conformal field theories. I consider two different geometric settings: in part I the topological A-model with embedding $x\colon \mathbb{R}\times S^1 \to \mathbb{CP}^1$, and in part II conformal, fermionic ghosts on the torus.

The A-model can be transformed such that the path integral yields a δ distribution on the moduli space of instantons. Integrating out the dependency on S^1, one obtains Morse theory on the universal cover $\widetilde{L\mathbb{CP}^1}$ of loop space. Its low-energy state space can be derived perturbatively in cells of this manifold, and can be modelled by the representations of the chiral de Rham complex. Assuming that the representation theory of the A-model and the chiral de Rham complex are identical, I consider the chiral de Rham complex in the following. The state spaces are local, induced representations of the symmetry generated by the gradient vector field of the Morse function. According to a conjecture of E. Frenkel, A. Losev and N. Nekrasov, a generalization of these local representations as distributions on $\widetilde{L\mathbb{CP}^1}$ leads to nonperturbative states of the theory. On these states, the Hamiltonian must be corrected by additional terms. I discuss the representation theory of the nonperturbative states and determine the terms which deform the Hamiltonian. They have a geometric significance as cohomology operators in a complex of globally extended local representation spaces. Eventually, I prove that a logarithmic extension of the chiral de Rham complex corresponds the additional terms in the Hamiltonian.

The conformal, fermionic ghosts of part II transform in irreducible representations of the monodromy group $\mathbb{Z}_2$. I show that the conformal field theory of these fields has to be logarithmically extended as soon as the representations of the monodromy goup are allowed to move freely on the parameter space $\mathbb{CP}^1 \setminus \{0,1,\infty\}$ of the torus. The triplet model constitutes a minimal logarithmic extension of this theory and is fundamental for my last chapter. Therein I obtain the spectral curve of $SU(2)$ Seiberg-Witten theory in terms of characters of the triplet model. Further, I trace back the prepotential to that model by expressing it as a function of the torus modulus of the spectral curve.

Keywords: Nonlinear Sigma Model, Logarithmic Conformal Ghosts, Seiberg Witten Theory

Contents

Notations

$*$	Field-state-correspondence, 93
$\prec$	$A \prec B$: the set A is a subset of codimension one in the closure of B.
$\tilde{1}(z)$	The logarithmic partner of $1(z)$, 100
$\Delta_T(\phi)$	Conformal weight of ϕ with respect to T
δ	Grothendieck-Cousin operator, 32
$[\cdot,\cdot]$, $\{\cdot,\cdot\}$	Graded commutator, anticommutator
$[f,g]_n(z)$	Field in the operator product expansion, 72
$\coprod$	Disjoint union
$\lvert\tilde{0}\rangle$	Logarithmic partner of $\lvert 0\rangle$, 100
$\lvert p\rangle_\pm$,$\lvert p,\bar{p}\rangle_\pm$, $\lvert p,\bar{p}\rangle$	Charged representations of the CSbc, 47, 49, 49
$\mathscr{A}^\epsilon$	Representation space of the Heisenberg Lie algebra, 61
CSbc	Conformal supersymmetric bc-system, 46
$\mathbb{C}_0$	$\mathbb{CP}^1 \setminus \{\infty\}$
$\mathbb{C}_\infty$	$\mathbb{CP}^1 \setminus \{0\}$
$\mathbb{C}^\times$	As a set $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, as a symmetry cf. pg. 20
$\mathbb{C}[\cdot]$	Polynomials
$\mathbb{C}((\cdot))$	Formal power series
$\mathbb{C}[[\cdot]]$	Power series
$D,\ D^*$	Complex unit disk with/without the point $\{0\}$
$\mathscr{D}$, $\mathscr{D}^*$	Test functions, distributions, 23
$\mathscr{D}_0$, $\mathscr{D}_\infty$	Test functions with compact support in $\mathbb{C}_0$, $\mathbb{C}_\infty$
$e,\ \bar{e}$	Extension of the perturbative representations, 34,
	extension field, 72
$\mathscr{E}_\lambda$	Legendre family, 95
$\mathscr{F}_0$, $\mathscr{F}_\infty$, $\mathscr{F}^\times$, $\mathscr{F}^1_\infty$	Holomorphic representations of the CSbc, 58, 59
$\underline{f}$	(Logarithmically) extended field, 30, 72
GCO	Grothendieck-Cousin operator
$\mathfrak{g}$, $\mathfrak{g}_\mathscr{O}$	GCO 32, nontrivial part of $\underline{\mathscr{O}} = O + \mathfrak{g}_\mathscr{O}$ 30
$\underline{\mathscr{H}}$	Globally defined states, 28
H_λ	Non-unitary Hamiltonian, 14
$H^{(\mathrm{pert})}$	Perturbative Hamiltonian, 18
$\mathscr{H}^{\mathrm{in}}_{c,n}$, $\mathscr{H}^{in}_{0,0}$, $\mathscr{H}^{\mathrm{in}}_{\infty,0}$	Perturbative spectrum of the Tbc, 44, 45, 46
$\mathscr{H}_0$	— " — Morse theory/CSbc, 22, 51
$\mathscr{H}_\infty$	— " — , 22, 51
Homogeneity r	A prefactor of $\lvert z\rvert^r,\ r \in \mathbb{R}$ in the field expansion, 45, 51
iff	"if and only if"

j^ϵ	Currents of the CSbc, 48
$j^\epsilon_V, j^\epsilon_A$	Vectorial and axial currents of the CSbc, 48
$J^\epsilon_\mu,\ J^\epsilon$	Currents of the Heisenberg Lie algebras, 61
$\mathfrak{J}^-$	Current of the bosonized bosons, 64
J_N	Current measuring the grading of $N(p,p)$, 65
$J(\phi)$	Charge of the field ϕ, 50
$\Lambda^{a,b}$	Basis of exterior forms, 23
LX	Loop space of X, 39
$\widetilde{LX}$	Universal cover of LX, 40
$\widetilde{LX}_n$	Sheet of LX in $\widetilde{LX}$, 41
$\widetilde{LX}_{c,n}$	Descending manifold with critical point x_c, 44
$M^\epsilon(p)\ ,\ \bar{M}^\epsilon(p)\ ,\ M^\epsilon(p,\bar{p})$	Charged representations of the CSbc, 48, 49
$\mathscr{M}_{\mathscr{E}}$	Parameter space of the Legendre Family 96
$\mathscr{M}(\alpha,\beta)$	Instanton moduli space, 10
μ	The A-model "gauge"-fieldstrenght 41, twist field on the torus 100
$v^\epsilon_{p,\bar{p}}$	Highest weight state of the Heisenberg Lie algebra, 61
$N(p)\ ,\bar{N}(\bar{p})\ ,\ N(p,\bar{p})$	Extended representations of the bosonized bosons, 64
$\overline{N}(p),\ \bar{\overline{N}}(\bar{p})\ ,\overline{N}(p,\bar{p})$	Perturbative representations — " — , 66
$N_L(p,\bar{p})$	Logarithmic extension of $\overline{N}(p-1,p-1)$, 67
$\mathscr{O}^{(\text{naive})}$	Naive operator, 30
OPA	Operator Product Algebra
OPE	Operator Product Expansion
PER	Physically Eligible Representation, 102
$\mathscr{P}_0$	Polynomial of the field modes in the CSbc, 54
$p,\ \bar{p}$	In part II, projection $\mathbb{T}^{n,m} \to \mathbb{CP}^1 \setminus \{e_i\}$, 89
$\mathfrak{q}$	Background charge, anomaly, 49
Q, Q_0	BRST charge in Morse theory, 12, in the Tbc, 37, 53
$Q(z,\bar{z})\ ,\ \mathscr{Q}(z)$	Superchargefield, 49, its holomorphic part, 48
SQM	Super quantum mechanics
Tbc	Topological bc-system, ungauged 38, gauged 42
$V^\epsilon(r,z)$	Fields of the Heisenberg Lie algebra, 62
X_n	Subspace $X_n \simeq X$ in $\widetilde{LX}_n$, 44
$X_{c,n}$	$X_n \cap \widetilde{LX}_{c,n}$, 44

Introduction 1

This thesis was initiated by my interest in the relation between geometry and physics. It was since I got to know the publication of V. G. Knizhnik [Kni87] that I wanted to investigate the geometric significance of the aspects which render a conformal field theory logarithmic.

Knizhnik considers holomorphic differential forms on algebraic surfaces which are branched coverings of $\mathbb{CP}^1$ and have a global $\mathbb{Z}_n$ monodromy group. The differential forms can be identified with conformal fermionic ghosts, and the monodromy group has an induced action on these fields, which thus fall into n irreducible representations. In the spirit of conformal field theory (CFT), these representations are realized by locating the conformal fields isomorphic to the respective highest weight vectors at the branch points. In mathematical terms, this amounts to restricting the differential forms to a neighborhood of a branch point and to considering representation theory thereon.

If the algebraic surface has branch points $e_i,\ i \in \{1,\ldots,2N\}$, $N \geq 2$, one may turn the surface into a family of topologically equivalent surfaces by allowing $2N-3$ branch points to vary over $\mathbb{CP}^1 \setminus \bigcup_{i=1}^{2N-3}\{e_i\}$. This helps to extract further geometric information, such as degeneracies when branch points are fusing, or periods, which satisfy differential equations with respect to the floating parameters.

Although my investigations started with the work of Knizhnik, I will discuss this setting in the second part of my thesis. There, I will consider the CFT realization of both, degeneracies and periods for the algebraic surface being a torus. The differential equation for its periods is realized as the nullstate condition for the odd representation of the monodromy group $\mathbb{Z}_2$. Therefore, the four-point function of the so-called twist field corresponding to this representation is proportional to the periods of the torus. In particular, it contains logarithms and the fusion of two branch points, which is simulated by the operator product expansion (OPE) of two such fields, yields a doublet representation of the symmetries of the conformal fermionic ghost system. The Hamiltonian is not diagonalizable on this doublet, which signifies that the CFT has to be extended to a logarithmic conformal field theory (LCFT). The minimalistic way to do this will lead to the triplet model, as explained by M. Flohr in [Flo98].

This setting has been the starting point for my publication with M. Flohr [VF07]. As the torus is the spectral curve of pure gauge, $SU(2)$ Seiberg-Witten theory, we wanted to express the prepotential in terms of characters of the triplet model. Although we only obtained the prepotential in terms of the torus modulus, which equals the ratio of twist field four-point functions, we have been able to determine the spectral curve by means of such characters. This will be the subject of chapter 9 in part II.

The origin for my second main project, described in part I of this thesis, is the work of E. Frenkel, A. Losev and N. Nekrasov [FLN06, FLN08], who investigated Morse theory and the topological A-model beyond their topological sectors. What is implied by those considerations?

(Cohomological) topological field theories deal with global geometric objects on manifolds, in particular with diffeomorphism invariants that are in the cohomology of some nilpotent operator Q, called Becchi-Rouet-Stora-Tyutin (BRST) charge due to its properties. It has an action on the fields and state spaces of the theory and the elements in its cohomology classes comprise what is called the topological sector of a field theory.

Under certain circumstances a field theory has in addition to its topological sector further "dynamical" states and observables. While the cohomology of Q is invariant under diffeomorphisms, this is not the case for the dynamical sector. Hence, the dynamical degrees of freedom should in principle describe part of the local geometry of the target or domain manifold.

In [FLN06], Frenkel, Losev and Nekrasov consider the situation described above for Morse theory with a first order Lagrangian on a Kähler manifold X with scaled metric $\lambda g,\ \lambda \in \mathbb{R}^{>0}$. The perturbative spectrum of this theory includes topological as well as dynamical states. If X is supplemented with an additional structure, these states have their support on the descending manifolds of the gradient vector field of the Morse function. Moreover, the submanifolds yield a disjoint cover of X, and so do the perturbative state spaces.

The local geometry of X can be accessed employing the dynamical states. For $\lambda \to \infty$, the Hamiltonian becomes the Lie derivative in direction of the gradient vector field of the Morse function. The perturbative state spaces which survive that limit turn into locally defined induced representations of the symmetry generated by the gradient field. This is, metaphorically, what an observer located on a descending manifold would expect to see. However, Frenkel, Losev and Nekrasov claim that there are nonperturbative effects through which the observer obtains additional insights into the local representations of the Hamiltonian on X . They propose that the nonperturbative state spaces are obtained by extending the perturbative state spaces as distributions to X and their analysis shows that the thus globalized representations are the local cohomology groups in a complex called the global Grothendieck-Cousin complex, [Kem78]. This complex has a cohomology operator, the Grothendieck-Cousin operator (GCO), which compounds the local representation spaces and appears as an additional term in the Hamiltonian. The observer is thus confronted with a Hamiltonian which can not be diagonalized on all dynamical states – a situation well known in the theory of logarithmic CFTs.

My initial interest in the work of Frenkel, Losev and Nekrasov [FLN06] arose from their proposal that the topological A-model in the large volume limit is an LCFT beyond its topological sector. In [FLN08], they reduce the A-model with embedding $x\colon \mathbb{R}^1 \times S^1 \to \mathbb{CP}^1$

to the Morse theory of [FLN06] by integrating out the dependence on S^1. In particular, one can derive the perturbative state spaces and it appears that they can be modelled by representation spaces of the conformal supersymmetric ghosts (CSbc) with target space $\mathbb{CP}^1$. It is now suggestive to assume that at least the representation theory of the A-model in the large volume limit equals that of the CSbc and the theories can, accordingly, be substituted.

Furthermore, Frenkel, Losev and Nekrasov propose the deformation of the Hamiltonian, but do not analyze the extension of the representation spaces in detail. Moreover, in order to prove their conjecture that the A-model is an LCFT in the large volume limit and beyond its topological sector, it is not sufficient to consider the underlying Morse theory. A logarithmic deformation of the CSbc has to be found, which yields the correct extensions of the perturbative representation spaces and adds the deformation terms to the Hamiltonian. It is only then, that the Grothendieck-Cousin operators can be interpreted as the zero modes of the logarithmic improvement terms which deform the energy momentum tensor. Parts of those considerations have been addressed in my second publication with M. Flohr [VF09].

As mentioned above, this thesis has two parts, the first treats the logarithmic extension of the CSbc underlying the A-model, the second is about fermionic ghosts on the torus and their relation to Seiberg-Witten theory. Before I start with an outline, I will briefly comment on the appendix, which serves to supplement the main part. In appendix A I summarize and specify the basic ingredients of a topological field theory [BBRT91, Wit82, Wit88a, Wit88b]. In appendix B.1 I briefly explain how the topological A-model is obtained by twisting an $\mathcal{N}=2$ supersymmetric sigma model and note down the supersymmetry of this theory [Mar05]. The last appendix C is the foundation of another publication, wherein I study the possibility to generalize the approach of Frenkel, Losev and Nekrasov [FLN08], by which they deform the Hamiltonian of the A-model, to a deformation of the associated CSbc.

Part I In the following chapter 2, I will start with a discussion of Morse theory. Therein, the geometric origin of the deformation operators is discussed and the conditions on the target space manifold are fixed. This chapter follows the publication of Frenkel, Losev and Nekrasov [FLN06], but some subtle points are treated in more detail. In particular this concerns the extension of the perturbative representation spaces. I will propose an alternative ansatz for the extension, which relies on a principle by which I can enlarge the representation spaces. This ansatz is applicable in the context of the A-model.

In chapter 3, I will introduce the A-model with target space $\mathbb{CP}^1$ and take the large volume limit. Reducing the thus obtained theory to Morse theory, I will derive the perturbative state spaces and explain why they can be modelled by the CSbc. Because the A-model is defined on $\mathbb{CP}^1$, it is necessary to make chart transitions. For the CSbc, these transitions are defined through the chiral de Rham complex, which I will also introduce. My method to derive the deformation of the Hamiltonian differs again from that of Frenkel, Losev and Nekrasov

[FLN08]. It relies crucially on bosonization, which I will discuss in detail. It will be important that the holomorphic and anti-holomorphic "halves" of the CSbc are considered together, not only because of anomalies occurring but also because the GCOs are composed of both parts. Indeed, I will explain that this composition constrains the representation spaces and the symmetries of the theory.

Having determined the perturbative representation spaces, their extensions, and the Grothendieck-Cousin operators that mediate between them, I will then move back from Morse theory to the conformal field theory. In chapter 4, I will use the method of Fjelstad et al. [FFH+02] to deform the CSbc logarithmically. I will do that in such a way that the representation spaces are extended consistently and that the GCOs are added to the Hamiltonian. This has an effect on the operator product algebra of the fields, but neither on the supersymmetry nor the conformal symmetry of the CSbc.

I will conclude this part of the thesis with a brief summary and discussion in chapter 5.

Part II In part two I will concentrate on the fermionic conformal ghosts on branched coverings of $\mathbb{CP}^1$ [Kni87]. After a brief motivation in chapter 6, I will specify the algebraic surfaces under consideration and introduce the conformal ghosts in chapter 7. Since they will have nontrivial operator product expansions in a neighborhood of a branch point it is necessary to extend the representation spaces by the representations of the monodromy group.

In the the subsequent chapter 8, I will derive by geometric arguments that the fermionic ghosts on the torus necessarily comprise a logarithmic conformal field theory. The minimal version is the triplet model [Flo98], which I will introduce in chapter 8.3.

In the last chapter 9, I will explain how the spectral torus of pure gauge Seiberg-Witten theory can be obtained from certain characters of the triplet model and note down an expression of the prepotential which is given completely in terms of quantities of this LCFT.

The thesis will be concluded with a summary and a discussion of open questions in the last chapter 10.

I

Supersymmetric Ghosts with Values on the Sphere

Morse Theory 2

This chapter has three parts. My starting point will be Morse theory on a general Riemannian surface X with scaled metric λg and symplectic form ω.

Firstly, I will prepare the topological sector of this theory by breaking CPT invariance and by making localization on the instantons explicit. This amounts to consecutively putting constraints on X. The constraints will be such that the instanton sectors are well defined and that the gradient field corresponding to the Morse function decomposes X into submanifolds, to each of which one can perturbatively associate a state space. Among those, there are excited states which are not scaled out in the large volume limit $\lambda \to \infty$.

Frenkel et al. proposed [FLN06] that the state spaces in the limit $\lambda \to \infty$, when generalized as distributions on X, comprise the nonperturbative low energy spectrum. In sections 2.4 and 2.5 I will discuss some consequences of this assumption for Morse theory on $\mathbb{CP}^1$, mainly following their publication but also with an additional discussion of the cohomology of the supercharge, as well as a different method for extending the state spaces as distributions. The most important observation will be that observables which include exterior derivatives are no longer diagonalizable on all states. In particular, this concerns the Hamiltonian and thus draws a similarity to logarithmic conformal field theories. Rather, those operators intermix the state spaces which formerly have been located in different charts.

Finally, I will discuss the physical and geometrical meaning of this sort of non-locality, which is due to the non-topological states.

This chapter will be concluded with a generalization of the toy model to a class of manifolds X and will be the basis for an understanding and analysis of the Morse theory underlying the topological A-model. My explanations rely mostly on [FLN06, BBRT91, Wit82].

2.1 The Path Integral Point of View

In terms of the structures just introduced, the Morse theory I will consider consists of a Riemannian surface X, a smooth embedding $x : \Sigma \subseteq \mathbb{R} \to X$, its Grassmann valued superpartner ψ and another Grassmann valued quantity π, which is the conjugate momentum of ψ. The Euclidean metric g on X is scaled by some parameter $\lambda \in \mathbb{R}^{>0}$ and, without loss of generality, I fix a connection D to be the Levi-Civita connection, defined with positive sign on $\frac{\partial}{\partial x^\mu}$: $D_\nu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\lambda} \Gamma^\lambda_{\nu\mu}$.

Let $f : X \to \mathbb{R}$ be Morse, i.e. single valued and with isolated critical points $x_c : df(x_c) = 0$, and denote further by $\mathrm{D}_t\psi^\mu = \frac{\mathrm{d}\psi^\mu}{\mathrm{d}t} + \Gamma^\mu_{\lambda\sigma}\frac{\mathrm{d}x^\lambda}{\mathrm{d}t}\ \psi^\sigma$ the pullback of D to Σ and by $\nabla^\mu f := g^{\mu\nu}\partial_\nu f$ the gradient of f. In local coordinates, the action I am interested in is

$$
\begin{aligned}
S_\lambda = \int_\Sigma \Big(\frac{1}{2}\lambda g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t} + \frac{1}{2}\lambda g^{\mu\nu}\partial_\mu f \partial_\nu f \\
+ \mathrm{i}\pi_\mu \nabla_t \psi^\mu - \mathrm{i}\pi_\mu \left(\mathrm{D}_\alpha \nabla^\mu f\right)\psi^\alpha + \frac{1}{2\lambda} R^{\mu\nu}_{\alpha\beta}\ \pi_\mu\pi_\nu\psi^\alpha\psi^\beta\Big)\mathrm{d}t.
\end{aligned}
\tag{2.1.1}
$$

In the following sections I will extract its topological sector, selecting either the instantons or anti-instantons and by specifying several conditions on X.

Since $\mathrm{d}f(x_c) = 0$, the Hessian $H(x)[\gamma] := \mathrm{D}_\gamma(\mathrm{d}f)(x)$, $\gamma \in T_xX$ does not depend on the choice of the connection at a critical point x_c. In local coordinates it reads $H_{\mu\nu}(x_c) = \partial_\mu\partial_\nu f(x_c)$. There exists a basis e_μ of tangent vectors at $T_{x_c}X$ in which it is diagonal with eigenvalues $\kappa_{c\,\mu}$: $H(x_c)\ e_\mu = \kappa_{c\,\mu}\ e_\mu$. The condition that the critical points are isolated is equivalent to the condition that $H(x_c)$ has no zero eigenvalues. Since the Hessian does not depend on the connection, it is reasonable to define an index for every critical point

$$
\mathrm{ind}(x_c) = \#\{\mu :\ \kappa_{c\,\mu} < 0\}\,,
\tag{2.1.2}
$$

which is a topological invariant.

In order to see what the classical solutions are, I will for a moment concentrate on the bosonic part. One can apply the so-called "Bogomlny trick" to find the absolute minima of the action:

$$
S_{\mathrm{bos}} = \int_\Sigma \left(\frac{\lambda}{2}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t} \mp \nabla^\mu f\right)^2 \pm \lambda \frac{\mathrm{d}f}{\mathrm{d}t}\right)\mathrm{d}t.
\tag{2.1.3}
$$

Since it was positive semi-definite before, I obtain a lower bound

$$
S_{\mathrm{bos}} \geq \left|\int_\Sigma \mathrm{d}f\right|,
\tag{2.1.4}
$$

which is satisfied by the gradient trajectories

$$
\frac{\mathrm{d}x^\mu}{\mathrm{d}t} \pm \nabla^\mu f = 0\,.
\tag{2.1.5}
$$

These are the classical bosonic solutions to $\delta S = 0$. There are three kinds, depending on the boundary conditions. The vacuum configurations are solutions of

$$
\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = 0 \quad \wedge \quad \nabla^\mu f(x) = 0\,,
\tag{2.1.6}
$$

which is satisfied by constant loops, i.e. the critical points x_c. If there exists more than one critical point, say $\{x_+, x_-\}$, there are also instanton $(-\nabla f\ )$ and anti-instanton configurations $(+\nabla f\ )$:

$$
\frac{\mathrm{d}x^\mu}{\mathrm{d}t} \pm \nabla^\mu f(x) = 0 \quad , \quad x(\pm\infty) = x_\pm
\tag{2.1.7}
$$

where w.l.o.g. I fixed some initial and final time. From (2.1.4) one can conclude that the instantons satisfy $f(x_+) > f(x_-)$ and the anti-instantons $f(x_+) < f(x_-)$.

2.1.1 Making CPT Breaking and Localization Manifest

The anti-instantons can be excluded from the classical minima by subtracting $\lambda \int \mathrm{d}f$ from the action (2.1.1). This term does not depend on the metric and is hence topological. It, however, breaks CPT invariance as one would expect for a theory without anti-instantons.[1]

In order to make the localization property manifest, I massage the action $S - \lambda \int \mathrm{d}f$ into a first order form, by introducing a Lagrangian multiplier p_μ. Viewed as part of the integration kernel $\exp\{-S\}$ in the path integral, I may now consider, equivalently to (2.1.1):

$$\begin{aligned} S_\lambda = \int_\Sigma \Big(&-\mathrm{i} p_\mu \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t} - g^{\mu\nu} \partial_\nu f \right) + \frac{1}{2\lambda} g^{\mu\nu} p_\mu p_\nu \\ &+ \mathrm{i}\pi_\mu \big(\mathrm{D}_t \psi^\mu - (\mathrm{D}_\alpha \nabla^\mu f) \psi^\alpha \big) + \frac{1}{2\lambda} R^{\mu\nu}_{\alpha\beta}\, \pi_\mu \pi_\nu \psi^\alpha \psi^\beta \Big) \mathrm{d}t . \end{aligned} \tag{2.1.8}$$

In the limit $\lambda \to \infty$, the integral kernel turns into a δ distribution on instanton moduli space, which makes localization explicit. Indeed, for finite λ, the instantons still contribute with a weight factor $\mathrm{e}^{-2\lambda |f(x_+) - f(x_-)|}$ to correlation functions, but for $\lambda \to \infty$ their contribution disappears. On the contrary, the instantons contribute with a constant weight factor 1 for any value of λ.

Let $v^\mu(x) := \nabla^\mu f(x)$ be the vector field associated with f and $p'_\mu := p_\mu + \Gamma^\lambda_{\mu\nu} \psi^\nu \pi_\lambda$. The action in the large volume limit can now be written as:

$$S_\infty = -\mathrm{i} \int_\Sigma \left(p'_\mu \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t} - v^\mu \right) - \pi_\mu \left(\frac{\mathrm{d}\psi^\mu}{\mathrm{d}t} - \psi^\alpha \partial_\alpha v^\mu \right) \right) \mathrm{d}t . \tag{2.1.9}$$

It is invariant under the following susy transformations

$$\begin{array}{ll|ll} [Q, x^\mu] = \psi^\mu, & [Q, \psi^\mu] = 0 & [Q^*, x^\mu] = 0, & [Q^*, \psi^\mu] = v^\mu \\ [Q, \pi_\mu] = p'_\mu, & [Q, p'_\mu] = 0 & [Q^*, \pi_\mu] = 0, & [Q^*, p'_\mu] = 0 \end{array} \tag{2.1.10}$$

and moreover, the Lagrangian is Q-exact, $L = -\mathrm{i}[Q, \pi_\mu \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t} - v^\mu \right)]$ and thus is the Hamiltonian.

This is roughly the model I am going to consider. However, I will need some more informations on the instanton moduli space, especially in order to find constraints on the target manifold. There will be serveral obstacles which have to be resolved and I will list them up, whenever I encounter one. In the following and for convenience, I will leave away the prime for p'_μ.

2.1.2 The Instanton Moduli Space

The instanton equation $\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = v^\mu(x)$ gives rise to a symplectormorphism of X, i.e. $L_v \omega = 0$:

$$\phi_v : \ X \times \Sigma \to X \qquad x \mapsto \phi_v(x, t) = x(t) , \tag{2.1.11}$$

[1] Though for the model under consideration CPT is really CT, I will follow the terminology of Frenkel, Losev and Nekrasov [FLN06]. For a more detailed discussion of CPT breaking, c.f. section 2.2.4.

where $x(t)$ is an instanton solution and $\phi_v(\cdot,t)$ determines a one parameter group in t. By means of this flow equation of v one can try to find a partition of X into submanifolds which is generated by the fixed points of v. These will be the descending X_c and ascending manifolds X^c :

$$X_c^{(c)} := \left\{ x \in X \ : \ \lim_{t \to \overset{(+)}{-}\infty} \phi_v(x,t) = x_c \right\}. \tag{2.1.12}$$

If x_c is a nondegenerate critical point and ϕ_v a diffeormorphism, they are indeed submanifolds [AR67, pg. 87f] and inherit the tangent spaces defined by the flow lines.

For the following reason I demand that a decomposition of X into descending and ascending manifolds exists. In section 2.2.4 I will explain that the state spaces will be localized around the fixed points of v. A decomposition of X in terms of, say, descending manifolds is useful because one can then canonically associate to each such submanifold a state space $\mathscr{F}_\alpha$ and these cover X. Therefore:

❏ The target manifold X has a (Bialynicki-Birula) decomposition
$X = \bigsqcup_{\alpha\in\mathscr{A}} X_\alpha = \bigsqcup_{\alpha\in\mathscr{A}} X^\alpha$ with respect to v.

The instanton moduli spaces are defined by means of descending and ascending manifolds

$$\mathscr{M}(\alpha,\beta) := X_\alpha \cup X^\beta, \tag{2.1.13}$$

and under further conditions it is possible to calculate the dimension of this moduli space. Let x_c be a critical point, I can choose local coordinates such that it is located at the origin. In its neighborhood I can approximate a solution of the instanton equation by a line element $y = x_c + x$ and by making a Taylor expansion around the critical point. This yields to lowest order $\mathrm{d}_t x^\mu - H^\mu_\nu(0)x^\nu = 0$, whith Hessian H evaluated at $x_c = 0$. Thus, locally around the fixed point, the directions along which H has positive eigenvalues span the tangent space of the descending manifold while the others span the tangent space of the ascending manifold. Therefore, at least in a neighborhood of a fixed point x_c, $TX_c \simeq \mathbb{R}^{\dim X - \mathrm{ind}(x_c)}$ or $\simeq \mathbb{C}^{\dim_{\mathbb{C}} X - \frac{1}{2}\mathrm{ind}(x_c)}$ while for the ascending manifold $TX^c \simeq \mathbb{R}^{\mathrm{ind}(x_c)}$ or $\simeq \mathbb{C}^{\frac{1}{2}\mathrm{ind}(x_c)}$. The generalization of this condition is as follows:

❏ Let $(f, X, \lambda g)$ allow for Morse-Smale transversality, i.e.
$\forall \ x \in \mathscr{M}(\alpha,\beta), \ \forall \ \alpha,\beta \ : \ \dim \ T_x X_\alpha + \dim \ T_x X^\beta - \dim \ X = \dim \ \left(T_x X_\alpha \cup T_x X^\beta\right)$.

One can now calculate

$$\dim_{\mathbb{R}} \mathscr{M}(\alpha,\beta) = \mathrm{ind}(\beta) - \mathrm{ind}(\alpha). \tag{2.1.14}$$

The Morse-Smale condition yiels a nice description of the tangent spaces of X in terms of instanton flow lines. Especially the dimensions of the instanton moduli spaces are natural numbers including zero, restricted by the dimension of the target manifold, and there are no dimensional degeneracies. Since it is expressed by the Morse indeces, the dimension of the instanton moduli space is a topological invariant. Morse-Smale transversality does further restrict the flow lines to move from fixed points with lower to fixed points with higher Morse index.

There is another, physically inspired way to calculate the dimension of the instanton moduli space [H⁺03, sec. 10.5.2]. Consider an instanton solution $x\colon\ \mathrm{d}_t x^\mu - v^\mu(x) = 0,\ x^\mu(-\infty) = x^\mu_\alpha,\ x^\mu(\infty) = x^\mu_\beta$. Again, I will move in the solution space of this differential operator to another solution $y = x + \eta z$, where $\eta > 0$ is an infinitesimally small number. The curve y is an instanton solution if the displacement z satisfies $\mathrm{D}_- z := (\ \mathrm{d}_t - H(x(t))\ )\, z = 0,\ z(\pm\infty) = 0$ to the order η. For every t I may choose a basis of eigenvectors of $H(x(t))$ with eigenvalues $\kappa_\mu(t)$ which spans the tangent space $T_{x(t)}X$. The operator D_- is diagonal in this basis and has homogeneous solutions

$$z_\mu(t) = e_\mu \exp\Big(\int_0^t \kappa_\mu(\tau)\mathrm{d}\tau\Big)\,, \tag{2.1.15}$$

where e_μ diagonalizes D_- at $t = 0$. These solutions have the correct boundary conditions if $\kappa_\mu(-\infty) > 0$ and $\kappa_\mu(\infty) < 0$.

There are two possible scenarios. The first is that the dimension of the solution space equals the dimension of the eigenspace of the Hessian. This is the case if none of the eigenvalues $\kappa_\mu(t)$ changes its sign from a negative to a positive value when passing from $t = -\infty$ to $t = \infty$. If this is satisfied, $\dim_{\mathbb{R}} \mathscr{M}(\alpha,\beta) = \mathrm{ind}(\beta) - \mathrm{ind}(\alpha) = \#\{\mu\colon\ \kappa_\mu(-\infty) > 0,\ \kappa_\mu(\infty) < 0\} = \dim\ker \mathrm{D}_-$. In the second scenario there exist eigenvalues which change their signs from negative to positve value. They belong to homogeneous solutions of the differential operator $\mathrm{D}_+ := \mathrm{d}_t + H(x(t))$. In that general case, the difference $\mathrm{ind}(\beta) - \mathrm{ind}(\alpha)$ can be written as

$$\dim_{\mathbb{R}} \mathscr{M}(\alpha,\beta) = \dim\ker \mathrm{D}_- - \dim\ker \mathrm{D}_+\,. \tag{2.1.16}$$

The operators $\mathrm{D}_\mp$ appear in the equations of motion for the fermions ψ^μ and π_μ, respectively. Under the assumption that the dimension of the instanton moduli space equals $\dim\ker \mathrm{D}_-$, it further equals the number of linear independent solutions of $\mathrm{D}_- \psi_{0,l} = 0,\ l = 1\ldots d,\ d = \dim \mathscr{M}(\alpha,\beta)$, whereas π_μ has no “zero modes”. This leads to the selection rule that observables have to contain a product $\prod_{l=1}^d \psi_{0,l}$, if the correlation function is not to be zero. The reason is that the path integral is a δ distribution on the homogeneous solutions of D_- and the instanton configurations x_0

$$\langle \mathscr{O} \rangle = \int_{\mathscr{M}(\alpha,\beta)} \prod_{l=1\ldots d} \psi_{0,l}\, \mathscr{O}|_{\mathscr{M}(\alpha,\beta)}\,. \tag{2.1.17}$$

An integral over Grassmann variables is zero if the integrand is not a volume form, and in the next section I will make clear that, indeed, the zero modes of ψ have a geometric meaning as differentials on X. From the discussion above I conclude that they are physically signifying the presence of instantons, and the number of fermionic insertions counts the dimension of their moduli space.[2]

[2] In the fermionic bc-system, that I will discuss in the next chapter, it will also be necessary to insert "zero-modes" in

2.2 The Canonical Point of View

The Morse action (2.1.9) has an immediate interpretation in terms of geometric quantities of the target manifold X. The best place to understand this is the canonical formulation of the theory. Reshuffeling the terms in (2.1.9), I can read off the classical Hamiltonian in the large volume limit[3]

$$H_\infty = v^\mu \ (\mathrm{i}p_\mu) + \psi^\alpha \partial_\alpha \ v^\mu (\mathrm{i}\pi_\mu)\,. \tag{2.2.1}$$

Reconsidering (2.1.10), an immediate choice how to quantize consists in relating the "field"-coordinates with geometric quantities in the following way:

$$\begin{array}{ll|ll} \text{bosons:} & & \text{fermions:} & \\ \hline x^\mu & x^\mu & \psi^\mu & \mathrm{d}x^\mu \\ \mathrm{i}p_\mu & \partial_\mu & \mathrm{i}\pi_\mu & \iota_\mu \end{array} \tag{2.2.2}$$

The Hamiltonian above and the supercharges Q and Q^* can now be rewritten as

$$Q = \mathrm{d}\,, \quad Q^* = \iota_v\,, \quad H_\infty = \mathscr{L}_v = \{Q, Q^*\}\,, \tag{2.2.3}$$

and they have a canonical action on differential forms on X. The geometric data satisfy the usual quantization rules $[p_\mu, x^\nu] = -\mathrm{i}\delta^\nu_\mu$, $[\pi_\mu, \psi^\nu] = -\mathrm{i}\delta^\nu_\mu$ for the superbracket, and in particular

$$Q = \mathrm{i}\psi^\mu p_\mu\,. \tag{2.2.4}$$

In the following I will reproduce the deformations described for the path integral ansatz for the canonical formalism of Morse theory. The idea behind this is to see what the spectrum of the Hamiltonian in the large volume limit looks like and to investigate if there remain well defined exited states in this limit. I will again start with the action (2.1.1) before taking the large volume limit and the target manifold $(X, \lambda g)$, endowed with an inner product on differential forms $\eta, \chi \in \Omega^\bullet(X)$

$$\langle \eta, \chi \rangle := \int_X (\star\ \bar{\eta}) \wedge \chi\,. \tag{2.2.5}$$

The bar denotes complex conjugation, if necessary, and $\star$ the Hodge operator.[4] The Hamiltonian corresponding to the action (2.1.1) with Morse function f is obtained from the

correlation functions. These do, however, not represent instantons because they are mappings between isomorphic representation spaces, cf. section 3.4.1 and section 8.3. On the contrary, instantons relate different vacuum configurations (they are highest weight vectors of different representations).

[3]This classical Hamiltonian is not bounded from below. However, in section 2.4, I will derive it from the canonically quantized Hamiltonian with $\lambda \neq 0$ by deforming the spectrum in a specific way, cf. [FLN06]. Thereby one obtains states which are not in the closure of $\Omega^\bullet_{\mathrm{d}}(X)$ with respect to the L^2 norm, but on which one can define an orthogonal pairing and whose eigenvalues with respect to the canonically quantized H_∞ are positive semidefinit (when considered perturbatively, c.f. section 2.5). Analogous will be satisfied for the A-model.

[4]On volume elements $\star\ \mathrm{d}x^{\mu_1} \wedge \cdots \mathrm{d}x^{\mu_k} = \frac{\sqrt{|g|}}{(\dim_{\mathbb{R}} X - k)!} \epsilon^{\mu_1 \cdots \mu_k}{}_{\nu_{k+1} \cdots \nu_{\dim X}}\ \mathrm{d}x^{\nu_1} \wedge \cdots \mathrm{d}x^{\nu_k}$ and $\epsilon_{\mu_1 \cdots \mu_{\dim_R X}} = +1$ for even permutations.

supercharges

$$
\begin{aligned}
Q &= \mathrm{d}_\lambda = \mathrm{e}^{-\lambda f}\mathrm{d}\mathrm{e}^{\lambda f} = \mathrm{d} + \lambda\ \mathrm{d}f\wedge\,,\\
Q^\dagger &= \mathrm{d}_\lambda^\dagger = \mathrm{e}^{\lambda f}\mathrm{d}^\dagger\mathrm{e}^{-\lambda f} = \frac{1}{\lambda}\mathrm{d}^\dagger + \iota_{\nabla f}\,,
\end{aligned}
\tag{2.2.6}
$$

as

$$
H = \Delta_\lambda = \frac{1}{2}\{Q, Q^\dagger\} = \frac{1}{2}\left(\lambda^{-1}\Delta + \lambda\|\mathrm{d}f\|^2 + K_f\right), \tag{2.2.7}
$$

where, $\|\mathrm{d}f\|^2 = \iota_{\nabla f}\mathrm{d}f$, $K_f = \mathscr{L}_{\nabla f} + \mathscr{L}^\dagger_{\nabla f}$, $\mathscr{L}^\dagger_{\nabla f} = \{\mathrm{d}^\dagger, \mathrm{d}f\}$ and $\Delta = \{\mathrm{d}, \mathrm{d}^\dagger\}$. Conjugation $\dagger$ is defined with respect to the inner product. Let me emphasize, that up to now CPT is not broken and the two supercharges are indeed conjugate. However, in the large volume limit CPT will be violated and this makes the difference between the dagger and the star, for instance for the supercharge in (2.2.3).

2.2.1 On the Cohomology

As I explained in the introduction and in appendix A, the topological states are in the cohomology of the supercharge Q. Under certain conditions on X, that I will concentrate on in this section, the cohomology of Q is isomorphic to the kernel of the Hamiltonian.

The supercharges above are obtained by a similarity transformation of d and $\mathrm{d}^\dagger$, and I can hence carry over the results on the de Rham differential to the more general situation in Morse theory, in particular that $H^\bullet_{\mathrm{d}_\lambda}(X) \simeq H^\bullet_{\mathrm{d}}(X)$. If X is a real manifold which is moreover compact, oriented and without boundary, there exists a unique Hodge decomposition

$$
\Omega^k_{\mathrm{d}_\lambda}(X) = \mathrm{d}_\lambda\Omega^{k-1}_{\mathrm{d}_\lambda}(X) \oplus \mathrm{d}^\dagger_\lambda\Omega^{k+1}_{\mathrm{d}_\lambda}(X) \oplus \Omega^k_{\Delta_\lambda}(X), \tag{2.2.8}
$$

where $\Omega^k_{\Delta_\lambda}(X)$ denotes the harmonic forms on X with respect to $H = \Delta_\lambda$ [Nak03]. If such a decomposition exists and moreover an inner product like (2.2.5) one can show that $H^\bullet_{\mathrm{d}_\lambda}(X) \simeq \Omega^\bullet_{\Delta_\lambda}(X)$.[5] Thus, in order to identify the cohomology of the supercharge with the ground states of the Hamiltonian it would be sensible to invoke that whenever X is real, it should also be compact, oriented and without boundary.

If X is a compact Kähler manifold there exist unique, orthogonal Hodge decompositions for the Dolbeault derivatives ∂_λ and $\bar\partial_\lambda$. Notice that in this case $\mathrm{d}_\lambda = \partial_\lambda + \bar\partial_\lambda$ and similar for the conjugate. Since $\Delta_{\mathrm{d}_\lambda} = 2\Delta_{\partial_\lambda} = 2\Delta_{\bar\partial_\lambda}$ [Nak03], one finds that $H^{p,q}_{\partial_\lambda}(X) \simeq \Omega^{p,q}_{\Delta_{\mathrm{d}_\lambda}}(X)$ and the same is true for the conjugate differential forms. Therefore:

- ❏ Let X be a compact Kähler manifold or, if real, compact, oriented and without boundary.

[5] Let $\omega \in \Omega^\bullet_{\Delta_\lambda}(X)$, then $\langle\omega, \Delta_\lambda\omega\rangle = 0 = \|\mathrm{d}_\lambda\omega\|^2 + \|\mathrm{d}^\dagger_\lambda\omega\|^2$ and this proves that a harmonic form is closed under d_λ and $\mathrm{d}^\dagger_\lambda$. The Hodge decomposition is orthogonal and therefore the harmonic forms are not exact with respect to d_λ.

The next section will clarify that the isomorphy between the cohomology of the supercharge and the kernel of the Hamiltionian will survive CPT breaking if $\lambda < \infty$. For $\lambda \to \infty$ this will still be true at least for $X = \mathbb{CP}^1$ and I will prove this in section 2.4.1.

2.2.2 Implementing CPT Breaking and Localization

The transformations I have done on the path integral in section 2.1.1 can be translated to the canonical point of view by considering correlation functions of topological observables and states

$$\begin{aligned}&\langle \omega,\ e^{(t_n-t_+)H}\mathcal{O}_n e^{(t_{n-1}-t_n)H}\dots e^{(t_1-t_2)H}\mathcal{O}_1 e^{(t_--t_1)H}\cdot\chi\rangle = \\ &\int_{X\times X}[\star\ \bar{\omega}(x_+)]\wedge\chi(x_-)\int_{\Sigma\to X:\ x(t_-)=x_-,\ x(t_+)=x_+}\mathcal{O}_n(t_n)\wedge\dots\wedge\mathcal{O}_1(t_1)e^{-S}.\end{aligned} \tag{2.2.9}$$

Since the topological sector is supposed to be invariant under subtracting the exact term $\int_{x_-}^{x_+}\mathrm{d}f = f(x_+) - f(x) + f(x) - f(x_-)$ from the action, this must have an effect on the operators and states. Expectation values of topological observables, calculated with an Hamiltonian in which CPT is manifestly broken by the additional term, must equal the undeformed expectation values. Therefore, the states and observables in the CPT-broken phase are subject to the following transformations of the physical counterparts[6]

$$\begin{array}{lll} \chi & \mapsto & e^{\lambda f}\chi \\ \star\ \bar{\omega} & \mapsto & e^{-\lambda f}\star\ \bar{\omega} \\ \mathcal{O} & \mapsto & e^{\lambda f}\mathcal{O}e^{-\lambda f} \end{array} \quad \text{and in particular} \quad \begin{array}{lll} Q & \mapsto & \mathrm{d} \\ Q^\dagger & \mapsto & Q^*_\lambda = 2\iota_v + \lambda^{-1}\mathrm{d}^\dagger \\ H & \mapsto & H_\lambda = \mathcal{L}_v + \frac{1}{2\lambda}\Delta \end{array} \tag{2.2.10}$$

Let me emphasize that *all* operators transform in the same way and the mappings above are not similarity transformations. Therefore, the new Hamiltonian is not self-conjugate any more and I rather put a $*$ than a $\dagger$.

One may now allow the transformed Hamiltonian not only to act on topological but also on dynamical states. Still, for finite values of λ, the new Hamiltonian has the same spectrum as H because the in-states have just gained a phase. In particular, the isomorphy between the supercharge cohomology and the ground states is still valid, though the theory is not unitary any more and the in- and out-states are no longer connected by an inner product (I will discuss the out states in section 2.2.4). The Morse theory with broken CPT and the one determined by (2.2.7) have the same cohomologies with respect to the supercharge, since $H^\bullet_{\mathrm{d}_\lambda} \simeq H^\bullet_{\mathrm{d}}$. Moreover, for finite λ, $H^\bullet_{\mathrm{d}} \simeq \Omega^\bullet_{\Delta_\lambda} \simeq \Omega^\bullet_{H_\lambda}$, such that $\dim\Omega^\bullet_{\Delta_\lambda} = \dim H_\lambda$. These dimensions are a topological invariants and thus should not be affected by taking $\lambda \to \infty$.

[6]The exponent $\mathrm{e}^{\lambda f} := \mathrm{e}^{\lambda(f(x)-f(x_-))}$ for the "ket" and $\mathrm{e}^{-\lambda f} = \mathrm{e}^{-\lambda(f(x)-f(x_+))}$ for the "bra".

2.2.3 The Instanton Moduli Space Revisited

According to the considerations of the last sections, the topological states are elements of the de Rham cohomology of $\mathbf{d}$. In the following I will consider observables $\hat{\omega}$ which can be identified with differential forms ω on X, substituting ψ^μ with $\mathrm{d}x^\mu$. Integrating out the conjugate momenta, the quantum mechanical Greens function between two critical points $x_\pm$ is

$$G^{(x_+,t_+)}_{(x_-,t_-)}[\hat{\omega}_1 \dots \hat{\omega}_n] = \int_{\mathcal{M}(-,+)} \mathrm{sgn}\ \det\left(\delta^\mu_\alpha \frac{\mathrm{d}}{\mathrm{d}t} - \partial_\alpha v^\mu\right) \bigwedge_{k=1\dots n} \phi_{*\,v}(\omega_k, t_k)\,. \tag{2.2.11}$$

By $\phi_{*\,v}(\omega_k, t_k)$ I denote the push forward of the diffeomorphism (2.1.11), evaluating ω along the flow lines, and I assume that these operators are time ordered.

The Partition Function One of the most famous of such Greens functions is the (supersymmetric) partition function

$$Z(T) = \int_X \delta(x_+ - x_-)\delta(\psi_+ - \psi_-) G^{(x_+,t_+)}_{(x_-,t_-)}[1] = \sum_{c\in\mathcal{A}} \mathrm{sgn}\ \det\left(-H^\mu_\nu(x_c)\right)\,. \tag{2.2.12}$$

The set $\mathcal{A}$ encompasses the critical points, $T = t_+ - t_-$ is the time period and the periodic boundary conditions cause localization on the flow lines that are loops, i.e. the vacuum configurations. The operator $\frac{\mathrm{d}}{\mathrm{d}t}$ does not contribute to the sign of the determinant because of these boundary conditions.[7] The supersymmetric partition function can also be written in terms of the Hamiltonian, using (2.2.9):

$$Z(T) = \mathrm{str}\ \mathrm{e}^{HT} = \mathrm{tr}\ (-)^F \mathrm{e}^{HT}\,, \tag{2.2.13}$$

where $(-)^F$ gives a minus sign on fermions (forms with odd degree) and plus on bosons (even degree). Since the excited eigenstates of H are always boson-fermion pairs due to supersymmetry, the partition function counts the difference in the number of fermionic and bosonic ground states $Z(T) = \mathrm{tr}_{\Omega^\bullet_{\Delta_\lambda}}\ (-)^F$. Thus, if X is such that the harmonic differential forms are isomorphic to the de Rham cohomology,

$$Z(T) = \sum_n (-)^n \dim_{\mathbb{R}} H^n_{Q_\lambda}(X,\mathbb{R})\,. \tag{2.2.14}$$

A careful reader might have objections against this derivation, because it is not obvious how to interpret the trace if CPT is broken. However, for finite values of λ, the in- and out-states are isomorphic and the spectrum of the Hamiltonian is basically the same, such that the equation above remains correct.

[7] This is nicely explained in [BBRT91]. Due to periodic boundary conditions one can make an expansion in Fourier modes $x^\mu(t) = \sum_{n\in\mathbb{Z}} x^\mu_n \mathrm{e}^{\mathrm{i}nt}$ and the same holds for the other coordinates. For simplicity let X be one dimensional. The Hessian is diagonal in the tangent basis of flow lines at x_c with eigenvalues λ_c. Hence, in that basis and at x_c, the sign of the determinant is: sgn det$\left(\prod_{n\in\mathbb{Z}}(-\mathrm{i}n + \lambda_c)\right)$. Only the zero mode contributes with a sign for the others square to a positive number.

Correlation Functions with Observables To be topological, more general correlation functions including observables have to be zero on Q_λ-exact observables. Notice, that $Q_\lambda = \mathrm{d}$ and using Stokes formula this implies the condition

$$\int_{\mathcal{M}(-,+)} \mathrm{d}\,\phi_{*\ \nu}(\omega, t) = 0\,. \tag{2.2.15}$$

This can be obtained by demanding that the boundary $\partial\mathcal{M}(-,+)$ vanishes. In the following I will, however, fix another property of X such that the integral yields zero.

In order to yield non-trivial correlation functions, the observables must have a total form degree of $\dim \mathcal{M}(-,+)$. In particular, if the dimension of $\partial\mathcal{M}(-,+)$ in the equation above was less than the form degree of $\phi_{*\ \nu}(\omega, t)$, the correlation function would also vanish, and this is what I am going to enforce in the following.

First, I have to ensure that $\partial\mathcal{M}(-,+)$ is a submanifold such that an integration of differential forms on this space is defined. In order to investigate $\partial\mathcal{M}(-,+)$, I take the closure of the descending and ascending manifolds X_- and X^+. Since X is compact these closures are compact. If the following condition holds

- ❏ The X_α and X^α are stratifications of X, i.e. $\overline{X}_\alpha = \cup_{\beta\in\mathscr{A}_{\geq\alpha}} X_\beta$ where $\mathscr{A}_{\geq\alpha}$ is the set of critical points with index greater or equal $\mathrm{ind}\ x_\alpha$ and similar $\overline{X}^\alpha = \cup_{\beta\in\mathscr{A}_{\leq\alpha}} X^\beta$ where now $\mathscr{A}_{\leq\alpha}$ counts lower indices

there is a canonical compactification of the instanton moduli spaces

$$\overline{\mathcal{M}}(-,+) = \left(\cup_{\alpha\in\mathscr{A}_{\geq -}} X_\alpha\right) \bigcap \left(\cup_{\beta\in\mathscr{A}_{\leq +}} X^\beta\right) \tag{2.2.16}$$

and thus their boundaries will be manifolds [Hut02].

If X is Kähler, the analysis is immediate. All indices are even valued, as one has a holomorphic and antiholomorphic part. The supercharge is $Q_\lambda = \partial + \bar{\partial}$ and raises the total form degree by one. Hence, under the correlation function and after invoking Stokes formula, the differential form has degree $(\dim \mathcal{M}(-,+) - 1)$. Because the compactified instanton moduli space can be rewritten as

$$\overline{\mathcal{M}}(-,+) = \bigcup_{\alpha_i\in\mathscr{A}_{>-}\ ,\ \beta_j\in\mathscr{A}_{<+}} \mathcal{M}(-,+) \times \mathcal{M}(-,\beta_j) \times \mathcal{M}(\alpha_i,\beta_j) \times \mathcal{M}(\beta_j,+)\,, \tag{2.2.17}$$

the boundary must also have even dimension, as it consists of instanton moduli spaces being glued together. Therefore, the correlation of an exact differential form must be zero in this case.

If X is a real manifold, the situation is more complicated and I know of no general argument. Due to that lack of knowledge I will restrict to

- ❏ The manifold X be Kähler.

2.2.4 The Out-States

The in- and out-states are related by a CPT transformation: $\mathscr{F}^{\mp}_{\text{out}} = \text{CPT} \cdot \mathscr{F}^{\pm}_{\text{in}}$, where + denotes particles and − anti-particles. Formally, an in-state can be written as

$$\omega_{\text{in}} = \int_{x(t):\ (-\infty,0],\ x(-\infty)=x_-,\ x(0)=x} \prod_i \mathcal{O}(t_i)\ \mathrm{e}^{-S_\lambda}\,, \tag{2.2.18}$$

where the boundary condition x_- defines a vacuum configuration, and CPT acts by conjugation $\omega \mapsto \star\ \bar{\omega}$ and time reversal. Thus, if the theory were unitary the out states would be of the form $\omega_{\text{out}} = \star\bar{\omega}_{\text{in}}$. Under that circumstances, there exists an hermitian inner product and the out-states can be identified with the in-states. However, in the case under consideration and due to the additional term, CPT acts non-trivially on the Lagrangian $L_\lambda(t) = L(t) - \lambda\ \mathrm{d}_t f(x(t))$, (2.1.8),

$$L_\lambda(t) \mapsto L_\lambda(-t) + 2\lambda\ \frac{\mathrm{d}}{\mathrm{d}t} f(x(-t))\,, \tag{2.2.19}$$

and the extra term indicates that the theory is not unitary.[8]

When decomposing the thus transformed Lagrangian in analogy with section 2.2.2, the out-states obtain a phase factor $\mathrm{e}^{-2\lambda f}$ and thus

$$\omega_{\text{out}} = \mathrm{e}^{-2\lambda f} \star\ \bar{\omega}_{\text{in}}\,. \tag{2.2.20}$$

For finite values of λ, the out-states are still isomorphic to the in-states, but in the limit $\lambda \to \infty$, this is not canonically valid.

The In-States in the Vicinity of a Critical Point

In section 2.1.2 I wrote that the states are localized around the critical points of f. This is definitely the case for the topological states. To see this I consider (2.2.7) and undertake the semiclassical analysis in analogy to Witten [Wit82].

Taking into account that the conjugate derivative in real coordinates and for an even dimensional manifold X is $\mathrm{d}^\dagger = -\iota_\mu \nabla^\mu$, the operator K_f can be written in a simpler way:

$$H = \frac{1}{2}\left(\lambda^{-1}\Delta + \lambda \|\mathrm{d}f\|^2 + H^\nu_\mu(x)\ [\mathrm{d}x^\mu, \iota_\nu]\right). \tag{2.2.21}$$

If $\lambda \to \infty$, the potential energy will grow, and this enforces the low energy states to localize around the critical points. In this case it is customary to undertake a Taylor expansion

[8] However, notice that for a vacuum configuration the extra term yields zero and CPT invariance is not affected.

around a critical point in order to study the low energy spectrum.[9] Thus, I choose local coordinates x (Riemann normal coordinates respectively Kähler normal coordinates [HIN02]), in which the critical point x_c is at the origin $x_c = 0$, the metric is approximately Euclidean, i.e. $g_{\mu\nu} = \delta_{\mu\nu}$ and $\partial_\lambda g_{\mu\nu}(0) = 0$, and the Hessian is diagonal, $H^\mu_\nu(0) = \delta^\mu_\nu\, \kappa_\mu$. The Hamiltonian can now be approximated as

$$\begin{aligned} 2H^{(\mathrm{pert})} &= \sum_\mu \left(-\lambda^{-1} \left(\partial_\mu\right)^2 + \lambda(\kappa_\mu x^\mu)^2 + \kappa_\mu [\mathrm{d}x^\mu, \iota_\mu] \right) + O(x^3) \\ &\simeq \sum_\mu \left(2\lambda^{-1} H^\mu_{\mathrm{bos}} - \kappa_\mu (-)^{F_\mu} \right) . \end{aligned} \tag{2.2.22}$$

The operator F_μ equals one if the differential form contains $\mathrm{d}x^\mu$ and zero, else. The bosonic part is just a sum over independent harmonic oscillators, and since $[H^\mu_{\mathrm{bos}}, (-)^{F_\mu}] = 0$ these operators can be diagonalized simultaneously. From the eigenvalues

$$E = \sum_\mu \left(|\kappa_\mu|(2n_\mu + 1) - \kappa_\mu (-)^{F_\mu} \right) , \quad n_\mu \in \mathbb{N} \cup \{0\} \tag{2.2.23}$$

one can conclude that the vacuum configurations are unique and the form degree must equal the index of x_c. Namely, $\kappa_\mu \neq 0$ since f is Morse, and $n_\mu = 0$ for vacuum configurations.

Let me conclude with some remarks. Firstly, for the class of target manifolds under consideration, the perturbative ground states equal the actual ground states. The reason is as follows: In general, the perturbative ground states might get lifted to massive states due to nonperturbative effects. However, there is a pairing of massive fermions and bosons due to supersymmetry. On a Kähler manifold, all ground states have an even form degree and a lift to an excited state would yield bosons, only. This does not conform with supersymmetry, such that on a Kähler manifold the number of critical points must equal the number of ground states. This does not mean that nonperturbative effects can not be observed on excited states.

Further, I would like to emphasize that due to the scaling of the metric with λ, there remain finite energy contributions in the large volume limit. These excited states do also localize on the ascending and descending manifolds. Namely, the in-states take the form (2.2.18), and when $\lambda \mapsto \infty$ they localize on the gradient trajectories. Since $x(-\infty)$ must be a

[9] Notice, however, that such an expansion destroys the kinematics of the theory. Further, it demands that a particular vacuum configuration is selected around which the Hamiltonian is expanded. This might promote the idea that it would be necessary to distinguish a "physical" from "other" vacua, while further it destroys the topological properties of the theory such as instantons. Behind these drawbacks, Taylor approximating around a fixed background has, is hidden the idea that for a theory on curved target spaces there should be distinguished a "free" from an "interacting" part in the Hamiltonian respectively the Lagrangian, just as is common in quantum theories on flat spaces. The careful reader will find that the approach of Frenkel, Losev and Nekrasov [FLN06], though it heavily relies on a proposal on the nonperturbative states and makes use of the Taylor ansatz in order to obtain the perturbative states, tries to overcome this rationality, cf. section 2.4. At least the nontrivial topolology will be preserved.

critical point x_c, these states have their support on the descending manifolds X_c. Therefore, the in-states are associated to the descending manifolds that cover X. By the same argument the excited out-states are supported on the ascending manifolds.

The ground states, extended by those excited states, will be focused on in the following. Before, I will briefly summarize the constraints on X that I have obtained.

2.3 Summary of the Constraints on X

In the last two sections, I have transformed a general Morse theory in such a way that the main ingredients which make a topological theory integrable are manifest: breaking of CPT invariance and localization. I have discussed the relation between the canonical and path integral point of view. I had to put several constraints on the target manifold X in order to achieve that there exists a topological sector. Now I would like to add a last constraint.

I always assumed that f is Morse and derived a vector field $v = \nabla f$ as a gradient of this function. In the situation of the A-model it will be important to reverse the logic and start from a given vector field v. For the transformations (2.2.10), the existence of such a potential is essential. It is in general not guaranteed that v can be expressed in terms of a gradient of a unique potential f. However, if X is a compact, simply connected, symplectic manifold and v is a symplectomorphism, one can invoke de Rham duality $H^1(X) \simeq H_1(X) = 0$ and conclude that $\omega := \iota_v g$ is an exact one-form $\omega = \mathrm{d}f$. Consequently, for every vector field v there exists a unique and single-valued function f such that $v = \nabla f$.

❏ Let X be compact and simply connected.

I have been as unrestrictive as possible and at the end of my discussions it appears that I had to put the same constraints as those used by Frenkel, Losev and Nekrasov [FLN06]. Here is the summary of the conditions:

① The target manifold X is a compact, simply connected, oriented Kähler manifold with Euclidean metric λg.

② There is a Morse function $f : M \to \mathbb{R}$ such that M has a Bialynicki-Birula decomposition by means of the descending and ascending manifolds.

③ The descending and ascending manifolds are Morse-Smale transversal.

④ The descending and ascending manifolds are stratifications of X.

The main side-effect of the transformations is that the theory is no longer unitary and therefore the out- and in-states are not related by an inner product. The in-states are supported on the descending manifolds X_c and for the vacuum states I used the argument of [Wit82] in order to see that their form degree equals the index of the fixed point x_c.

2.4 Morse Theory on $X = \mathbb{CP}^1$

In this section I am going to review the toy model considered in [FLN06]. Many features of the Morse theory underlying the topological A-model can already be studied by this example. The most important aspect will be that the Hamiltonian is not diagonalizable due to the excited states.

The toy model is defined on $X = \mathbb{CP}^1$ with inhomogeneous coordinates $z, \bar{z}$, endowed with the Fubini-Study metric $\lambda g = \lambda \frac{\mathrm{d}z \otimes \mathrm{d}\bar{z}}{(1+|z|^2)^2}$ and a Morse function $f = \frac{1}{4}\frac{|z|^2-1}{|z|^2+1}$. I do further assume that the topology of $\mathbb{CP}^1$ is the Zariski topology. The vector field associated with the Morse function is a generator of the $\mathbb{C}^\times$ symmetry of X, $v = z\partial_z + \bar{z}\partial_{\bar{z}}$.[10] It has fixed points $\{0, \infty\}$ and the corresponding descending manifolds are obtained from the flow equation $\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \zeta[z(t)]$, $\zeta = z\partial_z$. The point $\{0\}$ is repulsive with $\mathrm{ind}(0) = 0$ and has an associated descending manifold $X_0 = \mathbb{C}_0$, where $\mathbb{C}_0 = \mathbb{CP}^1 \setminus \{\infty\}$. The other fixed point $\{\infty\}$ is attractive with $\mathrm{ind}(\infty) = 2$ and descending manifold $\mathbb{C}_\infty = \{\infty\}$. The Hamiltonian before the transformations reads

$$H = -\frac{2}{\lambda}(1+|z|^2)^2 \partial_z \partial_{\bar{z}} + \frac{\lambda}{2}\frac{|z|^2}{(1+|z|^2)^2} + \frac{1-|z|^2}{1+|z|^2}(F_z + F_{\bar{z}} - 1)\,. \tag{2.4.1}$$

The rationale behind the work of Frenkel, Losev and Nekrasov [FLN06] is now as follows. It is not possible to derive the spectrum of the Hamiltonian H. Therefore, one may use the trick to break CPT invariance ($H \mapsto H_\lambda$) and move to the large volume limit $\lambda \to \infty$. The Hamiltonian will then be the Lie derivative in direction of the vector field v, cf. (2.2.10). The advantage is that this is a linear operator which is better tractable. However, if this operator is considered in its own right and independent from the physical Hamiltonian, it is not clear what the spectrum looks like. Firstly, since $H_\infty = \mathscr{L}_v$ is not bounded from below, one might get states with negative energy Eigenvalues. Secondly, it is an operator on differential forms on $\mathbb{CP}^1$ but it is not obvious what kind of differential forms should be allowed. If one allowed only smooth differential forms, due to the shape of the vector field v this would restrict the Eigenvectors of $\mathscr{L}_v$ to the space of constant differential forms and these have Eigenvalue zero. They would only cover the topological sector but not the dynamical. In order to overcome these difficulties, Frenkel et al. go back to the physical Hamiltonian H and consider its approximation as an harmonic oscillator (2.2.22) in the charts around the critical points of v. It turns out that the thus obtained eigenstates survive the large volume limit and become eigenstates of $\mathscr{L}_v$ with support on the descending manifolds. This is, however, only the perturbative spectrum and Frenkel et al. have to invoke a hypothesis on how to obtain the nonperturbative states, which I will explain at the end of this section.[11]

[10] The Lie algebra of $\mathbb{C}^\times$ is generated by $v = z\partial_z + \bar{z}\partial_{\bar{z}}$ and $u = \mathrm{i}(z\partial_z - \bar{z}\partial_{\bar{z}})$. The group elements are $\mathrm{e}^{\phi v}$ and $\mathrm{e}^{\phi u}$ with $\phi \in \mathbb{R}$.

[11] In their publication, Frenkel et al. [FLN06, pg. 7] claim that their approach should be viewed as an alternative to the usual Gaussian perturbation theory. Their method, they say, captures the nontrivial topology (and perhaps even

Below, I will derive the low energy states locally around the critical points in the charts $\mathbb{C}_0$ and $\mathbb{C}_\infty := \mathbb{CP}^1 \setminus \{0\}$. In the large volume limit, their supports turn out to be the descending manifolds. As already discussed in section 2.2.4, the perturbatively obtained ground states are the exact ground states for the global theory.

In order to treat the situation in the charts around $\{0\}$ and $\{\infty\}$ at the same time, I introduce a constant $k \in \{\pm 1\}$ that distinguishes if the the fixed point is attractive or repulsive. The respective Morse potential and its gradient are $v = k(z\partial_z + \bar{z}\partial_{\bar{z}})$, $f = \frac{1}{2}k|z|^2$ for both charts, where $k = +1$ simulates the fixed point $\{0\}$ and $k = -1$ the fixed point $\{\infty\}$. Notice that I neglected the constant in the Taylor expansion of f because it is irrelevant for the analysis of the spectrum of the Hamiltonian, the Morse potential does only enter the Hamiltonian in terms of ∇f. The coordinate z is already a Kähler normal coordinate in a neighborhood of $z = 0$, $g(z) = 1 + O(|z|^2)$, and the Hamiltonian is perturbatively given by (2.2.22)

$$H^{(\mathrm{pert})} = -\frac{2}{\lambda}\partial_z\partial_{\bar{z}} + \frac{\lambda}{2}\ k^2|z|^2 + k(F_z + F_{\bar{z}} - 1)\,. \tag{2.4.2}$$

The eigenfunctions are the Laguerre Polynomials

$$\Psi_{n,m} = \left(\pi(\lambda k)^{(n+m-1)} n!m!\right)^{-\frac{1}{2}}\ \mathrm{e}^{\frac{1}{2}\lambda|k|z\bar{z}}\partial_z^m\partial_{\bar{z}}^n\ \mathrm{e}^{-\lambda|k|z\bar{z}}\mathrm{d}z^p \wedge \mathrm{d}z^q\,, \tag{2.4.3}$$

with Eigenvalues $E_{n.m,p,q} = n + m + 1 + k(p + q - 1)$ and $n, m \in \mathbb{N} \cup \{0\}$, $p, q \in \{0, 1\}$.

When I apply the transformations (2.2.10) and (2.2.20), the sign of k matters. In analogy with [FLN06] I start with $k = 1$, i.e. $\{0\}$ is repulsive. The in- and out-states of the transformed theory are now

$$\begin{aligned}
\Psi_{n,m}^{(\mathrm{in},\lambda)} &= \frac{1}{\lambda^{n+m}}\ \mathrm{e}^{-\lambda z\bar{z}}\ \partial_z^m\partial_{\bar{z}}^n\ \mathrm{e}^{-\lambda z\bar{z}}\,,\\
\Psi_{n,m}^{(\mathrm{out},\lambda)} &= \frac{\lambda}{2\pi\ n!m!}\ \partial_z^n\partial_{\bar{z}}^m\ \mathrm{e}^{-\lambda z\bar{z}}\ \frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}\,.
\end{aligned} \tag{2.4.4}$$

Whith the normalization above, the limit $\lambda \to \infty$ makes sense and the resulting states will have the same Eigenvalues as the original ones.

If $k = -1$ and $\{0\}$ is attractive, the rôle of the in- and out-states are exchanged and hence, the in-state for an attractive fixed point is just the out-state above. Taking the large volume limit, the in-states become polynomials in z and $\bar{z}$. The out-states are functionals on the in-states, and a partial integration makes transparent that the exponential is a representation

the geometry) of the configuration space. Using the harmonic oscillator approximation, they, however, do rely on the Gaussian approximation and on an hypothesis about the nonperturbative state spaces. Although this is a slight drawback, I still find their attempt and results of great importance, in particular concerning the question of how to quantize quantum field theories on curved manifolds without destroying their kinematics and/or their topological properties such as instantons or symmetries between different vacuum configurations. By the Gaussian approximation, an interacting part is distinguished from a free part (leading to a linear equation of motion) of the theory. I consider it a necessary question to ask, if it makes sense at all, to make such a distinction in a quantum field theory on curved manifolds.

of the Dirac distribution. Therefore, when $\lambda \to \infty$,

$$\begin{aligned}\Psi^{(\mathrm{in},\lambda)}_{n,m} &\to z^n \bar{z}^m\,,\\ \Psi^{(\mathrm{out},\lambda)}_{n,m} &\to \frac{1}{n!m!}\partial_z^n \partial_{\bar{z}}^m\ \delta^{(2)}(z,\bar{z})\ \frac{\mathrm{i}}{2}\mathrm{d}z\wedge\mathrm{d}\bar{z}\,.\end{aligned} \tag{2.4.5}$$

The perturbative situation on $X = \mathbb{CP}^1$ is now as follows: On the descending manifold $\mathbb{C}_0$, the in-states are given by

$$\mathscr{H}_0 = \mathscr{F}_0 \otimes \bar{\mathscr{F}}_0\,,\quad \mathscr{F}_0 = \mathbb{C}[[z]] \otimes \wedge[[\mathrm{d}z]] \cdot 1|_{\mathbb{C}_0}\,,\quad \Delta_0 = 1|_{\mathbb{C}_0} \tag{2.4.6}$$

and Δ_0 is the vacuum configuration. The expression $\mathbb{C}[[\cdot]]$ denotes a power series and $\wedge$ the exterior product. The operators ∂_z and ι_z annihilate the vacuum $1|_{\mathbb{C}_0}$. The in-states associated with the descending manifold $\{\infty\}$ are elements of $\mathscr{H}_\infty = \mathscr{F}_\infty \otimes \bar{\mathscr{F}}_\infty$ with

$$\mathscr{H}_\infty = \mathbb{C}[[\partial_\omega, \partial_{\bar{\omega}}]] \otimes \wedge[[\iota_\omega, \iota_{\bar{\omega}}]] \cdot \Delta_\infty\,,\quad \Delta_\infty = \frac{\mathrm{i}}{2}\delta^2(\omega,\bar{\omega})\ \mathrm{d}\omega\wedge\mathrm{d}\bar{\omega}\,. \tag{2.4.7}$$

The local coordinate ω belongs to the chart $\mathbb{C}_\infty$ and $\frac{\mathrm{i}}{2}\delta^{(2)}(\omega,\bar{\omega})\ \mathrm{d}\omega\wedge\mathrm{d}\bar{\omega}$ is annihilated by ω and $\mathrm{d}\omega$.

For the out-states, the rôles of the state spaces are interchanged. The out-states at $\{0\}$ are the δ-distributions and take the form of the in-states at $\{\infty\}$. They localize on the ascending manifold $X^0 = \{0\}$. The out states at $\{\infty\}$ localize on the ascending manifold $X^\infty = \mathbb{C}_\infty$ and are given by polynomials. Moreover, there exist well defined pairings between the in- and out-states at the critical points. Indeed, the integral

$$\int_{X_c} \Psi_c^{(\mathrm{out},\infty)} \wedge \Psi_c^{(\mathrm{in},\infty)}\,,\qquad c\in\{0,\infty\} \tag{2.4.8}$$

yields a product of Kronecker symbols and thus has a finite value, whereby $\Psi_c^{(\cdot,\infty)}$ denotes an in- or out-state in the large volume limit on the descending manifold X_c, respectively ascending manifold X^c.

The perturbative state spaces above motivated Frenkel et al. to make an assumption about the nonperturbative state spaces, [FLN06].

❑ Frenkel, Losev and Nekrasov conjecture that the nonperturbative, low energy states are obtained by extending the perturbative states, as obtained by the Taylor approximation, as distributions on $\mathbb{CP}^1$.

In particular, this implies work on the polynomials. Their proposal can be motivated by three observations. Firstly, the state space around $\{\infty\}$ can immediately be considered as a space of distributions defined on $\mathbb{CP}^1$. From this point of view it would make sense to put the other state space on an equal footing. Secondly, the perturbative states obtained above are Eigenstates of the Hamiltonian $H_\infty = \mathscr{L}_v$, cf. eqn. (2.2.10), when restricted as an operator to the respective charts on which the state spaces live. This Hamiltonian is a linear operator which only indirectly depends on the metric by means of the condition that the gradient vector field is a symplectomorphism. Therefore, its Taylor approximation around a vacuum configuration simply equals its restriction to the chart of this point. This logic can be reversed, the "perturbative" Hamiltonian in a chart can be extended to the full Hamiltonian when its domain is extended to $\mathbb{CP}^1$. Therefore, one might assume that in an analogous way one obtaines the nonperturbative states from the perturbative ones by also extending their domain to $\mathbb{CP}^1$. And lastly, it would be nice to extend the definition of (2.4.8) to $\mathbb{CP}^1$.

The consequence of the conjecture above is that the "globalized" polynomials will be the source for the Hamiltonian being non-diagonalizable. This will be the subject of the following section.

2.4.1 Polynomial Distributions on $\mathbb{CP}^1$

Denote by $\mathscr{D}\otimes\Lambda^{a,b}$ the space of "test forms" on $\mathbb{CP}^1$, which is the space of smooth differential forms on $\mathbb{CP}^1$ with form degree (a,b) and compact support. In this section, I will extend the polynomial $z^\nu\bar{z}^\mu$ as a distribution on test functions for arbitrary $\nu,\mu\in\mathbb{C}$, $\nu-\mu\in\mathbb{Z}$, and similar as distribution forms dual to $\mathscr{D}\otimes\Lambda^{a,b}$. In particular, the vacuum state $1_{\mathbb{C}_0}$ can immediately be generalized by defining it to be the distribution form Δ_0 acting on a differential form $\eta\in\mathscr{D}\otimes\Lambda^{1,1}$ according to

$$\Delta_0(\eta)=\int_{\mathbb{C}_0}\eta\,. \tag{2.4.9}$$

In order to work out the extension for general polynomials, I will firstly concentrate on polynomials on $\mathbb{C}$. If the exponents n and m are allowed to be negative integers, they may have poles at $z=0$ and it will be necessary to regularize them and to generalize them as distributions on $\mathbb{C}$.

This situation will appear for $\mathbb{CP}^1$ in the chart $\mathbb{C}_\infty$ around $\{\infty\}$, and I will generalize the former discussion to this case. Thereby, the polynomials with support in $\mathbb{C}_0$ will be extended as distributions on $\mathbb{CP}^1$ in the sense defined for the polynomials on $\mathbb{C}$.

Most results of this section are obtained by using the definitions of Gel'fand and Shilov [GS64]. The extension to $\mathbb{CP}^1$ is handmade and the main results of this section (2.4.29) equals that of [FLN06, pg. 55], though I chose a different approach.

The Case $\mathbb{C}$

Let $\mathrm{d}^2z:=\frac{\mathrm{i}}{2}\mathrm{d}z\wedge\mathrm{d}\bar{z}$ and denote by $\int$ an integration over $\mathbb{C}$ with this measure. Let further $\nu,\ \mu\in\mathbb{C}$, $\nu-\mu\in\mathbb{Z}$ and $\mathscr{D}$ be the functions with compact support on $\mathbb{C}$. The polynomial in

$$\int z^\nu\bar{z}^\mu\phi,\quad \phi\in\mathscr{D},\ n:=\nu-\mu\in\mathbb{Z} \tag{2.4.10}$$

is analytic in ν, μ and locally integrable if the real part of $s := \nu+\mu$ is $\Re(s) > -2$, such that the integral above defines a distribution on test functions ϕ. One can understand this, writing the expression in angular coordinates

$$\int z^{\nu}\bar{z}^{\mu}\phi = \int_0^{\infty} r^{s+1}\left(\int_0^{2\pi}\phi(re^{\mathrm{i}\alpha}, re^{-\mathrm{i}\alpha})\; e^{\mathrm{i}n\alpha}\mathrm{d}\alpha\right)\mathrm{d}r. \tag{2.4.11}$$

If $\Re(s) = -2$ there might be a logarithmic pole and local integrability fails in a subset containing the origin. For integer values less than -2 there will be poles as explained below. In two steps I will generalize (2.4.10) as a distribution for more general values of ν and μ.

Analytic Continuation to $\Re(s) > -2-m,\ m \in \mathbb{N},\ s \notin \mathbb{Z}$ Firstly, it is possible to continue (2.4.10) analytically to $\Re(s) > -2-m,\ s \notin \mathbb{Z}$. Suppose that $\Re(s) > -2$ and add 0 in a way such that later the singularities for $\Re(s) > -2-m$ will be extracted:

$$\begin{aligned}\int z^{\nu}\bar{z}^{\mu}\phi = &\int_{|z|\le 1} z^{\nu}\bar{z}^{\mu}\left(\phi(z,\bar{z}) - \sum_{k+l=0}^{m-1}\frac{\phi^{(k,l)}(0,0)}{k!l!}z^k\bar{z}^l\right)\\ &+\int_{|z|>1} z^{\nu}\bar{z}^{\mu}\phi + 2\pi\sum_{k+l=0}^{m-1}\frac{\phi^{(k,l)}(0,0)}{k!l!}\frac{\delta_{l-k,n}}{k+l+s+2},\end{aligned} \tag{2.4.12}$$

where $\phi^{(k,l)}(z,\bar{z}) := \partial_z^k\partial_{\bar{z}}^l\phi(z,\bar{z})$. The last term is minus the insertion under the integral, integrated over in polar coordinates. The thus obtained equation above is analytic in ν, μ up to simple singularities at $s = -l-k-2 \wedge n = l-k$ or equivalently at $\nu = -k-1 \wedge \mu = -l-1$. Hence, it can be analytically continued. If further m is such that $-m-2 < \Re(s) < -m-1$, one can simplify this expression:

$$\int z^{\nu}\bar{z}^{\mu}\phi = \int z^{\nu}\bar{z}^{\mu}\left(\phi(z,\bar{z}) - \sum_{k+l=0}^{m-1}\frac{\phi^{(k,l)}(0,0)}{k!l!}z^k\bar{z}^l\right). \tag{2.4.13}$$

The point is, that in this case, the last term in (2.4.12) can be expressed as

$$-\int_{|z|>1} z^{\nu}\bar{z}^{\mu}\sum_{k+l=0}^{m-1}\frac{\phi^{(k,l)}(0,0)}{k!l!}z^k\bar{z}^l, \tag{2.4.14}$$

since $k+l+s+2 < 0$. In detail that can be seen in polar coordinates. It is now reasonable to define (2.4.10) as equation (2.4.13) if $\Re(s) < -2 \wedge s \notin \mathbb{Z}$, as one can always choose m as above.

Analytic Continuation to $s \in \mathbb{Z}_{<-1}$ The transition to $s \in \mathbb{Z}_{<-1}$ is done by subtracting the singular term, say at $s = -m-1$, and taking the limit $s \to -m-1$ with fixed $n = l-k$ or equivalently one can take the limit $\nu \to -k-1 \wedge \mu \to -l-1$. From (2.4.12) one can see, that this pole corresponds to $k+l=m-1$, which is the highest order term in

$$\begin{aligned}\int z^{-k-1}\bar{z}^{-l-1}\phi := \lim_{\substack{\nu\to -k-1\\ \mu\to -l-1}} & \int \left(z^{\nu}\bar{z}^{\mu} - 2\pi \frac{(-)^{k+l}}{k!l!}\frac{\delta^{(k,l)}(z,\bar{z})}{m+1+s}\right)\phi(z,\bar{z}) \\ = \int z^{-k-1}\bar{z}^{-l-1} & \left(\phi(z,\bar{z}) - \sum_{a+b=0}^{m-2}\frac{\phi^{(a,b)}(0,0)}{a!b!}z^{a}\bar{z}^{b}\right. \\ & \left. - \sum_{a+b=m-1}\frac{\phi^{(a,b)}(0,0)}{a!b!}z^{a}\bar{z}^{b}\theta(1-|z|)\right).\end{aligned} \tag{2.4.15}$$

This equation follows from (2.4.13). Firstly, one splits up the integral into an integration over $|z|\le 1$ and one over its complement. The term with polynomial degree $a+b=m-1$ under the integral over $|z|>1$ can be extracted and cancels to zero with the term subtracted in in first line of (2.4.15). Therefore, the theta-function appears, whereby $\theta(x)=1$ if $x\ge 0,\ x\in\mathbb{R}$ and 0 otherwise.

Differentiating It is important to notice that due to the appearance of the theta function in the case $s\in\mathbb{Z}_{<-1}$, differentiating is not a trivial task. Using the property of the derivative on distributions one obtains

$$\int \left(\partial_z z^{-k-1}\bar{z}^{-l-1}\right)\phi = \int \left((-k-1)z^{-k-2}\bar{z}^{-l-1} - \frac{2\pi(-)^{k+l}}{l!(k+1)!}\delta^{(k+1,l)}(z,\bar{z})\right)\phi, \tag{2.4.16}$$

and similar for $\partial_{\bar{z}}$.

The Case $\mathbb{CP}^1$

The polynomials (2.4.10) on $\mathbb{C}$ with positive exponents equal to the polynomial states (2.4.6) localized on $\mathbb{C}_0$. They are well defined on this chart, however not around $\{\infty\}$. In order to obtain the nonperturbative states by extending them as distributions on $\mathbb{CP}^1$, I will make two steps. Griffiths and Harris [GH78, pg. 373] suggest that distributions on general manifolds should be defined locally in charts. Therefore, I will firstly define the polynomials as distributions on test functions $\mathscr{D}_{0/\infty}$ with compact support in the charts $\mathbb{C}_{0/\infty}$ of $\mathbb{CP}^1$, whereby their action on $\mathscr{D}_\infty$ is of particular importance. Under this procedure, the generalized polynomials can be viewed as a direct sum of functionals, each of wich is defined as a distribution on test functions $\mathscr{D}_\infty$ and $\mathscr{D}_0$ respectively, i.e. $\mathscr{H}^{(\mathrm{in})} \xrightarrow{ext.} \mathscr{D}_0^* \oplus \mathscr{D}_\infty^*$. Secondly, by eqn. (2.4.31) I will define a pairing of out- and in-states, following Frenkel et al. [FLN06]. The nonperturbative states will thereby be defined on smooth differential forms on $\mathbb{CP}^1$, which are also test forms since $\mathbb{CP}^1$ is compact.

Let ϕ be an element in $\mathscr{D}_0\otimes\Lambda^{a,b}$ with support on the compact subset $\{0\}$. Consequently,

$$\int z^{\nu}\bar{z}^{\mu}\phi = \int_{|z|\le 1} z^{\nu}\bar{z}^{\mu}\phi + \int_{|z|>1} z^{\nu}\bar{z}^{\mu}\phi, \quad \int\cdot = \int_{\mathbb{CP}^1}\cdot, \tag{2.4.17}$$

is well defined.[12] Using the analysis for $X=\mathbb{C}$, the second term on the right hand side can further be given a meaning on test functions with support on the compact subset $\{\infty\}\in\mathbb{CP}^1$. Rewriting $z=\omega^{-1}$ in coordinates on $\mathbb{C}^\times$, the integral above yields $\int z^\nu \bar{z}^\mu\phi = \int_{|z|\le\epsilon} z^\nu\bar{z}^\mu\phi(z,\bar{z})+\int_{|\omega|<\epsilon^{-1}}\omega^{-\nu-2}\bar{\omega}^{-\mu-2}\phi(\omega^{-1},\bar{\omega}^{-1})$ and by means of (2.4.15) this expression can now be defined on $\phi\in\mathscr{D}_\infty\otimes\Lambda^{a,b}$. Setting w.l.o.g. $\epsilon=1$, the polynomial distribution acting on $\mathscr{D}_{0/\infty}$ splits into a direct sum

$$\int z^\nu\bar{z}^\mu\phi := \int_D \omega^{-\nu-2}\bar{\omega}^{-\mu-2}\hat{\phi}(\omega,\bar{\omega})+\int_D z^\mu\bar{z}^\nu\phi(z,\bar{z})\,, \tag{2.4.18}$$

whereby $\hat{\phi}(\omega,\bar{\omega}):=\phi(\omega^{-1},\bar{\omega}^{-1})$ and D is the unit disk around $\{0\}$. The first integral amounts to zero if $\phi\in\mathscr{D}_0$ while this is true for the second integral if $\phi\in\mathscr{D}_\infty$.

Differential Operators In order to analyze the action of differential operators on the thus generalized states, I will now introduce another notation in accordance with Frenkel, Losev and Nekrasov [FLN06]. Thus, denote every polynomial distribution (form) of the type (2.4.18) with $n,m\in\mathbb{N}$ by

$$\begin{aligned} |n,m,p,q\rangle_0 \ &\in (\mathscr{D}_0^*\oplus\mathscr{D}_\infty^*)\otimes\Lambda^{p,q},\quad p,q\in\{0,1\}\,,\\ |n,m,p,q\rangle_0\ [\phi] &:= \begin{cases} \int_D\omega^{-\nu-2}\bar{\omega}^{-\mu-2}\hat{\phi}(\omega,\bar{\omega})+\int_D z^\mu\bar{z}^\nu\phi(z,\bar{z})\,, \\ 0 \quad \text{if} \quad n,m<0\,,\ p,q>1 \end{cases}, \end{aligned} \tag{2.4.19}$$

and similarly the in-states build from δ distributions by

$$|n,m,p,q\rangle_\infty\ [\hat{\phi}] := \frac{\mathrm{i}}{2}\frac{(-)^{m+n}}{n!m!}\int\delta^{(m,n)}(\omega,\bar{\omega})\ \mathrm{d}\omega^p\wedge\mathrm{d}\bar{\omega}^q\wedge\ \hat{\phi}\,. \tag{2.4.20}$$

I can now generalize the notion of an exterior derivation on such distribution forms by means of

$$\partial|n,m,p,q\rangle_{0/\infty}\ [\phi] := (-)^{p+q+1}|n,m,p,q\rangle_{0/\infty}\ [\partial\phi]\,. \tag{2.4.21}$$

In order to calculate the derivative of (2.4.19) for the case $\phi\in\mathscr{D}_\infty$, I have to apply $\partial=\mathrm{d}\omega\wedge\partial_\omega$:

$$\begin{aligned} \partial|n,m,p,q\rangle_0\ [\phi] &= (-)^{p+q+1}|n,m,p,q\rangle_0\ [\partial_\omega\hat{\phi}(\omega,\bar{\omega})\ \mathrm{d}\omega^{a+1}\wedge\mathrm{d}\bar{\omega}^b] \\ &= (-)^{p+q}\ \frac{\mathrm{i}}{2}\int\left(\partial_\omega\omega^{-n-2p-2a}\bar{\omega}^{-m-2q-2b}\right) \\ &\qquad \times\mathrm{d}\omega^p\wedge\mathrm{d}\bar{\omega}^q\wedge\hat{\phi}(\omega,\bar{\omega})\ \mathrm{d}\omega^{a+1}\wedge\mathrm{d}\bar{\omega}^b\,. \end{aligned} \tag{2.4.22}$$

Without loss of generality, I set $p=a=0$ and keep the other degrees of freedom

$$\begin{aligned} \partial|n,m,p,q\rangle_0\ [\phi] &= \frac{\mathrm{i}}{2}\int\left(\partial_\omega\omega^{-n}\bar{\omega}^{-m-2q-2b}\right)\mathrm{d}\omega\wedge\mathrm{d}\bar{\omega}^q\wedge\hat{\phi} \\ &= \frac{2\pi(-)^{n+m-1}}{n!(m+2q+2b-1)!}\int\delta^{(n,m+2b+2q-1)}(\omega,\bar{\omega})\mathrm{d}\omega\wedge\mathrm{d}\bar{\omega}^q\wedge\hat{\phi} \\ &\qquad - n|n-1,m,p+1,q\rangle_0\ [\phi]\,. \end{aligned} \tag{2.4.23}$$

[12] For convenience I shift the test functions always to the right, also if they represent out-states. This will not have an effect on the results of the following sections.

For $\phi \in \mathscr{D}_0$ one obtains the first term on the right but with another sign. To summarize,

$$\begin{aligned}\partial|n,m,p,q\rangle_0 = & n|n-1,m,p+1,q\rangle_0\big|_{\mathscr{D}_0} - n|n-1,m,p+1,q\rangle_0\big|_{\mathscr{D}_\infty} \\ & + 2\pi|n,m+2q-1,p+1,q\rangle_\infty .\end{aligned} \tag{2.4.24}$$

Calculating the exterior derivative of (2.4.20) is not so technical, it turns out to be

$$\partial|n,m,p,q\rangle_\infty = -(n+1)|n+1,m,p+1,q\rangle_\infty \tag{2.4.25}$$

and the prefactor comes from the normalization of the state.

Another important differential operator is the interior product ι_ζ with some vector field $\zeta = z\partial_z$ (in local coordinates on $\mathbb{C}_0$). The point is, that the Hamiltonian is given by the Lie derivative on such polynomial distribution forms. Again, I make use of

$$\iota_\zeta|n,m,p,q\rangle_0\ [\phi] := (-)^{p+q+1}|n,m,p,q\rangle_0\ [\iota_\zeta\phi] . \tag{2.4.26}$$

Since $\zeta = -\omega\partial_\omega$ in $\mathbb{C}_\infty$, the action of the interior product is

$$\iota_\zeta|n,m,p,q\rangle_0 = \pm|n+1,m,p-1,q\rangle_0 , \quad \text{“}-\text{” on } \mathscr{D}_\infty \otimes \Lambda^{(a,b)} . \tag{2.4.27}$$

The action on a distribution $|n,m,p,q\rangle_\infty$ is derived analoguousely by means of some partial integration (again, I fix the non-trivial values $p = 1 = a$):

$$\begin{aligned}&\iota_\zeta|n,m,p,q\rangle_\infty\ [\hat{\phi}] = (-)^{p+q+1}|n,m,p,q\rangle_\infty\ [-\omega\hat{\phi}(\omega,\bar{\omega})\ \mathrm{d}\omega^{a-1}\wedge \mathrm{d}\bar{\omega}^b] \\ &= \frac{\mathrm{i}}{2}\frac{(-)^q}{n!m!}\int \delta(\omega,\bar{\omega})\mathrm{d}\omega^p\wedge\mathrm{d}\bar{\omega}^q\wedge\Big(-n\ \partial_\omega^n\partial_{\bar{\omega}}^m\hat{\phi}(\omega,\bar{\omega}) + \mathscr{O}(\omega)\Big)\mathrm{d}\omega^{a-1}\wedge\mathrm{d}\bar{\omega}^b \\ &= -|n-1,m,p-1,q\rangle_\infty\ [\hat{\phi}] .\end{aligned} \tag{2.4.28}$$

In the calculation above I used the fact that the delta function localizes on $\omega = 0$ and therefore the terms proportional to ω vanish. Now I can calculate the Lie derivative for any of the local test functions

$$\begin{aligned}\mathscr{L}_\zeta|n,m,p,q\rangle_0 &= (n+p)|n,m,p,q\rangle_0 - 2\pi|n+2p-1,m+2q-1,p,q\rangle_\infty , \\ \mathscr{L}_\zeta|n,m,p,q\rangle_\infty &= (n+1-p)|n,m,p,q\rangle_\infty .\end{aligned} \tag{2.4.29}$$

Thus, due to the extension as distributions, the operators including exterior differentials are in general not diagonal on $|n,m,p,q\rangle_0$. These states get mixed with states $|n,m,p,q\rangle_\infty$ on which the operators have a one-dimensional representation. In particular, the analytic extension of the excited states to $X = \mathbb{CP}^1$ makes it necessary that the spaces of in-states can not be considered independently, rather one has to take a direct sum of the extended state spaces $\underline{\mathscr{H}}_0 \oplus \underline{\mathscr{H}}_\infty$.[13] Here, the underline shall denote the state spaces extended as distributions.

[13]Notice, that by the proposal of Frenkel et al. on the nonperturbative states, the topological features of the theory are preserved, in particular all vacuum configurations are taken into account. In section 2.5, I will argue that also the instantons will be present and geometrically meaningful.

❑ If $\mathscr{H}$ is a perturbative state space related with some descending manifold, I will denote its extension to X as $\underline{\mathscr{H}}$.

In section 2.5, I will further specify the difference between the unextended and extended representation spaces and operators.

The Out-States as Dual States In order to allow an action of the in-states on the out-states, which are not all test functions, one has to define an adequate pairing. Thereby, the polynomial states will gain an action on smooth differential forms on $\mathbb{CP}^1$ while the splitting (2.4.18) will be preserved.

As explained in section 2.4, up to some normalization factor, the out-states are defined by the right hand sides of (2.4.19) with the rôle of the in-states exchanged [FLN06]

$$\begin{aligned} {}_\infty\langle n,m,p,q|[\phi] &= \frac{\mathrm{i}}{2}\begin{cases} \int_D z^{-n-2}\bar{z}^{-m-2}\ \mathrm{d}z^p\wedge\mathrm{d}\bar{z}^q\wedge\phi + \int_D \omega^n\bar{\omega}^m\ \mathrm{d}\omega^p\wedge\mathrm{d}\bar{\omega}^q\wedge\hat{\phi}\,, \\ 0 \qquad \text{if } n,m<0 \end{cases} \\ {}_0\langle n,m,p,q| &= \frac{\mathrm{i}}{2}\frac{(-)^{m+n}}{n!m!}\int \delta^{(m,n)}(z,\bar{z})\ \mathrm{d}z^p\wedge\mathrm{d}\bar{z}^q\wedge\ \phi\,. \end{aligned} \tag{2.4.30}$$

Thus, $|n,m,p,q\rangle_{0/\infty}$ are test forms if restricted to $\mathbb{C}_0$, and distribution forms in a neighborhood of $\{\infty\}$, whereas ${}_{0/\infty}\langle n,m,p,q|$ are test forms on $\mathbb{C}_\infty$ and distributions around $\{0\}$. For that reason, it makes sense to generalize the pairing for in and out states for distributions, setting [FLN06]

$$\int_X \Psi^{(\mathrm{out})}\wedge\Psi^{(\mathrm{in})} := \int_D \Psi^{(\mathrm{out})}\wedge\Psi^{(\mathrm{in})} + \int_{X-D}\Psi^{(\mathrm{out})}\wedge\Psi^{(\mathrm{in})}\,. \tag{2.4.31}$$

Assumed that the δ distributions split into a part of value 0 and the distribution, this pairing is fine on all combinations of nonperturbative states but on ${}_\infty\langle n,m,p,q|n',m',p',q'\rangle_0$. It is still true that no distribution is paired with another distribution, however, the distributional polynomials get evaluated on functions on which they are not defined. The way out is to set this pairing to zero, which equals the definition by Frenkel, Losev and Nekrasov, cf. [FLN06].[14] Under these circumstances,

$${}_i\langle n,m,p,q|n',m',p',q'\rangle_j = \delta_{n,n'}\delta_{m,m'},\delta_{p+p',1}\delta_{q+q',1}\delta_{i,j}\,,\quad i,j\in\{0,\infty\}\,. \tag{2.4.32}$$

and the nonperturbative states may act on smooth differential forms (when expanded in the charts). According to (2.4.31) the space of distributional polynomial states remains a direct sum, consisting of a distribution and a function. I will denote this property by

$$\underline{\mathscr{H}}_0 = \mathscr{H}_0\oplus\mathscr{H}_0^*\,, \tag{2.4.33}$$

[14] They define ${}_\infty\langle n,m,1,1|n',m',0,0\rangle_0 := PV\left(\int_{\epsilon<|z|<1} z^{n'-n-2}\bar{z}^{m'-m-2} + \int_{\epsilon'<|\omega|<1}\omega^{n-n'}\bar{\omega}^{m-m'}\right)$, whereby $PV(f(\epsilon,\epsilon'))$ equals the value of f which is independent of ϵ,ϵ' and w.l.o.g. I chose some values for the form degrees.

whereby $\mathscr{H}_0^*$ means the distributional part acting on $\mathscr{D}_\infty \otimes \Lambda^{a,b}$ by (2.4.15) and on polynomial functions according to the definition above, while $\mathscr{H}_0$ is the perturbative, unextended part localized on $\mathbb{C}_0$ and acting on functions on this chart. Notice that such a splitting is not necessary for the δ distributions which are already globally defined on $\mathbb{CP}^1$.

Cohomology of the Supercharge I will now fill in the missing details for my assertion in section 2.2.2, that the cohomology of the supercharge is not affected by taking $\lambda \to \infty$, and that it still equals the space of ground states.

The kernel of $Q_\infty = \partial + \bar{\partial}$ is generated by $\{|n,m,1,1\rangle_{0/\infty}, |0,0,0,0\rangle_0\}$. Among these, the states $|n,m,1,1\rangle_\infty$, $n,m \geq 1$ are in the image of Q_∞. For $n \geq 1$, one finds that $\partial[|n,m,0,1\rangle_0 + \frac{2\pi}{n}|n-1,m+1,1,1\rangle_\infty] = \pm n|n-1,m,1,1\rangle_0$ and similar for the antiholomorphic differential. Therefore $|n,m,1,1\rangle_0$, $\forall n,m \geq 0$ belongs to the image of the supercharge. Consequently, the cohomology of Q_∞ is effectively restricted to $\{|0,0,0,0\rangle_0, |0,0,1,1\rangle_\infty\}$, which are just the ground states. By a direct calculation one finds, that the solutions of $H_\infty|n,m,p,q\rangle_{0/\infty} = 0$, $H_\infty = \mathscr{L}_\zeta + \mathscr{L}_{\bar{\zeta}}$ equal the kernel of Q_∞, which proves the assertion above.

2.5 Interpretation of the Extension

Extending the states associated with the descending manifolds to distributions on X was the source for a sort of non-locality. Some state spaces which formerly were restricted to live in different charts, are now intermixed by operators containing exterior differentials. In this section, I will specify between what state spaces this happens. Moreover, this kind of non-locality can only be seen on the excited, non-topological states, and therefore must be analyzed as an effect of the broken topological phase. Therefore, certain aspects of the geometry of the target manifold should become visible. To tackle those, I will decouple the intermixing effect in the operators, extracting the mathematically responsible parts. My discussion follows Frenkel et al. [FLN06], but also includes my own interpretations, in particular that of non-locality as an instanton effect.

Perturbative States and Naive Operators

Perturbatively, the state spaces under consideration are associated with the descending manifolds and include the part of the low lying spectrum which has a finite energy spectrum in the limit $\lambda \to \infty$. I will call these the *perturbative* spaces of states. They seem to be independent from each other, in that they are locally defined on the descending manifolds and do not intermix under the action of observables. This changes for the excited states, as soon as they are extended to X.

Besides distinguishing the perturbative states from the extended ones, I will further introduce what I call *naive* operators. They act on the extended states as if they were acting on the perturbative ones. For instance, the naive Hamiltonian is diagonal on all extended states, $\mathscr{L}_\zeta^{(\text{naive})}|n,m,p,q\rangle_0 = (n+p)|n,m,p,q\rangle_0 \ \forall\ n,m,p,q$, whereas the full Hamiltonian can now be decomposed $\mathscr{L}_\zeta = \mathscr{L}_\zeta^{(\text{naive})} + \mathfrak{g}$. I will also define a representation of this Hamiltonian on the perturbative states in the following way. Instead of $\mathfrak{g}$, consider the operator $\delta := \mathfrak{g} \circ e$, wherein e denotes the extension $\mathscr{H}_i \xrightarrow{e} \underline{\mathscr{H}}_i$, $i \in \{0,\infty\}$. Consequently, δ acts on $\mathscr{H}_i$ and the full Hamiltonian can be represented on the perturbative states by $\mathscr{L}_\zeta + \delta$.

- ❏ For the rest of my thesis I will fix the following notation. Let $\mathscr{O}$ be an operator acting on the perturbative state space $\mathscr{H}$. I will denote the same operator, acting on the extended state space $\underline{\mathscr{H}}$ by $\underline{\mathscr{O}} = \mathscr{O} + \mathfrak{g}_{\mathscr{O}}$, wherein really $\mathscr{O} = \mathscr{O}^{(\text{naive})}$. For convenience I use this abuse of notation, it will always be possible to conclude from the context if $\mathscr{O}$ denotes the operator acting on $\mathscr{H}$ or $\mathscr{O}^{(\text{naive})}$, acting on $\underline{\mathscr{H}}$.

The additional operator $\mathfrak{g}$ is supposed to make *local* geometric aspects of the target space visible (in contrast to the global, topological invariants), and causes that the Hamiltonian is not reducible on all states: *non-reducibility of the Hamiltonian can be viewed as an effect of the broken topological phase.* More ventured, I am tempted to say that the additional term can be understood as an effect of target space gravity, since beyond the topological phase, invariance under diffeomorphisms is broken down to invariance under the isometries of some background metric.

The Local Geometry behind the Deformation Term

In order to understand what kind of geometry becomes visible in the deformation operator δ, I will now discuss its proper interpretation as a Grothendieck-Cousin operator (GCO), cf. [FLN06, Har67, Kem78, Har70].

The Hamiltonian $\mathscr{L}_v$ represents the action of $\phi_v(\cdot,t)$, induced on differential forms, cf. (2.1.11). Therefore, the perturbative state spaces can be interpreted as representations of the symmetry generated by the gradient vector field $v = z\partial_z + \bar{z}\partial_{\bar{z}}$ associated to the Morse function. The target manifold $X = \mathbb{CP}^1$ is thence covered by different representation spaces, each of which is supported on a descending (ascending) manifold.

Frenkel et al. [FLN06] had the idea to describe those local representations by means of sheaves on X.[15] Let X be endowed with the Zariski topology, then $X_0 = \mathbb{C}_0$ is an open subset while $X_\infty = X \setminus X_0$ is closed. The representation $\mathscr{H}_0$ can now be described as follows. The homogeneous rational functions $\mathscr{O}_X[n]_\infty$ on X that are regular except for a pole of order $n > 0$

[15] For a definition of sheaves and an introduction, cf. [GH78, Har70, Gat02].

at $\{\infty\}$ form a sheave on $\mathbb{CP}^1$. According to section 2.4.1, I can identify

$$\mathscr{H}_0 \setminus \{\Delta_0\} = \bigoplus_{|n-m|>0,\, n,m\geq 0} \Gamma(X_0, \mathscr{O}_X[n,m]_\infty)\,, \tag{2.5.1}$$

whereby $\mathscr{O}_X[n,m]_\infty = \mathscr{O}_X[n]_\infty \otimes \bar{\mathscr{O}}_X[m]_\infty$ and $\Gamma(U, \mathscr{O}_X[n,m]_\infty)$ denotes the sections of those polynomials, restricted to the open subset $U \subset X$.[16] In particular, the restriction to X_0 is injective, and the analysis of section 2.4.1 implies that the sequence

$$0 \to \bigoplus_{n,m>0} \Gamma(X, O_X[n,m]_\infty) \to \mathscr{H}_0 \setminus \{\Delta_0\} \xrightarrow{\delta} \mathscr{H}_\infty \setminus \{\Delta_\infty\} \to 0 \tag{2.5.2}$$

is exact. It summarizes the extension of the local (irreducible) representations to (non-reducible but indecomposable) representations defined globally X.

It would be nice, if not only $\mathscr{H}_0$ could be related with the theory of sheaves, but also $\mathscr{H}_\infty$. Since the support of $\mathscr{H}_\infty$ is a closed set but sheaves are defined on open sets, some generalization will be necessary. This will lead to the theory of local cohomology [Har67]. Let F be a sheaf on X, $Z \subset X$ a closed set and $U \subset X$ an open set such that $Z \subset U$. The support of a section $s \in F(U) := \Gamma(U, F)$ is $\{p \in U : s_p \neq 0\}$, where s_p is the germ of s in the stalk F_p.[17] The sections of F with support in Z are defined to be the subgroup $\Gamma_Z(X, F)$ of sections $F(U)$, whose support is in Z. The sections with support on closed subsets will be at the heart of the interpretation of $\mathscr{H}_\infty$.

The term “local cohomology” enters the work of Frenkel et al. [FLN06] through a publication of G. Kempf [Kem78], wherein the sequence (2.5.2) appears as an example in the introduction. A huge part of the paper is dedicated to an analysis of the following setting. Given a topological space X, filtered by closed subsets $X = Z_0 \supseteq Z_1 \supseteq \cdots Z_n \supset \emptyset$ and supplemented with a sheaf F. Kempf derives an exact sequence which he calls a “global Grothendieck-Cousin complex”:

$$0 \to \Gamma(X, F) \to H^0_{Z_0/Z_1} \xrightarrow{\delta_1} H^1_{Z_1/Z_2} \xrightarrow{\delta_2} H^2_{Z_2/Z_3} \xrightarrow{\delta_3} \cdots H^n_{Z_n} \to 0\,. \tag{2.5.3}$$

Here, I shortened $H^i_{Z_i/Z_{i+1}}(X, F) = H^i_{Z_i/Z_{i+1}}$, $H^n_{Z_n/\emptyset} = H^n_{Z_n}$, and the spaces $H^i_{Z_i/Z_{i+1}}$ denote (abstract) cohomology groups, associated with the quotient presheaf $\Gamma_{Z_i}(X, F)/\Gamma_{Z_{i+1}}(X, F)$. These are the so-called local cohomology groups.

By comparison, for the toy model on $X = \mathbb{CP}^1$ one has $F = \bigoplus_{n,m>0} \mathscr{O}_X[n,m]_\infty$ and the closed sets $X \supset \{\infty\} \supset \emptyset$. Consequently, $\mathscr{H}_\infty \setminus \{\Delta_\infty\}$ can be identified with the first local cohomology group $H^1_\infty(X, F)$. This is the mathematical answer to the question what sort of local geometry of X gets visible due to the excited states. Because the complex above is called Grothendieck-Cousin complex,

[16] The sections of $\Gamma(X_0, \mathscr{O}_X[n,m]_\infty)$ are polynomials in the inhomogeneous coordinates and thus obey the equivalence relation $\mathbb{C}^2 \setminus \{0\} \ni (f,g) \sim \lambda(f,g), f \in \mathbb{C} \setminus \{<0\}$ of the homogeneous coordinates. Therefore, I may take the direct sum.

[17] Let $\{U_i\}$ denote an open covering of X, a stalk F_p of F at $p \in X$ is the set of pairs (U_i, s_i), $p \in U_i$, whereby $s_i \in \Gamma(U_i)$ modulo $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. An equivalence class in F_p is called a germ, and I denoted it by s_p [Har70, Gat02].

- the operator δ is called the Grothendieck-Cousin operator (GCO). I will also denote the operator $\mathfrak{g}$ in $\delta = \mathfrak{g} \circ e$ as Grothendieck-Cousin operator, which I am considering will always be evident from the context.

Non-locality as an Effect of Instantons

The additional term δ has besides the geometric a further physical interpretation. It contains the nonperturbative effects due to the presence of instantons. Instantons, considered as tunneling solutions, can be viewed as non-local field configurations that procure some of the structure of the theory as defined in the chart around the repulsive fixed point $\{\infty\}$ to the one defined in the other chart around the attractive fixed point $\{0\}$. Since there are no anti-instantons this does not apply the other way around. This makes it obvious that one might consider the following: *The Grothendieck-Cousin operator δ mixes the state space $\mathcal{H}_0$ with $\mathcal{H}_\infty$, but not the other way around, and in that sense it mimics the instantons.*

Mixing of Holomorphic and Antiholomorphic Parts

A further speciality of the Grothendieck-Cousin operator is that it mixes the holomorphic and antiholomorphic parts. In particular, it contributes only on states which are not purely holomorphic or antiholomorphic. From (2.4.29) follows that $\ker\delta = \{|n,0,p,0\rangle_0, |0,m,0,q\rangle_0 : n,m \geq 0,\ p,q \in \{0,1\}\}$. For that reason, *as soon as the excited spectrum is considered, the theory can not be divided into an holomorphic and antiholomorphic "half"*. Just as the existence of non-diagonalizable operators, this is a typical characteristic of logarithmic conformal field theories [DF08].

2.6 Generalization to General Target Manifolds

In the following sections I will generalize the discussion to a larger class of manifolds X, again relying on [FLN06]. For convenience I will restrict my considerations to the in-states. Furthermore, I will restrict to Morse functions with the property that their gradient vector field equals $v = x^a\partial_a + x^{\bar{a}}\partial_{\bar{a}}$, where x^a and $x^{\bar{a}}$ are local coordinates on X.

2.6.1 The Perturbative State Spaces

The perturbative state spaces localize on the descending manifolds, thus I will first start with a generalization of those.

Let X_α be a descending manifold with critical point x_α which has an index $\mathrm{ind}(x_a) = \dim_{\mathbb{C}} X - n_\alpha$. By coordinates *along* X_α I understand (holomorphic) coordinates $x^1, \ldots, x^{n_\alpha}$

such that X_α is the hyperplane defined by the zero set of the complementary, *transversal* coordinates $x^{n_\alpha+1},\dots,x^{\dim_{\mathbb{C}}X}$. In the toy model, there exists one holomorphic coordinate z along $X_0 \simeq \mathbb{C}$ and no transversal coordinate, whereas $X_\infty = \{\infty\}$ is zero dimensional and has just a transversal coordinate z.

Now the perturbative state spaces can be generalized. In the toy model, the vacuum associated with X_0 was the characteristic function in the coordinate along X_0, whereas the vacuum associated with X_∞ was a Dirac distribution. This can be generalized as follows:

- ❏ A ground state Δ_α is a distribution form defined by $\int_X \Delta_\alpha \wedge \eta = \int_{X_\alpha} \eta|_{X_\alpha}$ on differential forms $\eta \in \Omega_{\mathrm{d}}(X)$.

Again, in the toy model, the excited states on X_0 are polynomials in the coordinates along X_α multiplied with the exterior algebra again along X_0. The excited states associated with X_∞ which has only transversal coordinates, are polynomials in interior derivatives and simple derivatives along the transversal coordinates. This is also canonically generalized:

- ❏ The excited states associated with X_α are given by $(\mathbb{C}[[x^a]] \otimes \wedge[[\mathrm{d}x^a]])_{a=1,\dots,n_\alpha} \otimes (\mathbb{C}[[\partial_a]] \otimes \wedge[[\iota_a]])_{a=n_\alpha+1,\dots,\dim_{\mathbb{C}} X} \cdot \Delta_\alpha$.

2.6.2 The Grothendieck-Cousin Operators

In order to determine the Grothendieck-Cousin operators for the more general case I will use two properties of δ as determined before.

The first property is that the Grothendieck-Cousin operator is a mapping between different representation spaces which are locally defined in charts of X, and that it appears in an exact sequence of the kind (2.5.3). This is, however, too general. In the situation of the toy model, the GCO is a mapping between two state spaces of relative codimension one, i.e. $\{\infty\} \prec \mathbb{C}_0 = \mathbf{codim}(\{\infty\}, \mathbb{C}_0^c) = 1$, where the uppercase c denotes taking the closure. In order to preserve this property, one must further constrain X and the sheaf F. I will not articulate those conditions and refer the reader to the publication of [Tch04]. Under the conditions explained there and which will always be satisfied in this theses, X and F are such that the Grothendieck-Cousin operators are mappings between representation spaces on descending manifolds with relative codimension one, $(Z_i \setminus Z_{i+1}) \prec (Z_{i-1} \setminus Z_i) = 1$. This restricts the state spaces between which Grothendieck-Cousin operators exist:

- ❏ The GCOs are mapping between perturbative state spaces whose descending manifolds have relative codimension one.

$$\exists\ \delta_i\ : H^{i-1}_{Z_{i-1}/Z_i} \to H^i_{Z_i/Z_{i+1}} \quad \Leftrightarrow \quad (Z_i \setminus Z_{i+1}) \prec (Z_{i-1} \setminus Z_i)\,. \tag{2.6.1}$$

The second property does not make use of the full geometric analysis described in section 2.4.1 and is more heuristic. The situation of the topological A-model I am going to introduce in the next chapter, will lead to an analysis of an infinite dimensional manifold. Thus, I do not know how to transfer the results above from its roots. When it comes to determine the GCOs, I will rather search after an adequate extension e of the perturbative representation spaces, such that I find operators $\mathfrak{g}$ which have the properties of cohomology operators on the extended complex. Thus, in order to determine the extension, I will make use of the following observation:

I have explained that the polynomials, extended as distributions, fall into a direct sum of functionals on test functions in charts – respectively via (2.4.31) this might be used for more general functions (2.4.33) if X is compact. The observation I will concentrate on is that the different functionals, naturally defined on the different charts of X, all have the same quantum numbers with respect to the naive Hamiltonian, f.i. in (2.4.33), $\mathscr{H}_0$ and $\mathscr{H}_0^*$ are degenerate. Therefore, I propose that the analytic extension e should be performed such that the local spectrum in a certain chart, for instance $\mathscr{H}_0$, is enlarged by adding the direct sum of the possible missing "dual" states, on which the naive Hamiltonian is degenerate. The mapping $\mathfrak{g}$ is then a mapping from this dual part onto the local cohomology group at the other chart, say around $\{\infty\}$:

- ❏ The GCOs act non-trivially on the "dual part" of the spectrum of the naive Hamiltonian, obtained by an extension of the state space

$$\mathscr{H}_{X_\alpha} \xrightarrow{e} \underline{\mathscr{H}}_{X_\alpha} = \mathscr{H}_{X_\alpha} \oplus \mathscr{H}^*_{X_\alpha} \xrightarrow{\mathfrak{g}} \mathscr{H}_{X_\beta} \to 0 , \tag{2.6.2}$$

  where $X_\beta \prec X_\alpha$ and $\mathscr{H}_{X_\alpha}$ denotes the states on which the symmetries of the theory become degenerate.

Instead of determining the Grothendieck-Cousin complex from the roots, in the following chapter I will make use of this heuristic recipe.

From the A-Model to Morse Theory 3

This chapter has again three parts. I will successively reproduce the situation of the last chapter for the topological A-model, reformulating it as an infinite sum of Morse theories of the kind just considered. Thereby, I will obtain its perturbative representation spaces. It will be possible to identify them with representations of conformal supersymmetric ghosts, which I will further substitute for the A-model. Bosonization of the conformal theory will enable me to derive the Grothendieck-Cousin operators and propose the extension of the perturbative state spaces. Due to the properties of the Grothendieck-Cousin operators it is then evident that if the topological A-model is a conformal field theory, it must be a logarithmic conformal field theory beyond its topological sector. The main reference of this chapter is the publication of Frenkel et al. [FLN08].

In the first part, I will massage the topological A-model, [Wit88b, Mar05, DVV91], into a first order form such that in the large volume limit, it yields a δ distribution on the instantons. The action thus obtained is that of a supersymmetric bc-system, and I will call it the *topological* supersymmetric bc-system (Tbc).

In the second part, 3.2 - 3.6, I will reverse the direction of analysis of [Wit88b] and derive the super quantum mechanics associated with the Tbc, as was done by Frenkel et al. [FLN08]. The result will be a theory that is not yet Morse and demands two further steps to reproduce the situation of the last chapter. I will discuss how to do that in section 3.2 and afterwards restrict my considerations to the target manifold $X = \mathbb{CP}^1$, cf. section 3.3. I will then derive the perturbative state spaces associated with the descending manifolds corresponding to the fixed points $\{0,\infty\} \in \mathbb{CP}^1$. They can be modeled by some conformal supersymmetric ghost system (CSbc) that I introduce in 3.4. In order to formulate the CSbc on $\mathbb{CP}^1$, it is necessary to implement chart transitions. Therefore, I have to further introduce the chiral de Rham complex, invented by Malikov et al. [MSV99], cf. section 3.5.1.

That the representation spaces of the Morse theory behind the Tbc can be modeled by a conformal field theory raises the question whether this could be true for the A-model itself. I will only touch lightly on that question, pg. 52f, and otherwise assume that the CSbc will simulate all aspects relevant for the perturbative low energy spectrum of Morse theory behind the A-model.

In the last part, starting with 3.6, I will extend the perturbative representations to the non-perturbative spectrum and introduce the infinite dimensional analogues of the Grothendieck-Cousin operators. This analysis is done for the CSbc, and I again assume that it generalizes to the A-model. The most important step will be to bosonize the CSbc. To do that, I will use and generalize the methods described in [FMS86, Fri85, FF91, FF90], cf. 3.6.2. This will enable me to analyze the algebraic properties of the representation theory for the perturbative and nonperturbative states of the Morse theory underlying the A-model. Some parts of that investigation have been published in [VF09]. My approach differs from that of Frenkel et al. [FLN08], who relied on a publication of Malikov [Bor01]. Motivated by a prior work of Frenkel and Losev [FL07], they proposed that the Grothendieck-Cousin operator is the zero mode of a particular field, which is part of a vertex algebra constituted by the CSbc after rewriting it in logarithmic coordinates and extending it by additional field zero modes. My approach will make use of the bosonized CSbc and of the method of logarithmic deformation invented by Fjelstad et al. [FFH$^+$02]. I will discuss the approach of Frenkel, Losev and Nekrasov and its relation to the method I have chosen in an appendix C.

3.1 Massaging the A-model

The A-model is a two dimensional field theory with an $\mathcal{N}=2$ ($\mathcal{N}=(2,2)$) worldsheet supersymmetry [Mar05], cf. appendix B.1. I will start with preparing the topological sector of this model and with the transformation of its integration kernel in the path integral to a delta distribution. For this purpose, let $\Sigma=\mathbb{CP}^1$ with local metric $h=\mathrm{d}z\otimes\mathrm{d}\bar{z}$ and volume form $\mathrm{d}^2z:=\frac{\mathrm{i}}{2}\mathrm{d}z\wedge\mathrm{d}\bar{z}$, as before. The indices μ, ν will denote local coordinates $\sigma^\mu\,:\,\sigma^1=t,\ \sigma^2=\sigma$ on Σ considered as a real manifold. The complex coordinates are $z=t+\mathrm{i}\sigma,\ \bar{z}=t-\mathrm{i}\sigma$. Further, I will need the epsilon symbol $\epsilon^{\bar{z}z}=-\epsilon^{z\bar{z}}=2\mathrm{i}$, as defined by $\frac{1}{2}\omega_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu=:\frac{1}{2}\omega_{\mu\nu}\epsilon^{\mu\nu}\cdot\mathrm{d}^2z$. The target manifold X be a simply connected, connected, compact Kähler manifold with metric λg. I denote its local holomorphic coordinates as x^a with small latin letters $a=1,\dots,\dim_{\mathbb{C}}X$ and similarly the anti-holomorphic coordinates as $x^{\bar{a}}$.

The A-model, without auxiliary fields, has the action

$$S=\int_\Sigma \mathrm{d}^2z\,\left\{\lambda g_{a\bar{b}}(\partial_z x^a\partial_{\bar{z}}x^{\bar{b}}+\partial_{\bar{z}}x^a\partial_z x^{\bar{b}}+\mathrm{i}\pi^a\mathrm{D}_z\psi^{\bar{b}}+\mathrm{i}\pi^{\bar{b}}\mathrm{D}_{\bar{z}}\psi^a)-\frac{1}{2\lambda}R_{a\bar{b}c\bar{d}}\ \pi^a\pi^{\bar{b}}\psi^c\psi^{\bar{d}}\right\},\quad(3.1.1)$$

where the embedding x is a Grassmann even and ψ a Grassmann odd scalar on Σ and with values in $x^*(T^{1,0}X)$, $\pi_a\in\Gamma(\Sigma,\Omega^{1,0}(\Sigma)\otimes x^*(\Omega^{1,0}(X)))$ is Grassmann odd and similar holds for $\pi_{\bar{a}}$.[1] The covariant derivative, for instance on ψ^a, is given by $\mathrm{D}_{\bar{z}}\psi^a=\partial_{\bar{z}}\psi^a+\Gamma^a_{bc}\partial_{\bar{z}}x^b\psi^c$. I will call the Grassmann odd fields fermions, though they have the wrong statistics.

Among others (cf. appendix B.1), this theory has a symmetry generated by

[1] The reader who is puzzled by the presence of λ^{-1} in the last term in (3.1.1) might consider the following. Take the usual action with metric g and not λg. Call the fermionic one form $\rho^{\bar{a}}$, its indices are lowered with $g_{a\bar{b}}$. Now introduce λg and identify $\pi^{\bar{a}}=\rho^{\bar{a}}$, where $\pi^{\bar{a}}$ is the corresponding field lowered by $\lambda g_{a\bar{b}}$. Then $R^a_{b\bar{c}d}\rho_a\rho^b\psi^{\bar{c}}\psi^d\simeq\lambda^{-1}\tilde{R}^a_{b\bar{c}d}\pi_a\pi^b\psi^{\bar{c}}\psi^d$ because $\rho_a=\lambda^{-1}(\lambda g_{a\bar{b}}\pi^{\bar{b}})$, whereas $\tilde{R}^a_{b\bar{c}d}=R^a_{b\bar{c}d}$ and I omitted the tilde in the action.

$\delta = \kappa^{++} Q_{++} + \kappa^{--} Q_{--}$:

$$
\begin{aligned}
&\delta x^a = \kappa^{++} \psi^a \,, && \delta x^{\bar{a}} = \kappa^{--} \psi^{\bar{a}} \,, \\
&\delta \psi^a = 0 \,, && \delta \psi^{\bar{a}} = 0 \,, \\
&\delta \pi^a = 2\mathrm{i}\kappa^{--}\ \partial_{\bar{z}} x^a + \kappa^{++}\ \Gamma^a_{bc} \pi^b \psi^c \,, && \delta \pi^{\bar{a}} = 2\mathrm{i}\kappa^{++}\ \partial_z x^{\bar{a}} + \kappa^{--}\ \Gamma^{\bar{a}}_{\bar{b}\bar{c}} \pi^{\bar{b}} \psi^{\bar{c}} \,.
\end{aligned}
\tag{3.1.2}
$$

From the transformation of the fermions one can conclude that the holomorphic embeddings $\partial_{\bar{z}} x^a = 0 = \partial_z x^{\bar{a}}$ are fixed points of that symmetry. These are called instantons, whereas the antiholomorphic ones, which are fixed points of another symmetry generator, are called anti-instantons. The nilpotent generator $Q_0 = Q_{++} + Q_{--}$ is independent of the geometry of the domain manifold in the sense that $[P_\mu, Q_0] = 0$, as can be derived from the relation $[Q_0, G_\mu] = P_\mu$, where G_μ is another supersymmetry generator, cf. appendix B.1.

The action above has more than just instantons as fixed points. In the following I will make localization on instanton configuration space manifest, in order to satisfy ⑤ of A. Therefore, I will again apply the Bogomolny trick and add a term which excludes the anti-instantons (i.e. antiholomorphic embeddings) from the global minima of the action. When I write the Lagrangian in first order form and integrate over the S^1 coordinate, the action will have the same shape as the Morse theory of the last chapter.

Excluding the Anti-Instantons

Consider the bosonic part of the action, it can alternatively be written as

$$
\int_\Sigma \mathrm{d}^2 z\ \left(2|\partial_z x^a|^2 - x^*(\omega_K)\right) \quad \text{or} \quad \int_\Sigma \mathrm{d}^2 z\ \left(2|\partial_{\bar{z}} x^a|^2 + x^*(\omega_K)\right) , \tag{3.1.3}
$$

where $\omega_K = \frac{\mathrm{i}}{2}\lambda g_{a\bar{b}}\ \mathrm{d}x^a \wedge \mathrm{d}x^{\bar{b}}$ is the Kähler form. Obviously, the action has both sorts of instantons as global minima. In order to exclude the anti-instantons I subtract $\int_\Sigma x^*(\omega_K)$ from the action above. The transformed action

$$
S_\lambda = \int_\Sigma \mathrm{d}^2 z\ \left(2\lambda g_{a\bar{b}}\ \partial_{\bar{z}} x^a \partial_z x^{\bar{b}} + \mathrm{i}\pi_a \mathrm{D}_{\bar{z}} \psi^a + \mathrm{i}\pi_{\bar{b}} \mathrm{D}_z \psi^{\bar{b}} - \frac{1}{2\lambda} R_{a\bar{b}c\bar{d}}\ \pi^a \pi^{\bar{b}} \psi^c \psi^d\right) \tag{3.1.4}
$$

does not have the full supersymmetry of the former one but still the symmetry generated by Q_{++} and Q_{--}.

The pullback $x^*(\omega_K)$ of the Kähler form is a volume form on Σ and hence topological with respect to the domain manifold. However, it is defined with respect to the target space metric λg and the question remains if it changes the topological sector of the theory. Since the Kähler form is closed, the integral $\int_\Sigma x^*(\omega_K) = \int_{x_*(\Sigma)} \omega_K$ does only depend on the cohomology class of $\beta := x_*(\Sigma) \in H_2(X, \mathbb{Z})$. Namely under a smooth mapping $f:\ X \to X$, $g_p(U,V) \mapsto g_{f(p)}(f_* U, f_* V)$ the homology classes are not changed. Thus, according to [Nak03], $x^*(\omega_K)$ and $(f \circ x)^*(\omega_K)$ are in the same cohomology class. Therefore, the integral above is invariant under a smooth change of the Kähler form, respectively the metric and the topological sector is not changed by excluding the anti-instantons.

By the choice of β, the instanton configuration spaces can be distinguished. A familiar way to make that visible in the action is to introduce the analogue of a theta angle. Instead of subtracting $\int_\Sigma x^*(\omega_K)$ from (3.1.1), one adds a closed, complex two form with real part proportional to the Kähler form $B = B_{a\bar{b}}\ dx^a \wedge dx^{\bar{b}} := \tau - \omega_K$ on X, $\tau = \tau_{a\bar{b}} \mathrm{d}x^a \wedge \mathrm{d}x^{\bar{b}}$. With this definition

$$S_{\tau,\bar{\tau}} = S_\lambda + \int_\Sigma\ x^*(\tau) \tag{3.1.5}$$

and the last term yields the "theta angle". Since τ is a closed differential form on X, the integral again depends only on the homology class β. In order to preserve τ, the limit $\lambda \to \infty$ is reformulated as the condition that $\bar{\tau}_{a\bar{b}} := B_{a\bar{b}} - \frac{\mathrm{i}\lambda}{2} g_{a\bar{b}}\ \ \to -\mathrm{i}\infty$, whilst $\tau = \text{const.}$. In the following, I will not make use of the theta angle τ.

First Order Formalism and the Supersymmetric bc-System

To make localization explicit, I introduce a Lagrangian multiplier $p = p_a\ \mathrm{d}z\mathrm{d}x^a + p_{\bar{a}}\ \mathrm{d}\bar{z}\mathrm{d}x^{\bar{a}}$ and rewrite the action in first order form

$$S_\lambda = \int_\Sigma \mathrm{d}^2 z\ \Big[-\mathrm{i}p_a \partial_{\bar{z}} x^a - \mathrm{i}p_{\bar{a}} \partial_z x^{\bar{a}} + \mathrm{i}\pi_a \mathrm{D}_{\bar{z}} \psi^a + \mathrm{i}\pi_{\bar{a}} \mathrm{D}_z \psi^{\bar{a}} + \frac{1}{2\lambda}\Big(g^{a\bar{b}} p_a p_{\bar{b}} - R_{a\bar{b}c\bar{d}}\ \pi^a \pi^{\bar{b}} \psi^c \psi^{\bar{d}} \Big) \Big]. \tag{3.1.6}$$

In the large volume limit $\lambda \to \infty$, the exponential of the action becomes a delta function on the instanton moduli spaces while the action itself becomes what is called a supersymmetric ghost or bc-system

$$S_\infty = \int_\Sigma \mathrm{d}^2 z\ \left(-\mathrm{i}p_a \partial_{\bar{z}} x^a - \mathrm{i}p_{\bar{a}} \partial_z x^{\bar{a}} + \mathrm{i}\pi_a \partial_{\bar{z}} \psi^a + \mathrm{i}\pi_{\bar{a}} \partial_z \psi^{\bar{a}} \right), \tag{3.1.7}$$

where I redefined $p'_a := p_a + \Gamma^b_{ac}\ \psi^c \pi_b$ and already left the prime away in the formula above. The supersymmetry takes the simple form

$$\begin{aligned} &[Q_0, x^a] = \psi^a, \quad &&[Q_0, x^{\bar{a}}] = \psi^{\bar{a}}, \\ &[Q_0, p_{\bar{a}}] = 0, \quad &&[Q_0, p_a] = 0, \\ &[Q_0, \pi_{\bar{a}}] = p_{\bar{a}}, \quad &&[Q_0, \pi_a] = p_a, \end{aligned} \tag{3.1.8}$$

in analogy with (2.1.10), and Q_0 plays the rôle of the BRST operator. In section 3.5.1 it will become clear in what respect Q_0 can be identified with the de Rham differential. The action S_∞ is Q_0-exact

$$S_\infty = \int_\Sigma \mathrm{d}^2 z\ [Q_0, -\mathrm{i}(\pi_a \partial_{\bar{z}} x^a + \pi_{\bar{a}} \partial_z x^{\bar{a}})], \tag{3.1.9}$$

and I will call it the topological bc-system (Tbc). It will be the main character in the following.

Remark: Let me conclude the large volume limit with a remark on the symmetries of the Tbc. The action (3.1.7) has an additional bosonic axial symmetry in analogy with (B.0.4), that the original action did not have. Therefore, it seems that in the large volume limit, the theory acquires an additional anomaly. In section 4.2 I will prove, that the bosonic axial symmetry will be broken by the Grothendieck-Cousin operators.

3.2 The Morse Theory behind the A-model

In analogy with Frenkel et al. [FLN08], I will now reverse the analysis of Witten [Wit88b] to obtain the super quantum mechanics (SQM) underlying the Tbc. It will differ in two aspects from the model of chapter 2. The target manifold will not be simply connected and the critical manifold of the Morse function will not be zero-dimensional, such that additional steps have to be taken to reduce the super quantum mechanics derived from the Tbc to the Morse theory discussed in the last chapter. Afterwards, I will restrict to the case $X = \mathbb{CP}^1$ in section 3.3.

To extract the Morse theory, let $\Sigma = \mathbb{R} \times S^1$ with local coordinates $z = t + \mathrm{i}\sigma$. For a fixed t, the embedding $x^a|_t(\sigma)$ is an element of loop space $LX := \{\gamma \in C^\infty(S^1, X) \ : \ \gamma \text{ is contractible}\}$ and can be represented by a Fourier series

$$x^a|_t(\sigma) = \sum_{n\in\mathbb{Z}} x^a_n \mathrm{e}^{-\mathrm{i}n\sigma}\,. \tag{3.2.1}$$

Similar holds for the other fields, for instance $p_a|_t(\sigma) = \sum_{n\in\mathbb{Z}} p_{an}\mathrm{e}^{\mathrm{i}n\sigma}$. The modes x^a_n are local coordinates on LX and one can reformulate the Tbc as a SQM on LX by integrating out the dependence on S^1. Up to irrelevant prefactors, the holomorphic part of the action yields

$$S_\infty = -\mathrm{i}\int \mathrm{d}t\Big(\ p_{a,-n}[\partial_t x^a_n - v^a_n(x)] - \pi_{a,-n}[\partial_t \psi^a_n - \psi^b_n \partial_b v^a(x)]\Big) \tag{3.2.2}$$

and similar holds for the antiholomorphic one. Summation over n is understood and $v^a_n(x)\partial_{an} := -n x^a_n \frac{\partial}{\partial x^a_n}$. The Lagrangian can be interpretedd as an infinite sum Lagrangians of the kind (2.1.9), if the v_n are interpreted as components of the gradient fields of a Morse function.

The gradient fields are associated with the generator of loop rotations ∂_σ. It is represented on the loops x by means of the vector field $v(x) = -\mathrm{i}\partial_\sigma x^a \partial_a + \mathrm{i}\partial_\sigma x^{\bar{a}}\partial_{\bar{a}}$, $\partial_a := \frac{\partial}{\partial x^a}$ and on the coordinates of LX by integrating over the parameter σ, $\int_{S^1} v^a(x)\partial_a = \sum_n v^a_n \partial_{an}$. Therefore, the fixed points of v are the constant loops, i.e. points on X. These are the zero modes x^a_0. Consequently, the fixed points of the gradient field are not isolated but comprise what is called a "critical manifold", which in the situation above is $X \subset LX$.

Another way to see this is by analyzing the spectrum of the Hessian $H_{aa\,n} = -n$. The coordinates x^a_n with $n > 0$ belong to negative eigenvalues and thus are coordinates on the ascending manifold, coordinates with $n<0$ belong to the descending manifolds while the zero modes x_0^a are coordinates at which the Hessian is indifferent.

The instanton equation can be written as the flow equation generated by the vector field v:

$$\partial_t x^a - v^a(x) = \partial_t x^a + \mathrm{i}\partial_\sigma x^a = 0\,, \tag{3.2.3}$$

which is nothing else but the condition of holomorphicity $\partial_{\bar z} x^a = 0$. In local coordinates of LX the instanton equation is

$$\partial_t x_n^a - v_n^a(x) = 0,\;\; v_n^a(x) = -n x_n^a\,. \tag{3.2.4}$$

However, the SQM above differs in two aspects from the one of the last chapter. Firstly, the critical points are not isolated and secondly, the target manifold LX is connected but not simply connected. This latter observation raises the question whether there exists a function f such that $\mathrm{d}f = \iota_v g_\gamma$. Here, g_γ is the induced Kähler metric $g_\gamma(\eta_1,\eta_2) := \int_{S^1} \lambda g|_\gamma(\eta_1(\sigma),\eta_2(\sigma))$, $\eta_{1/2} \in \Gamma(T_\gamma LX)$, $T_\gamma LX := \gamma^* TX$ are vector fields along the loop γ, and the contraction is understood as $\iota_v g_\gamma[\eta] = \int_{S^1} \lambda g|_\gamma(v(\sigma),\eta(\sigma))$. In the next section I will introduce a potential such that the vector field v can be obtained as its gradient. The potential will, however, not be single-valued on loop space.

3.2.1 The Potential

On a simply connected, symplectic manifold, every symplectomorphism can be expressed as a gradient of some potential.[2] The universal cover of loop space $\widetilde{LX} := \{(\gamma,\tilde\gamma) \mid \gamma \in LX,\ \tilde\gamma : D \to X \text{ s.t. } \gamma = \tilde\gamma|_{\partial D}\}/\sim$, where $\sim$ means equivalence under homotopy and D is the complex unit disk, is a simply connected and symplectic manifold (with the induced Kähler metric).

In the situation of the last chapter, I subtracted a term $-\lambda\int \mathrm{d}f$ to get rid of the anti-instantons. It trivially determines the Morse function. This motivates to try

$$f_\gamma(\tilde\gamma) := -\int_D \tilde\gamma^*(\omega_K) \tag{3.2.5}$$

as a candidate for the Morse function on $\widetilde{LX}$. Indeed, taking the exterior derivative and evaluating it in the direction of a smooth vector field $\eta \in T_\gamma LX$, one obtains an appropriate one form on the boundary $\mathrm{d}f_\gamma(\tilde\gamma)[\eta] = -\int_{S^1} \omega_K(\partial_\sigma\gamma,\eta) = \iota_v g_\gamma[\eta]$, while the orthogonal, radial direction does not contribute. However, the potential is only single-valued on $\widetilde{LX}$ but multivalued on LX, namely

$$f_\gamma(\tilde\gamma) = f_\gamma(\tilde\gamma') - \int_{S^2} (\tilde\gamma \bullet \tilde\gamma')^*(\omega_K) \tag{3.2.6}$$

[2] A symplectomorphism is a vector field v s.t. $\mathscr{L}_v\,\omega_K = d\iota_v\,\omega_K = 0$, with ω_K the symplectic form. If the manifold is simply connected, a closed one form is already exact and $\iota_v\omega_K = \mathrm{d}f$ for some f.

when two disks $\tilde{\gamma}$ and $\tilde{\gamma}'$ with the same boundary γ are glued together (which I denoted by the $\bullet$). The sphere S^2 is the generator of $H_2(X,\mathbb{Z})$ and counts the components of $u^{-1}(X)$, $X \subset LX$ in the universal cover $u : \widetilde{LX} \to LX$. More illustrative, in the case $X = \mathbb{CP}^1$ it counts the number of times the disks are wrapped around X.

That the potential is multi-valued on loop space has an impact on the space of states and I will discuss that in section 3.3.1. For the time being, let me note that under the mapping u, LX fans out into leaves in $\widetilde{LX}$, distinguished by $H_2(X,\mathbb{Z})$. According to Frenkel et al. [FLN08], I will denote these leaves as $\widetilde{LX}_n$, $n \in H_2(X,\mathbb{Z})$.

3.2.2 Isolating the Critical Points

I will now approach the second problem and isolate the critical points. This is done by deforming the instanton flow equation. The deformation will be such that the fixed point set is reduced to the points $\{0,\infty\} \in X$. Frenkel et al. do achieve this by introducing an additional target space symmetry into the action, which for the case $X = \mathbb{CP}^1$ will be a generator of the $\mathbb{C}^\times$ symmetry of X [FLN08].

The starting point is the supersymmetric bc-system (3.1.7) which I generalize in analogy to the Morse theory action (2.1.9)

$$S := \int_\Sigma \mathrm{d}^2 z \, \Big(-\mathrm{i} p_a [\partial_{\bar{z}} x^a + \mu V^a(x)] + \mathrm{i}\pi_a [\partial_{\bar{z}} \psi^a + \mu \partial_b V^a(x) \psi^b] + c.c. \Big), \tag{3.2.7}$$

where $\mu \in \mathbb{R}$. This step can be understood as a deformation of the vector field $v(x) = -\mathrm{i}\partial_\sigma x^a \partial_a + \mathrm{i}\partial_\sigma x^{\bar{a}} \partial_{\bar{a}}$ according to $v(x) \mapsto \mathcal{V}(x) = v(x) - \mu\ V(x)$. The instanton equation is changed to

$$\partial_{\bar{z}} x^a + \mu V^a(x) = \partial_t x^a - \mathcal{V}^a(x) = 0 \tag{3.2.8}$$

and its critical points are solutions of $\mathcal{V}^a(x) = 0$.

In order to approach the situation of the last chapter, it would be nice if in the situation $X = \mathbb{CP}^1$ these were again $\{0,\infty\} \in X$. This can be achieved by choosing the additional vector field to be $V(x) = x^a \partial_a + x^{\bar{a}} \partial_{\bar{a}}$, which is a generator of the $\mathbb{C}^\times$ symmetry of $\mathbb{CP}^1$. Assumed that the composite vector field $\mathcal{V}(x)$ is not degenerate, the critical manifold reduces to the intersection of the critical manifolds of V and v, which consists of the points $\{0\}$ and $\{\infty\} \in \mathbb{CP}^1$.

A deformation of the gradient vector field must be followed by a redefinition of the Morse function f

$$f_\gamma(\tilde{\gamma}) \mapsto -\int_D \tilde{\gamma}^*(\omega_K) - \mathrm{i}\mu \int_{S^1} H_V(\gamma,\sigma)\mathrm{d}\sigma, \tag{3.2.9}$$

where H_V is the solution of $\mathrm{d}H_V(\gamma,\sigma)[\eta] = \omega_K(V,\eta)$, $\eta \in T_\gamma LX$. The deformation term only depends on the boundary γ and, hence, does not contribute with an additional term to (3.2.6).

The Deformation as "Gauging" the Theory

In the case of the symmetry I have just implemented, the action further simplifies to

$$S = \int_\Sigma \mathrm{d}^2 z \; \left(-\mathrm{i} p_a (\partial_{\bar z} + \mu) x^a + \mathrm{i}\pi_a (\partial_{\bar z} + \mu)\psi^a + c.c. \right), \tag{3.2.10}$$

where μ now looks like a gauge connection. Frenkel et al. give this interpretation a meaning by reconsidering the original action as a quantum mechanical system [FLN08]. I will follow their discussion for the bosonic part which thus takes the form

$$S_{\text{bos}} = -\mathrm{i} \int_{\mathbb{R}} \left[\int_{S^1} (p_a \partial_t x^a + p_{\bar a} \partial_t x^{\bar a}) \mathrm{d}t \wedge \mathrm{d}\sigma - \mathrm{d}t H(x,p) \right], \tag{3.2.11}$$

with $H(x,p) = p[v],\;\; p[v] = \int_{S^1} \left(p_a(-\mathrm{i}\partial_\sigma x^a) + p_{\bar a}(\mathrm{i}\partial_\sigma x^{\bar a})\right)\mathrm{d}\sigma$. The Hamiltonian $H(x,p)$ couples to the one form $\mathrm{d}t$ on $\mathbb{R}$ and one might be tempted to consider the more general situation where it is a representation of some Lie algebra coupling to a gauge potential $A(t)\mathrm{d}t = A_L(t) H^L(x,p)\mathrm{d}t$ with $[H^L, H^M] = f_N^{LM} H^N$.[3]

In order to interpret the deformation as a sort of gauging, I let $X = \mathbb{CP}^1$ and choose $H^1 := p[V],\;\; H^2 := p[U]$ where $U(x) = \mathrm{i}(x^a\partial_a - x^{\bar a}\partial_{\bar a})$ is the $U(1) = \mathbb{R}/2\pi\mathbb{Z}$ generator on X and $V = x^a\partial_a + x^{\bar a}\partial_{\bar a}$. These Hamiltonians are indeed representations of the Lie algebra of $\mathbb{C}^\times$ with $[H^1, H^2] = 0$. The deformation of the action can then be interpreted as a deformation of the Hamiltonian $H(x,p)\mathrm{d}t \mapsto H(x,p)\mathrm{d}t - A_L(t)H^L\mathrm{d}t$ with $A_1 = \mu$ and $A_2 = \rho$. The form of the action (3.2.7), now including fermions, is reproduced when defining $A_{\bar z} := \mu + \mathrm{i}\rho,\;\; \mu, \rho \in \mathbb{R}$

$$\begin{aligned} S = \int_\Sigma \mathrm{d}^2 z \; \Big(&-\mathrm{i} p_a [\partial_{\bar z} x^a + A_{\bar z}\; V^a(x)] - \mathrm{i} p_{\bar a}[\partial_z x^{\bar a} + A_z \; V^{\bar a}(x)] \\ &+ \mathrm{i}\pi_a[\partial_{\bar z}\psi^a + A_{\bar z}\; \partial_b V^a(x)\psi^b] + \mathrm{i}\pi_{\bar a}[\partial_z \psi^{\bar a} + A_z\; \partial_{\bar b} V^{\bar a}(x)\psi^{\bar b}] \Big). \end{aligned} \tag{3.2.12}$$

and specifically, for the discussion above, when setting $\rho = 0$. For finite time evolutions, the holonomy of A is invariant under the $U(1)$ gauge transformation $\rho \mapsto \rho + \frac{2\pi n}{T},\;\; \mu \mapsto \mu$. However, the gauge field is not quantized and I will only use the name "gauged", if I want to explicitely distinguish the action (3.2.10), from now on called the "gauged" Tbc, from the action (3.1.7).

3.3 Perturbative Morse Description of the A-Model

From now on I will restrict my considerations to the case $X = \mathbb{CP}^1$. Furthermore, I will write x for the homolorphic and $\bar x$ the anti-holomorphic target space components and similar for the other fields. I assume that these coordinates are the inhomogeneous coordinates on $\mathbb{CP}^1$. The action I am going to consider is the deformed one (3.2.10) with $\mu \in (-1, 0)$.[4]

[3] The idea behind this is that $\exp\{\int A(t) H \mathrm{d}t\}$ can either be considered as a propagator, $A = 1$, or the holonomy of a gauge field.

[4] The gauge field component μ is not allowed to be an integer since otherwise $\mathcal{V}$ would be degenerate. This will become evident in equation (3.3.5).

In the consecutive section I will determine the perturbative state spaces of the underlying Morse theory. After I will start with some general discussion of the in-state spaces and then determine the state space located on the descending manifold with fixed point $\{0\} \in \mathbb{CP}^1$ in section 3.3.2. In order to derive the perturbative state space on the descending manifold with fixed point $\{\infty\} \in \mathbb{CP}^1$, it is necessary to make a chart transition, and I will explain how this works in section 3.3.3.

3.3.1 The Perturbative State Spaces

In the last chapter and particularly section 2.6, the perturbative state spaces associated with a descending manifold X_c have been obtained as

$$\big(\mathbb{C}[[x^\mu]] \otimes \wedge[[\mathrm{d}x^\mu]]\big)_{\mu=1,\dots,n_c} \otimes \big(\mathbb{C}[[\partial_\mu]] \otimes \wedge[[\iota_\mu]]\big)_{\mu=n_c+1,\dots,\dim_{\mathbb{R}} X} \cdot \Delta_c\,,$$

wherein $\mathrm{d}x^\mu$ are differential forms on and x^μ are the coordinates along X_c which has $\dim_{\mathbb{R}} X_c = n_c$, while the derivatives are in the transversal directions. The vacuum state Δ_c was the volume form on X_c, extended in the transversal directions as a distribution.

This situation carries over to the Morse theory behind the supersymmetric bc-system up to a peculiarity. Since LX is not simply connected, the perturbative state spaces and also the descending manifolds will be branched. On every leaf, the situation is however the same as in the toy model of the last chapter.

Branching of the State Spaces

In the Morse theory of chapter 2, the perturbative states corresponding to a descending manifold X_c have been obtained by solving $H^{(\mathrm{pert})}\Psi = E\Psi$, and taking the large volume limit of $\mathrm{e}^{\lambda f}\Psi$, cf. 2.4. These states should be related with those of the Morse theory behind the A-model with action $S_\lambda = S - \int_\Sigma x^*(\omega_K)$.

Including the points $\{\pm\infty\} \in \mathbb{R}$ such that $\Sigma \simeq S^2$, I can split the integral $\int_{S^2} x^*(\omega_K) = \int_D \tilde{\gamma}^{-\,*}(\omega_K) - \int_D \tilde{\gamma}^{+\,*}(\omega_K)$. Here, $(\tilde{\gamma}^- \bullet \tilde{\gamma}^+)^* = x^*$, $\tilde{\gamma}^-$ covers the hemisphere of $\mathbb{CP}^2$ including a repulsive fixed point and $\tilde{\gamma}^+$ covers the other hemisphere of X, including an attractive fixed point. Therefore, the ket states of the super quantum mechanics on loop space and associated with some descending manifold LX_c, are of the form

$$\Psi_0 = \mathrm{e}^{\int_D \tilde{\gamma}^{-\,*}(\omega_K)}\Psi \tag{3.3.1}$$

with Ψ a differential form on LX. Since the integrand is not a total derivative, Ψ_0 depends on the integration "path". In particular, from the discussion in section 3.2.1 follows that the states are homotopically distinguished by $H_2(X, \mathbb{Z})$, which measures how often Σ is wrapped around X. Consequently, one can distinguish a stack of Hilbert spaces by the winding number n via the relation

$$\Psi_n := \mathrm{e}^{\int_{n\in H_2(X,\mathbb{Z})}\tilde{\gamma}^{-\,*}(\omega_K)}\Psi_0 \quad , \quad \Psi_{n+m} = \mathrm{e}^{\int_{n\in H_2(X,\mathbb{Z})}\tilde{\gamma}^{-\,*}(\omega_K)}\Psi_m \,. \tag{3.3.2}$$

The full state space of in-states, corresponding to some critical point $x_c \in LX$, is the tensor product of the state spaces with a specific wrapping number,

$$\mathscr{H}_c^{\mathrm{in}} := \bigotimes_{n\in H_2(X,\mathbb{Z})} \mathscr{H}_{c,n}^{\mathrm{in}} \,. \tag{3.3.3}$$

However, since all states are isomorphic by a multiplication with

$$q^n := \mathrm{e}^{\int_{n\in H_2(X,\mathbb{Z})}\tilde{\gamma}^{-\,*}(\omega_K)} \,, \tag{3.3.4}$$

I will restrict my discussion to $\mathscr{H}_{c,0}^{\mathrm{in}}$.[5]

3.3.2 The Perturbative State Space on $\widetilde{LX}_{0,k}$

The operator q may serve to distinguish not only the leaves of the state spaces but also the instanton sectors (cf. pg. 37) and the leaves $\widetilde{LX}_k$. Therefore, I will associate the k^{th} instanton sector with the k^{th} branch and the k^{th} sector of the state space. Every leaf $\widetilde{LX}_k$ contains $X_k \simeq X$ and the preimages of the critical points with respect to $u : \widetilde{LX} \to LX$. Due to (3.3.2), the instanton equation looks the same on all leaves, and I will denote the descending manifolds corresponding to some preimage $x_{c,k} \in X_k$ of a critical point $x_c \in X$ by $\widetilde{LX}_{c,k}$. The perturbative state spaces will be associated with these descending manifolds.

The perturbative state spaces follow from the knowledge of the coordinates on the descending manifolds, c.f. section 2.6. Therefore, I consider the instanton equation (3.2.4) for the gauged Tbc in a neighborhood of $\{0\} \in X_{0,k}$

$$\mathrm{d}_t x_n - (-n-\mu)x_n = 0 \,, \quad \mu \in (-1,0) \,, \tag{3.3.5}$$

wherein the x_n are coordinates of $\widetilde{LX}_k$ for an arbitrary k. By means of the Hessian $H_n = -(n+\mu)$ one can distinguish the directions of the tangent space along the descending manifold $\widetilde{LX}_{0,k}$. They belong to positive eigenvalues and are thus the $\{x_n\}_{n\le 0}$, including the critical point $x_0 = 0$. The differential forms on $\widetilde{LX}_{0,k}$ are the modes $\{\psi_n\}_{n\le 0}$, and ψ_0 can be identified with the usual holomorphic differential form $\mathrm{d}x_0$ on the zero mode part $X_{0,k} \subset \widetilde{LX}_{0,k}$, $X_{0,k} \subset X_k$ of the descending manifold:

$$x_n \simeq x_n \quad , \quad \psi_n \simeq \mathrm{d}x_n \,. \tag{3.3.6}$$

The momenta, conjugate to x_n and ψ_n, $n \le 0$ are also coordinates along the descending manifold. These are the modes $\mathrm{i}p_{-n}$ and $\mathrm{i}\pi_{-n}$ with $n \ge 0$, and they may be identified with geometric data according to

$$\mathrm{i}p_{-n} \simeq \partial_n \quad , \quad \mathrm{i}\pi_{-n} \simeq \iota_n \,. \tag{3.3.7}$$

[5] Frenkel et al. considered a different operator q with τ in the exponent, cf. section 3.1 and [FLN08].

These coordinates satisfy the conditions for a canonical quantization $[p_n, x_m] = -\mathrm{i}\delta_{n,-m}$ and $[\pi_n, \psi_m] = -\mathrm{i}\delta_{n,-m}$. Consequently, the perturbative state space on $\widetilde{LX}_{0,k}$, $k = 0$, now including the antiholomorphic part, must contain the span

$$\mathscr{H}^{\mathrm{in}}_{0,0} = \mathbb{C}[x_n, \bar{x}_n, \psi_n, \bar{\psi}_n]_{n\leq 0} \otimes \mathbb{C}[p_n, \bar{p}_n, \pi_n, \bar{\pi}_n]_{n<0} \cdot \Delta_0 \,, \tag{3.3.8}$$

where

$$\begin{aligned} \Delta_0 &= \Xi_{\widetilde{LX}_{0,0}} (\psi_1 \psi_2 \cdots)(\bar{\psi}_1 \bar{\psi}_2 \cdots) \,, \\ \Xi_{\widetilde{LX}_{0,0}} &\sim \prod_{n>0, m\geq 0} \delta^{(2)}(x_n, \bar{x}_n) \delta^{(2)}(\psi_n, \bar{\psi}_n) \delta^{(2)}(p_m, \bar{p}_m) \delta^{(2)}(\pi_m, \bar{\pi}_m) \end{aligned} \tag{3.3.9}$$

acts like a characteristic function along $\widetilde{LX}_{0,0}$ and a distribution in the other coordinates. I have been carful with stating that the state space *contains* (3.3.8) and not with claiming that it equals this space. The reason is that I want to relate the perturbative state spaces of Morse theory to a conformal field theory. If in the spirit of Morse theory the field modes are interpreted as simple coordinates or differentials, it makes sense to allow for Taylor expansions and thus for power series. However, the representations of CFTs are usually spanned by polynomials [KR87]. Yet, if this related CFT will be formulated on $\mathbb{CP}^1$ this condition must be relaxed for the zero modes, cf. section 3.5.1.

An alternative way to identify the descending manifolds is to consider the instanton flow equation (3.2.3) for $x(z)$ in the gauged Tbc and after a change to radial coordintates $\omega = t + \mathrm{i}\sigma \mapsto \exp \omega \in \mathbb{C}^\times$

$$\left(\partial_{\bar{z}} + \frac{\mu}{\bar{z}}\right) x(z) = 0 \,. \tag{3.3.10}$$

To derive this, it is necessary to remember that $A = A_\omega \mathrm{d}\omega + A_{\bar{\omega}} \mathrm{d}\bar{\omega}$ and $A_{\bar{\omega}} = \mu$ transforms like a one form, $A_{\bar{z}} = A_{\bar{\omega}} \frac{\partial \bar{\omega}}{\partial \bar{z}}$. In particular, if I add the point $\{0\}$ to $\mathbb{C}^\times$ and consider the instanton flow equation of the Morse theory to the vacuum configuration $\{0\} \in X$ when $z \mapsto 0$ ($\Leftrightarrow t \mapsto -\infty$), i.e. invoking $x(0) = 0$, the solutions

$$x(z) = |z|^{-2\mu} \sum_{n\leq 0} x_n z^{-n} \,, \;\; x(0) = 0 \,, \;\; \mu \in (-1, 0) \tag{3.3.11}$$

reproduce the flow lines along the descending manifold and thus along the state space (3.3.8).[6] In the equation above I have scaled x with the "homogeneity" $|z|^{2\mu}$. It would have been sufficient to multiply $\bar{z}^\mu$, however, the solution x would then have been multi-valued. Single-valuedness of the fields and of correlation functions is a property demanded by conformal field theories, and I anticipated this in the solution above.

[6] The solutions ascending to $\{0\} \in X_{0,0}$ require a different boundary condition: $x(\infty) = 0$. Notice further, that closing $\mathbb{C}^\times$ to the disk $\mathbb{C}^\times \cup \{0\} \simeq D$ and demanding $x(0) = 0$ identifies $x \in LX$ with an element in $\widetilde{LX}$.

3.3.3 The Perturbative State Space on $\widetilde{LX}_{\infty,k}$

In order to derive the state space on $\widetilde{LX}_{\infty,0}$, it is at suggestive to make a coordinate transition for $x \in LX$

$$x(\sigma) \mapsto \tilde{x}(\sigma) = \tilde{x}_n \mathrm{e}^{-\mathrm{i}n\sigma} := [x(\sigma)]^{-1}, \tag{3.3.12}$$

where I define $x(\sigma)^{-1} = x_0^{-1} \sum_{n=0}^{\infty} (-)^n x_0^{-n} \Delta x(\sigma)^n$ by a Taylor expansion and with help of $\Delta x(\sigma) = \sum_{k \neq 0} x_k \mathrm{e}^{-\mathrm{i}k\sigma}$. For the only mode being inverted one has to assume that $x_0 \neq 0$. Notice, that the inverse $[x(\sigma)]^{-1}$ is well defined because x_0 has the meaning as a simple coordinate on $\mathbb{CP}^1$.

Under this coordinate transition, the instanton flow equation (3.2.8) is changed to

$$\partial_t \tilde{x}_n - (-n + \mu)\tilde{x}_n = 0, \tag{3.3.13}$$

or alternatively in radial coordinates $z = \exp t + \mathrm{i}\sigma$ for $\tilde{x}(z)$ to

$$\left(\partial_{\bar{z}} - \frac{\mu}{\bar{z}}\right)\tilde{x}(z) = 0. \tag{3.3.14}$$

This mirrors, that the action (3.2.10) is not invariant under coordinate changes.[7] In analogy to the discussion in the last section, I can now add the point $\{\infty\} = \{\tilde{x}_0 = 0\} \in X_{\infty,0}$ to $\mathbb{C}^\times$ and solve the instanton equation with boundary condition $\tilde{x}(0) = 0$ $(z \to 0 \Leftrightarrow t \to -\infty)$, in order to extract the coordinates along the descending manifold $\widetilde{LX}_{\infty,0}$. The single-valued solution for $\tilde{x}$ reads

$$\tilde{x}(z) = |z|^{2\mu} \sum_{n<0} \tilde{x}_n z^{-n}, \tag{3.3.15}$$

and similar holds for $\tilde{\psi}$. The other field modes along $\widetilde{LX}_{\infty,0}$ can now indirectly be obtained as the modes conjugate to those of $\tilde{x}$ and $\tilde{\psi}$. Therefore, the perturbative state space on $\widetilde{LX}_{\infty,0}$ equals

$$\mathscr{H}^{\mathrm{in}}_{\infty,0} = \mathbb{C}[\tilde{x}_n, \bar{\tilde{x}}_n, \tilde{\psi}_n, \bar{\tilde{\psi}}_n]_{n<0} \otimes \mathbb{C}[\tilde{p}_n, \bar{\tilde{p}}_n, \tilde{\pi}_n, \bar{\tilde{\pi}}_n]_{n\leq 0} \cdot \Delta_\infty, \tag{3.3.16}$$

with

$$\begin{aligned} \Delta_\infty &= \Xi_{\widetilde{LX}_{\infty,0}} (\psi_0 \psi_1 \cdots)(\bar{\psi}_0 \bar{\psi}_1 \cdots), \\ \Xi_{\widetilde{LX}_{\infty,0}} &\sim \prod_{n\geq 0, m>0} \delta^{(2)}(x_n, \bar{x}_n) \delta^{(2)}(\psi_n, \bar{\psi}_n) \delta^{(2)}(p_m, \bar{p}_m) \delta^{(2)}(\pi_m, \bar{\pi}_m). \end{aligned} \tag{3.3.17}$$

This fits with an analysis of the eigenvalues of the Hessian $\tilde{H}_n = -n + \mu$.

3.4 Relation to Conformal Supersymmetric Ghosts

On a first sight, these state spaces equal particular representations of the conformal supersymmetric bc-system (CSbc) with domain manifold $\mathbb{C}^\times$ and target space $\mathbb{C}$. I will first give

[7] The composition $x \mapsto x^{-1}$, $\mu \mapsto -\mu$ is a symmetry of the action.

a brief introduction to the CSbc which should clarify this relation. Afterwards, I am going to explain why I am careful with identifying the CSbc and the Tbc, though I will argue that the perturbative state spaces of the Morse theory underlying the gauged Tbc can be modelled by the CSbc.

I assume that the reader has a basic knowledge of CFTs, otherwise she or he may consult [Fri85, Gin88, Gab00].

3.4.1 The Conformal Supersymmetric bc-System

As long as it is not logarithmically extended [DF08], the CSbc is assumed to split into (equivalent) holomorphic and antiholomorphic halves. For the moment I will start with the holomorphic part.

Representation Theory

Let the domain manifold be $\mathbb{C}^\times$ with coordinates $z = \mathrm{e}^{t+\mathrm{i}\sigma}$ and the target space be $\mathbb{C}$. The CSbc consists of bosonic fields $x(z) = \sum_{n\in\mathbb{Z}} x_n z^{-n}$ and $p(z) = \sum_{n\in\mathbb{Z}} p_n z^{-n-1}$, whose modes define a Heisenberg algebra $[p_n, x_m] = -\mathrm{i}\delta_{n,-m}$, and of the superpartners $\psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n}$, and $\pi_n = \sum_{n\in\mathbb{Z}} \pi_n z^{-n-1}$ which comprises a Clifford algebra $[\pi_n, \psi_m] = -\mathrm{i}\delta_{n,-m}$.[8] There exists a whole stack of "charged" representations

$$x_n|p\rangle_- = 0 = \psi_n|p\rangle_+\,,\ n > -p \quad , \quad p_n|p\rangle_- = 0 = \pi_n|p\rangle_+\,,\ n \geq p \tag{3.4.1}$$

with $p \in \mathbb{Z}$ [FF91, Fri85]. In the case of the fermions, these representation spaces are equivalent because all highest weight states are related by

$$\begin{aligned} |p\rangle_+ &= \psi_{-p+1}\cdots\psi_0|0\rangle_+\,,\ p \geq 0\,,\\ |p\rangle_+ &= \mathrm{i}^p\ \pi_p\pi_{p+1}\cdots\pi_{-1}|0\rangle_+\,,\ p < 0\,. \end{aligned} \tag{3.4.2}$$

This does not hold for the bosonic representation spaces, as I am going to discuss in section 3.6.2. This observation will be of crucial importance for the existence of the Grothendieck-Cousin operators.

The representation spaces are graded by some bosonic and fermionic $U(1)$ currents $j^-(z) = -\mathrm{i} : x(z)p(z) :$ and $j^+(z) = -\mathrm{i} : \psi(z)\pi(z) :$, where normal ordering is defined in the $|0\rangle_\pm$ vacuum.[9] Under that condition, $|p\rangle_\epsilon$ has charge $-\epsilon p$, where $\epsilon = +1$ for fermions and -1 for the bosons. The field modes satisfy $[j^-_n, x_m] = -x_{n+m}$, $[j^-_n, p_m] = p_{n+m}$, $[j^+_n, \psi_m] = -\psi_{n+m}$, $[j^+_n, \pi_m] = \pi_{n+m}$ and the currents comprise Lie Heisenberg algebras $[j^\epsilon_n, j^\epsilon_m] = \epsilon\, n\delta_{n,-m}$. According to

[8] Remember, that $[\cdot,\cdot]$ denotes the superbracket.

[9] I use $:\cdot:$ as a $\mathbb{C}$-linear mapping such that $\lambda : a + b :=: \lambda a + \lambda b :, \lambda \in \mathbb{C}$.

Feigin and Frenkel [FF91], I will denote the thus graded representation spaces as $M^\epsilon(p) = \bigoplus_{l\in\mathbb{Z}} M^\epsilon(p)_l$, where l is the $U(1)$ charge.

To the algebra of the field modes corresponds the operator product algebra of the fields. It is represented on the p vacua by means of the operator product expansions (OPEs)

$$\begin{aligned} x(z)p(\omega) &= \frac{\mathrm{i}}{z-\omega}\left(\frac{z}{\omega}\right)^p, \quad & \psi(z)\pi(\omega) &= \frac{-\mathrm{i}}{z-\omega}\left(\frac{z}{\omega}\right)^p, \\ p(z)x(\omega) &= \frac{-\mathrm{i}}{z-\omega}\left(\frac{\omega}{z}\right)^p, \quad & \pi(z)\psi(\omega) &= \frac{-\mathrm{i}}{z-\omega}\left(\frac{\omega}{z}\right)^p. \end{aligned} \tag{3.4.3}$$

The Virasoro algebra is represented on these spaces by the energy momentum tensor

$$T(z) = \mathrm{i}\ :p(z)\partial_z x(z) - \pi(z)\partial_z\psi(z): \quad , \quad T(z) = \sum_{n\in\mathbb{Z}} T_n z^{-n-2}. \tag{3.4.4}$$

It can be obtained from the fields

$$\mathcal{G}(z) = \mathrm{i} :\pi(z)\partial_z x(z): \quad \text{and} \quad \mathcal{Q}(z) = \mathrm{i} : p(z)\psi(z): \tag{3.4.5}$$

by $T(z) = [\mathcal{Q}_0, \mathcal{G}(z)]$, where $\mathcal{Q}_0 = \oint_0 \mathcal{Q}(z)$. These fields together with the fermionic $U(1)$ charge define a twisted $\mathcal{N} = 2$ superconformal algebra [DVV91]. Since the bosonic and fermionic parts contribute with opposite central charges $c^\epsilon = -2\epsilon$, the composite system has central charge zero.

The basic fields have conformal weights

$$\Delta_T(x) = 0 = \Delta_T(\psi) \quad \text{and} \quad \Delta_T(p) = 1 = \Delta_T(\pi) \tag{3.4.6}$$

and the commutation relations with the Virasoro generators are $[T_n, x_m] = -(m+n)x_{m+n}$, $[T_n, p_m] = -mp_{n+m}$ and analogously for the fermions. In particular, one has $[j_0^\epsilon, T_0] = 0$ and the Hamiltonian respects the grading of the representation spaces $M^\epsilon(p)_l$.

The Antiholomorphic Part

The antiholomorphic currents necessarily have to be taken into account, when the CSbc gets related to the Tbc. Two reasons are that the Tbc has an anomaly free vectorial current and the central charge is zero. These effects can be achieved for the CSbc, only if the holomorphic and antiholomorphic parts are both considered.

I define the antiholomorphic currents to be

$$\bar{j}^+(\bar{z}) = +\mathrm{i} : \bar{\psi}(\bar{z})\bar{\pi}(\bar{z}):, \quad \bar{j}^-(\bar{z}) = +\mathrm{i} : \bar{x}(\bar{z})\bar{p}(\bar{z}):, \tag{3.4.7}$$

with representation spaces just as before. According to my choice of sign in that definition, the grading is, however, different, namely $\bar{M}^\epsilon(\bar{p}) = \bigoplus_{l\in\mathbb{Z}} \bar{M}^\epsilon(\bar{p})_l$, $\bar{j}_0^\epsilon\ |\bar{p}\rangle_\epsilon = \epsilon\bar{p}\ |\bar{p}\rangle_\epsilon$. Since

$$\begin{aligned} j_V^\epsilon(z,\bar{z}) &= j^\epsilon(z) + \bar{j}^\epsilon(\bar{z}), \\ j_A^\epsilon(z,\bar{z}) &= j^\epsilon(z) - \bar{j}^\epsilon(\bar{z}) \end{aligned} \tag{3.4.8}$$

are the vectorial and axial currents, respectively, the choice above invokes that the holomorphic-antiholomorphic representation spaces $M^\epsilon(p,\bar{p}) = \bigoplus_{l,s\in\mathbb{Z}} M^\epsilon(p)_l \otimes \bar{M}^\epsilon(\bar{p})_s$ are graded with respect to the vectorial currents. At this stage, this choice is a question of convenience, however, when the CSbc is logarithmically deformed, the bosonic axial symmetry will be broken, which I am going to explain in section 4.2.

Concerning the other fields in the antiholomorphic half, they are defined in complete analogy with the holomorphic scenario. The full Virasoro algebra acts on $M^\epsilon(p,\bar{p})$ by means of

$$\begin{aligned} T^-(z,\bar{z}) &= \mathrm{i} : \partial_z x(z) p(z) + \partial_{\bar{z}} \bar{x}(\bar{z}) \bar{p}(\bar{z}) :, \\ T^+(z,\bar{z}) &= \mathrm{i} : \partial_z \psi(z) \pi(z) + \partial_{\bar{z}} \bar{\psi}(\bar{z}) \bar{\pi}(\bar{z}) : \end{aligned} \tag{3.4.9}$$

under which the state $|p,\bar{p}\rangle_\epsilon := |p\rangle_\epsilon \otimes |\bar{p}\rangle_\epsilon$ has conformal weight

$$\Delta_{T^\epsilon}(|p,\bar{p}\rangle_\epsilon) = \frac{1}{2}\epsilon[p(p-1) + \bar{p}(\bar{p}-1)], \tag{3.4.10}$$

as follows from calculating $(T_0^\epsilon + \bar{T}_0^\epsilon)|p,\bar{p}\rangle$. Together with the supercharges

$$Q(z,\bar{z}) = \mathrm{i} : p(z)\psi(z) + \bar{p}(\bar{z})\bar{\psi}(\bar{z}) : \quad \text{and} \quad G(z,\bar{z}) = \mathrm{i} : \pi(z)\partial_z x(z) + \bar{\pi}(\bar{z})\partial_{\bar{z}} \bar{x}(\bar{z}) :, \tag{3.4.11}$$

the complete CSbc determines a twisted $\mathcal{N} = (2,2)$ superconformal algebra.

Ground States

The full, supersymmetric theory has several states with weight zero, i.e. all combinations of $|0\rangle_\pm$ and $|1\rangle_\pm$. However, only one of them, $|0,0\rangle := |0,0\rangle_- \otimes |0,0\rangle_+$, is a conformally invariant ground state. This can be seen by applying $T_{\pm 1}$. For instance, the state $|1,1\rangle$, whereby

$$|p,\bar{p}\rangle := |p,\bar{p}\rangle_- \otimes |p,\bar{p}\rangle_+, \tag{3.4.12}$$

has weight zero but is not invariant under $T_{\pm 1}$. A computation shows that $T_{-1}|1,1\rangle = \mathrm{i}(x_{-1}p_0 + \psi_{-1}\pi_0)|1,1\rangle \neq 0$, and similar for the antiholomorphic part.

Correlation Functions and Unitarity

Like the Tbc, the CSbc is not unitary. I will now discuss, how that can be understood as an effect of the anomaly $\mathfrak{q}$ of the currents

$$\begin{aligned} T(z) j^\epsilon(\omega) &= \frac{\mathfrak{q}}{(z-\omega)^3} + \frac{j^\epsilon(z)}{(z-\omega)^2} \\ [T_n, j_m^\epsilon] &= -m j_{n+m}^\epsilon + \frac{\mathfrak{q}}{2} n(n+1)\delta_{n,-m} \end{aligned} \quad , \qquad \mathfrak{q} = \epsilon. \tag{3.4.13}$$

Similar holds for the antiholomorphic part with $\bar{\mathfrak{q}} = -\epsilon$. The appearance of the anomaly for T_1 means that $j^\epsilon(z)$ is not invariant under $SL(2,\mathbb{C})/\mathbb{Z}_2$ transformations. Under a holomorphic transformation $z \mapsto f(z)$, the currents acquire an additional term

$$j^\epsilon(z) = j^\epsilon(f(z))\partial_z f + \frac{\mathfrak{q}}{2}\frac{\partial_z^2 f}{\partial_z f}. \tag{3.4.14}$$

The quantities that make non-unitarity manifest are the correlation functions. These are $\mathbb{C}$-bilinear mappings $(|q\rangle, \phi(z)|p\rangle) = {}_q\langle\phi(z)\rangle_p \in \mathbb{C}$, whereby ϕ is an arbitrary combination of quasi-primary fields and their field modes. This pairing is defined such, that the adjoint of $\phi(z)$ is obtained by the transformation $z \mapsto z^{-1}$, which maps an incoming to an outgoing field.[10] Moreover, it shall be $SL(2,\mathbb{C})$ invariant and respect the operator product algebra (OPA) in the sense that ${}_q\langle b(z)c(\omega)\rangle_p = \underbracket{b(z)c}(\omega)$ for appropriate q, p.[11] What is meant by "appropriate" will be clarified below.

The adjoint currents, in the sense above, are given by

$$\begin{array}{rclrcl} {j^\epsilon}^\dagger(\omega) &=& z^{-2} j^\epsilon(z^{-1}), & {\bar{j}^\epsilon}{}^\dagger(\omega) &=& \bar{z}^{-2}\bar{j}^\epsilon(\bar{z}^{-1}), \\ {j^\epsilon}^\dagger_k &=& -\mathfrak{q}\,\delta_{k,0} - j^\epsilon_{-k}, & {\bar{j}^\epsilon}{}^\dagger_k &=& -\bar{\mathfrak{q}}\,\delta_{k,0} - \bar{j}^\epsilon_{-k}. \end{array} \tag{3.4.15}$$

Due to the different sign of $\mathfrak{q} = \epsilon$ and $\bar{\mathfrak{q}} = -\epsilon$, the adjoint of the vectorial current remains anomaly free. If, however, the holomorphic part is considered separately, the anomalies due to $z \mapsto z^{-1}$ have to be compensated, if the correlation functions are supposed to be $SL(2,\mathbb{C})$ invariant. Therefore, they have to satisfy

$$(|q\rangle, j^\epsilon(z)\phi(\omega)|p\rangle) = ({j^\epsilon}^\dagger(\omega)|q\rangle, \phi(\omega)|p\rangle). \tag{3.4.16}$$

In particular, for the zero mode $({j_0^\epsilon}^\dagger|q\rangle, |p\rangle) = ([-\mathfrak{q}+q]|q\rangle, |p\rangle) \stackrel{!}{=} (|q\rangle, -p|p\rangle) = (|q\rangle, j_0^\epsilon|p\rangle)$, and the state dual to $|p\rangle$ is given by $(|-p+\mathfrak{q}\rangle, \cdot)$. In the following I will use the notation $\langle p| = (|q\rangle, \cdot)$, such that

$$\langle q|p\rangle = \delta_{q,-p+\mathfrak{q}}. \tag{3.4.17}$$

The same line of arguments holds if any combination of fields is inserted, and the non-trivial correlation functions are subject to

$$\mathrm{Corr}(q,p) = \left\{ {}_q\langle\phi(z)\rangle_p \; : \; J(\phi) = q + p - \mathfrak{q} \right\}, \tag{3.4.18}$$

whereby $J(\phi)$ denotes the total charge of that combination. The charge $\mathfrak{q}$ is called a background charge, it causes that the dual "bra" and "ket" states determine a pairing but not a scalar product.

[10] This conjugation shall not be confused with the definition of the dual states I have used in (2.2.5). The adjoint fields here are different, for they are not the antiholomorphic counter parts.

[11] Usually, one also demands that correlation functions be single valued. This can be achieved by including the antiholomorphic half, and the way how to do that is restricted by the demand to build a single-valued quantity.

3.4.2 Identifying the State Spaces

The Fock space of the CSbc in the representation on $|0,0\rangle$ equals

$$\mathscr{H}_0 = \mathbb{C}[x_n, \bar{x}_n, \psi_n, \bar{\psi}_n]_{n\le 0} \otimes \mathbb{C}[p_n, \bar{p}_n, \pi_n, \bar{\pi}_n]_{n<0} \cdot |0,0\rangle\,, \tag{3.4.19}$$

which seems to be identical with the perturbative state space (3.3.8) on $\widetilde{LX}_{0,0}$, when the field modes are related and under $\Delta_0 \simeq |0,0\rangle$. This is further promoted by the observation that upon canonical quantization, the loop space coordinates and field modes satisfy the same commutation relations, cf. pg. 44. However, the identification fails to be exact with respect to the quantum numbers of the fieldmodes and states.

Moreover, according to (3.4.1) and if the CSbc were considered on the chart of $\mathbb{CP}^1$ including the point $\{\infty\}$, the representation

$$\mathscr{H}_\infty = \mathbb{C}[\tilde{x}_n, \bar{\tilde{x}}_n, \tilde{\psi}_n, \bar{\tilde{\psi}}_n]_{n<0} \otimes \mathbb{C}[\tilde{p}_n, \bar{\tilde{p}}_n, \tilde{\pi}_n, \bar{\tilde{\pi}}_n]_{n\le 0} \cdot \widetilde{|1,1\rangle} \tag{3.4.20}$$

should structurally be identified with the perturbative state space of Morse theory (3.3.16), putting $\mathscr{H}^{\mathrm{in}}_{\infty,0} \simeq \mathscr{H}_\infty$ and $\Delta_\infty \simeq \widetilde{|1,1\rangle}$. It is, however, not yet clear how to define the CSbc on $\mathbb{CP}^1$ and, in particular, how to implement chart transitions. This has been tackled by Malikov, Schechtman and Vaintrob [MSV99], and will be the subject of section 3.5.1. Before I discuss this topic, I will extend the CSbc by introducing the homogeneities, appropriate to accomodate the quantum numbers. Moreover, I will briefly discuss the consequences it would have if one related the CSbc without homogeneity to the ungauged Tbc. This will touch the question if the Tbc can be identified with a CFT.

The CSbc with Homogeneity

For convenience, I will restrict my considerations to the chart around $0 \in \mathbb{CP}^1$. The Hamiltonian of the Morse description of the topological bc-system (3.2.10) is

$$\begin{aligned} H &= -\mathrm{i} \sum_{n\in\mathbb{Z}} (\mu + n)(x_n p_{-n} + \psi_n \pi_{-n} + \bar{x}_n \bar{p}_{-n} + \bar{\psi}_n \bar{\pi}_{-n}) \\ &= \sum_{n\in\mathbb{Z}} (\mathscr{L}_{V_n} + \mathscr{L}_{\bar{V}_n})\,, \quad V_n = -(\mu+n) x_n \partial_n\,,\ \bar{V}_n = \overline{(V_n)} \end{aligned} \tag{3.4.21}$$

and due to the shift by μ differs from $T_0 = T_0^+ + \bar{T}_0^+ + T_0^- + \bar{T}_0^-$. One can overcome this mismatch of energies by redefining the fields of the CSbc:

$$x(z) = \sum_{n\in\mathbb{Z}.} x_n z^{-n} |z|^{-2\mu}\,, \quad p(z) = \sum_{n\in\mathbb{Z}} p_n z^{-n-1} |z|^{2\mu}\,, \tag{3.4.22}$$

and similar for the fermions [FLN08]. As has been the case for the Morse theory, the fields are not holomorphic any more. Indeed, the equation of motion for the conformal field x with homogeneity μ equals the instanton equation of Morse theory $\left(\partial_{\bar{z}} + \frac{\mu}{\bar{z}}\right) x(z) = 0$. Furthermore, the boundary condition which selected the descending manifold for Morse theory has been $x(0) = 0$ and led to the expansion (3.3.11). In case of the CSbc, this boundary condition is realized by plugging in the representation $|0,0\rangle$ and considering the on-shell expansion of $x(z)$, i.e.

$$x(z)|0,0\rangle - = |z|^{-2\mu} \sum_{n\leq 0} x_n z^{-n} |0,0\rangle\,. \tag{3.4.23}$$

The field redefinitions introduce tadpoles due to the inhomogeneity. Calculating $T(z)T(\omega)$, one finds that the stress tensor should be corrected

$$T^{\epsilon}(z) \mapsto T^{\epsilon}(z) + \frac{\epsilon\mu(\mu+1)}{2z^2}\,, \quad \bar{T}^{\epsilon}(\bar{z}) \mapsto \bar{T}^{\epsilon}(\bar{z}) - \frac{\epsilon\mu(\mu+1)}{2\bar{z}^2}\,, \tag{3.4.24}$$

where T^{ϵ}, $\bar{T}^{\epsilon}$ are defined as before but with the redefined fields. However, the full stress tensor has no tadpoles and its zero mode equals the Hamiltonian of the Morse theory, $T_0 = H$. Indeed, $[T_0, x_n] = (-\mu - n)x_n$, $[T_0, p_n] = (\mu - n)p_n$ and similar for the other field modes. The highest weight states obtain new conformal weights of value $\Delta_{T^{\epsilon}}(|p\rangle_{\epsilon}) = \frac{\epsilon}{2}(p-\mu)(p-\mu-1)$, while the central charges for the bosons and the fermions are still the same. The $U(1)$ charges are also corrected by tadpoles,

$$j^{\epsilon}(z) \mapsto j^{\epsilon}(z) + \frac{\epsilon\mu}{z}\,, \quad \bar{j}^{\epsilon}(\bar{z}) \mapsto \bar{j}^{\epsilon}(\bar{z}) - \frac{\epsilon\mu}{\bar{z}}\,, \tag{3.4.25}$$

while the charge anomalies are not affected. Thus, the states $|p\rangle_{\epsilon}$ and $|\bar{p}\rangle_{\epsilon}$ have $U(1)$ charges of value $-\epsilon(p-\mu)$ and $\epsilon(\bar{p}-\mu)$, while the charges of the field modes are insensitive to μ.

Let me conclude that for the CSbc with homogeneity one may identify

$$\mathscr{H}^{\text{in}}_{0,0} \simeq \mathscr{H}_0\,, \qquad \Delta_0 \simeq |0,0\rangle\,, \tag{3.4.26}$$

and the field modes and states have the correct quantum numbers.

3.4.3 What if the Gauge Field is Absent?

Having stated a correspondence between the low energy spectrum of the gauged Morse theory on the descending manifold $\widetilde{LX}_{0,0}$ and the CSbc with homogeneity, one might now ask, if the CSbc with $\mu = 0$ were the appropriate theory to describe the Morse theory of the Tbc without gauge field? The Hamiltonians are identical and the field modes have the same energies. I will now argue, that such a relation fails, because the Tbc without gauge field has more topological states than the ordinary CSbc.

The Topological States of Morse Theory without Homogeneity

Since the Hessian is indefinite on the zero modes, these coordinates are neither transversal coordinates nor coordinates along the descending manifold. Moreover, they have zero energy and in principle may be multiplied to the ground states. Thus, there are not such strong constraints on the ground states as in the situation with gauge field.

A first consequence is that in the zero modes the ground states are smooth differential forms on $X = \mathbb{CP}^1$ with respect to the de Rham differential d, i.e. elements of $\Omega^\bullet_{\mathrm{d}}(X)$ [FLN08]. To comprise ground states in the sense of topological states, this space must be further restricted by the BRST condition $Q_0\Delta_0 = 0$. In analogy with (2.2.4) and in coordinates of loop space, the BRST charge for the Morse theory equals

$$Q_0 = \mathrm{i}\sum_{n\in\mathbb{Z}} (\psi_n p_{-n} + \bar{\psi}_n \bar{p}_{-n})\,. \tag{3.4.27}$$

In particular, its zero mode part can be identified with the usual de Rham derivative $\mathrm{d} = \partial + \bar{\partial}$ on X. Since $\mathbb{CP}^1$ has Betti numbers $\dim H^0_{\mathrm{d}}(X,\mathbb{R}) = \dim H^2_{\mathrm{d}}(X,\mathbb{R}) = 1$ and $\dim H^1_{\mathrm{d}}(X,\mathbb{R}) = 0$, all closed differential forms must have an even form degree, i.e. an even number of ψ_0, $\bar{\psi}_0$. Consequently, the zero mode part of the topological states comprises even graded differential forms on X, and there do not exist states with an odd number of ψ_0..

In contrast, if $\mu \in (-1,0)$, the zero modes have non-vanishing energy and are not subject to the restriction via Q_0. In particular, this signifies that the theory with gauge field includes differential forms with odd numbers of $\psi_0, \bar{\psi}_0$.

The Ungauged Morse Theory is not Canonically Related to the CSbc

The representation space for the CSbc consists of polynomials in the zero modes and not of smooth differential forms. However, in my oppinion this is not the main aspect which makes the difference to the Morse theory with $\mu = 0$, as claimed by Frenkel et al. in [FLN08, pg. 32]. As already mentioned on pg. 45, the zero modes will be allowed to appear in power series, when the CSbc is generalized to the chiral de Rham complex [MSV99]. Rather, the difference lies in the following observation. The ground states in the Morse type theory do not necessarily factorize into holomorphic and antiholomorphic (target space) coordinates, in general there do not exist holomorphic and antiholomorhpic functions h and $\bar{h}$ such that $f(x_0,\bar{x}_0)\psi_0^p\bar{\psi}_0^q = [h(x_0)\psi_0^p]\cdot[\bar{h}(\bar{x}_0)\bar{\psi}_0^q]$. In ordinary conformal field theories this is, however, the case because the Virasoro algebra factorizes. Therefore, the vacuum sector of the CSbc is smaller than that of the Tbc when $\mu = 0$.

That the holomorphic and antiholomorphic parts do not factorize is a property which is also typical for logarithmic conformal field theories. However to the best of my knowledge, this is still untypical for the ground states. At least it indicates that if the Tbc without gauge field is conformal, it can not be an ordinary conformal field theory.

3.5 Conformal Supersymmetric Ghosts on $\mathbb{CP}^1$

In the last section I have obtained the perturbative state spaces of the Morse theory underlying the Tbc. The most important observation has been that they can be modelled by representations of the conformal supersymmetric bc-system (CSbc). However, this relation had the drawback that the CSbc is not globally defined on $\mathbb{CP}^1$, such that I could not reproduce the chart transition of the Morse theory on the level of the CSbc.

I will now clarify how the CSbc can be formulated globally on $\mathbb{CP}^1$ and introduce the chiral de Rham complex [MSV99]. This section will conclude the analysis of the perturbative representation theory of the Morse theory underlying the Tbc.

3.5.1 The Chiral de Rham Complex

The chiral de Rham complex generalizes the usual de Rham complex on X to a larger complex $\Omega^\bullet_{\mathscr{Q}_0}(X)$, defined on a sheaf of vertex algebras on X. In the context of the A-model, it will be the Dolbeault complex with is generalized by the cohomology operator $\mathscr{Q}_0 = \partial + \mathrm{D}$, $[\partial, \mathrm{D}] = 0$, $\mathscr{Q}_0^2 = 0$. Hereby, ∂ denotes the holomorphic (Dolbeault) differential on X, and the vertex algebra under consideration is the holomorphic CSbc with homogeneity, cf. section 3.4.1. Its supercharge $\mathscr{Q}_0 = \mathrm{i}\sum_{n\in\mathbb{Z}} \psi_n p_{-n}$ will play the rôle of the generalized exterior differential.

Local Vertex Algebra of the CSbc

Consider the holomorphic CSbc with homogeneity and embedding $x : \Sigma \to \mathbb{C}_0 \subset X = \mathbb{CP}^1$. For convenience, I choose the representation to be $M^\epsilon(0)$ on $|0\rangle = |0\rangle_+ \otimes |0\rangle_-$.

The state space can be identified with the polynomials in the modes

$$\mathscr{P}_0 = \mathbb{C}[x_n, \psi_n,]_{n\leq 0} \otimes \mathbb{C}[p_n, \pi_n]_{n<0} \tag{3.5.1}$$

and one can define a so-called vertex operator, constituting an isomorphy between fields and states

$$\begin{aligned} Y(x_0, z) &= x(z) \quad , \quad Y(x_{-n}, z) = \frac{1}{n!}\partial_z^n x(z)\,,\ n<0\,, \\ Y(p_{-1}, z) &= p(z) \quad , \quad Y(p_{-n}, z) = \frac{1}{n!}\partial_z^n p(z)\,,\ n<-1\,, \end{aligned} \tag{3.5.2}$$

and similar for the other fields. For any monomial $y_1 \cdots y_k$ which is built by elements $y_i \in \{x_n, p_m, \psi_n, \pi_m\}_{n\leq 0, m<0}$ the vertex operator is generalized by means of

$$Y(y_1 \cdots y_k, z) =: Y(y_1, z) \cdots Y(y_k, z) :, \tag{3.5.3}$$

and this further extends to polynomials. In order to simplify notations, I will equivalently write $Y(y_1 \cdots y_k, z) = y_1 \cdots y_k(z)$.

Due to their transformation property under $\mathcal{Q}_0$ and their conformal weights, at least for $\mu = 0$, the zero modes can be identified with the geometric data on X, as has already been done for the Morse theory, cf. (3.3.6) and (3.3.7). On that grounds, it would be nice to extend the definition of the vertex algebra to power series in the zero modes. I will adopt the approach of [MSV99, pg. 449f] to the situation $\mu \neq 0$. Let $f(x_0)$ be a power series and define $Y(f(x_0), z)$ by the Taylor expansion

$$Y(f(x_0), z) := \sum_{n=0}^{\infty} \Delta x(z)^n \frac{1}{n!} \partial^n_{|z|^{-2\mu} x_0} f(|z|^{-2\mu} x_0), \quad \Delta x(z) = |z|^{-2\mu} \sum_{k \neq 0} x_k z^{-k}. \tag{3.5.4}$$

One can write $\Delta x(z)^n = \sum_{k \in \mathbb{Z}} c_k(|z|) z^{-k}$, wherein $c_k(|z|)$ is an infinite sum of monomials of the kind $\{|z|^{-2\mu} x_n\}_{n \neq 0}$. On any $|v\rangle \in \mathbb{C}[x_n, p_n, \psi_n, \pi_n]_{n<0} \otimes \mathbb{C}[[x_0, \psi_0]] \cdot |0\rangle$, $c_k(|z|)$ breaks down to a finite sum and thus $Y(f(x_0), z)$ is a well defined endomorphism on that space. The thus generalized fields can be multiplied by any polynomial field $g(y)(z)$, $y \in \{x_n, p_m, \psi_n, \pi_m\}_{n \leq 0, m<0}$

$$Y(g(y) f(x_0), z) =: Y(g(y), z) Y(f(x_0), z) : . \tag{3.5.5}$$

The inverse operation, to obtain a state given a field, works by

$$f(y) = Y(f(y), z)|_{\mu=0} \cdot |0\rangle \, \Big|_{z=0} , \tag{3.5.6}$$

where $Y(f(y), z)$ is an arbitrary field. Thus, Y defines an isomorphism between states and fields.

Local Extension of the de Rham Complex

Since the zero modes can be identified geometric data on X, the supercharge $\mathcal{Q}_0$ takes the required form $\mathcal{Q}_0 = \partial + \mathrm{d}_- + \mathrm{d}_+$, $\mathrm{d}_- := \sum_{n<0} p_{-n} \psi_n$ and $\mathrm{d}_+ = \sum_{n>0} p_{-n} \psi_n$ on $\mathcal{P}_0$. Malikov et al. [MSV99] prove, that there is a quasiisomorphism $(\Omega, \partial) \to (\mathcal{P}_0, \mathcal{Q}_0)$, where $\Omega = \mathbb{C}[x_0, \psi_0]$. That means, ∂ does only act on the subsector of the zero modes and commutes with $\mathrm{d}_\pm$ and the cohomologies are the same $H^\bullet_\partial(\Omega) \simeq H^\bullet_{\mathcal{Q}_0}(\mathcal{P}_0)$. The proof is made by successively calculating the cohomologies of d_+ and d_- and can be generalized to $\Omega = \mathbb{C}[[x_0, \psi_0]]$ and $\mathcal{P}_0 = \mathbb{C}[x_n, p_n, \psi_n, \pi_n]_{n<0} \otimes \mathbb{C}[[x_0, \psi_0]]$, cf. [MSV99, pg. 448]. Thus, locally, the de Rham complex generalizes to a complex of vertex algebras under $\mathcal{Q}_0$.

Chart Transitions

In order to extend the local setting to $\mathbb{CP}^1$, it is especially important to give the mapping $X_{0,0} \setminus \{0\} \simeq \mathbb{C}^\times \ni x_0 \mapsto x_0^{-1}$ a meaning on the level of fields.

Firstly, on the level of field zero modes p_0 acts as a derivative and thus a commutation with x_0^{-1} can be defined as $[p_0, x_0^{-1}] = -[p_0, x_0]x_0^{-2}$. Now, in analogy with (3.5.4), the field corresponding to x_0^{-1} can be declared to equal

$$Y(x_0^{-1}, z) = |z|^{2\mu} x_0^{-1} \sum_{n=0}^{\infty} (-)^n |z|^{2n\mu} x_0^{-n} \Delta x(z)^n, \tag{3.5.7}$$

where I define $\tilde{Y}(\tilde{x}_0, z) = Y(x_0^{-1}, z)$. For convenience, I will also use the notation $\tilde{Y}(\tilde{x}_0, z) = \tilde{x}(z) = |z|^{2\mu} \sum_{n\in\mathbb{Z}} \tilde{x}_n z^{-n}$. Notice, that in analogy with (3.4.2), the transformed field $\tilde{x}$ satisfies the equation of motion $(\partial_{\bar{z}} - \frac{\mu}{\bar{z}})\tilde{x}(z) = 0$.

In the same spirit as above, Malikov et al. generalize chart transitions of the other zero modes to chart transitions of fields. Let $f : x_0 \mapsto \phi_x = f(x_0)$ be an invertible coordinate transformation with $f \in \mathbb{C}[[x_0]]$. Since they can be related to geometric quantities on X, the other field zero modes transform according to

$$\begin{aligned} \phi_x &= f(x_0), & \phi_\psi &= \frac{\partial f}{\partial x_0}\psi_0, \\ \phi_p &= \frac{\partial f^{-1}}{\partial \phi_x} p_0 + \frac{\partial^2 f^{-1}}{\partial \phi_x^2}\frac{\partial f}{\partial x_0}\psi_0\pi_0, & \phi_\pi &= \frac{\partial f^{-1}}{\partial \phi_x}\pi_0. \end{aligned} \tag{3.5.8}$$

Here, Malikov et al. assume that the action corresponding to the CSbc equals (3.1.7), where p_a is rather $p'_a = p_a + \Gamma^b_{ac}\psi^c\pi_b$. The transformation of $\Gamma \mapsto \left(\frac{\partial f^{-1}}{\partial \phi_x}\right)^2 \frac{\partial f}{\partial x_0}\Gamma + \frac{\partial^2 f^{-1}}{\partial \phi_x^2}\frac{\partial f}{\partial x}$ explains why p_0 above does not transform homogeneously. The fields corresponding to the power series above are now defined to be

$$\begin{aligned} \phi_x(z) &= f(x_0)(z), & \phi_\psi(z) &=: \frac{\partial f}{\partial x_0}(z)\psi(z):, \\ \phi_p(z) &=: \frac{\partial f^{-1}}{\partial \phi_x}(z)p(z) + \frac{\partial^2 f^{-1}}{\partial \phi_x^2}\frac{\partial f}{\partial x_0}(z)\psi(z)\pi(z):, & \phi_\pi(z) &=: \frac{\partial f^{-1}}{\partial \phi_x}(z)\pi(z):. \end{aligned} \tag{3.5.9}$$

This definition is not obtained by simply using the vertex operator on the field modes above. The reason is twofold. Firstly, Y is not defined on π_0 and p_0 since they are not part of $\mathscr{P}_0$. Secondly, the definition is such that the transformed fields are again primary fields.

In a next step, that I will not reproduce, the authors verify that the transformed fields preserve the commutation rules (3.4.3). The ambitioned reader may check this for the following example, making use of the relation

$$f(x)(z)p(\omega) = \frac{\partial f}{\partial x_0}(\omega)\ \overline{x(z)p}(\omega) \tag{3.5.10}$$

and similar for $p(z)f(x)(\omega)$. In terms of the field modes, this amounts to $[p_0, f(x_0)] = [p_0, x_0]\partial_{x_0} f(x_0)$.

Example in Logarithmic Coordinates A particular example that I will make use of in appendix C is the CSbc in logarithmic coordinates $x_0 \mapsto \exp x_0$. The thus transformed fields are

$$\begin{aligned} \phi_x(z) &=: \mathrm{e}^{x(z)} :, & \phi_p(z) &=: \mathrm{e}^{-x(z)} \left[p(z) - \psi(z)\pi(z) \right] :, \\ \phi_\psi(z) &=: \mathrm{e}^{x(z)}\psi(z) :, & \phi_\pi(z) &=: \mathrm{e}^{-x(z)}\pi(z) : . \end{aligned} \tag{3.5.11}$$

A coordinate transition $\phi_x \mapsto \phi_x^{-1}$ changes the sign of the fields $\{x, p, \psi, \pi\}$ above.

The Vertex Operator Algebra in the New Fields The vertex algebra in terms of the fields in (3.5.9) is obtained in analogy to (3.5.2) and (3.5.3). The question is indeed not how the fields are constituted, but how to get back the field modes in the new coordinates. This is obtained by (3.5.6). In particular, for a monomial $y^1_{n_1} \cdots y^N_{n_N}$, where y^k_n is a field mode among $\mathscr{P}_0$, one can specify the corresponding states in the new coordinates according to

$$\phi_{y^1_{n_1} \cdots y^N_{n_N} |0\rangle} = [\phi_{y^1_{n_1}}(z)]_{n_1} \cdots [\phi_{y^N_{n_N}}(z)]_{n_N} \cdot |0\rangle\,, \tag{3.5.12}$$

where $[\phi_y(z)]_n$ denotes the field mode $(\phi_y)_n$ in the expansion $\phi_y(z) = |z|^{2\mu} \sum_{n\in\mathbb{Z}} (\phi_y)_n z^{-n-\Delta}$.

Important examples are the composite fields $\mathscr{Q}(z)$, $T(z)$, $\mathscr{G}(z)$ and $j^\pm(z)$. Take, for instance, $\phi_{\mathscr{Q}}(z) = \mathrm{i} : \phi_p(z)\phi_\psi(z) :$, according to the discussion above this field is obtained as $\phi_{\mathscr{Q}}(z) = Y(\mathrm{i}\phi_{p_{-1}}\phi_{\psi_0}, z)$. Is it possible to further express the field modes (state) in terms of the original ones and thereby obtain a formulation in terms of the original fields? In the new coordinates, the state corresponding to the supercharge reads

$$\begin{aligned} \phi_{\mathscr{Q}|0\rangle} = \mathrm{i} &\left(\frac{\partial f^{-1}}{\partial \phi_x} p + \frac{\partial^2 f^{-1}}{\partial \phi_x^2} \frac{\partial f}{\partial x_0} \psi\pi \right)_{-1} (z) \left(\frac{\partial f}{\partial x_0}\psi \right)_0 (z) \cdot |0\rangle = \\ \mathrm{i} &\left[\left(\frac{\partial f^{-1}}{\partial \phi_x} \right)_0 p_{-1} \right] \left[\left(\frac{\partial f}{\partial x_0} \right)_0 \psi_0 \right] \cdot |0\rangle \ + \\ &\mathrm{i} \left[\left(\frac{\partial f^{-1}}{\partial \phi_x} \right)_{-1} p_0 + \left(\frac{\partial f^{-1}}{\partial \phi_x} p + \frac{\partial^2 f^{-1}}{\partial \phi_x^2} \frac{\partial f}{\partial x_0} \psi \right)_{-1} \pi_0 \right] \left(\frac{\partial f}{\partial x_0} \right)_0 \psi_0 \cdot |0\rangle\,, \end{aligned}$$

where I noted down all modes that potentially contribute non-trivially. To normal order the expression above, I commute them to the right such that

$$\begin{aligned} \mathrm{i}(\phi_p)_{-1}(\phi_\psi)_0 \cdot |0\rangle \ &= \ \mathrm{i}p_{-1}\psi_0 \cdot |0\rangle \ + \\ &\left[\left(\frac{\partial^2 f}{\partial x_0^2} \right)_0 \left(\frac{\partial f^{-1}}{\partial \phi_x} \right)_{-1} \psi_0 + \left(\frac{\partial^2 f^{-1}}{\partial \phi_x^2} \frac{\partial f}{\partial x_0} \psi \right)_{-1} \left(\frac{\partial f}{\partial x_0} \right)_0 \right] \cdot |0\rangle\,. \end{aligned}$$

Here, I used (3.5.10) in order to calculate the commutator $[p_0, x_0]$. Now, the fact that $\left(\frac{\partial^2 f}{\partial x_0^2}\right)_0 = -\left(\frac{\partial f}{\partial x_0}\right)_0^3 \left(\frac{\partial^2 f^{-1}}{\partial \phi_x^2}\right)_0$, and $\left(\frac{\partial f^{-1}}{\partial \phi_x}\right)_{-1} \left(\frac{\partial f}{\partial x_0}\right)_0 = -\left(\frac{\partial f}{\partial x_0}\right)_{-1} \left(\frac{\partial f^{-1}}{\partial \phi_x}\right)_0$ allows to simplify the expression above, and one ends up with

$$\phi_{\mathscr{Q}}(z) = \mathscr{Q}(z) + \partial_z \left[\partial_{\phi_x} \left(\log \frac{\partial f^{-1}}{\partial \phi_x} \right) \phi_\psi(z) \right]. \tag{3.5.13}$$

In particular, since the "correction" to $\mathcal{Q}$ is only a derivative in z, the zero mode is invariant under a coordinate change, i.e. $\mathcal{Q}_0 = \phi_{\mathcal{Q}\,0}$, the cohomology charge of the chiral de Rham system must already globally defined on X.

This observation holds for the zero modes of the fermionic current, and also the stress tensor $T(z)$ is globally defined on X, as follows from:

$$\phi_{j^+}(z) = j^+(z) + \partial_z \log\left(\frac{\partial f}{\partial x_0}\right) \quad , \quad \phi_{\mathcal{G}}(z) = \mathcal{G}(z) \tag{3.5.14}$$

and $T(z) = [\mathcal{Q}_0, \mathcal{G}(z)]$. Consequently, the j_0^+ operator, that measures the fermionic charge, and the BRST operator are well defined on the chiral de Rham complex and j_0^+ determines a grading of the sheaf. The bosonic $U(1)$ current does not transform in a particular nice way, as the reader might want to check. In logarithmic coordinates one gets

$$\phi_{j^-}(z) = -j^+(z) - \mathrm{i}p(z), \qquad (\text{with } \phi_x(z) = \mathrm{e}^x(z))\,. \tag{3.5.15}$$

The CSbc on $\mathbb{CP}^1$

The outcome of the former sections is that I can locally write down the CSbc and apply chart transitions. In order to formulate the theory globally on $\mathbb{CP}^1$, the local vertex algebras have to be glued together.

Let

$$\mathcal{F}_0 := \mathbb{C}[[x_0, \psi_0]] \otimes \mathbb{C}[x_n, \psi_n]_{n<0} \otimes \mathbb{C}[p_n, \pi_n]_{n<0} \cdot |0\rangle\,. \tag{3.5.16}$$

together with Y be the CSbc on $\mathbb{C}_0$ and

$$\mathcal{F}_\infty = \mathbb{C}[[\tilde{x}_0, \tilde{\psi}_0]] \otimes \mathbb{C}[\tilde{x}_n, \tilde{\psi}_n]_{n<0} \otimes \mathbb{C}[\tilde{p}_n, \tilde{\pi}_n]_{n<0} \cdot |\tilde{0}\rangle \tag{3.5.17}$$

with $\tilde{Y}$ another CSbc on $\mathbb{C}_\infty$. To both, I can apply $x_0 \mapsto x_0^{-1} = \tilde{x}_0$, $\tilde{x}_0 \mapsto \tilde{x}_0^{-1} = x_0$ and formulate the theories on the overlap $\mathbb{C}^\times$. By means of (3.5.9), $Y \mapsto \tilde{Y}$ and vice versa, and the vertex algebras can be glued together

$$\begin{aligned} \mathcal{F}^\times &= \mathbb{C}[[x_0^{-1}, \psi_0]] \otimes \mathbb{C}[x_n, \psi_n]_{n<0} \otimes \mathbb{C}[p_n, \pi_n]_{n<0} \otimes \cdot |0\rangle \\ &\simeq \mathbb{C}[[\tilde{x}_0^{-1}, \tilde{\psi}_0]] \otimes \mathbb{C}[\tilde{x}_n, \tilde{\psi}_n]_{n<0} \otimes \mathbb{C}[\tilde{p}_n, \tilde{\pi}_n]_{n<0} \otimes \cdot |\tilde{0}\rangle\,. \end{aligned} \tag{3.5.18}$$

This heuristically concludes the interpretation of the CSbc as a sheaf on $\mathbb{CP}^1$.[12]

Sheaves with Support In order to discuss the chiral de Rham complex associated to the topological A-model it is necessary to extend the analysis to sections with support in closed

[12] For a rigorous prove that the CSbc on $\mathbb{CP}^1$ and more general manifolds X constitutes a sheaf, cf. [MSV99].

or locally closed subsets.[13] In particular, the perturbative state space on $\widetilde{LX}_{\infty,0}$ are modeled by

$$\mathscr{F}^1_\infty = \mathbb{C}[\tilde{x}_n, \tilde{\psi}_n]_{n<0} \otimes \mathbb{C}[\tilde{p}_n, \tilde{\pi}_n]_{n\leq 0} \cdot |\tilde{1}\rangle\,, \tag{3.5.19}$$

which is the holomorphic part of (3.4.20), and not by $\mathscr{F}_\infty$.While the fermionic part of that space can be identified with the one in $\mathscr{F}_\infty$, because all these representations are isomorphic (3.4.2), this is not true for the bosons.

I will not attempt to enlarge the analysis of the Chiral de Rham complex to (locally) closed subsets. I will rather assume that this can be done and that $\mathscr{F}_0$ and $\mathscr{F}^1_\infty$ are part of a sequence similar to (2.5.3) or (2.6.2).

3.6 Beyond the Perturbative Representations

In the last sections, I have described the perturbative state spaces of the A-model on target space $X = \mathbb{CP}^1$. While the ground states are already globally and nonperturbatively defined on X, the exited states may be sensitive to nonperturbative corrections which destroy their local character, 2.5. One distinguished place where these corrections appear is the Hamiltonian, and the main task in the following sections will be to determine the analogues of the Grothendieck-Cousin operators of chapter 2. Throughout my thesis, I will denote these analogous operators as "Grothendieck-Cousin operators", though the term may not be correct for the infinite dimensional setting.

In order to determine the Grothendieck-Cousin operators, I will bosonize the CSbc in the spirit of Feigin and Frenkel [FF90, FF91] and of Friedan, Martinec and Shenker [FMS86, Fri85]. Thereby, I obtain the GCOs in a specific formulation of the vertex algebra of the CSbc. As already mentioned, this description differs from the one used by Frenkel et al. [FLN08], and exctends the analysis of [FF90, FF91, FMS86, Fri85].

Moreover, I will discuss the interpretation of the GCOs as cohomology operators. In the bosonized description of the vertex algebras defined by (3.5.16) and (3.5.19), it will become transparent that the GCOs are the bosonic analogues of the screening operator for the purely fermionic bc-system, cf. [FFH+02].

3.6.1 Existence of Grothendieck-Cousin Operators

The Grothendieck-Cousin operators δ are mappings between the perturbative state spaces $\mathscr{H}_{0/\infty,n}$ subject to the condition (2.6.1):

$$\exists\ \delta\ : \mathscr{H}^{\mathrm{in}}_{\infty/0,n} \to \mathscr{H}^{\mathrm{in}}_{0/\infty,k} \quad \Leftrightarrow \quad \widetilde{LX}_{0/\infty,n} \prec \widetilde{LX}_{\infty/0,k}\,. \tag{3.6.1}$$

[13] A locally closed set is a set which is an intersection of an open with a closed set.

Therefore, one has to clarify which descending manifolds satisfy $\widetilde{LX}_{0/\infty,n} \prec \widetilde{LX}_{\infty/0,k}$. I owe Edward Frenkel a nice proof of the fact that $\widetilde{LX}_{\infty,n} \prec \widetilde{LX}_{0,n}$ and $\widetilde{LX}_{0,n+1} \prec \widetilde{LX}_{\infty,n}$.

The proof starts with reconsidering the situation of Morse theory on $\mathbb{CP}^1$ in section 2.4. The target manifold is defined as $\mathbb{CP}^1 := (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times$, where $\mathbb{C}^2 \setminus \{0\} \ni (f,g) \sim \lambda(f,g)$, $\lambda \in \mathbb{C}^\times$ are the homogeneous coordinates.[14] In terms of homogeneous coordinates, when identifying the vectors $(0,1)$ with $\{0\} \in \mathbb{CP}^1$ and $(1,0)$ with $\{\infty\} \in \mathbb{CP}^1$, one can describe now X_0 as the $\mathbb{C}^\times$ orbit of $(f,1)$ and X_∞ as the $\mathbb{C}^\times$ orbit of $(1,0)$. These reproduce the inhomogeneous coordinates for X_0 by $z = f \in \mathbb{C}$, whereas for X_∞ it is $\omega = 0$ and $X_\infty \simeq \{\infty\}$. One can now proof that $X_\infty \prec X_0$ by letting $f \neq 0$ and $(f,1) \sim (1,f^{-1}) \stackrel{f\to\infty}{\longrightarrow} (1,0)$.

The space $\widetilde{L\mathbb{CP}^1}$ can analogously be defined by $(\mathbb{C}[[z]] \times \mathbb{C}[[z]] - \{0\})/\mathbb{C}^\times[[z]]$ with vectors $\mathbb{C}[[z]] \times \mathbb{C}[[z]] \ni (f(z),g(z)) \sim \lambda(z)(f(z),g(z))$, $\lambda \in \mathbb{C}^\times[[z]]$. Here, $\mathbb{C}[[z]]$ denotes the space of power series in z with $f(z) = \sum_{n\leq 0} f_n z^{-n}$, where $z \in D$, and similar holds for $g(z)$.

In the situation under discussion $\mu \in (-1,0)$, and the descending manifolds $\widetilde{LX}_{0/\infty,n}$ correspond to solutions of the instanton equation with boundary condition $x(0) = 0$. As discussed in 3.4.2, in a neighborhood of $\{0\} \in \mathbb{CP}^1$ they read $x(z) = |z|^{-2\mu} \sum_{n\geq 0} x_{-n} z^n$ and $\widetilde{LX}_{0,k}$ has inhomogeneous coordinates $\{x_n\}_{n\leq 0}$. In a neighborhood around $\{\infty\}$ one has solutions $\tilde{x}(z) = |z|^{2\mu} \sum_{n\geq 1} x_{-n} z^n$ and inhomogeneous coordinates $\{\tilde{x}_n\}_{n\leq -1}$ on $\widetilde{LX}_{\infty,k}$, cf. section 3.3.3.

The descending manifold $\widetilde{LX}_{0,k}$ can now be described as the orbit of $(f(z),g(z))$ under $\mathbb{C}^\times[[z]]$, whereby

$$f(z) \ \in \ z^k |z|^{-\mu} \mathbb{C}[[z]], \quad g(z) = (1 + O(z)) z^k |z|^\mu \ \in \ z^k |z|^\mu \mathbb{C}[[z]].$$

Analogously, $\widetilde{LX}_{\infty,k}$ is obtained as the orbit of $(f(z),g(z))$ with

$$g(z) \ \in \ z^{k+1} |z|^\mu \cdot \mathbb{C}[[z]], \quad f(z) = (1 + O(z)) z^k |z|^{-\mu} \ \in z^k |z|^{-\mu} \mathbb{C}[[z]],$$

and g is proportional to an additional factor of z in order to yield the correct expansion index in $\tilde{x}(z) = |z|^{2\mu} \sum_{n\geq 1} x_{-n} z^n$. Moreover, I have assumed that $z \neq 0$ and scaled the power series by z^k in order to distinguish the index by $H_2(X,\mathbb{Z})$. Without loss of generality I set $\mu = 0$ and prove below that ❶ $\widetilde{LX}_{\infty,k} \prec \widetilde{LX}_{0,k}$ and ❷ $\widetilde{LX}_{0,k+1} \prec \widetilde{LX}_{\infty,k}$.

❶ Let $(f(z),g(z)) = z^k(f_k + O(z), 1 + O(z))$ be an element of $\widetilde{LX}_{0,k}$ with $f_k \neq 0$, then $(f(z),g(z)) \sim z^k(1 + O(z), f_k^{-1} + O(z)) \stackrel{g_k\to\infty}{\longrightarrow} z^k(1 + O(z), zh(z))$ with $h \in \mathbb{C}[[z]]$, and this is an element of $\widetilde{LX}_{\infty,k}$.

❷ Let $(f(z),g(z)) = z^k(1 + O(z), g_{k+1} z + O(z^2))$ be in $\in \widetilde{LX}_{\infty,k}$ with $g_{k+1} \neq 0$, then $(f(z),g(z)) \sim z^k(g_{k+1}^{-1} + O(z), z + O(z^2)) \stackrel{g_{k+1}\to\infty}{\longrightarrow} z^{k+1}(h(z), 1 + O(z))$, where $h(z) \in \mathbb{C}[[z]]$, and this is an element of $\widetilde{LX}_{0,k+1}$.

[14] In the former sections I have considered the descending manifolds $X_0 \simeq \mathbb{C}$ and $X_\infty \simeq \{\infty\}$ always in inhomogeneous coordinates.

To conclude, in the situation that $X = \mathbb{CP}^1$ and the gauge field is determined by $\mu \in (-1,0)$, there exist two sorts of Grothendieck-Cousin operators

$$\begin{aligned} \delta_1 &: \mathcal{H}^{\text{in}}_{\infty,n} \to \mathcal{H}^{\text{in}}_{0,n}, \\ \delta_2 &: \mathcal{H}^{\text{in}}_{0,n+1} \to \mathcal{H}^{\text{in}}_{\infty,n}. \end{aligned} \tag{3.6.2}$$

3.6.2 Chiral Bosonization

The method of chiral bosonization goes back to Friedan, Martinec and Shenker [FMS86] and starts with the holomorphic (or antiholomorphic) part of the CSbc. In the following, I will generalize this approach to the CSbc with homogeneity μ.

In order to treat the bosons and fermions in one and the same formalism, I rescale the fields of the CSbc in 3.4.1

$$\begin{aligned} \epsilon = -: &\quad x \mapsto b^-, \quad \mathrm{i}p \mapsto c^-, \\ \epsilon = +: &\quad \psi \mapsto b^+, \quad \mathrm{i}\pi \mapsto c^+, \end{aligned} \tag{3.6.3}$$

whereby the index ϵ discriminates bosons, $\epsilon = -$, from fermions, $\epsilon = +$. The basic idea of chiral bosonization is to express the Heisenberg and Clifford algebras and their representations in terms of Heisenberg Lie algebras $\mathcal{A}^\epsilon(h)$:

$$[J^\epsilon_n, J^\epsilon_m] = \epsilon\, n \delta_{n,-m} \tag{3.6.4}$$

with representation

$$J^\epsilon_n v^\epsilon_h = h\delta_{n,0} \cdot v^\epsilon_h, \;\; n \geq 0, \;\; h \in \mathbb{C}, \tag{3.6.5}$$

and equally for the antiholomorphic part. I define the fields corresponding to J^ϵ as

$$\begin{aligned} J^\epsilon_\mu(z) = J^\epsilon(z) + \frac{\epsilon\mu}{z}, &\quad J^\epsilon(z) J^\epsilon(\omega) = \frac{\epsilon}{(z-\omega)^2}, \\ \bar{J}^\epsilon_\mu(\bar{z}) = \bar{J}^\epsilon(\bar{z}) - \frac{\epsilon\mu}{\bar{z}}, &\quad \bar{J}^\epsilon(\bar{z}) \bar{J}^\epsilon(\bar{\omega}) = \frac{\epsilon}{(\bar{z}-\bar{\omega})^2}. \end{aligned} \tag{3.6.6}$$

The different signs for the holomorphic and antiholomorphic fields will be understandable when it comes to match the Heisenberg Lie algebras with the CSbc. The action of the Virasoro algebra on these representations is given by

$$T_{J^\epsilon}(z) = \epsilon : \frac{1}{2} J^\epsilon_\mu(z)^2 + \alpha_0 \partial_z J^\epsilon_\mu(z) : \quad , \quad \bar{T}_{\bar{J}^\epsilon}(\bar{z}) = \epsilon : \frac{1}{2} \bar{J}^\epsilon_\mu(\bar{z})^2 + \bar{\alpha}_0 \partial_{\bar{z}} \bar{J}^\epsilon_\mu(\bar{z}) : . \tag{3.6.7}$$

Taking the OPE between T_{J^ϵ} and J^ϵ_μ yields

$$T_{J^\epsilon}(z) J^\epsilon_\mu(\omega) = \frac{-2\alpha_0}{(z-\omega)^3} + \frac{J^\epsilon_\mu(z)}{(z-\omega)^2}, \tag{3.6.8}$$

and similar for the antiholomorphic situation. Thus, I set $\alpha_0 = -\frac{1}{2}\epsilon$, $\bar{\alpha}_0 = \frac{1}{2}\epsilon$, in order to obtain the same background charges as for the CSbc, cf. (3.4.13). Notice, that now

$$\begin{aligned} T_{J^\epsilon}(z) &= \frac{\epsilon}{2}(J^\epsilon(z)^2 - \epsilon\partial_z J^\epsilon(z)) + \frac{\mu}{z}J^\epsilon(z) + \frac{\epsilon}{2}\frac{\mu(\mu+1)}{z^2}, \\ \bar{T}_{J^\epsilon}(\bar{z}) &= \frac{\epsilon}{2}(\bar{J}^\epsilon(\bar{z})^2 + \epsilon\partial_{\bar{z}}\bar{J}^\epsilon(\bar{z})) - \frac{\mu}{\bar{z}}\bar{J}^\epsilon(\bar{z}) + \frac{\epsilon}{2}\frac{\mu(\mu+1)}{\bar{z}^2}. \end{aligned} \tag{3.6.9}$$

The central charge for the holomorphic as well as the antiholomorphic part is given by $c_{J^\epsilon} = (1-3\epsilon)$ and $v^\epsilon_{h,\bar{h}} := v^\epsilon_h \otimes v^\epsilon_{\bar{h}}$ is a highest weight vector with conformal weight $\Delta_{T_{J^\epsilon}+\bar{T}_{\bar{J}^\epsilon}}(v^\epsilon_{h,\bar{h}}) = \frac{1}{2}\epsilon[h(h+\epsilon) + \bar{h}(\bar{h}-\epsilon) + 2\mu(\mu+1)] + \mu(h-\bar{h})$ and charges $h+\epsilon\mu$, $\bar{h}-\epsilon\mu$.

Bosonization means to define an action of the Clifford and Heisenberg algebras on these spaces. Therefore, one introduces the operators

$$\begin{aligned} V^\epsilon(r,z) &=: \exp\left(r\phi^\epsilon(z)\right) := \mathrm{e}^{\epsilon r\phi^\epsilon_0}|z|^{2r\mu}z^{\epsilon r J^\epsilon_0}\sum_{n\in\mathbb{Z}} V^\epsilon_n(r)z^{-n} \\ &= \mathrm{e}^{\epsilon r\phi_0}|z|^{2r\mu}z^{\epsilon r J^\epsilon_0}\mathrm{e}^{-\epsilon r\sum_{n<0}\frac{J^\epsilon_n}{n}z^{-n}}\mathrm{e}^{-\epsilon r\sum_{n>0}\frac{J^\epsilon_n}{n}z^{-n}}, \quad r\in\mathbb{C}\setminus\{0\} \end{aligned} \tag{3.6.10}$$

and similar operators for the antiholomorphic field, whereby the bosonic scalar fields are

$$\begin{aligned} \phi^\epsilon(z) &= \mu\log\bar{z} + \epsilon\int^z J^\epsilon_\mu(\omega)\mathrm{d}\omega \ = \mu\log|z|^2 + \epsilon\left(\phi^\epsilon_0 + J^\epsilon_0\log z - \sum_{n\neq 0}\frac{J^\epsilon_n}{n}z^{-n}\right), \\ \bar{\phi}^\epsilon(\bar{z}) &= \mu\log z + \epsilon\int^{\bar{z}} \bar{J}^\epsilon_\mu(\bar{\omega})\mathrm{d}\bar{\omega} = -\mu\log|\bar{z}|^2 + \epsilon\left(\bar{\phi}^\epsilon_0 + \bar{J}^\epsilon_0\log\bar{z} - \sum_{n\neq 0}\frac{\bar{J}^\epsilon_n}{n}\bar{z}^{-n}\right) \end{aligned} \tag{3.6.11}$$

with $[\phi_0, J^\epsilon_n] = -\epsilon\delta_{n,0} = [\bar{\phi}_0, \bar{J}^\epsilon_n]$. The vertex algebra is defined by taking derivatives and products of the operators V^ϵ, just as for the CSbc. The OPE of two fields V^ϵ in the vacuum v^ϵ_h is

$$\begin{aligned} V^\epsilon(r,z)V^\epsilon(s,\omega) &= (z-\omega)^{\epsilon rs}|z|^{2r\mu}|\omega|^{2s\mu}z^{\epsilon rh}\omega^{\epsilon sh} \ :V^\epsilon(r,z)V^\epsilon(s,\omega):, \\ \bar{V}^\epsilon(r,\bar{z})\bar{V}^\epsilon(s,\bar{\omega}) &= (\bar{z}-\bar{\omega})^{\epsilon rs}|z|^{-2r\mu}|\omega|^{-2s\mu}\bar{z}^{\epsilon rh}\bar{\omega}^{\epsilon sh} \ :\bar{V}^\epsilon(r,\bar{z})\bar{V}^\epsilon(s,\bar{\omega}): \end{aligned} \tag{3.6.12}$$

the charge of V^ϵ can be read off from

$$J^\epsilon(z)V^\epsilon(r,\omega) = \frac{r}{z-\omega}V^\epsilon(r,\omega) + \frac{\epsilon}{r}\partial_\omega V^\epsilon(r,\omega) \tag{3.6.13}$$

to be of the value r for the holomorphic and also for the antiholomorphic field. Taking the OPE with the energy momentum tensors, their conformal weights read

$$\Delta_{T_{J^\epsilon}}(V^\epsilon(r,z)) = \frac{1}{2}\epsilon\ r(r+\epsilon), \ \Delta_{\bar{T}_{\bar{J}^\epsilon}}(\bar{V}^\epsilon(r,\bar{z})) = \frac{1}{2}\epsilon\ r(r-\epsilon). \tag{3.6.14}$$

In particular, the operator

$$\mathrm{e}^{\epsilon r\phi_0}: \mathscr{A}^\epsilon_{-\frac{1}{2}\epsilon}(h) \to \mathscr{A}^\epsilon_{-\frac{1}{2}\epsilon}(h+r), \quad v^\epsilon_h \mapsto v^\epsilon_{h+r}, \tag{3.6.15}$$

and hence also $V^\epsilon(r,z)$, are mappings between different representations of the Heisenberg Lie algebra. In addition, the shift operator correctly changes the conformal weight of the highest weight vector v^ϵ_h

$$[(T_{J^\epsilon})_0,\ \mathrm{e}^{\epsilon r\phi_0}]\cdot v^\epsilon_h=(\frac{1}{2}\epsilon\ r(r+\epsilon)+\epsilon r h)\cdot\mathrm{e}^{\epsilon r\phi_0}v^\epsilon_h, \tag{3.6.16}$$

and similar for the antiholomorphic operator. Therefore, introducing the operators V^ϵ makes an extension of the Heisenberg Lie modules to the modules $\bigoplus_{l\in\mathbb{Z}}\mathscr{A}^\epsilon_{-\frac{1}{2}\epsilon}(h+l)$ and $\bigoplus_{l\in\mathbb{Z}}\bar{\mathscr{A}}^\epsilon_{\frac{1}{2}\epsilon}(\bar{h}+l)$, necessary, whereby l distinguishes sectors of different $U(1)$ charges, measured by J^ϵ_0 and $\bar{J}^\epsilon_0$.

Bosonizing Fermions

In the fermionic case, the action of the Clifford algebra of the bc-system is generated by

$$\begin{aligned}
c^+(z)&\simeq V^+(+,z), & \bar{c}^+(\bar{z})&\simeq\bar{V}^+(-,\bar{z}),\\
b^+(z)&\simeq V^+(-,z), & \bar{b}^+(\bar{z})&\simeq\bar{V}^-(+,\bar{z}),
\end{aligned} \tag{3.6.17}$$

and $\bigoplus_{l\in\mathbb{Z}}\mathscr{A}^+_{-\frac{1}{2}}(-p+l)\simeq M^+(p)$ and $\bigoplus_{l\in\mathbb{Z}}\bar{\mathscr{A}}^+_{\frac{1}{2}}(\bar{p}+l)\simeq\bar{M}^+(\bar{p})$ [FF91].

Indeed, the fields above have the correct OPEs (3.4.3) including the homogeneity and, when I further identify

$$\begin{aligned}
j^+(z)+\frac{\mu}{z}&\simeq J^+_\mu(z), & \bar{j}^+(\bar{z})-\frac{\mu}{\bar{z}}&\simeq\bar{J}^+_\mu(\bar{z}),\\
T^+(z)&\simeq T_{J^+}(z), & \bar{T}^+(\bar{z})&\simeq\bar{T}_{\bar{J}^+}(\bar{z}),
\end{aligned} \tag{3.6.18}$$

also the correct charges and conformal weights. In particular, the vertex operators above act on $v^+_{-p,\bar{p}}$ like the original fields b^+ and c^+ on $|p,\bar{p}\rangle_+$. The field modes can be determined by the Fourier expansions, for instance for $V^+(-,z)$,

$$\begin{aligned}
V^+(-,z)v_{-p}&=|z|^{-2\mu}z^p\mathrm{e}^{-\phi_0}\sum_{n\le 0}V^+_n(-)z^{-n}v_{-p}\\
&=|z|^{-2\mu}\sum_{m\le -p}\mathrm{e}^{-\phi_0}V^+_{m+p}(-)z^{-m}v_{-p}
\end{aligned} \tag{3.6.19}$$

in analogy with $b^+(z)|p\rangle_+=|z|^{-2\mu}\sum_{n\le -p}b^+_n z^{-n}|p\rangle_+$, and similar holds for the other field mode $V^+(+,z)$. The field modes inherit the correct commutation relations from the OPEs. Moreover,

$$|p,\bar{p}\rangle_+\simeq v^+_{-p,\bar{p}} \tag{3.6.20}$$

and these states have the same conformal weight and axial and vectorial charges.

Bosonizing Bosons

In the bosonic case [FF91, FF90] one has to include an auxiliary fermionic bc-system because of the wrong central charge. Thus, I introduce fermionic scalars $\xi(z)$, $\bar{\xi}(\bar{z})$ and fermionic fields of weight one $\eta(z)$, $\bar{\eta}(\bar{z})$ (all these fields do not have a homogeneity). The currents and the stress tensor are defined as before, see section 3.4.1.

The operators

$$
\begin{aligned}
c^-(z) &\simeq V^-(+,z)\otimes\eta(z), & \bar{c}^-(\bar{z}) &\simeq V^-(-,\bar{z})\otimes\bar{\eta}(\bar{z}),\\
b^-(z) &\simeq V^-(-,z)\otimes\partial_z\xi(z)\ , & \bar{b}^-(\bar{z}) &\simeq V^-(+,\bar{z})\otimes\partial_{\bar{z}}\bar{\xi}(\bar{z})
\end{aligned}
\tag{3.6.21}
$$

have the correct OPE to define an action of the Heisenberg algebra on a subspace of

$$
N(p,\bar{p}) = \left(\bigoplus_{l\in\mathbb{Z}} \mathscr{A}^-_{\frac{1}{2}}(p+l)\otimes\mathscr{A}^+_{\eta\xi,-\frac{1}{2}}(l)\right)\otimes\left(\bigoplus_{s\in\mathbb{Z}}\bar{\mathscr{A}}^-_{-\frac{1}{2}}(-\bar{p}+s)\otimes\bar{\mathscr{A}}^+_{\eta\xi,+\frac{1}{2}}(s)\right), \tag{3.6.22}
$$

where I implicitly assumed that the auxiliary part may be bosonized as before. The adequate subspace will be determined in the next section. For convenience, whenever I consider the (anti)holomorphic part alone, I will use the notation ($\bar{N}(p)$) $N(\bar{p})$ in the following. The space $N(p)$ collects all possible Verma modules by the fields above and by their derivatives represented on the states $\ldots, v^-_{p+1}|-1\rangle_{\eta\xi},\ v^-_p|0\rangle_{\eta\xi},\ v^-_{p-1}|1\rangle_{\eta\xi},\ldots$.

To prove that the spaces above respect the OPE of the bosonic ghosts, on has to take into account that the bosonized fields are tensored s.t., $c^-(z)b^-(\omega)\simeq V^-(+,z)\partial_z\xi(z)V^-(-,\omega)\eta(\omega)$. Moreover, since the auxiliary part and the fields V^- have the same $U(1)$ charges in (3.6.21), the vertex algebra is graded with the same charges for the $\eta\xi$-system and the vertex operators as above, which explains the summation indices. For the same reason I may identify

$$
j^-(z)-\frac{\mu}{z}\ \simeq\ \mathfrak{J}^-(z)=\frac{1}{2}\left(J^-_\mu(z)+j_{\eta\xi}(z)\right),\quad \bar{j}^-(\bar{z})+\frac{\mu}{\bar{z}}\ \simeq\ \bar{\mathfrak{J}}^-(\bar{z})=\frac{1}{2}\left(\bar{J}^-_\mu(\bar{z})+\bar{j}_{\eta\xi}(\bar{z})\right). \tag{3.6.23}
$$

These currents measure the charge of the representation spaces. In section 4.2 I will argue, that the coupling of the auxiliary current with J^-_μ causes that the bosons do not introduce an additional anomaly into the theory. Similarly, the stress tensor of the bc-system acts like a sum of the stress tensors of the parts of the bosonized system

$$
T^-(z)\simeq T_{J^-}(z)\ -:\partial_z\xi(z)\eta(z):,\quad \bar{T}^-(\bar{z})\simeq\bar{T}_{\bar{J}^-}(\bar{z})\ -:\partial_{\bar{z}}\bar{\xi}(\bar{z})\bar{\eta}(\bar{z}):. \tag{3.6.24}
$$

The fields in (3.6.21) have the correct conformal weights and charges under these identifications and they comprise the relations (3.4.1) on $v^-_{p,-\bar{p}}\otimes|0,0\rangle_{\eta\xi}$. However, only if the bosonic axial symmetry was broken, one can determine states that have the same bosonic vectorial charge as the corresponding states of the non bosonized CSbc. Since the axial symmetry will be broken due to the GCOs, I will now assume this to be true. Under these circumstances and for $p = \bar{p}$, the state $|p, p\rangle_-$ has the same quantum numbers as $v^-_{p,-p} \otimes |0,0\rangle_{\eta\xi}$. Therefore, I will identify

$$|p, p\rangle_- \simeq v^-_{p,-p} \otimes |0,0\rangle_{\eta\xi} \,. \tag{3.6.25}$$

Notice, that only the diagonal ($p = \bar{p}$) representation spaces $N(p,p)$ will be relevant for an analysis of the A-model.

Grading of $N(p,p)$ The spaces $N(p,\bar{p})$ are graded by the zero modes of

$$J_N(z,\bar{z}) = \frac{1}{2}\left\{[J^-_\mu(z) - j_{\eta\xi}(z)] - [\bar{J}^-_\mu(\bar{z}) - \bar{j}_{\eta\xi}(\bar{z})]\right\}, \tag{3.6.26}$$

which further respect the grading by conformal weight and the fermion number. The current J_N generates a third symmetry besides the vectorial and axial symmetries, which is due to the extension of the bosons by the auxiliary fermions. Due to the combination of the currents J^-_μ and $j_{\eta\xi}$, J_N is anomaly free. Still, also this symmetry will be broken due to the Grothendieck-Cousin operators, which ensures that J_N does not enter the theory as an additional symmetry.

Possible Vacuum Representations The condition of zero conformal weight is satisfied by the states that consist of all possible combinations of $v^-_{p,q}|s,t\rangle_{\eta\xi}$ with $p,s,t \in \{0,1\}$ and $q \in \{0,-1\}$. Here is the collection of such states in the representation $N(1,1)$, that will become important in the following sections

$$\begin{array}{ccc} & v^-_{1,0}|0,1\rangle_{\eta\xi} & \\ v^-_{1,-1}|0,0\rangle_{\eta\xi} & & v^-_{0,0}|1,1\rangle_{\eta\xi} \\ & v^-_{0,-1}|1,0\rangle_{\eta\xi} & \end{array} \,. \tag{3.6.27}$$

The states in the middle have zero vectorial charge and comprise a doublet within $N(1,1)$. The state on the top has a vectorial charge of value 1, and the lowest state has charge -1. However, only the state $v^-_{1,-1}|0,0\rangle_{\eta\xi}$ is an element of the representation space of the bosonized bosons, as I will explain below. The state $v^-_{0,0}|1,1\rangle_{\eta\xi}$ will later obtain the interpretation as the logarithmic partner of $v^-_{0,0}|0,0\rangle_{\eta\xi} \in N(0,0)$.

A further remark has to be made. If J_N gets broken as a symmetry of the theory, there is no reason why $v^-_{0,0}|0,0\rangle_{\eta\xi}$ should be in a different multiplet than $v^-_{0,0}|1,1\rangle_{\eta\xi}$. Indeed, only then, those two states can be logarithmic partners, because there is no way to further decompose the two-dimensional representation of the Hamiltonian on these states by means of an additional symmetry.

Restriction of $N(p,\bar{p})$ The representation space $N(p,\bar{p})$ above is not yet the correct representation of the Heisenberg algebra defined by b^- and c^-. Due to the absence of the zero modes ξ_0 and $\bar{\xi}_0$, the vertex algebras must be contained in the intersection of the kernels of η_0 and $\bar{\eta}_0$ and the space $N(p,\bar{p})$ is too large.

In addition, from Feigin's and Frenkel's analysis in [FF91] it follows that the correct representation space for the holomorphic part (without loss of generality) coincides with the kernel of η_0: The kernel of η_0 is obtained by applying $(j_{\eta\xi})_n$, η_n with $n \in \mathbb{Z}$ and ξ_n, $n \neq 0$ to $|0\rangle_{\eta\xi}$. Consequently, the representation space of the bosonized bosons equals the kernel of η_0 if η, $\partial_z\xi$ and $j_{\eta\xi}$ can be expressed in terms of the fields b, c and V^-. This is possible by means of $j_{\eta\xi}(z) = -\partial_z\phi(z)$, $\partial_z\xi(z) = \frac{1}{2}\partial_z b^-(z) \otimes V^-(+,z)$ and $\eta(z) = \partial_z c^-(z) \otimes V^-(-,z)$. The same holds for the antiholomorphic fields.

Therefore, $M^-(p,p) \simeq \overline{N}(p,p)$, whereby the overline denotes the intersection of the equivalence classes $\overline{N}(p)$ and $\bar{\overline{N}}(p)$ of field operators modulo η_0 and $\bar{\eta}_0$, respectively.

This result yields a nice heuristic interpretation why the instanton effects are supposed to be found within the bosonic part of the CSbc and not within the fermionic. Due to the presence of c_0^+ and b_0^+ in the field operator algebra, the representations of the fermionic ghosts on v_0^+ and v_{-1}^+ are isomorphic. For the fermions, there exists only one fundamental vacuum, namely v_0^+ since it has the highest symmetry.[15] On the other hand, the bosonic representations on $v_0^- \otimes |0\rangle_{\eta\xi}$ and $v_1^- \otimes |0\rangle_{\eta\xi}$ are different, for ξ_0 is absent as a dynamical degree of freedom and η_0 is effectively set to zero in the operator algebra, as argued above. The bosonic ghosts can thus be considered to comprise dynamical degrees of freedom in the presence of different background vacua. For these reasons, the charged representations of the bosons may serve as a source for instantons, to be introduced additionally to the bosonic ghosts, interpolating between those backgrounds. These explanations will obtain an exact mathematical sense in terms of the Grothendieck-Cousin operators.

Summary of the Main Facta

In order to describe the perturbative state spaces of the gauged topological A-model in terms of bosonized bosons, it is sufficient to restrict the representation space to the diagonal situation $p = \bar{p}$. As a result, $M^-(p,p) \simeq \overline{N}(p,p)$ and the highest weight vector is now uniquely determined by $|p,p\rangle \simeq v_{p,-p} \otimes |0,0\rangle_{\eta\xi}$. In particular, only the state $v_{1,-1}^-|0,0\rangle$ in the diamond (3.6.27) is an element of $\overline{N}(1,1)$.

The perturbative state spaces for the A-model on $\mathbb{CP}^1$ can now be identified with the bosonized representation spaces

$$\begin{aligned} \mathscr{H}^{\text{in}}_{0,0} &= \mathscr{F}_0 \otimes \bar{\mathscr{F}}_0 \simeq [\bigoplus_{s,s'} \mathscr{A}^+_{-\frac{1}{2}}(s) \otimes \bar{\mathscr{A}}^+_{\frac{1}{2}}(s')] \otimes \overline{N}(0,0)\,, \\ \mathscr{H}^{\text{in}}_{\infty,0} &= \mathscr{F}^1_\infty \otimes \bar{\mathscr{F}}^1_\infty \simeq [\bigoplus_{s,s'} \mathscr{A}^+_{-\frac{1}{2}}(s) \otimes \bar{\mathscr{A}}^+_{\frac{1}{2}}(s')] \otimes \overline{N}(1,1)\,, \end{aligned} \tag{3.6.28}$$

[15] I will discuss the representation theory of the conformal ghost systems more detailed in section 8.3.

where I used that all fermionic representation spaces are equivalent (c.f. (3.4.2)), and consequently $\bigoplus_s \mathscr{A}^+_{-\frac{1}{2}}(s+p) \simeq \bigoplus_s \mathscr{A}^+_{-\frac{1}{2}}(s)$.

The stress tensor and fundamental fields are derived above. The supercharge, must also be composed by bosonic and fermionic fields. An immediate calculation proves that the fields

$$\mathscr{Q}(z) = V^+(-,z) \otimes V^-(+,z) \otimes \eta(z)\,, \quad \bar{\mathscr{Q}}(\bar{z}) = \bar{V}^+(+,\bar{z}) \otimes \bar{V}^-(-,\bar{z}) \otimes \bar{\eta}(\bar{z}) \tag{3.6.29}$$

have the correct OPEs with the bosonized fields to be identified with the supercharge $Q(z,\bar{z}) = \mathscr{Q}(z) + \bar{\mathscr{Q}}(\bar{z})$.

I will now approach the question what operators may serve to define the Grothendieck-Cousin operators.

3.6.3 The GCOs and the Cohomology Interpretation

By (3.6.15), the nilpotent operator $c_0^+ \simeq \mathrm{e}^{\phi_0^+}$ is a cohomology operator

$$\cdots \to \mathscr{A}^+_{-\frac{1}{2}}(-p) \xrightarrow{\mathrm{e}^{\phi_0^+}} \mathscr{A}^+_{-\frac{1}{2}}(-p+1) \to \cdots\,, \tag{3.6.30}$$

cf. [FFH$^+$02]. However, since it connects isomorphic representation spaces, this operator can not be the GCO mapping between $\mathscr{F}_0$ and $\mathscr{F}^1_\infty$. As just explained, the difference between the perturbative state spaces must be rooted in the bosonic sector.

The extension of $\overline{N}(p,\bar{p})$ to $N(p,p)$ by means of $\mathrm{e}^{\phi_0^-}\xi_0$ and $\mathrm{e}^{-\bar{\phi}_0^-}\bar{\xi}_0$ permits a nontrivial action of η_0 and $\bar{\eta}_0$. Thereby, one obtains a complex for the bosonic sector in analogy to the fermionic one, above. The rôle of c_0 for the purely fermionic bc-system is now played by the nilpotent operator $\eta_0\bar{\eta}_0 \,:\, N(p,\bar{p}) \to N(p-1,\bar{p}-1)$. Therefore [FF91], it can be interpreted as a cohomology operator for the complex

$$\cdots \to N(p,\bar{p}) \xrightarrow{\eta_0\bar{\eta}_0} N(p-1,\bar{p}-1) \to \cdots\,, \tag{3.6.31}$$

whose grading is measured by J_N, since $[J_{N\,0} + \bar{J}_{N\,0}, \eta_0\bar{\eta}_0] = -\eta_0\bar{\eta}_0$. Notice, that in principle I could define different complexes using other combinations of η_0 and $\bar{\eta}_0$ acting on $N(p,\bar{p})$, for instance $\eta_0 + \bar{\eta}_0$. However, for the representation spaces of the gauged A-model the relation $p = \bar{p}$ has necessarilty to be satisfied and this restricts the choice to $\eta_0\bar{\eta}_0$ up to a prefactor.

To specify the cohomology of $\eta_0\bar{\eta}_0$, I will now determine the image of this operator. Consider the complement $N(p,p)/\overline{N}(p,p)$ of $\overline{N}(p,p)$ in $N(p,p)$. Since $\overline{N}(p,p)$ denotes the intersection of the kernels of η_0 and $\bar{\eta}_0$ considered independently, this space must not be equal to the kernel of $\eta_0\bar{\eta}_0$. Indeed, it is just a subspace. For instance, $\overline{N}(1,1)$ does not include the states $v^-_{1,0} \otimes |0,1\rangle_{\eta\xi}$ and $v^-_{0,-1}|1,0\rangle_{\eta\xi}$ which are sent to zero by $\eta_0\bar{\eta}_0$. I will call the expression

$$N_L(p,p) = \left(\bigoplus_{l,s\in\mathbb{Z}} \mathscr{A}^-_{\frac{1}{2}}(l) \otimes \mathscr{A}^+_{\eta\xi,-\frac{1}{2}}(l) \otimes \bar{\mathscr{A}}^-_{-\frac{1}{2}}(s) \otimes \bar{\mathscr{A}}^+_{\eta\xi,\frac{1}{2}}(s) \right)_{\eta_0,\bar{\eta}_0=0} v^-_{p-1,-p+1}|1,1\rangle \tag{3.6.32}$$

the "logarithmic extension" of $\overline{N}(p-1,p-1)$. With this definition I can now split

$$\begin{aligned} N(p,p) &= N_L(p,p) \oplus \overline{N}(p,p) \oplus R(p,p)\,, \\ R(p,p) &= \left(\overline{N}(p) \otimes \bar{N}_L(p)\right) \oplus \left(N_L(p) \otimes \bar{\overline{N}}(p)\right), \end{aligned} \tag{3.6.33}$$

wherein $N_L(p)$ and $\bar{N}_L(p)$ signify the holomorphic respectively antiholomorphic half of (3.6.32). One can now extract the image of $\eta_0\bar{\eta}_0$, namely

$$\mathrm{im}_{\eta_0\bar{\eta}_0}(N_L(p,p)) = \overline{N}(p-1,p-1)\,. \tag{3.6.34}$$

Therefore, the p^{th} cohomology class of $\eta_0\bar{\eta}_0$ is

$$H^p_{\eta_0\bar{\eta}_0} = R(p,p)\,. \tag{3.6.35}$$

This result differs from the situation where only the holomorphic or antiholomorphic parts are considered. In the case when η_0 is taken for the cohomology operator, the cohomology of this operator is trivial.

As a consequence of the following discussion, the local cohomology spaces in the analogue of the Grothendieck-Cousin complex will, however, not be the cohomology spaces of $\eta_0\bar{\eta}_0$.

The First GCO δ_1

In section 2.6.2, I made two formal assumptions on the Grothendieck-Cousin operators. The first was, that it is a mapping between the perturbative spaces of states if the descending manifolds have relative codimension one. The second was the observation, that the Grothendieck-Cousin operator is basically acting on the "dual part" of the eigenstates of the naive Hamiltonian. In the Morse theory on $\mathbb{CP}^1$ this was obtained by extending its spectrum by the missing states with the same quantum numbers. I will make use of this in order to propose that $N(1,1)$ is the appropriate extension, cf. 34

$$0 \to \mathscr{H}^{\text{in}}_{\infty,0} \xrightarrow{e} \underline{\mathscr{H}}^{\text{in}}_{\infty,0} = M^+(0,0) \otimes N(1,1) \xrightarrow{\mathfrak{g}_1} \mathscr{H}^{\text{in}}_{0,0} \to 0\,. \tag{3.6.36}$$

I will restrict my consideration to the holomorphic part. The representation $N(1)$ is generated by the action of $\mathscr{N} = \{\eta_{-n}\mathrm{e}^{-\phi_0^-},\ \xi_{-n}\mathrm{e}^{\phi_0^-},\ \mathfrak{J}^-_{-n}\}_{n<0}$ on $v_1^- \otimes |0\rangle_{\eta\xi}$. The spectrum can in analogy with the fermionic bc-system [FFH+02] be framed by the extremal states

$$\begin{array}{ccc} v_0^-|1\rangle_{\eta\xi} \times & & \bullet\, v_1^-|0\rangle_{\eta\xi} \\ v_{-1}^-|2\rangle_{\eta\xi} \times & \cdots\cdots\cdots & \bullet\, v_2^-|-1\rangle_{\eta\xi} \\ v_{-2}^-|3\rangle_{\eta\xi} \times & \cdots\cdots\cdots & \bullet\, v_3^-|-2\rangle_{\eta\xi} \\ v_{-3}^-|4\rangle_{\eta\xi} \times & \cdots\cdots\cdots & \bullet\, v_4^-|-3\rangle_{\eta\xi} \\ \vdots & \cdots\cdots\cdots & \vdots \end{array}$$

The horizontal axis is scaled by the $U(1)$ charge of $\mathfrak{J}^-(z)$, while the vertical axis distinguishes the conformal weights. The states denoted by $\times$ are not contained in $\overline{N}(1)$, and I will now explain that they appear due to an extension by the "dual" states. Generalizing the recipe of section 2.6.2, those have to be chosen such that they have the same quantum numbers as have the extremal states in $\overline{N}(1)$.

An extremal state $v_r^-|s\rangle_{\eta\xi} \in \overline{N}(p),\ r,s \in \mathbb{Z}$ must be subject to the condition $r+s=p$. Moreover, it has conformal weight $-\frac{1}{2}r(r-1)+\frac{1}{2}s(s-1)$. The conformal weight is invariant under $r \mapsto -r+1$ and/or $s \mapsto -s+1$, while the grading is in general not invariant under those transformations. The cases in which the grading is preserved are values of r and s that solve $r+s=1$. Therefore, dual states in that sense only exist in the representation $\overline{N}(1)$. I will argue below, that this already covers the situation of the gauged A-model. Thus, for $p=1$ the dual states are exactly those, which extend $\overline{N}(1)$ to $N(1)$.

The cohomology operator $\eta_0\bar{\eta}_0$ for (3.6.31) has now the desired properties to be identified with $\mathfrak{g}_1$. Thus, up to a prefactor, which is chosen to fit with the results of the following chapter 4, I set

$$\delta_1 = 2\ \eta_0\bar{\eta}_0 \circ e, \quad \mathfrak{g}_1 = 2\eta_0\bar{\eta}_0\,, \tag{3.6.37}$$

whereby e denotes the extension $\overline{N}(1,1) \to N(1,1)$.

The Second GCO δ_2

The second GCO already follows from the discussion above. This can be seen by a method that I owe Edward Frenkel.

In section 2.5, I promoted the idea to interpret the GCOs as operators that mimic the instantons. Consequently, an observer on the chart $\widetilde{LX}_{0,0}$ and calculating with states $\mathscr{H}^{\mathrm{in}}_{0,0}$ gets some insight into the perturbative state spaces around $\{\infty\} \in X$. Because there are no anti-instantons, no states of $\mathscr{H}^{\mathrm{in}}_{0,0}$ will appear to an observer on $\widetilde{LX}_{\infty,0}$.[16]

In order to "see" the instantons that flow from $\{0\}$ to $\{\infty\}$, the observer has to move to the other hemisphere and consider the states $\mathscr{H}^{\mathrm{in}}_{\infty,0}$, where the instantons introduce states of $\mathscr{H}^{\mathrm{in}}_{0,1}$, cf. (3.6.2). This movement should not change the physics, and thus is invoked by the composite mapping $x \mapsto \tilde{x},\ \mu \mapsto -\mu$, which leaves the action (3.2.10) invariant. Also the flow equation remains structurally the same and turns into $(\partial_{\bar{z}} + \frac{\mu}{\bar{z}})\tilde{x} = 0$.

There is an additional effect on the state spaces which can not be seen from the action. Considering $x \mapsto x^{-1},\ \mu \mapsto -\mu$ and the instanton flow equations, one could conclude that $\mathscr{F}^1_\infty \to \mathscr{F}_\infty,\ \mathscr{F}_0 \to \mathscr{F}^1_0$, where the states are defined as in equations (3.5.16) and (3.5.19), respectively (in adequate coordinates). However, one has to take care of the fact that the state

[16] These would be mimicked by the presence of $\xi_0\bar{\xi}_0$ in the Hamiltonian.

spaces are weighted by $\exp\{\int_D \tilde{\gamma}^{-*}(\omega_K) + \int_{n\in H_2(X,\mathbb{Z})} \tilde{\gamma}^{-*}(\omega_K)\}$, cf. eqns. (3.3.1) and (3.3.2). Intuitively, a coordinate transformation has to move the disk D to the other hemisphere, which can be done by wrapping it once around $\mathbb{CP}^1$. Therefore, $x \mapsto x^{-1}$, $\mu \mapsto -\mu$ should be accompanied by the transformation $\int_D \tilde{\gamma}^*(\omega_K) \mapsto \int_D \tilde{\gamma}^*(\omega_K) + \int_{S^2} \tilde{x}^*(\omega_K)$, and this adds to the operator $q^n \mapsto q^{n+1}$. The theory is then rather invariant under $x \mapsto \tilde{x}$, $\mu \mapsto -\mu$ and an additional multiplication of the transformed spaces of in-states with q^{-1}.[17]

The second GCO can now be derived from δ_1. The reason is that if the theory is invariant under $x \mapsto \tilde{x}$, $\mu \mapsto -\mu$ and a multiplication of the states with q^{-1}, the globally defined Hamiltonian must also be invariant under this mapping. Therefore, under this transition, $\delta_1 \mapsto \delta_2$ such that

$$\delta_2 = 2\,\tilde{\eta}_0\tilde{\bar{\eta}}_0 \circ e, \quad \mathfrak{g}_2 = 2\tilde{\eta}_0\tilde{\bar{\eta}}_0 . \tag{3.6.38}$$

In that way, δ_2 is acting on $q^{-1}\mathscr{H}^{\text{in}}_{0,1} \simeq \mathscr{H}^{\text{in}}_{0,0}$. Because the GCOs are structurally the same, it is sufficient to restrict my investigations to δ_1, which I will do in the rest of my thesis.

3.6.4 Conclusion

In (3.6.28) I have summarized the perturbative state spaces that will serve as the CFT model for the representations of the Tbc with gauge field. The "ground" states of the A-model are identified with

$$\Delta_0 \;\simeq\; v^+_{0,0} \otimes v^-_{0,0}|0,0\rangle_{\eta\xi}\,, \quad \Delta_\infty \;\simeq\; v^+_{-1,1} \otimes v^-_{1,-1}|0,0\rangle_{\eta\xi} \tag{3.6.39}$$

The Grothendieck-Cousin operators appear in an extension of the perturbative state spaces that is analoguousely to that of pg. 34. If have noted down that extension for δ_1 in (3.6.36).

The Grothendieck-Cousin operators add to the Hamiltonian, that has an action on the nonperturbative representations according to pg. 30 :

$$\underline{H} = H + \mathfrak{g}_1 + \mathfrak{g}_2 \;\;\simeq\;\; \underline{T}_0 = T_0 + \bar{T}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 . \tag{3.6.40}$$

With these data, I conclude my analysis of the low-energy, nonperturbative Morse theory behind the gauged A-model. In the following chapter, I will extend the focus on the quantum mechanical operators to the fields. I will prove that a specific logarithmic transformation of the CSbc on $\mathbb{CP}^1$ adds the Grothendieck-Cousin operators to the Hamiltonian and further deforms the stress tensor and fields. The following analysis again shifts the attention back from Morse theory [FLN06, FLN08], to field theory [VF09].

[17] Because of (2.2.9), the operators are not affected by this transformation of q.

The A-Model beyond Topology 4

In the last chapter I have considered the Morse theory underlying the A-model in the large volume limit (Tbc). Using the recipe of chapter 2, I have derived its nonperturbative state spaces and the Grothendieck-Cousin operators mapping between them. The representation spaces have been modelled by a conformal supersymmetric bc-system (CSbc).

One of the main proposals of Frenkel et al. was that, if there corresponds a conformal field theory to the "gauged" Tbc, beyond the topological sector it must be a logarithmic conformal field theory [FLN06, FLN08]. However, they did not push forward their proposal and introduce the logarithmic CFT. This will be the subject in the following and conclude part one of my thesis. The chapter is grounded on and also extends my publication with M. Flohr, [VF09].

Firstly, I will accommodate a method by Fjelstad et al. [FFH+02], which allows for a logarithmic extension of conformal field theories. The extension will be such that the Virasoro algebra as well as supersymmetry are preserved and the Grothendieck-Cousin operators of section 4.2 are added to the Hamiltonian. The logarithmic deformation affects not only the Hamiltonian but also the operator product algebra (OPA) of the fields and the other modes of the stress tensor. I will discuss those effects and conclude the chapter with a proof that the logarithmic extension implies the extension of the perturbative state spaces $\mathscr{H}^{\mathrm{in}}_{0,0}$ and $\mathscr{H}^{\mathrm{in}}_{\infty,0}$ as described in section 3.6.3.

4.1 The Method of Logarithmic Deformation

Fjelstad et al. invented a constructive method to deform CFTs to logarithmic CFTs [FFH+02]. The main idea is to enlarge the representation space of any chiral (antichiral) CFT systematically, by introducing additional field modes and tensoring their representation space to the one of the CFT. Thereby, the stress tensor gains an additional term which acts on the tensored vector space such that some of the Virasoro generators yield higher-dimensional, non-reducible representations.

4.1.1 Extension of the Fields

Let $\mathscr{C}$ denote some chiral algebra of conformal fields and $\mathscr{F}$ the corresponding representation space with conformally invariant highest weight vector $|0\rangle_{\mathscr{F}}$. I will further require that there exists a fermionic field $E(z) \in \mathscr{C}$ of weight one such that $E_0|0\rangle_{\mathscr{F}} = 0$ and $E(z)E(\omega) = 0$. Fjelstad et al. deform the fields $f(z)\in\mathscr{C}$ by introducing an odd graded vector space $\mathscr{K}$ with operators ϵ and ρ and a vector $|0\rangle_{\mathscr{K}}\in\mathscr{K}$, such that $[\epsilon,\rho]=1_{\mathscr{K}}$ and $\rho|0\rangle_{\mathscr{K}}=0$ [FFH+02]. In order to have an isomorphism between fields and states, they define a new field $e(z)$

$$e(z)=1_{\mathscr{F}}\otimes\epsilon-\int^z E(\omega)\mathrm{d}\omega\otimes 1_{\mathscr{K}}\,,\qquad \int^z E(\omega)\mathrm{d}\omega=E_0\,\log\,z-\sum_{n\neq 0}\frac{E_n}{n}z^{-n} \tag{4.1.1}$$

corresponding to $|0\rangle_{\mathscr{F}}\otimes\epsilon|0\rangle_{\mathscr{K}}$. This "extension field" determines a deformation map on $f\in\mathscr{C}$

$$f(z)\mapsto\ \underline{f}(z)=\,:\exp\{-\rho e(0)\}:f(z)\,, \tag{4.1.2}$$

which extends the algebra of field modes by the additional zero modes ϵ and ρ. The action of e on a field $F(z)=f(z)\otimes\sigma,\ \sigma\in\mathrm{End}(\mathscr{K})$, is defined by means of the OPE

$$e(z)F(\omega)=\left(-[E,f]_1\log(\omega-z)+\sum_{n\geq 1}\frac{1}{n}\frac{[E,f]_{n+1}}{(z-\omega)^n}\right)\otimes\sigma\,, \tag{4.1.3}$$

wherein $[E,f]_n$ denotes the contribution with pole of order n in the OPE of E with f, i.e. $E(z)f(\omega)=\sum_{n\geq 0}\frac{[E,f]_n(\omega)}{(z-\omega)^n}$. In particular, the energy momentum tensor gets deformed to

$$T(z)\ \mapsto\ \underline{T}(z)=T(z)+\frac{\rho}{z}E(z)\,. \tag{4.1.4}$$

In my opinion, further extensions of the fields generating the symmetries of the theory should be made, which Fjelstad et al. did not take into account. Namely, for e to make sense as a field, ϵ should have the same quantum numbers as E, which imposes further conditions on ϵ and ρ. Suppose, for instance, that there exists a current j according to which E has some charge q_E. Only if this current is extended by an additional zero mode

$$j(z)\mapsto j(z)\otimes 1_{\mathscr{K}}+1_{\mathscr{F}}\otimes q_E\frac{\rho}{z}\,, \tag{4.1.5}$$

the field e has a well defined charge. From the commutation relation of ϵ with ρ then follows that ρ must have charge $-q_E$. These additional extensions are not an integral part in the deformation by the extension field e, however, in the case of the CSbc this will be the case, cf. section 4.2.

4.1.2 Extension of the Representation Theory

Due to the additional term, the Virasoro algebra has two-dimensional representations on certain composite fields

$$\Psi_f(z)=-:e(z)\underline{f}(z):\,. \tag{4.1.6}$$

Their OPE with the stress tensor yields[1]

$$\underline{T}(z)\Psi_f(\omega)=\sum_{m\geq 3}\frac{[E,\underline{f}]_{m-1}}{(z-\omega)^m}+\frac{\Delta_T(f)\Psi_f+[E,\underline{f}]_1}{(z-\omega)^2}+\frac{\partial_\omega\Psi_f}{z-\omega}\,, \tag{4.1.7}$$

[1] I thank J. Fuchs who pointed out to me that I have to use the definition of normal ordering and contraction for interacting fields, (i.e. fields that have not just one singular term proportional to the identity in the OPE): $\overline{a(z):b}c:(\omega)=\oint_\omega\frac{\mathrm{d}\zeta}{\omega-\zeta}(\overline{a(z)b}(\zeta)c(\omega)+(-)^{F_aF_b}b(\zeta)\overline{a(z)c}(\omega))$, cf. [DFMS97].

which means for the state space that the ground state has now a logarithmic partner $E_0^*\epsilon\cdot|0\rangle_{\mathscr{F}}\otimes|0\rangle_{\mathscr{K}}$, due to $[\epsilon,\rho]=1_{\mathscr{K}}$. Here, E_0^* is defined by $[E_0,E_0^*]=1$.

Indeed, this kind of logarithmic deformation causes an extension of the state spaces. Let $|0\rangle := |0\rangle_{\mathscr{F}}\otimes|0\rangle_{\mathscr{K}}$ and denote by $\mathscr{F}'$ the Fock representation of $\mathscr{C}$ on that vector. Obviously $\mathscr{F}\simeq\mathscr{F}'$. However, by the construction above, there is a new state $\epsilon|0\rangle$ corresponding to the extension field e, and a representation $\mathscr{F}''$ of $\mathscr{C}$ thereon. The extended representation space can be identified with $\underline{\mathscr{F}} := \mathscr{F}'\oplus\mathscr{F}''$ and the deformed fields mix $\mathscr{F}'$ and $\mathscr{F}''$. In section 4.2.5, the space $\mathscr{F}''$ will take the rôle of the "dual part" that extends the perturbative state space of the Morse theory behind the A-model.

4.1.3 The Fermionic bc-System

As a crucial example for the A-model, I will now consider the auxiliary $\eta\xi$-system of section 3.6.2 and apply to it the method of Fjelstad et al. [FFH+02].

The fields constituting the vertex algebra are deformed to

$$\begin{aligned}
\xi(z) &\quad\mapsto\quad \underline{\xi}(z)=\xi(z)+\rho\ \log z,\\
\eta(z) &\quad\mapsto\quad \underline{\eta}(z)=\eta(z),\\
T_{\eta\xi}(z) &\quad\mapsto\quad \underline{T}_{\eta\xi}(z)=T_{\eta\xi}(z)+\rho\ \eta(z)z^{-1},\\
j_{\eta\xi}(z) &\quad\mapsto\quad \underline{j}_{\eta\xi}(z)=j_{\eta\xi}(z)+\rho z^{-1}-\rho\ \eta(z)\log z,
\end{aligned}\tag{4.1.8}$$

and extended by the new field

$$e(z)=\epsilon-\int^z\eta(\omega)\mathrm{d}\omega.\tag{4.1.9}$$

The additional field modes ρ and ϵ satisfy $[\epsilon,\rho]=1_{\mathscr{K}}$ and $\rho|0\rangle_{\mathscr{K}}=0$ for some $|0\rangle_{\mathscr{K}}\in\mathscr{K}$, whereby $\mathscr{K}$ is an odd graded Vector space. They extend the state space of the original fermionic bc-system $M^+_{\eta\xi}(0)\to M^+_{\eta\xi}(0)\otimes\mathscr{K}$, $|0\rangle_{\eta\xi}\mapsto|0\rangle_{\eta\xi}\otimes|0\rangle_{\mathscr{K}}$. The CFT defined by the fields above exhibits logarithms in the OPE and a non-degenerate stress tensor

$$\begin{aligned}
\underline{\xi}(z)e(\omega) &= \log(z-\omega),\\
\underline{T}_{\eta\xi}(z)\Psi_\xi(\omega) &= \frac{0\cdot\Psi_\xi(\omega)+1}{(z-\omega)^2}+\frac{\partial_\omega\Psi_\xi(\omega)}{z-\omega},
\end{aligned}\tag{4.1.10}$$

wherein $\Psi_\xi(z)=-:e(z)\underline{\xi}(z):$ is the logarithmic partner of the identity operator on $M^+_{\eta\xi}(0)\otimes\mathscr{K}$. In particular, the extra term in the Hamiltonian

$$\underline{T}_{\eta\xi\,0}=T_{\eta\xi\,0}+\rho\ \eta_0\tag{4.1.11}$$

looks similar to the GCOs if ρ was adjusted to be $\bar\eta_0$ and the $\eta\xi$-system was identified with the auxiliary fermions of section 3.6.2. Before I adapt the deformation to this situation in the next section, a comment on the the OPE of $\underline{\xi}$ with e is indispensable.

Due to the logarithm, the correlator of ξ with η yields a multi-valued function. This can be resolved by including the antiholomorphic sector and restricting the variable $\bar{z}$, usually considered to be independent from z, to be the complex conjugate. Thus, the observation in the last chapter, that the GCOs mix up the holomorphic and antiholomorphic parts of the CSbc, fits with a typical situation in a CFT which exhibits logarithms in OPEs. The deformed fermionic bc-system canonically demands that the holomorphic and antiholomorphic parts are considered together. Still, for convenience I will often restrict my discussion to the holomorphic "half".

Moreover, the logarithm in the OPE of e with $\underline{\xi}$ causes that Möbius covariance is broken. Indeed, under $(z,\omega) \mapsto \mathrm{e}^{\lambda}(z,\omega)$, $\lambda \neq 0$, I find that $\underline{\xi}(z)e(\omega) \mapsto \log(\mathrm{e}^{\lambda}(z-\omega)) \neq \underline{\xi}(z)e(\omega)$. This signifies that e can not enter the conformal field theory as an additional dynamical field. It just serves to deform the field algebra and to extend the representation spaces.

4.2 Introducing the GCOs

I will now discuss, how the bosons of the CSbc can be logarithmically extended in a way, such that the Hamiltonian and extended representation spaces cover the situation of the Morse theory behind the A-model, cf. chapter 3. From section 3.6.2 it is already clear that the deformation has to be applied to the bosons of the CSbc. Above, I have further motivated that the auxiliary fermions will be the main characters.

In the following section, I will propose a specific logarithmic extension e and analyze its effects on the field algebra. The Hamiltonian will turn out nicely, and I will fill in the missing argument why the logarithmic deformation breaks the bosonic axial symmetry and the symmetry generated by J_N, cf. (3.6.26).

Section 4.2.5 concludes this analysis. Therein, I will explain that the field e does not only deform the field algebra but also extends the representation space in a way, such that the results of the last chapter are reproduced.

4.2.1 Extension of the Fields

In order to introduce the Grothendieck-Cousin operator $\mathfrak{g}_1$, I fix the representation of the bosonic bc-system to be $\overline{N}(1,1)$. The second GCO can be obtained after a chart transition of the CSbc to the other hemisphere and just in the same manner as described below.

The GCOs are mixing holomorphic and anti-holomorphic (target-space) coordinates. Therefore, I set $\mathscr{K} = \bar{M}^+_{\eta\xi}(0)$, $\bar{\mathscr{K}} = M^+_{\eta\xi}(0)$ and define the additional fields

$$\begin{aligned} e(z) =& \mathrm{e}^{-\bar{\phi}_0^-}\left(1_{M^+_{\eta\xi}} \otimes \bar{\xi}_0 - \int^z \eta(\omega)\mathrm{d}\omega \otimes 1_{\bar{M}^+_{\eta\xi}}\right), \\ \bar{e}(\bar{z}) =& \mathrm{e}^{\phi_0^-}\left(\xi_0 \otimes 1_{\bar{M}^+_{\eta\xi}} - 1_{M^+_{\eta\xi}} \otimes \int^{\bar{z}} \bar{\eta}(\bar{\omega})\mathrm{d}\bar{\omega}\right). \end{aligned} \tag{4.2.1}$$

By this means, the holomorphic part is extended by the antiholomorphic part and vice versa. Having introduced the field modes $\mathrm{e}^{\phi_0^-}$ and $\mathrm{e}^{-\bar{\phi}_0^-}$ does not only extend $\overline{N}(1,1)$ in the desired way, but it is also necessary because it is now a bosonic system to which I apply the deformation.

Defining the field transformations as

$$f(z,\bar{z}) \mapsto \underline{f}(z,\bar{z}) =: \exp\left[-e(0)\cdot \mathrm{e}^{\bar{\phi}_0^-}\bar{\eta}_0 - \mathrm{e}^{-\phi_0^-}\eta_0\cdot\bar{e}(0)\right] : f(z,\bar{z}), \tag{4.2.2}$$

the stress tensor of the $\eta\xi$-system is deformed to

$$\underline{T}_{\eta\xi}(z,\bar{z}) = \left(T_{\eta\xi}(z) + \frac{1}{z}\,\eta(z)\bar{\eta}_0\right) + \left(\bar{T}_{\eta\xi}(\bar{z}) + \frac{1}{\bar{z}}\,\eta_0\bar{\eta}(\bar{z})\right). \tag{4.2.3}$$

The deformation further implies

$$\underline{T}_{\eta\xi\;n} + \underline{\bar{T}}_{\eta\xi\;n} = T_{\eta\xi\;n} + \bar{T}_{\eta\xi\;n} + \eta_n\bar{\eta}_0 + \eta_0\bar{\eta}_n \tag{4.2.4}$$

on the field modes and leads to the desired result (3.6.40). As I have already mentioned, not only the Hamiltonian but also the other modes of the Virasoro generator are deformed. This effect is invisible in the Morse theory description, and I will therefore discuss some consequences at the end of this chapter. In the following, I will refer to the deformation terms in the stress tensor as "Grothendieck-Cousin fields", which I will denote by

$$\mathfrak{g}_1(z) = \frac{1}{z}\eta(z)\bar{\eta}_0, \quad \bar{\mathfrak{g}}_1(\bar{z}) = \frac{1}{\bar{z}}\eta_0\bar{\eta}(\bar{z}). \tag{4.2.5}$$

In addition, the transformation affects the bosonic fields in $\overline{N}(1,1)$

$$\begin{aligned} &\underline{b}^-(z) = V^-(-,z)\otimes\left(\partial_z\xi(z) - \bar{\eta}_0 z^{-1}\right), && \underline{\bar{b}}^-(\bar{z}) = \bar{V}^-(+,\bar{z})\otimes\left(\partial_{\bar{z}}\bar{\xi}(\bar{z}) + \eta_0\bar{z}^{-1}\right) \\ &\underline{c}^-(z) = V^-(+,z)\otimes\eta(z), && \underline{\bar{c}}^-(\bar{z}) = \bar{V}^-(-,\bar{z})\otimes\bar{\eta}(\bar{z}) \end{aligned} \tag{4.2.6}$$

and

$$\begin{aligned} &\underline{T}^-(z) = T^-(z) + \mathfrak{g}_1(z), && \underline{\bar{T}}^-(\bar{z}) = \bar{T}^-(\bar{z}) + \bar{\mathfrak{g}}_1(\bar{z}), \\ &\underline{j}_{\eta\xi}(z) = j_{\eta\xi}(z) - \log z\;\eta(z)\bar{\eta}_0, && \underline{\bar{j}}_{\eta\xi}(\bar{z}) = \bar{j}_{\eta\xi}(\bar{z}) + \log\bar{z}\;\eta_0\bar{\eta}(\bar{z}), \\ &\underline{J}^-(z,\bar{z}) = J^-(z,\bar{z}), && \underline{Q}(z,\bar{z}) = Q(z,\bar{z}), \end{aligned} \tag{4.2.7}$$

whereas the supercharge $Q(z,\bar{z}) = \mathscr{Q}(z) + \bar{\mathscr{Q}}(\bar{z})$, $\mathscr{Q}(z) = V^+(-,z)\otimes\eta(z)V^-(+,z)$ is not deformed, cf. eqn. (3.6.29). Hence, the topological sector of the theory is insensible to this procedure.

In addition, the zero mode of the vectorial current $\mathfrak{J}_V^- = (J^- + \bar{J}^-) + (\underline{j}_{\eta\xi} + \underline{\bar{j}}_{\eta\xi})$ is not corrected, which means that it still measures the same quantum numbers as the undeformed one. This is not only an incidental remark, there is another reason why the vectorial current is preferential. As explained before, for e and $\bar{e}$ to have well defined charges, the holomorphic and antiholomorphic currents have to be generalized. Consider the affected holomorphic auxiliary current $j_{\eta\xi}$. The charge of $\bar{\xi}_0$ is measured by $\bar{j}_{\eta\xi}$ and yields the same value as the charge of η under $j_{\eta\xi}$. Therefore, it must be completed by the antiholomorphic current in such a way, that the total auxiliary current is vectorial. Since the auxiliary current is coupled to J^- via (3.6.23), this is inherited by $\mathfrak{J}^-$. This explains my claim that for the particular deformation above, the extension of the symmetry generating fields and the extension by e, $\bar{e}$ is the same.

In order to further specify my comments on the symmetries of the deformed theory, I will now discuss how the logarithmic deformation indeed breaks all symmetries whose generators contain the axial current of the auxiliary $\eta\xi$-system. Moreover, I will consider if supersymmetry and the Virasoro algebra are affected.

4.2.2 Notes on the Symmetries

The axial symmetry of the auxiliary system is broken by the presence of the deformation term in the Hamiltonian. To see this, I calculate the commutator

$$\oint \mathrm{d}z\ [\eta_0\bar{\eta}_0, \underline{j}_{\eta\xi}(z)] \pm \oint \mathrm{d}\bar{z}\ [\eta_0\bar{\eta}_0, \underline{\bar{j}}_{\eta\xi}(\bar{z})] = -\eta_0\bar{\eta}_0 \pm \eta_0\bar{\eta}_0\,. \tag{4.2.8}$$

Therefore, only the zero mode of the vectorial current commutes with the deformed Hamiltonian, whereas this fails for the axial symmetry. This concludes the proof that the currents J_N of eqn. (3.6.26) and $\mathfrak{J}^- - \bar{\mathfrak{J}}^-$ of eqn. (3.6.23) do not comprise symmetries of the logarithmically deformed CSbc.

On the other hand, this is not true for supersymmetry and conformal symmetry. The reason is that besides in the expression $\underline{j}_{\eta\xi}$, only derivatives of the field ξ enter the extended field algebra. Since all deformation terms are proportional to zero modes of $\eta(z)$ and $\bar{\eta}(\bar{z})$, the logarithmic extension does not spoil the commutation relations and, hence, preserve supersymmetry and the Virasoro algebra.

The absence of ξ has two further consequences that I will now discuss.

4.2.3 Exceptional Logarithmic Partners

A first consequence is that the field $\Psi_{b^-}(z) = -:e(z)\underline{b}^-(z):$ has no logarithmic partner,[2]

$$\underline{T}^-(z)\Psi_{b^-}(\omega) = \frac{\mathrm{e}^{-\bar{\phi}_0^-}V^-(-,\omega)}{(z-\omega)^3} + \frac{\partial_\omega \Psi_{b^-}(\omega)}{z-\omega}\,. \tag{4.2.9}$$

[2] Due to the anomaly of the holomorphic current $j_{\eta\xi}$, (4.1.7) does not apply and one has to derive the OPE by hand.

On the other hand, $\Psi_{j_{\eta\xi}}(z) = -:e(z)j_{\eta\xi}(z):$, and other combinations $:\phi(z)\Psi_{j_{\eta\xi}}(z):$, ϕ a field in the CSbc, have logarithmic partners. In particular,

$$\underline{T}(z)\Psi_{j_{\eta\xi}}(\omega) = \frac{-e(\omega)}{(z-\omega)^3} + \frac{\Psi_{j_{\eta\xi}}(\omega) + \partial_\omega e(\omega)}{(z-\omega)^2} + \frac{\partial_\omega \Psi_{j_{\eta\xi}}(\omega)}{z-\omega}. \tag{4.2.10}$$

This turns the logarithmically deformed CSbc into an exceptional case among logarithmic conformal field theories. Namely, its $U(1)$ current breaks the $SL(2,\mathbb{C})$ symmetry and therefore, the logarithmically deformed CSbc is an example for an LCFT whose basic Jordan block is not a primary field [Flo03, pg. 4516].

4.2.4 On the Necessity to Deform the Fermions

In section 4.1 I have considered the $\eta\xi$-system in its own right and argued that the extension field e should not be part of the dynamical fields because it breaks Möbius covariance. Since ξ is not a field in the vertex algebra of the bosonized bosons, I can not exclude e and $\bar{e}$ from the dynamical fields by this argument. However, if I treated them as additional dynamical fields in the CSbc, I would expect that I also have to logarithmically deform the fermionic sector, in order to supply the extension fields with their supersymmetric partners. I denote the fermions as in the last chapter by b^+ and c^+, an extension as described in section 4.1.3 can be performed

$$\begin{aligned} e^+(z) = \bar{b}_0^+ - \int^z c^+(\omega)\mathrm{d}\omega, \qquad \bar{e}^+(\bar{z}) = b_0^+ - \int^{\bar{z}} \bar{c}^+(\bar{\omega})\mathrm{d}\bar{\omega}, \\ f^+(z,\bar{z}) \mapsto \underline{f}^+(z,\bar{z}) =: \exp\left[-e^+(0)\bar{c}_0 - c_0^+\bar{e}^+(0)\right] : f^+(z,\bar{z}) \end{aligned} \tag{4.2.11}$$

and the zero modes of the bosonic and fermionic extension fields are related by supersymmetry

$$[\mathcal{Q}_0, \mathrm{e}^{\phi_0^-}\xi_0] = \mathrm{e}^{-\phi_0^+} \simeq b_0^+, \quad [\mathcal{Q}_0, \mathrm{e}^{-\phi_0^+}] = \mathrm{e}^{\phi_0^-}\xi_0. \tag{4.2.12}$$

However, eqn. (4.1.10) forbids that e^+ and $\bar{e}^+$ can be considered as dynamical fields in the fermionic sector. Therefore, it is again impossible to interpret e and $\bar{e}$ as dynamical fields in the CSbc.

Since supersymmetry was already preserved without deforming the fermions, it is not demandatory that the fermions are logarithmically extended. On the other hand, to the best of my knowledge there is nothing to be said against it, and I will argue below that, if the reader wishes to logarithmically extend the fermions, this will not affect the representation theory of the CSbc and thus the results of chapter 3.

4.2.5 Extension of the State Space

Although $\xi(z)$, $\bar{\xi}(\bar{z})$ are not part of the dynamical fields, the zero modes ξ_0 and $\bar{\xi}_0$ are introduced by the extension fields e, $\bar{e}$ and thus extend the state space. I will now prove that the extension is as in equations (2.6.2) and (3.6.36):

$$\overline{N}(1,1) \xrightarrow{e,\bar{e}} N(1,1) \xrightarrow{\mathfrak{g}_1} \overline{N}(0,0). \tag{4.2.13}$$

Firstly, I will restrict my considerations to the auxiliary $\eta\xi$-system in order to illustrate two aspects. As stated above, this will show that a logarithmic deformation of the fermions in the CSbc does not interfere with the extension of the representation spaces. Furthermore, the essential rôle of the coupling between the bosonized bosons and the auxiliary fermions will become evident. Secondly, I will explain how the logarithmic extension indeed leads to (4.2.13). By an explicit calculation of the action of the Grothendieck-Cousin fields on that extended space, I will substantiate the impact of the additional field modes that are invisible in the Morse theory description.

According to the deformation rule (4.2.2), the fields e, $\bar{e}$ and their composite $e\bar{e}$ extend the ground state $|0,0\rangle_{\eta\xi}$ of the $\eta\xi$-system by the new states $\xi_0|0,0\rangle_{\eta\xi}$, $\bar{\xi}_0|0,0\rangle_{\eta\xi}$ and $\xi_0\bar{\xi}_0|0,0\rangle_{\eta\xi}$. This extends the representation space as described in section 4.1,

$$\bigoplus_{l,r} \mathscr{A}^+_{-\frac{1}{2}}(l) \otimes \bar{\mathscr{A}}^+_{\frac{1}{2}}(r) \to \left(\bigoplus_{l,s} \mathscr{A}^+_{-\frac{1}{2}}(l) \oplus \mathscr{A}^+_{-\frac{1}{2}}(s-1)\right) \otimes \left(\bigoplus_{r,m} \bar{\mathscr{A}}^+_{\frac{1}{2}}(r) \oplus \bar{\mathscr{A}}^+_{\frac{1}{2}}(m+1)\right). \tag{4.2.14}$$

In particular, the logarithmic partners are modelled on the representation space with highest weight state $\xi_0\bar{\xi}_0|0,0\rangle_{\eta\xi}$,

$$\underline{T}_{\eta\xi\,0}|1,1\rangle_{\eta\xi} = 0\cdot|1,1\rangle - |0,0\rangle_{\eta\xi}, \tag{4.2.15}$$

while $\underline{T}_{\eta\xi\,0}$ is diagonal on the other representations. Therefore, one would naively assume that the logarithmic extension of the original state space equals $M^+_{\eta\xi\,L}(1,1) = \bigoplus_{l,s}(\mathscr{A}^+_{-\frac{1}{2}}(l-1)\otimes \bar{\mathscr{A}}^+_{\frac{1}{2}}(s+1))$, in analogy with the bosonized bosons eqn. (3.6.32). This state space is, however, isomorphic to the one defined by the partner fields, $\bigoplus_{l,s}\mathscr{A}^+_{-\frac{1}{2}}(l)\otimes\bar{\mathscr{A}}^+_{\frac{1}{2}}(s)$, because ξ_0 and $\bar{\xi}_0$ are part of the field algebra. This is the reason, why the $\eta\xi$-system alone is not capable of explaining the different nature of $\mathscr{H}^{\text{in}}_{0,0}$ and $\mathscr{H}^{\text{in}}_{\infty,0}$.

Fortunately, the extension of the state space of the full supersymmetric bc-system is more complicated because the algebra of the auxiliary fermionic field does not factorize. The new highest weight states, introduced by e and $\bar{e}$, are rather

$$\left.\begin{array}{l} e^{\phi_0^-}\xi_0 \\ e^{-\bar{\phi}_0^-}\bar{\xi}_0 \\ e^{\phi_0^- - \bar{\phi}_0^-}\xi_0\bar{\xi}_0 \end{array}\right\} \cdot v^-_{1,-1}\otimes|0,0\rangle_{\eta\xi} = \left\{\begin{array}{l} v^-_{0,-1}\otimes|1,0\rangle_{\eta\xi} \\ v^-_{1,0}\otimes|0,1\rangle_{\eta\xi} \\ v^-_{0,0}\otimes|1,1\rangle_{\eta\xi} \end{array}\right., \tag{4.2.16}$$

and the extension fields fill in the missing states in the diamond (3.6.27). The algebra of field modes

$$\left.\bigoplus_{l,s\in\mathbb{Z}} \mathscr{A}^-_{\frac{1}{2}}(l)\otimes\mathscr{A}^+_{\eta\xi,-\frac{1}{2}}(l)\otimes\bar{\mathscr{A}}^-_{-\frac{1}{2}}(s)\otimes\bar{\mathscr{A}}^+_{\eta\xi,\frac{1}{2}}(s)\right|_{\eta_0,\bar{\eta}_0=0} \tag{4.2.17}$$

is now represented on those states, and

$$\overline{N}(1,1) \xrightarrow{e,\bar{e}} [\overline{N}(1) \oplus N_L(1)] \otimes [\bar{\overline{N}}(1) \oplus \bar{N}_L(1)] = N(1,1)\,, \tag{4.2.18}$$

wherein the logarithmic extension $N_L(1,1)$ of $\overline{N}(0,0)$ appears, cf. eqn. (3.6.32).

The Action of the Grothendieck-Cousin Operator

I can now substantiate the action of $\underline{T}_{\eta\xi}$ on $N_L(1,1) \otimes R(1,1)$, cf. section 3.6.2. Therefore, I consider the states

$$\begin{aligned} &\chi_0^{(l)} := \mathscr{O}(\mathfrak{J}^-)\eta_{r_1}\cdots\eta_{r_i}\xi_{k_1}\cdots\xi_{k_j}\cdot v_{i-j+1}|0\rangle_{\eta\xi}\,,\\ &\chi_1^{(l)} := \mathscr{O}(\mathfrak{J}^-)\eta_{r_1}\cdots\eta_{r_i}\xi_{k_1}\cdots\xi_{k_j}\cdot v_{i-j}|1\rangle_{\eta\xi}\,,\\ &r_1 < \cdots < r_i < 0\,, \quad k_1 < \cdots < k_i < 0\,, \quad l = i-j\,, \end{aligned} \tag{4.2.19}$$

wherein $\mathscr{O}(\mathfrak{J}^-)$ is a monomial in $\mathfrak{J}^-_{-n}$, $n > 0$. They are elements of the Virasoro module with fixed charge $l+\frac{1}{2}$, measured by $\mathfrak{J}^-_0$.[3] I will denote these modules by $\overline{N}(1)_l$ and $N_L(1)_l$, respectively, which immediately generalizes to the compositions $\overline{N}(1,1)_{l,\bar{l}}$, $N_L(1,1)_{l,\bar{l}}$ and $R(1,1)_{l,\bar{l}}$ by means of

$$\chi_{s,\bar{s}}^{(l,\bar{l})} := \chi_s^{(l)} \otimes \bar{\chi}_{\bar{s}}^{(\bar{l})}\,, \quad s,\bar{s} \in \{0,1\}\,. \tag{4.2.20}$$

The action of $\underline{T}_{\eta\xi\,n} = T_{\eta\xi\,n} + \eta_n\bar{\eta}_0$ on such states is as follows.

For the zero mode, which is the Grothendieck-Cousin operator, I obtain

$$\underline{T}_{\eta\xi\,0}\cdot\chi_{s,\bar{s}}^{(l,\bar{l})} = E_\chi\cdot\chi_{s,\bar{s}}^{(l,\bar{l})} - \mathscr{N}\bar{\mathscr{N}}\chi_{0,0}^{(l,\bar{l})}\,, \tag{4.2.21}$$

where I used $\mathscr{N} := (-)^{i+\bar{i}+j+\bar{j}}\delta_{s,\infty}$ and $\bar{\mathscr{N}} := (-)^{i+\bar{i}+j+\bar{j}}\delta_{\bar{s},\infty}$. The deformed Hamiltonian is non-diagonal only on the states in $N_L(1,1)$, as I have already discussed in section 3.6.3.

For the other modes of the stress tensor with $n \neq 0$, I find

$$\underline{T}_{\eta\xi\,n}\cdot\chi_{s,\bar{s}}^{(l,\bar{l})} = T_{\eta\xi\,n}\cdot\chi_{s,\bar{s}}^{(l,\bar{l})} + (-)^s\bar{\mathscr{N}}\eta_n\cdot\chi_{s,0}^{(l,\bar{l})}\,, \tag{4.2.22}$$

and $\underline{T}_{\eta\xi\,n}$ is in general not diagonal if the states are in $R(1,1) \oplus N_L(1,1)$.

For all modes of the Virasoro field it is true that the ground state $v^-_{1,-1}|0,0\rangle_{\eta\xi}$ is not sensible for the logarithmic extension, as it is annihilated by all modes of the Grothendieck-Cousin field.

[3] The value of $\frac{1}{2}$ is due to the fact that I consider solely the holomorphic part.

4.2.6 Conclusion

I have logarithmically deformed the CSbc in such a way that it includes the situation of the Morse theory behind the A-model in the large volume limit. Thereby, also the fields and their OPA was deformed, and I have discussed the effects on the symmetries of the CSbc. In particular, the stress tensor obtained improvement terms

$$\begin{aligned} \underline{T}(z,\bar{z}) &= T(z,\bar{z}) + \mathfrak{g}_1(z,\bar{z}) + \mathfrak{g}_2(z,\bar{z})\,, \\ \mathfrak{g}_1(z,\bar{z}) &= \eta(z)\bar{\eta}_0 + \eta_0\bar{\eta}(\bar{z})\,, \quad \mathfrak{g}_2(z,\bar{z}) = \tilde{\eta}(z)\bar{\tilde{\eta}}_0 + \tilde{\eta}_0\bar{\tilde{\eta}}(\bar{z})\,, \end{aligned} \tag{4.2.23}$$

which I called Grothendieck-Cousin fields. Above, I included the second of these fields that is determined by a chart transition. The Grothendieck-Cousin operators break the bosonic axial symmetry, as well as the symmetry J_N which distinguishes the chains in the complex of extended bosonic representation spaces, cf. section 3.6.3. For this reason, the states in $N_L(1,1)$ and the corresponding fields can be interpreted as the logarithmic partners of the states and fields in the representation $\overline{N}(0,0)$.

Summary and Conclusion 5

In my first part of this thesis I have investigated the geometric significance of the improvement terms in the Hamiltonian of the logarithmic conformal bc-system with target $X = \mathbb{CP}^1$. Taking the perspective of its underlying Morse theory on loop space, I may now conclude the following.

The zero modes of the improvement terms are the infinite dimensional analogues of local cohomology operators (Grothendieck-Cousin operators – GCOs) in a complex of extended representation spaces of the Hamiltonian, whereby extension means that the representation spaces are extended by their missing dual part in the sense discussed in section 2.6.2. Therefore, the logarithmic conformal bc-system on $\mathbb{CP}^1$ is a field theoretic application of the Grothendieck-Cousin complex as considered by G. Kempf [Kem78], an interpretation already discussed by Frenkel, Losev and Nekrasov in [FLN08].

The same authors interpreted the extension as the transition from perturbative to non-perturbative state spaces, by which the zero modes of the improvement terms gain a second interpretation. They mimic the instantons becoming visible in the dynamical sector of the theory. This interpretation is in addition promoted by the fact that the GCOs are mappings in a specific direction, which is determined by a filtration of the local representation spaces. This direction conforms with the direction into which the instantons flow with growing time.

I will now briefly summarize the steps I have taken.

Morse Theory and Induced Representations In chapter 2, I have considered Morse theory on a compact Kähler manifold X, cf. [FLN06]. It was necessary to constrain X in order to guarantee that a non-empty topological sector would exist. After several transformations which left the topological sector invariant, I could massage the action into a first order form, such that the path integral would manifestly localize on the instantons. In particular, this spoiled CPT invariance and the transformed theory lost its former unitarity.

The speciality of this Morse theory has been that the metric was scaled with some positive, real-valued parameter λ, and that, hence, it got possible to move in the moduli space of the theory. Two phases of Morse theory have been of special importance, the phase when $\lambda \neq \infty$ and the large volume limit $\lambda \to \infty$. For finite λ, the representation spaces of the Hamiltonian are isomorphic to the representation spaces of the unitary theory. In the large volume limit it is not possible to make such a statement in general, besides for the topological sector, which is insensitive to the value of λ.

The most important impact of the scaled metric was that the perturbative spectrum of the Hamiltonian included apart from the topological further dynamical states. For the situation that the target manifold is $X = \mathbb{CP}^1$, these perturbative state spaces survived the large volume limit and became induced representations of the symmetry generated by the gradient field of the Morse function.

The perturbative representation spaces were defined locally on the so-called descending manifolds. These are the submanifolds into which X is decomposed by means of the gradient vector field. Frenkel et al. claimed that if the local representation spaces were extended as distributions to X, they did comprise the nonperturbative low energy spectrum of the theory, cf. [FLN08]. I have extended the perturbative spectrum in a manner which differs from that used by Frenkel et al. [FLN06]. The Hamiltonian turned out to be no longer diagonal on the thus obtained representation spaces. I did then decompose it into a trivial part and an operator which is responsible for that effect. The thus obtained operator entangled the extended representation spaces and, by comparison, could be identified with the local cohomology operator (GCO) of a particular Grothendieck-Cousin complex [Kem78]. Therefore, the GCO makes it possible to take an insight into the structure of the induced representations of the symmetry generated by the gradient field of the Morse function. In particular, this is an insight into the excited spectrum of the Morse theory and thus an effect beyond the topological sector.

Due to the GCO the Hamiltonian is indecomposable on certain dynamical states and also mixes the holomorphic and antiholomorphic target space coordinates. These aspects are typical for logarithmic conformal field theories and it is, hence, reasonable to generalize this concept to two-dimensional field theories, [FLN08].

A Field Theory Application In chapter 3 I have considered the A-model with domain manifold $\Sigma = \mathbb{R} \times S^1$ and target space $X = \mathbb{CP}^1$. The target space was again supplemented with a metric scaled by λ, cf. [FLN08]. Since many physicists and mathematicians assume that there exists a point in the moduli space of this theory where it is conformal [FL07, MSV99, DVV91], it was a good starting point for generalizing the discussion of the last chapter to a field theory and, additionally, for analyzing the meaning of the Grothendieck-Cousin operators in a conformal field theory.

As in the situation of Morse theory, I transformed the A-model into a first order shape by breaking CPT invariance and taking the large volume limit. Under this treatment, the A-model took the form of a supersymmetric bc-system which I called the "topological bc-system" (Tbc). Structurally, it looks like the conformal supersymmetric bc-system (CSbc), and I assumed that the representation theory for both systems is the same.

Having integrated out the dependence of S^1, the Tbc turns into an infinite sum of super quantum mechanical theories on loop space LX, which look similar to the Morse theory considered before. In order to attain the full analogy, it was necessary to add another vector field to the gradient vector field, which ensured that the critical manifold reduced to singular points. Like Frenkel, Losev and Nekrasov [FLN08], I have called this procedure as "gauging" and denoted the thus obtained Tbc as the gauged Tbc. Moreover, in order to obtain a Morse function for the gradient vector field it was necessary to lift the theory from loop space to its universal cover.

The Morse function thus obtained was multi-valued on loop space. Therefore, the preimages of LX in its universal cover fanned out into infinitely many leaves, distinguished by homology classes in $H_2(X, \mathbb{Z})$. In the same manner, the perturbative state spaces and the descending manifolds were distinguished. However, the state spaces were isomorphic, and I could restrict my consideration to one of those sectors.

Analyzing the Hessian of the Morse function, I could determine the coordinates of the descending manifolds in this sector. Because of the analogy to the Morse theory of chapter 2, I then could note down the perturbative representation spaces which localize on these submanifolds. It turned out that they could be modeled by representation spaces of the CSbc.

In order to define the CSbc on $X = \mathbb{CP}^1$ it was necessary to explain how chart transitions work, and I introduced the chiral de Rham complex [MSV99] to close this gap.

To determine the Grothendieck-Cousin operators, I had to find the local representation spaces between which such operators intermediate. As it turned out, there exist two such operators which, however, are related by a chart transition composed with a redefinition of the additional vector field I had used to reduce the critical manifold to isolated points. Therefore, it was sufficient to discuss only one Grothendieck-Cousin operator.

In order to obtain this GCO, I assumed that I may substitute the CSbc for the A-model. Having adjusted and generalized the method of chiral bosonization [FMS86], I could derive a cohomology operator in a long exact sequence of particular state spaces. The perturbative state spaces of the Tbc are part of this sequence, and I could extend them in such a way that the GCOs have been extracted as the cohomology operators in the short exact sequences of perturbative state spaces. Thise GCOs deform the Hamiltonian of the CSbc and are non-diagonalizable on a subspace of dynamical state spaces.

In the last chapter I discussed the question which deformation of the CSbc corresponds to the deformation of the Hamiltonian by the Grothendieck-Cousin operator [VF09].

Logarithmic Deformation of the Chiral de Rham Complex The GCOs made it necessary to reconsider the chiral de Rham complex. I looked for a logarithmic extension of this theory which would produce the GCOs within the Hamiltonian and extend the state spaces in the appropriate way. For this purpose, I have successfully accommodated the method of logarithmic deformation invented by Fjelstad et al. [FFH+02].

Since it must be applied to the bosonic subsector of the CSbc, this raised the question if, due to supersymmetry, it was not necessary to further deform the fermionic part. I have argued that supersymmetry did not demand this. Nevertheless, if the fermionic part is in addition logarithmically deformed, this does not affect the representation theory of the CSbc.

Moreover, the logarithmic deformation did neither destroy the Virasoro algebra nor supersymmetry. Yet, it spoiled all anomalous symmetries by which the Tbc exceeded the A-model with finite values of λ. I consider this as an additional confirmation that the logarithmic deformation of the chiral de Rham complex might be necessary, if the dynamical sector of the Tbc is taken into account.

Another interesting aspect has been that the basic Jordan blocks in the doublets of logarithmic partners are always comprised by fields which are not primary. In this respect, the theory is exceptional among logarithmic conformal field theories [Flo03].

II

Conformal Fermionic Ghosts on the Torus

Motivation 6

In the last part of my thesis I have investigated the conformal supersymmetric bc-system with target manifold $\mathbb{CP}^1$. Under the assumptions that this theory describes the topological A-model in the large volume limit and that it has a particular nonperturbative spectrum on the descending manifolds of its underlying Morse theory, it became necessary to logarithmically deform this CFT. The improvement terms in the stress tensor thereby inherited an interpretation as local cohomology operators and of instanton contributions.

This time I will consider a different geometric setting, which again gives rise to a logarithmic extension, now of the fermionic conformal bc-system.[1] In this setting, the CFT has target space $\mathbb{C}$, whereas the domain manifold is an algebraic surface $\mathbb{T}^{n,m}$ with global monodromy group $\mathbb{Z}_n$ as a branched covering of $\mathbb{CP}^1$. This situation has been discussed by V. Knizhnik [Kni87] for the non-logarithmic situation, and extended to the triplet model, in case that $\mathbb{T}^{n,m}$ is the torus, by M. Flohr [Flo98]. The triplet model [Kau95, GK96, Gab03], is not the same LCFT as the one I have discussed in the context of the A-model. It includes the situation of the last chapter but also exceeds it, in particular it contains additional twisted representations which mimic the branch points.

In the following chapters I will discuss two topics related with this setting. Firstly, I will argue from a purely geometric point of view that a logarithmic extension of the bc-system on the torus is unavoidable. Secondly, since the torus is the spectral curve of pure gauge, $SU(2)$ Seiberg-Witten (SW) theory [SW94], I will reduce the prepotential and the spectral curve of this theory to quantities in the triplet model [VF07].

In chapter 7 I will introduce the bc-system on the algebraic surfaces $\mathbb{T}^{n,m}$ along the lines of [Kni87]. The monodromy group will be responsible for additional, twisted representations which mimic the branch points.

In the following chapter 8, I will restrict my considerations to the case that the algebraic surface is a torus. Since the twisted representations mimic the branch points, there will exist a geometric argument why the bc-system must be logarithmic. This works by relating the Legendre family, which is a one parameter family of tori, to the nullvector condition of the twist fields. The minimal logarithmic CFT containing these representations is the triplet model which I will briefly introduce.

The last chapter 9 will be on pure gauge, $SU(2)$ Seiberg-Witten theory. After some introductionary remarks, I will explain how its spectral curve can be expressed in terms of triplet

[1] Since I will only treat this theory in the following, I will often refer to it as "the bc-system".

characters and how the prepotential can be obtained as a function of the torus modulus. Since this modulus equals the ratio of the four-point functions of the twist fields it is possible to determine the prepotential, and therefore this particular Seiberg-Witten theory, by means of quantities of the triplet model.

Fermionic Ghosts on Algebraic Curves 7

In this chapter I will summarize how Knizhnik formulates the conformal fermionic bc-system on a specific class of algebraic surfaces which are branched coverings of $\mathbb{CP}^1$, [Kni87]. Their monodromy group acts on the fields which thereby fall into irreducible representations. The highest weight vectors of those representations can again be related with conformal (twist) fields that simulate the effects of the branch points.

7.1 The Algebraic Surfaces

Every compact Riemannian surface can be obtained from a zero set of some polynomial in two variables by an inclusion of finitely many points [Fre09]. Therefore, I will trade such algebraic surfaces for compact Riemannian surfaces in the following. Particularly, I am interested in the class of polynomials

$$\mathbb{T}^{n,m} = \left\{ (y,x) \in \mathbb{C} \times (\mathbb{CP}^1 \setminus \{e_i\}) \ : P(y,x) = y^n - \prod_{i=1}^{nm} (x - e_i) = 0 \right\}, \ n, m \in \mathbb{N}, \tag{7.1.1}$$

with $e_i \neq e_j$, $\forall i \neq j$, and in those describing elliptic curves, subject to the restriction $n = 2$ and $m = 2$. I am particularly interested in the elliptic curves, because they become tori when compactified and the spectral curve of pure gauge Seiberg-Witten theory with $SU(2)$ gauge group is a torus.

The projection $p: \ (y,x) \mapsto x$, yields a covering (locally biholomorphic mapping) of $\Sigma = \mathbb{CP}^1 \setminus \{e_i\}$ by $\mathbb{T}^{n,m}$, and the Monodromy group has a global representation on differential forms on Σ due to the global $\mathbb{Z}_n$ symmetry. In an open neighborhood $U(e)$ of a branch point $e \in \Sigma$, there exists an open set $V(e) \subset p^{-1}(U(e))$ and biholomorphic mappings ϕ_V and $\tilde{\phi}_U$, such that the following diagram commutes, [Fre09]

$$\begin{array}{cccccc} \mathbb{T}^{n,m} : & V(e) & \stackrel{\phi_V}{\to} & D^* & z & \\ & p \downarrow & & \downarrow \tilde{p} & \downarrow & . \\ \Sigma : & U(e) & \stackrel{\tilde{\phi}_U}{\to} & D^* & z^n & \end{array} \tag{7.1.2}$$

Hereby, D^* denotes the unit disk without the point e, which I set to 0 without loss of generality. Therefore, in a neighborhood of a branch point e, the covering looks like $\tilde{p}(z) = e + z^n$ with inverse

$$\tilde{p}^{-1}(z) = (z - e)^{1/n}. \tag{7.1.3}$$

By $(z-e)^{1/n}$ I denote the whole stack of the n solutions to this equation, and which I label by $l \bmod n$, $l \in \mathbb{N}$. Whenever I want to distinguish a special root, I will denote it by $(z-e)^{1/n}|_{V_l}$.

When compactifying the algebraic curve, the mapping ϕ_V is analytically extended to the symbol $p^{-1}(e)$ by setting $\phi_V(p^{-1}(e)) = 0$. For this reason, though it is not quite correct, I will call $(V(e), \phi_V)$ a chart around $p^{-1}(e)$.

In the following I will describe how Knizhnik introduces the fermionic bc-system on the leaves of the covering and how the branch points introduce a stack of local representations of the theory on additional background fields.

7.2 The Fermionic bc-System on $\mathbb{T}^{n,m}$

Knizhnik defines a fermionic bc-system on the algebraic surface $\mathbb{T}^{n,m}$. It consists of a scalar fields b and a one-form c which he considers in the representation on $|0\rangle$, cf. section 3.4.1. These fields describe the purely holomorphic (and purely antiholomorphic) differential forms on the surface.[1] Due to the local biholomorphism, one can consider these fields on the different sheets l and in local coordinates z on Σ. For instance $b^{(l)}(z) = b \circ p^{-1}|_{V_l}(z)$, where V_l is an open subset of the l^{th} sheet, not including a branch point.[2] Similar holds for the one-form c. These fields have an action which due to the local biholomorphisms can be formulated on Σ

$$S^{(l)} = \int_\Sigma \mathrm{d}^2 z \; c^{(l)}(z) \partial_{\bar{z}} b^{(l)}(z) \,. \tag{7.2.1}$$

Accordingly, the total state space is a tensor product of n equivalent highest weight states, in particular

$$|0\rangle = \bigotimes_{l=0}^{n-1} |0\rangle_l \,. \tag{7.2.2}$$

On every sheet, the stress tensor is defined as in section 3.4.1 and the same holds for the fields. In particular, their operator product expansion yields

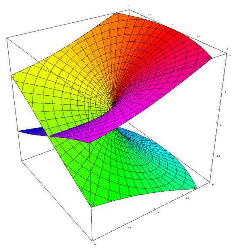
$$b^{(l)}(z) c^{(l')}(z') = \frac{\delta_{l,l'}}{z - z'} \,. \tag{7.2.3}$$

7.2.1 Around the Branch Points

Since analytic transitions between all sheets are possible in a chart around a branch point, this situation is more delicate. To visualize this, I depicted the Riemannian surface of $\sqrt{z}$, below.

[1] I will only consider fermionic fields b and c in this part of my thesis. Therefore, I will omit the index + used in section 3.4.1.

[2] In a chart I will allow myself the abuse of notation to equivalently denote by z a local coordinate on Σ or its preimage on $\mathbb{T}^{n,m}$.



Let $U(e)$ be a neighborhood of a branch point e. The different paths between the sheets, along which functions on $\mathbb{T}^{n,m}$ can be analytically continued, can be classified by means of the monodromy group related to e. It is defined as follows.

Let $\gamma \in \pi_1(U(e), z_0)$ be a closed path starting and ending at z_0 and enclosing at most the branch point e, and denote by $\tilde{\gamma}_l$ the (unique) lift of γ starting on the l^{th} sheet at q_l, $\tilde{p}(q_l) = z_0$.[3] The monodromy group permutes the elements of the fiber $p^{-1}(z_0) = \{q_0, \cdots, q_{n-1}\}$ and is defined by the action

$$\mu_\gamma \cdot q_l = \tilde{\gamma}_l(1)\,. \tag{7.2.4}$$

It is isomorphic to the group of roots defined by $q_l \mapsto q_{(l+k) \bmod n} = \mathrm{e}^{\frac{2\pi i k}{n}} q_l$, $k \in \mathbb{Z}_n$, and thus to $\mathbb{Z}_n$.

The monodromy group induces a representation on the fields by means of

$$\hat{\mu}_\gamma \cdot b(q_l) = b(\tilde{\gamma}_l(1))\,, \tag{7.2.5}$$

and similar for c. In a chart without branch point, the points q_l can again be projected on Σ such that this relation holds for fields $b^{(l)}$ and $c^{(l)}$.

Since $\mathbb{T}^{n,m}$ is globally $\mathbb{Z}_n$ symmetric, the representation of the monodromy group can be diagonalized simultaneously for every branch point. This is obtained by the Fourier transformations

$$\begin{aligned} b_k(z) &= \sum_{l=0}^{n-1} e_{k+1-n}(l) b^{(l)}(z) \\ c_k(z) &= \sum_{l=0}^{n-1} \bar{e}_{k+1-n}(l) c^{(l)}(z) \end{aligned} \quad , \qquad e_m(l) = \mathrm{e}^{2\pi i l \frac{m}{n}}\,, \;\; k = 0, \ldots, n-1\,. \tag{7.2.6}$$

The monodromy group now introduces the boundary conditions

$$\hat{\mu} \;:\quad b_k(z) \mapsto \mathrm{e}^{-2\pi i \frac{k+1-n}{n}} b_k(z)\,, \qquad c_k(z) \mapsto \mathrm{e}^{+2\pi i \frac{k+1-n}{n}} c_k(z)\,, \tag{7.2.7}$$

and the n different Fourier transformations distinguish n different irreducible representations of this group. The domain of b_k and c_k is $p^{-1}(U) = \bigsqcup_{l \in \{0,\ldots,n-1\}} V_l$, where U does not contain a branch point. While before it was reasonable to separate the fields together with the different sheets, the idea to entangle them in one equation is natural in a neighborhood of a branch point. The most important consequence is that the currents can now be defined also in a neighborhood of a branch point and as the single-valued fields

$$j_k(z) = -:b_k(z) c_k(z):\,. \tag{7.2.8}$$

[3] Composing such loops defined with respect to different branch points, one can generate all possible loops enclosing one or several branch points. Therefore, and due to the global $\mathbb{Z}_n$ symmetry, it is sufficient that I restrict my discussion to one branch point.

Operator Product Expansions Since the sheets of the algebraic surface are overlapping in a neighborhood of a branch point, the fields may have nontrivial OPEs in this region. To see this, Knizhnik starts with two local fields $b^{(l)}(z)$ and $c^{(l')}(\omega)$, $z \in p^{-1}(U)|_{V_l}$, $\omega \in p^{-1}(U)|_{V_{l'}}$, which are located close to a branch point e. Applying a chart transition to a neighborhood of e, $z \mapsto y = (z-e)^{1/n}|_{V_l(e)}$ and $\omega \mapsto y' = (\omega - e)^{1/n}|_{V_{l'}(e)}$ one ends up with

$$b^{(l)}(z)c^{(l')}(\omega) = \frac{n^{-1}}{z-\omega} \sum_{r=0}^{n-1} \left(\frac{y'}{y}\right)^{r+1-n} . \tag{7.2.9}$$

Here, I used that in the presence of a branch point $b^{(l)}(y)c^{(l')}(y') = (y-y')^{-1}$, even if $l \neq l'$. In order to apply this to the fields in the Fourier expansion, I will use that the basis elements $e_m(l)$ define a scalar product

$$e_m \cdot \bar{e}_s = \sum_{l=0}^{n-1} e_{m-s}(l) = \begin{cases} n & \text{if } \exists\ t \in \mathbb{Z}\ :\ tn = m-s \\ 0 & \text{else} \end{cases} \tag{7.2.10}$$

which can be applied to b_k and c_k. Combining it with the OPEs above, one ends up with

$$b_k(z) \cdot c_{k'}(\omega) = \delta_{k,k'} \frac{1}{z-\omega} \sum_{r=0}^{n-1} \left(\frac{y'}{y}\right)^{r+1-n} \tag{7.2.11}$$

This quantity has to respect the transformation (7.2.7), in particular letting z encircle e, this must result in a phase shift of b_k. Indeed, the product above yields a factor $(y^n)^{-\frac{r+1-n}{n}} \mapsto \mathrm{e}^{-2\pi \mathrm{i}\frac{r+1-n}{n}}(y^n)^{-\frac{r+1-n}{n}}$, which restricts r to $r \stackrel{!}{=} k$, and the sum collapses to this single term. Extending y^n around y'^n, one obtains

$$b_k(z) \cdot c_{k'}(\omega) = \left(\frac{1}{z-\omega} - \frac{\frac{k+1-n}{n}}{\omega - e} + :b_k(\omega)c_k(\omega): + O(z-\omega) \right) . \tag{7.2.12}$$

For $k = k'$, this result should be compared with the definition of the current

$$j_k(\omega) = \lim_{z\to\omega} [-b_k(z)c_k(\omega) + (z-\omega)^{-1}] . \tag{7.2.13}$$

Therefore, Knizhnik concludes that the additional term due to the branch point indicates the presence of some background field, serving as a source for the additional charge q_k

$$j_{q_k}(z) = j_k(z) + \frac{q_k}{z-e}, \qquad q_k = \frac{k+1-n}{n}, \quad k = 1, \ldots, n-1 . \tag{7.2.14}$$

7.2.2 The Twisted Representations

Motivated by the discussion above, I will now extend the representation theory of section 3.4.1 to charges with values in the rational numbers, such that

$$b_k(z)c_k(\omega)|q_k\rangle = (z-\omega)^{-1} \left(\frac{\omega}{z}\right)^{q_k} |q_k\rangle . \tag{7.2.15}$$

Here, I assume that normal ordering is again defined with respect to $|0\rangle$ and $b_{k\,n}|q_k\rangle = 0$, $n > 0$, $c_{k\,n}|q_k\rangle = 0$, $n \geq 0$. This representation is meant to exist locally in a chart around a branch point e which I have set to $e = 0$.

Due to the monodromy, the fields in the different sectors are supposed to have a new series expansion in this representation

$$b_k(z) = \sum_{n\in\mathbb{Z}} b_{k\,n}\, z^{-n-q_k}\,, \qquad c_k(z) = \sum_{n\in\mathbb{Z}} c_{k\,n}\, z^{-n+q_k-1}\,, \tag{7.2.16}$$

which must have an impact on the composite fields. Take for instance the the stress tensor. Firstly, it acquires additional terms

$$T_{q_k}(z) = T_k(z) + \frac{1}{2}\frac{q_k(q_k+1)}{z^2} \tag{7.2.17}$$

due to the OPE above. Secondly, it is build from b_k and c_k which are now in the representation (7.2.16) on $|q_k\rangle$. Therefore, the modes gain a shift by the charge q_k

$$T_{q_k\,m} = \sum_{n\in\mathbb{Z}} (n-q_k) : b_{k\,-n}\, c_{k\,n+m} : \; + \frac{1}{2} q_k(q_k+1)\delta_{m,0}\,. \tag{7.2.18}$$

and the field modes have new conformal weights $[T_{q_k\,0}, b_{k\,n}] = (-n-q_k)\, b_{k\,n}$ and $[T_{q_k\,0}, c_{k\,n}] = (-n+q_k)\, c_{k\,n}$. On the other hand, $[j_{q_k\,0}, b_{k\,n}] = -\, b_{k\,n}$ and $[j_{q_k\,0}, c_{k\,n}] = c_{k\,n}$, as before, and the $U(1)$ charges are not affected. The state $|q_k\rangle$ has charge q_k, conforming with the discussion in the last section, and conformal weight $\frac{1}{2}q_k(q_k+1)$, also cf. section 3.4.2.

To conclude this section on the representation theory of the bc-system on $|q_k\rangle$, notice that the $U(1)$ current behaves under Möbius transformations as in equation (3.4.14). Therefore, the representation on $|q_k\rangle$ is not unitary and it inherits the background charge $\mathfrak{q} = 1$ already obtained in section 3.4.1.

Twist Fields

From the CFT point of view there should correspond a unique field to this representation which has the same quantum numbers and which is fixed at the position of the branch point. Formally, I will denote this isomorphism by the mapping $*$: $\mu_{q_k}(0) * |0\rangle = |q_k\rangle$ wherein $\mu_{q_k}(0)$ is the field corresponding to $|q_k\rangle$ and $|0\rangle = \bigotimes_{l=0}^{n-1} |0\rangle_l$.[4] For convenience, I will omit the $*$ in a correlator and write $\cdots \mu_{q_k}(0) * |0\rangle = \cdots \mu_{q_k}(0)\rangle_0$, cf. section 3.4.1.

In order to represent a branch point, $\mu_{q_k}(0)$ should respect the monodromy property of the fields b_k and c_k, i.e.

$$\begin{aligned} b_k(\mathrm{e}^{2\pi\mathrm{i}} z)\mu_{q_k}(0) &= \mathrm{e}^{-2\pi\mathrm{i} q_k} b_k(z)\mu_{q_k}(0)\,, \\ c_k(\mathrm{e}^{2\pi\mathrm{i}} z)\mu_{q_k}(0) &= \mathrm{e}^{2\pi\mathrm{i} q_k} c_k(z)\mu_{q_k}(0)\,. \end{aligned} \tag{7.2.19}$$

[4] Formally, if V_0 is the vector space generated from $|0\rangle$ and the fields b, c, V_{q_k} denotes the vector space on whch the μ_{q_k} are represented, $* \in \mathrm{End}(V_{q_k}) \times V_0 \to V_{q_k}$.

Consequently, the boundary conditions (7.2.7) are represented on the bc-system by means of these fields. If the induced boundary conditions are non-trivial, i.e. $q_k \notin \mathbb{Z}$, it is common to call μ_{q_k} a "twist field" [Gin88] and the representation of the bc-system on the respective state $|q_k\rangle$ a "twisted representation".

The monodromy condition imposed on the field μ_{q_k} allows for a whole stack of twist fields with charge $q_k + n\ ,\ n \in \mathbb{Z}$, called "excited twist fields". For instance the operator

$$\mu_{q_k-1}(0) = \mu_{q_k}(0)\ b_{k\,0} \tag{7.2.20}$$

defines a field of charge $q_k - 1$ and with conformal weight $\frac{1}{2}q_k(q_k+1) - q_k$. Similarly, other excited twist fields can be generated by an action of the modes of b_k and c_k. However, because they are in the same representation of the monodromy group, all these excited twist fields belong to the same representation on $|q_k\rangle$. The operator $\mu_{q_k}(0)\ b_{k\,0}$ is special since it formally can be identified with $\mu_{q_k}(0)|1\rangle$, whereby $|1\rangle$ is the second possible, however not conformally invariant, vacuum representation in the CSbc. It played the rôle of the logarithmic partner of $|0\rangle$ in section 4.1.3. This time, however, the conformal weights of $\mu_{q_k}(0)$ and $\mu_{q_k}(0)\ b_{k\,0}$ are not the same and both fields can not be logarithmic partners.

7.2.3 Conclusion

Due to the action of the monodromy group and in addition to the representation on the conformally invariant state $|0\rangle$, the fermionic bc-system on $\mathbb{T}^{n,m}$ falls into n representations, each of which is comprised by the fields b_k and c_k, $k \in \{0, \dots, n-1\}$, with the field algebra described by (7.2.15) and represented on $\mu_{q_k}(e)$ respectively $|\mu_{q_k}\rangle$. These representations are locally defined in the sense that the fields $\mu_{q_k}(e)$ are fixed at a branch point e and the operator product algebra (7.2.15) is defined in a neighborhood of this point. However, since the monodromy group is $\mathbb{Z}_n$ for every branch point, it is sufficient to consider the representation theory in a chart including a single branch point. The currents j_k defined by the fields in these representations are single-valued on Σ and yield the same quantum numbers for any value of k. This is not true for the stress tensor, which measures different weights depending on the particular representation.

On Twist Fields and Torus Periods 8

It is the achievement of M. Flohr to have related the twisted bc-system on $\mathbb{T}^{2,2}$ to $SU(2)$ SW theory, [Flo98, Flo04]. Thereby, he took three crucial steps. Firstly, Flohr "released" the twist fields and considered the branch points as dynamical degrees of freedom on $\mathbb{CP}^1$. As a consequence, the question arised how the operator product algebra gets enlarged when OPEs between these fields are taken into account and which fields must be added in order to close this algebra. The answer to this question was the second step Flohr had taken, he proposed that the bc-system on the torus should be identified with the so-called triplet model [GK96, Roh96, Kau95]. Finally, he argued that if the bc-system on the torus is identified with the triplet model it is possible to describe the main data of $SU(2)$ SW in terms of correlation functions of this theory.

In this chapter I will motivate the choice of the triplet model but take a more geometric approach than that of Flohr. From this will follow that it is necessary to release the twist fields in order to describe the fundamental parameters of the torus (its periods and their ratio). As a consequence I will then further deduce that the bc-system on the torus must be extended to a logarithmic CFT, and the triplet model will be the minimalistic extension.

In the first section, I will release the branch points and transform the algebraic curve $\mathbb{T}^{2,2}$ into the "Legendre family". This formulation is canonical in order to study small movements in the moduli space of tori. In particular, the periods of the tori satisfy a hypergeometric differential equation in the moduli parameter [CMSP03].

In the following section 8.2, I will identify this differential equation with the nullvector condition on the twist field $\mu_{-\frac{1}{2}}$ [Flo98, Flo04, Flo03, Gab03], which again relies on the possibility that the branch points may vary. This will explain why it is necessary to extend the bc-system to an LCFT.

The chapter will be concluded with a brief discussion of the representation theory of the bc-system and a brief introduction of the triplet model as the minimalistic logarithmic extension includeing the twist fields.

8.1 The Legendre Family

The algebraic curve $\mathbb{T}^{2,2}$ can be transformed into a polynomial of third order

$$\mathscr{E}_\lambda \;:\quad y^2(z;\lambda) = z(z-1)(z-\lambda)\,, \quad \lambda \in \mathbb{CP}^1 \setminus \{\infty, 0, 1\} \tag{8.1.1}$$

by means of $SL(2,\mathbb{C})$ transformations of z and y.[1] Indeed, every compact Riemannian surface of genus one is the set of zeros of a polynomial of this form for some λ [Jos02, FB00]. Therefore, the moduli spaces of the two descriptions of tori are equivalent, $\mathbb{T}^{2,2} \simeq \mathscr{E}_\lambda$. The branch points are now positioned at $\{\infty, 0, 1, \lambda\}$, and $\mathscr{E}_\lambda$ can be considered to be parametrized by $\lambda \in \mathbb{CP}^1 \setminus \{\infty, 0, 1\}$. This makes the Legendre family particularly nice to study variations of the corresponding equivalence classes of tori as functions of λ, or to study the singularities of $\mathscr{E}_\lambda$ which are evident in terms of λ. I will denote the space $\mathscr{M}_\mathscr{E} = \mathbb{CP}^1 \setminus \{\infty, 0, 1\}$ as the moduli space of the Legendre family $\mathscr{E}_\lambda$, with coordinate λ.

8.1.1 Relation to the Lattice Torus

In what sense can a variation in λ evoke a movement between different equivalence classes of tori? The canonical parameter to distinguish or identify equivalence classes of tori is the ratio τ of the periods of a torus in the lattice description.

Below I will argue that each non-singular member of the Legendre family is equivalent to a lattice torus

$$\mathbb{C}/L_\lambda - \{[0]\}, \qquad L_\lambda = \{\ m\Pi_D(\lambda) + n\Pi(\lambda)\,,\ \tau(\lambda) = \pm\frac{\Pi_D(\lambda)}{\Pi(\lambda)}\,,\ \Im(\tau) > 0,\ m, n \in \mathbb{Z}\ \}, \tag{8.1.2}$$

whereby the choice of sign in the definition of τ is such as to customize $\Im(\tau) > 0$ [FB00]. Without loss of generality I will assume that after some rescaling of the periods I may choose the plus sign. The periods of L_λ are described in terms of cohomology classes of $\mathscr{E}_\lambda$. The differential form

$$\varpi(z;\lambda) = \frac{\mathrm{d}z}{y(z;\lambda)} \tag{8.1.3}$$

is holomorphic and without zeros on $\mathscr{E}_\lambda$.[2] Therefore, it is closed with respect to the de

[1] Without loss of generality, $e_4 \neq 0$. Apply the following transformations and some redefinition of y

$$z \mapsto \frac{e_4 z}{z + e_4^{-1}} \quad \Rightarrow \quad y^2 \mapsto {y'}^2 = \prod_{i=1}^{3}(e_4 - e_i)(z - u_1)(z - u_2)(z - u_3)\,, \quad u_i = e_i[e_4(e_4 - e_i)]^{-1}\,.$$

The change of variables $z \mapsto (u_1 - u_2)z + u_1$ and another appropriate redefinition of y' yield the desired result, whereby $\lambda = \frac{u_3 - u_1}{u_1 - u_2}$.

[2] This is most obvious in the Weierstrass formulation of $\mathscr{E}_\lambda$, [FB00]. Let L_λ be the lattice corresponding to $\mathscr{E}_\lambda$. One may again redefine $\mathscr{E}_\lambda$ by $z \mapsto 4^{1/3}z + \frac{\lambda+1}{3}$ which yields the Weierstrass normal form

$$\begin{aligned} &X(g_2, g_3):\quad y^2 = 4z^3 - g_2 z - g_3\,,\quad y, z \in \mathbb{C} \\ &g_2 = \frac{4^{1/3}}{3}(\lambda^2 - \lambda + 1)\,,\quad g_3 = \frac{1}{27}(\lambda + 1)(2\lambda^2 - 5\lambda + 2)\,. \end{aligned} \tag{8.1.4}$$

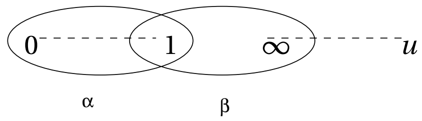
This curve is called Weierstrass normal form because the Weierstrass function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L_\lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right), \tag{8.1.5}$$

satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z) - g_2\wp(z) - g_3\wp(z)\,. \tag{8.1.6}$$

Rham differential and has a well defined cohomology class. By means of de Rham duality, this cohomology class can be defined to be the dual of some homology class in $H_1(\mathscr{E}_\lambda, \mathbb{Z})$, which, without loss of generality, is generated by the cycles as depicted below,



and with intersection number 1. Denote by α^* and β^* the basis for $H^1(\mathscr{E}_\lambda, \mathbb{Z})$ dual to α and β, i.e. $\int_\alpha \alpha^* = 1$, $\int_\alpha \beta^* = 0$. The cohomology class of ϖ is given by an expansion in this basis as

$$[\varpi] = \alpha^* \int_\alpha \varpi + \beta^* \int_\beta \varpi . \tag{8.1.7}$$

Thus, if $[\gamma] \in H_1(\mathscr{E}_\lambda, \mathbb{Z})$, $[\gamma] = m\alpha + n\beta \ , \ m, n \in \mathbb{Z}$ one finds that

$$\int_{[\gamma]} [\varpi] = m \int_\alpha \varpi + n \int_\beta \varpi . \tag{8.1.8}$$

Since the elliptic integrals like $\int_\alpha \varpi$ take their values on L_λ (cf. the explanation in the footnote on pg. 96), I can identify

$$\Pi_D(\lambda) = \int_\alpha \varpi \quad , \quad \Pi(\lambda) = \int_\beta \varpi \tag{8.1.9}$$

and interpret (8.1.8) as the representative for $[\gamma]$ on $\mathbb{C}/L_\lambda$.

8.1.2 A Differential Equation for the Periods

The homotopy class $[\varpi(\lambda)]$ and, if the cycles are fixed, also the periods Π_D and Π, satisfy a hypergeometric differential equation

$$\lambda(\lambda-1)\frac{\mathrm{d}^2\varpi(\lambda)}{\mathrm{d}\lambda^2} + (2\lambda-1)\frac{\mathrm{d}\varpi(\lambda)}{\mathrm{d}\lambda} + \frac{1}{4}\varpi(\lambda) = 0 , \tag{8.1.10}$$

whereby ϖ is the representative of $[\varpi]$ and the differential equation is zero up to exact forms.

The following nice proof is taken from [CMSP03]. The quantity $[\varpi(\lambda)] = \Pi_D(\lambda)\alpha^* + \Pi(\lambda)\beta^*$ can be interpreted as a differential form on

$$H^1(\mathscr{E}, \mathbb{Z}) := \bigcup_{\lambda \in \mathbb{CP}^1 \setminus \{\infty, 0, 1\}} H^1(\mathscr{E}_\lambda, \mathbb{Z}) . \tag{8.1.11}$$

The Weierstrass function is periodic in Π and Π_D and is defined on $\mathbb{C}/L_\lambda$. It induces a conformal equivalence between $X(g_2, g_3)$ and $\mathbb{C}/L_\lambda - \{[0]\}$, via $[z] \mapsto (\wp(z), \wp'(z))$, whereby $[0]$ is taken out since $\wp$ has a pole at this point [FB00]. Let $\gamma(t)$ be a curve on $\mathbb{C}/L_\lambda$ which does not pass a zero of $\wp'$. Omitting $[\cdot]$ for convenience, $\mathrm{d}\gamma(t) = \frac{\wp'(\gamma)}{\wp'(\gamma)}\mathrm{d}\gamma = \frac{\mathrm{d}\wp(\gamma)}{\wp'(\gamma)}$, and the elliptic integral $E(\gamma) = \int_\gamma \frac{\mathrm{d}\wp}{\wp'}$ is formally the inverse of $\wp$, mapping $X(g_2, g_3)$ to $\mathbb{C}/L_\lambda$. This integrand, restricted to a curve which is not passing a zero of $\wp'$, is a holomorphic one form and thus closed. It can be identified with ϖ on $\mathscr{E}_\lambda$.

The derivative $\partial_\lambda = \frac{\mathrm{d}}{\mathrm{d}\lambda}$ denotes the the covariant differential on this space, whereby the connection is chosen such that α^* and β^* are (locally) constant. Then, formally, $\partial_\lambda[\varpi(\lambda)] = \partial_\lambda \Pi_D(\lambda)\alpha^* + \partial_\lambda \Pi(\lambda)\beta^* = [\partial_\lambda \varpi(\lambda)]$. For this relation to make sense, one has to prove that $\partial_\lambda \varpi(\lambda)$ is indeed a representative of a cohomology class of $\mathscr{E}$. Take the representative $\varpi(\lambda)$, then

$$\partial_\lambda \varpi(\lambda) = \frac{1}{2}[z(z-1)(z-\lambda)^3]^{-\frac{1}{2}}\mathrm{d}z \tag{8.1.12}$$

is a meromorphic one-form. However, its pole has a multiplicity greater equal two at $(y,z) = (0,\lambda) =: P$, such that it nevertheless defines a cohomology class. Namely, in a neighborhood of P, $y(z)$ is invertible and one can write $y^2 = \frac{h(y)}{\lambda(\lambda-1)}(z(y)-1)$, whereby $h(y)$ is holomorphic in y and $h(0) = 1$. Solving for z and expanding $h(y)^{-1}$ around $y = 0$ yields $z = \lambda + O(y^2)$. Now, with $y^2(z) = p(z)$ one has $\varpi = 2\frac{\mathrm{d}y}{\partial_z p(z)}$, and inserting the approximation for z yields $\varpi = 2\frac{\mathrm{d}y}{\lambda(\lambda-1)} + O(y^{-2})$. Thus, plugging in again $z - \lambda = O(y^2)$,

$$\partial_\lambda \varpi(\lambda) = \frac{1}{2}\frac{\varpi(\lambda)}{z-\lambda} = \frac{\mathrm{d}y}{\lambda(\lambda-1)(z-\lambda)} + O(y^{-2}) \sim \frac{\mathrm{d}y}{\lambda(\lambda-1)y^2} + O(y^{-3}). \tag{8.1.13}$$

The following remarks conclude the proof. By Stokes theorem, the residuum of a one-form depends only on the cohomology class. Therefore, the sequence

$$0 \to H^1(\mathscr{E},\mathbb{Z}) \overset{restriction}{\longrightarrow} H^1(\mathscr{E}\setminus\{P\},\mathbb{Z}) \overset{\oint_P}{\to} 0. \tag{8.1.14}$$

is exact and $\partial_\lambda \varpi(\lambda)$ is a cohomology class on $\mathscr{E}$ and not just on $\mathscr{E}\setminus\{P\}$. Since ϖ and $\partial_\lambda \varpi$ are both cohomology classes and $H^1(\mathscr{E},\mathbb{Z})$ has two generators, every other cohomology class can be expanded in these two. In particular

$$A(\lambda)\partial_\lambda^2 \varpi + B(\lambda)\partial_\lambda \varpi + C(\lambda)\varpi = 0, \tag{8.1.15}$$

modulo an exact form. A calculation reveals that $f = [z(z-1)(z-\lambda)^{-3}]^{\frac{1}{2}}$ satisfies $\mathrm{d}f = (z-1)\partial_\lambda \varpi + z\partial_\lambda \varpi - 2z(z-1)\partial_\lambda^2 \varpi$. Using $z = z - \lambda + \lambda$ in this equation and $(z-\lambda)\partial_\lambda \varpi = \frac{1}{2}\varpi$, $(z-\lambda)\partial_\lambda^2 \varpi = \frac{3}{2}\partial_\lambda \varpi$ yields the differential equation for the periods.

8.1.3 Solutions for the Periods

This differential equation is a special case of the hypergeometric equation

$$\left(\lambda(\lambda-1)\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} + [(a+b+1)\lambda - c]\frac{\mathrm{d}}{\mathrm{d}\lambda} + ab\right)F = 0, \tag{8.1.16}$$

with $a = b = \frac{1}{2}$ and $c = 1$. Its solutions are the hypergeometric functions $F(a,b;c|\lambda)$, classified for instance in [E+85]. In the case under consideration, the solution space may be spanned by the functions

$$F_1(\lambda) = F(\tfrac{1}{2},\tfrac{1}{2};1|\lambda) \quad , \quad F_2(\lambda) = \mathrm{i}F(\tfrac{1}{2},\tfrac{1}{2};1|1-\lambda). \tag{8.1.17}$$

Erdelyi defines the function $F(\frac{1}{2},\frac{1}{2};1|\lambda)$ by an integral representation which yields an analytic, single-valued function on $\mathbb{C}\setminus\mathbb{R}^{\geq 0}$ [E+85]. Its local form in a neighborhood of $\lambda = 0$ equals

$$F(\tfrac{1}{2},\tfrac{1}{2};1|\lambda) = \frac{1}{\pi}\sum_{n=0}^{\infty}\left(\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2})n!}\right)^2 [k_n - \log(1-\lambda)](1-\lambda)^n\,, \tag{8.1.18}$$

whereby $|1-\lambda| < 1$, $|\arg(1-\lambda)| < \pi$ and

$$k_n = 2\psi(n+1) - 2\psi(\tfrac{1}{2}+n)\,, \quad \psi(\lambda) = \partial_\lambda \log\Gamma(\lambda)\,. \tag{8.1.19}$$

In this shape (8.1.18), it is evident that the solutions F_1 and F_2 of the differential equation for the periods have logarithmic singularities at $\lambda = 1$ and $\lambda = 0$, respectively.

Both solutions F_1 and F_2 get, however, mixed whenever λ passes the branch cut between 0 and 1. The results are again taken from [E+85], who used the relation

$$\frac{1}{2}\pi F_1(\lambda) - \frac{\mathrm{i}}{2}\log(1-\lambda)F_2(\lambda) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2})n!}\right)^2 k_n(1-\lambda)^n\,, \tag{8.1.20}$$

to obtain

$$\mu_0 : \begin{pmatrix}F_1\\F_2\end{pmatrix} \mapsto \begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}\begin{pmatrix}F_1\\F_2\end{pmatrix}, \quad \mu_1 : \begin{pmatrix}F_1\\F_2\end{pmatrix} \mapsto \begin{pmatrix}1 & -2\\0 & 1\end{pmatrix}\begin{pmatrix}F_1\\F_2\end{pmatrix}, \tag{8.1.21}$$

whereby μ_0 and μ_1 denote the operation of encircling the branch points 0 and 1, once. The group generated by the matrices above is called the "global monodromy group" of $\mathcal{E}_\lambda$ [CMSP03]. Due to the monodromy property, the choice of the solutions F_1 and F_2 has no fundamental meaning. Indeed, given the lattice defined by the periods F_1, F_2, all lattices in the orbit of the monodromy group are identical. For this reason, the periods corresponding to different algebraic surfaces are classified by the global monodromy groups and vice versa.

8.2 LCFT-fication of the Legendre Family

The Legendre family has a floating branch point, whereas in Knizhniks approach all branch points were fixed. Therefore, in order to find a field theoretic expression for the periods, I will now reinvestigate the fermionic bc-system on $\mathbb{T}^{2,2}$ and reformulate the branch points as dynamical degrees of freedom. Behind this work stands a pile of publications on the LCFT at $c = -2$, on my table are stacked up in particular the references [Flo98, Flo03, Flo04, Kau95, Gab03, Gur93].

Until now, the bc-system on $\mathbb{T}^{2,2}$ consists of two different local representations $|q_{0/1}\rangle$ in every chart which contains a branch point, and one globally defined representation on $|0\rangle$ with support on Σ. The following list summarizes the representations and fields I have discussed in chapter 7:

reps.	charges J	weights Δ	fields	domain
$\vert 0\rangle$	0	0	$1(z)$	Σ
$\vert\tilde{0}\rangle := \vert q_1\rangle$	0	0	$\tilde{1}(e_i) := \mu_0(e_i)$	$\{e_i\}$
$\vert\mu\rangle := \vert q_0\rangle$	$-\frac{1}{2}$	$-\frac{1}{8}$	$\mu(e_i) := \mu_{-\frac{1}{2}}(e_i)$	$\{e_i\}$
$\vert\sigma\rangle := b_{0\,0}\vert q_0\rangle$	$-\frac{3}{2}$	$\frac{3}{8}$	$\sigma(e_i) := b_{0\,0}\,\mu_{-\frac{1}{2}}(e_i)$	$\{e_i\}$

(8.2.1)

Only, the latter two rows denote twist fields, whereas the first representations have trivial monodromy. Notice that the untwisted representations have the same quantum numbers and might be logarithmic partners, whereas this is not true for the twist fields. The dynamical fields represented on these spaces are the fields $b(z)$ and $c(z)$. I have distinguished their representations by an index k such that for instance $b(z)$ denoted the representation on $|0\rangle$ and $b_k(z)$ the representation on $|q_k\rangle$. For convenience I will now drop this index.

It is necessary to release the fields representing the branch points in order to reproduce the situation of the Legendre family. The branch point coordinates and corresponding fields may then move on $\mathbb{CP}^1$, and the background fields become additional dynamical quantities. In this sense, the corresponding local representations become global representations on Σ and by a conformal transformation of the algebraic surface as described in the last section, one may identify $\{e_i\}_{i=1,\dots,4} = \{0, 1, \infty, \lambda\} \in \mathbb{CP}^1$, $\lambda \in \mathcal{M}_{\mathcal{E}}$.

As soon as twist the fields related to the branch points are released, the question arises what the operator product algebra looks like. In particular, I would like to be able to calculate correlation functions of the kind

$$\left\{ \langle \prod_{l=1}^{s} \mathcal{O}_l \prod_{i=1}^{n} \phi_i(z_i) \prod_{j=1}^{m} \mu_{q_{k_j}}(\omega_j)\rangle \neq 0\,, \quad \begin{array}{l} z_i \in \Sigma\,,\ \omega_j \in \mathbb{CP}^1 \\ \sum_i J(i) + \sum_j J(j) + \sum_k J(l)] = -1 \end{array} \right\}, \tag{8.2.2}$$

whereby ϕ_i can be $b(z)$ or $c(z)$ and $\langle\cdot\rangle = {}_0\langle\cdot\rangle_0$, cf. section 3.4.1. The condition $\sum_i J(i) + \sum_j J(j) + \sum_k J(l) = -1$ is necessary to cancel the background charge $\mathfrak{q} = 1$. This is accomplished by the operators $\mathcal{O}_l$, which denote any non-dynamical quantities and which I will call "screening operators", for this reason. For instance, b_0 is a screening operator in $\langle b_0 1(z)\rangle = \langle 0|1\rangle = 1$.

8.2.1 A Hypergeometric Equation for the Twist Fields

For the moment, I am interested in correlation functions including the twist fields $\mu_{-\frac{1}{2}}$. They are promising candidates to simulate the periods of the Legendre family because they introduce some monodromy and, hence, mimic the non-trivial behaviour of the branch points.

To calculate correlation functions, it is helpful to search for restrictions such as nullvector conditions. Indeed, the representation $|\mu\rangle$ satisfies a nullvector condition at level 2

$$(T_{-2} + 2T_{-1}^2)|\mu\rangle = 0\,, \tag{8.2.3}$$

which signifies that the four-point function has to satisfy a hypergeometric differential equation [Gur93, Flo03],

$$\begin{aligned} \langle c_0\ \mu(\infty)\mu(1)\mu(0)\mu(\lambda)\rangle &= \lambda^{\frac{1}{4}}(\lambda-1)^{\frac{1}{4}}F(\lambda)\,, \\ \lambda(\lambda-1)\frac{\mathrm{d}^2F(\lambda)}{\mathrm{d}\lambda^2}+(2\lambda-1)\frac{\mathrm{d}F(\lambda)}{\mathrm{d}\lambda}+\frac{1}{4}F(\lambda) &= 0\,. \end{aligned} \tag{8.2.4}$$

Thus, up to a prefactor, the four-point function of the μ fields reproduces the periods of the Legendre family and, without loss of generality, I choose the two solutions to be F_1 and F_2 as in (8.1.3). The corresponding four point functions now equals

$$\langle c_0\ \mu(\infty)\mu(1)\mu(0)\mu(\lambda)\rangle_k = \lambda^{\frac{1}{4}}(\lambda-1)^{\frac{1}{4}}F_k(\lambda)\,,\quad k\in\{1,2\}\,, \tag{8.2.5}$$

and should be compared with

$$\Pi(\lambda) = F(\tfrac{1}{2},\tfrac{1}{2},1|\lambda)\,,\quad \Pi_D(\lambda) = \mathrm{i}F(\tfrac{1}{2},\tfrac{1}{2},1|\lambda)\,. \tag{8.2.6}$$

Consequently, the correlation functions above and the periods of the Legendre family define equivalent tori and their quotient yields the same fundamental parameter[3]

$$\tau(\lambda) = \frac{\Pi_D(\lambda)}{\Pi(\lambda)} = \frac{\langle c_0\ \mu(\infty)\mu(1)\mu(0)\mu(\lambda)\rangle_2}{\langle c_0\ \mu(\infty)\mu(1)\mu(0)\mu(\lambda)\rangle_1} \tag{8.2.7}$$

Applying the monodromy group (8.1.21), I can redefine the periods without changing the underlying lattice torus. In this respect, the "conformal blocks" in the correlation functions are not uniquely determined.

8.2.2 The Necessity of a Logarithmic Extension

The necessity for a logarithmic extension of the bc-system on the torus can now be seen from the operator product expansion between the twist fields, which was originally derived by V. Gurarie [Gur93]. To explain this, I will, however, follow a publication of M. Gaberdiel in [Gab03]. A general solution of the nullvector condition equals

$$F(\lambda) = A\ F_1(\lambda)+B\ [F_1(\lambda)\log(\lambda)+H(\lambda)]\,, \tag{8.2.8}$$

whereby F_1 and H are regular at $z=0$, and I used (8.1.20) as well as $\lambda\mapsto 1-\lambda$ to reformulate $F_2(\lambda)=\frac{\mathrm{i}}{\pi}\big(F_1(\lambda)\log\lambda+H\big)$. In the expression above it is immediate that the OPE between two fields μ must contain logarithms and splits into two parts. Namely, if two of the fields in the four-point function are shifted to a neighborhood of infinity and treated as a background field $\Omega(\infty)$, the correlation function still has to respect the OPE by its definition. Thus,

$$\mu(z)\mu(\omega) = (z-\omega)^{\frac{1}{4}}(\phi_1(\omega)+\phi_2(\omega)\log(z-\omega))\,, \tag{8.2.9}$$

[3]Two tori are equivalent, iff their lattices differ by some nonzero complex number $L = aL'$, $a\in\mathbb{C}\setminus\{0\}$. This is more general than saying that two tori are identical, i.e. $L=L'$. The identical tori are related by the global monodromy group, cf. section 8.1.

with $A = \langle\Omega(\infty)\phi_1(0)\rangle$, $B = \langle\Omega(\infty)\phi_2(0)\rangle$. Gaberdiel uses a further trick which allows to determine the fields ϕ_i. He lets λ encircle 0 in the OPE with the other twist fields shifted nearby infinity, which yields

$$\langle\Omega(\infty)\mathrm{e}^{2\pi\mathrm{i}T_0}\mu(\lambda)\mu(0)\rangle = \lambda^{\frac{1}{4}}(A + 2\pi\mathrm{i}B + B\log(\lambda)). \tag{8.2.10}$$

Thus, with $\phi_i|0\rangle =: |\phi_i\rangle$ he obtains

$$T_0|\phi_2\rangle = 0, \quad T_0|\phi_1\rangle = |\phi_2\rangle. \tag{8.2.11}$$

I have encountered such an equation already in (4.1.10) and thus may conclude that the fermionic bc-system on the torus unavoidably has to be logarithmically extended, whereby $\phi_2(z) = 1(z)$ and $\phi_1(z) = \Psi_b(z)$, cf. chapter 4. The fields $1(z)$ and $\Psi_b(z)$ have the same conformal weights and $U(1)$ charges, as is demanded for logarithmic partners of the Virasoro algebra. In (8.2.1) already appears a set of fields and representations subject to that constraint. Therefore, I claim that for the fermionic bc-system on $\mathscr{E}_\lambda$,

$$\begin{aligned} &\mu(z)\mu(\omega) = (z-\omega)^{\frac{1}{4}}(\tilde{1}(\omega) + 1(\omega)\log(z-\omega)), \\ &\tilde{1}(z) = \Psi_b(z), \quad |\tilde{0}\rangle = b_0|0\rangle\otimes\epsilon|0\rangle_{\mathcal{K}}, \end{aligned} \tag{8.2.12}$$

and all fields in the untwisted sector have to be logarithmically extended in analogy with chapter 4.

8.3 The Triplet Model

The triplet model is an LCFT which contains the logarithmically extended untwisted sector as well as the twisted representations [GK96, Roh96, Kau95]. To the best of my knowledge, this model is in addition the LCFT whose operator product algebra closes on the representations noted down in (8.2.1) with a minimal amount of additional representations added. Its basic ingredient is an additional symmetry which restricts and controls the possible representations. In order to make this explicit, I will comment on the means which restrict the representation spaces of a conformal field theory. Therefore, I will firstly introduce what I understand under a physically eligible representation, and thereafter discuss the impact of the additional symmetries and nullstate conditions which lead to the triplet model.

8.3.1 Symmetries and Representations

The OPE of the twist fields could be reconstructed due to a nullstate condition which made it necessary to extend the representation of the fermionic bc-system on $|\mu\rangle$ by $|0\rangle$ and $|\tilde{0}\rangle$. Behind this stands a general feature of CFTs. Since the fields and states are supposed to be isomorphic, obtaining knowledge of the operator product algebra of the fields and studying the possible representation spaces are two sides of the same medal. This knowledge is basically deduced from nullstates and symmetries. To explain how this works, I must specify what I understand under a "physically relevant" representation space.

In section 3.4.1, I have defined a collection of representations on charged states $|p\rangle$, however, not all of them are "physically" reasonable. In the situation of a CFT for instance, the "physical" representation spaces should be build on states which preserve conformal symmetry. This condition would have restricted the bc-system to be build solely on $|0\rangle$, which is the only $SL(2,\mathbb{C})$ invariant state among the states $|p\rangle, p \in \mathbb{Z}$. On the other hand, since the bc-system breaks unitarity, it is not possible to construct a thoroughly "physical" theory, anyway, and it was necessary to include the dual state $\langle 1|$ to account for the background charge. Therefore, I will restrict the representations as follows:

Restriction by Symmetries:

Let $\mathscr{C} \cup \mathscr{S}$ be some operator product algebra of holomorphic fields, whereby I have extracted the part $\mathscr{S}$ consisting of the symmetry generators T, for the interior Virasoro symmetry, and $S^a(z)$, $a = 1 \ldots A$ for additional exterior symmetries. These symmetries are subject to $[S^a_0, T_0] = 0$, $[S^a_0, S^b_0] = 0$, and I assume that there exists a unique $SL(2,\mathbb{C})$ invariant state $|0\rangle$, on which they are diagonal. In the spirit of the consequences a logarithmic deformation along the lines of [FFH⁺02] implies, I understand by a *physically eligible representation* (PER) of $\mathscr{C} \cup \mathscr{S}$ a multiplet $M(\phi)_K$ of vectors $|\phi, k\rangle$, $k = 1, \ldots, K$ subject to the following conditions:

① Representation of the OPA: In the representation on $M(\phi)_K$, the fields in $\mathscr{C}$ have a mode expansion

$$\underline{\Phi}(z) = \Phi^{(\text{naive})}(z) + \tilde{\Phi}(z), \quad \Phi^{(\text{naive})}(z) = \sum_{n \in \mathbb{Z}} \Phi_n z^{-n-\Delta_T(\Phi)}$$

whereby $\tilde{\Phi} \in \mathrm{End}(M(\phi)_K)((z, z^{-1}))[\log z]$. For all k, $|\phi, k\rangle$ is annihilated by Φ_n , $\tilde{\Phi}_n$, $n > 0$.[4] The set of states $\{|\phi, k\rangle \in M(\phi)_K \ : \ \tilde{\Phi}(z)|\phi, k\rangle = 0\}$ is not empty. The operator product algebra of the fields in $\mathscr{C}$ is represented on $|\phi, k\rangle \ \forall \ k$.

② Interior Symmetry: On every $|\phi, k\rangle$, the field $\underline{T}$ can be decomposed as

$$\underline{T}(z) = T^{(\text{naive})}(z) + \mathfrak{g}(z),$$

such that the action of the field modes in $T^{(\text{naive})} = \sum_{n \in \mathbb{Z}} T^{(\text{naive})}_n z^{-n-2}$ on $|\phi, k\rangle$ does not lead out of the k^{th} sector, and the zero mode is diagonal. The other field $\mathfrak{g}(z) \in \mathrm{End}(M_K)((z, z^{-1}))$ permutes the elements of the multiplet. Hence, the eigenvalue Δ_ϕ of $T^{(\text{naive})}_0$ is a quantum number of $M(\phi)_K$. Moreover, I assume that the OPA of $\underline{T}$ with the

[4] In order to avoid indices which are not integers, I do not assume that the field modes Φ_n have conformal weight $-n$.

fields in $\mathscr{C}$ is preserved and that there exists some $|\phi,k\rangle \in \{|\phi,k\rangle \in M(\phi)_K \;:\; \tilde{\Phi}(z)|\phi,k\rangle = 0\}$ such that $\mathfrak{g}(z)|\phi,k\rangle = 0$.

③ Exterior Symmetry: I assume that the fields $S^a(z) \in \mathscr{S}$ have expansions and representations of the kind

$$\underline{S}^a(z) = S^{a\,(\text{naive})}(z) + \mathfrak{g}_{S^a}(z) \tag{8.3.1}$$

with $\mathfrak{g}^a_S \in \mathrm{End}(M_K)((z,z^{-1}))[\log z]$. The fields $S^{a\,(\text{naive})}$ and $\mathfrak{g}_{S^a}$ shall enjoy analogue properties as were demanded for $\underline{T}$ on any $|\phi,k\rangle$. The conditions $[S^{a\,(\text{naive})}_0, T^{(\text{naive})}_0] = 0$ and $[S^{a\,(\text{naive})}_0, S^{b\,(\text{naive})}_0] = 0$ shall be valid such that the eigenvalues of $S^{a\,(\text{naive})}_0$ are quantum numbers of $M(\phi)_K$. I do further assume that there exists some "non-logarithmic" states $|\phi,k\rangle \in \{|\phi,k\rangle \in M(\phi)_K \;:\; \tilde{\Phi}(z)|\phi,k\rangle = 0 = \mathfrak{g}(z)|\phi,k\rangle\}$ subject to $\mathfrak{g}_{S^a}(z)|\phi,k\rangle = 0 \;\forall a$.

④ Field-State Correspondence: There exists an isomorphism $*$ such that $\phi_k(0) * |0\rangle = |\phi,k\rangle \;\forall\; k$ defines an element $\phi_k \in \mathrm{End}(M_k)((z,z^{-1}))[\log z]$. I assume that the symmetry generators $\underline{T}(z)$ and $\underline{S}^a(z)$ have OPEs with the fields ϕ_k which take the generic form for the naive fields and do not lead out of the representation. Consequently, ϕ_k has the same quantum numbers as $|\phi,k\rangle$ with respect to $T^{(\text{naive})}$ and $S^{a\,(\text{naive})}$. The OPE of $\mathfrak{g}(z)$ and $\mathfrak{g}_{S^a}(z)$ with ϕ_k, contains fields $\phi_{k'},\; k \neq k'$ or their derivatives, corresponding to the action of those operators on a state $|\phi,k\rangle$.

⑤ Irreducibility: If there exist isomorphic representations $M_K \simeq M'_K$, i.e. the field algebra $\mathscr{C}$ contains a bijective linear mapping between the modules generated from $\mathscr{C}$ on these spaces, I treat them as equivalence classes and "choose" the set of vectors which is annihilated by the maximal amount of symmetry generators $T^{(\text{naive})}{}_n$, $S^{(\text{naive})}{}^a_n$, $n \in \mathbb{Z}$ as representative.

By this means it is clear that the PERs are also representations of certain symmetries and can thus be classified and restricted.

Some Examples The holomorphic- antiholomorphic fermionic bc-system of chapter 4 has four PERs if the zero modes $b, \bar{b}$ and $c, \bar{c}$ are excluded. The states $|0,0\rangle$ and $|1,1\rangle$ yield a doublet, the off-diagonal states $|0,1\rangle$ and $|1,0\rangle$ singlet representations. The representations with higher charge are not PERs, because there exist modes in the field algebra which act as isomorphisms. If, as described in chapter 4, it is logarithmically extended, the PERs are preserved. The reason is that even though the modes b_0 and $\bar{b}_0$ enter the extension fields, these can not add to the field algebra, for it would break Möbius covariance, cf. section 4.1.3.

Similar arguments for other scenarios lead to the following tabular, wherein "+" denotes the theory with zero modes, "−" the theory without zero modes and all states without tilde or ϵ are non-logarithmic:

bc-system	unextended PERs	logarithmically extended PERs
holo +	$\|0\rangle$	$\{\|0\rangle, \|\tilde{0}\rangle\}$
holo −	$\|0\rangle\,,\ \|1\rangle$	$\{\|0\rangle, \|\tilde{0}\rangle\}\,,\ \|0\rangle \otimes \epsilon\|0\rangle_{\mathcal{K}}\,,\ \|1\rangle \otimes \|0\rangle_{\mathcal{K}}$
holo-anti +	$\|0,0\rangle$	$\{\|0,0\rangle, \|1,1\rangle\}$
holo-anti −	$\{\|0,0\rangle, \|1,1\rangle\}\,,\ \|0,1\rangle\,,\ \|1,0\rangle$	$\{\|0,0\rangle, \|1,1\rangle\}\,,\ \|0,1\rangle\,,\ \|1,0\rangle$

Restriction by Nullstates

This examplifies that the condition of irreducibility puts constraints on the theory. Another example is the twist state $|\mu\rangle$, which has a potential subrepresentation on a nullstate, cf. section 8.2.1. This state was, however, identical to zero, such that the submodule generated by it already was excluded. Still, it may happen that there are subrepresentations on vectors $|N\rangle \in \mathrm{span}_{\mathbb{C}}\left\{\prod_{n,i} \phi_{i\,n_i}|0\rangle\right\}:\ \phi_{i\,n_i} \in \mathscr{C} \cup \mathscr{S},\ n_i < 0\}$ which do not vanish identically. The modules build on such vectors must be divided out, which is effectively the same as setting $|N\rangle = 0$. This must be accompanied by the condition that any correlation function which includes the field $N(z)$ corresponding to $|N\rangle$ must vanish, and this is equivalent to requiring that in the representation on any $M(\phi)_K$

$$\mathfrak{g}_N(z)|\phi,k\rangle = 0\,,\quad N_0^{(\mathrm{naive})}|\phi,k\rangle = 0\,,\quad \forall k\,. \tag{8.3.2}$$

If $N_0^{(\mathrm{naive})}$ is constituted by the zero modes of certain symmetry generators, this restricts the possible eigenvalues of those generators and thus the possible representation spaces.

8.3.2 Realization of the Triplet Model

The triplet model results from an additional $SU(2)$ symmetry in the logarithmic fermionic bc-system.[5] The additional symmetry introduces new nullstate conditions and thus restricts the PERs [GK96, Roh96, Kau95].

The $su(2)$ Lie algebra is realized in terms of the (naive part of the) zero modes of the field generators W_n^a corresponding to the fields[6]

$$\begin{aligned} W^1(z) &= -\partial_z^2 e(z)\partial_z e(z)\,,\\ W^2(z) &= \frac{1}{2}\left[\partial_z^2 e(z)\partial_z \underline{b}(z) + \partial_z^2 \underline{b}(z)\partial_z e(z)\right]\,,\\ W^3(z) &= -\partial_z^2 \underline{b}(z)\partial_z \underline{b}(z)\,. \end{aligned} \tag{8.3.3}$$

[5] This also works for the non-logarithmic fermionic bc-system without zero modes, which is a special case.

[6] For the logarithmic case, one may set $b_0 = 0 = c_0$, ad libitum. If the non-logarithmic situation is considered, set in addition $\epsilon = \rho = 0$.

The field modes W_n^a extend the Virasoro algebra by

$$\begin{aligned}
[\underline{T}_m, \underline{T}_n] &= (m-n)\underline{T}_{m+n} - \frac{1}{6}m(m^2-1)\delta_{m,-n}\,,\\
[\underline{T}_m, W_n^a] &= (2m-n)W^a_{m+n}\,,\\
[W_m^a, W_n^b] &= g^{ab}\Big(2(m-n)\Lambda_{m+n} + \frac{1}{20}(m-n)(2m^2+2n^2-mn-8)\underline{T}_{m+n}\\
&\qquad -\frac{1}{120}m(m^2-1)(m^2-4)\delta_{m,-n}\Big)\\
&\quad + f_c^{ab}\left(\frac{5}{14}(2m^2+2n^2-3mn-4)W^c_{m+n} + \frac{12}{5}V^c_{m+n}\right),
\end{aligned} \tag{8.3.4}$$

whereby $\Lambda(z) =: \underline{T}^2(z): -\frac{3}{10}\partial_z^2\underline{T}(z)$ and $V^a(z) =: \underline{T}(z)W^a(z): -\frac{3}{14}\partial_z^2 W^a(z)$. The metric is symmetric with $g^{ab} = \delta^{ab}$ and the structure constants are those of $su(2)$, namely $f_c^{ab} = \mathrm{i}\epsilon^{abc}$.

Gaberdiel and Rhosiepe also note down the nullstates which are decisive for the determination of the possible representations. The condition that the zero modes of the naive part of the corresponding nullfield on a PER be zero yields

$$\Delta_\phi^2(8\Delta_\phi+1)(8\Delta_\phi-3)(\Delta_\phi-1)\ |\phi,k\rangle = 0 \tag{8.3.5}$$

for arbitrary multiplets $M(\phi)_K$, and is accompanied by

$$[W_0^a, W_0^b]^{(\text{naive})}\ |\phi,k\rangle = \frac{2}{5}(6\Delta_\phi-1)f_c^{ab}\ W^{c\,(\text{naive})}_{0}\ |\phi,k\rangle\,. \tag{8.3.6}$$

Consequently, the only allowed PERs fall into representations of $su(2)$ and are states with highest weights $\{0,-\frac{1}{8},\frac{3}{8},1\}$. This extends the representations listed in (8.2.1) in a minimalistic way.

8.3.3 Characters

In the next chapter I will determine the prepotential of pure gauge, $SU(2)$ Seiberg-Witten in terms of some characters of the triplet model. Therefore, I will conclude this chapter by quoting the ones relevant for my considerations.

H. G. Kausch, [Kau95], proposed that certain primary fields in the Kac table, for instance those in the “augmented” minimal model $c_{6,3}$ with conformal weights

$$\Delta_{r,s} = \frac{1}{8}((2r-s)^2-1), \quad 0<r<3,\ 0<s<6, \tag{8.3.7}$$

can be identified with the fields appearing in the non-logarithmic triplet model. Indeed, the fields in the augmented minimal model have the correct quantum numbers and the field which by such is the analogue of μ also has the correct nullstate condition, cf. [Flo03, RRS08].

By this analogy, Kausch concluded that the characters of the non-logarithmic triplet model are those of the augmented minimal model

$$\begin{aligned} \chi_{-\frac{1}{8}}(q) &= \frac{\Theta_{0,2}(q)}{\eta(q)}, & \chi_{\frac{3}{8}}(q) &= \frac{\Theta_{2,2}(q)}{\eta(q)}, \\ \chi_0(q) &= \frac{1}{2}\left(\frac{\Theta_{1,2}(q)}{\eta(q)} - \eta^2(q)\right), & \chi_1(q) &= \frac{1}{2}\left(\frac{\Theta_{1,2}(q)}{\eta(q)} + \eta^2(q)\right), \end{aligned} \tag{8.3.8}$$

with Jacobi-Riemann theta functions $\Theta_{r,s}(q) = \sum_{n\in\mathbb{Z}} q^{\frac{(2kr+s)^2}{4s}}$, Dedekind η function defined by $\eta(q) = q^{\frac{1}{24}} \prod_{n\in\mathbb{N}}(1-q^n)$ and $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. The parameter τ is the modulus of some lattice torus.

These characters were completed by [GK96, Flo96] to match with the logarithmically extended triplet model. However, I will not make use of the additional characters and refer the interested reader to the literature just cited.

Now, I have everything together to relate the triplet model to Seiberg-Witten theory.

Relation to Seiberg-Witten Theory 9

In this chapter I will determine the spectral torus of pure gauge, $SU(2)$ Seiberg-Witten theory in terms of characters of the triplet model. Moreover, I will obtain the prepotential as a function of the torus modulus τ, which can be expressed as the ratio of the four-point functions of the twist field μ in this theory, (8.2.7). It follows, that this specific Seiberg-Witten theory is completely determined by the triplet model.

Firstly, I will start with a brief introduction to Seiberg-Witten theory and discuss its spectral curve. The relation to the triplet model will be discussed in section 9.2 and summarizes the results of [VF07].

9.1 Some Words on Seiberg-Witten Theory

In [SW94], N. Seiberg and E. Witten derived the full prepotential $\mathscr{F}$ (including instantons) of the low energy effective action for $\mathscr{N}=2$ supersymmetry with gauge group $SU(2)$. In terms of $\mathscr{N}=1$ fields, this theory is described by a family of Lagrangians

$$\mathscr{L}_A = \frac{1}{8\pi}\Im\left(\int \mathrm{d}^4\theta\ \bar{A}A_D + \int \mathrm{d}^2\theta\ \tau(A)W^\alpha W_\alpha\right), \quad A_D = \frac{\mathrm{d}\mathscr{F}(A)}{\mathrm{d}A},\ \tau = \frac{\mathrm{d}^2\mathscr{F}(A)}{\mathrm{d}A^2}. \tag{9.1.1}$$

The spacetime metric has a Minkowskian (mostly minus) signature and, with the exception that I use another normalization for the Pontrjagin index, $\frac{1}{8\pi^2}\int_{S^4} F\wedge F \in \mathbb{Z}$ [Ber96], I stick to the conventions of [Bil96]. The prepotential $\mathscr{F}$ is holomorphic in the expectation value A of the $\mathscr{N}=1$ chiral multiplet $\langle\Phi\rangle = \frac{1}{2}A\sigma_3$.

The Lagrangian above has its domain on the effective vacuum configurations while the massive Goldstone bosons are integrated out. By the term "effective vacuum" I mean that for nonvanishing values of $\langle\Phi\rangle$ the $SU(2)$ gauge symmetry is broken to $U(1)$ and the thus obtained field configurations do not enjoy the full symmetry of the theory. Furthermore, as soon as the scalar field is in an effective vacuum configuration, all other particles have the same property for they belong to the same $\mathscr{N}=2$ multiplet.

In the following I will only motivate the basic geometric facts which lead to the spectral curve of this theory and to its interpretation as a torus. The reader interested in the details, is refered to the literature [Bil96, SW94, DP99, Ler97]. Afterwards, I will relate the spectral torus to the triplet model.

9.1.1 The Spectral Curve of SW Theory

There is a remnant of the larger $SU(2)$ symmetry hidden behind the choice of A, namely under rotations by π around the first or second axis of the gauge group, $A \mapsto -A$ and these are equivalent gauge configurations. Thus, rather than $\langle\Phi\rangle$, it is reasonable to consider the Casimir $\langle \mathrm{tr}\, \Phi^2\rangle$ as a gauge invariant parameter. If ϕ is the scalar field in the chiral multiplet Φ, the Casimir yields some $u = \langle \mathrm{tr}\, \phi^2\rangle$. The parameter space of $u \in \mathbb{C}$ constitutes the moduli space of gauge inequivalent effective vacua $\mathcal{M}_{SW}$, and thus of the family $\mathcal{L}_{A(u)} = \mathcal{L}_u$. Formally one can add $\{\infty\}$ to $\mathcal{M}_{SW}$, which is a singular point for $\mathcal{L}_u$.

In general, the moduli space $\mathcal{M}_{SW}$ has singularities at those values of u at which the effective action is not defined or inadequate to describe the massless sector. Besides $\{\infty\}$, these are the points u at which massive field modes which have been integrated out turn massless, cf. [SW94, Bil96, DP99, Ler97].

Seiberg and Witten argued [SW94] that there should exist two additional singular points $\{s, -s\} \in \mathcal{M}_{SW}$, such that

$$\mathcal{M}_{SW} = \mathbb{CP}^1 \setminus \{\infty, s, -s\}\,. \tag{9.1.2}$$

The parametrization in terms of $u = \langle \mathrm{tr}\, \phi^2\rangle$ seems to make the setting more difficult. The reason is that the inverse of $\langle\phi\rangle \mapsto \langle \mathrm{tr}\, \phi^2\rangle$ has two roots in terms of u. Indeed, the analysis of Seiberg and Witten revealed that the paramtetrization in $\langle\phi\rangle$ yields a two-sheeted covering of $\mathcal{M}_{SW}$. Therefore, A, A_D and in particular $\mathcal{F}$ are not single-valued in u.

The particle spectrum for Seiberg-Witten theory is bound to satisfy the mass formula [DP99]

$$Z(u) = na(u) + ma_D(u)\,, \tag{9.1.3}$$

whereby a and a_D are the scalar field components in A and A_D, respectively, n corresponds to an electric charge and m to a magnetic charge. By this means, the spectrum can be read off from some lattice torus. In addition, $\Im(\tau(u)) > 0$ by requiring that $\Im(\tau)$ shall serve as a metric on the space of vacuum configurations a and a_D [Bil96, SW94]. The relation above (9.1.3) is the spectral torus describing the massive particles in Seiberg-Witten theory. The singularities in $\mathcal{M}_{SW}$ correspond to those values of a and a_D for which the torus becomes singular.

9.1.2 Modular Transformations

The spectral torus does only deserve its name "torus", if it is possible to prove that the physics behind it is invariant under modular $SL(2,\mathbb{Z})$ transformations. As already mentioned in section 8.2.1, the orbit of a lattice torus under $SL(2,\mathbb{Z})$ collects all equivalent tori. Thus, I will in the following explain that the partition function of Seiberg Witten theory is modular invariant.

The Lagrangians

$$\mathscr{L}_A = \frac{1}{8\pi}\left(\int \mathrm{d}^2\theta \; \Im\left[\tau(A) W^\alpha W_\alpha\right] + \frac{1}{2}\int \mathrm{d}\theta^4 \begin{pmatrix} A_D \\ A \end{pmatrix}^\dagger I \begin{pmatrix} A_D \\ A \end{pmatrix}\right), \quad I = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \tag{9.1.4}$$

are invariant under

$$\begin{pmatrix} A_D \\ A \end{pmatrix} \mapsto M(n)\begin{pmatrix} A_D \\ A \end{pmatrix}, \quad M(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \; n \in \mathbb{Z}. \tag{9.1.5}$$

While $M^\dagger I M = I$, one obtains a shift of the coupling constant $\tau = \frac{\theta(u)}{2\pi} + \frac{4\pi\mathrm{i}}{g^2(u)}$

$$\tau = \frac{\mathrm{d}A_D}{\mathrm{d}A} \mapsto \tau + n \tag{9.1.6}$$

which adds an, however, irrelevant term to the theta angle

$$\tau(u) + n = \frac{\theta(u) + 2n\pi}{2\pi} + \frac{4\pi\mathrm{i}}{g^2(u)}. \tag{9.1.7}$$

To see this, I have used the conventions of Bilal, $W^\alpha W_\alpha|_{\theta^2} = \frac{1}{4}(F_{\mu\nu} - \mathrm{i}\tilde{F}_{\mu\nu})(F^{\mu\nu} - \mathrm{i}\tilde{F}^{\mu\nu}) + \ldots$ [Bil96] and the observation that since $\frac{1}{8\pi^2}\int_{S^4} F \wedge F \in \mathbb{Z}$ the shift does not contribute to the partition function[1]

$$Z[u] = \exp\{\int \mathrm{d}^4x \; \mathrm{i}\mathscr{L}_u\}. \tag{9.1.8}$$

The partition function is further invariant under a duality which inverts the gauge coupling. This is obtained by a Legendre transformation

$$\mathscr{F}_D(A_D) = \mathscr{F}(A) - AA_D, \tag{9.1.9}$$

such that

$$\tau_D(A_D) = -\frac{\mathrm{d}A}{\mathrm{d}A_D} = -\frac{1}{\tau(A)}, \tag{9.1.10}$$

whilst the action looks structurally as before with new conjugate coordinate $\partial_{A_D}\mathscr{F}_D = -A$. How this transformation is implemented for the $\mathscr{N} = 1$ formulation of the theory is discussed in full detail in [Bil96, SW94]. Physically, it constitutes an analytic extension of $\mathscr{F}$ to the strong (respectively low) coupling regime. From another point of view, the action of the second generator exchanges the rôles of a_D and a and thus magnetic and electric charges.

For me it was important to note that the partition sum build from the Lagrangians $\mathscr{L}_u$ is indeed invariant under the elliptic modular group

$$SL(2,\mathbb{Z}) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle. \tag{9.1.11}$$

The action of this group is thus well defined on the spectral torus which consequently deserves its name.

[1] This is an abuse of denotation. The partition function is rather $Z = \int_{\mathscr{M}_{SW}} Z[u]\mathrm{d}u$, for some appropriate measure $\mathrm{d}u$ on $\mathscr{M}_{SW}$.

It is now suggestive to reinterpret the family of Lagrangians $\mathscr{L}_u$ and substitute the parameter $A(u)$ by the torus modulus $\tau(u)$. Thereby, $\mathscr{L}_u \mapsto \mathscr{L}_\tau$ and the family of Lagrangians gets parametrized over the space of inequivalent tori. This would be a first step towards a CFT approach to Seiberg-Witten theory.

9.2 The Spectral Curve and Triplet Characters

In the following, I will explain how the family of Lagrangians $\mathscr{L}_u$ can be reformulated in terms of τ. This was one main part in my publication with M. Flohr, [VF07]. At this time, we searched after an expression of $\mathscr{F}$ in terms of characters of the triplet model, which was the second main part. This was encouraged by some former work of Flohr on a correspondence between Seiberg-Witten theory and the triplet model [Flo04, Flo98] and by a publication of W. Nahm [Nah96]. In his papers, Flohr could express the spectral curve in terms of correlation functions of the triplet model. Nahm, on the other hand, proposed that it should be possible to combine a and a_D into a modular form of weight -1, for which he noted down the following expression in terms of τ:

$$c(\tau) = a_D(u(\tau)) - \tau a(u(\tau)) \sim \frac{\eta^2(\frac{\tau}{2})}{\eta^4(\tau)}. \tag{9.2.1}$$

It is not possible to express c in terms of characters of ordinary CFTs, since they have have modular weight zero. On the other hand, the characters χ_0 and χ_1 of the triplet model contain both a term η^2 which has modular weight one. Therefore, it seemed reasonable to try to obtain c in terms of characters of the triplet model. Indeed, we could determine c in terms of characters of the triplet model but not the prepotential.

I will now explain by which steps c could be articulated solely by means of triplet characters and by which the prepotential $\mathscr{F}$ could be determined as a function of τ.

9.2.1 The Spectral Curve in Terms of τ

The Moduli space $\mathscr{M}_{SW} = \mathbb{CP}^1 \setminus \{\infty, \pm s\}$ of $\mathscr{L}_u$ conforms with the moduli space of the spectral torus, as follows from section 9.1.1. Therefore, it is reasonable to relate to the spectral torus an algebraic curve of the form

$$\tilde{y}^2 = (z-s)(z+s)(z-u). \tag{9.2.2}$$

In analogy with the discussion in section 8.1.1, one can define a differential one-form

$$\tilde{\omega}(z;u) = \frac{\mathrm{d}z}{\tilde{y}(z;u)} \tag{9.2.3}$$

with respect to the curve above, fix two branch cuts $[\infty \cdots u]$ and $[-1 \cdots 1]$ and a choice of cycles, and derive the periods integrating over $\tilde{\omega}$. In order to make use of the results of sections 8.1.2 and 8.1.3, I substitute $z = 2z - 1$ under the corresponding integrals. This transforms the algebraic curve above into the Legendre form such that

$$\begin{array}{l}\tilde{\Pi}_D(\lambda) = (2s)^{-\frac{1}{2}} \int_\alpha \varpi(\lambda) \\ \tilde{\Pi}(\lambda) = (2s)^{-\frac{1}{2}} \int_\beta \varpi(\lambda)\end{array} \quad , \qquad \lambda(u) = \frac{u+s}{2s}\,, \tag{9.2.4}$$

with ϖ as defined in (8.1.3). The periods thus obtained can be expressed in terms of (8.1.17) and define a torus lattice with moduli parameter

$$\tau(\lambda) = \mathrm{i}\frac{F(\frac{1}{2},\frac{1}{2};1|1-\lambda)}{F(\frac{1}{2},\frac{1}{2};1|\lambda)}\,, \tag{9.2.5}$$

wherein λ is a function of u. Notice, that τ can directly be related to the triplet model and be derived by means of the twist field four-point functions (8.2.7).

In [E+85, Vol. 2, pg. 354f], I have found several choices for the inverse $\lambda(\tau)$ of (9.2.5). Since all of them are connected by modular (i.e. $SL(2,\mathbb{Z})$) transformations, I chose without loss of generality

$$\lambda(\tau) = \left(\frac{\theta_3(\tau)}{\theta_2(\tau)}\right)^4\,, \tag{9.2.6}$$

whereby

$$\theta_2(\tau) = 2\sum_{n=0}^{\infty} q(\tau)^{\frac{1}{2}(n+\frac{1}{2})^2}\,, \quad \theta_3(\tau) = 1 + 2\sum_{n=1}^{\infty} q(\tau)^{\frac{1}{2}n^2}\,, \quad \theta_4(\tau) = 1 + 2\sum_{n=1}^{\infty} (-)^n q^{\frac{1}{2}n^2} \tag{9.2.7}$$

are the Jacobi theta functions and as before $q = \exp\{2\pi\mathrm{i}\tau\}$, cf. section 8.3.3. This choice of λ is in concordance with the publications [HK07, ABK08], which appeared during the time when M. Flohr and I published our work. Given λ, one obtains u by means of the relation in (9.2.4) and, after some Maple gymnastics, it was possible to express this quantity in terms of the Dedekind η function [VF07]

$$u(\tau) = \frac{s}{8}\left(\left(\frac{\eta(\frac{\tau}{4})}{\eta(\tau)}\right)^8 + 8\right)\,. \tag{9.2.8}$$

Substituting this for u yields a new parametrization of the family of Lagrangians $\mathscr{L}_u$ by τ.

The Periods of the Spectral Curve

The question remains, what a and a_D look like in terms of τ. The periods $\tilde{\Pi}_D$ and $\tilde{\Pi}$ are not identical with a and a_D, however they are related by means of the modulus τ, demanding that it equals the modulus of the spectral curve

$$\tau = \frac{\Pi_D}{\Pi} \stackrel{!}{=} \frac{\mathrm{d}a_D}{\mathrm{d}a} \;\Leftrightarrow\; \Pi_D(u) = \partial_u a_D(u)\,,\; \Pi(u) = \partial_u a(u)\,. \tag{9.2.9}$$

Thus, a and a_D can be derived from a one-form ϖ_{SW}, called the Seiberg-Witten differential, which satisfies $\partial_u \varpi_{SW} = \tilde{\omega}(u)$. Integrating this condition, one ends up with

$$a_D(u) = \oint_\alpha \varpi_{SW}(u), \quad a(u) = \oint_\beta \varpi_{SW}(u), \quad \varpi_{SW} = \frac{(z-u)\mathrm{d}z}{\tilde{y}} + \text{exact}. \tag{9.2.10}$$

The solutions to these integrals have been derived in different ways. One is by noting that for fixed contours, the periods a_D and a satisfy again some Hypergeometric differential equation which yields [Ler97]

$$\begin{aligned} a_D(u) &= \frac{\mathrm{i}}{4}\sqrt{s}\left(\frac{u^2}{s^2}-1\right) F\left(\frac{3}{4},\frac{3}{4};2\,\middle|\,1-\frac{u^2}{s^2}\right), \\ a(u) &= \sqrt{\frac{u}{2}} F\left(-\frac{1}{4},\frac{1}{4};1\,\middle|\,\frac{s^2}{u^2}\right). \end{aligned} \tag{9.2.11}$$

Substituting the result for $u(\tau)$, this gives the spectral curve in terms of τ.

The Spectral Curve in Terms of Triplet Characters

The second main result of [VF07] was the modular one-form c, cf. (9.2.1), expressed by characters of the triplet model. It is already clear that the denominator of this quantity must contain $\chi_1 - \chi_0$, since it is a modular form of weight one. After some trials and errors with series expansions in Maple, I could prove that

$$c(\tau) = \frac{\mathrm{i}\sqrt{s}}{\pi}\left(\frac{\chi_{-\frac{1}{8}} - \chi_{\frac{3}{8}}}{\chi_1 - \chi_0}\right) \tag{9.2.12}$$

with the characters as in (8.3.8). This expression equals the one proposed by Nahm, cf. (9.2.1) and [Nah96]. Thus, up to the explicit parameter τ, I have obtained a and a_D in terms of characters, namely

$$a(\tau) = -\frac{\mathrm{d}c(\tau)}{\mathrm{d}\tau}, \quad a_D(\tau) = \left(1-\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\right)c(\tau). \tag{9.2.13}$$

Below, I will argue that the full prepotential can now be written as a function of τ.

The Prepotential in Terms of τ

M. Matone derived in [Mat95] the relation:

$$\mathscr{F}(u) = \frac{1}{2}a(u)a_D(u) - \mathrm{i}\pi u. \tag{9.2.14}$$

This works as follows. The periods of the spectral curve (9.1.3) can be transformed under $SL(2,\mathbb{Z})$, which leads to

$$aA_D + bA = \tilde{A}_D = \frac{\mathrm{d}\tilde{\mathscr{F}}}{\mathrm{d}A}\frac{\mathrm{d}A}{\mathrm{d}\tilde{A}}. \tag{9.2.15}$$

Integrating this expression, I find that

$$\tilde{\mathscr{F}} = \frac{1}{2}acA_D^2 + \frac{1}{2}bdA^2 + bcAA_D + \mathscr{F}. \tag{9.2.16}$$

The combination

$$\mathscr{F}(a) - \frac{1}{2}aa_D \tag{9.2.17}$$

is invariant under the monodromy group of the spectral curve, which is generated by

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \quad M_s = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{-s} = M_s^{-1} \cdot M_\infty. \tag{9.2.18}$$

This group can be determined by expanding (9.2.11) around $u_0 \in \{\infty, \pm s\}$ and by letting u encircle each of these points, i.e. $u - u_0 \mapsto \exp\{2\pi \mathrm{i}\}(u - u_0)$, [Bil96, SW94, DP99, Ler97]. Since (9.2.17) is invariant under the monodromy group, it can be identified with u, which parametrizes the equivalence class of periods a, a_D under this group.

Inserting the results on a and a_D above and that on u, (9.2.8), I end up with

$$\mathscr{F}(\tau) = \frac{1}{2}\left[\tau\left(\frac{\mathrm{d}c(\tau)}{\mathrm{d}\tau}\right)^2 - c(\tau)\frac{\mathrm{d}c(\tau)}{\mathrm{d}\tau}\right] - \frac{\mathrm{i}\pi s}{8}\left[\left(\frac{\eta(\frac{\tau}{4})}{\eta(\tau)}\right)^8 + 8\right]. \tag{9.2.19}$$

Thus, in our paper [VF07], Flohr and I obtained all basic quantities of $SU(2)$ Seiberg-Witten theory, including the instanton contributions, in terms of τ. In particular, we determined the spectral curve by means of characters of the triplet model.

Conclusion 10

In this thesis, I have been concerning myself with geometry as a source for a logarithmic deformation of conformal field theories. In this context I have been investigating two different geometric scenarios.

The first has been the conformal supersymmetric bc-system on $\mathbb{R}^1 \times S^1$ with target manifold $\mathbb{CP}^1$. The source for its logarithmic deformation is the extension of its local representation spaces to spaces of distribution forms on $\mathbb{CP}^1$. In particular, the bosons had to be logarithmically deformed, because it turned out that they describe the different vacuum sectors which are compounded by the instantons.

The second has been the purely fermionic conformal bc-system, with domain on a branched covering of $\mathbb{CP}^1$ and with global monodromy group. This time, the target space is $\mathbb{C}$ and the source for the logarithmic deformation consists in the twisted representations of the monodromy group.

In order to conclude my work, I will now bundle the questions which remained open and deserve further investigation from my point of view.

Bosons on Branched Coverings It would be interesting, also with an eye towards the supersymmetric conformal bc-system, to study bosonic ghosts on branched coverings. The representations of the monodromy group are analogous to those of the fermions, and the operator product algebra is also quite similar. If the algebraic surface is again a torus, it might be the case that the four-point function of the bosonic twist fields also reveals information about its periods for the following reason. It would be valuable, if there was a way to not only bosonize the bosonic ghosts but also the bosonic twist fields. Since the bosonized ghosts must be extended by an auxiliary fermionic system, I could imagine that similar works for the twist fields, such that the situation might again be reduced to considerations of fermionic ghosts on the torus.

Holomorphic Mappings between Compact Riemannian Surfaces The two scenarios that I have considered might be related by another publication of Frenkel and Losev [FL07]. There, the authors consider the CSbc with domain and target manifold $\mathbb{CP}^1$. In general, the holomorphic functions (i.e. solutions to the instanton equation) can be classified in three types: constant functions, meromorphic functions and functions with higher ramifications.

Frenkel and Losev claim that the transition from the conformal CSbc with target $\mathbb{C}^\times$ to the conformal CSbc with target $\mathbb{CP}^1$ must be accompanied by an inclusion of meromorphic functions. Therefore, the solutions to the instanton equation must exceed the subspace of constant vacuum configurations. Consequently, Frenkel and Losev interpret the additional meromorphic functions as instanton effects.

They further propose that the CSbc on $\mathbb{CP}^1$ can be modelled by the CSbc on $\mathbb{C}^\times$, if the action of the latter is enlarged by additional operators. These operators would then mimic the extension of the vacuum configurations to meromorphic functions. In [FLN08], the same authors proposed that those deformation terms in the action are identical to the Grothendieck-Cousin fields.

In appendix C, I have tried to prove that the approach of Frenkel and Losev [FL07] to the CSbc on $\mathbb{CP}^1$ is isomorphic to my approach in part one of this thesis. This was only successful for the Grothendieck-Cousin operator and the representation spaces. In particular, I could not determine an isomorphy between the respective Grothendieck-Cousin fields.

It would be favorable if the isomorphy did exist and could be proven.

From the Large Volume Limit Back to Physics If the extended representation spaces of the theories considered in part one of my thesis are indeed the nonperturbative state spaces, a new kind of perturbation theory should be possible, which does neither destroy the kinematics, induced by the curved target space, nor the topological features – in particular one retains all vacuum solutions, not running into the putative factual constraint to select a particular background. This new perturbation theory consists in varying the scaling paramter λ of the metric, thus moving away from the large volume limit in the moduli space of metrics. Frenkel, Losev and Nekrasov suggest to check if the non-diagonal representations of the Hamiltonian disappear for finite values of λ, by which the anti-instantons get reanimated, [FLN06, pg. 89f]. There is an even more important reason for trying this kind of perturbation theory. The state spaces in the large volume limit have been obtained by a succession of transformations of the physical spectrum of the *unitary* Morse theories underlying the models under consideration. The rationale was to derive the perturbative spectrum, multiply it by some exponential by which unitarity is broken, go to the large volume limit and derive the nonperturbative states by a conjecture. The way back to physics would consequently be to turn the scaling parameter finite and divide the proposed nonperturbative states by the exponential which broke unitarity. It is inevitable to apply the perturbation theory described above in order to obtain information about the former physical theory. If the conjecture on the nonperturbative state spaces was correct, one would thereby gain information on the nonperturbative sector of the (more) physical theory.

The Prepotential of Seiberg-Witten Theory Maybe I was wrong and, after all, it is possible to express the moduli parameter u, cf. (9.2.8), of pure gauge $SU(2)$ Seiberg-Witten theory in terms of characters of the triplet model. At least, I did not prove the contrary. One should look for combinations of the characters that are invariant under the monodromy group (9.2.18) of the spectral torus.

The Partition Function of Seiberg-Witten Theory It would be nice if the partition function of pure gauge $SU(2)$ Seiberg-Witten theory could be written in terms of characters of some CFT. In [NO03], N. Nekrasov and A. Okounkov claim that the dual partition function equals a correlation function of free fermions, and possibly the corresponding CFT can be specified.

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Topological Field Theories A

In this chapter, I will specify what I understand under the topological sector of a field theory. This short summary is along the lines of [BBRT91, Wit82, Wit88a, Wit88b].

Let (X,g) be a symplectic, oriented Riemannian manifold with Euclidean metric g, (Σ,h) another such manifold and $x:\Sigma \to X$ an embedding. The fields will be sections of some $\mathbb{Z}_2$ graded vector bundle over Σ, and I assume that there exists an action for the field theory. The ingredients of the topological sector are:

① An operator Q, the BRST charge, wich is odd graded and globally defined on X and Σ. The BRST charge has a nilpotent action on the fields and state spaces.

② Topological state spaces and topological observables in the cohomology of the BRST charge. Furthermore, I assume that the state spaces have dual vector spaces and a well defined pairing. The cohomology of Q is invariant under smooth variations of the metrics g and h.

③ Even graded and Q-exact fields T_g and T_h, the stress tensors with respect to X and Σ. In other words, the Lagrangian must be a combination of terms that are Q-exact or metric independent.

④ Correlation functions which can be obtained from a path integral. They vanish if one plugs into them a Q-exact observable and Q-closed fields.

⑤ A transformation of the action into a first order form by which the toplogical sector localizes on the vacuum configurations and exclusively on the instantons.

What consequences follow from these attributes? If $\Sigma \subseteq \mathbb{R}\times M$, there exists a generator of time translations $H = \int_M T_{h\,00}$. This operator is Q-exact and all correlation functions of Q-closed fields vanish if it is inserted. Consequently, the topological sector does not contain dynamical fields.

For the same reason, if a topological and a Q-exact observable is inserted into the correlation function and one varies it with respect to the metrics h or g, the variation vanishes. Therefore, the values of the correlation functions in the topological sector do not depend on the metrics defined on Σ and X. In physics, such diffeomorphism invariants are called "topolgical invariants", and the topological sector of a field theory is said to be generally covariant. In this thesis, I use the term topological in this sense.

Provided that the action is Q-exact, the topological sector is invariant under global scale transformations of h and g, namely for any set of topological observables the variation of the path integral in the scaling parameter yields a correlation function of a Q-exact operator. Theories with Q-exact actions are called *cohomological*, and I will only deal with this class.

Due to invariance under global scale transformations, the correlation functions localize on the classical solutions, and the topological sector is semiclassically exact.

Invariance under global scalings does not signify that the theory is conformally invariant. This additionally requires invariance under analytic *local* rescalings of the respective metric.

From the Sigma to the A-Model B

In this section I want to note down the symmetries of the $N = (2,2)$ supersymmetric sigma model and explain how the A-model is derived by the twisting procedure, cf. [Mar05]. Let the conventions be as in chapter 3. The topological A-model and the sigma model with $\mathcal{N} = (2,2)$ worldsheet supersymmetry differ in the spin of the fermionic fields and otherwise have the same action (3.1.1). The supersymmetry is generated by $Q_{\alpha I}$, where $I = +,-$ are the indeces of the R-charge and $\alpha = +,-$ the Lorentz indeces of the $U(1)$ Lorentz symmetry:

$$[Q_{\alpha+}, Q_{\beta-}] = \gamma^{\mu}_{\alpha\beta} P_{\mu}, \qquad [J^{(e)}, Q_{\pm I}] = \pm\frac{1}{2} Q_{\pm I}. \tag{B.0.1}$$

The bracket is a supercommutator and $J^{(e)}$ is the generator of Lorentz transformations. The gamma matrices are $\gamma^1_{\alpha\beta} = \delta_{\alpha\beta}$ and $\gamma^2_{\alpha\beta} = \mathrm{diag}(\mathrm{i}, -\mathrm{i})$, and the superfields transform under $\delta = \kappa^{\alpha I} Q_{\alpha I}$ ($\kappa^{\alpha I}$ is a Grassmann valued constant), cf. [Mar05, pg 73]:

$$\begin{aligned}
\delta x^a &= \kappa^{++}\ \psi^a + \kappa^{-+}\ \pi^a, & \delta x^{\bar{a}} &= \kappa^{--}\ \psi^{\bar{a}} + \kappa^{+-}\ \pi^{\bar{a}}, \\
\delta\psi^a &= 2\mathrm{i}\kappa^{+-}\ \partial_z x^a - \kappa^{-+}\Gamma^a_{bc}\pi^b\psi^c, & \delta\psi^{\bar{a}} &= 2\mathrm{i}\kappa^{-+}\ \partial_{\bar{z}} x^{\bar{a}} - \kappa^{+-}\ \Gamma^{\bar{a}}_{\bar{b}\bar{c}}\pi^{\bar{b}}\psi^{\bar{c}}, \\
\delta\pi^a &= 2\mathrm{i}\kappa^{--}\ \partial_{\bar{z}} x^a + \kappa^{++}\ \Gamma^a_{bc}\pi^b\psi^c, & \delta\pi^{\bar{a}} &= 2\mathrm{i}\kappa^{++}\ \partial_z x^{\bar{a}} + \kappa^{--}\ \Gamma^{\bar{a}}_{\bar{b}\bar{c}}\pi^{\bar{b}}\psi^{\bar{c}}.
\end{aligned} \tag{B.0.2}$$

These are the supersymmetries of the sigma model.

The internal R-symmetry allows for an axial and a non-anomalous vectorial fermionic $U(1)$ current:

$$\begin{aligned}
J^{(v)}_z &= -\mathrm{i}\lambda : g_{a\bar{b}}\pi^{\bar{b}}\psi^a : \quad , \quad & J^{(v)}_{\bar{z}} &= +\mathrm{i}\lambda : g_{a\bar{b}}\pi^a\psi^{\bar{b}} :, \\
J^{(a)}_z &= -\mathrm{i}\lambda : g_{a\bar{b}}\pi^{\bar{b}}\psi^a : \quad , \quad & J^{(a)}_{\bar{z}} &= -\mathrm{i}\lambda : g_{a\bar{b}}\pi^a\psi^{\bar{b}} : .
\end{aligned} \tag{B.0.3}$$

They generate rotations of the fermions

$$\begin{aligned}
&\text{vect:} \quad (\pi_{\bar{a}}, \psi^a) \mapsto \mathrm{e}^{\mathrm{i}\theta}(\pi_{\bar{a}}, \psi^a) \quad , \quad (\pi_a, \psi^{\bar{a}}) \mapsto \mathrm{e}^{-\mathrm{i}\theta}(\pi_a, \psi^{\bar{a}}) \\
&\text{axial:} \quad (\psi^a, \psi^{\bar{a}}) \mapsto \mathrm{e}^{\mathrm{i}\theta}(\psi^a, \psi^{\bar{a}}) \quad , \quad (\pi_a, \pi_{\bar{a}}) \mapsto \mathrm{e}^{-\mathrm{i}\theta}(\pi_a, \pi_{\bar{a}}).
\end{aligned} \tag{B.0.4}$$

The supercharges transform according to these symmetries as

$$\begin{aligned}
[J^{(v)}_z{}_0, Q_{+\pm}] &= \pm Q_{+\pm} \quad , \quad & [J^{(v)}_{\bar{z}}{}_0, Q_{-\pm}] &= \mp Q_{-\pm}, \\
[J^{(a)}_z{}_0, Q_{+\pm}] &= \pm Q_{+\pm} \quad , \quad & [J^{(a)}_{\bar{z}}{}_0, Q_{-\pm}] &= \pm Q_{-\pm},
\end{aligned} \tag{B.0.5}$$

such that in particular $[J^{(a)}_0, Q] = Q$ and the cohomology of Q is graded by the axial charge. In general, the axial $U(1)$ symmetry is (partially) broken.

B.1 Twisting/Gauging the Sigma Model

I will now specify the fields for the sigma model, the A-model can then be obtained by a redefinition of the Lorentz generator$J^{(e)}$. This procedure is called twisting or gauging.

To make the transformation properties of the fermionic fields under Lorentz transformations explicit, I will introduce the spin-connection ω, pretending that Σ is not flat. The fermions have now the properties $\pi^a_{\bar{z}},\ \psi^a \in \Gamma(\Sigma, S^{\pm} \otimes x^*(TX))$ and $\pi^{\bar{a}}_z,\ \psi^{\bar{a}} \in \Gamma(\Sigma, S^{\pm} \otimes x^*(T\bar{X}))$. The bar over the latter tangent bundle denotes a section into the anti-holomorphic part, $S^{\pm}$ are the spinor bundles of positive and negative chirality and Γ means a section. The fields ψ^a and $\pi^{\bar{a}}_z$ have spin $+\frac{1}{2}$ and the other fermions have spin $-\frac{1}{2}$. The connection on $S^{\pm} \otimes x^*(TX) \to \Sigma$ is obtained by $\mathrm{D} = \mathrm{D}^{(S)} \otimes 1_{x^*(TX)} + 1_S \otimes x^*(\mathrm{D}^{(TX)})$, for instance

$$\mathrm{D}_{\bar{z}}\psi^a = \partial_{\bar{z}}\chi^a + \frac{\mathrm{i}}{2}\omega_{\bar{z}}\psi^a + \Gamma^a_{bc}\partial_{\bar{z}}x^b\psi^c . \tag{B.1.1}$$

Under the vectorial symmetry, ψ^a and $\pi^a_{\bar{z}}$ transform with weight $+\frac{1}{2}$ while the others have weight $-\frac{1}{2}$ and the bosons are invariant. The transformation properties of the supercharges are listed below, and I included already the effect of redefining the Lorentz group:

	$U_e(1) \times U_v(1)$	$U_{e'}(1) \times U_v(1)$
Q_{++}	$(+\frac{1}{2}, +1)$	$(0, +1)$
Q_{-+}	$(-\frac{1}{2}, +1)$	$(1, +1)$
Q_{+-}	$(+\frac{1}{2}, -1)$	$(-1, -1)$
Q_{--}	$(-\frac{1}{2}, -1)$	$(0, -1)$

(B.1.2)

This redefinition is according to $J^{(e')} := J^{(e)} - \frac{1}{2}J^{(v)}$.[1] Since it is not possible to discriminate either of the $U(1)$ symmetries, this redefinition is an equivalence relation of the theory in case Σ is flat. One then still has the full supersymmetry. However, when passing to non-flat domain manifolds, only the scalar supercharges survive, for they do not depend on the metric or any related quantities such as the Levi-Civita connection.

After twisting it is reasonable to define new symmetry charges, a scalar and a one form on Σ, as follows:

$$Q := Q_{++} + Q_{--} \quad , \quad G_z := Q_{+-} \ , \ G_{\bar{z}} := Q_{-+} . \tag{B.1.3}$$

They are subject to the propery $Q^2 = 0,\ [Q, G_\mu] = P_\mu$ and define the topological algebra of the thus obtained A-model with BRST charge Q. The fermions have a new spin with respect to $J^{(e')}$. The field ψ is a Grassman valued scalar field while $\pi = \pi_{za}\ \mathrm{d}z\mathrm{d}x^a + \pi_{\bar{z}\bar{a}}\ \mathrm{d}\bar{z}\mathrm{d}x^{\bar{a}}$ is a selfdual one-form. This explains why twisting is the same as coupling the theory to the $U_v(1)$ current (i.e. “gauging” the theory) according to $S \mapsto S + \frac{1}{4}\int_\Sigma h^{\mu\nu}\omega_\mu J^{(v)}_\nu$. With respect to

[1] The choice of sign is for convenience and follows [Mar05, Wit88b].

Q, the fields now transform with $\delta := \kappa Q$, $\kappa^{--} = \kappa = \kappa^{++}$, $k^{\pm\mp} = 0$ and the rest can be read off tabular B.0.2:

$$\begin{aligned} &\delta x^a = \kappa \chi^a && \delta x^{\bar{a}} = \kappa \psi^{\bar{a}} \\ &\delta \psi^a = 0 && \delta \psi^{\bar{a}} = 0 \\ &\delta \pi^a_{\bar{z}} = 2\mathrm{i}\kappa\ \partial_{\bar{z}} x^a + \kappa\ \Gamma^a_{bc} \pi^b_{\bar{z}} \psi^c && \delta \pi^{\bar{a}}_z = 2\mathrm{i}\kappa\ \partial_z x^{\bar{a}} + \kappa\ \Gamma^{\bar{a}}_{\bar{b}\bar{c}} \pi^{\bar{b}}_z \psi^{\bar{c}} \end{aligned} \tag{B.1.4}$$

From that tabular one also finds that there is a fermionic fixed point on the holomorphic $\partial_{\bar{z}} x^a = \partial_z x^{\bar{a}} = 0$ embeddings. These are called instantons.[2]

[2]For $J^{(e')} = J^{(e)} + \frac{1}{2} J^{(v)}$, the BRST charge would be $Q = Q_{+-} + Q_{-+}$ and localization is on the anti-instantons $\partial_z x^a = \partial_{\bar{z}} x^{\bar{a}} = 0$.

The Toric CSbc - Unfinished C

Frenkel et al. [FLN08] use a different representation of the CSbc in order to derive the Grothendieck-Cousin operators. It goes back to a publication of Borisov [Bor01] and has two promising features. Firstly, the fields in the CSbc are not bosonized and the assumed Grothendieck-Cousin field is also expressed in terms of the original fields. Secondly, it is linked to another work of Frenkel with Losev [FL07], in which they already proposed that the Tbc on $\mathbb{CP}^1$, considered as a CSbc, should be deformed beyond its topological sector.

In [VF09] I used Frenkels and Losevs formalism in addition to the one described in sections 3.6.2 and 3.6.3. Thereby, I wanted to match my results with those of Frenkel et al. in [FLN08, FL07]. Because it concerned my own investigations, I will briefly discuss the question if both approaches are isomorphic. Unfortunately, I could not identify the Grothendieck-Cousin fields, whereas I might have found a positive result for their zero modes, the Grothendieck-Cousin operators.

C.1 Deformation by Holomorphic Completion

There exists another paper of Frenkel with Losev [FL07], wherein the authors consider the Tbc without "gauge" field. One of the subjects was the question, how to tackle that theory if formulated on nontrivial target spaces. The idea of the authors was as follows.

Frenkel and Losev started with the assumption that if $\Sigma = \mathbb{CP}^1$ and $X = \mathbb{C}/2\pi\mathrm{i}\mathbb{Z}$, the Tbc is an ordinary CSbc. Since Σ is compact, the solutions of the instanton equation $\partial_{\bar{z}} x = 0$ are the constant embeddings, which they interpret as vacuum configurations. Thus, this scenario only allows to take insight into the topological sector.

If, however, X was compactified to $\mathbb{CP}^1$, there appear further nontrivial holomorphic mappings, cf. [Jos02], which Frenkel and Losev consequently interpret as instanton solutions beyond the topological regime. It is not clear if the Tbc with target $\mathbb{CP}^1$ is conformal. However, Frenkel and Losev they assumed that this is the case if the target space is $\mathbb{C}/2\pi\mathbb{Z}$. Therefore, they searched after a method which allows to reduce the situation of $X = \mathbb{CP}^1$ to the free CSbc on $\mathbb{C}/2\pi\mathrm{i}\mathbb{Z}$, however, now deformed by additional operators. These operators supposedly give an insight into the dynamical sector of the Tbc and, hence, must inherit some information about the local geometry of the Tbc on $\mathbb{CP}^1$.

By taking out of Σ sets of pairs of zeros and poles $\omega_k^{\pm}$, Frenkel and Losev supplemented the constant holomorphic by meromorphic embeddings, constant as $\mathbb{CP}^1 \setminus \{\omega_k^{\pm}\} \to \mathbb{C}/2\pi\mathrm{i}\mathbb{Z}$ and with simple poles and zeros at $\omega_k^\pm$, $k \in \mathbb{N}$. Thus, they end up with a stack of coverings $x\colon \mathbb{CP}^1 \to \mathbb{CP}^1$, distinguished by the number k of singular points of x. Notice, however, that the coverings are not branched since Frenkel and Losev did neglect the embeddings with higher ramification.

Frenkel and Losev interpreted the meromorphic functions as a generalization of the CSbc by an inclusion of instantons, whereby the degree k measures the instanton sector. Since the singularities of those functions should appear in their vacuum expectation values, Frenkel and Losev concluded that the action of the CSbc with target $\mathbb{C}/2\pi\mathrm{i}\mathbb{Z}$ must be deformed. In order to analyze that, they made a chart transition to logarithmic coordinates as described in section 3.5.1. This is also reasonable because the equivalence classes $\mathbb{C}/2\pi\mathrm{i}\mathbb{Z}$ are naturally expressed by means of the exponential. The vacuum expectation value of an instanton solution should now yield

$$\langle \phi_x(z) \rangle_{S+\delta S} = c + \sum_{i=1}^{n} [\log(z-\omega_i^+) - \log(z-\omega_i^-)]\,, \tag{C.1.1}$$

where $S+\delta S$ is the deformed CSbc action. Frenkel and Losev proposed that this change in the action is caused by an additional term

$$\delta L(z,\bar z) = -\lambda[\Psi_+(z,\bar z) + \Psi_-(z,\bar z)]\pi(z)\bar\pi(\bar z) \quad , \quad \lambda = 1\,, \tag{C.1.2}$$

with $\Psi_\pm(z,\bar z) = \Psi_\pm(z)\bar\Psi_\pm(\bar z)$, $\Psi_\pm(z) = \exp\{\pm\mathrm{i}\int^z p(\omega)\mathrm{d}\omega\}$ and, similar, $\bar\Psi_\pm(\bar z) = \exp\{\pm\mathrm{i}\int^{\bar z} \bar p(\bar\omega)\mathrm{d}\bar\omega\}$. Because λ is dimensionless, this deformation can be interpreted as a movement in the moduli space of conformal theories.

By means of a method of Zamolodchikov [Zam89], Frenkel and Losev calculated the impact of these deformations on general fields $F(z)$ of the CSbc. This amounts to applying the Stokes-Green theorem (integral of motion) to[1]

$$\partial_{\bar z} F_\delta(z,\bar z) = \oint_z \mathrm{d}\zeta\ \delta L(\zeta,\bar z) F(z). \tag{C.1.3}$$

Of particular interest are the deformations of the stress tensor and the supercharge. A calculation reveals that the stress tensor is not deformed, whereas the integral of motion for the supercharge yields

$$\tilde Q = \oint \left\{\mathrm{d}z\ Q_\delta(z,\bar z) + \mathrm{d}\bar z\ \ [\Psi_+(z,\bar z) - \Psi_-(z,\bar z)]\,\bar\pi(\bar z)\right\}, \tag{C.1.4}$$

which is similar to the expression in [FL07, pg. 67].

Frenkel et al. refer to these results in their later work [FLN08, pg. 97]. They propose that the zero modes of the operators in (C.1.2)

$$\mathrm{i}\pi(z)\Psi_-(z)\,, \quad -\mathrm{i}\bar\pi(z)\bar\Psi_+(z)\,, \tag{C.1.5}$$

[1]This integral of motion is the first order correction (in λ) to $\partial_{\bar z} F = 0$ [Zam89]. In principle, since λ is dimensionless, one has to include corrections to all orders.

are identical with the cohomology operators $\eta_0\bar{\eta}_0$ in the context of chiral bosonization, and moreover with the Grothendieck-Cousin operators [FLN08, pg. 93f]. They conclude that the supercharge in the context of their later work is deformed just the same way as in (C.1.4), [FLN08, pg. 97].

Since the integral of motion (C.1.3) does not introduce the Grothendieck-Cousin operators, I looked for another CFT method that would deform the stress tensor in the appropriate way and also the supercharge according to (C.1.4). This was the method by Fjelstad et al. [FFH+02], that I used in chapter 4. By that means, I derived a deformation of the stress tensor and of the supercharge which was similar to [FL07, FLN08], cf. [VF09]. In the same publication I could also argue, that the cohomology of the deformed supercharge on the state space is not changed by the deformation. Thus, everything seemed to be nice.

However, I did not check if the assumed Grothendieck-Cousin field of (C.1.2) is well defined on the charged representation spaces, which is mandatory. Nor did I really extend Borisovs vertex algebra to charged representations and then prove isomorphism to the representations I have considered in chapter 3. Some steps into that direction I have done, however only superficially, in [VF09], and in this chapter I wanted to complete them. However, I could not determine either the representation spaces correctly, or the fields in (C.1.2) can not be the Grothendieck-Cousin fields, though their zero modes satisfy the properties of the Grothendieck-Cousin operators.

C.2 The Cohomology Operators in Logarithmic Coordinates

In order to simplify my discussion, I will set the homogeneity μ to zero.

The CSbc in logarithmic coordinates, cf. section 3.5.1, does not cover the situation of the CSbc on $\mathbb{C}/2\pi\mathrm{i}\mathbb{Z}$. Since the exponential is invariant under $2\pi\mathrm{i}\mathbb{Z}$, the field algebra should be extended by some winding number operator Ω and its conjugate Ω^* : $[\Omega,\Omega^*]=1$. This yields Borisovs' vertex algebra [Bor01], which is constituted by

$$\begin{aligned} \phi_x(z) &=: \mathrm{e}^{W(z)}:, & \phi_{\mathrm{i}p}(z) &=: \mathrm{e}^{-W(z)}[-\partial_z U(z)+j^+(z)]:, \\ \phi_\psi(z) &=: \mathrm{e}^{W(z)}\psi(z):, & \phi_{\mathrm{i}\pi}(z) &= \mathrm{i}:\mathrm{e}^{-W(z)}\pi(z):, \end{aligned} \tag{C.2.1}$$

and the symmetrie fields

$$\begin{aligned} \phi_{j^+}(z) &= j^+(z)+\partial_z W(z), & \phi_{j^-}(z) &= -j^+(z)+\partial_z U(z), \\ \phi_{\mathscr{G}}(z) &= \mathrm{i}:\pi(z)\partial_z W(z):, & \phi_{\mathscr{Q}}(z) &= \mathscr{Q}(z)+\partial_z\psi(z), \end{aligned}$$
$$\phi_T(z) = -:\partial_z W(z)\partial_z U(z)+\mathrm{i}\partial_z\psi(z)\pi(z):. \tag{C.2.2}$$

Above I used $\mathscr{Q}(z)=-\mathrm{i}:\partial_z U(z)\psi(z):$ and

$$U(z)=\Omega^*-\mathrm{i}\int'^{z} p(\omega)\mathrm{d}\omega \quad , \quad W(z)=\Omega\log z+x(z), \tag{C.2.3}$$

and the prime at the integral means that no additional "integration constant" should be introduced.

Borisov interprets U and W as the scalar fields related to certain "currents" of bosons on a two dimensional lattice, such that in analogy with (3.6.11) $W(z) = -\int^z J^{(1)}(\omega)\mathrm{d}\omega$ and $U(z) = -\int^z J^{(2)}(\omega)\mathrm{d}\omega$, with $J^{(1)}(z) = -\Omega z^{-1} - \partial_z x(z)$ and $J^{(2)}(z) = \mathrm{i}p(z)$. The Heisenberg Lie commutation relations are only satisfied between $J^{(1)}$ and $J^{(2)}$, $[J^{(1)}_n, J^{(2)}_m] = -n\delta_{n,-m}$. Further, $[U_0, J^{(1)}_0] = -[\Omega^*, \Omega] = 1$, as is expected for "bosonic" currents, while $[W_0, J^{(2)}_0] = [x_0, \mathrm{i}p_0] = -1$. According to the idea to interpret the currents as two components on a lattice, it is now reasonable to consider fields $V(l,s,z) =: \mathrm{e}^{lW(z)+sU(\omega)} :,\ l,s \in \mathbb{Z}$.

I will call the vertex algebra defined by (C.2.1), (C.2.2) and extended by $V(0,s,z)$ as the *toric* CSbc.

Remark In the context of the chiral de Rham complex, the introduction of $V(0,s,z)$ means that one has to generalize the state space further to power series in the zero modes p_0. This is a first instance wherein Borisovs' construction exeeds the usual CSbc.

Representation Spaces

In order to include charged representations, I define $|p,q|l,s\rangle \in F(p,q|l,s) := F(p|l,s) \otimes M^+(q)$ and try the Ansatz

$$\begin{aligned}
&x_n\ |p,q|l,s\rangle = 0\,,\ n > -p\,,\ n \neq 0\,, && p_n\ |p,q|l,s\rangle = 0\,,\ n \geq p\,,\ n \neq 0\ ,\\
&\mathrm{i}p_0\ |p,q|l,s\rangle = l\ |p,q|l,s\rangle\,, && \Omega\ \ |p,q|l,s\rangle = s\ |p,q|l,s\rangle\,,\\
&\psi_n\ |p,q|l,s\rangle = 0\,,\ n > -q\,, && \pi_n\ |p,q|l,s\rangle = 0\,,\ n \geq q\,.
\end{aligned} \tag{C.2.4}$$

This exceeds the discussion of Borisov [Bor01] who considered the situation $p = q = 0$. It will now be necessary to see if the operator product algebra is well defined on the representations above.

Firstly, the representation spaces for the toric CSbc must include states that are isomorphic to $V(l,s,z)$. This isomorphism is obtained by $\exp\{l'x_0\}|p,q|l,s\rangle = |p,q|l+l',s\rangle$ and $\exp\{s'\Omega^*\}|p,q|l,s\rangle = |p,q|l,s+s'\rangle$. In the language of vertex operators,

$$Y\big(|0,0|l,0\rangle, z\big) = \exp\{lW(z)\} \quad , \quad Y\big(|0,0|0,s\rangle, z\big) = \exp\{sU(z)\}\,. \tag{C.2.5}$$

This makes explicit that the vertex algebra defined by (C.2.1) does not lead out of a specific representation with a fixed value of s, since it does not include Ω^*. I will denote by $\overline{F}(p,q|l,s)$ the vertex algebra of these fields with fixed value s and Ω^* excluded. Moreover, I define normal ordering in the field modes to be taken with respect to $|0,0|0,0\rangle$.

In the representation $F(p,q|l,s)$, the fields of (C.2.1) have the OPEs

$$\phi_x(z)\phi_{\mathrm{i}p}(\omega) = \frac{-1}{z-\omega}\left(\frac{z}{\omega}\right)^{p+s}, \quad \phi_\psi(z)\phi_{\mathrm{i}\pi}(\omega) = \frac{-1}{z-\omega}\left(\frac{z}{\omega}\right)^{q+s}. \tag{C.2.6}$$

When acting on a highest weight state, the mode expansions of the fields inherits the inhomogeneity in terms of a shift in the index, for instance

$$\phi_x(z)|p,q|l,s\rangle = z^s \sum_{n=0}^{\infty} c_n(|z|)z^{-n}|p,q|l,s\rangle = \mathrm{e}^{x_0} \sum_{n\leq -p-s,\ n\neq 0} (\phi_x)_{n+s} z^{-n}|p,q|l,s\rangle \tag{C.2.7}$$

and similar for the other fields. In particular, up to the special rôle of x_0, when $s=0$, the field mode expansion equals that for the CSbc. Thus, the CSbc has a representation on the representation spaces above. The OPEs between the symmetry fields and the dynamical fields (C.2.1) follow accordingly.

The conformal weights and $U(1)$ charges of the highest weight states equal

$$\begin{aligned} \Delta_{\phi_T}(|p,q|l,s\rangle) &= -\frac{1}{2}p(p-1) + \frac{1}{2}q(q-1) + ls\,, \\ (\phi_{j^-})_0\ |p,q|l,s\rangle = q - l\,, &\qquad (\phi_{j^+})_0\ |p,q|l,s\rangle = -q + s\,, \end{aligned} \tag{C.2.8}$$

and the operators measuring these quantum numbers commute with each other. The field $V(l',s',z)$ shifts the conformal weight of $|p,q|l,s\rangle$ by

$$T_0 \cdot \mathrm{e}^{l'x_0 + s'\Omega^*}|p,q|l,s\rangle = (ls' + l's)\ |p,q|l+l',s+s'\rangle\,, \tag{C.2.9}$$

and has a bosonic and fermionic $U(1)$ charge of value $-l'$ and 0, respectively. In the subsector with $s=0$ and Ω^* excluded, all fields in (C.2.1) have the same conformal weights and $U(1)$ charges as the fields of the usual CSbc, wich follows from the OPEs and section 3.5.1, and there is not operator leading out of that representation.

OPEs of the Operators $V(l,s,z)$

If I restrict my discussion to the conformal vacuum $|0,0|0,0\rangle$, I can derive an OPE between the fields $V(l,s,z)$:

$$\mathrm{e}^{sU(z)}\mathrm{e}^{lW(\omega)} = (z-\omega)^{-ls} : \mathrm{e}^{sU(z)}\mathrm{e}^{lW(\omega)} :, \quad \text{in} \quad F(0,0|0,0)\,. \tag{C.2.10}$$

It turns out, however, that I am not able to tackle the OPE in the charged representation spaces in any reasonable way. Namely, if $p\neq 0$, I find that

$$\exp\left[-\mathrm{i}\int^z \underbrace{p(\zeta)x}(\omega)\mathrm{d}\zeta\right] = \exp\left[-\int^z \left(\frac{\omega}{\zeta}\right)^p \frac{\mathrm{d}\zeta}{\zeta-\omega}\right]. \tag{C.2.11}$$

Remark It seems that the charged representations that I have defined do not lead to nice results for the OPE betweeen e^{lW} and e^{sU}.

Identification of the CSbc

Due to the results above, the CSbc is a subsector of the toric CSbc with $s = 0$ and Ω^* excluded. I will now identify the bosonic and fermionic parts of the CSbc within $\overline{F}(p,q|l,0)$. Notice, that the term "identification", signified by "$\simeq$", is only appropriate up to the special rôle played by x_0.

The representations $\overline{F}(p,q|l,0)$ are graded by the bosonic and fermionic $U(1)$ charges,

$$\overline{F}(p,q|l,0) = \bigoplus_{n,m\in\mathbb{Z}} \overline{F}(p,q+n|l+n-m,0)\,, \tag{C.2.12}$$

whereby n and m count the fermionic and bosonic charges, respectively. I made no distinction between $\bigoplus_n M^+(q)_n$ and $\bigoplus_n M^+(q-n)$, since the fermionic representation spaces are all isomorphic, cf. (3.4.2).

The Fermionic Subsector The fermionic part of the CSbc appears in the toric CSbc as the subspace $\overline{F}(0,q|q,0) \simeq M^+(q)$. Indeed, ϕ_ψ and $\phi_{\mathrm{i}\pi}$ have the correct OPE on $|0,q|q,0\rangle$ and the appropriate quantum numbers with respect to $T^+(z)$ and $\phi_{j^+}(z)$. In particular, this holds for $|0,q|q,0\rangle$, such that I set $|0,q|q,0\rangle \simeq |q\rangle_+ \in M^+(q)$.

The Bosonic Subsector The bosonic subsector is given by $\overline{F}(p,0|-p,0) \simeq \overline{N}(p)$. Namely, the fields have the correct OPE on $|p,0|-p,0\rangle$ and the quantum numbers as expected, such that I set $|p,0|-p,0\rangle \simeq v_p^- \otimes |0\rangle_{\eta\xi} \in \overline{N}(p)$.

The Grothendieck-Cousin Operators

In order to derive the Grothendieck-Cousin operators, I used the recipe to extend the bosonic representation space by the missing degenerate part, cf. sections 2.6.2 and 3.6.3. The affected representation space takes now the form $\overline{F}(1,0|-1,0)$ and I have to look for a state that has the same quantum numbers as the hightes weight vector $|1,0|-1,0\rangle$.

The states $|p,0|-p,0\rangle$, $|-p+1,0|p-1,0\rangle$ and $|p-1,1|-p+1,1\rangle$ do all have the same conformal weight, but only $|p,0|-p,0\rangle$ and $|p-1,1|-p+1,1\rangle$ have the same $U(1)$ charges (both with respect to the bosonic and the fermionic charge). Therefore, the analogue of $\mathrm{e}^{\phi_0^-}\xi_0 : \overline{N}(1) \to N(1)$ should be the mapping $e_0 : |1,0|1,0\rangle \mapsto |1,1|1,1\rangle$. Moreover, I propose that the logarithmic extension $N_L(1)$ is now the representation of (C.2.1) on $|0,1|0,1\rangle$, and I will denote that by $F_L(1,0|-1,0)$,

In analogy with the discussion in section 3.6.3, I am looking for an operator $\mathfrak{g}$, such that

$$\begin{array}{cccc} \overline{F}(1,0|-1,0) \ni & |1,0|-1,0\rangle & \xrightarrow{e_0} & |0,1|0,1\rangle & \in F_L(1,0|-1,0) \\ & & & \downarrow \mathfrak{g} & \\ & & & |0,0|0,0\rangle & \in \overline{F}(0,0|0,0) \end{array} . \tag{C.2.13}$$

The operator

$$\mathfrak{g} = \mathrm{i}\pi_0 \mathrm{e}^{-\Omega^*} \tag{C.2.14}$$

does the job. Moreover, it satisfies $\oint_0 \mathrm{d}\omega\ [\mathfrak{g}, \phi(\omega)] = 0$ for all fields ϕ in (C.2.1). Therefore, the sequence

$$\cdots \to F(p,0|-p,0) \xrightarrow{\mathfrak{g}} F(p-1,0|-p+1,0) \to \cdots \tag{C.2.15}$$

is exact, whereby $F(p,0|-p,0) = \overline{F}(p,0|-p,0) \oplus F_L(p,0|-p,0)$ are the extended representation spaces.

In that respect, it is reasonable to identify $\mathfrak{g}$ with the cohomology operator η_0 in section 3.6.3, and with the Grothendieck-Cousin operator.

The Grothendieck-Cousin Field

To generalize the operator above to the Grothendieck-Cousin field, it is at hand to try the Ansatz

$$\mathrm{i} : \pi(z)\mathrm{e}^{-U(z)} : . \tag{C.2.16}$$

Indeed, when the fields ϕ of (C.2.1) are in the representation $\overline{F}(0,0|0,0)$, one may calculate the OPEs by means of (C.2.10) and derive that

$$\oint_z \mathrm{d}\omega\ \mathrm{i} : \pi(z)\mathrm{e}^{-U(z)} : \phi(\omega) = 0 . \tag{C.2.17}$$

For instance, use

$$\mathrm{i}\pi(z)\mathrm{e}^{-U(z)}\phi_{\mathrm{i}p}(\omega) = -\frac{\mathrm{i} : \pi(\omega)\mathrm{e}^{-U(\omega)-W(\omega)} :}{(z-\omega)^2} . \tag{C.2.18}$$

This calculation, however, turns nontrivial if the representation space is charged, cf. (C.2.11). For that reason, I could not derive the Grothendieck-Cousin field in terms of Borisovs' vertex algebra.

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