Risk Management and Regulation of Financial Institutions

Von der Fakultät für Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover
zur Erlangung des akademischen Grades

Doktorin der Naturwissenschaften
– Dr. rer. nat. –

genehmigte Dissertation von

M.Sc. Anna-Maria Hamm

2019

Referent: Prof. Dr. Stefan Weber
Korreferent: Prof. Dr. Ralf Korn
Korreferent: Prof. Dr. Ludger Overbeck
Tag der Promotion: 20. Juni 2019

Risk Management and Regulation of Financial Institutions

Anna-Maria Hamm

Leibniz Universität Hannover

Abstract

Insurance and financial products, companies and markets are highly complex. An understanding of the inherent upside and downside risk requires suitable tools for a detailed analysis. In addition, several crises in the history of the financial system have shown that powerful regulatory frameworks are indispensable in order to guarantee that products, firms and markets provide benefits to the society. These issues are the focus of this thesis.

Classical monetary risk measures are functionals that quantify the downside risk of positions. They facilitate a better understanding of the risks in products, companies and markets, and they are an important basis for regulation – in particular in the context of capital requirements. Risk measures have been studied intensively over the past twenty years. The present thesis focuses on the following aspects:

- From a practical point of view, the implementation of risk measures in the context of Monte Carlo simulations is an important issue; for a certain class of risk measures, we design and evaluate their efficient estimation via a stochastic root finding algorithm.
- The thesis contributes to the development of risk measures and the evaluation of their merits and disadvantages. Classical risk measures typically evaluate exogenous positions. We investigate feedback from trading and price impact and suggest suitable liquidity-adjusted risk measures. We also consider risk measurement in networks of firms and investigate the issues of optimal capital allocation and optimal risk sharing between entities within a network. We find that firms may hide a substantial portion of their downside risk if they use V@R-based risk measures as a basis for their capital requirements.
- We investigate the impact of insurance premium taxation. This tax on many insurance products differs from the standard tax scheme: the value-added tax.
- Finally, we focus on a specific functional of the upside and downside risk, the market consistent embedded value and its components within an asset-liability management model; this requires a combination of different valuation approaches and an integration of actuarial and financial perspectives.

Keywords: risk management, regulation, monetary risk measures, (solvency) capital requirements, corporate networks, optimal risk sharing, network risk, (set-valued) capital allocation, liquidity risk, insurance premium tax, asset-liability management, market consistent embedded value, stochastic root finding, value at risk, average value at risk, range value at risk, utility-based shortfall risk, optimized certainty equivalents, distortion risk measures

Contents

Introduction

Motivation. Insurance and financial products, companies and markets are highly complex. An understanding of the inherent upside and downside risk requires suitable tools for a detailed analysis. In addition, several crises in the history of the financial system have shown that powerful regulatory frameworks are indispensable in order to guarantee that products, firms and markets provide benefits to the society. These issues are the focus of this thesis.

Classical monetary risk measures are functionals that quantify the downside risk of positions. They facilitate a better understanding of the risks in products, companies and markets, and they are an important basis for regulation – in particular in the context of capital requirements. Risk measures have been studied intensively over the past twenty years. The present thesis focuses on the following aspects:

- From a practical point of view, the implementation of risk measures in the context of Monte Carlo simulations is an important issue; for a certain class of risk measures, we design and evaluate in Chapter 1 of this thesis their efficient estimation via a stochastic root finding algorithm.

- Chapters 2 - 4 of the thesis contribute to the development of risk measures and the evaluation of their merits and disadvantages. Classical risk measures typically evaluate exogenous positions. We investigate feedback from trading and price impact and suggest suitable liquidity-adjusted risk measures. We also consider risk measurement in networks of firms and investigate the issues of optimal capital allocation and optimal risk sharing between entities within a network. We find that firms may hide a substantial portion of their downside risk if they use V@R-based risk measures as a basis for their capital requirements.

In the last two chapters of the thesis, we investigate two topics that were motivated by discussions with practitioners:

- Chapter 5 investigates the impact of insurance premium taxation. This tax on many insurance products differs from the standard tax scheme: the value-added tax.

- Like the first part of this thesis, Chapter 6 looks at functionals for the evaluation of risk. Here, we focus on a specific functional of the upside and downside risk, the market consistent embedded value and its components within an asset-liability management model; this requires a combination of different valuation approaches and an integration of actuarial and financial perspectives.

Outline. The thesis is structured as follows:

1. In Chapter 1, we design and analyze Monte Carlo methods for the estimation of an important and broad class of convex risk measures that is constructed on the basis of optimized certainty equivalents (OCEs). This family of risk measures – originally introduced in Ben-Tal & Teboulle (2007) – includes, among other risk measures, e.g., the entropic risk measure or average value at risk. The computation of OCEs involves a stochastic optimization problem which can be reduced to a stochastic root finding problem via a first-order condition. We describe suitable algorithms and illustrate their properties in numerical case studies.

2. Risk measures were originally applied to exogenous random variables. The actions and interactions of market participants do, however, influence the value of assets and liabilities, and richer models are an attempt to capture these effects. An important type of risk, especially during times of crises, is the liquidity risk due to price impact of trades. As observed by Acerbi & Scandolo (2008), this type of risk requires adjustments to classical portfolio valuation and risk measurement. The key contribution of Chapter 2 is the construction of a new, cash-invariant liquidity-adjusted risk measure that can naturally be interpreted as a capital requirement. We clarify the difference between our construction and the one of Acerbi & Scandolo (2008) in the framework of capital requirements using the notion of eligible assets, as introduced by Artzner, Delbaen & Koch Medina (2009). Numerical case studies illustrate how price impact and limited access to financing influence the risk measurements. We apply stochastic root finding algorithms – as proposed in Chapter 1 – in order to compute the liquidity-adjusted average value at risk and liquidity-adjusted utility-based shortfall risk.

3. Chapter 3 considers risk of networks of firms. In such a setting, new phenomena arise that cannot be observed in models of single firms, for example, risk sharing or systemic interaction. We propose a unified framework for the regulation of corporate networks. Its cornerstone is the new notion of a set-valued network risk measure that quantifies network risk by the set of vectors of additional capital requirements that lead to acceptable regulatory outcomes. In this setting, we analyze capital allocations that are optimal from the point of view of the network's management while respecting regulatory requirements at the same time. We show that the Euler allocation principle can be embedded into our set-valued setting, and we analyze capital allocations in numerical examples. Since capital allocations interfere with management strategies, including asset-liability management strategies and internal capital transfers, we also study their impact on optimal capital allocations. Numerical case studies indicate that consolidated balance sheets can be mimicked via optimal management strategies.

4. If risk capital of a network of firms is not computed on the basis of a consolidated balance sheet, but defined as the sum of the capital requirements of the sub-entities,

risk sharing becomes highly important. While coherent risk measures are not problematic, downside risk can be hidden when V@R-based risk measures are used. In Chapter 4, we discuss the impact of risk sharing and asset-liability management on capital requirements. Our analysis contributes to the evaluation of the merits and deficiencies of different risk measures. In particular, we highlight that the class of V@R-based risk measures allows for a substantial reduction of the total capital requirement in corporate networks that share risks between entities. We provide case studies that complement previous theoretical results and demonstrate their practical relevance. For large networks, optimal asset-liability management is often contrary to those strategies that are desirable from a regulatory point of view.

5. In Chapter 5, we analyze the impact of insurance premium taxation. In many countries insurance premiums are subject to an insurance premium tax that replaces the common value-added tax (VAT) used for most products and services. Insurance companies cannot deduct VAT paid on inputs from premium tax; also corporate buyers of insurance cannot deduct premium tax payments from VAT on their outputs. Such deductions would be allowed, if insurance premiums were subject to VAT instead of insurance tax. We investigate the impact of the premium tax on insurance companies, insurance holders and government revenues from multiple perspectives. We explicitly compare tax systems with premium tax and tax systems that allow deductions. We find that the competitiveness of corporate buyers of insurance, the ruin probabilities of insurance firms and their solvency capital are hardly affected by the tax system. In contrast, the tax system has a significant influence on the cost of insurance, insurance demand, government revenues and the profitability of insurance firms.

6. Chapter 6 provides an introduction to the market consistent embedded value (MCEV) within the context of an asset-liability management model that probabilistically models both the asset and the liability side. This highlights the intertwined actuarial and financial valuation of the MCEV. We present a unified framework that shows how a martingale measure for the risk-neutral valuation of replicable risks and the statistical measure for the computation of the cost of capital of non-hedgeable risks are combined. The definition of the cost of capital is based on monetary risk measures.

7. Appendix B is an additional, technical chapter that discusses the notion of redistribution risk measures. This new definition provides a unifying framework for liquidity-adjusted risk measures and network risk measures.

Literature. Chapters 1, 2, 5 & 6 are based on the following publications:

- The work presented in Chapter 1 was previously published in *Proceedings of the 2013 Winter Simulation Conference* as *Stochastic Root Finding for Optimized Certainty Equivalents* by *Hamm, Salfeld & Weber (PhD advisor)*. The conceptional ideas were suggested by Stefan Weber. The displayed figures were created by Thomas Salfeld. All authors developed the formalism, discussed the results and jointly wrote the paper.

- The work presented in Chapter 2 was previously published in *Mathematics and Financial Economics* as *Liquidity-Adjusted Risk Measures* by *Weber (PhD advisor), Anderson, Hamm, Knispel, Liese & Salfeld.* The conceptional ideas were suggested by Stefan Weber. Figures of liquidity-adjusted portfolio values were created by Thomas Salfeld. I carried out the numerical case studies based on the findings in Chapter 1. Moreover, I embedded the liquidity-adjusted risk measure into the framework of redistribution risk measures, see Appendix B. All authors discussed the results and jointly wrote the paper.

- The work presented in Chapter 5 was previously published in *European Actuarial Journal* as *The Impact of Insurance Premium Taxation* by *Degelmann, Hamm & Weber (PhD advisor).* All authors conceived and planned the studies and jointly wrote the paper.

- A German version of the work presented in the introductory part of Chapter 6, Sections 6.1 & 6.2 was previously published in *Der Aktuar* under the title *Market Consistent Embedded Value – eine praxisorientierte Einführung* by *Becker, Cottin, Fahrenwaldt, Hamm, Nörtemann & Weber (PhD advisor).* All authors contributed equally to this published paper. All extensions in Chapter 6 were developed by myself in discussion with my supervisor.

Chapters 3 & 4 are based on a working paper and a preprint:

- The work presented in Chapter 3 is based on the working paper *Network Risk, Network Regulation, and Network Optimization* by *Hamm, Knispel & Weber (PhD advisor)* that is still work in progress. All authors contributed equally to this chapter.

- The work presented in Chapter 4 is based on the preprint *Optimal Risk Sharing in Insurance Networks – An Application to Asset-Liability Management* by *Hamm, Knispel & Weber (PhD advisor).* The final version is accepted for publication in *European Actuarial Journal.* All authors contributed equally to this chapter.

Other references on which the presented research is based are discussed in detail in the individual chapters of the thesis.

1 | Stochastic Root Finding for Optimized Certainty Equivalents

The original version of this chapter was previously published in *Proceedings of the 2013 Winter Simulation Conference*, pp. 922-932, 2013, see Hamm, Salfeld & Weber (2013).

Global financial markets require suitable techniques for the quantification of the downside risk of financial positions. The seminal papers of Artzner, Delbaen, Eber & Heath (1999) and Föllmer & Schied (2002) triggered an intensive scientific discussion about sensible risk measures with economically meaningful properties. However, there are not many contributions in the literature on the Monte Carlo simulation and implementation of risk measures – an issue of crucial importance for the implementation in practice.

This chapter introduces a new Monte Carlo simulation technique for an important and broad class of convex risk measures, the optimized certainty equivalents (OCE). This family of risk measures was suggested by Ben-Tal & Teboulle (2007) and includes, among others, the entropic risk measure and average value at risk. Average value at risk, also known as tail value at risk, conditional value at risk, or expected shortfall, plays already an important role in practice and provides, for example, the basis for the Swiss Solvency Test for insurance firms.

The computation of OCE-risk measures involves the solution of a stochastic optimization problem. The key contribution of the present chapter is to realize that by a first-order condition the Monte Carlo estimation of OCE-risk measures can be reduced to a two-step-procedure: the first step consists in solving a stochastic root finding problem, the second step amounts to a standard Monte Carlo simulation of a value function at the argument that was computed in the first step. Our observation allows us to use stochastic root finding techniques for the purpose of risk measurement that could previously only be applied to a very specific class of risk measures, i.e., utility-based shortfall risk, see Dunkel & Weber (2010), and Hamm (2012) for a preliminary extension of their results. This chapter shows that stochastic root finding possesses much higher relevance for financial risk management than would previously have been expected in view of the existing literature.

The chapter is organized as follows: Section 1.1 reviews the (now standard) axioms of risk measures, see Artzner et al. (1999) and Föllmer & Schied (2002). Section 1.2 presents the OCE-risk measures. We review examples (see Ben-Tal & Teboulle (2007)) illustrating that well-known risk measures like the entropic risk measure and the average value at risk are special cases of OCEs, corresponding to an exponential and piecewise linear utility function, respectively. The computation of OCEs requires the solution of a stochastic optimization problem that can be reduced to a stochastic root finding problem under suitable conditions. We present suitable algorithms in Section 1.3. The risk measures discussed in this chapter and the suggested algorithms are illustrated in the context of numerical case studies in Section 1.4. We calculate both the entropic risk measure and the average value at risk, as well as a third OCE that does not correspond to a classical risk measure and is defined in terms of quartic utility. Section 1.5 concludes with a short summary of the main findings.

1.1 | Risk Measures

For the convenience of the reader, the current section reviews standard notions of the theory of static risk measures. For a detailed description, we refer to the excellent book by Föllmer & Schied (2011). We assume that there is one time period characterized by the current time $t = 0$ and the future $t = 1$. Letting $(\Omega, \mathcal{F})$ be a measurable space of possible scenarios, a financial position is modeled by a measurable mapping $X : \Omega \to \mathbb{R}$. The value $X(\omega) \in \mathbb{R}$ for a specific $\omega \in \Omega$ represents the discounted net future value of X if the scenario ω occurs. We fix a family $\mathcal{X}$ of financial positions, assuming that $\mathcal{X}$ is a vector space containing the constants. Our aim is to summarize the risk of any financial position $X \in \mathcal{X}$ by a number $\rho(X) \in \mathbb{R}$.

Definition 1.1.1. A map $\rho : \mathcal{X} \to \mathbb{R}$ is called *monetary risk measure*, if it satisfies the following conditions for $X, Y \in \mathcal{X}$:

(i) *Monotonicity:* If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.

(ii) *Cash-invariance:* $\rho(X + m) = \rho(X) - m$, for all $m \in \mathbb{R}$.

This definition formalizes that (i) risk should increase, if positions become worse, and that (ii) risk is measured on a monetary scale.

Definition 1.1.2. A monetary risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is called *convex risk measure*, if it satisfies the following condition for $X, Y \in \mathcal{X}$ and $\lambda \in [0, 1]$:

$$\textit{Convexity: } \rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y).$$

Convexity formalizes that diversification of financial positions does not increase risk measurements.

Definition 1.1.3. A convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is called *coherent risk measure*, if it satisfies the following condition for $X \in \mathcal{X}$:

$$\textit{Positive homogeneity: } \text{If } \lambda \geq 0, \text{ then } \rho(\lambda X) = \lambda\rho(X).$$

Example 1.1.4. We recall three well-known risk measures.

(i) Value at risk: For $\lambda \in (0,1)$ we define

$$\mathrm{V@R}_\lambda(X) := \inf\{m \in \mathbb{R} \mid P(X+m<0) \leq \lambda\}.$$

The value at risk (V@R) at level λ yields the smallest monetary amount m that needs to be added to the financial position X in order to avoid that the probability of a loss exceeds λ. V@R is generally not convex, but only positively homogeneous.

(ii) Average value at risk: For $\lambda \in (0,1)$ we define the coherent risk measure

$$\mathrm{AV@R}_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda \mathrm{V@R}_\nu(X)\ d\nu,$$

which corresponds, under weak technical conditions, to the conditional expectation of a loss beyond the $\mathrm{V@R}_\lambda(X)$. The average value at risk (AV@R) is also known as conditional value at risk.

(iii) Utility-based shortfall risk: Letting $l : \mathbb{R} \to \mathbb{R}$ be an increasing, non-constant, convex loss function and λ the threshold level, we define the convex risk measure

$$\mathrm{UBSR}_{l,\lambda}(X) := \inf\ \{m \in \mathbb{R} \mid \mathbb{E}[l(-(X+m))] \leq \lambda\}.$$

Utility-based shortfall risk (UBSR) of a financial position X equals the smallest monetary amount m that needs to be added to X in order to avoid that the expected utility $-\mathbb{E}[l(-(X+m))]$ is less than the threshold level $-\lambda$. The special case $l(-X) = e^{-\beta X}$ admits the representation

$$\mathrm{UBSR}_{l,\lambda}(X) = \frac{1}{\beta}\left(\log\left(\mathbb{E}\left[e^{-\beta X}\right]\right) - \log(\lambda)\right),$$

see Appendix A, Lemma A.0.10 (i) for details.

1.2 | Optimized Certainty Equivalents

The current section summarizes important results on OCEs that provide the basis for the algorithms suggested below. For further details, we refer to the original reference Ben-Tal & Teboulle (2007).

In the following, we let $u : \mathbb{R} \to [-\infty, \infty)$ be a concave and non-decreasing utility function satisfying $u(0) = 0$, $u(x) \geq 0\ \forall x \geq 0$, and $u(x) < x\ \forall x$.

Definition 1.2.1. Letting u be a utility function as above and $X \in L^\infty$ be a random variable, the *optimized certainty equivalent (OCE)* is defined by the map $\mathrm{OCE}_u : L^\infty \to \mathbb{R}$ with

$$\mathrm{OCE}_u(X) = \sup_{\eta \in \mathbb{R}}\ \{\eta + \mathbb{E}\left[u(X-\eta)\right]\}. \tag{1.1}$$

As shown by Ben-Tal & Teboulle (2007), the negative of an OCE defines a convex risk measure. This is stated by the following corollary.

Corollary 1.2.2. *Under the assumptions of Definition 1.2.1,*

$$\rho(X) := -OCE_u(X), \quad X \in L^\infty,$$

defines a convex risk measure.

OCE thus yields a simple method for the construction of convex risk measures associated to a utility function u. As demonstrated in Ben-Tal & Teboulle (2007), the optimal η^* that maximizes (1.1) satisfies a first-order condition – provided that u is continuously differentiable and strictly concave. The first-order condition is given by the equation

$$\mathbb{E}\left[u'\left(X-\eta^*\right)\right] = 1. \tag{1.2}$$

This equation immediately leads to a stochastic root finding problem. An alternative approach would consist in a gradient-based simulation optimization procedure, as described in Kim (2006), Hong & Nelson (2009), Fu (2006), Fu (2008), Andradóttir (1998), and Henderson & Nelson (2006). In the current setting, the derivative of the function u that defines the risk measure can be computed directly.

Note that the derivation of Equation (1.2) involves an exchange of the order of differentiation and integration. If $X \in L^\infty$ and u is continuously differentiable and strictly concave, $u'(X-\eta)$ is almost surely bounded from above and below for any $\eta \in \mathbb{R}$ which justifies the first-order condition. In the general case, the assumptions of the differentiation lemma need to be verified, see, e.g., Lemma 16.2. in Bauer (2001).

Example 1.2.3. (i) If we choose a utility function of the form $u(t) = 1 - e^{-\beta t}$, $\beta > 0$, the optimal η^* – computed according to (1.2) – coincides with the negative of utility-based shortfall risk with loss function $l(-X) = e^{-\beta X}$ and threshold level $\lambda = \frac{1}{\beta}$. The risk measure defined in Corollary 1.2.2 is thus given by

$$\rho(X) = \mathrm{UBSR}_{l,\frac{1}{\beta}}(X) - \frac{\beta - 1}{\beta}$$

For $\beta = 1$, this risk measure coincides with the entropic risk measure with parameter 1, i.e.,

$$\rho_1(X) = \log\left(\mathbb{E}\left[e^{-X}\right]\right).$$

We refer to Appendix A, Lemma A.0.10 (ii) for a detailed computation.

(ii) For the utility function $u(t) = \min\{0, \alpha\, t\}$, $\alpha > 1$, the optimal η^* calculated according to (1.2) coincides with the negative of V@R at level $\frac{1}{\alpha}$. To be more precise, it is

$$\eta^* = F_X^{-1}\left(\frac{1}{\alpha}\right)$$

where $F_X^{-1}(\lambda) = q_X(\lambda)$ is the λ-quantile of the random variable X. Hence, according to Example A.0.3 (ii) (see Appendix A), it is $\eta^* = -\text{V@R}_{\frac{1}{\alpha}}(X)$. The risk measure defined in Corollary 1.2.2 is thus given by

$$\rho(X) = \text{AV@R}_{\frac{1}{\alpha}}(X).$$

Again, we refer to Appendix A, Lemma A.0.10 (iii) for a detailed computation.

Monte Carlo Methods. The main contribution of this chapter is a new Monte Carlo method for the computation of OCE-risk measures that are defined according to Corollary 1.2.2. The first-order condition (1.2) allows for the application of stochastic root finding methods. These can be used instead of stochastic optimization procedures that are directly applied to the defining Equation (1.1). We will explain two alternative approaches to compute the OCE-risk measure on the basis of stochastic root finding.

Method 1: According to Equation (1.2) the optimal η^* in (1.1) is the unique root of the function

$$g(\eta) = \mathbb{E}\left[u'\left(X - \eta\right)\right] - 1. \tag{1.3}$$

The root η^* can thus be computed by a stochastic root finding scheme, as explained in the next section. In a second step, the associated OCE and the associated OCE-risk measure can be estimated using a standard Monte Carlo procedure, since

$$\text{OCE}_u(X) = \eta^* + \mathbb{E}\left[u\left(X - \eta^*\right)\right].$$

Method 2: Alternatively, both the optimal η^* in (1.1) and the corresponding OCE could jointly be estimated by solving a 2-dimensional stochastic root finding problem. Denoting $\text{OCE}_u(X) = \xi^*$, the pair (ξ^*, η^*) satisfies

$$\begin{aligned} 0 &= \xi^* - \eta^* - \mathbb{E}\left[u\left(X - \eta^*\right)\right] \\ 0 &= \mathbb{E}\left[u'\left(X - \eta^*\right)\right] - 1 \end{aligned}$$

However, a 2-dimensional root finding algorithm turns out to be less efficient than Method 1. This is not surprising, since the equation characterizing η^* does not depend on ξ^*. It is thus preferable to estimate η^* first, and to compute ξ^* only for an approximately correct value of η^*. Intermediate computations of ξ^* for less precise estimates of η^* do not accelerate the computation of η^* – as it would be the case in a fully coupled system. For this reason, we will concentrate on Method 1 in the following sections.

1.3 | Stochastic Approximation

In this section, we provide a brief survey of stochastic root finding algorithms and explain how these can be applied to the computation of OCEs and the corresponding risk measures.

For further information on stochastic root finding, we refer to Dunkel & Weber (2010) and the references therein. A very detailed description of the topic is provided by Kushner & Yin (2003).

Robbins-Monro Algorithm. Our purpose is to construct an algorithm yielding a sequence of estimators (η_n), $n \in \mathbb{N}$, which converges quickly to the sought root η^* of the function $g(\eta)$ defined in (1.3). The OCE can then be obtained in a second step by a Monte Carlo procedure.

In many situations, stochastic root finding will not be based on naive Monte Carlo procedures, but on sophisticated variance reduction techniques. It is thus advisable to describe the algorithms in a way that allows for variance reduction. This approach is, for example, explained in Dunkel & Weber (2010). Moreover, in practice, random numbers are typically drawn uniformly on the interval $(0, 1)$. We will thus use uniform random variables as the starting point for the simulation scheme.

For practical and mathematical reasons, we design algorithms that are restricted to a bounded domain. Letting $-\infty < a < \eta^* < b < \infty$, we define for $t \in \mathbb{R}$ the projection

$$\pi(t) = \begin{cases} a, & t \leq a, \\ t, & a < t < b, \\ b, & b \leq t. \end{cases}$$

Assume that for any $\eta \in [a, b]$ there exists a known function $Y_\eta : (0, 1) \to \mathbb{R}$ such that for any on $(0, 1)$ uniformly distributed random variable U, we have

$$g(\eta) = \mathbb{E}\left[Y_\eta(U)\right].$$

The Robbins-Monro algorithm for the estimation of η^* is constructed as follows:

1. Choose
   - a constant $\gamma \in \left(\frac{1}{2}, 1\right]$,
   - a constant $c > 0$, and
   - a starting point $\eta_1 \in [a, b]$.
2. Calculate the sequence of estimators recursively:

$$\eta_{n+1} = \pi\left(\eta_n + \frac{c}{n^\gamma} Y_{\eta_n}(U_n)\right), \quad n \in \mathbb{N},$$

   for a sequence (U_n) of independent, unif(0,1)-distributed random variables.

In the case considered in the chapter, a simple choice of Y_η could consist in $Y_\eta(U) = 1 - u'(q_X(U) - \eta)$, where q_X denotes the quantile function of X.

Theorem 1.3.1. *Suppose that the function $g(\eta)$ is well-defined and finite for all $\eta \in [a, b]$. Moreover, assume that $\sup_{\eta \in [a,b]} Var(Y_\eta(U_n)) < \infty$. Then, the sequence $(\eta_n)_n$ converges P-almost surely to the root η^* of function* (1.3).

Besides the consistency of the algorithm, its convergence rate can be further analyzed on the basis of limit theorems for the rescaled quantities $\sqrt{n^\gamma}\,(\eta_n - \eta^*)$, $n \in \mathbb{N}$. For precise conditions, we refer to Dunkel & Weber (2010).

Theorem 1.3.2. *Under suitable conditions, the rescaled quantities exhibit the following asymptotic behavior:*
If $\gamma = 1$, *then*

$$\sqrt{n}\,(\eta_n - \eta^*) \to \mathcal{N}\left(0, \frac{-c^2\,Var\,(Y_{\eta^*}(U))}{2cg'\,(\eta^*) + 1}\right).$$

If $\gamma \in \left(\frac{1}{2}, 1\right)$, *then*

$$\sqrt{n^\gamma}\,(\eta_n - \eta^*) \to \mathcal{N}\left(0, \frac{-c\,Var\,(Y_{\eta^*}(U))}{2g'\,(\eta^*)}\right).$$

1.4 | Numerical Case Studies

The aim of the current section is to illustrate the properties of Method 1 in numerical case studies. We focus on three examples of OCEs: the entropic risk measure, average value at risk, and an OCE defined in terms of quartic utility. In the first two cases, we analyze the example in which X is standard normally distributed. In the third case in which the OCE corresponds to quartic utility, we investigate the impact of the properties of the distribution. We provide results for X being distributed either according to a normal distribution or a Student's t-distribution. Both distributions are assumed to have the same mean and variance. Observe that t-distributions are heavy tailed.

The estimation of the root η^* of (1.3) is compared for different Robbins-Monro algorithms (RM). We generated results from the recursive algorithms for values $n = 100, 300, 1\,000, 3\,000, 10\,000, 30\,000$. For each recursion depth n, we repeated the simulation $N = 10^4$ times in order to obtain an empirical distribution of the RM-estimator of the root. The starting point η_1 of each run was sampled uniformly from the interval $[a, b]$, where a and b are the lower and upper projection-bounds. In the cases of the entropic risk measure and average value at risk, we set $[a, b] := [\eta^* - 5, \eta^* + 5]$; in the case of quartic utility, we choose $[a, b] := [\eta^* - 10, \eta^* + 10]$. In real-world applications η^* is, of course, unknown. Choosing the bounds a and b will in this case require pre-estimates of η^* before running the algorithms. The fact that we assumed that the projection interval is symmetric about η^* is not crucial for our results. The Robbins-Monro algorithm was simulated for the parameters $\gamma = 0.7$ and $\gamma = 1$ setting $c = 1$.

1.4.1 | Entropic Risk Measure

Let u be the exponential utility function $u(t) = 1 - e^{-2t}$. Assuming that X is standard normally distributed, we can compute the optimal η^* exactly using the formulae in Example 1.2.3 (i), i.e., $\eta^* = -\frac{\log(2)}{2} - 1 \approx -1.34657$. This yields $\mathrm{OCE}_u(X) = -\frac{\log(2)}{2} - \frac{1}{2} \approx -0.846574$. Consequently, this example provides a good basis for testing our methods.

Figure 1.1 illustrates the distributions of 10^4 runs of the stochastic root finding algorithm for each recursion depth. The drop shapes in the charts are generated based on a smooth kernel density estimation. The estimated PDFs are mirrored along the η-axis (vertical axis) and scaled in the width. For increasing step size, the distributions are getting more Gaussian like with decreasing variance.

For $n = 100$, the right panel ($\gamma = 1$) of Figure 1.1 shows a bimodal distribution. The same effect can be observed in the other examples presented below. This is due to the fact that the initial root estimates hit the bounds of the simulation interval quite frequently. If γ is large, the correction terms of the iterative scheme decrease faster than for smaller γ. The initial accumulation of mass at the boundaries of the projection interval converges to the sought root as n grows. For $\gamma = 1$, a bimodal distribution remains visible in the histograms.

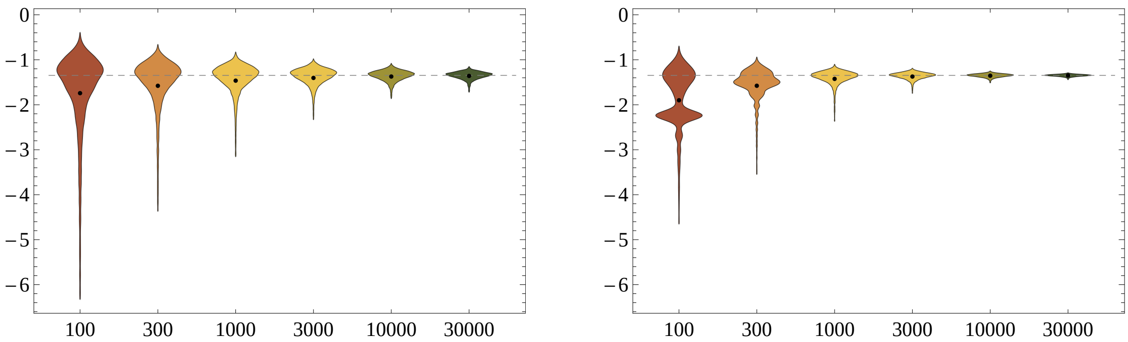


Figure 1.1: Distribution charts for the RM algorithm with recursion depth $n \in \{100, ..., 30\,000\}$ and (left: $\gamma = 0.7$, right: $\gamma = 1.0$) for the exponential utility function. The black dots represent the expectations of the distributions and the gray dashed line marks the exact value of the root.

Table 1.1 shows the mean and the variance of the simulation depending on the number of iterations and the constant γ. The bias decreases in both cases in a qualitatively similar way, but the variance decreases faster for $\gamma = 1$.

	$\gamma = 0.7$		$\gamma = 1$	
iteration steps n	mean	variance	mean	variance
100	-1.8373	0.2214	-1.9000	0.5051
300	-1.7296	0.0946	-1.5774	0.2096
1000	-1.4639	0.2552	-1.4245	0.0798
3000	-1.4032	0.1083	-1.3725	0.0251
10000	-1.3722	0.0445	-1.3543	0.0047
30000	-1.3589	0.0195	-1.3493	0.0017

Table 1.1: Mean and variance of the empirical distribution of η_n depending on the number of iteration steps n and the constant γ.

1.4.2 | Average Value at Risk

Let u be defined according to Example 1.2.3 (ii), i.e., $u(t) = \alpha \min\{0, t\}$. As explained above, we will calculate η^* as the negative of value at risk at level $\frac{1}{\alpha}$ according to the first-order condition. Afterwards, we can simulate the OCE which yields the (negative) average value at risk at level $\frac{1}{\alpha}$. We set $\alpha = 20$.

Figure 1.2 illustrates the distributions of 10^4 runs of the stochastic root finding algorithm for a given iteration size between 100 and 30 000. The figure shows that the variance decreases rapidly and that for larger number of iterations the performance of the algorithm is better for $\gamma = 1$. Mean and variance of the empirical distributions are provided in Table 1.2.

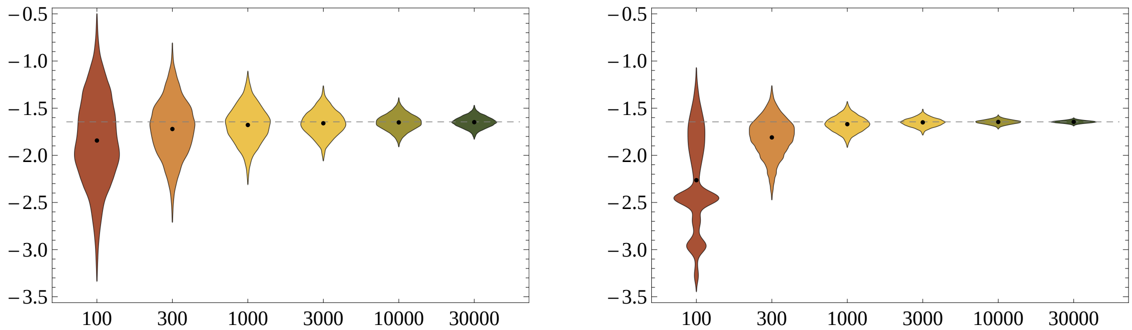


Figure 1.2: Distribution chart for the RM algorithm with recursion depth $n \in \{100, ..., 30\,000\}$ and (left: $\gamma = 0.7$, right: $\gamma = 1.0$) for the piecewise linear utility function. The black dots represent the mean of the distributions and the gray dashed line marks the exact value of the root.

	$\gamma = 0.7$		$\gamma = 1$	
iteration steps n	mean	variance	mean	variance
100	-1.8432	0.2219	-2.2636	0.2590
300	-1.7211	0.0927	-1.8101	0.0413
1000	-1.6782	0.0388	-1.6696	0.0066
3000	-1.6601	0.0176	-1.6504	0.0020
10000	-1.6509	0.0073	-1.6462	0.0006
30000	-1.6484	0.0035	-1.6453	0.0002

Table 1.2: Mean and variance of the empirical distribution of η_n depending on the number of iteration steps n and the constant γ.

Figure 1.3 illustrates a path of the calculation of the OCE which is the (negative) AV@R for the piecewise linear utility function. The path was generated by simple Monte Carlo simulation.

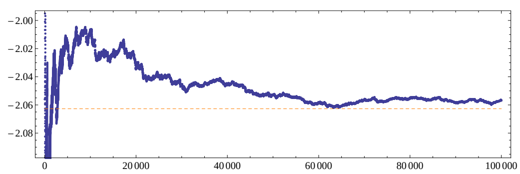


Figure 1.3: A path of a Monte Carlo estimation of the OCE for the piecewise linear utility function. The dashed orange line represents the exact value.

The precision of the Monte Carlo estimation of the OCE or corresponding risk measure does, of course, depend on the precision of the estimate of η^*. For a fixed simulation depth

n, both bias and variance contribute to the error in the estimation of η^*. Figure 1.4 illustrates how bias and variance in the estimation of η^* influence the average estimate of (negative) AV@R in step 2 of the simulation procedure.

The variance of the estimation of η^* can be reduced to zero, if we replace its random estimator by the mean of the empirical distributions shown in Figure 1.2. The error of the resulting estimate of η^* for various recursion depths is then essentially only due to the bias of the stochastic root finding algorithms. The purple dots in Figure 1.4 show the corresponding Monte Carlo estimates for (negative) AV@R, if the mean of the empirical distribution is used as input parameter η^* in step 2 of the computation of the OCE.

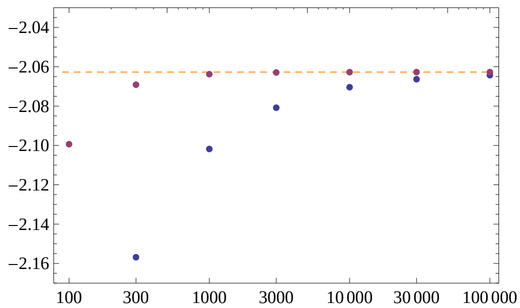


Figure 1.4: Convergence of the Monte Carlo estimation of the OCE for η having the empirical distribution (blue) or being Dirac distributed at the mean of the empirical distributions (purple).

In practice, the estimation of η^* will typically not be repeated several times. A single estimate will be used as the input parameter for the Monte Carlo simulation of the OCE. This corresponds to sampling η^* from the empirical distributions shown in Figure 1.2. The blue dots in Figure 1.4 show the sample average of the corresponding Monte Carlo estimates for (negative) AV@R. As expected, the comparison in Figure 1.4 shows that the bias of the estimator of (negative) AV@R is significantly larger in the second case.

1.4.3 | Quartic Utility

Assume that

$$u(t) := \begin{cases} 1 - (t-1)^4, & \text{if } t \leq 1, \\ 1, & \text{otherwise.} \end{cases}$$

We compare two different distributions for the financial position X. In the first case, we assume that X is normally distributed with variance $\sigma^2 = \frac{5}{3}$; in the second case, we suppose that X has a Student's t-distribution with $\nu = 5$ degrees of freedom. Both random variables have the same mean and variance, but different tail behavior. In both cases, the expression $\eta + \mathbb{E}\left[u\left(X - \eta\right)\right]$ can be written as a closed-form expression of η. This allows us to obtain the exact value of the OCE. We compute that $\eta^* = -2.16359$ and $\mathrm{OCE}_u(X) = -1.6511$ in the first case, i.e., if X is normally distributed. In the second case, i.e., if X is Student's t-distributed, we get $\eta^* = -3.73624$ and $\mathrm{OCE}_u(X) = -5.93075$. As expected, the heavy tailed Student's t-distribution does indeed lead to a higher OCE-risk measure. Detailed computations are provided in Appendix D, Section D.1.

Figures 1.5 & 1.6 illustrate the empirical distributions of the estimates of η^* which are sampled according to RM algorithms. The projection interval was set to $\eta^* \pm 10$. The figures show that for quartic utility and finite sample sizes the estimation errors can be smaller for $\gamma = 0.7$ than for $\gamma = 1$. Finite sample properties might, thus, deviate from the asymptotic behavior suggested by the limit theorems above.

Figure 1.6 illustrates that the algorithm might actually perform quite badly for heavy tailed distributions like the Student's t-distribution. Variance reduction techniques are in this case a crucial tool that needs to be employed in order to secure reasonable simulation results. For a discussion of this insight in a different context, we refer to Dunkel & Weber (2010).

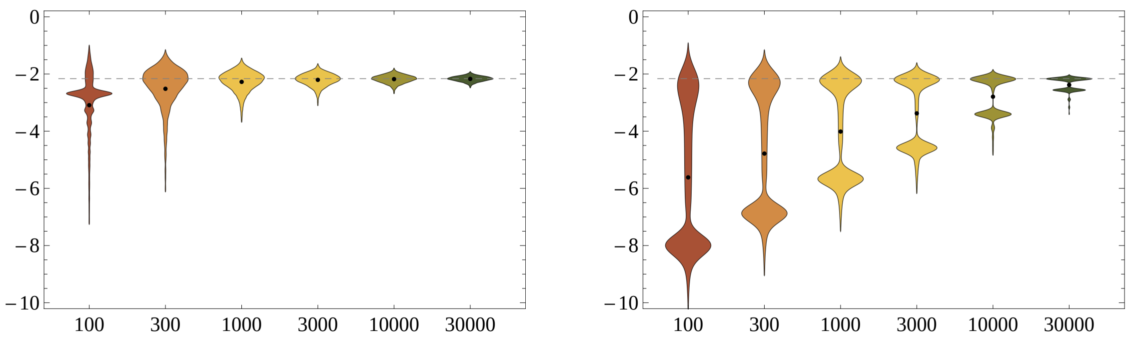


Figure 1.5: Distribution chart for the RM algorithm with recursion depth $n \in \{100, ..., 30\,000\}$ and (left: $\gamma = 0.7$, right: $\gamma = 1.0$) for the piecewise quartic utility function and a normally distributed random variable.

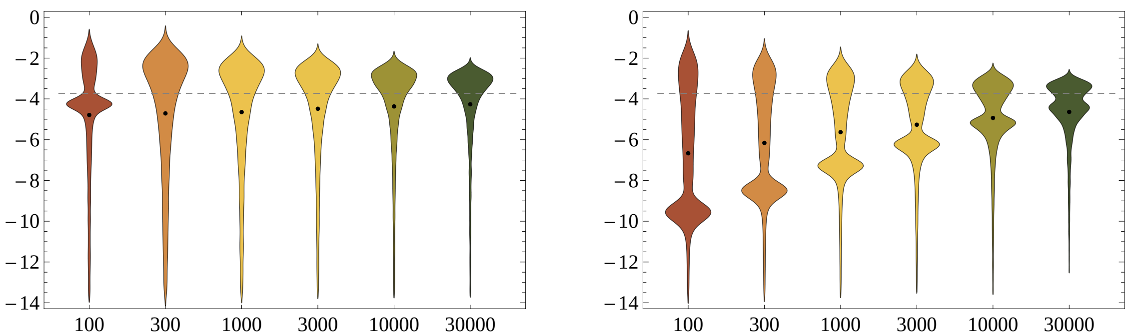


Figure 1.6: Distribution chart for the RM algorithm with recursion depth $n \in \{100, ..., 30\,000\}$ and (left: $\gamma = 0.7$, right: $\gamma = 1.0$) for the piecewise quartic utility function and a Student's t-distribution with $\nu = 5$ degrees of freedom.

1.5 | Conclusion

A large family of OCE-risk measures has been introduced by Ben-Tal & Teboulle (2007) which includes well-known examples like average value at risk and the entropic risk measure. The OCE-risk measures can be computed by solving a stochastic optimization problem. Under weak conditions, by a first-order condition, the calculation can be reduced to a two step procedure: a stochastic root finding problem in the first step, and the computation of an expected value in the second step. In this chapter, we suggest algorithms for the estimation of OCE-risk measures and test them numerically.

The performance of the algorithms depends largely on the properties of the distribution of the financial position whose risk is estimated. The performance of the algorithms is reasonable for distributions with bounded variance and light tails. However, the suggested simulation procedures are usually not optimal, if distributions are heavy tailed. In this case, it seems to be crucial to employ variance reduction techniques in order to improve performance.

The suggested techniques provide promising algorithms for the estimation of a broad class of convex risk measures based on utility theory. An application to liquidity risk is provided in Chapter 2. Future research needs to investigate how the performance of the algorithms can be improved in the case of distributions with high variance or heavy tails and how they can efficiently be applied to real-world examples.

2 | Liquidity-Adjusted Risk Measures

The original version of this chapter was previously published in *Mathematics and Financial Economics* **7**(1), pp. 69-91, 2013, see Weber, Anderson, Hamm, Knispel, Liese & Salfeld (2013).

Liquidity risk played a major role during many crises that have been observed during the last decades. Its impact was clearly apparent in the recent credit crisis (e.g., the failures of *Bear Stearns* and *Lehman Brothers*), and also during the collapse of *Long Term Capital Management* in 1998. Proper financial regulation and risk management requires appropriate concepts that enable the quantification of liquidity risk. Various aspects of liquidity risk have extensively been investigated during recent years, see, e.g., Cetin, Jarrow & Protter (2004), Cetin & Rogers (2007), Jarrow & Protter (2005), Astic & Touzi (2007), and Pennanen & Penner (2010). For further references, we refer to a survey article by Schied & Slynko (2011).

A key instrument to control the risk of financial institutions are suitable *measures of the downside risk*. These constitute an important basis for reporting, regulation, and management strategies. This chapter suggests a *liquidity-adjusted measure of the downside risk*, focusing on two key aspects of liquidity risk: access to financing and price impact of trades. These issues receive particular attention in the context of new regulatory standards, such as *Basel III* and *Solvency II*. The proposed liquidity-adjusted risk measure provides a unified framework beyond the current implementations in practice.

Our approach builds on a recent contribution of Acerbi & Scandolo (2008). While classical asset pricing theory assumes that the value of a portfolio is proportional to the number of its assets, Acerbi & Scandolo (2008) argue that, if access to financing is limited and the size of trades impacts prices, the linearity assumption breaks down; classical valuation should be replaced by liquidity-adjusted valuation.

A situation like this occurs, for example, if a fund or bank needs to execute a large block trade. In this case, the realized average price depends on the liquidity of the market as well as the chosen trading strategy. This phenomenon is called price impact. Its magnitude is affected by the specific structure of supply and demand, or, equivalently, the shape of the order book, if the trades are settled on an exchange. The importance of this type of liquidity risk differs among agents and is governed by the particular situation of the investor: adverse price impact is, of course, only relevant, if a particular trade must indeed be quickly executed. Investors with short-term obligations who are subject to strict budget constraints and have no access to cheap external funding might be forced to engage in fire sales. As a consequence, they might tremendously be hurt by price impact; their liquidity risk is large. In contrast, investors with deep pockets will almost not be affected by steep supply-demand curves. They can hold assets over very long time horizons, sell only a few assets simultaneously, and wait until a good price can be realized.

For the convenience of the reader, we review the approach of Acerbi & Scandolo (2008) in Section 2.1. More specifically, we consider an investor with an asset portfolio in a one-period economy. Limited access to financing is modeled by constraints on borrowing and short selling, or by more general portfolio constraints. At the same time, the investor is faced with temporary short-term obligations which could, for example, be associated with margin calls or withdrawals from customers. In this situation, the investor might be required to liquidate a fraction of her portfolio in order to avoid default. Price impact of orders is explicitly modeled by supply-demand curves. The liquidity-adjusted portfolio value modifies the classical mark-to-market value by accounting for the losses that occur from the forced liquidation of a fraction of the investor's portfolio.

Liquidity-adjusted risk measures can be constructed on the basis of the liquidity-adjusted value. Acerbi & Scandolo (2008) suggested to measure liquidity-adjusted risk by computing a standard monetary risk measure for the liquidity-adjusted value. The resulting liquidity-adjusted risk measure ρ^{AS} is convex, but in general not cash-invariant anymore, and does not possess a natural interpretation as a capital requirement. The key contribution of this chapter is the construction of a new, cash-invariant liquidity-adjusted risk measure ρ^V that can conveniently be interpreted as a capital requirement, see Section 2.2. Our definition endows ρ^V with a clear operational meaning: it equals the smallest monetary amount that needs to be added to a financial portfolio to make it acceptable. At the same time, ρ^V provides a rationale for convex cash-invariant risk measures, if price impact is important.

Section 2.2.2 further clarifies the difference between our construction ρ^V and the risk measure ρ^{AS}. For this purpose, we employ the theoretical framework of capital requirements and *eligible assets*, as introduced by Artzner et al. (2009). Section 2.3 illustrates in the context of numerical case studies how price impact and limited access to financing influence the liquidity-adjusted risk measurements.[1] Section 2.4 concludes with a short summary of our main findings and suggestions for future research.

2.1 | Liquidity Risk and Portfolio Values

For convenience, the current section recalls the deterministic notion of liquidity-adjusted portfolio value, a concept that was originally proposed by Acerbi & Scandolo (2008). We slightly modify their definition by introducing liquidity constraints that are easily interpretable and by imposing additional portfolio constraints.

[1] For a preliminary analysis on the estimation of liquidity risk, see Hamm (2012).

2.1.1 | Maximal Mark-to-Market and Liquidation Values

The price of an asset depends on the quantity that is traded. Following, e.g., Cetin et al. (2004) and Jarrow & Protter (2005), we capture this fact by supply-demand curves.

Definition 2.1.1 (Marginal supply-demand curve, best bid, best ask)**.**

(i) Setting $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$, a function $m : \mathbb{R}_* \to \mathbb{R}_+$ is called a *marginal supply-demand curve (MSDC)*, if m is decreasing. We denote by $\mathcal{M}$ the convex cone of all MSDCs.

(ii) The numbers $m^+ := m(0+)$ and $m^- := m(0-)$ are called the *best bid* and *best ask*, respectively. Their difference $\Delta m := m^- - m^+ \geq 0$ corresponds to the *bid-ask spread.*

An MSDC m models the current prices of a financial asset or, equivalently, the state of its 'order book' – capturing the dependence of prices on the actual quantities that are traded. If large amounts of an asset are sold, the average price of one unit of the asset will typically be smaller than for small amounts. Conversely, if large amounts are bought, the average price will typically be larger than for small amounts of the asset.

For any number of assets $x \in \mathbb{R}_*$, the price of an infinitesimal additional amount is captured by the marginal price $m(x)$. As a consequence, an investor selling $s \in \mathbb{R}_+$ units of the asset will receive the proceeds

$$\int_0^s m(x)\, dx.$$

Conversely, if the investor buys $|s| \in \mathbb{R}_+$ units of the asset, she will pay $\int_{-|s|}^0 m(x)\, dx$, thus 'receive' $\int_0^s m(x)\, dx \leq 0$.

We endow the convex cone $\mathcal{M}$ of all MSDCs with a canonical metric for which two MSDCs are close to each other if the corresponding proceeds are close to each other for any number of assets.[2]

A financial market of multiple assets is characterized by a collection of MSDCs.

Definition 2.1.2 (Spot market, portfolio)**.**

(i) A *spot market* of N risky assets ($N \in \mathbb{N}$) is a vector

$$\bar{m} = (m_0, m_1, \dots, m_N) \in \mathcal{M}^{N+1}.$$

We will always assume that asset 0 corresponds to cash and set $m_0 \equiv 1$.

[2] The convex cone $\mathcal{M}$ can be endowed with the metric

$$d_{\mathcal{M}}(m_1, m_2) := |m_1^- - m_2^-| + |m_1^+ - m_2^+| + \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\int_{-n}^{n} |\hat{m}_1(x) - \hat{m}_2(x)| dx \,\wedge 1\right), \quad m_1, m_2 \in \mathcal{M},$$

where we use the auxiliary function

$$\hat{m}(x) := \begin{cases} m(x) - m^+, & x > 0, \\ 0, & x = 0, \\ m(x) - m^-, & x < 0. \end{cases}$$

In the sequel, all topological properties of $\mathcal{M}$ are based on this metric.

(ii) A *portfolio* in a spot market of N risky assets is a vector

$$\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N) = (\xi_0, \xi) \in \mathbb{R}^{N+1}$$

whose entries specify the number of assets.

Notation 2.1.3. For $k \in \mathbb{R}$, $\bar{\xi} = (\xi_0, \xi) \in \mathbb{R}^{N+1}$, we write $k + \bar{\xi} := (\xi_0 + k, \xi)$.

In practice, portfolios are frequently marked-to-market at the best ask and best bid. We will call this value the *maximal mark-to-market*. The maximal mark-to-market is a hypothetical value of a portfolio that cannot always be realized in practice. In fact, unless there are no liquidity effects, the mark-to-market value typically differs from the *liquidation value*, i.e., from the income or cost of an immediate liquidation of the portfolio. The liquidation value does not only depend on the best bid and best ask, but on the whole supply-demand curve.

Definition 2.1.4 (Def. 4.6 & 4.7 in Acerbi & Scandolo (2008))**.** Let $\bar{\xi} \in \mathbb{R}^{N+1}$ be a portfolio in a spot market of N risky assets.

(i) The *liquidation value* of $\bar{\xi}$ is given by

$$L(\bar{\xi}, \bar{m}) \ := \ \sum_{i=0}^{N} \int_0^{\xi_i} m_i(x)\ dx \ = \ \xi_0 + \sum_{i=1}^{N} \int_0^{\xi_i} m_i(x)\ dx.$$

(ii) The *maximal mark-to-market value* of a portfolio $\bar{\xi}$ is given by

$$U(\bar{\xi}, \bar{m}) \ := \ \xi_0 + \sum_{i=1}^{N} m_i^{\pm}(\xi_i) \cdot \xi_i,$$

where $m_i^{\pm}(\xi_i) = \begin{cases} m_i^+, & \text{if } \xi_i \geq 0, \\ m_i^-, & \text{if } \xi_i < 0. \end{cases}$

Remark 2.1.5. Acerbi & Scandolo (2008) call the function U the *uppermost mark-to-market value.*

The following remark summarizes useful properties of L and U.

Remark 2.1.6 (Properties, see Section 4 in Acerbi & Scandolo (2008))**.**

(i) L and U are continuous on $\mathbb{R}^{N+1} \times \mathcal{M}^{N+1}$.

(ii) L and U are concave functions of their first argument (i.e., the portfolio) that are differentiable on $\mathbb{R} \times \mathbb{R}^N_*$.

(iii) Let $\bar{\xi} \in \mathbb{R}^{N+1}$ be some portfolio and $\bar{m} \in \mathcal{M}^{N+1}$ a spot market.

- If $\lambda \geq 1$, then $L(\lambda\bar{\xi}, \bar{m}) \leq \lambda L(\bar{\xi}, \bar{m})$. If $0 \leq \lambda \leq 1$, then $L(\lambda\bar{\xi}, \bar{m}) \geq \lambda L(\bar{\xi}, \bar{m})$.
- U is positively homogeneous, i.e., for $\lambda \geq 0$, we have that $U(\lambda\bar{\xi}, \bar{m}) = \lambda U(\bar{\xi}, \bar{m})$.

(iv) U and L are fully decomposable.[3]

(v) $\forall k \in \mathbb{R}$: $U(k+\bar{\xi},\bar{m}) = k + U(\bar{\xi},\bar{m})$, $L(k+\bar{\xi},\bar{m}) = k + L(\bar{\xi},\bar{m})$.

An investor can buy and sell assets and thereby change her portfolio at prevailing market prices. Letting $\bar{\xi} \in \mathbb{R}^{N+1}$ be a portfolio, an investor can liquidate a sub-portfolio $(0,\gamma)$, $\gamma \in \mathbb{R}^N$, changing the cash position of the portfolio by the liquidation value $L((0,\gamma),\bar{m})$ of the sub-portfolio $(0,\gamma)$. Any portfolio which is attainable from $\bar{\xi}$ thus has the form:

$$\left(\xi_0 + \sum_{i=1}^{N} \int_0^{\gamma_i} m_i(x)\ dx, \xi - \gamma\right), \quad \gamma \in \mathbb{R}^N.$$

Definition 2.1.7. We denote by $\mathcal{A}(\bar{\xi},\bar{m})$ the set of all portfolios which are attainable from $\bar{\xi}$ in the spot market $\bar{m}$.

2.1.2 | Liquidity and Portfolio Constraints

Classical portfolio theory assumes that the value of a portfolio does not depend on its owner. The value is a linear function of the number of assets. Acerbi & Scandolo (2008) argue convincingly that this standard approach is not correct if prices depend on the quantities traded and if investors have at the same time limited access to financing:

Investors typically need to fulfill short-term obligations, but cannot always quickly borrow liquidity on financial markets. If short of cash, they need to liquidate a part of their portfolio. The average prices, however, at which investors can sell (or buy) assets depend in the presence of price impact on the quantities that are traded. In this sense, the portfolio value depends on the specific financial situation of the investor as well as on the supply-demand curves of the assets.

In order to model these effects, we will characterize the investor by two different constraints that can be observed in real markets: *liquidity constraints* and *portfolio constraints*. Liquidity constraints signify short-time payments an investor needs to make. Portfolio constraints refer, for example, to borrowing and short selling constraints.

Liquidity Constraints. We consider a one period economy with time points $t = 0, 1$. An owner of an asset portfolio will typically receive certain payments, or is required to fulfill certain financial obligations – including, for example, items like rent payments, maintenance costs, coupons, or margin payments. The total amount of these cash flows will affect the cash position of the investor. For modeling purposes, we will assume that payments occur at the end of the time horizon, i.e., at $t = 1$, and are given by a function of the assets other than cash that the investor holds at time 1.

Definition 2.1.8. The *short-term cash flows* (SCF) are a function $\phi : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ such that $\phi(0_N) = 0$. We will write $\phi \in \mathrm{SCF}$.

[3] A function $f : \mathbb{R}^{N+1} \to \mathbb{R}$ is *fully decomposable*, if $f(x_0, x_1, \ldots, x_N) = \sum_{i=0}^{N} f_i(x_i)$ for functions $f_i : \mathbb{R} \to \mathbb{R}$, $i = 0, 1, \ldots, N$.

An investor is required to be sufficiently liquid at the end of the time horizon $t = 1$: she must own enough cash to cover any obligations due. Otherwise, the investor will default. Typically, an investor has a borrowing constraint that prevents her from obtaining an arbitrarily large amount of cash. The difference between liquid and illiquid portfolios is captured by the following definition.

Definition 2.1.9. Let $\phi \in \mathrm{SCF}$, and $a \in \mathbb{R}$. The set of *liquid portfolios* which are attainable from $\bar{\xi} \in \mathbb{R}^{N+1}$ is defined by

$$\mathcal{L}(\bar{\xi}, \bar{m}, \phi, a) = \left\{ \bar{\eta} \in \mathcal{A}(\bar{\xi}, \bar{m}) : \eta_0 + \phi(\eta) \geq a \right\}.$$

The pair (ϕ, a) is called a *liquidity constraint.*

The number a is typically negative and signifies the maximal amount the investor can borrow. We assume in this case that the investor is prohibited to borrow more than $|a|$. The value $\phi(\eta)$ signifies the short-term cash flows associated with a portfolio $\bar{\eta} \in \mathcal{A}(\bar{\xi}, \bar{m})$. These cash flows plus available cash must exceed a.

Remark 2.1.10. Definition 2.1.9 assumes that short-term cash flows are not directly determined from the original portfolio $\bar{\xi}$, but from liquid portfolios that can be attained from $\bar{\xi}$. Alternatively, one could, of course, assume that short-term cash flows are associated with the original portfolio and modify the theory accordingly. The economic interpretations of these two conceivable alternatives differ slightly:
Our convention essentially assumes that short-term cash flows are due at the beginning of a time period immediately after the investor decides about the composition of the portfolio (that needs to satisfy the constraints). Alternatively, one could assume that short-term cash flows are due at the end of a time period.

Stylized examples of liquidity constraints are proportional margin constraints and convex constraints.

Example 2.1.11. (i) *Proportional margin constraints*: The obligations due to holding the assets other than cash are proportional to the number of assets on which the investor is short, i.e.,

$$\phi(\eta) = -\sum_{i=1}^{N} \alpha_i \cdot \eta_i^-, \quad \alpha_i \geq 0, \quad i = 1, \ldots, N.$$

(ii) *Convex constraints*: $\phi \leq 0$ is a concave function with $\phi(0_N) = 0$. Proportional margin constraints are a special case of convex constraints.

Remark 2.1.12. Acerbi & Scandolo (2008) introduce the concept of a "liquidity policy" which is a convex and closed subset $\mathcal{C} \subseteq \mathbb{R}^{N+1}$ such that

(i) $\bar{\eta} \in \mathcal{C} \quad \Rightarrow \quad \bar{\eta} + b := \bar{\eta} + (b, 0_N) \in \mathcal{C} \;\; \forall b > 0.$

(ii) $\bar{\eta} = (\eta_0, \eta) \in \mathcal{C} \quad \Rightarrow \quad (\eta_0, 0_N) \in \mathcal{C}.$

The constraints in Example 2.1.11 correspond to special cases of liquidity policies. Conversely, if $\inf\{\eta_0 : (\eta_0, \eta) \in \mathcal{C}\} > -\infty$ for all $\eta \in \mathbb{R}^N$, a liquidity policy $\mathcal{C}$ induces a liquidity constraint by setting

$$\begin{aligned} a &= \inf\{\eta_0 : (\eta_0, 0_N) \in \mathcal{C}\}, \\ \phi(\eta) &= -\inf\{\eta_0 : (\eta_0, \eta) \in \mathcal{C}\} + a, \end{aligned}$$

with the usual convention that $\inf \emptyset = \infty$.

Portfolio Constraints. Real investors are also restricted by other constraints that limit the feasibility of trading strategies. These portfolio constraints are typically formulated in terms of a non-empty, closed, convex set $\mathcal{K} \subseteq \mathbb{R}^N$. It is required that $\eta \in \mathcal{K}$ for any admissible portfolio $\bar{\eta} = (\eta_0, \eta)$ at the end of the time horizon, $t = 1$. We suppose that $0_N \in \mathcal{K}$, i.e., holding cash only is acceptable, as long as the borrowing constraint $\eta_0 \geq a$ is satisfied.

Example 2.1.13. (i) *Unconstrained case*: $\mathcal{K} = \mathbb{R}^N$.

(ii) *Constraints on short selling*: $\mathcal{K} = [-q_1, \infty) \times [-q_2, \infty) \times \cdots \times [-q_N, \infty)$ for $q_i \geq 0$, $i = 1, \ldots, N$.

(iii) *Cone constraints*: $\mathcal{K}$ is a non-empty, closed, convex cone in $\mathbb{R}^N$.

2.1.3 | The Value of a Portfolio

An investor who owns a portfolio $\bar{\xi} \in \mathbb{R}^{N+1}$ at time $t = 0$ might need to liquidate a fraction of her portfolio in order to meet the liquidity and portfolio constraints at time $t = 1$: short-term payments need to be made, but borrowing and short selling are typically restricted. The liquidation of assets will, however, typically not occur at the best bid, unless supply-demand curves are horizontal. The supply-demand curve determines the proceeds of any transaction, and both average and marginal prices are functions of the number of assets that are traded. The liquidity-adjusted value that we define in this section takes these issues into account. Our definition of the portfolio value follows conceptually the ideas of Acerbi & Scandolo (2008). Liquidity-adjusted risk measures are, however, defined differently, see Section 2.2.1.

Although often used in practice, the maximal mark-to-market value is an artificial quantity. Measuring the value of a portfolio by its maximal mark-to-market has at least one important disadvantage: liquidity effects are completely neglected. If supply-demand curves were horizontal, then the maximal mark-to-market value could indeed be interpreted as the value of a portfolio. In reality, however, supply-demand curves are typically not horizontal which complicates the situation significantly. When liquidity and portfolio constraints are absent, the maximal mark-to-market value can be interpreted as a market-based approximation to the long-run value of a portfolio. If, however, short-term obligations and portfolio constraints are present – possibly forcing investors to liquidate a fraction of their assets, the maximal mark-to-market value becomes an inadequate approximation of the portfolio value.

The approach that we follow in this chapter requires that the mark-to-market value must only be used as an approximation of the portfolio value if a portfolio satisfies all liquidity and portfolio constraints. If a portfolio does *not* satisfy these constraints, we *require* that a suitable fraction of the portfolio is liquidated, *before* the mark-to-market value is computed. This procedure – originally suggested by Acerbi & Scandolo (2008) – thereby assigns a cost to illiquidity. The value of a portfolio is then given by the maximal mark-to-market value, after a suitable part of the original portfolio has been liquidated. For the formal definition, we consider again an economy with two dates $t = 0, 1$.

Definition 2.1.14. The value of a portfolio $\bar{\xi} \in \mathbb{R}^{N+1}$ under the liquidity constraint (ϕ, a) and the portfolio constraint $\mathcal{K}$ is given by

$$V(\bar{\xi}, \bar{m}) = V(\bar{\xi}, \bar{m}, \phi, a, \mathcal{K}) = \sup\{U(\bar{\eta}, \bar{m}) : \bar{\eta} \in \mathcal{L}(\bar{\xi}, \bar{m}, \phi, a) \cap (\mathbb{R} \times \mathcal{K})\}. \tag{2.1}$$

Remark 2.1.15. If ϕ is concave, the valuation problem amounts to maximizing the concave function $U(\cdot, \bar{m})$ on the convex set of attainable liquid portfolios $\mathcal{L}(\bar{\xi}, \bar{m}, \phi, a) \cap (\mathbb{R} \times \mathcal{K})$. If ϕ models obligations of the investor (which is typically the most interesting case), then ϕ will be non-positive.

Assumption 2.1.16. From now on, we will always assume that the SCF ϕ are concave and non-positive. This will be captured by the following definition.

Definition 2.1.17. We denote by Φ the family of all concave and non-positive functions on $\mathbb{R}^N$. We endow Φ with the uniform distance d_∞, i.e., if $\phi, \psi \in \Phi$, then $d_\infty(\phi, \psi) = \sup_{x \in \mathbb{R}^N} |\phi(x) - \psi(x)|$. The set Φ is called the *family of concave short-term cash flows.*

Proposition 2.1.18. *Suppose that Assumption 2.1.16 holds. Then, the value map V has the following properties:*

(i) The maximal mark-to-market dominates the value map:

$$V(\bar{\xi}, \bar{m}) \leq U(\bar{\xi}, \bar{m}).$$

This implies, in particular, that $V(\bar{\xi}, \bar{m}) < \infty$.

(ii) Suppose that $L(\bar{\xi}, \bar{m}) \geq a$. Then, $V(\bar{\xi}, \bar{m}) > -\infty$ (or, equivalently, $\mathcal{L}(\bar{\xi}, \bar{m}, \phi, a) \cap (\mathbb{R} \times \mathcal{K}) \neq \emptyset$), and $V(\bar{\xi}, \bar{m}) \geq L(\bar{\xi}, \bar{m})$.

(iii) Concavity: For $\alpha \in [0, 1]$ and $\bar{\xi}^1, \bar{\xi}^2 \in \mathbb{R}^{N+1}$, we have

$$V(\alpha\bar{\xi}^1 + (1-\alpha)\bar{\xi}^2, \bar{m}) \geq \alpha V(\bar{\xi}^1, \bar{m}) + (1-\alpha) V(\bar{\xi}^2, \bar{m}).$$

(iv) Translation-supervariance: For all $k \geq 0$ and $\bar{\xi} \in \mathbb{R}^{N+1}$, we have

$$V(\bar{\xi} + k, \bar{m}) \geq V(\bar{\xi}, \bar{m}) + k. \tag{2.2}$$

(v) *Monotonicity: If* $\bar{\xi} \leq \bar{\eta}$, *then* $V(\bar{\xi}, \bar{m}) \leq V(\bar{\eta}, \bar{m})$.

Proof. The proof is given in Section 2.5. □

Remark 2.1.19. Suppose that the portfolio constraint $\mathcal{K}$ can be expressed in terms of r convex functions $\psi_1, \ldots, \psi_r : \mathbb{R}^N \to \mathbb{R}$, i.e.,

$$\eta \in \mathcal{K} \qquad \Longleftrightarrow \qquad \psi_1(\eta) \leq 0, \ldots, \psi_r(\eta) \leq 0.$$

This condition is obviously satisfied for the cases presented in Example 2.1.13. In this situation, the portfolio value (2.1) can be characterized via Lagrange multipliers. Indeed, a portfolio $\bar{\eta}$ is attainable from $\bar{\xi}$, if

$$\eta_0 - \xi_0 - \sum_{i=1}^{N} \int_0^{\xi_i - \eta_i} m_i(x)\ dx = 0.$$

This constraint can be replaced by an inequality constraint that does not affect the value in (2.1). The objective is thus to determine the supremum of $U(\bar{\eta}, \bar{m})$ for varying $\bar{\eta}$ under the following $r + 2$ inequality constraints:

1. *Attainability*: $\nu_1(\bar{\eta}) := \eta_0 - \xi_0 - \sum_{i=1}^{N} \int_0^{\xi_i - \eta_i} m_i(x)\ dx \leq 0$.
2. *Liquidity constraint*: $\nu_2(\bar{\eta}) := a - \eta_0 - \phi(\eta) \leq 0$.
3. *Portfolio constraints*: $\psi_1(\eta) \leq 0, \ldots, \psi_r(\eta) \leq 0$.

This is a standard optimization problem, see, e.g., Section 28 in Rockafellar (1970). If the short-term cash flows and the portfolio constraints are fully decomposable, the optimization problem can be reduced to $N + 1$ one-dimensional unconstrained optimization problems and the determination of a Karush-Kuhn-Tucker vector. The assumption of decomposability greatly simplifies the analysis and is not too unrealistic to capture examples in practice.

2.2 | Liquidity Risk and Risk Measures

Definition (2.1) of the liquidity-adjusted value of a portfolio does not yet involve any randomness. So far, the portfolio value is a deterministic function of both the deterministic supply-demand curves and the deterministic short-term cash flows. We will now assume that at least one of these quantities is not revealed when portfolio risk is measured.

Consider again an economy with two dates $t = 0, 1$. The portfolio $\bar{\xi} \in \mathbb{R}^{N+1}$ whose risk needs to be measured is given at time $t = 0$. Again, liquidity and portfolio constraints are imposed at time 1. These do possibly require that a suitable fraction of the portfolio is liquidated. At time $t = 0$, the constraints are typically not completely known, since supply-demand curves and, possibly, short-term cash flows are random quantities that are not revealed until time $t = 1$. As a consequence, no action needs to be taken by the investor until time 1. At time 1, however, liquidity and portfolio constraints need to be respected, once the realizations of supply-demand curves and short-term cash flows are known. Pathwise the liquidity-adjusted value of the portfolio is then again defined by (2.1), but becomes now a random variable that is measurable with respect to the information that is available at time 1.

The goal of the current section is to define a liquidity-adjusted risk of the portfolio. The sought risk measure will be given as a *capital requirement* in Section 2.2.1, i.e., the smallest monetary amount that needs to be added to the portfolio at time 0 to make it acceptable. This approach ensures that the liquidity-adjusted risk measure is indeed a generalized convex monetary risk measure on the set of portfolios.

Section 2.2.2 illuminates the difference between our approach and the approach of Acerbi & Scandolo (2008) on liquidity-adjusted risk measures within a general framework of risk measures associated to capital requirements, see Artzner et al. (2009). The key concept is the notion of an *eligibile asset* which provides the reference point with respect to which capital requirements are computed.

2.2.1 | Liquidity-Adjusted Risk

Impact of Randomness. Let $(\Omega, \mathcal{F})$ denote a measurable space which models uncertainty, and let P be a given probability measure on $(\Omega, \mathcal{F})$. We will endow the metric spaces (Φ, d_∞) and $(\mathcal{M}, d_\mathcal{M})$ with the corresponding Borel-σ-algebras. Note that Φ and $\mathcal{M}$ are thus Standard-Borel spaces.

Definition 2.2.1 (Random MSDC, random SCF)**.**

(i) A *random marginal supply-demand curve* (MSDC) is a vector $\bar{m} = (1, m_1, \dots, m_N)$ of measurable mappings $m_i : \Omega \to \mathcal{M}$, $i = 1, \dots, N$. The vector $\bar{m} = (1, m_1, \dots, m_N)$ of random MSDCs corresponds to a *random spot market* of N risky assets.

(ii) A *random short-term cash flow* (SCF) is a measurable mapping $\phi : \Omega \to \Phi$.

If MSDCs and SCFs are random, then the liquidity-adjusted value of a portfolio $\bar{\xi} \in \mathbb{R}^{N+1}$ as defined in (2.1) is a random variable that needs to be computed for almost all scenarios $\omega \in \Omega$.

Definition 2.2.2. Let $\bar{m}$ be a random MSDC, ϕ a random SCF, $a \in \mathbb{R}$, and $\mathcal{K}$ a portfolio constraint. The *random (liquidity-adjusted) value* of the portfolio $\bar{\xi} \in \mathbb{R}^{N+1}$ is defined by

$$\Omega \to \mathbb{R} \cup \{-\infty\}, \quad \omega \mapsto V(\bar{\xi}, \bar{m}(\omega), \phi(\omega), a, \mathcal{K}).$$

We will sometimes simply write $V(\bar{\xi})$ for the random value of $\bar{\xi}$.

Our goal is to measure the risk of a portfolio $\bar{\xi}$ in terms of random values of cash-adjusted portfolios. The following assumption ensures that $L(\bar{\xi}, \bar{m}), U(\bar{\xi}, \bar{m})$ belong to $L^\infty := L^\infty(\Omega, \mathcal{F}, P)$ for all $\bar{\xi} \in \mathbb{R}^{N+1}$.

Assumption 2.2.3. For all $i = 1, \dots, N$ and $x \in \mathbb{R}_*$: $m_i(x) \in L^\infty$.

Liquidity-Adjusted Risk Measure. For the definition of liquidity-adjusted risk measures, we fix a convex risk measure ρ on L^∞, as described in Section 4.3 in Föllmer & Schied (2011). It is well-known that ρ induces an acceptance set $\mathcal{A}$ from which it can be recovered as a capital requirement, see, e.g., Section 4.1 in Föllmer & Schied (2011).

Acerbi & Scandolo (2008) suggest measuring the liquidity-adjusted risk of a portfolio $\bar{\xi} \in \mathbb{R}^{N+1}$ by

$$\rho^{\mathrm{AS}}(\bar{\xi}) := \rho(V(\bar{\xi})),$$

i.e., by applying a classical risk measure to the liquidity-adjusted value. We propose an alternative definition of a liquidity-adjusted risk measure which is based on the notion of capital requirements. In contrast to the liquidity-adjusted risk measure of Acerbi & Scandolo (2008), our liquidity-adjusted risk measure remains cash-invariant and thus measures risk on a monetary scale.

Definition 2.2.4. The liquidity-adjusted risk of a portfolio $\bar{\xi}$ is defined as

$$\rho^V(\bar{\xi}) := \inf\{k \in \mathbb{R} : V(k + \bar{\xi}) \in \mathcal{A}\}.$$

Definition 2.2.4 defines liquidity-adjusted risk as the smallest monetary amount that has to be added to the portfolio at time 0 such that its liquidity-adjusted random value at time 1 is acceptable for the risk measure ρ. Since the liquidity-adjusted value incorporates price effects as well as the size of short-term cash flows and access to financing, the liquidity-adjusted risk measure will quantify these influences. We will illustrate this in the context of a numerical case study in Section 2.3 and provide comparative statics.

Remark 2.2.5. The mapping ρ^V in Definition 2.2.4 is given by a redisitribution risk measure in the sense of Definition B.1.3 (i) (see Appendix B). Indeed, letting

$$f : \mathbb{R}^{N+1} \to \mathcal{X}(\mathbb{R}), \quad f(\bar{\xi}) = V(\bar{\xi}),$$

the liquidity risk of a portfolio $\bar{\xi}$ can be measured by

$$\tilde{\rho}(f; \bar{\xi}) = \inf\{k \in \mathbb{R} \,|\, V(k + \bar{\xi}) \in \mathcal{A}\},$$

see Example B.2.2, and thus $\rho^V(\bar{\xi}) = \tilde{\rho}(f; \bar{\xi})$.

It is easy to see that for a given risk measure ρ our liquidity-adjusted risk and the one suggested by Acerbi & Scandolo (2008) have the same sign; however, the absolute value of ρ^{AS} is always larger than the one of ρ^V.

Proposition 2.2.6. *If $\bar{\xi} \in \mathbb{R}^{N+1}$ is a portfolio, then $\rho^V(\bar{\xi}) \in \mathbb{R}$. Moreover,*

$$|\rho^V(\bar{\xi})| \leq |\rho^{\mathrm{AS}}(\bar{\xi})|, \tag{2.3}$$

and $\rho^V(\bar{\xi})$ and $\rho^{\mathrm{AS}}(\bar{\xi})$ have the same sign, if $\rho^V(\bar{\xi}) \neq 0$.

Proof. The proof is given in Section 2.5. □

The mapping ρ^V defines a liquidity-adjusted risk measure that is cash-invariant and that can be interpreted as a capital requirement.

Theorem 2.2.7. *The mapping $\rho^V : \mathbb{R}^{N+1} \to \mathbb{R}$ is inverse monotone and convex as well as cash-invariant in the following sense:*

$$\rho^V(\bar{\xi} + k) = \rho^V(\bar{\xi}) - k \quad \textit{for all } k \in \mathbb{R}.$$

The acceptance set

$$\mathcal{A}^V := \{\bar{\xi} \in \mathbb{R}^{N+1} : V(\bar{\xi}) \in \mathcal{A}\}$$

is convex, and ρ^V can be recovered from $\mathcal{A}^V$ as a capital requirement:

$$\rho^V(\bar{\xi}) = \inf\{k \in \mathbb{R} : \bar{\xi} + k \in \mathcal{A}^V\}.$$

The mapping ρ^V is called liquidity-adjusted risk measure*.*

Proof. The proof is given in Section 2.5. □

For a numerical implementation of liquidity-adjusted risk measures, the following implicit equation is useful.

Theorem 2.2.8. *Let ρ be a convex risk measure that is continuous from above, and assume that P-almost surely $\bar{\eta} \mapsto V(\bar{\eta})$ is continuous on the interior of its essential domain. Suppose that $\bar{\xi}$ is a portfolio such that $\bar{\xi} + \rho^V(\bar{\xi})$ is P-almost surely in the interior of the essential domain of V. Then, the liquidity-adjusted risk $\rho^V(\bar{\xi})$ is equal to the unique solution $k \in \mathbb{R}$ of the equation*

$$0 = \rho(V(\bar{\xi} + k)). \tag{2.4}$$

Proof. The proof is given in Section 2.5 □

2.2.2 | Liquidity-Adjusted Risk Measures and Capital Requirements

The definitions of our liquidity-adjusted risk measure and the one introduced by Acerbi & Scandolo (2008) can be embedded into a conceptual framework that was provided by Artzner et al. (2009). Their paper describes the process of measuring risk as follows:

> Measuring the risk of a portfolio of assets and liabilities by determining the minimum amount of capital that needs to be added to the portfolio to make the future value "acceptable" has now become a standard in the financial service industry. [...]
>
> [This approach requires to specify] a traded asset in which the supporting capital may be invested (the "eligible asset" [...]). [...]
>
> The minimum required capital will of course depend on the definition of acceptability, but also on the choice of the eligible asset.

The notion of capital requirements and *eligible assets* facilitates a comparison between ρ^V and ρ^{AS}. As suggested in Artzner et al. (2009), we specify a set of acceptable positions at the end of the time horizon and then compare different eligible assets.[4] For a given eligible asset, the implied liquidity-adjusted risk measure is the smallest number of shares of the eligible asset that need to be added to the portfolio to make its liquidity-adjusted value acceptable.

Suppose that we are in the situation described in the previous Section 2.2.1. For the purpose of formally defining liquidity-adjusted capital requirements relative to an eligible asset, we enlarge the initial spot market $\bar{m}$ by one further asset $e \in \mathcal{M}$ to be interpreted as the eligible asset. The extended spot market is thus given by the vector $\tilde{m} = (\bar{m}, e) \in \mathcal{M}^{N+2}$, an extended portfolio is a vector $\tilde{\xi} = (\bar{\xi}, k) \in \mathbb{R}^{N+2}$. We do not impose any further restrictions on the eligible asset e, i.e., we allow, for example, redundancy if $e \equiv 1$.

Risk of an original portfolio $\bar{\xi}$ is measured in units of the eligible asset. For this reason, the eligible asset should not distort the original liquidity constraints. This can be formalized by assuming that the short-term cash flows in the extended market are given by

$$\tilde{\phi}(\omega)((\eta, k)) := \phi(\omega)(\eta) \quad (\omega \in \Omega, \eta \in \mathbb{R}^N, k \in \mathbb{R}).$$

Objects associated with the extended market will be marked by a tilde, i.e., $\tilde{m}$, $\tilde{L}$, $\tilde{U}$, $\tilde{\mathcal{L}}$, $\tilde{\phi}$, $\tilde{\mathcal{K}}$, $\tilde{V}$, in order to differentiate between the original and the extended market.

Definition 2.2.9. Liquidity-adjusted risk of a portfolio $\bar{\xi}$ *relative to the eligible asset e* is defined as the smallest number of assets that need to be added to a portfolio to make its liquidity-adjusted value acceptable, i.e.,

$$\rho^e(\bar{\xi}) := \inf\{k \in \mathbb{R} : \tilde{V}((\bar{\xi}, k)) \in \mathcal{A}\}. \tag{2.5}$$

The following proposition is an immediate consequence of the definitions and an appropriately modified proof of Theorem 2.2.7.

Proposition 2.2.10. *The mapping $\rho^e : \mathbb{R}^{N+1} \to \mathbb{R}$ has the following properties:*

- Inverse monotonicity*: If $\bar{\xi} \leq \bar{\eta}$, then $\rho^e(\bar{\xi}) \geq \rho^e(\bar{\eta})$.*
- Convexity: *For all $\alpha \in [0, 1]$ and $\bar{\xi}, \bar{\eta} \in \mathbb{R}^{N+1}$, we have*

$$\rho^e(\alpha\bar{\xi} + (1-\alpha)\bar{\eta}) \leq \alpha\rho^e(\bar{\xi}) + (1-\alpha)\rho^e(\bar{\eta}).$$

Both liquidity-adjusted risk measures ρ^V and ρ^{AS} can be recovered from ρ^e for suitably chosen e:

(i) ρ^V corresponds to the special case $e \equiv 1$ and the choice $\tilde{\mathcal{K}} = \mathcal{K} \times \mathbb{R}_+$. This is apparent from the definitions. ρ^V can thus be interpreted as the smallest monetary amount

[4] An alternative approach, suggested by Filipović (2008), investigates the effect of a change of numéraire on risk measures. In this case, the acceptance set of nominal final values is dependent on the numéraire. For our comparison between ρ^V and ρ^{AS}, the closely related framework of Artzner et al. (2009) is preferable since it fixes an acceptance set independently of the eligible asset.

that needs to be added to the portfolio at time 0 to make its liquidity-adjusted value acceptable.

(ii) ρ^{AS} is not a special case of ρ^e for an appropriate e, but a limiting case for a suitably chosen sequence of eligible assets.

To state this more precisely, we consider the portfolio constraint $\tilde{\mathcal{K}} = \mathcal{K} \times \mathbb{R}$ and the family

$$e_\varepsilon(x) := \begin{cases} 0, & \text{if } x > \varepsilon, \\ 1, & \text{if } -\varepsilon \leq x \leq \varepsilon, \\ 2, & \text{if } x < -\varepsilon, \end{cases} \qquad (\varepsilon > 0)$$

of "random" MSDCs for the eligible asset. This choice signifies that selling more than ε units of the eligible asset as well as buying more than ε units is suboptimal. In the limiting case $\varepsilon \downarrow 0$, the eligible asset becomes completely illiquid. The limiting case formalizes in the context of Artzner et al. (2009) that ρ^{AS} is the smallest monetary amount that needs to be added to the liquidity-adjusted value at time 1 to make it acceptable.

Proposition 2.2.11. *Suppose that P-almost surely $\bar{\eta} \to V(\bar{\eta})$ is continuous on the interior of its essential domain and that the reference risk measure ρ is continuous from above and below. Let ρ^{e_ε} denote the capital requirement* (2.5) *relative to the eligible asset e_ε. Suppose that $\bar{\xi} \in \mathbb{R}^{N+1}$ is P-almost surely in the interior of the essential domain of V. Then, we have*

$$\rho^{\mathrm{AS}}(\bar{\xi}) = \lim_{\varepsilon \downarrow 0} \rho^{e_\varepsilon}(\bar{\xi}).$$

Proof. The proof is given in Section 2.5. □

Remark 2.2.12. Acerbi & Scandolo (2008) argue that risk measures should be coherent, if no liquidity risk is present. Liquidity-adjusted risk measures should thus be coherent in the extreme case of a spot market in which all MSDCs are horizontal.

Consider again the setup of Section 2.2.1 – now *assuming* that ρ is coherent. The liquidity-adjusted risk measure ρ^V will indeed be coherent, if the MSDCs in the spot market are horizontal. In the case of general MSDCs, ρ^V will be convex and cash-invariant. Example 2.2.13 illustrates that ρ^V will typically not anymore be positively homogeneous, if MSDCs are not horizontal. Conceptually, the risk measure ρ^V provides *a rationale for convex risk measures*, if price impact is important. At the same time, ρ^V measures risk as the minimal cash amount that makes the future value of the position acceptable.

In contrast to ρ^V, the liquidity-adjusted risk measure ρ^{AS} does not preserve cash-invariance. In the extreme case of horizontal MSDCs, ρ^{AS} is also coherent. If MSDCs are not horizontal, the risk measure ρ^{AS} of Acerbi & Scandolo (2008) is convex, but *not* cash-invariant.

Example 2.2.13. Consider a spot market with only one risky asset whose MSDC is given by

$$m_1(x) := \begin{cases} 1 - x, & \text{if } x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Suppose furthermore that there are no portfolio constraints and that the liquidity constraint is given by $\phi(\eta_1) = -|\eta_1|$ and $a = 0$. The coherent risk measure ρ is given by minus the expectation operator.

In this framework, the portfolio $\bar{\xi} = (0, 1)$ satisfies $\rho^V(\bar{\xi}) = -0.5$. If ρ^V would be positively homogeneous, then $\rho^V(2\bar{\xi}) = -1$. However, adding the capital injection of $-0.9 > -1 = 2\rho^V(\bar{\xi})$ to the scaled portfolio $2\bar{\xi}$ yields the portfolio value $V((-0.9, 2)) = -\infty < 0$, hence $\rho^V(2\bar{\xi}) > 2\rho^V(\bar{\xi})$.

2.3 | Numerical Case Studies

Equation (2.4) provides a convenient characterization of liquidity-adjusted risk that we will now exploit for the specific reference risk measure *utility-based shortfall risk*, see Weber (2006), Giesecke, Schmidt & Weber (2008), and Föllmer & Schied (2011). In this setting, we will see how the various ingredients of the framework, such as the MSDC, liquidity and portfolio constraints, affect the liquidity-adjusted risk of a portfolio. For comparison, we will also investigate the risk measures value at risk and average value at risk, see Appendix A for their definitions.

Definition 2.3.1. For a given convex loss function[5] l and an interior point z in the range of l, we define a convex acceptance set by

$$\mathcal{A} := \{X \in L^\infty : \mathbb{E}[l(-X)] \leq z\}.$$

The risk measure $\rho : L^\infty \to \mathbb{R}$ defined by

$$\rho(X) := \inf\{k \in \mathbb{R} : X + k \in \mathcal{A}\}$$

is called *utility-based shortfall risk* (UBSR).

UBSR is a distribution-based convex monetary risk measure which is continuous from above and below, see, e.g., Chapter 4.9 in Föllmer & Schied (2011) for a detailed discussion of basic properties, its robust representation, and its relation to expected utility theory. Moreover, it is easy to check that $y = \rho(X)$ is the unique solution of the equation

$$\mathbb{E}\left[l(-X - y)\right] = z.$$

This implicit characterization reduces the computation of UBSR to a stochastic root finding problem, and it is thus particularly useful for the numerical estimation of the downside risk, cf. Chapter 1. Combined with Theorem 2.2.8, it provides the basis for an efficient algorithm to compute the liquidity-adjusted utility-based shortfall risk UBSR^V.

Corollary 2.3.2. *Assume that P-almost surely $\bar{\eta} \mapsto V(\bar{\eta})$ is continuous on the interior of its essential domain, and let $\bar{\xi}$ be a portfolio such that P-almost surely $\bar{\xi} + UBSR^V(\bar{\xi})$ is*

[5] An increasing, non-constant function $l : \mathbb{R} \to \mathbb{R}$ is called a *loss function*.

in the interior of the essential domain of V. Then, $UBSR^V(\bar{\xi})$ is equal to the unique root $k^* \in \mathbb{R}$ of the function

$$g : \mathbb{R} \to \mathbb{R}, \quad k \mapsto \mathbb{E}\left[l(-V(\bar{\xi}+k)) - z\right].$$

The complexity of the random supply-demand curves and constraints will typically require a numerical evaluation of the value of the function g at a given argument $k \in \mathbb{R}$. To obtain a solution of the root finding problem described above, we use two stochastic approximation algorithms, the Robbins-Monro and the Polyak-Ruppert algorithm. For a short review of the Robbins-Monro algorithm, we refer to Chapter 1, Section 1.3; for a detailed analysis of suitable algorithms the reader is refered to Dunkel & Weber (2010). Liquidity-adjusted value at risk is approximated as the root $k^* \in \mathbb{R}$ of the function

$$g : \mathbb{R} \to \mathbb{R}, \quad k \mapsto \mathbb{E}\left[\mathbb{1}_{\{V(\bar{\xi}+k)<0\}} - \lambda\right], \qquad \lambda \in (0,1),$$

using stochastic root finding techniques. Liquidity-adjusted average value at risk is estimated as a negative optimized certainty equivalent as provided in Chapter 1, Example 1.2.3 (ii).
The numerical results are obtained for the following specifications of our market model.

Portfolio Construction. In order to illustrate the interplay of price effects, limited access to financing, and convex risk measures, we consider a financial market with three assets: cash and two risky assets indexed by $i = 1, 2$. We fix a portfolio $\bar{\xi}$ consisting of zero cash, a short position of three shares of asset $i = 1$, and a long position of four shares of asset $i = 2$, i.e., $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$.

For the purpose of comparative statics, we compare different random supply-demand curves. Specifically, we assume that the financial market of the risky assets ($i = 1, 2$) is characterized by exponential marginal supply-demand curves $m_i(x) = h_i \cdot e^{-bx}$ with $b, h_1, h_2 > 0$. The slope b of the exponent is treated as a model parameter, while h_i, $i = 1, 2$, are modeled as random variables. We compare three values of b: 0.005 (which can essentially be considered as a value of 0), 0.5 and 1. The parameter $b = 0.005$ corresponds to a market with essentially no price impact, $b = 0.5$ corresponds to a medium-size price impact, and $b = 1$ to a large price impact. The stochastic parameters h_i, $i = 1, 2$, have a shifted Beta distribution $h_i - s \sim M \cdot \text{Beta}(2, 4)$. For the portfolio $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$, we choose $(s, M) = (25, 6)$. The parameters (s, M) shift and rescale the original Beta distribution such that the support of h_i, $i = 1, 2$, equals the interval $[25, 31]$. We compare the results for three different dependence structures of the assets $i = 1$ and $i = 2$: comonotonicity, independence, and countermonotonicity.

Limited access to financing becomes particularly important, if the absolute values of negative short-term cash flows are large. We use proportional margin constraints

$$\phi(\xi) = -\alpha \cdot \xi_1^- - \alpha \cdot \xi_2^-$$

for various values of α. The larger α, the larger is the absolute value of the short-term cash flows, and the more important are the liquidity constraints. The parameter a in Definition 2.1.9 is set to -0.6.

The last ingredient of our specification are the portfolio constraints. We fix short selling constraints $\mathcal{K} = [-q_i, \infty)^2$ for $q_i \geq 0$, $i = 1, 2$. The values of parameter q_i are set to 4, which prohibits short selling 4 or more assets.

Liquidity-Adjusted Portfolio Values. The liquidity-adjusted portfolio value is a function of the realizations of the random parameters h_i, $i = 1, 2$. In the case of comonotonicity and countermonotonicity, this function effectively depends only on one parameter, since the realization of h_1 is a monotonic function of the realization of h_2.

Figure 2.1 displays the liquidity-adjusted portfolio value $V(\bar{\xi})$ as a function of $h_1 = h_2 =: h$ in the comonotonic case for the portfolio $(\xi_0, \xi_1, \xi_2) = (0, -3, 4)$. We focus on the parameter values $b = 0.5$ and $\alpha \in \{5, 10, 15, 17\}$. Again, $b = 0.5$ corresponds to medium-size price impact. The size of the short-term cash flows induced by short asset positions increases with α. Increasing α thus leads to lower liquidity-adjusted values of the portfolio. If $\alpha = 17$ – corresponding to particularly high short-term cash flows – the constraints cannot be satisfied anymore for low values of h and a default occurs. In this case, the liquidity-adjusted value equals $-\infty$. In the figure, this discontinuity is emphasized by an orange dot.

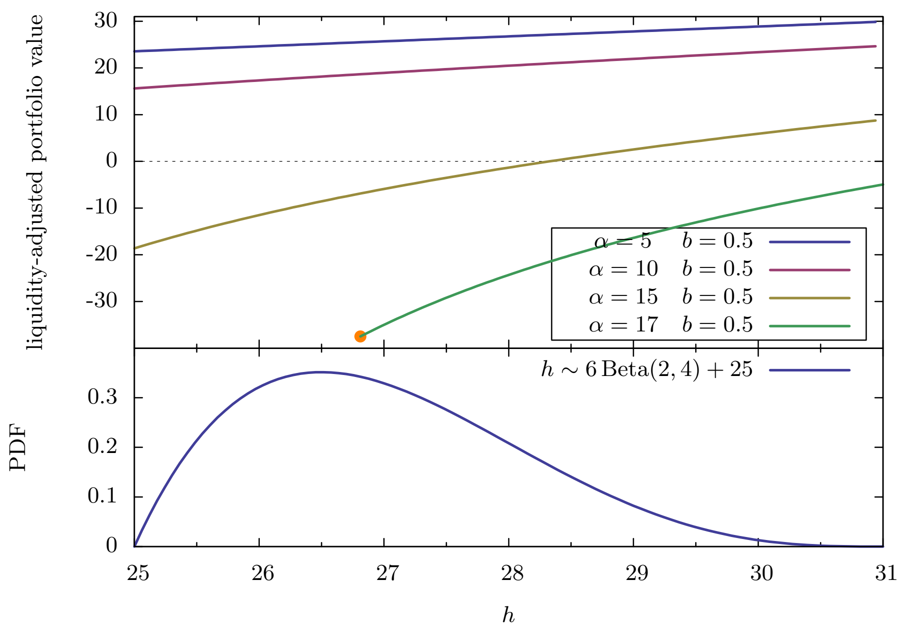


Figure 2.1: Upper part: Liquidity-adjusted portfolio values as a function of the asset price $h = h_1 = h_2$ for the portfolio $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$ with fixed b and varying α. Lower part: PDF of the asset price h, where $h \sim 6 \cdot \text{Beta}(2, 4) + 25$.

Comonotonicity implies that increasing h increases both the prices of asset $i = 1$ and asset $i = 2$. For $h \in [25, 31]$, the marginal price for buying or selling the first infinitesimal unit of assets $i = 1, 2$ is at least 25. Short positions are, however, associated with short-term cash flows of α per unit. If $\alpha \in \{5, 10, 15, 17\}$, it turns out to be suboptimal to reduce the size of the short position in asset $i = 1$ by buying shares when computing the optimal portfolio $\bar{\eta}^*$ in Eq. (2.1), since $\alpha \leq 25$ in these cases. This is confirmed numerically for portfolios with a finite liquidity-adjusted value showing that $-4 < \eta_1^* < -3$. At the same time, we observe that the investor optimally sells more units of asset $i = 2$ than asset $i = 1$. The rational for this phenomenon is that the short position in asset $i = 1$ incurs an additional temporary cost which is caused by the short-term cash flows.

Both assets are optimally sold, and the liquidity-adjusted value of the portfolio increases with h, the multiplicative factor of the supply-demand curves. For smaller values of h, more assets need to be sold in order to satisfy the liquidity constraint. The average price that can be achieved in this case is smaller, because supply-demand curves are downward sloping for $b = 0.5$. At the same time, the obligation associated with the short position becomes relatively more important, since more assets of the long position are optimally liquidated. This explains why the liquidity-adjusted value of the portfolio is a concave function of h in all cases $\alpha \in \{5, 10, 15, 17\}$. For $\alpha = 5$, i.e., a modest liquidity constraint, the optimal portfolio $\bar{\eta}^*$ in Eq. (2.1) is quite insensitive to changes in h. In this case, we have, for example, $\bar{\eta}^* = (-15.9, -3.3, 3.6)$ for $h = 25$ and $\bar{\eta}^* = (-15.6, -3.2, 3.7)$ for $h = 31$. Only a small amount of shares needs to be liquidated. The liquidity-adjusted value is thus an almost linear function of h. The liquidity constraint is stronger for larger α implying that more shares need to be sold. If h becomes smaller, prices decrease and the amount of shares that is sold needs to be increased. Since average prices decrease with the number of shares sold, even more shares need to be sold in order to fulfill the liquidity constraint. The concavity of the liquidity-adjusted value as a function of h is, hence, more pronounced for larger α. For $\alpha = 15$, we obtain, for example, optimal portfolios $\bar{\eta}^* = (55.9, -3.8, 0.8)$ for $h = 25$ and $\bar{\eta}^* = (55.2, -3.7, 2.2)$ for $h = 31$. Optimal portfolios for further values of α and h are provided in Table 2.1.

$\alpha = 5 \quad b = 0.5$				
h	$V(\bar{\xi})$	η_0^*	η_1^*	η_2^*
25	23.55	15.92	-3.30	3.61
26	24.63	15.86	-3.29	3.63
27	25.69	15.80	-3.28	3.64
28	26.76	15.75	-3.27	3.66
29	27.81	15.70	-3.26	3.67
30	28.86	15.66	-3.25	3.69
31	29.91	15.62	-3.24	3.70

$\alpha = 15 \quad b = 0.5$				
h	$V(\bar{\xi})$	η_0^*	η_1^*	η_2^*
25	-18.63	55.95	-3.77	0.78
26	-11.50	55.96	-3.77	1.17
27	-5.92	55.90	-3.76	1.47
28	-1.33	55.78	-3.75	1.71
29	2.54	55.63	-3.74	1.91
30	5.91	55.44	-3.73	2.08
31	8.90	55.24	-3.72	2.22

Table 2.1: Liquidity-adjusted portfolio values $V(\bar{\xi})$ and the corresponding optimal portfolios $(\eta_0^*, \eta_1^*, \eta_2^*)$ as functions of h in the comonotonic case. These results refer to the portfolio $\bar{\xi} = (0, -3, 4)$ and parameters $\alpha \in \{5, 15\}$ and $b = 0.5$.

Figure 2.2 illustrates the liquidity-adjusted portfolio value as a function of the asset price $h = h_1 = h_2$ for fixed $\alpha = 10$ (short-term cash flows) and varying b (price impact). Increasing the price impact b does, of course, decrease the portfolio value. Again, comonotonicity implies that increasing h increases both the prices of asset $i = 1$ and asset $i = 2$. At the same time, it remains suboptimal to reduce the size of the short position in asset

$i = 1$ by buying shares, since also in this case short-term cash flows per share $\alpha = 10$ are smaller than the lower bound 25 for the marginal price of buying or selling the first infinitesimal unit of asset $i = 1$. If $b = 0.005$, there is essentially no price impact and the liquidity-adjusted portfolio value is almost linear in h. If b is increased, the price impact becomes larger. Due to a non-negligible liquidity constraint for $\alpha = 10$, cash is required and shares of the assets need to be sold. Again, if h is smaller, prices decrease and the amount of shares that needs to be sold is increased. Since average prices decrease with the number of shares sold, even more shares need to be sold in order to fulfill the liquidity constraint. The concavity of the liquidity-adjusted value as a function of h becomes more pronounced for larger price impact b. For $b = 1$, we obtain, for example, optimal portfolios $\bar{\eta}^* = (37.8, -3.8, 1.1)$ for $h = 25$ and $\bar{\eta}^* = (35.6, -3.6, 2.8)$ for $h = 31$.

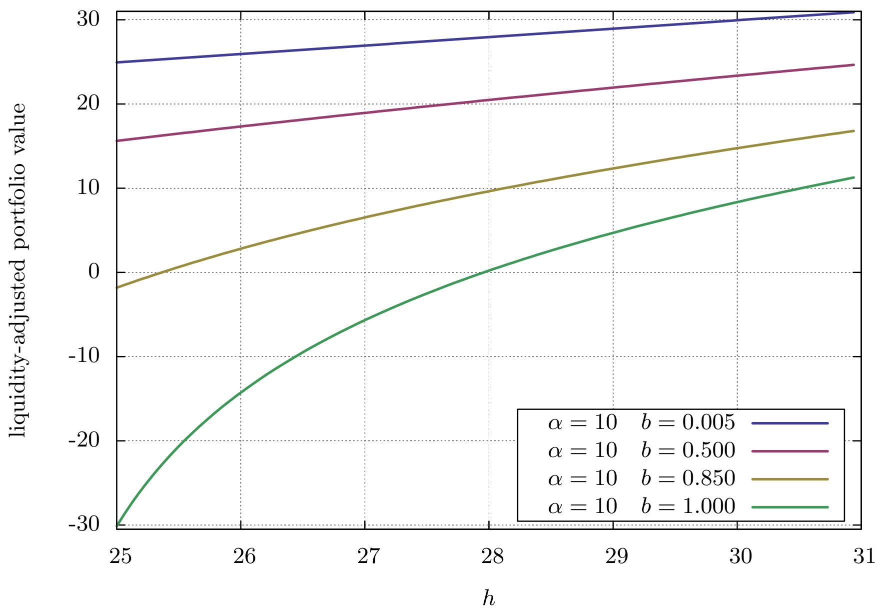


Figure 2.2: Liquidity-adjusted portfolio value as a function of $h = h_1 = h_2$ for the portfolio $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$ with $\alpha = 10$ and varying b.

Table 2.2 below shows the means and variances of the liquidity-adjusted portfolio value for $b \in \{0.005, 0.5, 1\}$ and $\alpha \in \{5, 10, 15, 20\}$. The qualitative behavior of these quantities can already be inferred from Figures 2.1 & 2.2, if the distribution of h is given as displayed in the lower part of Figure 2.1: the mean decreases and the variance increases for increasing α (liquidity constraints) and increasing b (price impact).

In contrast to Figures 2.1 & 2.2, Figure 2.3 displays the liquidity-adjusted portfolio value in the countermonotonic case. In this situation, h_1 and h_2 are decreasing functions of each other. We plot the liquidity-adjusted portfolio value as a function of h_2, the multiplicative factor of the supply-demand curve of long asset $i = 2$. If h_2 increases, prices of asset $i = 2$ increase, while prices of the short asset $i = 1$ decrease. Like Figure 2.1, Figure 2.3 shows the liquidity-adjusted portfolio value for $b = 0.5$ and $\alpha \in \{5, 10, 15, 17\}$. Qualitatively, the findings are very similar in both cases. Again, it is optimal to liquidate shares of both assets. The main difference is that the range of liquidity-adjusted value of the portfolio changes for varying h_2 is significantly larger in the countermonotonic case. For large h_2 (associated with a small h_1), the long position in η_2^* is valuable, but the absolute value of the negative mark-to-market η_1^* is smaller than in the comonotonic case. An analogous argument applies, if h_2 is small. In this case, the long position has a smaller value than for large h_2, while the short position constitutes a larger obligation.

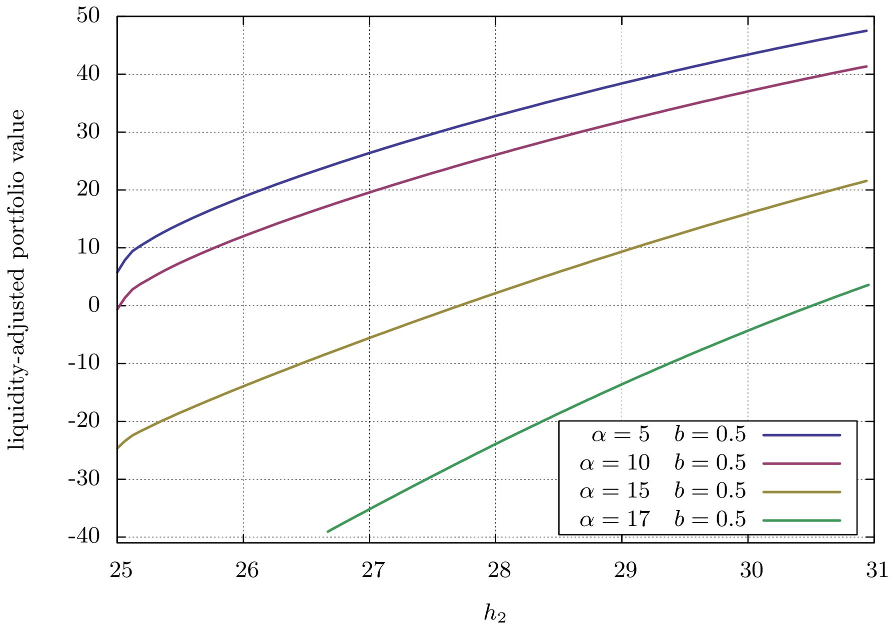


Figure 2.3: Liquidity-adjusted portfolio value $V(\bar{\xi})$ in the countermonotonic case as a function of h_2 for $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$ with fixed b and varying α.

For further comparison, we also consider a different portfolio which contains 45 units of cash, a short position of five shares of asset $i = 1$ and a long position of seven shares of asset $i = 2$, i.e., $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (45, -5, 7)$. Corresponding to comonotonic dependence, we assume that $h = h_1 = h_2$. In contrast to the other example, we suppose that the range of h is given by $[20, 40]$. In this case, the short selling constraint requires the investor to buy at least one share of asset $i = 1$. A similar effect would also occur in other examples, if α was large compared to h_1.

Figure 2.4 displays the liquidity-adjusted portfolio value of the portfolio $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (45, -5, 7)$ as a function of $h = h_1 = h_2$ for fixed α (short-term cash flows) and varying b (price impact). The liquidity-adjusted value does, of course, decrease with increasing price effect b. We observe that the liquidity-adjusted value is not necessarily a monotonically increasing function. For a market with a medium size price effect ($b = 0.5$), the liquidity-adjusted portfolio value still increases with increasing h, while it decreases for large h in the case of large price impacts ($b \in \{0.95, 1\}$). This can be understood by inspecting the optimal portfolios $\bar{\eta}^*$. In all cases, both the liquidity constraint and the short selling constraint are binding. In order to satisfy the short selling constraint, the investor buys exactly one share of asset $i = 1$. The price of this share increases with h and needs to be financed by selling asset $i = 2$. For $b = 0.95$, the position in asset $i = 2$ needs to be reduced from 7 shares to 6.5 shares if $h = 20$, but to 3 shares if $h = 40$. This diminishes the liquidity-adjusted portfolio value.

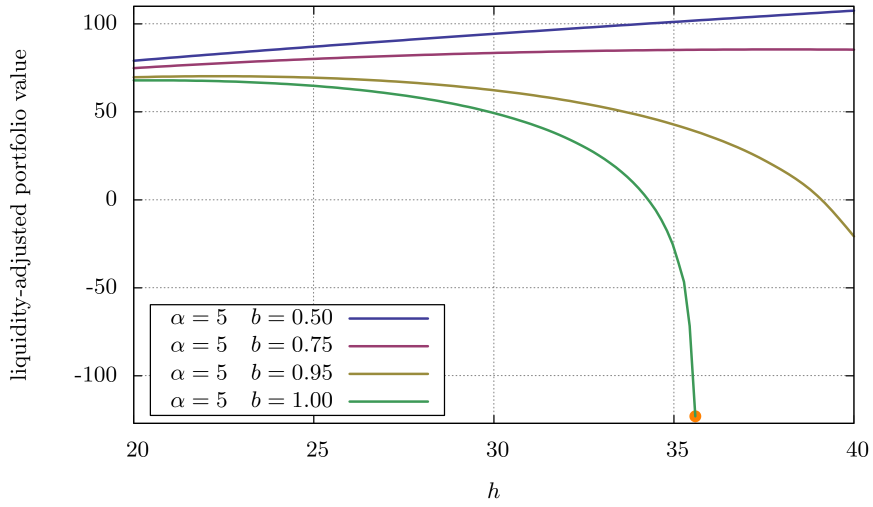


Figure 2.4: Liquidity-adjusted portfolio value as a function of $h = h_1 = h_2$ for the portfolio $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (45, -5, 7)$ with fixed α and varying b.

Liquidity-Adjusted Risk Measures. In this paragraph, we focus again on the portfolio $\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$ and assume that the stochastic parameters h_i, $i = 1, 2$, have a shifted Beta distribution $h_i - 25 \sim 6 \cdot \text{Beta}(2, 4)$. In all cases, we ran $n = 5,000$ independent simulations yielding an empirical distribution of liquidity-adjusted portfolio values or liquidity-adjusted risk measures. The individual samples of the portfolio values are solutions to the optimization problem described in Section 2.1.3. We computed moments of the empirical distributions. We estimated liquidity-adjusted V@R at level 5%, AV@R at level 5% as well as UBSR with an *exponential loss function* $l_{\exp}(x) = \exp(0.5x)$ and with threshold level $z = 0.05$, see Appendix A, Lemma A.0.10 (i). All risk measures were computed both according to the approach of Acerbi & Scandolo (2008) and according to our approach, see Definition 2.2.4; the estimates were labeled by (AS) and (V), respectively. The results are documented in Table 2.2 (h_1 and h_2 comonotonic), Table 2.3 (h_1 and h_2 independent) & Table 2.4 (h_1 and h_2 countermonotonic).

The liquidity-adjusted portfolio values have generally lower variance in the comonotonic case than in the countermonotonic case. The independent case exhibits intermediate values. The mean of the liquidity-adjusted value always decreases with larger short-term cash flows and larger price impact.

Tables 2.2, 2.3 & 2.4 demonstrate that all risk measures detect the increase of liquidity risk as b (price impact) and α (short-term cash flows/liquidity constraints) increase. As proven in Proposition 2.2.6, the absolute value of our liquidity-adjusted risk measure is indeed always smaller than the one suggested by Acerbi & Scandolo (2008). Furthermore, in some cases the liquidity-adjusted risk measure according to Acerbi & Scandolo (2008) becomes infinite, while our liquidity-adjusted risk measure is still finite. The reason is that our risk measure computes the cash amount that needs to be added to the position at time

					b=0.005				
		mean	variance	V@R(V)	V@R(AS)	AV@R(V)	AV@R(AS)	UBSR(V)	UBSR(AS)
α	5	27.0	1.1	-25.4	-25.5	-25.2	-25.3	-20.7	-20.8
	10	27.0	1.1	-25.2	-25.4	-25.1	-25.3	-20.6	-20.7
	15	26.9	1.1	-25.1	-25.3	-24.9	-25.2	-20.4	-20.6
	20	26.7	1.1	-24.8	-25.2	-24.6	-25.0	-20.2	-20.4
					b=0.5				
		mean	variance	V@R(V)	V@R(AS)	AV@R(V)	AV@R(AS)	UBSR(V)	UBSR(AS)
α	5	25.7	1.2	-17.1	-24.1	-17.0	-23.9	-14.5	-19.4
	10	18.9	2.7	-8.3	-16.5	-7.4	-16.2	-6.6	-12.3
	15	-6.4	27.7	3.9	14.9	4.4	16.2	4.7	17.7
	20	$-\infty$	∞	17.9	∞	18.0	∞	54.2	∞
					b=1				
		mean	variance	V@R(V)	V@R(AS)	AV@R(V)	AV@R(AS)	UBSR(V)	UBSR(AS)
α	5	24.0	1.5	-11.1	-22.2	-10.2	-22.0	-9.6	-17.7
	10	-7.0	59.5	2.7	20.8	3.1	23.7	3.1	25.7
	15	$-\infty$	∞	18.5	∞	18.5	∞	18.7	∞
	20	$-\infty$	∞	34.3	∞	34.4	∞	41.7	∞

Table 2.2: Liquidity-adjusted risk measures for the portfolio $\bar{\xi} = (0, -3, 4)$ in the comonotonic case $h = h_1 = h_2$.

					b=0.005				
		mean	variance	V@R(V)	V@R(AS)	AV@R(V)	AV@R(AS)	UBSR(V)	UBSR(AS)
α	5	27.1	28.9	-17.7	-18.4	-16.1	-16.5	-14.5	-14.6
	10	27.0	28.9	-17.3	-18.4	-15.9	-16.5	-14.4	-14.5
	15	26.9	28.9	-17.2	-18.3	-15.6	-16.4	-14.3	-14.4
	20	26.8	28.9	-17.1	-18.1	-15.6	-16.2	-14.1	-14.3
					b=0.5				
		mean	variance	V@R(V)	V@R(AS)	AV@R(V)	AV@R(AS)	UBSR(V)	UBSR(AS)
α	5	25.8	28.9	-12.3	-17.1	-11.0	-15.2	-10.3	-13.3
	10	19.0	30.1	-5.1	-10.4	-4.0	-8.5	-3.7	-6.6
	15	-6.0	51.1	5.2	16.6	6.5	18.1	6.3	19.6
	20	$-\infty$	∞	18.3	∞	30.2	∞	21.1	∞
					b=1				
		mean	variance	V@R(V)	V@R(AS)	AV@R(V)	AV@R(AS)	UBSR(V)	UBSR(AS)
α	5	24.1	29.1	-8.1	-15.4	-7.0	-13.6	-6.8	-11.6
	10	-6.2	64.3	3.2	18.6	4.6	20.6	4.0	23.0
	15	$-\infty$	∞	18.3	∞	20.0	∞	18.7	∞
	20	$-\infty$	∞	34.1	∞	36.7	∞	34.4	∞

Table 2.3: Liquidity-adjusted risk measures for the portfolio $\bar{\xi} = (0, -3, 4)$ in the independent case.

					b=0.005				
		mean	variance	V@R(V)	V@R(AS)	AV@R(V)	AV@R(AS)	UBSR(V)	UBSR(AS)
α	5	26.8	54.2	-14.9	-14.9	-13.2	-13.1	-12.3	-12.3
	10	26.8	54.2	-14.8	-14.9	-13.1	-13.0	-12.2	-12.2
	15	26.7	54.2	-14.7	-14.8	-13.0	-13.0	-12.0	-12.1
	20	26.5	54.3	-14.5	-14.6	-12.8	-12.8	-11.8	-12.0
					b=0.5				
		mean	variance	V@R(V)	V@R(AS)	AV@R(V)	AV@R(AS)	UBSR(V)	UBSR(AS)
α	5	25.5	54.2	-11.2	-13.6	-8.7	-11.8	-12.3	-12.3
	10	18.8	55.0	-4.4	-6.9	-1.8	-5.1	-12.2	-12.2
	15	-6.0	71.2	6.0	18.8	7.6	20.4	7.0	21.2
	20	$-\infty$	∞	18.8	∞	28.0	∞	21.1	∞
					b=1				
		mean	variance	V@R(V)	V@R(AS)	AV@R(V)	AV@R(AS)	UBSR(V)	UBSR(AS)
α	5	23.8	54.3	-7.4	-12.0	-5.1	-10.2	-5.7	-9.3
	10	-5.8	67.8	3.8	17.8	5.2	19.1	4.5	20.1
	15	$-\infty$	∞	18.2	∞	19.5	∞	18.6	∞
	20	$-\infty$	∞	34.0	∞	34.6	∞	34.2	∞

Table 2.4: Liquidity-adjusted risk measures for the portfolio $\bar{\xi} = (0, -3, 4)$ in the countermonotonic case.

0 in order to make it acceptable. The position can thus still be modified such that default is prevented. As explained in Section 2.2.2, the approach of Acerbi & Scandolo (2008) computes the amount of cash that needs to be added to the liquidity-adjusted value at time 1 to make it acceptable. In the event of default, this amount will be infinitely large. The *ex post* inflow of cash cannot prevent a default once it has occurred. See also Appendix B for a discussion of this issue.

Figure 2.5 presents the realized liquidity-adjusted portfolio values in the comonotonic case in a histogram. As an example, we display a market with medium-size price impact $b = 0.5$ and short-term cash flows parameterized by $\alpha = 10$ and $\alpha = 15$, respectively.

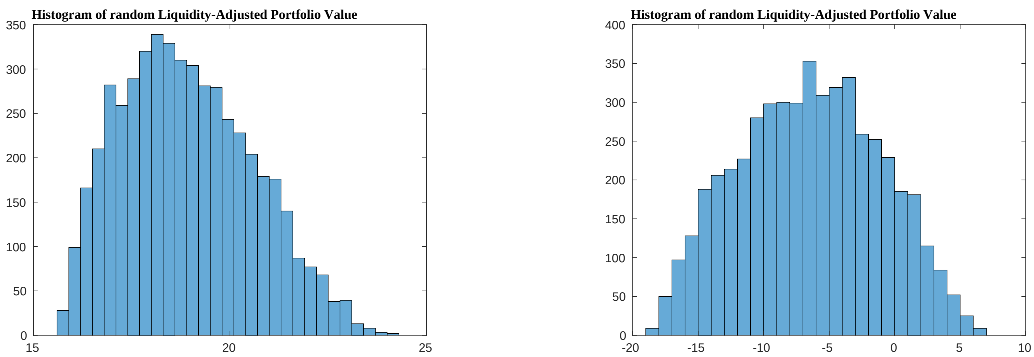


Figure 2.5: Histogram of liquidity-adjusted portfolio value by price impact $b = 0.5$ (left: $\alpha = 10$, right: $\alpha = 15$) and 5,000 simulations in the comonotonic case $h = h_1 = h_2$.

Figures 2.6 & 2.7 show the stochastic root finding procedures for both liquidity-adjusted V@R and liquidity-adjusted UBSR in the same setting, i.e., $b = 0.5$ and $\alpha = 10$, $\alpha = 15$, respectively.

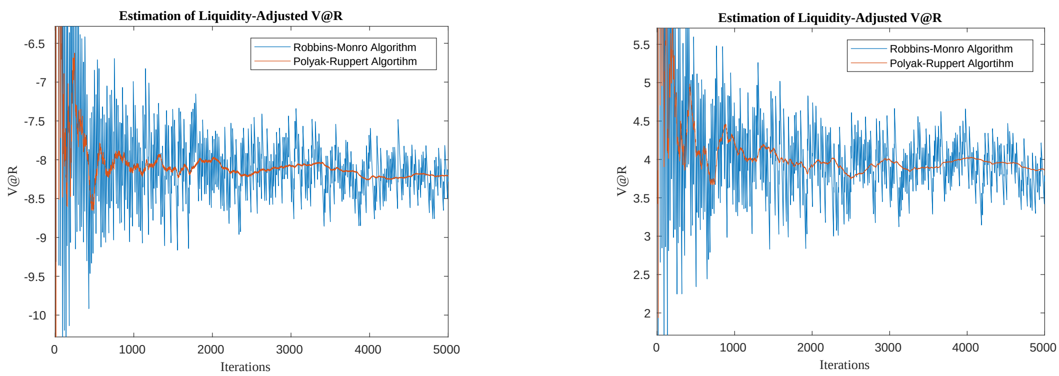


Figure 2.6: Stochastic root finding for liquidity-adjusted V@R by price impact $b = 0.5$ (left: $\alpha = 10$, right: $\alpha = 15$) and 5,000 iterations in the comonotonic case $h = h_1 = h_2$.

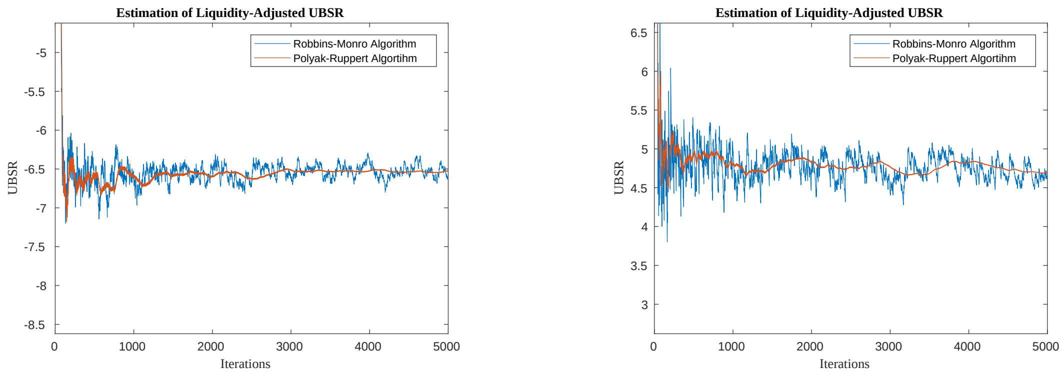


Figure 2.7: Stochastic root finding for liquidity-adjusted UBSR by price impact $b = 0.5$ (left: $\alpha = 10$, right: $\alpha = 15$) and 5,000 iterations in the comonotonic case $h = h_1 = h_2$.

When conducting our numerical experiments, we also noticed that variance reduction techniques become important when computing liquidity-adjusted risk measures if price effects and short-term cash flows are large. Suitable techniques are described in Dunkel & Weber (2007) and Dunkel & Weber (2010). The effective implementation in the context of measuring risk in highly illiquid markets constitutes an interesting direction for future research.

2.4 | Conclusion

We propose liquidity-adjusted risk measures in the context of a static one-period model. Main drivers are two dimensions of liquidity risk, namely price impact of trades and limited access to financing. The suggested cash-invariant risk measures are based on the notion of capital requirements and provide a simple method to properly managing portfolio risk by injecting an appropriate amount of capital upfront. Our analysis is based on the notion of liquidity-adjusted portfolio valuation that was originally developed by Acerbi & Scandolo (2008). The presented numerical case studies apply the stochastic root finding algorithm provided in Chapter 1. Moreover, the proposed liquidity-adjusted risk measure can be defined in terms of a redistribution risk measure as suggested in Appendix B.

Our approach is quite stylized, and it remains an important topic for future research to investigate how liquidity-adjusted valuation and risk measurement can successfully be implemented in practice. In particular, the random supply-demand curves and the liquidity constraints of the model would have to constitute appropriate proxies of reality. This requires the design and detailed analysis of suitable estimation procedures.

Two further topics are important and might be promising for future research. First, a dynamic extension of the current framework could provide a more realistic approach to measuring liquidity-adjusted risk. Second, liquidity-adjusted risk measures might contribute to the theory of portfolio choice. Modified objective functions or constraints that integrate the results of this chapter will lead to different optimal investments which do not ignore the important dimension of liquidity risk anymore.

2.5 | Appendix: Proofs

In this section, we provide the proofs of the results presented in Sections 2.1.3 & 2.2.

Proof of Proposition 2.1.18.

Proof. The proofs of (i), (iii) and (iv) are similar to Proposition 3 and Theorem 1 in Acerbi & Scandolo (2008).

(ii) can be shown as follows: If $L(\bar{\xi},\bar{m}) \geq a$, then $(L(\bar{\xi},\bar{m}), 0_N) \in \mathcal{L}(\bar{\xi},\bar{m},\phi,a) \cap (\mathbb{R}\times\mathcal{K})$, since $0_N \in \mathcal{K}$. Thus, $V(\bar{\xi},\bar{m}) \geq L(\bar{\xi},\bar{m}) \geq a > -\infty$.

In order to verify (v), suppose that $\bar{\xi} \leq \bar{\eta}$ and recall that attainable portfolios $\bar{\mu} \in \mathcal{A}(\bar{\xi},\bar{m})$, $\bar{\nu} \in \mathcal{A}(\bar{\eta},\bar{m})$ take the form

$$\bar{\mu} = \left(\xi_0 + \sum_{i=1}^{N} \int_0^{\alpha_i} m_i(x)\ dx, \xi - \alpha\right), \quad \bar{\nu} = \left(\eta_0 + \sum_{i=1}^{N} \int_0^{\beta_i} m_i(x)\ dx, \eta - \beta\right) \quad (\alpha, \beta \in \mathbb{R}^N).$$

We associate to any $\bar{\mu} \in \mathcal{L}(\bar{\xi},\bar{m},\phi,a) \cap (\mathbb{R}\times\mathcal{K})$ the vector $\bar{\nu}$ corresponding to $\beta = \eta - \xi + \alpha \geq \alpha$, i.e.,

$$\bar{\nu} = \left(\eta_0 + \sum_{i=1}^{N} \int_0^{\eta_i - \xi_i + \alpha_i} m_i(x)\ dx, \xi - \alpha\right).$$

Then, $\bar{\nu}$ belongs to $\mathcal{L}(\bar{\eta},\bar{m},\phi,a) \cap (\mathbb{R}\times\mathcal{K})$, and we have the inequality $U(\bar{\mu},\bar{m}) \leq U(\bar{\nu},\bar{m})$. This implies $V(\bar{\xi},\bar{m}) \leq V(\bar{\eta},\bar{m})$. □

Proof of Proposition 2.2.6.

Proof. Since $L(\bar{\xi},\bar{m}) \in L^\infty$, there exists $k \in \mathbb{R}$ such that both $L(k+\bar{\xi},\bar{m}) = k + L(\bar{\xi},\bar{m}) \geq a$ and $L(k+\bar{\xi},\bar{m}) = k + L(\bar{\xi},\bar{m}) \in \mathcal{A}$. Thus, by Prop. 2.1.18, $L(k+\bar{\xi},\bar{m}) \leq V(k+\bar{\xi}) \leq U(k+\bar{\xi},\bar{m}) \in L^\infty$. Hence, $V(k+\bar{\xi}) \in \mathcal{A}$. This implies that $\rho^V(\bar{\xi}) < \infty$. Moreover, observe that $V(k+\bar{\xi}) \leq U(k+\bar{\xi},\bar{m}) = k + U(\bar{\xi},\bar{m})$. Thus,

$$\rho^V(\bar{\xi}) = \inf\{k : V(k+\bar{\xi}) \in \mathcal{A}\} \geq \inf\{k : U(k+\bar{\xi},\bar{m}) \in \mathcal{A}\} = \rho(U(\bar{\xi},\bar{m})) > -\infty,$$

since $U(\bar{\xi},\bar{m}) \in L^\infty$.

Estimate (2.3) is a consequence of the translation-supervariance (2.2) of $V(\bar{\xi})$. Indeed, if $\rho^V(\bar{\xi}) > 0$, then we have $V(k+\bar{\xi}) \notin \mathcal{A}$ for any fixed $k \in (0, \rho^V(\bar{\xi}))$. Since $V(k+\bar{\xi}) \geq k + V(\bar{\xi})$, this yields $k + V(\bar{\xi}) \notin \mathcal{A}$, hence $\rho^{\mathrm{AS}}(\bar{\xi}) \geq k > 0$. Letting k increase to $\rho^V(\bar{\xi})$, we obtain $\rho^V(\bar{\xi}) \leq \rho^{\mathrm{AS}}(\bar{\xi})$ for $\rho^V(\bar{\xi}) > 0$.

Conversely, $\rho^V(\bar{\xi}) < 0$ implies that $V(k+\bar{\xi}) \in \mathcal{A}$ for any fixed $k \in (\rho^V(\bar{\xi}), 0)$. Here, translation-supervariance yields the estimate $V(k+\bar{\xi}) \leq k + V(\bar{\xi})$. Thus, $k + V(\bar{\xi}) \in \mathcal{A}$, and so we have $\rho^{\mathrm{AS}}(\bar{\xi}) \leq k < 0$. Taking the limit $k \downarrow \rho^V(\bar{\xi})$, this translates into $\rho^{\mathrm{AS}}(\bar{\xi}) \leq \rho^V(\bar{\xi})$ for $\rho^V(\bar{\xi}) < 0$. □

Proof of Theorem 2.2.7.

Proof. Letting $\bar{\xi}, \bar{\eta} \in \mathbb{R}^{N+1}$ and $m \in \mathbb{R}$, we obtain

$$\rho^V(\bar{\xi} + m) = \inf\{k : V(k + \bar{\xi} + m) \in \mathcal{A}\} = \rho^V(\bar{\xi}) - m,$$

which proves the cash-invariance of ρ^V. Suppose now that $\bar{\xi} \leq \bar{\eta}$. Then, $V(k+\bar{\xi}) \leq V(k+\bar{\eta})$ for any $k \in \mathbb{R}$. Thus,

$$V(k + \bar{\xi}) \in \mathcal{A} \quad \Rightarrow \quad V(k + \bar{\eta}) \in \mathcal{A},$$

since $\mathcal{A}$ is the acceptance set of the risk measure ρ. Hence, $\rho^V(\bar{\eta}) \leq \rho^V(\bar{\xi})$.
In order to prove convexity, we fix $\alpha \in [0, 1]$ and $\bar{\xi}, \bar{\eta} \in \mathbb{R}^{N+1}$. For all $k_1, k_2 \in \mathbb{R}$ such that $V(k_1 + \bar{\xi}), V(k_2 + \bar{\eta}) \in \mathcal{A}$, convexity of the acceptance set $\mathcal{A}$ yields that $\alpha V(k_1 + \bar{\xi}) + (1 - \alpha)V(k_2 + \bar{\eta}) \in \mathcal{A}$. Since V is concave by Prop. 2.1.18, we have

$$\alpha V(k_1 + \bar{\xi}) + (1 - \alpha)V(k_2 + \bar{\eta}) \leq V(\alpha(k_1 + \bar{\xi}) + (1 - \alpha)(k_2 + \bar{\eta})),$$

hence

$$V(\alpha k_1 + (1 - \alpha)k_2 + \alpha\bar{\xi} + (1 - \alpha)\bar{\eta}) \in \mathcal{A}.$$

This implies $\alpha k_1 + (1 - \alpha)k_2 \geq \rho^V(\alpha\bar{\xi} + (1 - \alpha)\bar{\eta})$. Taking the limits $k_1 \downarrow \rho^V(\bar{\xi})$ and $k_2 \downarrow \rho^V(\bar{\eta})$, we obtain convexity of ρ^V. □

Proof of Theorem 2.2.8.

Proof. Let $\rho^V(\bar{\xi}) = k$. The cash-invariance of ρ^V implies

$$0 = \rho^V(\bar{\xi}) - k = \rho^V(\bar{\xi} + k) = \inf\{m \in \mathbb{R} : V(\bar{\xi} + k + m) \in \mathcal{A}\}. \tag{2.6}$$

Since V is increasing in the portfolio and $\mathcal{A}$ is an acceptance set, we have $V(\bar{\xi}+k+m) \in \mathcal{A}$ for all $m > 0$. Thus, $\rho(V(\bar{\xi} + k)) = \lim_{m \searrow 0} \rho(V(\bar{\xi} + k + m)) \leq 0$.

Suppose that $\rho(V(\bar{\xi}+k)) < -\varepsilon < 0$ for $\varepsilon > 0$, i.e., $\rho(V(\bar{\xi}+k) - \varepsilon) < 0$ because of cash-invariance. Since V is P-almost surely continuous on the interior of its essential domain and increasing in the portfolio vector, there exists $k' < k$ such that $V(\bar{\xi}+k') \geq V(\bar{\xi}+k) - \varepsilon$. By inverse monotonicity of the risk measure ρ, we get that $\rho(V(\bar{\xi}+k')) \leq \rho(V(\bar{\xi}+k) - \varepsilon) < 0$, i.e., $V(\bar{\xi} + k') \in \mathcal{A}$ – contradicting (2.6). Thus, $\rho(V(\bar{\xi} + k)) = 0$.

Uniqueness of the solution of (2.4) can be shown as follows: Suppose there exist two solutions $k' > k$ to the equation. Letting $k' - k =: c > 0$, we get

$$\rho(V(\bar{\xi} + k')) = \rho(V(\bar{\xi} + k + c)) \leq \rho(V(\bar{\xi} + k) + c) = \rho(V(\bar{\xi} + k)) - c = -c < 0,$$

a contradiction. Here, the inequality follows from the translation-supervariance of V, see Prop. 2.1.18, as well as from the cash-invariance and monotonicity of ρ. □

Proof of Proposition 2.2.11.

Proof. Let $\bar{\xi} \in \mathbb{R}^{N+1}$ be a given portfolio and $k \in \mathbb{R}$ the units of the eligible asset e_ε with $\varepsilon > 0$. We are going to show that

$$V(\bar{\xi}) + k \leq \tilde{V}((\bar{\xi}, k)) \leq V(\bar{\xi} + \varepsilon) + k + \varepsilon \quad \text{for all } k \in \mathbb{R}, \varepsilon > 0, \tag{2.7}$$

where $\tilde{V}$ depends on ε implicitly. Indeed, for any $\bar{\eta} = (\eta_0, \eta) \in \mathcal{L}(\bar{\xi}, \bar{m}, \phi, a) \cap (\mathbb{R} \times \mathcal{K})$, the portfolio

$$\tilde{\mu} := \left(\eta_0 + \int_0^\varepsilon e_\varepsilon(x)\, dx, \eta, k - \varepsilon\right) \in \mathbb{R}^{N+2}$$

is attainable from $(\bar{\xi}, k)$, belongs to $\tilde{\mathcal{L}}((\bar{\xi}, k), (\bar{m}, e_\varepsilon), \tilde{\phi}, a) \cap (\mathbb{R} \times \tilde{\mathcal{K}})$ and satisfies $\tilde{U}(\tilde{\mu}, (\bar{m}, e_\varepsilon)) = U(\bar{\eta}, \bar{m}) + k$. This yields the first inequality $V(\bar{\xi}) + k \leq \tilde{V}((\bar{\xi}, k))$.

In order to verify the second inequality in (2.7), note first that buying more as well as selling more than ε units of the eligible asset decreases $\tilde{U}(\cdot, (\bar{m}, e_\varepsilon))$. For the relevant portfolios $\tilde{\eta} = (\bar{\eta}, \eta_{N+1}) \in \mathcal{L}((\bar{\xi}, k), (\bar{m}, e_\varepsilon), \phi, a) \cap (\mathbb{R} \times \mathcal{K} \times [k - \varepsilon, k + \varepsilon])$, we have $\tilde{U}(\tilde{\eta}, (\bar{m}, e_\varepsilon)) = U(\bar{\eta}, \bar{m}) + \eta_{N+1} \leq U(\bar{\eta}, \bar{m}) + k + \varepsilon$. Note that $\bar{\eta}$ is an element of $\mathcal{L}(\bar{\xi} + \delta, \bar{m}, \phi, a) \cap (\mathbb{R} \times \mathcal{K})$ for some $\delta \in [-\varepsilon, \varepsilon]$. This implies

$$\tilde{V}((\bar{\xi}, k)) \leq V(\bar{\xi} + \delta) + k + \varepsilon \leq V(\bar{\xi} + \varepsilon) + k + \varepsilon,$$

and so we have shown (2.7). The inequality (2.7) translates into

$$\rho(V(\bar{\xi})) \geq \rho^{e_\varepsilon}(\bar{\xi}) \geq \rho(V(\bar{\xi} + \varepsilon)) - \varepsilon.$$

Since ρ is continuous and V is continuous at $\bar{\xi}$, letting ε tend to 0 yields

$$\lim_{\varepsilon \downarrow 0} \rho^{e_\varepsilon}(\bar{\xi}) = \rho(V(\bar{\xi})) = \rho^{\mathrm{AS}}(\bar{\xi}).$$

□

3 | Network Risk, Network Regulation, and Network Optimization

This chapter is based on a working paper that is still work in progress, see Hamm, Knispel & Weber (2019*a*).

Regulation schemes such as Basel III, Solvency II, or the Swiss Solvency Test rely on capital requirements that provide a buffer against potential losses and thereby serve to ensure a financial firm's solvability. In internal models, such solvency capital requirements (SCR) are computed in terms of a risk measure applied to the simulated distribution of the future net asset value (NAV). The required capital depends on the one hand on the underlying risk measure. While the regulation scheme Solvency II for the European insurance sector is based on the risk measure value at risk, the Swiss Solvency Test employs the average value at risk, a coherent risk measure that in contrast to value at risk does not penalize economical meaningful diversification and measures risk in the tail. On the other hand, regulatory guidelines may leave room for interpretation such that different SCR definitions are compliant with regulatory requirements.

In this chapter, we do not limit the computation of capital requirements to an individual firm, but consider a corporate *network* of $n \in \mathbb{N}$ sub-entities where each sub-entity $i = 1, \ldots, n$, has its own balance sheet. The network setting allows for interaction among the entities and, in particular, internal capital transfers are possible. In this case, capital requirements also depend on the aggregation method for regulatory purposes. For example, balance sheets can be consolidated among all entities by summation, and then the network is treated as one legal entity. We propose a unified framework for the regulation of firm networks and their capital requirements. Capital requirements are derived from a set-valued network risk measure, consisting of the following ingredients:

(i) Terminal net asset values E: These are modeled by a random field depending on initial equity $e \in \mathbb{R}^n$ and additional capital $k \in \mathbb{R}^n$. The terminal NAV for each entity is determined with respect to the corporate structure of the network.

(ii) Regulatory aggregation function Λ and acceptance sets $(\mathcal{A}_j)_{j=1,\ldots,m}$: The regulatory aggregation function maps terminal NAVs of the entities to final *regulatory* outcomes that need to be acceptable for regulatory purposes. The acceptance sets of scalar monetary risk measures define criteria for terminal regulatory outcomes to be acceptable.

This setting is flexible and includes a wide variety of regulatory frameworks used in practice. The network risk measure is the set of vectors of additional capital requirements such that terminal regulatory outcomes are acceptable:

$$R^{\Lambda}(E;e) := \{k \in \mathbb{R}^n \mid \Lambda_j(E(e,k)) \in \mathcal{A}_j \, \forall j = 1,\dots,m\}.$$

From a regulator's perspective, any of these allocations can be chosen in order to meet the regulatory requirements. However, from the perspective of the network's management, it is efficient to identify optimal capital allocations, i. e., for example, those allocations that minimize the total cost of capital or those that maximize the overall performance of the network. Management objectives can be formalized by an objective function, and so we face the problem of optimizing the objective function among all acceptable capital allocations.

In practice, terminal net asset values are affected by management strategies, including asset-liability management (ALM) strategies as well as internal capital transfer (ICT) agreements. We thus model these strategies explicitly and identify ALM- and ICT-strategies that are feasible and optimize the objective function of the network's management while respecting regulatory requirements at the same time.

Outline. The chapter is organized as follows: In Section 3.1, we introduce our set-valued framework for the regulation of firm networks based on capital requirements. For this purpose, we first review a notion of solvency capital requirements for individual firms, derived from the natural requirement of acceptability in the theory of monetary risk measures. Second, we propose the concept of a set-valued network risk measure to generalize this approach to network regulation.

In the context of network regulation, the problem of optimal capital allocation is analyzed in Section 3.2. First, Section 3.2.1 defines the notion of an optimal set-valued capital allocation principle, depending on the objective function of the network's management, and provides solutions for a variety of regulatory frameworks. Second, we show in Section 3.2.2 that the classical Euler allocation principle corresponds to a special case of our set-valued setting. Third, Section 3.2.3 illustrates the computation of set-valued capital allocations via a grid search algorithm in numerical examples.

In Section 3.3, we extend our framework by including management strategies that affect the evolution of terminal NAVs. Section 3.3.1 introduces both ALM- and ICT-strategies explicitly and specifies the network's constrained optimization problem. The numerical case studies in Section 3.3.2 analyze the impact of different regulatory systems on the optimal management strategy by computing an optimal investment strategy in the financial market, optimal internal transfer agreements, and an optimal acceptable capital endowment. Overall, our case studies indicate that consolidated balance sheets can be mimicked via optimal management strategies. Section 3.4 concludes with a summary.

Literature. The analysis of risk management in corporate networks under regulatory constraints is an ongoing field of research; see Remark 8 in Weber (2018) for a discussion on the classification of networks vs. groups. Based on the Swiss Solvency Test, Keller (2007) and Luder (2007) provide an overview on insurance group risk management. Both papers are rather of qualitative nature. Filipović & Kupper (2008*b*) provide a sound mathematical framework and propose a bottom up approach based on legal entities and capital transfer agreements. Their approach is an alternative to the traditional two step procedure of, first, computing the aggregate risk and, second, splitting this risk and allocating capital to the entities. In contrast to the chapter at hand, Filipović & Kupper (2008*b*) consider scalar-valued risk measures and a fixed management function. For an application of their model in the context of the Swiss Solvency Test, we refer to Filipović & Kupper (2007), see also Filipović & Kunz (2008) for further case studies.

For group regulation, a framework based on set-valued risk measures - similar to our approach - is proposed in Haier, Molchanov & Schmutz (2016), see, e. g., Jouini, Meddeb & Touzi (2004), Hamel & Heyde (2010), and Hamel, Heyde & Rudloff (2011) for an overview on set-valued risk measures. The authors consider a group as acceptable for regulatory purposes, if there exists an admissible intragroup transfer such that the terminal wealth of each entity becomes acceptable. Additionally required capital can be added to terminal equity only (insensitive case). In contrast, our setting also allows for capital injections to initial equity endowments (sensitive case), and it includes both internal capital transfer agreements and asset-liability management strategies. Our set-valued network risk measure is a generalization of the concept of set-valued systemic risk measures introduced in Feinstein, Rudloff & Weber (2017).

Capital allocation principles based on scalar-valued risk measures as well as their properties have been the subject of intense scientific research. The axiomatic theory of coherent risk measures introduced in Artzner et al. (1999) was, for the first time, applied to insurance by Artzner (1999). Denault (2001) extends the concept of coherent risk measures to a coherent allocation of risk capital to portfolios based on game theory approaches. Tasche (2000), Tasche (2008) and Kalkbrener (2005) discuss the Euler capital allocation in detail. While Tasche (2000) and Tasche (2008) focus on performance measurement, Kalkbrener (2005) provides an axiomatic approach to capital allocations. Buch & Dorfleitner (2008) combine the concepts of coherent risk measures by Artzner et al. (1999) and coherent capital allocations by Denault (2001) with the Euler principle and derive certain axiom pairs that are shown to be equivalent. The Euler allocation can be embedded into our set-valued framework, and so this chapter also contributes to the methods of capital allocation.

To the best of our knowledge, the combination of asset-liability management strategies and internal capital transfer agreements in the context of network optimization under regulatory constraints in a set-valued framework is new to the literature.

3.1 | Network Regulation

Capital requirements provide a buffer against downside risk and serve to ensure a firm's financial solvability. *Solvency capital requirements* (SCR) are thus a key instrument of financial regulation schemes for insurance companies or banks. In this chapter, we focus on firms where the stochastic solvency balance sheet is derived by projections within *internal models*, for example, in the sense of the Solvency II Directive 2009/138/EC for the European insurance sector. In particular, insurance firms computing their SCRs with the Solvency II Standard Formula are not in the scope of this chapter.

In this section, we introduce an abstract framework for the regulation of financial firm networks and for the computation of their SCRs based on a set-valued *network risk measure*. The network risk measure consists of three ingredients: *terminal net asset values*, a *regulatory aggregation function*, and *acceptability criteria related to monetary risk measures*. This setting is flexible and includes a wide variety of regulatory frameworks used in practice.

For the convenience of the reader, we begin with a short review on SCRs for individual firms.

3.1.1 | Regulation of Individual Insurance Firms

To sketch the SCR computation for an individual firm, consider a one period economy with two dates, say $t = 0, 1$. Time 0 is interpreted as today, time 1 as the future time horizon of the regulation scheme. Note that Solvency II and the Swiss Solvency Test rely on a time horizon of one year.

Throughout this chapter, the set of financial positions at time 1 whose risk needs to be assessed is a vector space of random variables on a probability space $(\Omega, \mathcal{F}, P)$ that contains the constants. For example, $\mathcal{X}(\mathbb{R}) = L^0(\mathbb{R})$ is the family of random variables, $\mathcal{X}(\mathbb{R}) = L^\infty(\mathbb{R})$ is the subspace of bounded random variables, or $\mathcal{X}(\mathbb{R}) = L^p(\mathbb{R})$ is the subspace of p-integrable random variables. Analogously, we use later on the notation $\mathcal{X}(\mathbb{R}^n)$, where $\mathcal{X}(\mathbb{R}^n) \subseteq L^0(\mathbb{R}^n)$ is a subspace of the family of n-dimensional random vectors. By sign convention, negative values correspond to debt or losses.

For individual entities, the SCR computation is based on two key components, *stochastic balance sheet projections* capturing the random evolution of the firm's equity over a given time horizon, and a scalar *monetary risk measure* ρ that measures risk on a monetary scale or, equivalently, defines acceptability of financial positions in terms of its acceptance set:

$$\mathcal{A}^\rho := \{X \in \mathcal{X}(\mathbb{R}) \mid \rho(X) \leq 0\}.$$

Standard examples include the *value at risk* (V@R) and *average value at risk* (AV@R), see, e. g., Artzner et al. (1999), Föllmer & Schied (2011), Föllmer & Weber (2015) for detailed reading, or Section 3.5 as well as Appendix A for a short review of key facts and examples on risk measures.

In $t = 0$, assets, liabilities and the net asset value (or book value of equity) are deterministic and denoted by $a, l, e = a - l \in \mathbb{R}$, respectively. In $t = 1$, assets, liabilities and their difference - referred to as *net asset value* (NAV) - are the random variables denoted by $A, L, E = A - L \in \mathcal{X}(\mathbb{R})$, respectively, cf. Table 3.1.

$t = 0$

Assets	**Liabilities**
a	$e = a - l$
	l

$t = 1$

Assets	**Liabilities**
A	$E = A - L$
	L

Table 3.1: Balance sheet of an insurance company for different points in time.

These quantities can be derived from stochastic balance sheet projections within sophisticated internal models that rely extensively on Monte Carlo simulations. We set

$$X := (A - a) - (L - l) = E - e$$

for the random change in wealth of the company over the considered time horizon.

From the perspective of a financial supervisory authority, it is natural to call an insurance firm solvent if the future NAV E is acceptable with respect to a prescribed scalar risk measure ρ and its acceptance set $\mathcal{A}^\rho$, i. e., $E \in \mathcal{A}^\rho$. Note that

$$E \in \mathcal{A}^\rho \quad \Leftrightarrow \quad e + X \in \mathcal{A}^\rho \quad \Leftrightarrow \quad \rho(e + X) \leq 0 \quad \Leftrightarrow \quad \rho(X) = \rho(E - e) \leq e.$$

Defining $\mathrm{SCR} = \rho(X)$, acceptability of E is equivalent to $\mathrm{SCR} \leq e$, i. e., the firm's initial equity is sufficient to cover the solvency capital requirement. This construction can be applied for arbitrary monetary risk measures, and in particular for the industry standard V@R. The following example, taken from Chapter 4 (cf. Example 4.1.1), clarifies the link to Solvency II regulation.

Example 3.1.1. For Solvency II regulation, Recital 64 of the Directive 2009/138/EC states that capital must be sufficient to prevent ruin with probability 99.5% on a one-year time horizon, i. e., $P(E < 0) \leq \alpha$ with $\alpha = 0.005$. This condition is equivalent to $E \in \mathcal{A}^{\mathrm{V@R}_{0.005}}$, where $\mathcal{A}^{\mathrm{V@R}_\alpha} = \{X \in \mathcal{X}(\mathbb{R}) \mid P(X < 0) \leq \alpha\}$ denotes the acceptance set of value at risk. Hence, a canonical SCR definition in the context of Solvency II is

$$\mathrm{SCR} := \mathrm{V@R}_{0.005}(X) = \mathrm{V@R}_{0.005}(E - e) = e + \mathrm{V@R}_{0.005}(E) = e - q_E^+(0.005),$$

where q_E^+ denotes the upper quantile function of E, see Appendix A, Definition A.0.2. Note, however, that, §101(2) of the Directive 2009/138/EC supports the definition in terms of the so-called *mean value at risk* which is widely used in practice.

3.1.2 | Regulation of Firm Networks

Let us now extend these ideas to the regulation of firm networks. For this purpose, we first introduce the new concept of a set-valued network risk measure. In a nutshell, a network risk measure is defined by the set of vectors that specify for each entity within the network the amount of additional required capital such that terminal net asset values generated by the network are acceptable for regulatory purposes. Based on the network risk measure, we describe a solvency condition for networks.

3.1.2.1 | Network Risk Measure

Consider a network, e. g., an insurance network, consisting of $n < \infty$ sub-entities. We denote by $N = \{1, \dots, n\}$ the set of entities, i. e., $i \in N$ represents sub-entity i of the network. Conditional on any initial equity allocation in the network, the sub-entities generate random NAVs (or equity values) at terminal time due to the evolution of their balance sheets. Regulatory aggregations of these terminal NAVs need to be acceptable for regulatory authorities, and if necessary, entities need to add additional capital in order to ensure acceptability. We compute those vectors of additional capital leading to acceptable outcomes by the set-valued network risk measure, defined in terms of the following ingredients:

(i) Terminal net asset values: The joint balance sheets of the entities evolve over the given time horizon, starting with an initial equity allocation $e \in \mathbb{R}^n$ in the network. Given both initial equity allocation and additional required regulatory capital $k \in \mathbb{R}^n$, terminal NAVs are random, and they may depend on asset-liability management strategies. If these are held constant, then the following random field formalizes the dependence on initial equity and required capital:

$$E : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^n), \quad (e, k) \mapsto E(e, k).$$

The random field E is chosen by the network.

We assume that a higher initial equity as well as higher additional capital lead to higher random terminal NAVs of the sub-entities. Thus, we consider random fields that are non-decreasing:

- If $c_i \leq e_i, \forall i \in N, e, c \in \mathbb{R}^n$, then $E_i(c, k) \leq E_i(e, k), \forall i \in N$.
- If $c_i \leq k_i, \forall i \in N, k, c \in \mathbb{R}^n$, then $E_i(e, c) \leq E_i(e, k), \forall i \in N$.

The space of non-decreasing random fields is denoted by $\mathcal{Y}$, i. e., $E \in \mathcal{Y}$.

(ii) Network regulation: The network regulation consists of two more ingredients:

- Regulatory aggregation function: Let $1 \leq m < \infty$ and

$$\Lambda : \mathbb{R}^n \to \mathbb{R}^m, \quad l \mapsto \Lambda(l).$$

The regulatory aggregation function is increasing and maps terminal NAVs of sub-entities to terminal regulatory values that need to be acceptable for regulatory purposes. The function Λ is chosen by the regulator.

– Acceptance criteria: Acceptability is specified by acceptance sets $(\mathcal{A}_j)_{j=1,\dots,m}$ of monetary risk measures. Acceptability and acceptance sets are prescribed by the regulator.

These components allow to define allocations of additional required capital $k \in \mathbb{R}^n$ such that terminal NAVs are acceptable for regulatory purposes. The set of all these capital allocations is called a set-valued *network risk measure*. Our formal definition is a generalization of the systemic risk measure defined in Feinstein et al. (2017).

Definition 3.1.2. Let $\mathcal{P}(\mathbb{R}^n;\mathbb{R}^n_+) := \{A \subseteq \mathbb{R}^n \mid A = A + \mathbb{R}^n_+\}$ be the collection of upper sets with ordering cone $\mathbb{R}^n_+$. Let Λ be a regulatory aggregation function, $E \in \mathcal{Y}$ a random field of terminal NAVs and $e \in \mathbb{R}^n$ an initial equity allocation. We call the function

$$R^\Lambda : \mathcal{Y} \times \mathbb{R}^n \to \mathcal{P}\left(\mathbb{R}^n;\mathbb{R}^n_+\right)$$

a *network risk measure*, if for some acceptance sets $\mathcal{A}_j \subseteq \mathcal{X}(\mathbb{R})$ of scalar monetary risk measures ρ_j, $j = 1,\dots,m$:

$$R^\Lambda\left(E;e\right) := \{k \in \mathbb{R}^n \mid \Lambda_j\left(E(e,k)\right) \in \mathcal{A}_j \, \forall j = 1,\dots,m\}.$$

Let us first summarize examples for terminal NAVs. Note that, in particular, internal capital transfers - defined by the network itself - are determined by the choice of E.

Example 3.1.3. (i) Basically, we distinguish two types of terminal NAVs:

1. **Sensitive case:** In the sensitive case, required capital $k \in \mathbb{R}^n$ is added to the *initial* equity $e \in \mathbb{R}^n$. Thus, the evolution of balance sheets does not only depend on the initial equity, but also on required capital added in $t = 0$. Terminal NAVs are given by

$$E : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^n), \quad E(e,k) = E(e+k,\eta),$$

where η is the neutral element with respect to the conjunction of the arguments. In the sequel, we focus on terminal NAVs, where initial equity e and additional capital k are combined additively only. In this case, the neutral element is given by the zero vector denoted by $\mathbf{0} = (0,0,\dots,0) \in \mathbb{R}^n$, and so we have

$$E : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^n), \quad E(e,k) = E(e+k,\mathbf{0}). \tag{3.1}$$

We denote the random field in Eq. (3.1) by $\overline{E}$.

2. **Insensitive case:** In the insensitive case, required capital $k \in \mathbb{R}^n$ is added to *terminal* equity values in $t = 1$. In contrast to the sensitive case, additional

capital cannot affect the evolution of balance sheets anymore. Terminal NAVs evolve according to

$$E : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^n), \quad E(e, k) = E(e, \eta) + k,$$

where η is the neutral element with respect to the conjunction of initial equity and additional capital. With the same arguments as in 1., we obtain

$$E : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^n), \quad E(e, k) = E(e, \mathbf{0}) + k. \tag{3.2}$$

We denote the random field in Eq. (3.2) by $\underline{E}$.

For a critical discussion on the difference between the sensitive and the insensitive case, in particular, in terms of interpretation, we refer to Appendix B.

(ii) Let $e \in \mathbb{R}^n$ be the initial equity allocation, and let $X \in \mathcal{X}(\mathbb{R}^n)$ denote the random change in wealth of the network. Then, the NAVs of entities evolve to

$$e + X = (e_1 + X_1, e_2 + X_2, \ldots, e_n + X_n) \in \mathcal{X}(\mathbb{R}^n).$$

However, $e_i + X_i$ does not necessarily coincide with the *terminal* NAV of sub-entity i. Indeed, these values can be shaped by capital transfer agreements between network's entities such that terminal NAVs are determined after redistribution of equity. This redistribution, in turn, may depend on the additional required capital $k \in \mathbb{R}^n$. The *terminal net asset value* of sub-entity i is the stochastic outcome of the random field $E_i(e, k)$ and may capture capital transfers inside the network. Examples include:

1. Stand-alone firms without any internal capital transfers:

$$E^1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^n), \quad E^1(e, k) = X + e + k.$$

2. Profit and Loss Transfer Agreements within networks:

$$E^2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^n),$$
$$E_i^2(e, k) = \sum_{d=1}^{n} a_d^i \cdot \max\{X_d + e_d + k_d, 0\} + \sum_{d=1}^{n} b_d^i \cdot \min\{X_d + e_d + k_d, 0\},$$

$i \in N$, where $\sum_{i=1}^{n} a_d^i = \sum_{i=1}^{n} b_d^i = 1$, $d = 1, \ldots, n$. Here, $a_d^i > 0$ is the part of *profits* generated by entity d that is transferred to entity i. Analogously, $b_d^i > 0$ is the part of the *losses* generated by entity d that is captured by entity i.

The random fields for terminal NAVs in the sensitive and insensitive case, $i \in N$, take the form

$$\overline{E}^1(e,k) = \underline{E}^1(e,k) = E^1(e,k),$$
$$\overline{E}^2(e,k) = E^2(e,k) \quad \text{and}$$
$$\underline{E}_i^2(e,k) = E_i^2(e,\mathbf{0}) + k_i = \sum_{d=1}^{n} a_d^i \cdot \max\{X_d + e_d, 0\} + \sum_{d=1}^{n} b_d^i \cdot \min\{X_d + e_d, 0\} + k_i.$$

Secondly, we discuss examples for regulatory aggregation functions Λ. Note that aggregation functions (as well as risk measures and their acceptance sets) are prescribed by the regulatory scheme or the local supervisory authority. In particular, the choice of Λ determines whether consolidation of balance sheets is allowed or not.

Example 3.1.4. Regulatory aggregation functions include:

(a) If each sub-entity is considered individually, then the regulatory aggregation function is the identity, i. e.,

$$\Lambda^1 : \mathbb{R}^n \to \mathbb{R}^n, \quad \Lambda^1(l) = l.$$

Applying Λ^1 to the random field E^2 in Example 3.1.3 (ii) corresponds to a regulatory framework of partial consolidation where sub-entities consolidate their balance sheets, but for the regulatory authority each legal entity needs to be still acceptable. The network risk measure with respect to Λ^1 is given by

$$R^{\Lambda^1}(E;e) = \{k \in \mathbb{R}^n \mid E_i(e,k) \in \mathcal{A}_i, \, \forall i \in N\}.$$

This network regulation allows for the application of an individual scalar-valued risk measure to each entity in the network.

(b) Instead of the legal entity approach in (a), the regulator may prescribe a consolidated approach. Here, the network of n sub-entities is structured such that m consolidating sub-networks arise, $1 \leq m < n$. The regulatory aggregation function is given by

$$\Lambda^2 : \mathbb{R}^n \to \mathbb{R}^m, \quad l \mapsto \Lambda^2(l).$$

If, for example, the individual positions of the entities in the sub-networks are aggregated by summation, then the aggregation function takes the form

$$\Lambda^{2.1} : \mathbb{R}^n \to \mathbb{R}^m, \quad \Lambda^{2.1}(l) = \left(\sum_{i=1}^{n_1} l_i, \sum_{i=n_1+1}^{n_2} l_i, \ldots, \sum_{i=n_{m-1}+1}^{n_m} l_i \right),$$

where $n_j \in N, n_m = n$. The network risk measure with respect to $\Lambda^{2.1}$ is given by

$$R^{\Lambda^{2.1}}(E;e) = \left\{ k \in \mathbb{R}^n \,\middle|\, \sum_{i=n_{j-1}+1}^{n_j} E_i(e,k) \in \mathcal{A}_j, \, \forall j = 1, \ldots, m \right\}, \quad (n_0 = 0, n_m = n).$$

As special case of Λ^2, we obtain a network that is considered as one legal entity by setting $m = 1$. If the aggregation mechanism is given by summation, then the regulatory aggregation function becomes

$$\Lambda^{2.2} : \mathbb{R}^n \to \mathbb{R}, \quad \Lambda^{2.2}(l) = \sum_{i=1}^{n} l_i.$$

This leads to the network risk measure

$$R^{\Lambda^{2.2}}(E;e) = \left\{ k \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} E_i(e,k) \in \mathcal{A} \right\}.$$

In contrast to network regulation based on the regulatory aggregation function Λ^1 in (a), only one scalar-valued risk measure is sufficient to measure the total network risk.

Let us emphasize that a regulatory framework that aggregates the balance sheets of individual entities by summation allows for subsidization within the networks, i. e., losses of sub-entities can be subsidized by gains of the remaining entities.

(c) Suppose that the regulator is blind for several sub-networks that are hidden under the regulatory screen, i. e., not all but only some sub-entities or sub-networks need to be acceptable. Examples for these regulatory aggregation functions are

$$\Lambda^3 : \mathbb{R}^n \to \mathbb{R}^h, \quad \Lambda^3(l) = (l_1, \ldots, l_h), \quad 1 \le h < n,$$

or

$$\Lambda^{3.1} : \mathbb{R}^n \to \mathbb{R}^h, \quad \Lambda^{3.1}(l) = \left(\Lambda_1^{2.1}(l), \ldots, \Lambda_h^{2.1}(l) \right), \quad 1 \le h < m.$$

Here, Λ^3 and $\Lambda^{3.1}$ are projections to certain components. If $h = 1$, then only one entity, e. g., the parent company, or one sub-network needs to generate acceptable outcomes. Applying $\Lambda^{3.1}$, for example, to the random field E^2 in Example 3.1.3 (ii), yields a regulatory framework of partial consolidation where sub-networks are individually regulated but within the sub-networks the entities consolidate their balance sheets. We obtain the corresponding network risk measures

$$R^{\Lambda^3}(E;e) = \{ k \in \mathbb{R}^n \mid E_j(e,k) \in \mathcal{A}_j, \, \forall j = 1, \ldots, h \}$$

and

$$R^{\Lambda^{3.1}}(E;e) = \left\{ k \in \mathbb{R}^n \;\middle|\; \sum_{i=n_{j-1}+1}^{n_j} E_i(e,k) \in \mathcal{A}_j, \, \forall j = 1, \ldots, h \right\}, \quad (n_0 = 0, n_h = m).$$

In these frameworks of network regulation, only entities $1, \ldots, h$ or sub-networks $1, \ldots, h$ need to fulfill acceptability conditions. In Feinstein et al. (2017), $R^{\Lambda^3}(E;e)$ with $h = 1$ is called *systemic risk measure.*

In contrast to $R^{\Lambda^{3.1}}(E;e)$, the network risk measure $R^{\Lambda^{2.1}}(E;e)$ in (b) captures the contribution of each sub-entity to the total network. On the other hand, $R^{\Lambda^{3.1}}(E;e)$ incorporates only those entities being part of one of the sub-networks $1,\dots,h$. Other entities or sub-networks are not visible for the regulatory authority.

Table 3.2 summarizes specific settings for the network risk measure that are mainly used throughout this chapter. In particular, this overview demonstrates that the abstract concept of a set-valued network risk measure is flexible and rich enough to account for a wide variety of regulatory frameworks used in practice.

Regulatory framework	**Ingredients for network risk measure**
Stand-alone approach	$E(e,k) = E^1(e,k)$, $m = n$, $\Lambda(l) = \Lambda^1(l)$
Legal entity approach	$E(e,k) = \overline{E}(e,k)$, $m = n$, $\Lambda(l) = \Lambda^1(l)$
Consolidated approach	$E(e,k) = \overline{E}(e,k)$, $m = 1$, $\Lambda(l) = \Lambda^{2.2}(l)$
Legal entity approach (insensitive)	$E(e,k) = \underline{E}(e,k)$, $m = n$, $\Lambda(l) = \Lambda^1(l)$
Consolidated approach (insensitive)	$E(e,k) = \underline{E}(e,k)$, $m = 1$, $\Lambda(l) = \Lambda^{2.2}(l)$

Table 3.2: Overview on regulatory frameworks.

If terminal NAVs are set to the random field E^1 in the legal entity or legal entity approach (insensitive), these approaches coincide with the stand-alone approach (see Example 3.1.3 (ii)).

3.1.2.2 | Basis Properties and Comparison of Network Risk Measures

Let us now compare the regulatory frameworks in Table 3.2 and discuss some general properties of our network risk measure. Proposition 3.1.5 shows how the legal entity and consolidated approach are linked to each other. It demonstrates that if regulatory authorities measure the network risk based on individual balance sheets, then capital requirements are stricter than in the consolidated approach where the single balance sheets are simply aggregated by summation.

Proposition 3.1.5. *Assume that for all entities $i \in N$ acceptability is determined by the same subadditive risk measure ρ. Then, for both sensitive and insensitive terminal net asset values,*

$$R^{\Lambda^1}(E;e) \subseteq R^{\Lambda^{2.2}}(E;e).$$

Proof. The proof is given in Section 3.7. □

Proposition 3.1.6 exhibits in which sense sensitive and insensitive terminal NAVs are linked to each other.

Proposition 3.1.6. *Assume that terminal net asset values are translation-supervariant in the initial equity for all $i \in N$.*

(i) Let $k \in \mathbb{R}^n_+$*. If* $k \in R^{\Lambda^1}(\underline{E}; e)$*, then* $k \in R^{\Lambda^1}(\overline{E}; e)$*.*

If each entity of the network has to add capital, each capital allocation that is acceptable in the legal entity approach (insensitive) is also acceptable in the legal entity approach. Hence, in the insensitive case, capital requirements are stricter.

(ii) Let $k \in \mathbb{R}^n_-$*. If* $k \in R^{\Lambda^1}(\overline{E}; e)$*, then* $k \in R^{\Lambda^1}(\underline{E}; e)$*.*

If each entity of the network can remove capital, each capital allocation that is acceptable in the legal entity approach is also acceptable in the legal entity approach (insensitive). Hence, in the sensitive case, capital requirements are more prudent.

Proof. The proof is given in Section 3.7. □

Remark 3.1.7. (i) The network risk measure in Definition 3.1.2 is a redistribution risk measure in the sense of Definition B.1.3 (ii) in Appendix B. Indeed, letting terminal NAVs be given by the sensitive random field in (3.1), and setting

$$f : \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^m), \quad f(e) = \Lambda(\overline{E}(e, \mathbf{0})),$$

the network risk can be measured by

$$\tilde{\rho}(f; e) = \{k \in \mathbb{R}^n \mid \Lambda_j\left(E(e + k, \mathbf{0})\right) \in \mathcal{A}_j \, \forall j = 1, \ldots, m\}.$$

Thus, $R^{\Lambda}(\overline{E}; e) = \tilde{\rho}(f; e)$.

(ii) For non-decreasing random fields E and increasing Λ, property (ii) in Remark 3.5.1 (see Section 3.5) implies that $l + c \in R^{\Lambda}(E; e)$ whenever $l \in R^{\Lambda}(E; e)$ and $c \in \mathbb{R}^n_+$. Hence, increasing the capital allocation does not affect the regulator's acceptability, see also Appendix B, Remark B.3.5.

The following proposition provides general properties of our network risk measure.

Proposition 3.1.8. *Let* Λ *be a regulatory aggregation function,* $E \in \mathcal{Y}$ *a random field of terminal net asset values,* $e \in \mathbb{R}^n$ *an initial equity allocation, and* R^{Λ} *a network risk measure. The following properties are satisfied:*

(i) Cash-invariance: If $\overline{E}$ *is sensitive (see Eq.* (3.1)*), then*

$$R^{\Lambda}(\overline{E}; e + l) = R^{\Lambda}(\overline{E}; e) - l, \quad \forall l \in \mathbb{R}^n.$$

(Adding a fixed capital vector to initial equity reduces the additional capital exactly by this amount.)

(ii) Monotonicity: Let $F \in \mathcal{Y}$ *be another random field such that* $F_i(e, k) \le E_i(e, k)$*,* $\forall i \in N$*. Then,*

$$R^{\Lambda}(F; e) \subseteq R^{\Lambda}(E; e).$$

(Capital requirements are stricter if future wealth levels decrease.)

(iii) *Convex values: If $\underline{E}$ is insensitive (see Eq.* (3.2)*), Λ_j is concave for all $j = 1, \ldots, m$, and $(\mathcal{A}_j)_{j=1,\ldots,m}$ is convex, then R^Λ is a convex subset of $\mathbb{R}^n$, i. e., if $k, l \in R^\Lambda(\underline{E}; e)$, then*

$$\alpha\, k + (1-\alpha)\, l \in R^\Lambda(\underline{E}; e), \ \forall \alpha \in [0,1].$$

(Convex combinations of acceptable allocations are again acceptable.)

Proof. The proof is given in Section 3.7. □

Remark 3.1.9. In the insensitive case, $R^\Lambda(\underline{E}; e)$ is cash-invariant if $\underline{E}$ is translation-invariant in the initial equity. In the sensitive case, property (iii) in Prop. 3.1.8 is satisfied, if $\mathbb{R}^n \to \mathcal{X}(\mathbb{R}^n), (e,k) \mapsto E(e,k)$ is concave in the initial equity, i. e.,

$$\begin{aligned}\alpha\overline{E}_i(e,k) + (1-\alpha)\overline{E}_i(e,l) &= \alpha\, E_i(e+k, \mathbf{0}) + (1-\alpha)\, E_i(e+l, \mathbf{0}) \\ &\le E_i(e+\alpha k + (1-\alpha) l, \mathbf{0}) = \overline{E}_i(e, \alpha k + (1-\alpha) l),\end{aligned}$$

for all $i \in N, \alpha \in [0,1]$. Since Λ is increasing, it is $\Lambda_j(\alpha\overline{E}(e,k) + (1-\alpha)\overline{E}(e,l)) \le \Lambda_j(\overline{E}(e, \alpha k + (1-\alpha) l))$, $\forall j = 1, \ldots, m$, and the claim follows.

The solvency condition for individual firms outlined in Section 3.1.1 can be translated to network regulation as follows: The network is solvent if and only if the values of the regulatory aggregation function that depend on terminal equity are acceptable, i. e.,

$$\Lambda_j(E(e,k^*)) \in \mathcal{A}_j, \ \forall j = 1, \ldots, m \ \Leftrightarrow \ k^* \in R^\Lambda(E; e).$$

In the sensitive case, this is equivalent to the solvency condition

$$\mathbf{0} \in R^\Lambda\left(\overline{E}; e + k^*\right),$$

due to cash-invariance. In the insensitive case, the equivalence only holds, if $\underline{E}$ is translation-invariant in the initial equity, see Remark 3.1.9. Hence, the network is solvent if initial equity endowments are sufficient in order to generate acceptable terminal values.

Our set-valued framework is a sophisticated concept for the regulation of networks since risk measurement and capital allocation are not split into two processes. Instead, capital allocation is inherent in the risk measurement procedure which, in particular, incorporates the structure of the network. In our framework, no artificial capital allocation method, such as proportional allocation or the covariance principle (see Section 3.6), has to be specified. However, the covariance principle is included within our framework and corresponds - as shown in Section 3.2.2 - to a particular choice of the ingredients of our network risk measure.

If capital requirements are not-unique, then the regulatory authority can choose among vectors of acceptable capital endowments $R^\Lambda(E, e)$. The network itself, however, is interested in choosing the *optimal* capital allocation subject to an overall management strategy. We analyze this optimization problem, including the objectives of the network (optimizing management strategies) as well as regulatory guidelines (acceptability of terminal values) in Sections 3.2 & 3.3.

3.2 | Capital Allocation in Networks

In this section, we focus on capital allocations that are optimal from the point of view of the network's management while respecting regulatory requirements at the same time. These allocations may, for example, reflect the objective of minimizing the total (cost of) additional capital among all acceptable allocations or maximizing the overall performance of the network. Since all acceptable capital allocations are - *per definitionem* - collected in the network risk measure R^{Λ}, *optimal* capital allocations correspond to a particular subset of R^{Λ} depending on the management's objective function. Our setting leads to a general set-valued approach of capital allocation which allows for a wide variety of choices for terminal net asset values, the regulatory aggregation function, acceptance sets and the management's objective function, see Section 3.2.1. In Section 3.2.2, we show that for a specific choice of these ingredients the well-known Euler capital allocation principle is obtained. Finally, numerical case studies illustrate our findings in Section 3.2.3.

3.2.1 | Set-Valued Capital Allocation

Let us now extend our framework in Section 3.1.2 by including the objectives of the network's management. Thus, we consider the following components:

(i) Terminal net asset values: $E : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^n)$, $(e, k) \mapsto E(e, k)$.

(ii) Network regulation via a regulatory aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$, $l \mapsto \Lambda(l)$, and acceptance sets $(\mathcal{A}_j)_{j=1,\dots,m}$.

(iii) Objective function: For a given allocation of additional capital $k \in \mathbb{R}^n$ within the network, the objective function

$$\mu : \mathbb{R}^n \to \mathbb{R}, \quad k \mapsto \mu(k),$$

characterizes the *objective of the network's management.*

Management objectives may include, for example, the minimization of the network's total capital requirement

$$\mu : \mathbb{R}^n \to \mathbb{R}, \quad \mu(k) = \sum_{i=1}^{n} k_i, \tag{3.3}$$

or the maximization of the expected return on capital. To obtain a unified minimization problem of objective functions later on, the latter objective is described with negative sign:

$$\mu : \mathbb{R}^n \to \mathbb{R}, \quad \mu(k) = -\frac{\sum_{i=1}^{n} \mathbb{E}\left[E_i(e,k)\right]}{\sum_{i=1}^{n} (e_i + k_i)}. \tag{3.4}$$

The key idea is that optimal set-valued capital allocation principles identify certain subsets of the acceptable allocations according to an objective function defined by the network's management. Let us define optimal set-valued capital allocation principles based on network risk measures.

Definition 3.2.1. Let R^Λ be a network risk measure in the sense of Definition 3.1.2, $E \in \mathcal{Y}$, $e \in \mathbb{R}^n$, and μ an objective function. For $\mathcal{R} \subseteq R^\Lambda(E;e)$, an *optimal set-valued capital allocation principle* is defined by

$$A^{\Lambda,\mu} : \mathcal{Y} \times \mathbb{R}^n \to \mathcal{R}, \quad A^{\Lambda,\mu}(E;e) = \underset{l \in R^\Lambda(E;e)}{\arg\min}\, \mu(l).$$

Note that the set $A^{\Lambda,\mu}(E;e)$ is set- or single-valued depending on whether the optimization has a unique solution or not.

Proposition 3.2.2 and Remark 3.2.3 provide optimal set-valued capital allocations for the regulatory frameworks in Table 3.2.

Proposition 3.2.2. *Let the management's objective function be given by (3.3). For the regulatory frameworks in Table 3.2, optimal set-valued capital allocations are:*

(i) Stand-alone approach:

$$A^{\Lambda^1,\mu}(E^1;e) = \begin{pmatrix} \mathrm{SCR}_1 - e_1 \\ \mathrm{SCR}_2 - e_2 \\ \vdots \\ \mathrm{SCR}_n - e_n \end{pmatrix}$$

(The optimal additional capital allocation is unique and given by the solvency capital requirement for individual firms less initial equity.)

(ii) Legal entity approach:

$$A^{\Lambda^1,\mu}(\overline{E};e) = \arg\min \left\{ \sum_{i=1}^n l_i \,\middle|\, l \in \mathbb{R}^n : \rho_i\left(E_i(e+l,\mathbf{0})\right) \le 0,\ \forall i \in N \right\}.$$

(If terminal NAVs are given by the random field E^1, then optimal set-valued capital allocations coincide with the allocation in (i).)

(iii) Consolidated approach:

$$A^{\Lambda^{2.2},\mu}(\overline{E};e) = \arg\min \left\{ \sum_{i=1}^n l_i \,\middle|\, l \in \mathbb{R}^n : \rho\left(\sum_{i=1}^n E_i(e+l,\mathbf{0})\right) \le 0 \right\}.$$

(If terminal NAVs are given by the random field E^1, then optimal set-valued capital allocations coincide with the allocations in (v).)

(iv) Legal entitiy approach (insensitive):

$$A^{\Lambda^1,\mu}(\underline{E}; e) = \begin{pmatrix} \rho_1(E_1(e, \mathbf{0})) \\ \rho_2(E_2(e, \mathbf{0})) \\ \vdots \\ \rho_n(E_n(e, \mathbf{0})) \end{pmatrix}$$

(If terminal NAVs are given by the random field E^1, then optimal set-valued capital allocations coincide with the allocation in (i).)

(v) Consolidated approach (insensitive):

$$A^{\Lambda^{2.2},\mu}(\underline{E}; e) = \left\{ l \in \mathbb{R}^n \,\middle|\, \rho\left(\sum_{i=1}^{n} E_i(e, \mathbf{0})\right) = \sum_{i=1}^{n} l_i \right\}.$$

(Any capital allocation such that the sum of additional capital equals the risk of the aggregated balance sheets is optimal.)

Proof. The proof is given in Section 3.7. □

Remark 3.2.3. By concritizing $\overline{E}$ by $\overline{E}^2$ in Proposition 3.2.2 (ii), we obtain optimal set-valued capital allocations for the regulatory framework of the partially consolidated approach (legal entity based). Moreover, setting $\Lambda(l) = \Lambda^{3.1}(l)$, optimal set-valued capital allocations for the partially consolidated approach (consolidation based) can be derived.

The achievement of management objectives - defined as optimization of the objective function - clearly depends on the regulatory framework. Corollary 3.2.4 shows that the optimization of the objective function in a regulatory framework allowing for the consolidation of balance sheets is less challenging than in a legal entity approach.

Corollary 3.2.4. *Assume that the same subadditive risk measure ρ is applied to each entity $i \in N$. In this case, the optimal value of the management's objective function in the consolidated approach is a lower bound for the optimal value of the objective function in the legal entity approach, i. e.,*

$$\min_{k \in R^{\Lambda^1}(E;e)} \mu(k) \geq \min_{k \in R^{\Lambda^{2.2}}(E;e)} \mu(k).$$

In particular, this holds for both sensitive and insensitive terminal net asset values.

Proof. The proof is given in Section 3.7. □

3.2.2 | Special Case of Euler Capital Allocation

In this section, we show that the *Euler allocation principle* - also known as *gradient allocation principle* - can be embedded into our set-valued setting. More precisely, the Euler allocation corresponds to the consolidated approach (insensitive) combined with the management's objective function (3.3) seeking to minimize the total additional capital, see Corollary 3.2.6 below.

The Euler capital allocation principle is based on Euler's well-known theorem on homogeneous functions and can be applied as follows in our setting, see, e. g., McNeil, Frey & Embrechts (2015) and Buch & Dorfleitner (2008): Let $W \subset \mathbb{R}^n \backslash \{0\}$ be a cone of weights such that $\mathbf{1} = (1, \dots, 1) \in W$. For $w \in W$, consider the regulatory aggregation function Λ^w defined by

$$\Lambda^w : \mathbb{R}^n \to \mathbb{R}, \quad \Lambda^w(l) = \sum_{i=1}^{n} w_i l_i.$$

Let ρ be some risk measure defined on a set $\mathcal{M}$ such that $\{\Lambda^w (E(e, \mathbf{0})) : w \in W\} \subseteq \mathcal{M}$, and let us define the *risk measure function*

$$r_\rho : W \to \mathbb{R}, \quad w \mapsto r_\rho(w) = \rho\left(\Lambda^w\left(E(e, \mathbf{0})\right)\right).$$

Note that the risk measure function is positively homogeneous, if ρ is a positively homogeneous risk measure. Indeed, for any $\lambda > 0$, we have

$$r_\rho(\lambda w) = \rho\left(\Lambda^{\lambda w}\left(E(e, \mathbf{0})\right)\right) = \rho\left(\sum_{i=1}^{n} \lambda w_i E_i(e, \mathbf{0})\right) = \lambda \rho\left(\Lambda^w\left(E(e, \mathbf{0})\right)\right) = \lambda r_\rho(w).$$

In the sequel, the random field that computes terminal NAVs with regulatory additional capital $k = \mathbf{0}$ is denoted by $E^{\mathbf{0}}$:

$$E^{\mathbf{0}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^n), \quad (e, k) \mapsto E(e, \mathbf{0}). \tag{3.5}$$

In this case, terminal equity values depend on initial equity only.

Definition 3.2.5. Suppose that the risk measure function r_ρ is differentiable in every $w \in W$. The *Euler allocation principle* is given by

$$A_\nabla : \mathcal{Y} \times W \to \mathbb{R}^n, \quad (E^{\mathbf{0}}, w) \mapsto \begin{pmatrix} A_{\nabla,1}\left(E^{\mathbf{0}}, w\right) \\ \vdots \\ A_{\nabla,n}(E^{\mathbf{0}}, w) \end{pmatrix}$$

with

$$A_{\nabla,i}(E^{\mathbf{0}}, w) := w_i \frac{\partial r_\rho}{\partial w_i}(w).$$

For $w = \mathbf{1}$, Definition 3.2.5 leads to

$$A_{\nabla,i}(E^{\mathbf{0}}, \mathbf{1}) = \frac{\partial r_\rho}{\partial w_i}(\mathbf{1}),$$

and so Euler's Theorem on homogeneous functions yields the *full capital allocation* property

$$\rho(\Lambda^{\mathbf{1}}\left(E(e,\mathbf{0})\right)) = r_\rho(\mathbf{1}) = \sum_{i=1}^{n} \frac{\partial r_\rho}{\partial w_i}(\mathbf{1}) = \sum_{i=1}^{n} A_{\nabla,i}(E^{\mathbf{0}},\mathbf{1}). \tag{3.6}$$

Now we embed the Euler capital allocation principle into our set-valued framework of capital allocation.

Corollary 3.2.6. *Let us consider the consolidated approach (insensitive) in Table 3.2 and the management's objective function (3.3) seeking to minimize the total additional capital. If ρ is a positively homogeneous risk measure, then*

$$A_{\nabla}(E^{\mathbf{0}},\mathbf{1}) \in A^{\Lambda^{2.2},\mu}\left(\underline{E};e\right).$$

Proof. The proof is given in Section 3.7. □

The Euler allocation principle is based on a linear regulatory aggregation function that aggregates terminal insensitive equity values by summation. It is a useful tool for capital allocation when the full allocation property is required. This is the case when capital can flow without any constraints between entities and hence, an aggregated position resulting from the summation of individual positions can be considered. In general, e. g., for insurance networks, this is not possible since the capital flow between entities is based on legally binding capital transfer agreements. For example, usually the losses of one entity cannot be subsidized by gains of another. Our general framework of set-valued capital allocation is much more flexible and allows for different regulatory aggregation functions beyond simple summation of positions.

For further reading, Section 3.6 provides some well-known examples of the Euler capital allocation for different scalar-valued risk measures. Moreover, Appendix C.2 generalizes the Euler principle - that is a priori restricted to differentiable and positively homogeneous risk measures - to the *subgradient allocation* for general convex risk measures. This allocation method relies on the robust representation of convex risk measures, cf., e. g., Föllmer & Schied (2011), Section 4.2, for detailed reading and Appendix C.1 for a short review.

3.2.3 | Numerical Case Studies

Let us now illustrate the computation of (optimal) set-valued capital allocations in numerical examples. In Section 3.2.3.1, we first analyze network risk measures and optimal set-valued capital allocations for a fictitious insurance network and different regulatory frameworks. Secondly, Section 3.2.3.2 shows how capital requirements can be reduced by capital transfers among the entities. The numerical results are based on a modification of the grid search algorithm introduced in Feinstein et al. (2017).

Throughout these case studies, we limit the discussion to a network with $n = 2$ entities. Recall that $X_i \in \mathcal{X}(\mathbb{R})$ denotes the random change in wealth of entity, $i = 1, 2$, and $X = (X_i)_{i=1,2} \in \mathcal{X}(\mathbb{R}^2)$. We assume that the initial equity endowment is $e_i = 0$, $i = 1, 2$, and that $X \in \mathcal{X}(\mathbb{R}^2)$ follows a two-dimensional normal distribution with expected value vector μ and covariance matrix Σ given by

$$\mu = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 1 & 0.75 \\ 0.75 & 2 \end{pmatrix}.$$

3.2.3.1 | Optimal Capital Allocations for Different Regulatory Frameworks

Consider the random field $E^1 \in \mathcal{Y}$ (see Example 3.1.3 (ii)) that captures the terminal net asset values in the network:

$$E^1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathcal{X}(\mathbb{R}^2), \quad E^1(e,k) = X + e + k.$$

In the sequel, we focus on network risk measures for the following regulatory frameworks:

(1) For the regulatory aggregation function $\Lambda^1 : \mathbb{R}^2 \to \mathbb{R}^2$, $\Lambda^1(l) = l$, the network risk measure takes the form

$$\begin{aligned} R^{\Lambda^1}(E^1; e) &= \{k \in \mathbb{R}^2 \,|\, \rho_i(X_i + e_i + k_i) \leq 0,\, i = 1,2\} \\ &= \{k \in \mathbb{R}^2 \,|\, \rho_i(X_i + e_i) \leq k_i,\, i = 1,2\}, \end{aligned}$$

cf. Example 3.1.4 (a). Note that this regulatory framework captures the stand-alone, the legal entity and the legal entity approach (insensitive), due to the specific choices of the random field and the regulatory aggregation function.

(2) Suppose that the regulatory aggregation function is given by $\Lambda^{2.2} : \mathbb{R}^2 \to \mathbb{R}$, $\Lambda^{2.2}(l) = l_1 + l_2$. By Example 3.1.4 (b), the network risk measure is given by

$$\begin{aligned} R^{\Lambda^{2.2}}(E^1; e) &= \left\{ k \in \mathbb{R}^2 \,\middle|\, \rho\left(\sum_{i=1}^{2} X_i + e_i + k_i\right) \leq 0 \right\} \\ &= \left\{ k \in \mathbb{R}^2 \,\middle|\, \rho\left(\sum_{i=1}^{2} X_i + e_i\right) \leq \sum_{i=1}^{2} k_i \right\}. \end{aligned}$$

This regulatory framework captures both the consolidated and the consolidated approach (insensitive).

(3) Let us finally consider the regulatory aggregation function

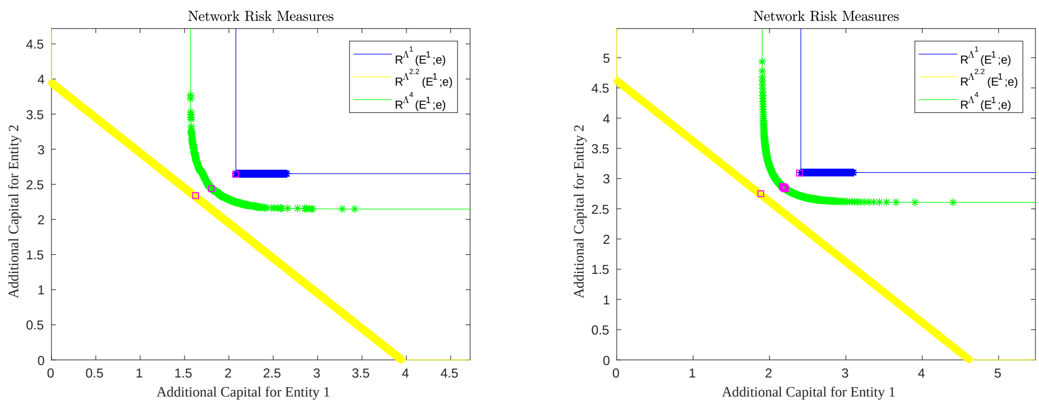
$$\Lambda^4 : \mathbb{R}^2 \to \mathbb{R}, \quad \Lambda^4(l) = \sum_{i=1}^{2} \min\{l_i, 0\} + r.$$

In contrast to $\Lambda^{2.2}$, the regulatory aggregation function Λ^4 adds up losses only and so losses of one sub-entity cannot be subsidized by gains of another. The amount $r > 0$ reflects the advantage of being part of the network. For example, r can be interpreted as a fixed overall capital buffer of the network or a regulatory bonus value for the network. In this case, the network risk measure is given by

$$\begin{aligned} R^{\Lambda^4}(E^1;e) &= \left\{ k \in \mathbb{R}^2 \,\middle|\, \rho\left(\sum_{i=1}^{2} \min\{X_i + e_i + k_i, 0\} + r\right) \le 0 \right\} \\ &= \left\{ k \in \mathbb{R}^2 \,\middle|\, \rho\left(\sum_{i=1}^{2} \min\{X_i + e_i + k_i, 0\}\right) \le r \right\}. \end{aligned}$$

For our case studies, we fix the parameter $r = 0.5$.

Network Risk Measures. Acceptability is specified with respect to the two standard risk measures value at risk (V@R) and average value at risk (AV@R) at level $\lambda = 0.005$, cf. Appendix A. While capital requirements in the regulation scheme Solvency II are based on V@R, the Swiss Solvency Test applies the coherent risk measure AV@R. We compute $R^{\Lambda^1}(E^1;e)$, $R^{\Lambda^{2.2}}(E^1;e)$ and $R^{\Lambda^4}(E^1;e)$ for both risk measures. Figures 3.1 (a) & (b) display the boundaries of the corresponding sets.



(a) (b)

Figure 3.1: Network risk measure for different regulatory systems using V@R in (a) and AV@R in (b).

Note first that

$$R^{\Lambda^1}(E^1;e) \subseteq R^{\Lambda^4}(E^1;e) \subseteq R^{\Lambda^{2.2}}(E^1;e),$$

in line with the results in Proposition 3.1.5. Hence, all capital requirements that are acceptable in setting (1) are also acceptable for the settings (2) & (3). Insofar, the stand-alone approach (1) leads to the strictest additional capital requirements. In contrast, in settings (2) & (3), the entities of the network are not considered separately. This leads to a larger set of acceptable additional capital. Since, in setting (2), the entities consolidate gains and losses completely, $R^{\Lambda^{2.2}}(E^1;e)$ is the largest set. Comparing Figures 3.1 (a) & (b), we see that in all settings the application of AV@R leads to higher additional capital requirements than the application of V@R – as expected.

Optimal Set-Valued Capital Allocations. Let us now compute optimal set-valued capital allocations $A^{\Lambda,\mu}$. For this purpose, we consider again the management's objective function in (3.3), i. e., $\mu : \mathbb{R}^2 \to \mathbb{R}$, $\mu(k) = k_1 + k_2$. By Proposition 3.2.2, optimal set-valued capital allocations for V@R are[1]

$$\begin{aligned}
A^{\Lambda^1,\mu}(E^1;e) &= \operatorname*{arg\,min}_{l \in R^{\Lambda^1}(E^1;e)} \mu(l) = (2.0758, 2.6428),\\
A^{\Lambda^{2.2},\mu}(E^1;e) &= \operatorname*{arg\,min}_{l \in R^{\Lambda^{2.2}}(E^1;e)} \mu(l) = \{(k_1,k_2) \in \mathbb{R}^2 \mid -k_1 + \mathrm{V@R}_\lambda(X_1+X_2) = k_2\},\\
A^{\Lambda^4,\mu}(E^1;e) &= \operatorname*{arg\,min}_{l \in R^{\Lambda^4}(E^1;e)} \mu(l) = (1.8704, 2.3586).
\end{aligned}$$

For AV@R, we obtain[2]

$$\begin{aligned}
A^{\Lambda^1,\mu}(E^1;e) &= \operatorname*{arg\,min}_{l \in R^{\Lambda^1}(E^1;e)} \mu(l) = (2.3919, 3.0898),\\
A^{\Lambda^{2.2},\mu}(E^1;e) &= \operatorname*{arg\,min}_{l \in R^{\Lambda^{2.2}}(E^1;e)} \mu(l) = \{(k_1,k_2) \in \mathbb{R}^2 \mid -k_1 + \mathrm{AV@R}_\lambda(X_1+X_2) = k_2\},\\
A^{\Lambda^4,\mu}(E^1;e) &= \operatorname*{arg\,min}_{l \in R^{\Lambda^4}(E^1;e)} \mu(l) = (2.2010, 2.8910).
\end{aligned}$$

Observe that $A^{\Lambda^{2.2},\mu}(E^1;e)$ is not unique, but contains a line segment. The Euler capital allocation picks a certain point on this line, i. e., the Euler capital allocation is an element of $R^{\Lambda^{2.2}}(E^1;e)$. More precisely, cf Corollary 3.2.6,

$$A_\nabla(E^{\mathbf{0}}, \mathbf{1}) \in A^{\Lambda^{2.2},\mu}(E^1;e).$$

For V@R and AV@R, we obtain

$$A_\nabla^{\mathrm{V@R}}(E^{\mathbf{0}}, \mathbf{1}) = (1.6250, 2.3392) \quad \text{and} \quad A_\nabla^{\mathrm{AV@R}}(E^{\mathbf{0}}, \mathbf{1}) = (1.8857, 2.7490)$$

via the covariance principle, see Section 3.6. These allocations are marked by a magenta square in Figures 3.1 (a) & (b).

3.2.3.2 | Reduction of Network Capital Requirements by Capital Transfers

The following example shows that network capital requirements can be reduced by capital transfers that are defined by derivatives on the random terminal net asset values of the entities. To this end, we explicitly incorporate *fixed* capital transfers among the entities.

Consider the following random field $\overline{E}^2 \in \mathcal{Y}$ (see Example 3.1.3 (ii)) that captures the

[1] Our implemented algorithm approximates a set of optimal set-valued capital allocations $A^{\Lambda^4,\mu}(E^1;e) = \left\{ \begin{pmatrix}1.885\\2.344\end{pmatrix}, \begin{pmatrix}1.884\\2.345\end{pmatrix}, \begin{pmatrix}1.883\\2.346\end{pmatrix}, \begin{pmatrix}1.882\\2.347\end{pmatrix}, \begin{pmatrix}1.854\\2.376\end{pmatrix}, \begin{pmatrix}1.853\\2.376\end{pmatrix}, \begin{pmatrix}1.852\\2.377\end{pmatrix} \right\}$. We simply computed the average to get a unique allocation.

[2] The approximated set of our algorithm is
$A^{\Lambda^4,\mu}(E^1;e) = \left\{ \begin{pmatrix}2.215\\2.877\end{pmatrix}, \begin{pmatrix}2.211\\2.881\end{pmatrix}, \begin{pmatrix}2.207\\2.885\end{pmatrix}, \begin{pmatrix}2.203\\2.889\end{pmatrix}, \begin{pmatrix}2.199\\2.893\end{pmatrix}, \begin{pmatrix}2.195\\2.897\end{pmatrix}, \begin{pmatrix}2.191\\2.901\end{pmatrix}, \begin{pmatrix}2.187\\2.905\end{pmatrix} \right\}$.

terminal NAVs in the network:

$$E^2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathcal{X}(\mathbb{R}^2),$$
$$E_i^2(e,k) = \sum_{d=1}^{2} a_d^i \cdot \max\{X_d + e_d + k_d, 0\} + \sum_{d=1}^{2} b_d^i \cdot \min\{X_d + e_d + k_d, 0\}, \quad i = 1, 2,$$

where $\sum_{i=1}^2 a_d^i = \sum_{i=1}^2 b_d^i = 1,\ d = 1, 2$. Recall that $a_d^i > 0$ is the part of profits generated by entity d that is transferred to entity i, while $b_d^i > 0$ is the part of the losses generated by entity d that is captured by entity i. Insofar, these vectors determine fixed capital transfers among the sub-entities. In our case study, we assume $a^1 = b^1$ and $a^2 = (1-a_1^1, 1-a_2^1) = b^2$, i. e., the part of gains and the part of losses that are generated by entity d and captured by entity i coincide.

We compute the network risk measure in the partially consolidated approach (legal entity based) with respect to V@R at level $\lambda = 0.005$:

$$R^{\Lambda^1}(\overline{E}^2; e) = \{k \in \mathbb{R}^2 \mid \mathrm{V@R}_{0.005}(E_i^2(e + k, \mathbf{0})) \le 0, i = 1, 2\}.$$

Figure 3.2 displays the boundaries of the set-valued network risk measure as well as optimal set-valued capital allocations for several choices of a^1.

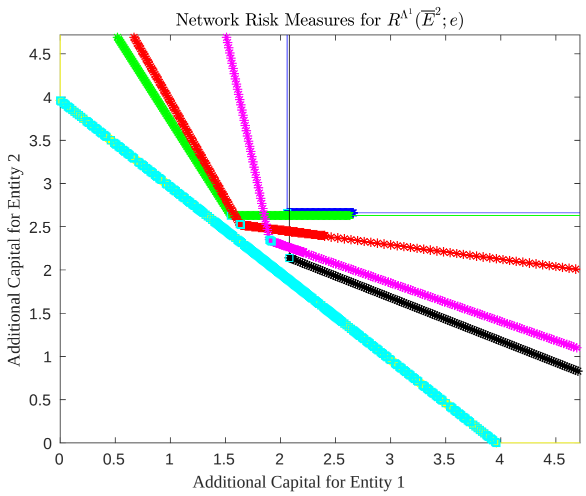


Figure 3.2: Network risk measure $R^{\Lambda^1}(\overline{E}^2; e)$ for $a^1 = (1,1)$ (yellow), $a^1 = (1,0)$ (blue), $a^1 = (1,0.5)$ (green), $a^1 = (0.5,0)$ (black), $a^1 = (0.9,0.4)$ (red), $a^1 = (0.4,0.9)$ (magenta). Optimal set-valued capital allocations with respect to the management's objective function in (3.3) are marked by a cyan square.

Setting $a^1 = (1,1)$, i. e., all gains and losses are transferred to entity 1, the random field E^2 takes the form

$$E : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathcal{X}(\mathbb{R}^2), \quad E(e,k) = \begin{pmatrix} E_1^1(e,k) + E_2^1(e,k) \\ 0 \end{pmatrix},$$

which corresponds – in combination with Λ^1 – to the consolidated approach. For $a^1 = (1,0)$, we obtain the random field E^1. Hence, this choice of a^1 yields the stand-alone approach. In the case $a^1 = (1, 0.5)$, entity 1 can be interpreted as the parent company of sub-entity 2. All gains and losses of entity 1 retain within the firm, whereas 50% of the gains and losses of entity 2 are passed to the parent company. By this construction, entity 1, i. e., the parent company, can reduce its capital requirement. The same interpretation holds for $a^1 = (0.5, 0)$. Here, entity 2 is the parent company and takes 50% of gains and losses of sub-entity 1. Hence, entity 2 can reduce its capital requirement compared to the stand-alone case. The values $a^1 = (0.9, 0.4)$ and $a^1 = (0.4, 0.9)$ define further splitting rules for gains and losses among the sub-entities. In comparison to the stand-alone case, both entities can reduce their required capital if gains and losses are shared.

We observe that the smallest (strictest) set of additional capital requirements is obtained in the stand-alone case, i. e., $a^1 = (1, 0)$, while the largest set of additional capital requirements corresponds to the consolidated approach, i. e., $a^1 = (1, 1)$. In all other cases, we obtain subsets of the consolidated and supersets of the stand-alone approach. Thus – considering the management's objective function in (3.3) (total capital minimization) – it is efficient for the network to consolidate "as much as possible". Usually, there are restrictions on the free movement of capital such that the splitting rule $a^1 = (1, 1)$ is not applicable. However, our case studies in Section 3.3 indicate that consolidated balance sheets can be mimicked via optimal management strategies.

3.3 | Network Optimization

In this section, we model the network's optimization problem explicitly. In contrast to Section 3.2, we now allow to *choose* management strategies including *asset-liability management* (ALM) strategies and *internal capital transfers* (ICTs). Thus, firms can not only adjust their equity by additional capital, but also affect the evolution of terminal equity via internal transactions, investments in the financial market, and underwriting decisions, see Section 3.3.1. While so far management strategies were held constant and only capital allocations were varied, we now include optimal ALM- and ICT-strategies that optimize the network's management function subject to regulatory constraints. Numerical case studies illustrate our results and indicate that consolidated balance sheets can be mimicked via optimal management strategies, see Section 3.3.2.

3.3.1 | The Network's Optimization Problem

Let $\mathcal{Z}$ be the set of management strategies, see below for specific examples. With this additional ingredient, terminal net asset values and the management's objective function depend on $z \in \mathcal{Z}$, and so the extended framework consists of the following components:

(i) Terminal net asset values: To account for the dependency on the management strategy $z \in \mathcal{Z}$, the random terminal NAVs are denoted by

$$E : \mathcal{Z} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^n), \quad (z, e, k) \mapsto E(z, e, k).$$

(ii) Network regulation via a regulatory aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$, $l \mapsto \Lambda(l)$, and acceptance sets $(\mathcal{A}_j)_{j=1,\dots,m}$.

(iii) Objective function: Depending on the management strategy $z \in \mathcal{Z}$, the function

$$\mu : \mathcal{Z} \times \mathbb{R}^n \to \mathbb{R}, \quad (z,k) \mapsto \mu(z,k),$$

characterizes the objective of the network's management.

We call a management strategy $z \in \mathcal{Z}$ *trivial*, if z leads to nil-tranfers within the network. The set of trivial strategies is denoted by $\mathcal{Z}^{tri}$. The following proposition shows that including non-trivial strategies allows for a reduction of the total solvency capital requirement.

Proposition 3.3.1. *Let ρ_j be monetary risk measures, $j = 1, \dots, m$. If asset-liability management strategies are held constant, then*

$$R^{\Lambda,tri}(E;e) \subseteq R^{\Lambda,\mathcal{Z}}(E;e),$$

where

$$R^{\Lambda,tri}(E;e) := \{k \in \mathbb{R}^n \mid \Lambda_j(E(z,e,k)) \in \mathcal{A}_j \text{ for } j = 1,\dots,m, z \in \mathcal{Z}^{tri}\}$$

and

$$R^{\Lambda,\mathcal{Z}}(E;e) := \{k \in \mathbb{R}^n \mid \Lambda_j(E(z,e,k)) \in \mathcal{A}_j \text{ for } j = 1,\dots,m, z \in \mathcal{Z}\}.$$

In particular, this holds for both sensitive and insensitive terminal net asset values.

Proof. The proof is given in Section 3.7. □

In the remaining part of the chapter, we simply write $R^{\Lambda}(E;e)$ instead of $R^{\Lambda,\mathcal{Z}}(E;e)$ whenever any management strategy $z \in \mathcal{Z}$ is admissible. If trivial strategies are allowed only, then we write $R^{\Lambda,\text{tri}}(E;e)$.

Remark 3.3.2. Note that $R^{\Lambda,\text{tri}}(E;e) = R^{\Lambda}\left(E^1;e\right)$ for the random field E^1 defined in Example 3.1.3 (ii). Hence, $R^{\Lambda,\text{tri}}(E;e)$ encodes the network risk measurement without any internal cash-flows. Choosing, for example, the regulatory aggregation function Λ^1, the set $R^{\Lambda^1,\text{tri}}(E;e)$ is given by

$$R^{\Lambda^1,\text{tri}}(E;e) = \begin{pmatrix} \rho_1\left(E_1^1\left(z^1,e,\mathbf{0}\right)\right) \\ \rho_2\left(E_2^1\left(z^1,e,\mathbf{0}\right)\right) \\ \vdots \\ \rho_n\left(E_n^1\left(z^1,e,\mathbf{0}\right)\right) \end{pmatrix} + \mathbb{R}^n_+, \quad z^1 \in \mathcal{Z}^{tri}.$$

The following corollary shows that in a regulatory framework allowing for trivial management strategies only, the objective function of the network's management can be less efficiently optimized than in a general framework.

Corollary 3.3.3. *Let μ be the objective function of the management. The optimal value function of the network's management in a regulatory framework allowing for trivial management strategies only is an upper bound for the optimal value function of the network's management in a general framework, i. e.,*

$$\min_{\substack{z\in\mathcal{Z},\\ k\in R^{\Lambda}(E;e)}} \mu(z,k) \leq \min_{\substack{z\in\mathcal{Z}^{tri},\\ k\in R^{\Lambda,tri}(E;e)}} \mu(z,k).$$

In particular, this is true for both sensitive and insensitive terminal net asset values.

Proof. The proof is given in Section 3.7. □

In our setting, a network can manage its terminal NAV E by implementing both ALM strategies and ICT agreements. The aim is to optimize some objective function μ over all admissible strategies z under regulatory constraints imposed by a regulatory aggregation function Λ and acceptance sets $(\mathcal{A}_j)_{j=1,\dots,m}$. Thus, the network needs to identify ALM- and ICT-strategies that are feasible and optimize the objective function of the network's management while respecting regulatory constraints at the same time.

In the sequel, we set $\mathcal{Z} = \Theta \times \mathcal{T}$, where Θ is the set of ALM strategies and $\mathcal{T}$ denotes the set of ICT strategies. Let us first describe these sets in detail.

ALM Strategies. Let us assume that the network's firms can invest in a financial market with a finite number D of financial assets. At time $t = 0$, asset prices are deterministic and denoted by s_d, $d = 1, \dots, D$. Asset prices evolve randomly over the given time horizon. The set of admissible ALM strategies is given by a subset

$$\Theta(e,l,s) \subseteq \left\{ x \in \mathbb{R}^{n\times D} \;\middle|\; \sum_{d=1}^{D} x_{id}s_d = e_i + l_i,\ \forall i \in N \right\},$$

where x_{id} is the number of shares invested in asset d by entity i over the given time horizon, l_i denotes the liabilities of the i-th entity in $t = 0$, and the sum $e_i + l_i$ equals the total asset amount of the balance sheet for entity i in $t = 0$. Hence, we assume that each entity fully allocates its capital in the financial market. The subset $\Theta(e, l, s)$ may reflect further investment constraints, e. g., upper and lower bounds on the investment in certain assets. In this case, the set of admissible ALM strategies takes the form

$$\Theta(e,l,s) = \left\{ x \in \mathbb{R}^{n\times D} \;\middle|\; \sum_{d=1}^{D} x_{id}s_d = e_i + l_i,\ \bar{b}_{id} \geq x_{id} \geq \underline{b}_{id},\ \forall i \in N \right\}.$$

ICT Strategies. Depending on the vector L of the entities random liabilities at $t = 1$, the set $\mathcal{T}(L)$ contains all strategies $x \in \mathbb{R}^{n\times n}$ defining the capital transfer from entity i to j given by the contract $\xi_{ij}(L, x)$. If $\xi_{ij}(L, x)$ is positive, then entity j receives the amount $\xi_{ij}(L, x)$ from entity i. Conversely, if $\xi_{ij}(L, x)$ is negative, then entity j pays the amount $\xi_{ij}(L, x)$ to entity i. The case $\xi_{ij}(L, x) = 0$ corresponds to no internal transfer from i to j. Note that $\xi_{ii}(L, x) = 0$ for all $i \in N$. All legally binding contracts are given by the matrix function

$$\xi : \mathcal{X}(\mathbb{R}^n_+) \times \mathbb{R}^{n\times n} \to \mathcal{X}(\mathbb{R}^{n\times n}), \quad (L,x) \mapsto \xi(L,x).$$

The set of admissible ICT strategies is given by

$$\mathcal{T}(L) \subseteq \left\{ x \in \mathbb{R}^{n\times n} \,\middle|\, x_{ij} = x_{ji}, \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_{ij}(L,x) = 0, \xi_{ii}(L,x) = 0,\ i \in N \right\}.$$

The constraints describe that the transfers are financed by the network and that there are no transfers within one entity. Thus, the matrix $\xi(L,x)$ is skew symmetric with zeros on its diagonal. Since there are no transfers within one entity, we set $x_{ii} := 0$ for all $i \in N$. The set of trivial ICT strategies is given by

$$\mathcal{T}^{\text{tri}}(L) = \left\{ x \in \mathbb{R}^{n\times n} \,\middle|\, x_{ij} = x_{ji}, \xi_{ij}(L,x) = 0\, \forall i,j \right\}.$$

Note that $\mathcal{T}(L)$ may reflect further constraints on admissible ICT strategies, e. g., no profits from ICTs:

$$\mathcal{T}(L) = \left\{ x \in \mathbb{R}^{n\times n} \,\middle|\, x_{ij} = x_{ji}, \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_{ij}(L,x) = 0, \xi_{ii}(L,x) = 0, \sum_{i=1}^{n} \xi_{ij}(L,x) \le L_j \right\}.$$

Example 3.3.4. For insurance networks, reinsurance agreements provide typical examples of contracts $\xi_{ij}(L,x)$ from firm i to firm j.

(i) Quota share reinsurance (proportional internal transfers):

$$(L,x) \mapsto \xi_{ij}(L,x) = x_{ij} L_j.$$

Entity i pays a portion x_{ij} of the liabilities of entity j, while entity j pays $(1-x_{ij})L_j$. In this case, the set of admissible ICTs is given by

$$\mathcal{T}^{\text{QSR}}(L) = \left\{ x \in \mathbb{R}^{n\times n} \,\middle|\, x_{ij} = x_{ji}, \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_{ij}(L,x) = 0, \xi_{ii}(L,x) = 0, x_{ij} \in [0,1] \right\}.$$

A trivial ICT strategy is given by the zero-matrix.

(ii) Stop loss reinsurance (non-proportional internal transfers):

$$(L,x) \mapsto \xi_{ij}(L,x) = (L_j - x_{ij})^+.$$

Entity i pays that part of the liabilities of entity j exceeding a barrier x_{ij}. Entity j pays $\min\{L_j, x_{ij}\}$. The set of admissible ICTs is given by

$$\mathcal{T}^{\text{SLR}}(L) = \left\{ x \in \mathbb{R}^{n\times n} \,\middle|\, x_{ij} = x_{ji}, \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_{ij}(L,x) = 0, \xi_{ii}(L,x) = 0, x_{ij} \in [0, ||L_j||_\infty] \right\}.$$

A trivial ICT strategy is given by

$$x^{\text{tri}} = \begin{pmatrix} 0 & ||L_2||_\infty & ||L_3||_\infty & \cdots & ||L_n||_\infty \\ ||L_2||_\infty & 0 & ||L_3||_\infty & \cdots & ||L_n||_\infty \\ ||L_3||_\infty & ||L_3||_\infty & 0 & \cdots & ||L_n||_\infty \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ ||L_n||_\infty & ||L_n||_\infty & ||L_n||_\infty & \cdots & 0 \end{pmatrix} \in \mathcal{T}^{\text{tri}}(L).$$

Remark 3.3.5. In contrast to our setting, Haier et al. (2016) consider admissible internal strategies depending on terminal equity E:

$$\mathbb{R}^n_- \subseteq \mathcal{T}(E) \subseteq \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n x_i \leq 0 \right\},$$

i. e., internal tranfers are financed by the network and nil-tranfers are possible. Haier et al. (2016) do not specify the optimal legally binding internal contracts in $t = 0$ explicitly, but concentrate on the outcome of admissible contracts for each entity in $t = 1$. Thus, their internal strategy is a vector specifying the amount of capital added to terminal equity by internal transfers for each entity in $t = 1$, i. e., this corresponds to an insensitive approach in our setting. In our model, the internal strategy is a matrix defining the contracts from entity i to entity j in $t = 0$ such that network management is implemented in an optimal way.

Taking into account ALM- and ICT-strategies, the network's management faces the following optimization problem under constraints:

Problem 3.3.6. The constrained optimization problem of a network is given by

$$\min_{\vartheta,\pi,k} \mu(\vartheta, \pi, k)$$

such that $\vartheta \in \Theta(e + k, l, s)$, $\pi \in \mathcal{T}(L)$, $k \in R^\Lambda(E; e)$.

Note that in the insensitive case the corresponding ALM constraint is given by $\vartheta \in \Theta(e, l, s)$.

For a review on constrained optimization problems, we refer, e. g., to Freund (2004), see also Giorgi & Kjeldsen (2014) for a collection of seminal papers on this topic. In this section, we compare the results for different regulatory frameworks in the corollaries below and illustrate the optimizing strategies in numerical case studies in Section 3.3.2.

In analogy to Corollary 3.2.4, the first corollary shows that in a regulatory framework allowing for the consolidation of balance sheets, the minimal value of the management function is less or at most equal to the minimal value of the management function in a regulatory system applying the legal entity approach.

Corollary 3.3.7. *Assume that the same subadditive risk measure ρ is applied to each entity $i \in N$.*

(i) If terminal net asset values are sensitive, then

$$\min_{\substack{\vartheta\in\Theta(e+k,l,s),\\ \pi\in\mathcal{T}(L),\\ k\in R^{\Lambda^{2.2}}(\overline{E};e)}} \mu(\vartheta,\pi,k) \;\leq\; \min_{\substack{\vartheta\in\Theta(e+k,l,s),\\ \pi\in\mathcal{T}(L),\\ k\in R^{\Lambda^{1}}(\overline{E};e)}} \mu(\vartheta,\pi,k).$$

(ii) If terminal net asset values are insensitive, then

$$\min_{\substack{\vartheta\in\Theta(e,l,s),\\ \pi\in\mathcal{T}(L),\\ k\in R^{\Lambda^{2.2}}(\underline{E};e)}} \mu(\vartheta,\pi,k) \;\leq\; \min_{\substack{\vartheta\in\Theta(e,l,s),\\ \pi\in\mathcal{T}(L),\\ k\in R^{\Lambda^{1}}(\underline{E};e)}} \mu(\vartheta,\pi,k).$$

Corollary 3.3.8 states that including non-trivial strategies reduces the minimal value of the management function, cf. Corollary 3.3.3. Note that in a regulatory framework applying the consolidated approach via the regulatory aggregation function $\Lambda^{2.2}$, ICT strategies are not relevant.

Corollary 3.3.8. *If terminal net asset values are sensitive, then*

$$\min_{\substack{\vartheta\in\Theta(e+k,l,s),\\ \pi\in\mathcal{T}^{tri}(L),\\ k\in R^{\Lambda,tri}(\overline{E};e)}} \mu(\vartheta,\pi,k) \;\geq\; \min_{\substack{\vartheta\in\Theta(e+k,l,s),\\ \pi\in\mathcal{T}(L),\\ k\in R^{\Lambda}(\overline{E};e)}} \mu(\vartheta,\pi,k).$$

If terminal net asset values are insensitive, then

$$\min_{\substack{\vartheta\in\Theta(e,l,s),\\ \pi\in\mathcal{T}^{tri}(L),\\ k\in R^{\Lambda,tri}(\underline{E};e)}} \mu(\vartheta,\pi,k) \;\geq\; \min_{\substack{\vartheta\in\Theta(e,l,s),\\ \pi\in\mathcal{T}(L),\\ k\in R^{\Lambda}(\underline{E};e)}} \mu(\vartheta,\pi,k).$$

3.3.2 | Numerical Case Studies

The following two case studies provide numerical solutions to Problem 3.3.6 for the two objective functions (3.3) and (3.4). We analyze ALM- as well as ICT-strategies that optimize the management's objective function while respecting regulatory constraints at the same time.

Both case studies are based on a common setting: Consider a simple insurance network with $n = 2$ entities endowed with initial equity $e = (8, 2)$. We assume ICTs from entity 1 to entity 2 and compare the impact of trivial ICTs, ICTs based on quota share reinsurance and on stop loss reinsurance (cf. Example 3.3.4) on the optimal management strategy.

In addition to ICT strategies, both entities can also invest in a financial market in order to meet their future liabilities. For illustration purposes, we limit the discussion to a financial market with two assets: a risk-free bond and a risky stock. The bond is specified by the initial value $s_1 = 3$ and a zero interest rate. The future stock price is assumed to follow - at least approximately - a normal distribribution with parameters $\mu = 4$ and $\sigma^2 = 0.5$, i. e., $S_2 \sim \mathcal{N}(4, 0.5)$. We set $s_2 = 4$. For the future liabilities of entity i, L_i, we assume a shifted and scaled Beta distribution $L_i - t_i \sim M_i \cdot \mathrm{Beta}(2, 4)$, $i = 1, 2$, with

$(t_1, M_1) = (30, 10)$ and $(t_2, M_2) = (25, 6)$. In this case, the support of L_1 equals $[30, 40]$ and $L_2 \in [25, 31]$. The terminal net asset value E is given by the difference of assets and liabilities, and it depends on the ALM- and ICT-strategy.

Regulatory acceptability constraints are specified by the subadditive risk measure average value at risk at level $\lambda = 0.05$, i. e., $\rho(\Lambda_j(E(z,e,k))) = \mathrm{AV@R}_{0.05}(\Lambda_j(E(z,e,k)))$ for $j = 1, \ldots, m$. We solve Problem 3.3.6 for the following regulatory systems (cf. Table 3.2) and constraints:

(a) Legal entity approach:

$$\vartheta \in \Theta(e+k, l, s),\ \pi \in \mathcal{T}^{\bullet}(L),\ k \in R^{\Lambda^1}(\overline{E}; e),\ k_i \in [-e_i, e_i],$$

where $\bullet \in \{\text{QSR, SLR, tri}\}$. The last constraint ensures that each entity cannot remove or borrow more capital than the initial amount.

(b) Legal entity approach (insensitive):

$$\vartheta \in \Theta(e, l, s),\ \pi \in \mathcal{T}^{\bullet}(L),\ k \in R^{\Lambda^1}(\underline{E}; e),\ k_i \in [-\mathbb{E}[E_i((\vartheta,\pi), e, \mathbf{0})], \mathbb{E}[E_i((\vartheta,\pi), e, \mathbf{0})]]\,,$$

where $\bullet \in \{\text{QSR, SLR, tri}\}$. The last constraint ensures that each entity cannot remove or borrow more capital than the expected terminal amount.

(c) Consolidated approach:

$$\vartheta \in \Theta(e+k, l, s),\ k \in R^{\Lambda^{2.2}}(\overline{E}; e),\ k_i \in [-e_i, e_i].$$

(d) Consolidated approach (insensitive):

$$\vartheta \in \Theta(e, l, s),\ k \in R^{\Lambda^{2.2}}(\underline{E}; e),\ k_i \in [-\mathbb{E}[E_i((\vartheta,\pi), e, \mathbf{0})], \mathbb{E}[E_i((\vartheta,\pi), e, \mathbf{0})]]\,.$$

Note that in the consolidated approaches, ICTs always vanish and thus do not need to be considered.

Minimization of the Total Additional Capital. Suppose first that the management's objective is to minimize the total additional capital:

$$\mu(\vartheta, \pi, k) = k_1 + k_2.$$

We ran the optimization $1,000$ times and calculated the mean of the minimizers and minimal values. Each optimization step relies on $100,000$ simulations of the random variables. The initial values for the Matlab algorithm are set to

$$\vartheta_{11} = 12,\ \vartheta_{12} = 0.3,\ \vartheta_{21} = 9,\ \vartheta_{22} = 0.1,\ \pi_{12}^{\text{QSR}} = 0.5,\ \pi_{12}^{\text{SLR}} = 27,\ k_1 = k_2 = 0.$$

Tables 3.3 & 3.4 display the optimal ALM strategy in blue, the optimal ICT strategy in green and the additional required capital in red. While Table 3.3 provides the results for insensitive terminal NAVs, the results for sensitive terminal NAVs are shown in Table 3.4. In both cases, we observe that the sum of additional required capital in the consolidated regulatory framework is a lower bound, while the sum of the total capital in the case of trivial ICT strategies serves as an upper bound. The total capital requirement when quota share- or stop loss-reinsurance is included is between these bounds. The optimal internal contract for capital transfer in the case of insensitive terminal equity values is given by $\xi_{12}(L,\pi) = 0.22 \cdot L_2$ and $\xi_{12}(L,\pi) = (L_2 - 24.94)^+$, respectively, meaning that for the first contract entity 1 pays 22% of entity 2's liabilities. In the second contract, liabilities of entity 2 exceeding the amount of 24.95 are captured by entity 1. All other liabilities retain within entity 2. Including these ICT strategies reduces the regulatory capital requirement. In particular, the network's consolidated balance sheet can be nearly mimicked via a stop loss reinsurance contract from entity 1 to entity 2. For the ALM strategy, we observe that practically nothing is invested in the risky asset. Instead, both entities invest in the risk-free bond. In nearly all cases, both entities can reduce their terminal capital in order to fulfill the regulatory constraints. Only for $\mathcal{T}^{\text{tri}}(L)$, entity 2 has to add $k_2 = 0.3813$ and for $\mathcal{T}^{\text{QSR}}(L)$, entity 1 has to add $k_1 = 1.99$.

Problem	Minimizer	Minimal Value
$\mathcal{T}^{\text{tri}}(L)$	$(\vartheta, \pi_z, k) = (13.7674; 0.0079; 9.6607; 0.0045; \pi_z; -4.0309; 0.3813)$ $\pi_{\text{QSR}} = 0, \quad \pi_{\text{SLR}} = 31$	-3.6495
$\mathcal{T}^{\text{QSR}}(L)$	$(\vartheta, \pi, k) = (13.7671; 0.0079; 9.6619; 0.0036; 0.2222; 1.9999; -6.1480)$	-4.1480
$\mathcal{T}^{\text{SLR}}(L)$	$(\vartheta, \pi, k) = (13.7658; 0.0089; 9.6666; 0.0000; 24.9464; -1.3390; -4.0535)$	-5.3925
Consolidated	$(\vartheta, k) = (13.7723; 0.0041; 9.6612; 0.0041; -4.6691; -0.7448)$	-5.4139

Table 3.3: Optimal strategy for minimizing total required capital when terminal NAVs are insensitive.

The optimal internal contract for capital transfer in the case of sensitive terminal equity values is given by $\xi_{12}(L,\pi) = 0.08 \cdot L_2$ and $\xi_{12}(L,\pi) = (L_2 - 27)^+$, respectively, meaning that for the first contract, entity 1 pays 8% of entity 2's liabilities. In the second contract, liabilities of entity 2 exceeding the amount of $\mathbb{E}[L_2] = 27$ are captured by entity 1. All other liabilities retain within entity 2. Including these ICT strategies reduces the regulatory capital requirement. For the ALM strategy, we observe the same as in the insensitive case. In particular, for $\mathcal{T}^{\text{SLR}}(L)$, entity 2 invests in the risk-free bond only. In nearly all cases, both entities can reduce their initial capital in order to fulfill the regulatory constraints in the future. Only for $\mathcal{T}^{\text{tri}}(L)$, entity 2 has to add $k_2 = 0.3811$. We point out that for $\mathcal{T}^{\text{QSR}}(L)$ and $\mathcal{T}^{\text{SLR}}(L)$, entity 2 reduces its initial equity to zero.

Problem	Minimizer	Minimal Value
$\mathcal{T}^{\text{tri}}(L)$	$(\vartheta, \pi_z, k) = (12.4244; 0.0074; 9.7875; 0.0046; \pi_z; -4.0308; 0.3811)$ $\pi_{\text{QSR}} = 0, \quad \pi_{\text{SLR}} = 31$	-3.6496
$\mathcal{T}^{\text{QSR}}(L)$	$(\vartheta, \pi, k) = (13.1554; 0.0073; 8.9943; 0.0042; 0.0811; -1.8381; -2.0000)$	-3.8381
$\mathcal{T}^{\text{SLR}}(L)$	$(\vartheta, \pi, k) = (12.6684; 0.0089; 9.0000; 0.0000; 27.0001; -3.2920; -2.0000)$	-5.2920
Consolidated	$(\vartheta, k) = (12.1776; 0.0045; 9.4505; 0.0045; -4.7828; -0.6307)$	-5.4135

Table 3.4: Optimal strategy for minimizing total required capital when terminal NAVs are sensitive.

Figure 3.3 compares the total network capital requirement for sensitive and insensitive terminal NAVs. When individual entities without any transfers are considered as well as in the case of consolidated balance sheets, i. e., free movement of capital without any constraints or legally binding contracts, the required capital almost coincides. Since for quota share- and stop loss-reinsurance capital can be reduced, even more capital can be taken out of the network in the insensitive case. Hence, in that case sensitive terminal NAVs lead to stricter capital requirements.

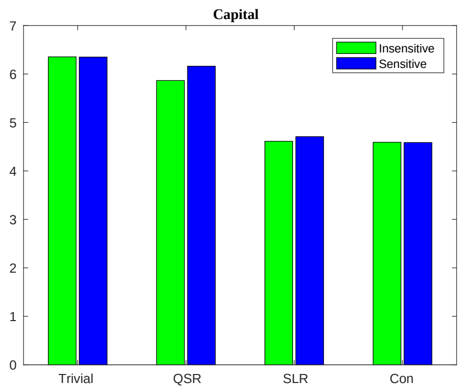


Figure 3.3: Network capital requirement for minimizing the total capital for sensitive and insensitive terminal NAVs.

Maximization of the Expected Return on Capital. Suppose now that the management's objective is to maximize the total expected return on capital. Recalling from (3.4) the sign convention to obtain a minimization problem, the management's objective is described by the objective function

$$\mu(\vartheta, \pi, k) = -\frac{\sum_{i=1}^{2} \mathbb{E}[E_i((\vartheta, \pi), e, k)]}{\sum_{i=1}^{2} e_i + k_i}$$

Again, we ran the optimization $1,000$ times with $100,000$ simulations of random variables per optimization step. The initial values for the Matlab algorithm are set to

$$\vartheta_{11} = 11,\ \vartheta_{12} = 2,\ \vartheta_{21} = 8,\ \vartheta_{22} = 1,\ \pi_{12}^{\text{QSR}} = 0.5,\ \pi_{12}^{\text{SLR}} = 27,\ k_1 = k_2 = 0.$$

In comparison to the first case study, we make the following observations in the optimal strategies: Both entities adjust their ALM strategy by investing more capital in the risky stock in order to improve the expected return. ICTs from entity 1 to entity 2 decrease for insensitive terminal NAVs. More precisely, entity 1 pays 11% of the liabilities of entity 2, when a quota share reinsurance is present. For a stop loss reinsurance, liabilities of entity 2 exceeding the amount of $\pi_{12} = 25.58$ are captured by entity 1. For sensitive terminal NAVs, ICTs stay at the same level as in the previous example. Since the objective is to maximize the expected return on capital, in all cases, initial or terminal equity is increased, respectively.

Problem	Maximizer	Maximal Value
$\mathcal{T}^{\text{tri}}(L)$	$(\vartheta, \pi_z, k) = (10.9316; 2.1346; 8.4175; 0.9368; \pi_z; 0.7246; 1.4445)$ $\pi_{\text{QSR}} = 0, \quad \pi_{\text{SLR}} = 31$	1.00
$\mathcal{T}^{\text{QSR}}(L)$	$(\vartheta, \pi, k) = (11.2198; 1.9185; 7.9977; 1.2517; 0.1134; 1.9021; 0.4105)$	1.00
$\mathcal{T}^{\text{SLR}}(L)$	$(\vartheta, \pi, k) = (11.1146; 1.9974; 8.2563; 1.0578; 25.5877; 1.1189; 1.2411)$	1.00
Consolidated	$(\vartheta, k) = (11.0170; 2.0707; 8.0662; 1.2003; 0.0373; 0.0481)$	1.00

Table 3.5: Optimal strategy for maximizing the expected return on capital when terminal NAVs are insensitive.

Problem	Maximizer	Maximal Value
$\mathcal{T}^{\text{tri}}(L)$	$(\vartheta, \pi_z, k) = (11.1028; 2.4809; 9.1888; 0.7057; \pi_z; 1.8989; 1.3894)$ $\pi_{\text{QSR}} = 0, \quad \pi_{\text{SLR}} = 31$	1.00
$\mathcal{T}^{\text{QSR}}(L)$	$(\vartheta, \pi, k) = (11.1268; 2.0425; 8.1451; 1.1728; 0.0850; 0.2172; 0.1263)$	1.00
$\mathcal{T}^{\text{SLR}}(L)$	$(\vartheta, \pi, k) = (11.0481; 2.0715; 8.2981; 1.0894; 26.7343; 0.0973; 0.2519)$	1.00
Consolidated	$(\vartheta, k) = (11.0227; 2.0758; 8.0701; 1.2035; 0.0376; 0.0242)$	1.00

Table 3.6: Optimal strategy for maximizing the expected return on capital when terminal NAVs are sensitive.

Comparing total required capitals in Figure 3.4 particularly illustrates the decrease of capital for sensitive terminal NAVs when ICTs are included in the network structure. By choosing an optimal strategy, the capital requirement can be reduced efficiently and approximates the lower bound of the consolidated regulatory framework.

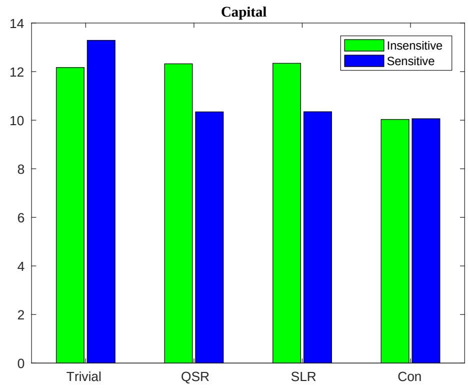


Figure 3.4: Network capital requirement for maxmizing the expected return on capital for sensitive and insensitive terminal NAVs.

3.4 | Conclusion

In this chapter, we provide a unified framework for the regulation of corporate networks and their solvency capital requirements. Capital requirements are defined in terms of a set-valued network risk measure, depending on terminal net asset values, a regulatory aggregation function, and acceptability criteria related to scalar-valued monetary risk measures. Network risk is measured by the set of vectors of additional capital requirements such that terminal net asset values are acceptable for regulatory purposes. In particular, the solvability of corporate networks is directly linked to acceptability.

For regulatory purposes different capital allocations may be acceptable, and so the network's management can choose - depending on its objective function - an optimal asset allocation that satisfies regulatory requirements at the same time. We provide an optimal set-valued capital allocation principle for a variety of regulatory frameworks. In particular, the Euler allocation principle can be embedded into our setting and coincides with the consolidated approach (insensitive). Capital allocations interfere with management strategies including asset-liability management (ALM) strategies and internal capital transfers (ICTs) defined by the network's management. To analyze their impact on capital requirements and optimal capital allocations, we include ALM and ICT strategies explicitly. Numerical case studies indicate that consolidated balance sheets can be mimicked via optimal management strategies.

A question for future research concerns the explicit characterization of optimal set-valued capital allocations for different regulatory systems and management's objective functions beyond the Euler allocation. Moreover, the suitability of certain capital allocations for performance measurement should be analyzed in order to evaluate their economic efficiency.

3.5 | Appendix: Risk Measures and Their Acceptance Sets

We denote by $\mathcal{X}$ a vector space of measurable, real-valued functions on a measurable space $(\Omega, \mathcal{F})$ that contains the constants. If P is a probability measure on $(\Omega, \mathcal{F})$, typical examples of $\mathcal{X}$ are L^p-spaces, $p \in [1, \infty]$, where P-almost sure equal functions are identified with each other.
A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is called *monetary risk measure* if it satisfies

1. *Monotonicity:* $X, Y \in \mathcal{X}, X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$
2. *Cash-invariance:* $X \in \mathcal{X}, m \in \mathbb{R} \Rightarrow \rho(X + m) = \rho(X) - m$

Property 1 states that the risk of a position Y is smaller than the risk of a position X, if the future value of Y is at least X. Property 2 states that risk is measured on a monetary scale: If m Euro are added to X, then the risk of X is exactly reduced by this amount.

In particular, any monetary risk measure corresponds to its *acceptance set*, $\mathcal{A} = \{X \in \mathcal{X} \,|\, \rho(X) \leq 0\}$, from which it can be recovered via

$$\rho(X) = \inf\{m \in \mathbb{R} \,|\, X + m \in \mathcal{A}\}.$$

Thus, a monetary risk measure can be viewed as a capital requirement: $\rho(X)$ is the minimal capital that has to be added to the position X to make it acceptable.

Remark 3.5.1. The defining properties of an acceptance set $\mathcal{A}$ are given by (see, e.g., Föllmer & Schied (2011), Section 4.1, and Feinstein et al. (2017), Section 2.2):

(i) $\inf\{m \in \mathbb{R} \,|\, m \in \mathcal{A}\} > -\infty$.
(Not any deterministic monetary amount is acceptable.)

(ii) If $X \in \mathcal{A}, Y \in \mathcal{X}(\mathbb{R})$ and $Y(\omega) \geq X(\omega)\ \forall \omega \in \Omega$, then $Y \in \mathcal{A}$.
(Positions that dominate acceptable positions are again acceptable.)

(iii) $\mathcal{A}$ is closed in $\mathcal{X}(\mathbb{R})$.

For further properties and examples of monetary risk measures, we refer to Appendix A.

3.6 | Appendix: Examples of the Euler Allocation Principle

To be self-contained, we briefly recall the well-known examples for Euler capital allocations for the scalar-valued risk measures applied in the numerical case studies in Section 3.2.3.

(i) Let ρ be based on the standard deviation, i. e.,

$$\rho\left(\Lambda^w\left(E(e, \mathbf{0})\right)\right) = -\mathbb{E}\left[\Lambda^w\left(E(e, \mathbf{0})\right)\right] + \gamma\sqrt{\mathrm{Var}\left(\Lambda^w\left(E(e, \mathbf{0})\right)\right)}, \quad \gamma > 0.$$

Then, the Euler capital allocation is given by the *covariance principle*, i. e.,

$$A_{\nabla,i}(E^{\mathbf{0}}, w) = -\mathbb{E}\left[w_i E_i(e, \mathbf{0})\right] + \gamma \frac{\mathrm{Cov}\left(w_i E_i(e, \mathbf{0}), \Lambda^w(E(e, \mathbf{0}))\right)}{\sqrt{\mathrm{Var}\left(\Lambda^w\left(E(e, \mathbf{0})\right)\right)}},$$

see, e. g., Kalkbrener (2005) and Buch & Dorfleitner (2008).

(ii) Let ρ be the value at risk at level $\lambda \in (0,1)$, i. e.,

$$\rho\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right) = \inf\{m \in \mathbb{R} \mid P\left(\Lambda^w\left(E(e,\mathbf{0})\right) + m < 0\right) \leq \lambda\} = -q^+_{\Lambda^w(E(e,\mathbf{0}))}(\lambda).$$

Then, subject to technical conditions, the Euler capital allocation is given by

$$A_{\nabla,i}(E^{\mathbf{0}}, w) = -\mathbb{E}[w_i E_i(e,\mathbf{0}) \mid \Lambda^w\left(E(e,\mathbf{0})\right) = -\mathrm{V@R}_\lambda\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right)],$$

see, e. g., Tasche (2000) and McNeil et al. (2015). For Gaussian random variables and $\lambda \leq 0.5$, the Euler capital allocation coincides with the covariance principle with $\gamma = \Phi^{-1}(1-\lambda)$, cf. Appendix A, Remark A.0.6.

(iii) Let ρ be the average value at risk at level $\lambda \in (0,1)$, i. e.,

$$\rho\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right) = \frac{1}{\lambda}\int_0^\lambda \mathrm{V@R}_\alpha\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right)\, d\alpha.$$

In this case, the Euler capital allocation takes the form

$$A_{\nabla,i}(E^{\mathbf{0}}, w) = -\mathbb{E}[w_i E_i(e,\mathbf{0}) \mid -\Lambda^w\left(E(e,\mathbf{0})\right) \geq \mathrm{V@R}_\lambda\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right)],$$

see, e. g., Tasche (2004), Prop. 5, and McNeil et al. (2015). For Gaussian random variables, the Euler capital allocation is given by the covariance principle with $\gamma = \frac{\phi(\Phi^{-1}(1-\lambda))}{\lambda}$, cf. Appendix A, Remark A.0.6.

3.7 | Appendix: Proofs

In this section, we provide the proofs of the results presented in Sections 3.1, 3.2 & 3.3.

Proof of Proposition 3.1.5.

Proof. Assume that $k^* \in R^{\Lambda^1}(E;e) = \{k \in \mathbb{R}^n \mid \rho\left(E_i(e,k)\right) \leq 0 \;\forall i \in N\}$. Subadditivity of ρ implies

$$\rho\left(\sum_{i=1}^n E(e,k^*)\right) \leq \sum_{i=1}^n \rho\left(E_i(e,k^*)\right) \leq 0,$$

hence

$$k^* \in \left\{k \in \mathbb{R}^n \mid \rho\left(\sum_{i=1}^n E_i(e,k)\right) \leq 0\right\} = R^{\Lambda^{2.2}}(E;e).$$

This shows $R^{\Lambda^1}(E;e) \subseteq R^{\Lambda^{2.2}}(E;e)$. □

Proof of Proposition 3.1.6.

Proof. (i) Let $k^* \in \mathbb{R}^n_+$ be an additional capital allocation in the case of insensitive terminal NAVs, i. e.,

$$k^* \in R^{\Lambda^1}(\underline{E}; e) = \left\{k \in \mathbb{R}^n_+ \mid \rho_i\left(E_i(e, \mathbf{0}) + k_i\right) \leq 0 \ \forall i \in N\right\}.$$

Since terminal NAVs are assumed to be translation-supervariant in the initial equity, we have $E_i(e, \mathbf{0}) + k_i^* \leq E_i(e + k^*, \mathbf{0})$ for all $i \in N$, and so monotonicity of the monetary risk measures implies

$$\rho_i\left(E_i(e + k^*, \mathbf{0})\right) \leq \rho_i\left(E_i(e, \mathbf{0}) + k_i^*\right) \leq 0 \quad \text{for all } i \in N.$$

Hence, $k^* \in \left\{k \in \mathbb{R}^n_+ \mid \rho_i\left(E_i(e + k, \mathbf{0})\right) \leq 0 \ \forall i \in N\right\} = R^{\Lambda^1}(\overline{E}; e)$.

(ii) The proof is analogous to the proof in (i).

□

Proof of Proposition 3.1.8.

Proof. (i) Since $\overline{E}$ is sensitive, we have

$$\begin{aligned} R^{\Lambda}(\overline{E}; e + l) &= \left\{k \in \mathbb{R}^n \mid \Lambda_j\left(\overline{E}(e + l, k)\right) \in \mathcal{A}_j, \ j = 1, \ldots, m\right\} \\ &= \left\{k \in \mathbb{R}^n \mid \Lambda_j\left(E(e + l + k, \mathbf{0})\right) \in \mathcal{A}_j, \ j = 1, \ldots, m\right\}. \end{aligned}$$

Setting $b = l + k$ leads to

$$\begin{aligned} R^{\Lambda}(\overline{E}; e + l) &= \left\{b - l \in \mathbb{R}^n \mid \Lambda_j\left(E(e + b, \mathbf{0})\right) \in \mathcal{A}_j, \ j = 1, \ldots, m\right\} \\ &= \left\{b \in \mathbb{R}^n \mid \Lambda_j\left(\overline{E}(e, b)\right) \in \mathcal{A}_j, \ j = 1, \ldots, m\right\} - l = R^{\Lambda}(\overline{E}; e) - l. \end{aligned}$$

(ii) Let $F_i(e, k) \leq E_i(e, k)$, $i \in N$. Since Λ is increasing, it is $\Lambda_j\left(F(e, k)\right) \leq \Lambda_j\left(E(e, k)\right)$ for all $j = 1, \ldots, m$. Hence, $\Lambda_j\left(F(e, k)\right) \in \mathcal{A}_j$ implies $\Lambda_j\left(E(e, k)\right) \in \mathcal{A}_j$, $j = 1, \ldots, m$, due to Remark 3.5.1 (ii). Thus, $R^{\Lambda}(F; e) \subseteq R^{\Lambda}(E; e)$.

(iii) Let $\alpha \in [0, 1]$ be fixed. Since $k, l \in R^{\Lambda}(\underline{E}; e)$, it is $\Lambda_j\left(\underline{E}(e, k)\right), \Lambda_j\left(\underline{E}(e, l)\right) \in \mathcal{A}_j$, $j = 1, \ldots, m$. Due to convexity of $\mathcal{A}_j$, we have $\alpha\Lambda_j\left(\underline{E}(e, k)\right) + (1 - \alpha)\Lambda_j\left(\underline{E}(e, l)\right) \in \mathcal{A}_j$ for all j. The concavity of Λ_j yields

$$\alpha\Lambda_j\left(\underline{E}(e, k)\right) + (1 - \alpha)\Lambda_j\left(\underline{E}(e, l)\right) \leq \Lambda_j\left(\alpha\underline{E}(e, k) + (1 - \alpha)\underline{E}(e, l)\right),$$

and this implies

$$\begin{aligned} \Lambda_j\left(\alpha\underline{E}(e, k) + (1 - \alpha)\underline{E}(e, l)\right) &= \Lambda_j\left(E(e, \mathbf{0}) + \alpha\, k + (1 - \alpha)\, l\right) \\ &= \Lambda_j\left(\underline{E}(e, \alpha\, k + (1 - \alpha)\, l\right) \in \mathcal{A}_j \text{ for all } j. \end{aligned}$$

From the definition of R^{Λ}, we conclude $\alpha k + (1 - \alpha)l \in R^{\Lambda}(\underline{E}; e)$.

□

Proof of Proposition 3.2.2.

Proof. (i) For the stand-alone approach the network risk measure takes the form

$$\begin{aligned} R^{\Lambda^1}(E^1; e) &= \{k \in \mathbb{R}^n \mid X_i + e_i + k_i \in \mathcal{A}_i, \forall i \in N\} \\ &= \{k \in \mathbb{R}^n \mid \rho_i(X_i + e_i) \leq k_i, \forall i \in N\}. \end{aligned}$$

This leads to the optimal set-valued capital allocation

$$\begin{aligned} A^{\Lambda^1,\mu}(E^1; e) &= \arg\min \left\{ \sum_{i=1}^n l_i \,\middle|\, l \in \mathbb{R}^n : \rho_i(X_i + e_i) \leq l_i, \forall i \in N \right\} \\ &= \{l \in \mathbb{R}^n \mid \rho_i(X_i + e_i) = l_i, \forall i \in N\} \\ &= \begin{pmatrix} \rho_1(X_1 + e_1) \\ \rho_2(X_2 + e_2) \\ \vdots \\ \rho_n(X_n + e_n) \end{pmatrix} = \begin{pmatrix} \mathrm{SCR}_1 - e_1 \\ \mathrm{SCR}_2 - e_2 \\ \vdots \\ \mathrm{SCR}_n - e_n \end{pmatrix} \end{aligned}$$

(ii) The network risk measure in the legal entity approach is given by

$$\begin{aligned} R^{\Lambda^1}(\overline{E}; e) &= \{k \in \mathbb{R}^n \mid E_i(e + k, \mathbf{0}) \in \mathcal{A}_i, \forall i \in N\} \\ &= \{k \in \mathbb{R}^n \mid \rho_i(E_i(e + k, \mathbf{0})) \leq 0, \forall i \in N\}, \end{aligned}$$

and this yields the optimal set-valued capital allocation

$$A^{\Lambda^1,\mu}(\overline{E}; e) = \arg\min \left\{ \sum_{i=1}^n l_i \,\middle|\, l \in \mathbb{R}^n : \rho_i(E_i(e + l, \mathbf{0})) \leq 0\, \forall i \in N \right\}.$$

(iii) For the consolidated approach the network risk measure takes the form

$$\begin{aligned} R^{\Lambda^{2.2}}(\overline{E}; e) &= \left\{ k \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n E_i(e + k, \mathbf{0}) \in \mathcal{A} \right\} \\ &= \left\{ k \in \mathbb{R}^n \,\middle|\, \rho\left(\sum_{i=1}^n E_i(e + k, \mathbf{0}) \right) \leq 0 \right\}. \end{aligned}$$

Thus, the optimal set-valued capital allocation is given by

$$A^{\Lambda^{2.2},\mu}(\overline{E}; e) = \arg\min \left\{ \sum_{i=1}^n l_i \,\middle|\, l \in \mathbb{R}^n : \rho\left(\sum_{i=1}^n E_i(e + l, \mathbf{0}) \right) \leq 0 \right\}.$$

(iv) The network risk measure in the legal entity approach (insensitive) becomes

$$R^{\Lambda^1}(\underline{E}; e) = \{k \in \mathbb{R}^n \mid E_i(e, \mathbf{0}) + k_i \in \mathcal{A}_i, \forall i \in N\}$$

$$= \{k \in \mathbb{R}^n \mid \rho_i(E_i(e, \mathbf{0})) \le k_i, \forall i \in N\}.$$

This leads to the optimal set-valued capital allocation

$$A^{\Lambda^1,\mu}(\underline{E}; e) = \arg\min \left\{ \sum_{i=1}^n l_i \,\middle|\, l \in \mathbb{R}^n : \rho_i(E_i(e, \mathbf{0})) \le l_i, \forall i \in N \right\}$$
$$= \begin{pmatrix} \rho_1(E_1(e, \mathbf{0})) \\ \rho_2(E_2(e, \mathbf{0})) \\ \vdots \\ \rho_n(E_n(e, \mathbf{0})) \end{pmatrix}$$

(v) For the consolidated approach (insensitive) the network risk measure is given by

$$R^{\Lambda^{2.2}}(\underline{E}; e) = \left\{ k \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n E_i(e, \mathbf{0}) + k_i \in \mathcal{A} \right\}$$
$$= \left\{ k \in \mathbb{R}^n \,\middle|\, \rho\left(\sum_{i=1}^n E_i(e, \mathbf{0})\right) \le \sum_{i=1}^n k_i \right\},$$

and so we derive the optimal set-valued capital allocation

$$A^{\Lambda^{2.2},\mu}(\underline{E}; e) = \arg\min \left\{ \sum_{i=1}^n l_i \,\middle|\, l \in \mathbb{R}^n : \rho\left(\sum_{i=1}^n E_i(e, \mathbf{0})\right) \le \sum_{i=1}^n l_i \right\}$$
$$= \left\{ l \in \mathbb{R}^n \,\middle|\, \rho\left(\sum_{i=1}^n E_i(e, \mathbf{0})\right) = \sum_{i=1}^n l_i \right\}.$$

□

Proof of Corollary 3.2.4.

Proof. The claim follows from $R^{\Lambda^1}(E; e) \subseteq R^{\Lambda^{2.2}}(E; e)$, cf. Prop. 3.1.5. □

Proof of Corollary 3.2.6.

Proof. Due to Eq. (3.6), it is $\sum_{i=1}^n A_{\nabla,i}(E^{\mathbf{0}}, \mathbf{1}) = \rho\left(\sum_{i=1}^n E_i(e, \mathbf{0})\right)$. From Prop. 3.2.2 (v), we obtain

$$A_\nabla(E^{\mathbf{0}}, \mathbf{1}) \in \left\{ l \in \mathbb{R}^n \,\middle|\, \rho\left(\sum_{i=1}^n E_i(e, \mathbf{0})\right) = \sum_{i=1}^n l_i \right\} = A^{\Lambda^{2.2},\mu}(\underline{E}; e)$$

for the particular choice $l_i = A_{\nabla,i}(E^{\mathbf{0}}, \mathbf{1})$. □

Proof of Proposition 3.3.1.

Proof. Let $k^* \in R^{\Lambda,\mathrm{tri}}(E; e)$, i. e., $\Lambda_j(E(z, e, k^*)) \in \mathcal{A}_j$ for all $j = 1, \dots, m$, and $z \in Z^{tri}$. Since $\mathcal{Z}^{tri} \subseteq \mathcal{Z}$ for each $z^1 \in \mathcal{Z}^{tri}$, we can choose $z^2 \in \mathcal{Z}$ such that $E_i(z^1, e, k^*) \le E_i(z^2, e, k^*)$ for all $i \in N$. Since Λ is increasing, it is $\Lambda_j(E(z^1, e, k^*)) \le \Lambda_j(E(z^2, e, k^*))$ for all j and hence, $\Lambda_j(E(z^1, e, k^*)) \in \mathcal{A}_j$ implies $\Lambda_j(E(z^2, e, k^*)) \in \mathcal{A}_j$ due to Remark 3.5.1

(ii). Thus, $k^* \in \{k \in \mathbb{R}^n \,|\, \Lambda_j(E(z,e,k)) \in \mathcal{A}_j \ \forall j = 1,\ldots,m, z \in \mathcal{Z}\} = R^{\Lambda,\mathcal{Z}}(E;e)$. Hence, $R^{\Lambda,\text{tri}}(E;e) \subseteq R^{\Lambda,\mathcal{Z}}(E;e)$. □

Proof of Corollary 3.3.3.

Proof. By Prop. 3.3.1, we have $R^{\Lambda,\text{tri}}(E;e) \subseteq R^{\Lambda}(E;e)$. Together with $\mathcal{Z}^{tri} \subseteq \mathcal{Z}$, this yields the claim. □

4 | Optimal Risk Sharing in Insurance Networks: An Application to Asset-Liability Management

This chapter is based on the working paper *Optimal Risk Sharing in Insurance Networks: An Application to Asset-Liability Management* by Hamm, Knispel & Weber. A revised version will appear in the *European Actuarial Journal*, see Hamm, Knispel & Weber (2019*b*).

Capital requirements of insurance companies or banks serve as a protection of policyholders, banking customers, and creditors. They provide a buffer against downside risk, i. e., the adverse random fluctuations of the financial resources of a company. In internal models of financial institutions such capital requirements are computed on the basis of the simulated distribution of the firm's book value of equity at a finite time horizon, also called the future net asset value. The resulting capital requirements depend on the risk measures that are used. While the regulation scheme Solvency II is based on the risk measure value at risk (V@R), the Swiss Solvency Test employs a coherent risk measure, average value at risk (AV@R). The influence of risk measures on capital requirements as well as their properties have been the subject of intense scientific research over the last twenty years, see, e. g., Föllmer & Schied (2004) or Föllmer & Weber (2015).

In this chapter, we discuss the impact of risk sharing and asset-liability management on capital requirements. This investigation will contribute to the evaluation of the merits and deficiencies of different risk measures. In particular, we highlight that the class of V@R-based risk measures, as defined in Weber (2018), allows for a substantial reduction of the total capital requirement in corporate networks that share risks between entities. We provide case studies that complement the theoretical analysis of Embrechts, Liu & Wang (2018) and Weber (2018), and illustrate their practical relevance. In addition, we refine in Section 4.1.2.2 the tail allocations suggested in these papers to ensure that downside risk is shared in an approximately symmetric manner. The analysis of optimal risk sharing within a model for asset-liability management is – to the best of our knowledge – new to the literature.

The chapter is structured as follows: Section 4.1.1 first reviews the notion of solvency capital requirements, emphasizing that the definition commonly used in practice deviates from an alternative definition that is naturally derived from the notion of acceptability in the theory of monetary risk measures. Second, in Section 4.1.2, we consider risk sharing between entities. If networks of companies are not required to report a total capital requirement on the basis of a consolidated balance sheet, risk sharing may serve as an instrument to reduce required capital. We review the general risk sharing problem and the notion of inf-convolutions, summarize theoretical results for distortion risk measures of Embrechts et al. (2018) or Weber (2018), and refine the allocations of the distribution's tail within the network. Section 4.2 introduces a model setting that admits the joint analysis of asset-liability management and risk sharing. The general structure is described in Section 4.2.1.1. In the context of a Black-Scholes asset model and for deterministic liabilities, the inf-convolutions and capital requirements are explicitly computed for three important examples: average value at risk and the two V@R-type risk measures value at risk and range value at risk. More sophisticated models are then analyzed on the basis of Monte Carlo case studies. Section 4.2.2 describes how distributions and parameters are chosen, and how we calibrate value at risk, average value at risk and range value at risk in order to allow for a meaningful comparison of these risk measures. In Section 4.2.3, we analyze three case studies of different complexity: a) Assets are modeled by a Black-Scholes market, liabilities are deterministic. b) Liabilities may be random; different types of dependence between assets and liabilities are investigated. c) An additional left-tailed asset is available. We find that corporate networks may largely hide downside risk, if capital requirements are computed on the basis of V@R-type risk measures. For large networks, optimal asset-liability management is often contrary to those strategies that are desirable from a regulatory point of view. The results are quite striking and thus we discuss this issue in detail.

Literature. The general problem of optimal risk sharing is an ongoing field of research. Barrieu & El Karoui (2005) and Barrieu & El Karoui (2008) introduced the inf-convolution in order to formulate the risk sharing problem among agents with *convex* risk measures. They show that the inf-convolution of two convex risk measures is again a convex risk measure. The optimal structure to the optimal risk sharing problem is explicitly derived when agents have *dilated* risk measures, i. e., $\rho_\gamma(Z) = \frac{1}{\gamma}\,\rho\,(\gamma Z)$. Jouini, Schachermayer & Touzi (2008) show that for *law-invariant* monetary utility functions (i. e., law-invariant convex risk measures) the set of Pareto optimal comonotone allocations is non-empty. Acciaio (2007) considers *non-necessarily monotone* monetary functionals and characterizes optimal solutions in that case. Moreover, the existence of such solutions is proven. The author introduces the *best* monotone approximation of non-monotone functionals, where the resulting optimization corresponds to the inf-convolution with constraint $\sum_{i=1}^{n} Z_i \leq Z$ defined by Filipović & Kupper (2008*a*) and Filipović & Svindland (2008). Explicit calculations of optimal risk sharing rules for particular cases are given. Further risk sharing strategies for special cases of two or three agents can be found in Acciaio (2005). The case of non-necessarily monotone, law- and cash-invariant convex functions is also considered by Filipović & Svindland (2008). The authors prove that the capital and risk allocation problem always admits a solution via contracts whose payoffs are defined as increasing Lipschitz-continuous functions of the total risk. Embrechts et al. (2018) solve the optimal risk sharing problem for range value at risk and state some robustness results on optimal allocations. Weber (2018) provides a risk sharing rule for distortion risk measures and embeds the risk sharing problem into the context of corporate networks, for example, insurance networks. We also refer to Weber (2018) for a discussion on the classification of networks vs. groups. Weber's (2018) general framework includes the results of Embrechts et al. (2018) as special cases and constitutes the basis for our comparison of the impact of different risk measures on regulatory capital requirements.

4.1 | Capital Regulation and Network Risk Minimization

4.1.1 | Capital Requirements

Capital requirements are a cornerstone of regulation schemes such as *Basel III* for banks, *Solvency II* for European insurance companies, or the *Swiss Solvency Test* for insurance companies in Switzerland. The key idea is that financial firms should hold a buffer for potential losses that ensures the firm's financial solvability and thereby serves to protect customers, policyholders, and other counterparties. The computation of such a capital requirement - in the sequel named *solvency capital requirement* (SCR) - typically involves two components: *stochastic balance sheet projections* capturing the random evolution of the firm's equity over a given time horizon, and a *monetary risk measure* that quantifies the inherent risk on a monetary scale or, equivalently, specifies acceptability of financial positions, e. g., from the perspective of a financial supervisory authority, a rating agency, or the board of management.

To formalize the SCR computation in a stylized manner, let us consider an atomless probability space $(\Omega, \mathcal{F}, P)$ and a one period economy with two dates, say $t = 0, 1$. Time 0 is interpreted as today, time 1 as the future time horizon of the regulation scheme, e. g., one year in case of Solvency II. We denote by $\mathcal{X}$ the set of financial positions at time 1 whose risk needs to be assessed. By sign convention, negative values correspond to debt or losses. Throughout this chapter, $\mathcal{X}$ is a vector space of random variables on $(\Omega, \mathcal{F}, P)$ that contains the constants.

Assets and Liabilities. At time $t = 0, 1$, the economic values of assets and liabilities of a financial firm according to the solvency balance sheet are denoted by A_t and L_t, respectively, and the book value of equity or net asset value (NAV) is then derived as $E_t = A_t - L_t$. Note that the quantities A_0, L_0, E_0 at $t = 0$ are deterministic, while their counterparts A_1, L_1, E_1 at $t = 1$ are typically not known in advance, but random. Mathematically, the values of assets and liabilities A_1, L_1 and the resulting equity E_1 are modeled as real-valued random variables on the given probability space $(\Omega, \mathcal{F}, P)$. In practice, these quantities can be derived from stochastic balance sheet projections within sophisticated *internal models* that rely extensively on Monte Carlo simulations.

Solvency Capital Requirement. Regulatory guidelines typically describe requirements on the SCR computation verbally, but do not provide an exact and unique SCR definition

in mathematical terms. In particular, as illustrated in Example 4.1.1 below for Solvency II regulation, regulatory requirements can be contradictory, leaving considerable room for interpretation.
In this chapter, we focus on two different SCR definitions:

$$\begin{aligned} \mathrm{SCR}_{\mathcal{A}}(E_1) &:= \rho(E_1 - E_0), \\ \mathrm{SCR}_{\mathrm{mean}}(E_1) &:= \rho(E_1 - \mathbb{E}[E_1]), \end{aligned}$$

where ρ denotes a monetary risk measure with acceptance set $\mathcal{A}$ such as *value at risk* (V@R), *average value at risk* (AV@R) or *range value at risk* (RV@R), see Appendix A for a short review on monetary risk measures, and in particular Example A.0.3 for the definition of V@R, AV@R and RV@R.

While $\mathrm{SCR}_{\mathcal{A}}(E_1)$ evaluates the risk of the random capital increment $E_1 - E_0$ over the given time horizon, the alternative definition $\mathrm{SCR}_{\mathrm{mean}}(E_1)$ refers to the firm's centered equity $E_1 - \mathbb{E}[E_1]$ at time 1. Also note that

$$\mathrm{SCR}_{\mathcal{A}}(E_1) = E_0 + \rho(E_1) \quad \text{and} \quad \mathrm{SCR}_{\mathrm{mean}}(E_1) = \mathbb{E}[E_1] + \rho(E_1),$$

due to cash-invariance of the monetary risk measure ρ.

From a conceptional point of view, the definition $\mathrm{SCR}_{\mathcal{A}}$ corresponds to a regulator's perspective, and it is based on the natural requirement that equity E_1 at time 1 should be acceptable with respect to a prescribed monetary risk measure ρ, i. e.,

$$E_1 \in \mathcal{A} \quad \Leftrightarrow \quad \rho(E_1) \leq 0.$$

For $\mathrm{SCR}_{\mathcal{A}}(E_1) = E_0 + \rho(E_1)$, acceptability of the firm's equity E_1 is equivalent to $\mathrm{SCR}_{\mathcal{A}}(E_1) \leq E_0$, i. e., the firm's equity is sufficient to cover the solvency capital requirement. In practice, however, it is a common approach to consider only unexpected losses, in particular for market risks and underwriting risks. This leads to the alternative definition $\mathrm{SCR}_{\mathrm{mean}}$.

Example 4.1.1. For Solvency II regulation, Recital 64 of the Directive 2009/138/EC states that capital must be sufficient to prevent ruin with probability 99.5% on a one-year time horizon, i. e., $P(E_1 < 0) \leq \alpha$ with $\alpha = 0.005$. This condition is equivalent to $E_1 \in \mathcal{A}_{\mathrm{V@R}_{0.005}}$, where $\mathcal{A}_{\mathrm{V@R}_{0.005}}$ denotes the acceptance set of value at risk defined in Appendix A, Eq. (A.1). Hence, a canonical SCR definition in the context of Solvency II is

$$\mathrm{SCR}_{\mathcal{A}}(E_1) := \mathrm{V@R}_{0.005}(E_1 - E_0) = E_0 + \mathrm{V@R}_{0.005}(E_1) = E_0 - q^+_{E_1}(0.005),$$

where $q^+_{E_1}$ denotes the upper quantile function of E_1, see Appendix A, Definition A.0.2.

Contradicting, §101(2) of the Directive 2009/138/EC prescribes that the SCR "shall cover only unexpected losses“, and that "it shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5 % over a one-year period." This supports the definition in terms of the so-called *mean value at risk*

$$\mathrm{SCR}_{\mathrm{mean}}(E_1) := \mathrm{V@R}_{0.005}(E_1 - \mathbb{E}[E_1]) = \mathbb{E}[E_1] + \mathrm{V@R}_{0.005}(E_1) = \mathbb{E}[E_1] - q^{+}_{E_1}(0.005)$$

which is widely used in practice. Both definitions are consistent to specific regulatory requirements, but lead, however, to different solvency capital requirements.

Financial institutions are typically owned by shareholders with limited liability. The free surplus - given as equity less SCR - can be distributed as dividends to the shareholders. Consequently, shareholders and the management board are interested in reducing the SCR via appropriate techniques. In the sequel, we focus on this problem from a network's perspective.

4.1.2 | The Risk Sharing Problem of the Network

4.1.2.1 | Inf-Convolutions

Consider a financial network that consists of n entities that are all individually subject to capital regulation. We suppose that the solvency capital requirement of entity $i = 1, 2, \ldots, n$ is computed based on a monetary risk measure ρ^i, and we write $\mathrm{SCR}^i_{\mathcal{A}}$ and $\mathrm{SCR}^i_{\mathrm{mean}}$, respectively, to differentiate between the two SCR definitions for entity i.

The network's balance sheet is obtained by consolidating the individual balance sheets of its sub-entities. Denoting from now on by A_t and L_t the total consolidated assets and liabilities at times $t = 0, 1$, the network's total equity is given by $E_t = A_t - L_t$, $t = 0, 1$. The corporate network now uses at time $t = 0$ legally binding transfer agreements to modify the equities at time $t = 1$. The resulting new allocation is denoted by $\left(E^i\right)_{i=1,\ldots,n}$, where $\sum_{i=1}^n E_1^i = E_1$ and $\sum_{i=1}^n E_0^i = E_0$. In this situation, the total SCR of the network is given by

$$\sum_{i=1}^{n} \mathrm{SCR}^i_{\mathcal{A}}(E_1^i) = E_0 + \sum_{i=1}^{n} \rho^i(E_1^i) \quad \text{and} \quad \sum_{i=1}^{n} \mathrm{SCR}^i_{\mathrm{mean}}(E_1^i) = \mathbb{E}[E_1] + \sum_{i=1}^{n} \rho^i(E_1^i),$$

respectively. This definition relies on the assumption that the firm's individual SCRs are added up to obtain the network's SCR. In particular, this means that the network's SCR is not computed based on a consolidated solvency balance sheet.

For both SCR definitions, the minimization of the network's SCR is equivalent to the minimization of $\sum_{i=1}^n \rho^i\left(E_1^i\right)$. In other words, for a fixed number of n firms the problem of the network consists in the design of optimal transfers that minimize $\sum_{i=1}^n \rho^i\left(E_1^i\right)$. We thus face the optimal risk sharing problem

$$\Box_{i=1}^n \rho^i(E_1) = \inf\left\{ \sum_{i=1}^{n} \rho^i\left(E_1^i\right) \,\middle|\, \sum_{i=1}^{n} E_1^i = E_1,\ E_1^1, \ldots, E_1^n \in \mathcal{X} \right\}, \tag{4.1}$$

also known as *inf-convolution*. Let us write

$$\Box_{i=1}^n \mathrm{SCR}^i_{\mathcal{A}}(E_1) = E_0 + \Box_{i=1}^n \rho^i(E_1) \quad \text{and} \quad \Box_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}(E_1) = \mathbb{E}[E_1] + \Box_{i=1}^n \rho^i(E_1) \tag{4.2}$$

for the corresponding solvency capital requirements.

Remark 4.1.2. Let ρ be a coherent risk measure and assume that $\rho^i = \rho$ for any firm $i = 1, \ldots, n$, $n \in \mathbb{N}$. In this case, optimal risk sharing and splitting the risk within the network to more firms do not reduce the total network's risk, i. e.,

$$\square_{i=1}^{n} \rho(E_1) = \rho(E_1) \quad \text{for all } n \in \mathbb{N}.$$

Indeed, for all decompositions $E_1 = E_1^1 + \ldots + E_1^n$, subadditivity yields

$$\rho(E_1) = \rho\left(\sum_{i=1}^{n} E_1^i\right) \leq \sum_{i=1}^{n} \rho(E_1^i),$$

and this lower bound is attained for $E_1^i = \alpha^i E_1$, $i = 1, \ldots, n$, with $\alpha^1 + \ldots + \alpha^n = 1$.

4.1.2.2 | Risk Sharing with Distortion Risk Measures

In the context of distortion risk measures, problem (4.1) is discussed in Weber (2018). The risk measures V@R, AV@R and RV@R belong to this class of risk measures. Theorem 4.1.5 provides an upper bound to the solution and an allocation that attains this bound. The results in Weber (2018) characterize under which conditions the bound is attained and generalize the work of Embrechts et al. (2018).

Definition 4.1.3. An increasing function $g : [0, 1] \to [0, 1]$ with $g(0) = 0$ and $g(1) = 1$ is called a *distortion function*. If P is a probability measure on $(\Omega, \mathcal{F})$, then

$$c^g(A) := g(P(A)), \quad A \in \mathcal{F},$$

defines a capacity. The risk measure

$$\rho^g(X) := \int (-X) dc^g,$$

defined as the Choquet integral with respect to c^g, is called *distortion risk measure*.

As special cases, Weber (2018) introduces the class of V@R-type distortion risk measures.

Definition 4.1.4. Consider the class of distortion functions g such that

$$g(x) = 0, \quad \forall x \in [0, \alpha],$$
$$g(x) > 0, \quad \forall x \in (\alpha, 1],$$

for some $\alpha \in [0, 1)$. The number α is called the *parameter* of g and

$$\hat{g}(x) = \begin{cases} g(x + \alpha), & 0 \leq x \leq 1 - \alpha, \\ 1, & 1 - \alpha < x, \end{cases}$$

is the *active part* of g. If the parameter $\alpha > 0$, then ρ^g is called a V@R-*type distortion risk measure*.

The risk measures V@R and RV@R are of V@R-type, AV@R is not. This is shown in Table 4.1.

Risk Measure	$\mathrm{V@R}_\alpha$	$\mathrm{AV@R}_\beta$	$\mathrm{RV@R}_{\alpha,\beta}$
g (x) =	$\begin{cases} 0, & 0 \le x \le \alpha, \\ 1, & \alpha < x. \end{cases}$	$\begin{cases} \frac{x}{\beta}, & 0 \le x \le \beta, \\ 1, & \beta < x. \end{cases}$	$\begin{cases} 0, & 0 \le x \le \alpha, \\ \frac{x-\alpha}{\beta}, & \alpha < x \le \alpha+\beta, \\ 1, & \alpha+\beta < x. \end{cases}$
Type	V@R-type	**Not** V@R-type	V@R-type

Table 4.1: Distortion functions for the risk measures V@R, AV@R and RV@R for $\alpha, \beta > 0$ with $\alpha + \beta \le 1$.

The solution to the optimal risk sharing problem (4.1) minimizes the network's total risk. The minimizer is an allocation $\left(E^i\right)_{i=1,\dots,n}$ with $\sum_{i=1}^n E_1^i = E_1$ and $\sum_{i=1}^n E_0^i = E_0$.

Theorem 4.1.5 (Weber (2018), Theorem 2.4)**.** *Let $E_1 \in L^\infty$ and $n \in \mathbb{N}$. By $g^1, g^2, \dots, g^n$, we denote left-continuous distortion functions with parameters $\alpha_1, \alpha_2, \dots, \alpha_n \in [0,1)$, and define $d = \sum_{i=1}^n \alpha_i$. We set $\rho^i = \rho^{g_i}$, i. e., ρ^i is the distortion risk measure associated with the distortion function g^i, $i = 1, 2, \dots, n$. Define the left-continuous functions*

$$f = \min\left\{\widehat{g^1}, \widehat{g^2}, \dots, \widehat{g^n}\right\}, \qquad g(x) = \begin{cases} 0, & 0 \le x \le d \wedge 1, \\ f(x-d), & d \wedge 1 < x \le 1. \end{cases}$$

Note that $g \equiv 0$, if $d \ge 1$. In particular, g is not necessarily a distortion function with $g(1) = 1$. We set $\mathrm{V@R}_\lambda := \mathrm{V@R}_1 = -\operatorname{ess\,sup}$ for $\lambda \ge 1$.

1. *There exist $E_1^1, E_1^2, \dots, E_1^n \in L^\infty$ such that $\sum_{i=1}^n E_1^i = E_1$ and*

$$\sum_{i=1}^n \rho^i(E_1^i) = \int_{[0,1]} \mathrm{V@R}_\lambda(E_1) g(d\lambda) + (g(1) - 1) \operatorname{ess\,sup} E_1.$$

If $d \ge 1$, this equation can be simplified and we obtain

$$\sum_{i=1}^n \rho^i(E_1^i) = -\operatorname{ess\,sup} E_1.$$

2. *The allocation $(E_1^i)_{i=1,2,\dots,n}$ can be constructed as follows: Let*

$$Y := E_1 - \operatorname{ess\,sup} E_1 \le 0.$$

There exists a random variable U, uniformly distributed on $[0,1]$, such that $Y = -\mathrm{V@R}_U(Y)$. For $i = 1, 2, \dots, n$, we set

$$r_i(\lambda) = \begin{cases} 1, & i = \inf\{j : \hat{g}_j(1-\lambda) = f(1-\lambda)\}, \\ 0, & \textit{else}, \end{cases}$$

$(\lambda \in [0,1])$ and $R_i(y) = -\int_0^{|y|} r_i(\lambda)d\lambda$. We define $\tilde{Y} = Y \cdot \mathbb{1}_{\{U \geq d\}}$ and $\tilde{E}_1^i = R_i(\tilde{Y})$. For $i = 1, 2, \ldots, n$, we set

$$E_1^i \quad = \quad Y \cdot \mathbb{1}_{\left\{\sum_{l=1}^{i-1} \alpha_l \,\leq\, U \,<\, \sum_{l=1}^{i} \alpha_l\right\}} \; + \; \tilde{E}_1^i \; + \; \frac{\text{ess sup } E_1}{n} \tag{4.3}$$

If $d \geq 1$, this equation can be simplified and we obtain

$$E_1^i = Y \cdot \mathbb{1}_{\left\{\sum_{l=1}^{i-1} \alpha_l \,\leq\, U \,<\, \sum_{l=1}^{i} \alpha_l\right\}} \; + \; \frac{\text{ess sup } E_1}{n}$$

Remark 4.1.6. Theorem 4.1.5 can be generalized to unbounded random variables, see Weber (2018).

Tail Allocation. Theorem 4.1.5 characterizes a particular solution to (4.1), but for V@R-based risk measures multiple solutions are admissible. V@R-based risk measures ignore the extreme tail. This implies that the tail part of the distribution of E_1 that is hidden via risk sharing can be allocated to different entities in various ways. While V@R-based risk measurements remain invariant under these re-allocations of the tail, other quantities that are important from the perspective of the single entities may change, e.g., the profit of the individual firms in the network.

In contrast to Embrechts et al. (2018) and Weber (2018), we construct an alternative tail allocation; this also minimizes the network's total risk, but provides a fairer allocation of the extreme downside risk from the perspective of the single firms. In (4.3), the terms

$$Y \cdot \mathbb{1}_{\left\{\sum_{l=1}^{i-1} \alpha_l \,\leq\, U \,<\, \sum_{l=1}^{i} \alpha_l\right\}}$$

can, more generally, simply be replaced by

$$Y \cdot s_i(U)$$

for càdlàg functions $s_i : [0, d] \to \{0, 1\}$, $\int_0^d s_i(u)du = \alpha_i$, $i = 1, 2, \ldots, n$, with $\sum_{i=1}^n s_i \equiv 1$.

Remark 4.1.7. Suppose Y has a continuous density. Then, fair allocations of the downside risk among all entities can be constructed as follows:
Consider a sequence of n-tupels of functions $(s_i^m)_{i=1,2,\ldots,n}$, $m = 1, 2, \ldots$, as defined above. We set

$$S_i^m : [0,1] \to \mathbb{R}^{\mathbb{R}} \cup \{\dagger\}, \; S_i^m(x) = \begin{cases} \text{id}, & s_i^m(x) = 1, \; 0 \leq x \leq d, \\ \dagger, & \text{else.} \end{cases}$$

and define probability measures

$$\mu_i^m((a,b]) := \frac{1}{\alpha_i} P\left(S_i^m(U) \circ Y \in (a,b]\right), \quad -\infty < a < b < \infty.$$

There exists a sequence $(s_i^m)_{i=1,2,\ldots,n}$ $(m = 1, 2, \ldots)$ such that μ_i^m converges weakly as $m \to \infty$ to a limit that is independent of $i = 1, 2, \ldots, n$. An example of such a sequence is given by

$$s_i^m = \begin{cases} 1, & \frac{k}{m} \cdot d + \frac{\sum_{l=1}^{i-1} \alpha_l}{m} \ \leq \ x \ < \ \frac{k}{m} \cdot d + \frac{\sum_{l=1}^{i} \alpha_l}{m} \text{ for some } k = 0, 1, \dots, m-1, \\ 0, & \text{else.} \end{cases}$$

In this case, μ_i^m converges weakly as $m \to \infty$ to the conditional distribution $P^Y(\cdot \,|\, U < d)$.

Special Cases. For the particular distortion risk measures V@R, AV@R and RV@R, we recover the results in Embrechts et al. (2018), Theorem 2.

Example 4.1.8. For any $E_1 \in \mathcal{X}$ and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \geq 0$, $n \in \mathbb{N}$, we have

(i) $\square_{i=1}^n \mathrm{V@R}_{\alpha_i}(E_1) = \mathrm{V@R}_{\sum_{i=1}^n \alpha_i}(E_1)$,

(ii) $\square_{i=1}^n \mathrm{AV@R}_{\beta_i}(E_1) = \mathrm{AV@R}_{\max\{\beta_1,\dots,\beta_n\}}(E_1)$,

(iii) $\square_{i=1}^n \mathrm{RV@R}_{\alpha_i,\beta_i}(E_1) = \mathrm{RV@R}_{\sum_{i=1}^n \alpha_i, \max\{\beta_1,\dots,\beta_n\}}(E_1)$.

Note that the optimal risk sharing problem (4.1) can be combined with other management actions. For example, the network may adjust its structure by increasing the number of firms over longer time horizons, or the network may optimize its asset allocation to further reduce its total risk and its total SCR (cf. Section 4.2). In particular, Weber (2018) shows that for V@R-type risk measures and sufficiently large n, the corporate network can find a capital allocation such that

$$\square_{i=1}^n \rho^i(E_1) = -\operatorname{ess\,sup} E_1, \tag{4.4}$$

corresponding to the best case scenario. Downside risk can thus completely be hidden within corporate network structures.

V@R is a special case of a V@R-type distortion risk measure, and hence our observations are relevant in the context of Solvency II. In contrast, they do not apply to the Swiss Solvency Test that uses the coherent risk measure AV@R as the basis for capital regulation (cf. Remark 4.1.2).

4.2 | An Application to Asset-Liability Management

This section provides numerical case studies on optimal risk sharing. In Section 4.2.1, we introduce an asset-liability management (ALM) model for networks. Entities can implement various (static) asset allocation strategies over a one-year time horizon. Within this framework, we analyze three case studies of different complexity:

1. Assets are modeled by a Black-Scholes market, liabilities are deterministic.
2. Liabilities may be random; different types of dependence between assets and liabilities are investigated.
3. An additional left-tailed asset is available.

For these cases, we quantify the impact of the number n of sub-entities in the network on the network's minimal risk $\square_{i=1}^{n}\rho^i(E_1)$ and on the solvency capital requirements $\square_{i=1}^{n}\mathrm{SCR}^i_{\mathcal{A}}(E_1)$ and $\square_{i=1}^{n}\mathrm{SCR}^i_{\mathrm{mean}}(E_1)$. We demonstrate how asset-liability management can further reduce the minimal network risk. We focus on three different risk measures: V@R, AV@R and RV@R.

4.2.1 | Asset-Liability Management Model

4.2.1.1 | General Asset-Liability Model

Consider an ALM model with finite time horizon 1. We assume that the network's firms can invest in a financial market with a finite number $K \geq 1$ of liquidly traded assets. We denote by A^k_t, the price of one share of asset $k = 1, \dots, K$, and by L_t the consolidated liabilities at time $t \in [0, 1]$, respectively. At $t = 0$, the network decides – in a static manner – how to invest in the different assets in the period $t \in [0, 1]$ by determining an *asset allocation strategy* $\delta \in \mathbb{R}^K$ with

$$\delta^k \geq 0 \quad \text{and} \quad \sum_{k=1}^{K} \delta^k = 1,$$

where δ^k denotes the fraction of the total asset amount of the balance sheet invested in asset k. The corresponding numbers of shares held in the assets $k = 1, \dots, K$ are given by

$$\eta^k(\delta) = \delta^k \cdot \frac{E_0 + L_0}{A^k_0}$$

where E_0 is the net asset value – or book value of equity – at time 0. Afterwards the net asset value, calculated as the difference of total assets and liabilities, is a function of the asset allocation strategy and takes the form

$$E_t(\delta) = \sum_{k=1}^{K} \eta^k(\delta) A^k_t - L_t, \quad t \in [0, 1].$$

As a consequence, both risk $\rho(E_1(\delta))$ and return $E_1(\delta)/E_0 - 1$ depend on the strategy δ.

4.2.1.2 | Basis Asset-Liability Model

As a simple reference model we consider a Black-Scholes market and a single deterministic liability.

Asset Model. The financial market model consists of two liquidly tradable primary products: one riskless asset (*savings account*) and one risky asset (*stock*). Their price processes $(A^1_t)_{t\in[0,1]}$, $(A^2_t)_{t\in[0,1]}$ are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}, P)$ and follow the classical *Black-Scholes model*, i. e.,

- *Savings account*: $A^1_t = \exp(rt)$, $t \in [0, 1]$, with interest rate r,

- *Stock:* $A_t^2 = A_0^2 \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t)$, $t \in [0,1]$, with $A_0^2 \in (0,\infty), \mu \in \mathbb{R}, \sigma \in (0,\infty)$,

where $(W_t)_{t\in[0,1]}$ denotes a one-dimensional Wiener process. Note that $\mathbb{E}[A_t^2] = A_0^2 \exp(\mu t)$. For the remaining part of the chapter, we assume that the risk-free interest rate r equals zero, i. e., $A_t^1 = 1$, $t \in [0,1]$.

Liability Model. We assume that the insurance network sells a pure endowment with maturity 1 only. The network's premium income in $t = 0$ is denoted by π. The liabilities are deterministic, and the actuarial interest rate is assumed to be zero. Consequently, the actuarial reserve is a constant, i. e., $L_t = \pi$, $t \in [0,1]$.

In this basis setting, the net asset value is given by

$$E_t(\delta) = \eta^1(\delta)A_t^1 + \eta^2(\delta)A_t^2 - L_t = \eta^1(\delta) + \eta^2(\delta)A_t^2 - \pi \quad (t \in [0,1])$$

for any asset allocation $\delta \in \mathbb{R}^2, \delta^2 \geq 0, \delta^1 = 1 - \delta^2 \geq 0$. Randomness is driven only by the terminal stock value A_1^2. This allows us to derive the minimal risk capital

$$\Box_{i=1}^n \rho^i \left(E_1(\delta)\right)$$

for the three risk measures V@R, AV@R and RV@R in closed form.

Corollary 4.2.1. *Let $\rho^i = \mathrm{RV@R}_{\alpha_i,\beta_i}$, $\alpha_i, \beta_i \geq 0$, be the risk measure of network's entity i, $i = 1,\ldots,n$, and define $\alpha = \alpha_1 + \ldots + \alpha_n$, $\beta = \max\{\beta_1,\ldots,\beta_n\}$. Let $\delta \in \mathbb{R}^2$ be a fixed asset allocation strategy of the network. If $\alpha + \beta \leq 1$, then optimal risk sharing yields*

$$\begin{aligned}\Box_{i=1}^n \mathrm{RV@R}_{\alpha_i,\beta_i}(E_1(\delta)) &= \mathrm{RV@R}_{\alpha,\beta}(E_1(\delta))\\ &= -\eta^2(\delta)\, A_0^2\, e^{\mu}\, \tfrac{1}{\beta}\, \left(\Phi\left(\Phi^{-1}(\alpha+\beta) - \sigma\right) - \Phi\left(\Phi^{-1}(\alpha) - \sigma\right)\right)\\ &\quad -\eta^1(\delta) + \pi,\end{aligned}$$

where Φ denotes the cumulative distribution function of the standard normal distribution. In particular, the minimal SCRs take the form

$$\begin{aligned}\Box_{i=1}^n \mathrm{SCR}^i_{\mathcal{A}}(E_1(\delta)) &= \eta^2(\delta)\, A_0^2\, \left(1 - e^{\mu}\tfrac{1}{\beta}\left(\Phi\left(\Phi^{-1}(\alpha+\beta) - \sigma\right) - \Phi\left(\Phi^{-1}(\alpha) - \sigma\right)\right)\right),\\ \Box_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}(E_1(\delta)) &= \eta^2(\delta)\, A_0^2\, \left(e^{\mu} - e^{\mu}\tfrac{1}{\beta}\left(\Phi\left(\Phi^{-1}(\alpha+\beta) - \sigma\right) - \Phi\left(\Phi^{-1}(\alpha) - \sigma\right)\right)\right).\end{aligned}$$

Proof. The proof is given in Section 4.4. □

As a byproduct, Corollary 4.2.1 provides the corresponding results for V@R and AV@R.

Corollary 4.2.2. *Let $\delta \in \mathbb{R}^2$ be the network's asset allocation strategy.*

(i) Let ρ^i be given by $\mathrm{V@R}_{\alpha_i}$, $\alpha_i \in (0,1)$, $i = 1,\ldots,n$, and set $\alpha = \alpha_1 + \ldots + \alpha_n$.

If $\alpha \leq 1$, then

$$\square_{i=1}^{n} \mathrm{V@R}_{\alpha_i}(E_1(\delta)) = -\eta^2(\delta)\, A_0^2\, e^{\mu}\, \exp\left(\Phi^{-1}(\alpha)\sigma - \tfrac{\sigma^2}{2}\right) - \eta^1(\delta) + \pi.$$

Moreover,

$$\begin{aligned} \square_{i=1}^{n} \mathrm{SCR}_{\mathcal{A}}^{i}(E_1(\delta)) &= \eta^2(\delta)\, A_0^2 \left(1 - e^{\mu} \exp\left(\Phi^{-1}(\alpha)\sigma - \tfrac{\sigma^2}{2}\right)\right), \\ \square_{i=1}^{n} \mathrm{SCR}_{\mathrm{mean}}^{i}(E_1(\delta)) &= \eta^2(\delta) A_0^2 \left(e^{\mu} - e^{\mu} \exp\left(\Phi^{-1}(\alpha)\sigma - \tfrac{\sigma^2}{2}\right)\right). \end{aligned}$$

(ii) Let ρ^i be given by $\mathrm{AV@R}_{\beta_i}$, $\beta_i \in (0,1)$, $i = 1, \ldots, n$, and define $\beta = \max\{\beta_1, \ldots, \beta_n\}$. If $\beta \leq 1$, then

$$\square_{i=1}^{n} \mathrm{AV@R}_{\beta_i}(E_1(\delta)) = -\eta^2(\delta)\, A_0^2\, e^{\mu}\, \tfrac{1}{\beta}\, \Phi\left(\Phi^{-1}(\beta) - \sigma\right) - \eta^1(\delta) + \pi.$$

In particular, we have

$$\begin{aligned} \square_{i=1}^{n} \mathrm{SCR}_{\mathcal{A}}^{i}(E_1(\delta)) &= \eta^2(\delta)\, A_0^2 \left(1 - e^{\mu}\, \tfrac{1}{\beta} \Phi\left(\Phi^{-1}(\beta) - \sigma\right)\right), \\ \square_{i=1}^{n} \mathrm{SCR}_{\mathrm{mean}}^{i}(E_1(\delta)) &= \eta^2(\delta)\, A_0^2 \left(e^{\mu} - e^{\mu}\, \tfrac{1}{\beta} \Phi\left(\Phi^{-1}(\beta) - \sigma\right)\right). \end{aligned}$$

Proof. The proof is given in Section 4.4. □

4.2.2 | Parameterization

Let us now summarize our standing assumptions on the parameterization.

Remark 4.2.3. For V@R-type risk measures and for sufficiently many sub-entities, the network can reduce its total risk substantially, as described in Eq. (4.4). If the best case is unbounded, total risk will be equal to $-\infty$. Our case studies below rely on simulation methods with a finite number of samples. In all numerical experiments we run $500,000$ simulations. If the best case is unbounded, the sampled best case will always be a finite number, and it will thus not be possible to reproduce Eq. (4.4). For this reason, we modify all distributions in the extreme tails such that they will be of bounded support. This enables a simulation-based analysis of Eq. (4.4) and related results in our case studies. To be more precise, asset distributions are modified by setting asset values above the 99.95%-quantile to the 99.95%-quantile.

Analogously, we also modify liability distributions by setting liability values above the 99.95%-quantile to the 99.95%-quantile, and below the 0.05%-quantile to the 0.05%-quantile. This procedure will be applied to a liability distribution in Section 4.2.3.2 despite the fact that its support is bounded. This is done in order to avoid settings that require more sophisticated rare event simulation. The resulting, simplified model is well suited to numerically illustrate the effect of network size and ALM-strategies on risk. Alternative models in which extreme tail events possess a large influence on the outcome of simulations require more care in terms of simulation techniques for rare events. This is, however, not the focus of our analysis. We thus concentrate on the described setting.

Details for used distributions are discussed below. The relevant numerical values are provided in Table 4.2.

	Lognormal distr.	Beta distr.	Stable distr.
0.9995-quantile	66.2512	0.9713	3.2209
0.0005-quantile	not relevant	0.7792	not relevant

Table 4.2: Adjustment of distributions.

Parameterization of the Basis Model. For the asset side, we assume that the initial stock price is given by $A_0^2 = 30$ and that the stock price dynamics is determined by the drift $\mu = \ln(35/30) \approx 0.1542$ (i. e., $\mathbb{E}[A_1^2] = 35$) and the volatility $\sigma = 0.2$. As discussed in Remark 4.2.3, we bound the asset value by its 99.95%-quantile which equals 66.2512, see Table 4.2, by modifying its distribution as explained. Interest rates of the savings account are assumed to be zero. On the liability side, we assume that the network's premium income in $t = 0$ is $\pi = 90$. Since the liabilities are deterministic, this implies $L_0 = L_1 = \pi = 90$.

The initial equity value is set to $E_0(\delta) = 30$. In this case, the total asset amount of the balance sheet is given by $E_0(\delta) + L_0 = 120$. The network's asset allocation $\delta \in \mathbb{R}^2$ is assumed to be fixed and set to $\delta^1 = 0.75$ and $\delta^2 = 0.25$, i. e., we obtain the corresponding numbers of shares

$$\eta^1(\delta) = 90, \quad \eta^2(\delta) = 1.$$

Note that this asset allocation yields the terminal equity

$$E_1(\delta) = \eta^1(\delta) + \eta^2(\delta)A_1^2 - \pi = A_1^2$$

proportional to the stock value. In particular, for the given positive drift μ, we have

$$\mathbb{E}[E_1(\delta)] = \mathbb{E}[A_1^2] = A_0^2 \exp(\mu) > A_0^2 = E_0,$$

i. e.,

$$\mathrm{SCR}_{\mathcal{A}}(E_1(\delta)) = E_0 + \rho(E_1(\delta)) < \mathbb{E}[E_1(\delta)] + \rho(E_1(\delta)) = \mathrm{SCR}_{\mathrm{mean}}(E_1(\delta))$$

for any monetary risk measure ρ.

Parameterization of Risk Measures. Our case studies compare and analyze the effect of optimal risk sharing for three different risk measures: V@R, AV@R and RV@R. We assume that within the network all firms use the same risk measure with the same parameters, i. e.,

(a) $\rho^i = \mathrm{V@R}_\alpha$, $\alpha \in (0, 1)$, for all $i = 1, \dots, n$,

(b) $\rho^i = \mathrm{AV@R}_\beta$, $\beta \in (0, 1)$, for all $i = 1, \dots, n$,

(c) $\rho^i = \mathrm{RV@R}_{\gamma,\varepsilon}$, $\gamma, \varepsilon \in (0,1)$, for all $i = 1, \dots, n$.

This situation might result from a management decision to apply a unified risk measurement approach within the network, or it could be enforced by regulatory requirements if all firms are subject to the same regulation scheme.

For value at risk, we choose the level $\alpha = 0.1$, and we fix $\gamma = 0.05$ for the range value at risk. To ensure comparability of results between the three risk measures, the remaining parameters β, ε are calibrated such that for $X \sim \mathcal{N}(0,1)$ with cumulative distribution function Φ and probability density function ϕ

$$\mathrm{V@R}_\alpha(X) = \mathrm{AV@R}_\beta(X) = \mathrm{RV@R}_{\gamma,\varepsilon}(X). \tag{4.5}$$

For this purpose, we use that $\mathrm{V@R}_\alpha(X) = -q_X^+(\alpha) = -\Phi^{-1}(\alpha)$. This leads to

$$\begin{aligned}\mathrm{AV@R}_\beta(X) &= \tfrac{1}{\beta}\int_0^\beta \mathrm{V@R}_\alpha(X)\,d\alpha = -\tfrac{1}{\beta}\int_0^\beta \Phi^{-1}(\alpha)\,d\alpha \\ &= -\tfrac{1}{\beta}\int_{-\infty}^{\Phi^{-1}(\beta)} y\phi(y)\,dy = -\tfrac{1}{\beta}\phi(\Phi^{-1}(\beta)),\end{aligned}$$

due to the substitution $y = \Phi^{-1}(\alpha)$ and $\phi'(y) = y\phi(y)$ (see also Appendix A, Lemma A.0.5), and

$$\begin{aligned}\mathrm{RV@R}_{\gamma,\varepsilon}(X) &= \tfrac{1}{\varepsilon}\int_\gamma^{\gamma+\varepsilon} \mathrm{V@R}_\alpha(X)\,d\alpha = -\tfrac{1}{\varepsilon}\int_\gamma^{\gamma+\varepsilon} \Phi^{-1}(\alpha)\,d\alpha \\ &= -\tfrac{1}{\varepsilon}\left(\phi\left(\Phi^{-1}(\gamma+\varepsilon)\right) - \phi\left(\Phi^{-1}(\gamma)\right)\right).\end{aligned}$$

Solving Eq. (4.5) with these formulae for given α and γ numerically, yields the following parameters:

$\mathrm{V@R}_\alpha$	$\mathrm{AV@R}_\beta$	$\mathrm{RV@R}_{\gamma,\varepsilon}$
$\alpha = 0.1$	$\beta = 0.2456$	$\gamma = 0.05, \varepsilon = 0.1072$

Table 4.3: Parameterization of risk measures.

Remark 4.2.4. Observe that the chosen quantile levels in Remark 4.2.3 and Table 4.2 are in the extreme tail of the distributions, if compared to the parameters of the risk measures in Table 4.3. Consequently, also within our modified model with adjusted distributions the chosen risk measures are non-trivial functionals of the tails.

4.2.3 | Numerical Case Studies

4.2.3.1 | Unsophisticated Network vs. Sophisticated Network

Let us first consider the basis ALM model with deterministic liabilities. The first row in Tables 4.4–4.6 displays the risk capital $\rho(E_1(\delta))$ and the corresponding SCRs for V@R, AV@R and RV@R of a single firm. This corresponds to the consolidated case and can be interpreted as an unsophisticated network. All values are almost equal across different risk measures due to the applied standardization of the risk levels $\alpha, \beta, \gamma, \varepsilon$, although $E_1(\delta)$ follows a lognormal distribution instead of a standard normal distribution.

Sophisticated networks may, firstly, adjust their structure by increasing the number of entities n. For a fixed number n of firms, the corporate network will, secondly, design optimal intra-network capital transfers that minimize the total risk in Eq. (4.1). The second and the third row in Tables 4.4–4.6 quantify the effect on risk capital and on the corresponding SCRs for $n = 5$ and $n = 10$ firms:

- For the two risk measures of V@R-type, V@R and RV@R, we observe that downside risk can be reduced significantly by optimal capital transfers that hide the tail risk. For n sufficiently large, the corporate network could even determine a capital allocation such that
$$\square_{i=1}^{n} \rho^i(E_1(\delta)) = -\operatorname{ess\,sup} E_1(\delta),$$
corresponding to the best case scenario. This requires $n \cdot \alpha \geq 1$ for the risk measure $\mathrm{V@R}_\alpha$ and $n \cdot \gamma \geq 1$ for $\mathrm{RV@R}_{\gamma,\varepsilon}$ (cf. Theorem 4.1.5). For $\mathrm{V@R}_\alpha$ with $\alpha = 0.1$, this condition is already satisfied for a number of firms $n \geq 10$, and the simulations provide the expected result.

- In contrast, for the coherent risk measure AV@R, optimal risk sharing does, of course, not reduce the risk capital – hiding tail risk is not possible.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^{n} \mathrm{V@R}_\alpha^i\,(E_1(\delta))$	$\square_{i=1}^{n} \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^{n} \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n = 1$	34.9982	-26.5577	3.4423	8.4405
$n = 5$	34.9982	-34.3060	-4.3060	0.6922
$n = 10$	34.9982	-66.2512	-36.2512	-31.2530

Table 4.4: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{V@R}_{0.1}$; deterministic liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^{n} \mathrm{AV@R}_\beta^i\,(E_1(\delta))$	$\square_{i=1}^{n} \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^{n} \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n = 1, 5, 10$	34.9982	-26.6784	3.3216	8.3198

Table 4.5: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{AV@R}_{0.2456}$; deterministic liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^{n} \mathrm{RV@R}_{\gamma,\varepsilon}^i\,(E_1(\delta))$	$\square_{i=1}^{n} \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^{n} \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n = 1$	34.9982	-26.5722	3.4278	8.4260
$n = 5$	34.9982	-30.9523	-0.9523	4.0459
$n = 10$	34.9982	-35.2473	-5.2473	-0.2491

Table 4.6: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{RV@R}_{0.05,0.1072}$; deterministic liabilities.

4.2.3.2 | Random Liabilities

We extend the basis ALM model by including random liabilities. The insurance network is assumed to sell pure endowment contracts only, i. e., a product depending on the random future life time of the insurees. The idiosyncratic risk of individuals becomes irrelevant in a very large pool, but the systematic risk, random mortality, does not average out. This is the focus of the case study.

For insured persons aged x, we denote by p_x^* and p_x their one-year *actuarial* survival probability and their one-year *random* survival probability, respectively. We use the assumption that the actuarial survival probability p_x^* is the *best estimate* of the random survival probability in the sense that $\mathbb{E}[p_x] = p_x^*$ and that p_x^* does not yet include any margin for unexpected losses, i. e., deviations from the expected value. In this case, for a sum insured $L > 0$, the premium is calculated as $\pi = L \cdot p_x^*$, and the random liabilities at time $t = 1$ are given by

$$L_1 = L \cdot p_x = \frac{p_x}{p_x^*}\pi.$$

The last term corresponds to the actuarial reserve adapted to mortality by an appropriate multiplier, the ratio of random and actuarial survival probability, as introduced in Chapter 6, Section 6.3.

For zero interest rates, the network's random equity at time $t = 1$ is then given by

$$E_1(\delta) = \eta^1(\delta) + \eta^2(\delta)A_1^2 - L_1 = \eta^1(\delta) + \eta^2(\delta)A_1^2 - \frac{p_x}{p_x^*}\pi. \tag{4.6}$$

The extended model (4.6) reduces to the basis ALM model if $p_x \equiv p_x^*$ is deterministic. For a monetary risk measure ρ, the risk

$$\rho\left(E_1(\delta)\right) = \rho\left(\eta^2(\delta)A_1^2 - \frac{p_x}{p_x^*}\pi\right) - \eta^1(\delta)$$

accounts for both the network's asset risk and biometric risk, i. e., the longevity of policyholders.

We analyze the network's optimal risk sharing strategy for three different dependence structures of assets (stock) and liabilities: *independence*, *comonotonicity*, and *countermonotonicity*. These dependencies are illustrated in Figure 4.1. We do not claim that pure comonotonicity and countermonotonicity are realistic, but study them to illustrate the implications of particularly extreme forms of dependence.

Independent assets and liabilities do not affect each other. In the comonotonic case, asset and liability values change in the same direction. In particular, increasing liabilities are associated with increasing asset values such that increasing costs for the insurer are hedged by gains in the financial market. In contrast, countermonotonic assets and liabilities correspond to a scenario in which increasing liability values are associated with decreasing asset values. The first situation could, for example, correspond to a scenario of joint technical and medical innovation with both increased wealth and longevity. The second situation could be associated with medical innovation and longevity coupled with

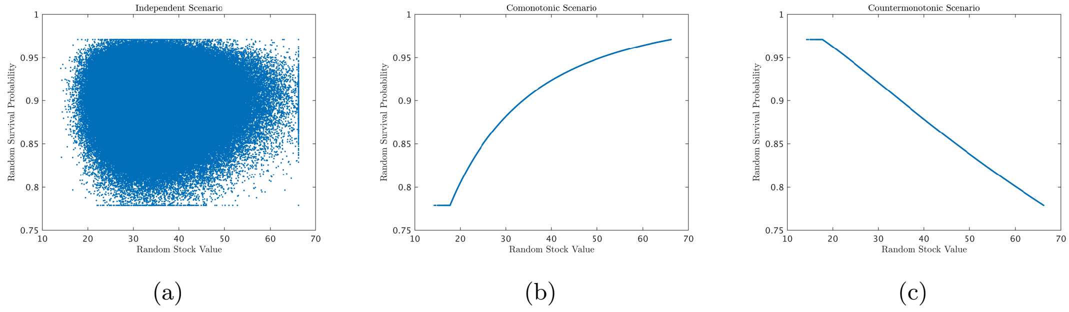


Figure 4.1: Dependence structures: The stock value and the survival probability are (a) independent, (b) comonotonic, and (c) countermonotonic.

an aging population that liquidates assets to generate liquidity. Countermonotonic assets and liabilities are problematic, since high insurance claims occur together with low asset values and yield a low book value of equity of insurers. In the worst case, the liabilities might not be covered by the asset value anymore.

For the numerical results, we rely on the parameterization of Section 4.2.2. In addition, we assume a sum insured $L = 100$, $p_x^* = 0.9$ and

$$p_x \sim \mathrm{Beta}(92.1842, 10.2427),$$

see Chapter 6, Section 6.3.2, for more details on the applied Beta distribution. Then, $\mathbb{E}[p_x] = p_x^* = 0.9$, and hence $\mathbb{E}[L_1] = \pi = L_0$. As described in Remark 4.2.3, we modify this distribution in the tails. The relevant quantiles are shown in Table 4.2. For the sake of comparison to the basis case in Section 4.2.3.1, we calibrate the asset allocation δ^1, δ^2 with $\delta^1 + \delta^2 = 1$ such that for a network with a single firm only and for independent assets and liabilities $\mathrm{V@R}_\alpha(E_1(\delta))$ coincides with the basis ALM model. This yields

$$\delta^1 = 0.8355, \;\; \delta^2 = 0.1645,$$

i. e., the fraction δ^1 in the savings account is now higher. This is not surprising since random mortality increases risk which needs to be offset by a reduction of the stock investment. As a consequence, the return decreases as well; the expected value of the future net asset value is

$$\mathbb{E}[E_1(\delta)] = (E_0 + L_0)\left(\delta^1 \mathbb{E}\left[\frac{A_1^1}{A_0^1}\right] + \delta^2 \mathbb{E}\left[\frac{A_1^2}{A_0^2}\right]\right) - \mathbb{E}[L_1] = (E_0 + L_0)\left(\delta^1 + \delta^2 \exp(\mu)\right) - \pi.$$

The following tables summarize the numerical results.

Case Study I - Independent Stock and Liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{V@R}_\alpha^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n=1$	33.2871	-26.5578	3.4422	6.7292
$n=5$	33.2871	-32.8938	-2.8938	0.3932
$n=10$	33.2871	-65.9336	-35.9336	-32.6465

Table 4.7: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n=1,5,10$ firms; risk measure: $\mathrm{V@R}_{0.1}$; independent assets and liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{AV@R}_\beta^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n=1,5,10$	33.2871	-26.6392	3.3608	6.6479

Table 4.8: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n=1,5,10$ firms; risk measure: $\mathrm{AV@R}_{0.2456}$; independent assets and liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{RV@R}_{\gamma,\varepsilon}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n=1$	33.2871	-26.5707	3.4293	6.7163
$n=5$	33.2871	-30.2036	-0.2036	3.0835
$n=10$	33.2871	-33.6313	-3.6313	-0.3443

Table 4.9: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n=1,5,10$ firms; risk measure: $\mathrm{RV@R}_{0.05,0.1072}$; independent assets and liabilities.

Case Study II - Comonotonic Stock and Liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{V@R}_\alpha^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n=1$	33.2883	-31.6892	-1.6892	1.5991
$n=5$	33.2883	-32.5983	-2.5983	0.6901
$n=10$	33.2883	-46.7250	-16.7250	-13.4367

Table 4.10: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n=1,5,10$ firms; risk measure: $\mathrm{V@R}_{0.1}$; comonotonic assets and liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{AV@R}_\beta^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n=1,5,10$	33.2883	-31.7290	-1.7290	1.5593

Table 4.11: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n=1,5,10$ firms; risk measure: $\mathrm{AV@R}_{0.2456}$; comonotonic assets and liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{RV@R}_{\gamma,\varepsilon}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n=1$	33.2883	-31.6960	-1.6960	1.5923
$n=5$	33.2883	-32.0096	-2.0096	1.2787
$n=10$	33.2883	-32.8229	-2.8229	0.4654

Table 4.12: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n=1,5,10$ firms; risk measure: $\mathrm{RV@R}_{0.05,0.1072}$; comonotonic assets and liabilities.

Case Study III - Countermonotonic Stock and Liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^{n}\text{V@R}_{\alpha}^{i}\,(E_1(\delta))$	$\square_{i=1}^{n}\text{SCR}_{\mathcal{A}}^{i}\,(E_1(\delta))$	$\square_{i=1}^{n}\text{SCR}_{\text{mean}}^{i}\,(E_1(\delta))$
$n = 1$	33.2905	-24.1566	5.8434	9.1340
$n = 5$	33.2905	-32.5717	-2.5717	0.7188
$n = 10$	33.2905	-65.9336	-35.9336	-32.6431

Table 4.13: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\text{V@R}_{0.1}$; countermonotonic assets and liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^{n}\text{AV@R}_{\beta}^{i}\,(E_1(\delta))$	$\square_{i=1}^{n}\text{SCR}_{\mathcal{A}}^{i}\,(E_1(\delta))$	$\square_{i=1}^{n}\text{SCR}_{\text{mean}}^{i}\,(E_1(\delta))$
$n = 1, 5, 10$	33.2905	-24.3028	5.6972	8.9877

Table 4.14: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\text{AV@R}_{0.2456}$; countermonotonic assets and liabilities.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^{n}\text{RV@R}_{\gamma,\varepsilon}^{i}\,(E_1(\delta))$	$\square_{i=1}^{n}\text{SCR}_{\mathcal{A}}^{i}\,(E_1(\delta))$	$\square_{i=1}^{n}\text{SCR}_{\text{mean}}^{i}\,(E_1(\delta))$
$n = 1$	33.2905	-24.1817	5.8183	9.1088
$n = 5$	33.2905	-28.9309	1.0691	4.3596
$n = 10$	33.2905	-33.5930	-3.5930	-0.3025

Table 4.15: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\text{RV@R}_{0.05,0.1072}$; countermonotonic assets and liabilities.

In the consolidated case – corresponding to an unsophisticated network consisting of a single firm only – and for all three risk measures V@R, AV@R and RV@R, the associated risk capital $\rho(E_1(\delta))$ reflects the different dependence structures in the following sense: The highest risk capital is attained for the countermonotonic case, the lowest risk capital is observed for the comonotonic case, while the risk capital for independent assets and liabilities is between the values of the two extreme dependency structures.

In analogy to Section 4.2.3.1, the numerical results illustrate for all three dependence structures that optimal capital transfers within a sophisticated network hide the downside risk, if capital regulation is based on V@R-type risk measures such as V@R and RV@R. In contrast, there is no reduction of risk capital by optimal risk sharing for the coherent risk measure AV@R. For V@R-type risk measures, the different levels of risk capital for the countermonotonic and independent case disappear for increasing n. The difference of the inf-convolutions $\square_{i=1}^{n}\text{V@R}_{\alpha}^{i}\,(E_1(\delta))$ in the countermonotonic and the independent case decreases from 2.4 for $n = 1$ to 0.3 for $n = 5$ and finally to 0 for $n = 10$. Similarly, the difference of the inf-convolutions $\square_{i=1}^{n}\text{RV@R}_{\gamma,\varepsilon}^{i}\,(E_1(\delta))$ in the countermonotonic and the independent case decreases from 2.4 for $n = 1$ to 1.3 for $n = 5$ and finally to nearly 0 for $n = 10$. Observe that $\square_{i=1}^{10}\text{V@R}_{\alpha}^{i}\,(E_1(\delta))$ equals -65.93 for both the countermonotonic and the independent case, corresponding to the best case – as known from Eq. (4.4).

4.2.3.3 | Left-Tailed Assets

In this section, we consider again the basis ALM model with deterministic liabilities as described in Section 4.2.1, but extend the financial market by including a third *left-tailed*, also called *left-skewed*, asset with price process $(A_t^3)_{t\in[0,1]}$. This asset is characterized by a skewed distribution with the possibility of losses and – in comparison to the stock – a higher downside risk. More precisely, its price process is modeled by

$$A_t^3 = A_0^3 \exp(\zeta t) + Z - \mathbb{E}[Z], \quad t \in (0,1],$$

where the initial value $A_0^3 > 0$ is a fixed constant, $\zeta > 0$ is a rate of exponential growth, and Z is a random variable with stable distribution.

Definition 4.2.5. A random variable Z has a *stable distribution* $\mathcal{S}(a,b,c,d)$ with parameters $a \in (0,2], b \in [-1,1], c \in (0,\infty), d \in \mathbb{R}$, i. e., $Z \sim \mathcal{S}(a,b,c,d)$, if its characteristic function is given by

$$\mathbb{E}\left[e^{isZ}\right] = \begin{cases} \exp\left(-c^{\alpha}|s|^{a}\left[1 + ib\,\text{sign}(s)\tan\frac{\pi a}{2}\left(\left(c|s|^{1-a} - 1\right)\right)\right] + ids\right), & a \neq 1, \\ \exp\left(-c|s|\left[1 + ib\,\text{sign}(s)\tan\frac{2}{\pi}\left(c|s|\right)\right] + ids\right), & a = 1. \end{cases}$$

For the numerical case study, we fix $A_0^3 = 1$, $\zeta = 0.3$ and assume that

$$Z \sim \mathcal{S}(1.5, -1, 1, 0)$$

is independent from the stock price process $(A_t^2)_{t\in[0,1]}$. Figure 4.2 shows the probability density function of Z.

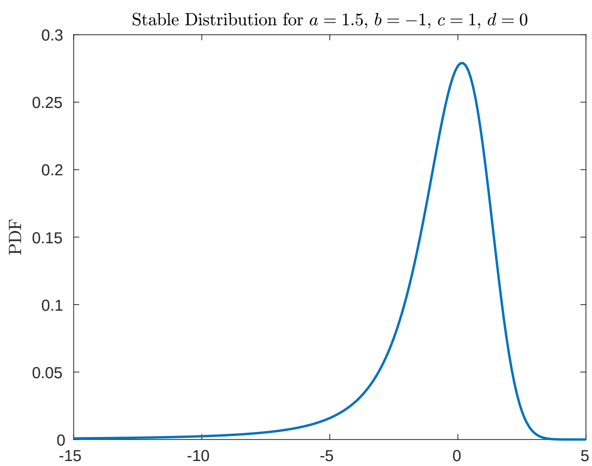


Figure 4.2: PDF of $Z \sim \mathcal{S}(1.5, -1, 1, 0)$.

Again, as explained in Remark 4.2.3, we modify the distribution such that it is bounded from above. Table 4.2 shows the new upper bound of 3.22 at the 99.95%-quantile of the original distribution. Note that

$$\mathbb{E}\left[\frac{A_1^3}{A_0^3}\right] \approx \exp(\zeta) > \exp(\mu) \approx \mathbb{E}\left[\frac{A_1^2}{A_0^2}\right]$$

for the parameters $\zeta = 0.3$ and $\mu = 0.1542$, i. e., the expected return of the left-tailed asset exceeds the expected return of the stock, compensating for the higher risk of this position. For three assets, the book value of equity at terminal time 1 is given by

$$E_1(\delta) = \eta^1(\delta) + \eta^2(\delta)A_1^2 + \eta^3(\delta)A_1^3 - \pi = (E_0 + L_0)\left(\delta^1 \frac{A_1^1}{A_0^1} + \delta^2 \frac{A_1^2}{A_0^2} + \delta^3 \frac{A_1^3}{A_0^3}\right) - \pi,$$

where $\delta \in \mathbb{R}^3, \delta^1, \delta^2, \delta^3 \geq 0, \delta^1 + \delta^2 + \delta^3 = 1$. Further, $\eta^1(\delta)$ and $\eta^2(\delta)$ are as before and $\eta^3(\delta)$ denotes the number of left-tailed assets bought at time $t = 0$. Thus, a higher fraction δ^3 yields a higher expected terminal net asset value $\mathbb{E}[E_1(\delta)]$, but is associated with a higher downside risk.

Case Study I - Fixed Asset Allocation Including a Left-Tailed Asset. Let us first analyze the impact of the left-tailed asset on network risk minimization for a fixed asset allocation $\delta \in \mathbb{R}^3$, where a small fraction $\delta^3 = 0.01$ is invested in the left-tailed asset. For the sake of comparison to the basis case in Section 4.2.3.1, we calibrate the remaining fractions δ^1, δ^2 with $\delta^1 + \delta^2 + \delta^3 = 1$ such that for the consolidated case, i. e., a network with only a single firm, $\mathrm{V@R}_\alpha(E_1(\delta))$ coincides with the basis case. This yields the allocation

$$\delta^1 = 0.73901,\ \delta^2 = 0.2510,\ \delta^3 = 0.01.$$

In analogy to Section 4.2.3.1, the numerical results in Tables 4.16, 4.17 & 4.18 illustrate that optimal capital transfers within a sophisticated network hide the downside risk, if capital regulation is based on V@R-type risk measures such as V@R and RV@R.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{V@R}_\alpha^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n = 1$	35.4378	-26.5577	3.4423	8.8801
$n = 5$	35.4378	-35.1833	-5.1833	0.2545
$n = 10$	35.4378	-71.8246	-41.8246	-36.3867

Table 4.16: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{V@R}_{0.1}$; additional left-tailed asset.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \mathrm{AV@R}_\beta^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_\mathcal{A}^i\,(E_1(\delta))$	$\square_{i=1}^n \mathrm{SCR}_{\mathrm{mean}}^i\,(E_1(\delta))$
$n = 1, 5, 10$	35.4378	-25.4473	4.5527	9.9905

Table 4.17: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\mathrm{AV@R}_{0.2456}$; additional left-tailed asset.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{RV@R}^i_{\gamma,\varepsilon}(E_1(\delta))$	$\square_{i=1}^n \text{SCR}^i_{\mathcal{A}}(E_1(\delta))$	$\square_{i=1}^n \text{SCR}^i_{\text{mean}}(E_1(\delta))$
$n = 1$	35.4378	-26.5512	3.4488	8.8866
$n = 5$	35.4378	-31.5879	-1.5879	3.8499
$n = 10$	35.4378	-36.1717	-6.1717	-0.7339

Table 4.18: Expected equity, minimized network risk capital and corresponding SCRs for a split into $n = 1, 5, 10$ firms; risk measure: $\text{RV@R}_{0.05,0.1072}$; additional left-tailed asset.

Case Study II - Optimizing the Asset Allocation. In the second step, we fix the fraction $\delta^1 = 0.75$ invested in the savings account and vary the fraction δ^3 held in the left-tailed asset (and $\delta^2 = 1-\delta^1-\delta^3$, respectively) in the range $[0, 0.25]$. The left boundary point $\delta^3 = 0$ corresponds to the basis ALM model in Section 4.2.3.1, i. e., there is no investment in the left-tailed asset and the full remaining fraction $\delta^2 = 0.25$ of asset amount of the balance sheet is invested in the stock. As an anchor point, the first row in Tables 4.19–4.21 coincides with the numerical results of the basis ALM model (cf. Tables 4.4–4.6). For the right boundary point $\delta^3 = 0.25$, the fraction of 0.25 invested initially in the stock is completely replaced by the left-tailed asset.

Table 4.19 displays the expected terminal net asset value and the risk capital for the consolidated case, i. e., an unsophisticated network, for varying δ^3. A higher fraction δ^3 increases the expected terminal equity $\mathbb{E}[E_1(\delta)]$, thus the expected profit of the network. At the same time, substituting the stock by the left-tailed asset substantially increases risk capital for all three risk measures V@R, AV@R and RV@R. The risk measure AV@R is most sensitive to the re-allocation between stock and left-tailed asset.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{V@R}^i_{\alpha}(E_1(\delta))$	$\square_{i=1}^n \text{AV@R}^i_{\beta}(E_1(\delta))$	$\square_{i=1}^n \text{RV@R}^i_{\gamma,\varepsilon}(E_1(\delta))$
$\delta^3 = 0$	34.9982	-26.6784	-26.6822	-26.5722
$\delta^3 = 0.05$	36.0977	-21.6074	-12.7634	-21.2202
$\delta^3 = 0.1$	37.1972	-10.8346	7.1190	-10.0190
$\delta^3 = 0.15$	38.2967	0.7034	27.6693	1.9167
$\delta^3 = 0.2$	39.3962	12.4657	48.4084	14.0584
$\delta^3 = 0.25$	40.4958	24.2745	69.2267	26.2904

Table 4.19: Expected equity and minimized network risk capital for V@R, AV@R and RV@R for a split into $n = 1$ firms; additional left-tailed asset.

Tables 4.20 & 4.21 show the relevant quantities for a sophisticated network which splits into $n = 5$ or $n = 10$ entities. Returns increase with δ^3, i. e., the fraction in the left-tailed asset, and are independent of n. However, with increasing n for the two V@R-type risk measures, V@R and RV@R, required capital decreases substantially. The effect of reduction is stronger for $\delta^3 > 0$ than in the basis ALM model corresponding to $\delta^3 = 0$. For $\delta^3 = 0$, the difference in $\square_{i=1}^n \text{V@R}^i_{\alpha}(E_1(\delta))$ for $n = 1$ and $n = 10$ is equal to 39.6, but for $\delta^3 = 0.25$ equal to 190.9. In the case of RV@R, the corresponding differences are smaller but qualitatively similar, i. e., 8.7 and 93.8, respectively. In particular, the increase of risk capital for a single firm driven by investments in the left-tailed asset can completely be compensated within a sophisticated network.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{V@R}_\alpha^i\,(E_1(\delta))$	$\square_{i=1}^n \text{AV@R}_\beta^i\,(E_1(\delta))$	$\square_{i=1}^n \text{RV@R}_{\gamma,\varepsilon}^i\,(E_1(\delta))$
$\delta^3 = 0$	34.9982	-34.3060	-26.6784	-30.9523
$\delta^3 = 0.05$	36.0977	-39.6707	-12.7634	-33.7124
$\delta^3 = 0.1$	37.1972	-45.4260	7.1190	-34.6340
$\delta^3 = 0.15$	38.2967	-50.9053	27.6693	-34.9642
$\delta^3 = 0.2$	39.3962	-56.2971	48.4084	-35.1112
$\delta^3 = 0.25$	40.4958	-61.6229	69.2267	-35.1748

Table 4.20: Expected equity and minimized network risk capital for V@R, AV@R and RV@R for a split into $n = 5$ firms; additional left-tailed asset.

	$\mathbb{E}[E_1(\delta)]$	$\square_{i=1}^n \text{V@R}_\alpha^i\,(E_1(\delta))$	$\square_{i=1}^n \text{AV@R}_\beta^i\,(E_1(\delta))$	$\square_{i=1}^n \text{RV@R}_{\gamma,\varepsilon}^i\,(E_1(\delta))$
$\delta^3 = 0$	34.9982	-66.2512	-26.6784	-35.2473
$\delta^3 = 0.05$	36.0977	-86.1500	-12.7634	-41.0915
$\delta^3 = 0.1$	37.1972	-106.0488	7.1190	-47.8593
$\delta^3 = 0.15$	38.2967	-125.9476	27.6693	-54.4418
$\delta^3 = 0.2$	39.3962	-145.8646	48.4084	-60.9784
$\delta^3 = 0.25$	40.4958	-166.5765	69.2267	-67.5032

Table 4.21: Expected equity and minimized network risk capital for V@R, AV@R and RV@R for a split into $n = 10$ firms; additional left-tailed asset.

But the numerical results are even more striking. For $n = 1$, all risk measures indicate that investments into the left-tailed asset increase risk. The coherent risk measure AV@R is invariant under an increase of the number of entities. But, for $n = 5$ and $n = 10$, both V@R-type risk measures lead to decreasing measurements of total risk if the fraction δ^3 invested in the left-tailed asset is increased. In the case of V@R, total risk $\square_{i=1}^n \text{V@R}_\alpha^i\,(E_1(\delta))$ increases for $n = 1$ from -26.7 for $\delta^3 = 0$ to 24.3 for $\delta^3 = 0.25$, but decreases for $n = 5$ from -34.3 for $\delta^3 = 0$ to -61.6 for $\delta^3 = 0.25$, and for $n = 10$ from -66.3 for $\delta^3 = 0$ to -166.6 for $\delta^3 = 0.25$. A similar phenomenon is observed for RV@R, but less significant. Total risk $\square_{i=1}^n \text{RV@R}_{\gamma,\varepsilon}^i\,(E_1(\delta))$ increases for $n = 1$ from -26.6 for $\delta^3 = 0$ to 26.3 for $\delta^3 = 0.25$, but decreases for $n = 5$ from -31.0 for $\delta^3 = 0$ to -35.2 for $\delta^3 = 0.25$, and for $n = 10$ from -35.2 for $\delta^3 = 0$ to -67.5 for $\delta^3 = 0.25$.

Figure 4.3 illustrates the impact of the fraction δ^3 invested in the left-tailed asset on $\mathbb{E}[E_1(\delta)]$, $\square_{i=1}^n \rho^i\,(E_1(\delta))$, $\square_{i=1}^n \text{SCR}_{\mathcal{A}}^i\,(E_1(\delta))$ and $\square_{i=1}^n \text{SCR}_{\text{mean}}^i\,(E_1(\delta))$ for $n = 1, 5, 10$ firms.

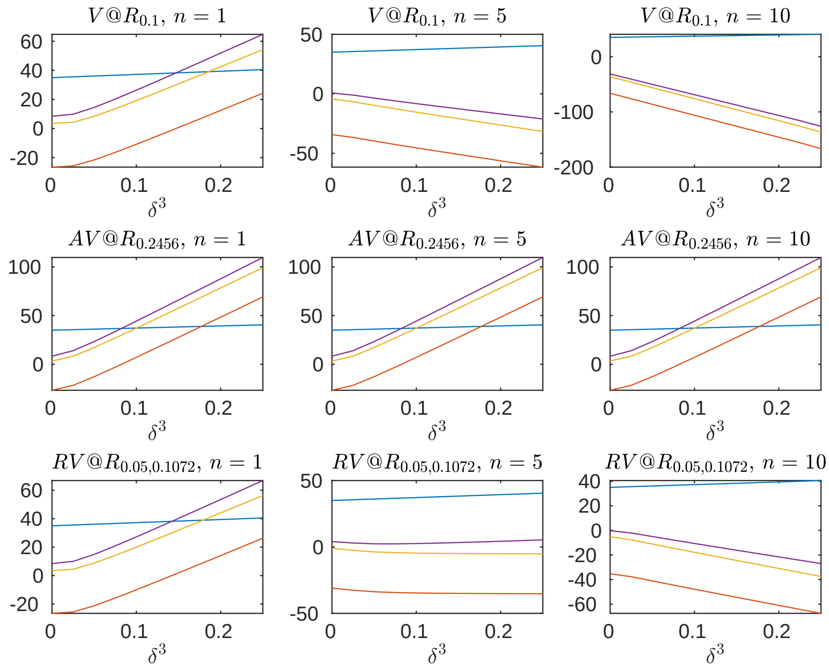


Figure 4.3: $\mathbb{E}[E_1(\delta)]$ in blue, $\square_{i=1}^n \rho^i\,(E_1(\delta))$ in red, $\square_{i=1}^n \mathrm{SCR}^i_{\mathcal{A}}\,(E_1(\delta))$ in yellow and $\square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}\,(E_1(\delta))$ in purple.

Optimal risk sharing for V@R-type risk measures suggests that the network's management should invest as much as possible in the left-tailed asset and provides incentives for highly risky investments, i. e., – from a regulatory point of view – for risk mismanagement. In fact, the left-tailed asset is associated with a significant downside risk, as indicated by the coherent risk measure AV@R. In contrast to V@R-type risk measures, an asset allocation decision based on this coherent risk measure would avoid too large investments in the left-tailed asset.

4.3 | Conclusion

Unless a consolidated solvency balance sheet is required, corporate networks may largely hide their total risk, if downside risk is quantified by risk measures of V@R-type – which includes the industry's standard risk measure value at risk. More precisely, a corporate network consisting of sufficiently many firms can largely reduce its total solvency capital requirement via optimal intra-network capital transfers and asset-liability management strategies. The size of capital reduction is increasing in the number n of firms in the network. If n is sufficiently large, the network can design a capital allocation such that the optimal network risk $\square_{i=1}^n \rho^i(E_1)$ coincides with $-\operatorname{ess\,sup} E_1$, corresponding to the best case scenario.

This chapter illustrates the impact of optimal intra-network capital transfers embedded into a general asset-liability management model, allowing for different asset allocation strategies, random liabilities with different dependencies between assets and liabilities, and investments in a left-tailed asset. The numerical case studies show that V@R-type risk measures provide incentives for risky investments. In contrast, if risk management is based on the coherent risk measure average value at risk, downside risk cannot be hidden and misleading incentives are not present.

4.4 | Appendix: Proofs

In this section, we provide the proofs of the results presented in Section 4.2.1.2.

Proof of Corollary 4.2.1.

Proof. By Example 4.1.8 (iii), we have

$$\square_{i=1}^{n}\mathrm{RV@R}_{\alpha_i,\beta_i}(E_1(\delta)) = \mathrm{RV@R}_{\alpha,\beta}(E_1(\delta))$$

for $\alpha = \alpha_1 + \ldots + \alpha_n$, $\beta = \max\{\beta_1, \ldots, \beta_n\}$. It is thus enough to show that

$$\mathrm{RV@R}_{\alpha,\beta}(E_1(\delta)) = -\eta^2(\delta)\, A_0^2\, e^{\mu}\, \tfrac{1}{\beta}\, \left(\Phi\left(\Phi^{-1}(\alpha+\beta)-\sigma\right) - \Phi\left(\Phi^{-1}(\alpha)-\sigma\right)\right) - \eta^1(\delta) + \pi. \tag{4.7}$$

To this end, note first that

$$\begin{aligned} \mathrm{RV@R}_{\alpha,\beta}(E_1(\delta)) &= \mathrm{RV@R}_{\alpha,\beta}\left(\eta^2(\delta)A_1^2 + \eta^1(\delta) - \pi\right) \\ &= \eta^2(\delta)\,\mathrm{RV@R}_{\alpha,\beta}(A_1^2) - \eta^1(\delta) + \pi, \end{aligned} \tag{4.8}$$

since $\mathrm{RV@R}_{\alpha,\beta}$ is cash-invariant and positively homogeneous. Hence, it remains to compute

$$\mathrm{RV@R}_{\alpha,\beta}(A_1^2) = \tfrac{1}{\beta}\int_{\alpha}^{\alpha+\beta} \mathrm{V@R}_{\gamma}(A_1^2)\, d\gamma = \tfrac{1}{\beta}\int_{\alpha}^{\alpha+\beta} -q_{A_1^2}(\gamma)\, d\gamma.$$

Using the quantile transformation rule for $A_1^2 = f(W_1)$ with the increasing function $f(x) = A_0^2 \exp(\mu - \frac{1}{2}\sigma^2 + \sigma x)$ combined with the fact that $q_X(\gamma) = \mathbb{E}[X] + \Phi^{-1}(\gamma)\sqrt{\mathrm{Var}}(X)$ for any normally distributed X (see Appendix A, Lemma A.0.5 (i)), we obtain

$$\begin{aligned} \mathrm{RV@R}_{\alpha,\beta}(A_1^2) &= \tfrac{1}{\beta}\int_{\alpha}^{\alpha+\beta} -q_{A_1^2}(\gamma)\, d\gamma = \tfrac{1}{\beta}\int_{\alpha}^{\alpha+\beta} -A_0^2\, e^{\mu-\frac{1}{2}\sigma^2+\sigma q_{W_1}(\gamma)}\, d\gamma \\ &= \tfrac{1}{\beta}\int_{\alpha}^{\alpha+\beta} -A_0^2\, e^{\mu-\frac{1}{2}\sigma^2+\Phi^{-1}(\gamma)\sigma}\, d\gamma = -A_0^2\, e^{\mu-\frac{1}{2}\sigma^2}\, \tfrac{1}{\beta}\int_{\alpha}^{\alpha+\beta} e^{\Phi^{-1}(\gamma)\sigma}\, d\gamma. \end{aligned}$$

Substituting $y = \Phi^{-1}(\gamma)$ with $dy = (1/\phi(\Phi^{-1}(\gamma))d\gamma$ in terms of the density ϕ of the standard normal distribution, leads to

$$\begin{aligned} \mathrm{RV@R}_{\alpha,\beta}(A_1^2) &= -A_0^2\, e^{\mu-\frac{1}{2}\sigma^2}\, \tfrac{1}{\beta}\int_{\Phi^{-1}(\alpha)}^{\Phi^{-1}(\alpha+\beta)} e^{\sigma y}\, \phi(y)\, dy \\ &= -A_0^2\, e^{\mu-\frac{1}{2}\sigma^2}\, \tfrac{1}{\beta}\int_{\Phi^{-1}(\alpha)}^{\Phi^{-1}(\alpha+\beta)} e^{\sigma y}\, \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\, dy \\ &= -A_0^2\, e^{\mu}\, \tfrac{1}{\beta}\int_{\Phi^{-1}(\alpha)}^{\Phi^{-1}(\alpha+\beta)} \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(y-\sigma)^2}\, dy \\ &= -A_0^2\, e^{\mu}\, \tfrac{1}{\beta}\int_{\Phi^{-1}(\alpha)-\sigma}^{\Phi^{-1}(\alpha+\beta)-\sigma} \phi(y)\, dy \\ &= -A_0^2\, e^{\mu}\, \tfrac{1}{\beta}\left(\Phi\left(\Phi^{-1}(\alpha+\beta)-\sigma\right) - \Phi\left(\Phi^{-1}(\alpha)-\sigma\right)\right). \end{aligned}$$

Together with (4.8) this proves (4.7). Since

$$\mathbb{E}[E_1(\delta)] = \eta^2(\delta)\, A_0^2\, e^{\mu} + \eta^1(\delta) - \pi,$$

the formulae for $\square_{i=1}^n \mathrm{SCR}^i_{\mathcal{A}}(E_1(\delta))$ and $\square_{i=1}^n \mathrm{SCR}^i_{\mathrm{mean}}(E_1(\delta))$, respectively, follow from (4.2) immediately. □

Proof of Corollary 4.2.2.

Proof. Recalling that the limiting cases of $\mathrm{RV@R}_{\alpha,\beta}$ correspond to $\mathrm{V@R}_\alpha$ for $\beta \to 0$ and $\mathrm{AV@R}_\beta$ for $\alpha \to 0$, the claim follows from Example 4.1.8 and Corollary 4.2.1. More precisely, we have

$$\begin{aligned}
\mathrm{V@R}_\alpha(E_1(\delta)) &= \lim_{\beta\to 0} \mathrm{RV@R}_{\alpha,\beta}(E_1(\delta)) \\
&= \lim_{\beta\to 0} \Big(-\eta^2(\delta)\, A_0^2\, e^{\mu}\, \tfrac{1}{\beta} \Big(\Phi\Big(\Phi^{-1}(\alpha+\beta) - \sigma\Big) - \Phi\Big(\Phi^{-1}(\alpha) - \sigma\Big)\Big) \\
&\qquad -\eta^1(\delta) + \pi\Big) \\
&= -\eta^2(\delta)\, A_0^2\, e^{\mu} \Big(\tfrac{d}{d\beta}\Phi\Big(\Phi^{-1}(\alpha+\beta) - \sigma\Big)\Big) - \eta^1(\delta) + \pi \\
&= -\eta^2(\delta)\, A_0^2\, e^{\mu}\, \tfrac{\phi(\Phi^{-1}(\alpha)-\sigma)}{\phi(\Phi^{-1}(\alpha))} - \eta^1(\delta) + \pi \\
&= -\eta^2(\delta)\, A_0^2\, e^{\mu} \exp\Big(\sigma\Phi^{-1}(\alpha) - \tfrac{\sigma^2}{2}\Big) - \eta^1(\delta) + \pi.
\end{aligned}$$

In the same manner, we derive

$$\begin{aligned}
\mathrm{AV@R}_\beta(E_1(\delta)) &= \lim_{\alpha\to 0} \mathrm{RV@R}_{\alpha,\beta}(E_1(\delta)) \\
&= \lim_{\alpha\to 0} \Big(-\eta^2(\delta)\, A_0^2\, e^{\mu}\, \tfrac{1}{\beta} \Big(\Phi\Big(\Phi^{-1}(\alpha+\beta) - \sigma\Big) - \Phi\Big(\Phi^{-1}(\alpha) - \sigma\Big)\Big) \\
&\qquad -\eta^1(\delta) + \pi\Big) \\
&= -\eta^2(\delta)\, A_0^2\, e^{\mu}\, \tfrac{1}{\beta}\, \Phi\Big(\Phi^{-1}(\beta) - \sigma\Big) - \eta^1(\delta) + \pi,
\end{aligned}$$

since $\lim_{\alpha\to 0}\Phi^{-1}(\alpha) = -\infty$ and $\lim_{\alpha\to 0}\Phi(\Phi^{-1}(\alpha) - \sigma) = 0$. □

5 | The Impact of Insurance Premium Taxation

The original version of this chapter was previously published in *European Actuarial Journal* **8**(1), pp. 127-167, 2018, see Degelmann, Hamm & Weber (2018).

In many countries insurance premiums are subject to insurance premium tax that replaces the common value-added tax (VAT) used for most products and services. This is, for example, mandatory according to EU-law. In contrast to VAT, premium tax does not permit any deductions: first, insurance companies cannot deduct VAT paid on inputs from premium tax; second, corporate buyers of insurance cannot deduct their premium tax payments from VAT on their outputs, although the insurance contracts are an input to their production. As a consequence, insurance premium tax leads to a higher taxation than VAT if the same tax rate is applied. In this chapter, we investigate the impact of premium tax on insurance companies, insurance holders and government revenues from multiple perspectives.

Subject to premium tax are insurance premiums only. In the case of insurance companies, these are approximately equal to the total revenues of these firms. VAT – in contrast – is not charged on revenues, but on the value added which is smaller than revenues.

Premium tax rates and VAT rates vary across countries. Tax rates may also be different for different types of products. Information on tax rates for individual countries and products can be found in European Commission (2017), Insurance Europe (2016), and Bundesministerium der Justiz und für Verbraucherschutz (2017). In Germany, the VAT rate and the premium tax rate coincide for most types of products and are generally both equal to 19%.

In absolute terms, government revenues from VAT are much larger than revenues from premium tax due to a larger tax base. In 2015, VAT revenues in Germany were 159,015 million EUR – corresponding to 23.6% of total tax revenues; premium tax revenues during the same year were equal to 12,419 million EUR, i.e., 1.8% of total tax revenues or 7.8% of VAT revenues, see Bundesfinanzministerium (2016).

On the individual level of both the providers and buyers of insurance, premium tax may lead to higher total tax payments. We quantify this effect in Section 5.1. As already explained, on the one hand, a provider of insurance cannot deduct VAT paid on input goods from premium tax. On the other hand, commercial buyers of insurance cannot deduct the incurred premium tax from VAT on their outputs. While Section 5.1 provides an analysis from the point of view of individual tax payers, Section 5.2 calculates the impact on overall tax revenues. More specifically, a tax system with insurance tax is compared to one in which insurance tax is replaced by VAT. If the VAT rate was unchanged, total tax revenues would decrease by 16 billion EUR. An equivalent VAT rate of 89.2% is computed that leads to the original total tax revenues.

After analyzing the basic differences between VAT and premium tax, we provide in Sections 5.3 – 5.6 a broader perspective on the topic by comparing the impact of different tax systems on insurance demand, the competitiveness of corporate buyers of insurance, ruin probabilities of insurance firms, and solvency capital. In Section 5.3, we model corporate buyers of insurance as utility maximizers that can choose their optimal level of insurance. The total cost of insurance depends on the tax system that is implemented. We provide case studies that illustrate potential consequences on the demand for insurance. These show that a change in the tax system from insurance to value-added tax increases the insurance demand. Section 5.4 investigates the competitiveness of corporate buyers of insurance. In contrast to Section 5.3 it is assumed that the amount of insurance that is bought is constant, but that its cost depends on the tax system. If an insurance tax is replaced by a VAT with the same rate, the costs are effectively reduced. If these savings are completely passed on to the buyer of the output products, the demand for these products increases. This is explicitly quantified in two numerical case studies. While Section 5.4 assumes that tax savings are used to reduce the price of output products, Sections 5.5 & 5.6 suppose that savings are retained by the insurance company. In Section 5.5, we generalize the classical Cramér-Lundberg model by including tax payments in order to study the impact on ruin probabilities. Finally, Section 5.6 computes how solvency capital requirements change that are, e.g., implemented under the regulatory framework of Solvency II or the Swiss Solvency Test. In summary, we find that the competitiveness of corporate buyers of insurance, the ruin probabilities of insurance firms and their solvency capital are hardly affected by the tax system. In contrast, the tax system has a significant influence on the cost of insurance, insurance demand, government revenues and the profitability of insurance firms. Section 5.7 concludes with a discussion and suggestions for further research.

Literature. Holzheu (1997) and Holzheu (2000) suggest an accounting methodology in order to compute basic quantities that characterize the impact of a premium tax. Sections 5.1 & 5.2 build on this methodology. Our sections are, however, based on current data and constitute a necessary prerequisite for the other parts of this chapter. Straubhaar (2006) provides a qualitative analysis of the impact of an increased premium tax that is complemented by a regression analysis in order to obtain quantitative estimates. Schrinner (1997) qualitatively discusses the impact of premium tax and provides related accounting figures besides a preliminary economic analysis. Another preliminary analysis can be found in Hildebrandt (2013).

5.1 | Impact on Insurance Companies and Insurance Holders

Insurance premium tax and value-added tax lead to different tax expenses related to insurance services. To begin with, we provide a descriptive analysis on the basis of aggregate accounting data that quantifies the impact of different tax systems. Our computations follow the methodology described in Holzheu (1997) & Holzheu (2000), using data from 2011 – 2015 provided in BaFin (2011-2015).

Impact on Insurance Companies. We consider an insurance company with earnings of *gross premiums* denoted by $\pi \in \mathbb{R}$. Gross premiums are before the deduction of reinsurance. For the purpose of our comparisons, we define *taxed premiums* as the sum of gross premiums π and the tax that is charged from the policyholders for their insurance contracts, i.e., either premium tax or value-added tax – depending on the (real or hypothetical) tax system that we consider.

We denote the prevailing *VAT rate* by $\tau_{VAT} \geq 0$. The input of the production of insurance contracts is always taxed according to VAT. *Taxed input*[1] $\bar{L}$ and *untaxed input* L can thus be converted into each other:

$$\bar{L} = (1 + \tau_{VAT})L.$$

We define the *rate of input* α as the ratio of untaxed input and untaxed gross premiums earned:

$$\alpha = \frac{L}{\pi}$$

Tax legislation in many countries typically prohibits the deduction of VAT paid on inputs from premium tax, but would allow a deduction if instead VAT was also paid on outputs. The amount that would be deducted in this case equals

$$\bar{L} - L = \tau_{VAT}\alpha\pi. \tag{5.1}$$

We now estimate this quantity that is typically not directly reported by insurance companies. Untaxed inputs can roughly be estimated as gross premiums earned plus capital income minus total losses and costs including taxes. For Germany, the required data were obtained from the annual reports of *Bundesanstalt für Finanzdienstleistungsaufsicht (BaFin)*.

Example 5.1.1. On the basis of BaFin-data for the years 2011 – 2015 (see BaFin (2011-2015)), the mean rate of input[2] equals $\alpha = 5.2\%$. With τ_{VAT} =19%, this implies a difference of untaxed and taxed inputs equal to $\bar{L} - L \approx 0.99\% \cdot \pi$.

[1] In this chapter, taxed variables are marked with a bar $^{-}$.

[2] Input ratios were estimated on the basis of an aggregated stylized income statement of insurance companies provided by BaFin. Input ratios were computed for single years, then added and finally averaged over time. For the detailed computation, we refer to Section 5.9.

Equivalent Value-Added Tax Rate. The current tax system charges VAT on the production inputs of insurance firms, but premium tax on their outputs, i.e., on insurance contracts. A deduction of VAT on inputs from premium tax is not permitted. What is the hypothetical VAT rate on the value added generated by the insurance contracts that leads to the same tax payments as insurance tax? We call this counterfactual VAT the equivalent value-added tax. We stress that the VAT rate on all other goods and services remains unchanged in this gedankenexperiment. We simply analyze a modified basis of assessment of the tax that is charged on insurance contracts, holding the tax revenue constant. In the case of premium tax, the basis of assessment are the premiums; in the case of the equivalent VAT, the basis of assessment is the value added generated by the insurance industry.

We denote the *premium tax rate* by $\tau_{PT} \geq 0$ and the insurance's value added before taxes by $\widetilde{W}$. The value added before taxes is estimated as the sum of acquisition costs and administrative expenses, profits before taxes and changes in equalization provisions and similar provisions. In order to account for the fraction of the value added that is indirectly generated by reinsurers' share of gross premiums earned, we add the difference between the gross technical result and net technical result. From BaFin-data 2011 – 2015,[3] we compute

$$\widetilde{W} \approx 31.3\% \cdot \pi.$$

The equivalent value-added tax rate that leads to the same tax revenue can be calculated by the change of basis of assessment equation,

$$\widetilde{\tau_{VAT}}\widetilde{W} = \tau_{PT}\pi \;\Leftrightarrow\; \widetilde{\tau_{VAT}} = \tau_{PT} \cdot \frac{\pi}{\widetilde{W}}$$

Example 5.1.2. For many types of insurance contracts the premium tax rate in Germany equals $\tau_{PT} = 19\%$. With $\pi/\widetilde{W} \approx 1/31.3\% = 3.19$ as calculated above, we obtain an equivalent VAT rate of

$$\widetilde{\tau_{VAT}} \approx 60.7\%.$$

Remark 5.1.3. Value added before taxes varies among different lines of insurance. Equivalent VAT rates are displayed in Table 5.1.[4]

Impact on Insurance Holders. Next, we consider a hypothetical tax system in which VAT can be deducted from premium tax, and premium tax from VAT. In this case, the basis of assessment of the premium tax are the gross premiums earned, but counterfactual deductions are admissible. We assume that all tax savings are passed to a corporate buyer of insurance contracts. The latter are treated as an input good to the buyer's production, allowing for a deduction of incurred premium tax from VAT on outputs of the corporate customer. Total tax savings can thus be decomposed into two components: a) the VAT on

[3]Value added was calculated for every year, summed up and averaged over time. For the detailed computation, we refer to Section 5.9.

[4]For the detailed computation of the quantities, we refer to Section 5.9.

Class	**Value added ratio**	**Equivalent VAT rate**
Accident	39%	49%
Public liability	37%	51%
Car total	21%	89%
Defense	37%	51%
Fire	35%	55%
Household	43%	45%
Residential building	32%	59%
Credit and guarantee	37%	51%
Total	31%	61%

Table 5.1: Equivalent VAT rates for different lines of insurance.

the input goods of insurance firms deducted from premium tax, b) premium tax deducted from the VAT on the output goods of the corporate buyer of insurance.

We compute the size of these tax savings. For this purpose, we denote the total revenue (or business volume) of the corporate insurance holder by $U \in \mathbb{R}$ and her untaxed cost of insurance by $V \in \mathbb{R}$. The insurance ratio of the company is defined as

$$\beta := \frac{V}{U} \in [0, 1]. \tag{5.2}$$

Lemma 5.1.4. *If VAT and premium tax can be deducted from each other, if taxes on outputs are higher than on inputs and if all tax savings are passed on to a corporate buyer of insurance, then this firm has tax savings of*

$$E = (\tau_{VAT}\ \alpha + \tau_{PT})\, \beta U. \tag{5.3}$$

Proof. The proof is given in Section 5.8. □

Example 5.1.5. Setting the input ratio to α =5.2% as in Example 5.1.1 and the insurance ratio to β =0.5%,[5] we obtain

$$E = 0.1\% \cdot U,$$

i.e., the total savings are only approximately 10 basis points of the business volume of the company. The parameter β depends on the industry sector of the corporate insurance holder. According to Swiss Re, sigma No 5/2012 (sigma (2012), p. 17), it varies between 0.1% and 1.4% for different US industries.

In summary, we estimated for insurance firms in Germany that their mean rate of input is about 5% of gross premiums, implying a difference between untaxed and taxed inputs of about 1% of gross premiums. For a premium tax of 19%, an equivalent VAT rate depends on the line of insurance and ranges from about 50% for accident insurance to about 90% for car insurance. Finally, we considered a counterfactual tax system in which

[5] As a rough approximation of β, we use an estimate that is provided in Swiss Re, sigma No 5/2012 (sigma (2012), p. 16) for the US market. Since the main purpose of Example 5.1.5 is to provide an estimate of the order of the impact of a modified tax system on corporate insurance costs, precise knowledge of β is not required.

VAT and premium tax can be deducted from each other and estimated for Germany that tax savings of corporate policyholders would amount to about 10 basis points of their total revenues.

5.2 | Impact on Tax Revenues

We will now discuss the impact on total tax revenues, if premium tax is replaced by VAT. First, we compute the modified tax revenues. Second, we calculate an equivalent VAT which leads to the same total tax revenues. Our findings build on the results of the previous section. The methodology is motivated by Holzheu (2000), p. 76ff.

Comparison of Tax Revenues. Let Π be the total national untaxed gross insurance premiums earned, $\widetilde{W}$ the total value added before taxes of the corresponding insurance companies, and α their input ratio. The tax revenue S related to insurance contracts in a tax system with premium tax can be split into three parts:

1. VAT of insurance companies on their inputs: This amount cannot be deducted from premium tax. It can be computed according to Eq. (5.1).

2. Premium tax.

3. VAT on taxed insurance premiums: The costs of the outputs of corporate buyers of insurance are increased by the premium tax. This is implicitly reflected in their prices and leads to additional value-added tax revenues.

Total tax revenues are given by adding up the three parts:

$$S = (\tau_{VAT}\alpha + \tau_{PT}) \cdot \Pi + \tau_{VAT} \cdot (1 + \tau_{PT}) \cdot \Pi_G.$$

Here, Π_G denotes the untaxed insurance premiums of corporate insurance holders. Policies of private customers are included in Π but not in Π_G, since they are not indirectly charged with additional VAT.

Suppose now that insurance premiums are not subject to a premium tax but to a value-added tax. In this case, these taxes are fully deductable and double-taxation is avoided. The total relevant tax revenue thus amounts to $\tilde{S} := \tau_{VAT}\widetilde{W}$. In particular, if we assume that $\tau_{PT} \geq \tau_{VAT}$ and $\widetilde{W} < \Pi$, then

$$\tilde{S} = \tau_{VAT}\widetilde{W} < \tau_{PT}\Pi < S.$$

Changing the tax system from premium tax to VAT thus leads to lower total tax revenues, if the corresponding tax rates are equal.

Equivalent Value-Added Tax Rate. As before, we compute an equivalent VAT rate $\widetilde{\tau_{VAT}}$ that leads to the same tax revenues, but now also incorporates taxes on premium tax paid by corporate policyholders. As in the previous section, we assume that the VAT rate on all other goods and services remains unchanged in this thought experiment. We only focus on those tax revenues that are directly related to insurance contracts as explained above.

Lemma 5.2.1. *We denote by* $g := \frac{\Pi_G}{\Pi}$ *the ratio of untaxed corporate insurance premiums over total untaxed insurance premiums. The equivalent value-added tax rate* $\widetilde{\tau_{VAT}}$ *is given by*

$$\widetilde{\tau_{VAT}} = (\tau_{VAT}\alpha + \tau_{PT} + \tau_{VAT}(1+\tau_{PT})g) \cdot \frac{\Pi}{\widetilde{W}}$$

Proof. The proof is given in Section 5.8. □

Example 5.2.2. The average value added before taxes of German insurance firms during the period 2011 – 2015[6] amounts to

$$\widetilde{W} \approx 31.3\% \cdot \Pi.$$

If we assume that the fraction of premiums of corporates is g =35%,[7] we obtain for $\tau_{VAT} = \tau_{PT} = 19\%$ and $\alpha = 5.2\%$[8] an equivalent value-added tax rate of

$$\widetilde{\tau_{VAT}} \approx 89.2\%.$$

Finally, let us consider a modification of the tax system in which premium tax is replaced by VAT. This would, in particular, imply that both VAT on insurance companies' inputs and VAT on premium tax paid by corporate customers are deductible. For the purpose of illustrating the size of this effect, we suppose that the German premium tax of 19% is replaced by VAT of 19%. This would decrease total German tax revenues by approximately

$$\begin{aligned} S - \tau_{VAT} \cdot \widetilde{W} &= (\tau_{VAT}\alpha + \tau_{PT} + \tau_{VAT}(1+\tau_{PT})g - \tau_{VAT} \cdot 31.3\%) \cdot \Pi = (27.9\% - 5.9\%) \cdot \Pi \\ &= 22\% \cdot \Pi \end{aligned}$$

Taking $\Pi \approx$ EUR 75 billion (corresponding to German gross premiums earned in 2015), tax revenues would decrease by approximately EUR 16 billion, if the tax system was changed.

5.3 | Impact on Insurance Demand

The current German premium tax leads to an additional tax burden for insurance contracts. In this section, we investigate the impact of the tax system on insurance demand. Insurance demand is endogenously modeled in a classical expected utility framework. For proportional insurance, we compute the optimal demand maximizing the expected utility of the policyholder.

Model 5.3.1 (Insurance Demand)**.** We consider a proportional insurance contract over a fixed time horizon. The initial wealth of the insurance holder is denoted by $w > 0$. Over

[6] Compare Footnote 3.

[7] This number quantifies the fraction for Germany in 2010 according to sigma (2012), p. 10, Table 3. 35% of the total non-life premium income was generated by corporate buyers of insurance contracts.

[8] See Example 5.1.1.

the time interval, the insurance holder incurs a *random loss* $X \in L^1(\mathbb{R}_+)$ where $L^1(\mathbb{R}_+)$ denotes the space of integrable real-valued random variables on some probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}_+$. The insurance contract is characterized by the parameter $\nu \in [0,1]$ which is the fraction of the loss that is covered by the insurance. The premium for *full insurance* is $\pi \in \mathbb{R}_+$; the premium for *partial insurance* of a fraction ν of the loss X is $\nu \cdot \pi$.

The terminal endowment of the insurance holder as a function of ν is

$$X_\nu = w - X + \nu(X - \pi) = (1-\nu)(w - X) + \nu(w - \pi).$$

Buyers of insurance can choose the fraction ν according to their preferences. This fraction is computed as the solution to a utility maximization problem of the policyholder, see, e.g., Chapter 2 in Föllmer & Schied (2011).

As a first example, we consider a Bernoulli utility function with *constant absolute risk aversion* (CARA). This function has the form $u_1^\kappa(x) = 1 - e^{-\kappa x}$ with $\kappa > 0$. Another example is a Bernoulli utility function with *hyperbolic absolute risk aversion* (HARA), given by $u_2^\lambda(x) = \frac{1}{\lambda} x^\lambda$ for $\lambda \in (0,1)$. The limiting case $\lambda = 0$ corresponds to logarithmic utility. The Arrow-Pratt-coefficients of absolute risk aversion are κ for u_1^κ and the hyperbolic function $x \mapsto (1-\lambda)/x$ for u_2^λ, explaining the terminology. In the case of HARA utility, we will always assume that $X \leq w$, for logarithmic utility $X < w$.

Problem 5.3.2 (Expected Utility Maximization)**.** Let $S \subseteq \mathbb{R}$ be convex and assume that $u : S \to \mathbb{R}$ is a Bernoulli utility function, i.e., a function that is strictly concave, strictly increasing and continuous on S. Suppose that the support $\operatorname{supp} X_\nu$ is contained in S and that $u(X_\nu)$ is integrable with respect to P for all $\nu \in [0,1]$. Then, the optimal insurance contract is characterized by the maximizer $\nu^* \in [0,1]$ of the expected utility

$$\nu \mapsto \mathbb{E}\left[u(X_\nu)\right].$$

A necessary condition for an interior solution $\nu \in (0,1)$ is given by the *first-order condition*

$$\frac{\partial}{\partial \nu} \mathbb{E}\left[u(X_\nu)\right] = 0.$$

We compare different tax regimes for two examples of loss distributions, a Bernoulli and a Gamma distribution. In the case of a Bernoulli distribution, we assume that a loss $\hat{x} > 0$ occurs with probability $p \in (0,1)$, and no loss with probability $1-p$, i.e., $X \sim \operatorname{Ber}(\hat{x}, p)$. For HARA utility, we assume that $\hat{x} \leq w$, for logarithmic utility $\hat{x} < w$. For a Gamma distribution with parameters $\xi, \mu > 0$ and density

$$f_{\xi,\mu}(x) = \frac{\mu^\xi}{\Gamma(\xi)} x^{\xi-1} e^{-\mu x} \mathbb{1}_{(0,\infty)}(x), \quad x \in \mathbb{R},$$

we use the notation $\Gamma(\xi, \mu)$. Note that $\Gamma(\cdot)$ denotes the ordinary gamma function. The Gamma distribution with unbounded support will only be considered in the case of CARA utility.

Remark 5.3.3. The following result is a simple consequence of Föllmer & Schied (2011), Proposition 2.39: Let u : dom $u \to \mathbb{R}$ be a Bernoulli utility function. We assume that $\mathbb{R}_+ \subseteq$ dom u, $X \leq w$ and $\pi \leq w$. Then, the following assertions hold:

(i) We have $\nu^*(\pi) = 1$ if $\pi \leq \mathbb{E}[X]$, and $\nu^*(\pi) > 0$ if $\pi \leq w - c_X$, where c_X is the *certainty equivalent* given by the equation $\mathbb{E}[u(X)] = u(c_X)$.

(ii) If u is differentiable, then

$$\nu^*(\pi) = 1 \;\Leftrightarrow\; \pi \leq \mathbb{E}[X]$$

and

$$\nu^*(\pi) = 0 \;\Leftrightarrow\; \pi \geq w - \frac{\mathbb{E}[(w-X)u'(w-X)]}{\mathbb{E}[u'(w-X)]}$$

If $\pi < w - \frac{\mathbb{E}[(w-X)u'(w-X)]}{\mathbb{E}[u'(w-X)]}$, then $\nu^*(\pi) > 0$.

A risk-averse buyer purchases full insurance, if and only if the premium does not exceed the expected loss. Insurers will, however, always charge premiums that are larger in order to avoid ruin. In this case, full insurance is never optimal.
For the special case of Bernoulli-distributed random variables, Schrinner (1997) discussed

$$\nu^*(\pi) \begin{cases} = 1, & \text{if } \pi \leq \mathbb{E}[X], \\ < 1, & \text{if } \pi > \mathbb{E}[X], \end{cases}$$

in the context of premium tax. He argued that higher premium taxes lead to higher premiums and therefore to a larger deviation of the premium from the expected loss, which results in less demand for insurance.

Theorem 5.3.4. *The solutions to Problem 5.3.2 for specific utility functions and loss distributions are as follows:*

(i) CARA-utility: *Consider the Bernoulli utility* $u(x) = u_1^{\kappa}(x), \kappa > 0$.

- *Assume that losses are Bernoulli-distributed, i.e.,* $X \sim Ber(\hat{x}, p)$. *Then, the optimal insurance contract is characterized by*

$$\nu^*(\pi) = \begin{cases} 0, & \pi \;\geq\; \frac{p\hat{x}e^{\kappa\hat{x}}}{1-p+pe^{\kappa\hat{x}}}, \\ 1 - \frac{1}{\kappa\hat{x}} \ln\left(\frac{\frac{1}{p}-1}{\frac{\hat{x}}{\pi}-1}\right), & \frac{p\hat{x}e^{\kappa\hat{x}}}{1-p+pe^{\kappa\hat{x}}} \;>\; \pi \;>\; p\hat{x}, \\ 1, & p\hat{x} \;\geq\; \pi. \end{cases}$$

- *Assume that losses are Gamma-distributed, i.e.,* $X \sim \Gamma(\xi, \mu)$, *and assume that* $\kappa < \mu$. *Then, the optimal insurance contract is given by*

$$\nu^*(\pi) = \begin{cases} 0, & \pi \;\geq\; \frac{\xi}{\mu-\kappa}, \\ 1 + \frac{\xi}{\pi\kappa} - \frac{\mu}{\kappa}, & \frac{\xi}{\mu-\kappa} \;>\; \pi \;>\; \frac{\xi}{\mu}, \\ 1, & \frac{\xi}{\mu} \;\geq\; \pi. \end{cases}$$

(ii) HARA-utility: *Consider the Bernoulli utility $u(x) = u_2^\lambda(x)$, $\lambda \in (0,1)$. We set $\zeta = \frac{1}{1-\lambda}$.*

- *Assume that losses are Bernoulli-distributed, i.e., $X \sim Ber(\hat{x}, p)$. We suppose that $0 < \hat{x} \leq w$. Then, the optimal insurance contract is*

$$\nu^*(\pi) = \begin{cases} 0, & \pi \geq \frac{p\hat{x}w^{1-\lambda}}{pw^{1-\lambda}+(1-p)(w-\hat{x})^{1-\lambda}}, \\ \frac{\pi^\zeta(1-p)^\zeta(\hat{x}-w)+p^\zeta(\hat{x}-\pi)^\zeta w}{\pi^\zeta(1-p)^\zeta(\hat{x}-\pi)+p^\zeta(\hat{x}-\pi)^\zeta \pi}, & \frac{p\hat{x}w^{1-\lambda}}{pw^{1-\lambda}+(1-p)(w-\hat{x})^{1-\lambda}} > \pi > p\hat{x}, \\ 1, & p\hat{x} \geq \pi. \end{cases}$$

(iii) Logarithmic utility: *Consider the logarithmic utility $u_2^0(x) = \log(x)$, i.e., the limiting case of HARA-utility for $\lambda = 0$.*

- *Assume that losses are Bernoulli-distributed, i.e., $X \sim Ber(\hat{x}, p)$. We suppose that $0 < \hat{x} < w$. Then, the optimal insurance contract is characterized by*

$$\nu^*(\pi) = \begin{cases} 0, & \pi \geq \frac{p\hat{x}w}{w+\hat{x}(p-1)}, \\ \frac{\pi(w-\hat{x})-p\hat{x}(w-\pi)}{\pi(\pi-\hat{x})}, & \frac{p\hat{x}w}{w+\hat{x}(p-1)} > \pi > p\hat{x}, \\ 1, & p\hat{x} \geq \pi. \end{cases}$$

Proof. The proof is given in Section 5.8. □

Remark 5.3.5. Although the technical conditions of bounded support of the random variable and differentiability of the utility function are not fulfilled in all cases of Theorem 5.3.4, we point out that the provided thresholds for the premium still coincide with those given in Remark 5.3.3 (ii). For detailed computations, we refer to Appendix D, Section D.2.2.

In order to gauge the effect of a modified tax on insurance demand, we now compute the modification of the effective premiums for different tax systems.

Lemma 5.3.6. *We denote the taxed premium in a system with premium tax by $\bar{\pi}_{PT}$. Assume now the counterfactual situation that VAT and premium tax can be deducted from each other and that at the same time all tax savings are passed to a corporate buyer of insurance. In this case, the effective premium equals*

$$\bar{\pi}_{VAT} = \gamma \cdot \bar{\pi}_{PT}, \qquad \gamma := 1 - \frac{\tau_{VAT}\alpha + \tau_{PT}}{1 + \tau_{PT}},$$

where α denotes the input ratio.

Proof. The proof is given in Section 5.8. □

Example 5.3.7. For an input ratio $\alpha = 5.2\%$ and $\tau_{VAT} = \tau_{PT} = 19\%$, we obtain $\gamma \approx 0.832$, i.e., the effective premium reduces to 83.2% of the original premium, if deduction is permitted.

Before we can analyze the impact of alternative tax systems on demand, we need to specify how premiums are calculated net of taxes. We focus on two examples of classical premium principles, namely the expected value principle and the standard deviation principle, see, e.g., Chapter 12 in Schmidt (2009). We also investigated the semi-standard deviation principle which leads to similar results as the standard deviation principle; for this reason it is not included in the case studies below. However, we provide the corresponding formulae. Untaxed premiums with safety loading $\delta > 0$ are given in Table 5.2 for the considered loss distributions; see Appendix D, Section D.2.3, for detailed computations. Note that $\underline{\Gamma}(\cdot,\cdot)$ denotes the upper incomplete gamma function. Adjusting tax payments, optimal insurance contracts can finally be computed according to Theorem 5.3.4.

Premium Principle	$X \sim \mathbf{Ber}(\hat{x}, p)$	$X \sim \boldsymbol{\Gamma}(\xi, \mu)$
Expected Value Principle	$p\hat{x}(1+\delta)$	$\frac{\xi}{\mu}(1+\delta)$
Standard Deviation Principle	$p\hat{x}\left(1+\delta\sqrt{\frac{1-p}{p}}\right)$	$\frac{\xi}{\mu}\left(1+\delta\frac{1}{\sqrt{\xi}}\right)$
Semi-Standard Deviation Principle	$p\hat{x}\left(1+\delta\frac{1-p}{\sqrt{p}}\right)$	$\frac{\xi}{\mu}\left(1+\delta\frac{1}{\xi}\sqrt{\frac{1}{\Gamma(\xi)}(\xi^{\xi}\,e^{-\xi}+\xi\,\underline{\Gamma}(\xi,\xi))}\right)$

Table 5.2: Computation of untaxed premiums.

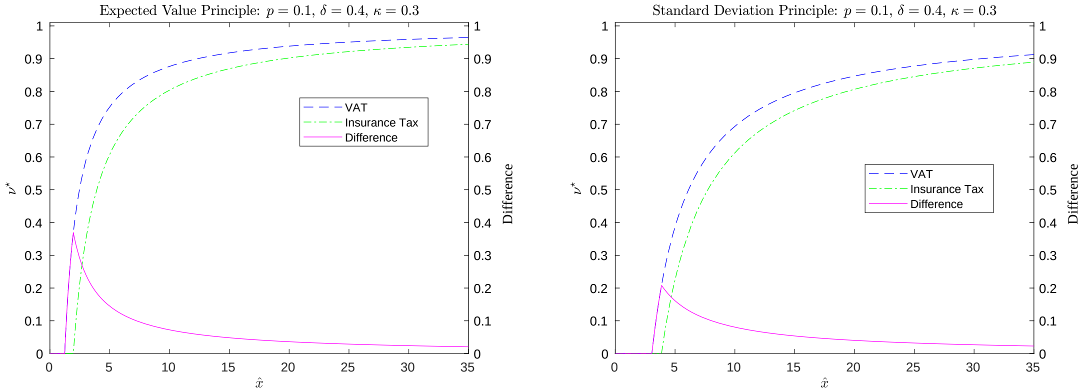
The following examples analyze the impact of different tax systems on insurance demand. In all case studies, we assume $\tau_{PT} = 19\%$ and $\gamma = 0.832$ according to Example 5.3.7.

Example 5.3.8. In the first numerical example, we consider losses $X \sim \text{Ber}(\hat{x}, p)$ and a policyholder with CARA-utility $u(x) = u_1^{\kappa}(x), \kappa > 0$. We choose $p = 0.1$ and vary $\hat{x}$. Risk aversion is set to $\kappa = 0.3$, and the safety loading equals $\delta = 0.4$.

Figure 5.1 displays optimal insurance contracts ν^* for the two different tax systems. In the case of CARA-utility, these do not depend on the initial endowment of the policyholder. As expected, if deduction is permitted, the demand for insurance is increased. The difference in demand initially increases for small loss sizes $\hat{x}$ and decreases towards a small level for larger loss sizes. Comparing Figures 5.1 (a) & (b), we observe similar shapes of the functions for both premium principles. Due to higher premiums for the standard deviation principle, the optimal demand for insurance is smaller than in the case of the expected value principle.

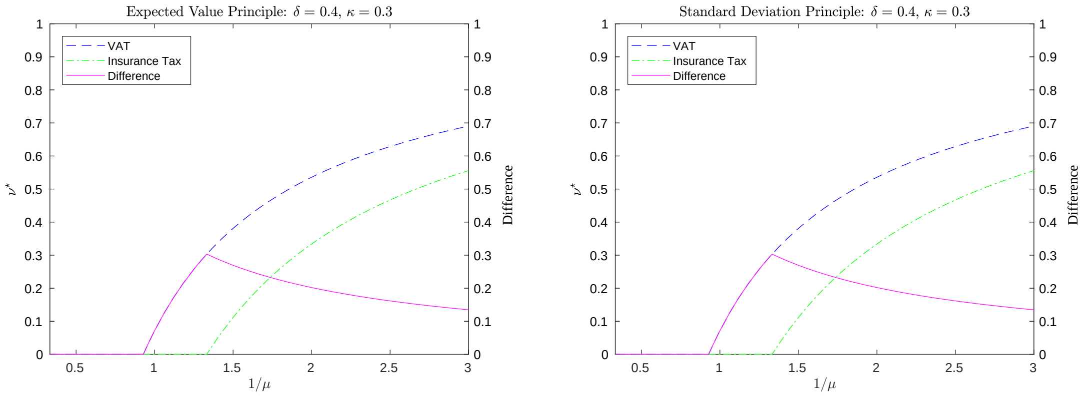
Example 5.3.9. In the second numerical example, we consider losses $X \sim \Gamma(\xi, \mu)$ and a policyholder with CARA-utility $u(x) = u_1^{\kappa}(x), \kappa > 0$. We choose $\xi = 1$ and vary $1/\mu$. Risk aversion is again set to $\kappa = 0.3$, and the safety loading equals $\delta = 0.4$.

Figure 5.2 displays optimal insurance contracts ν^* for the two different tax systems. Again, if deduction is permitted, the demand for insurance is increased. The difference in demand is zero for small expected loss sizes $1/\mu \leq 0.92$, increases for $1/\mu \in (0.92, 1.33)$, and decreases for larger losses.



(a) (b)

Figure 5.1: Insurance demand in Example 5.3.8 for expected value and standard deviation principle.



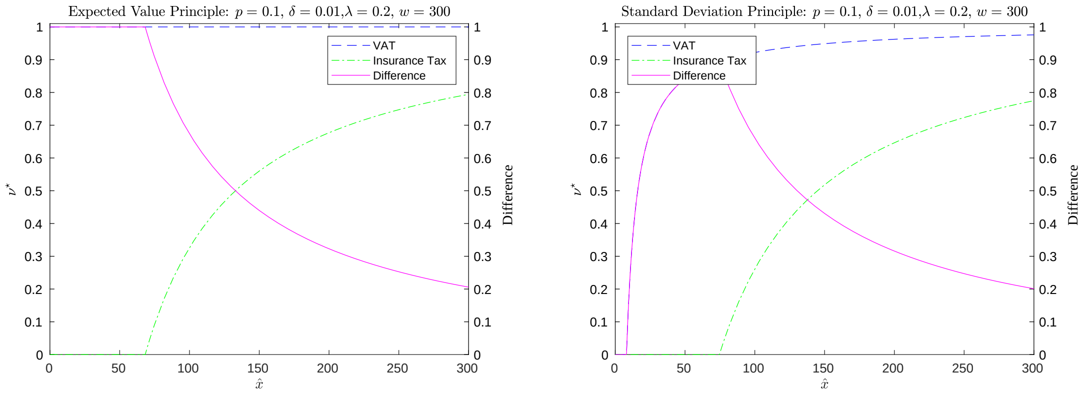
(a) (b)

Figure 5.2: Insurance demand in Example 5.3.9 for expected value and standard deviation principle.

Example 5.3.10. In the third numerical example, we consider losses $X \sim \text{Ber}(\hat{x}, p)$ and a policyholder with HARA-utility $u(x) = u_2^\lambda(x), \lambda \in (0, 1)$. We choose $p = 0.1$ and vary $\hat{x}$. We set $\lambda = 0.2$. The safety loading equals $\delta = 0.01$. Initial wealth is $w = 300$.

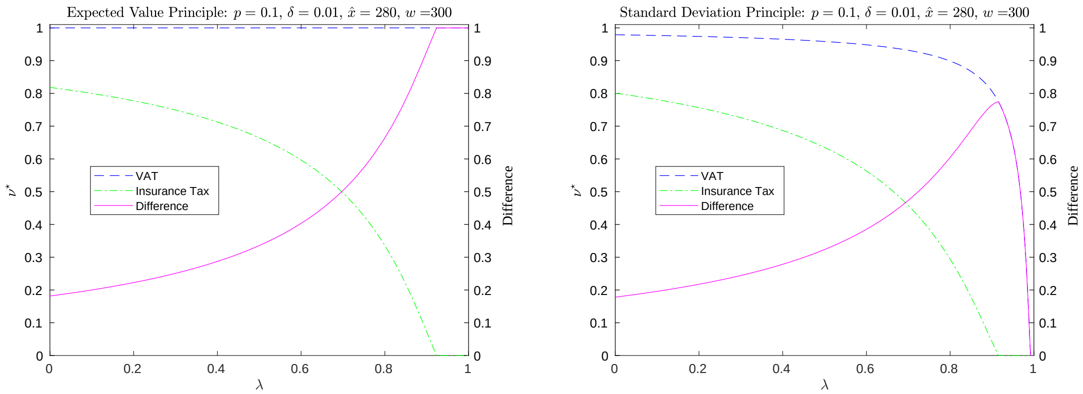
Figure 5.3 displays optimal insurance contracts ν^* for the two different tax systems. Again, we obtain that the demand for insurance is increased, if deduction is permitted. The difference in demand is large for small loss sizes $\hat{x}$ and decreases for larger losses. It remains larger than 0.2 for all values of $\hat{x}$ in the case of the expected value principle, respectively, for all values of $\hat{x} \geq 10.2$ in the case of the standard deviation principle.

Example 5.3.11. Finally, we consider the same situation as in Example 5.3.10, but keep $\hat{x} = 280$ fixed and vary λ. Risk aversion decreases with increasing λ. Figure 5.4 displays optimal insurance contracts ν^* for the two different tax systems. With VAT, insurance demand stays close to 1 for small values of λ. Premium tax leads to a higher cost of insurance, and insurance demand is significantly lower. In the case of premium tax, insurance demand decreases to 0 as risk aversion goes to 0, i.e., λ approaches 1. In the case of VAT, this effect occurs only if premiums are computed according to the standard deviation principle.



(a) (b)

Figure 5.3: Insurance demand in Example 5.3.10 for expected value and standard deviation principle.



(a) (b)

Figure 5.4: Insurance demand in Example 5.3.11 for expected value and standard deviation principle.

In summary, our case studies show that the tax system may have a substantial impact on insurance demand of corporate buyers of insurance. The size of this effect depends on the loss distribution and the utility of the policyholder, in particular, on the size of potential losses and the risk aversion of the policyholder.

5.4 | Impact on Competitiveness

The current tax system in Germany with premium tax does not allow that premium tax and VAT are deducted from each other. If such deductions were permitted, as, for example, in an hypothetical tax system that charges VAT on insurance contracts (instead of premium tax), overall tax payments would be reduced.

Consider a corporate insurance holder. We compare two tax systems: a realistic tax system with premium tax, and a counterfactual tax system that permits the full deduction of VAT and premium tax from each other. We assume that the resulting tax savings lead to a reduction of the sales prices of the corporate policyholder. The reduced prices increase the relative competitiveness of a domestic firm that benefits from a modified tax system in contrast to its international competitors.

We design a stylized model that captures this effect and allows its quantification. There are two firms that produce different goods $i = 1, 2$ that they sell for prices p_i, $i = 1, 2$. We assume that demand for the two goods in the economy is the solution to a utility maximization of a representative consumer.

Problem 5.4.1 (Utility Maximization)**.** The utility function of the representative consumer with budget $w \in \mathbb{R}_+$ is denoted by $u : \mathcal{X} \to \mathbb{R}$, $\mathcal{X} = \mathbb{R}_+ \times \mathbb{R}_+$. The consumer's demand $x^* = (x_1^*, x_2^*)$ solves her utility maximization problem

$$x^* \in \operatorname{argmax}_{x \in \mathbb{R}_+^2} u(x_1, x_2)$$

subject to her budget constraint

$$p_1 x_1 + p_2 x_2 = w.$$

The following preliminary lemma computes the price reduction when the tax system is changed.

Lemma 5.4.2. *Let α be the rate of input of insurance companies, and let β be the insurance ratio of company 1 as defined in Eq.* (5.2)*. If company* 1 *is a domestic company, a modification of the domestic tax system, as described in Lemma 5.1.4, decreases the price of its product by $\theta_{\alpha,\beta} \cdot p_1$, where*

$$\theta_{\alpha,\beta} := (\tau_{VAT}\alpha + \tau_{PT})\,\beta,$$

and p_1 denotes the original price with premium tax. I.e., the unit price decreases to $\tilde{p}_1 = (1 - \theta_{\alpha,\beta}) \cdot p_1$.

Proof. The proof is given in Section 5.8. □

In two case studies, we will now illustrate how a modification of the tax system may change product demand. In the first example, the representative consumer has a utility function of Cobb-Douglas type, in the second with constant elasticity of substitution.

5.4.1 | Cobb-Douglas Utility Function

We recall that a *Cobb-Douglas Utility Function* has the form

$$u_a(x_1, x_2) := x_1^a x_2^{1-a}, \qquad a \in (0, 1).$$

Solving the optimization problem 5.4.1, one obtains the solution

$$x_1^{(a)} = \frac{aw}{p_1}, \qquad x_2^{(a)} = \frac{(1-a)w}{p_2}$$

We compare the change in competitiveness of a domestic and a foreign firm, if the domestic system is changed as described before. For this purpose, we assume that company 1 is domestic and company 2 foreign. The price of the product of company 2 is p_2 and fixed, but the price of the product of company 1 is a function of the domestic tax system.

Lemma 5.4.3 (Shift in Demand)**.** *Let p_1 be the original price of product 1 and $x_1^{(a)}$ the corresponding demand. Suppose that $\tilde{p}_1$ is the price of product 1 after modifying the tax system, see Lemma 5.4.2, and $\tilde{x}_1^{(a)}$ the corresponding demand. Setting $\Delta x_1^{(a)} = \tilde{x}_1^{(a)} - x_1^{(a)}$, we obtain*

$$\frac{\Delta x_1^{(a)}}{x_1^{(a)}} = \frac{\theta_{\alpha,\beta}}{1-\theta_{\alpha,\beta}}$$

Proof. The proof is given in Section 5.8. □

The relative shift in demand does neither depend on the available budget w nor the preference parameter a.

Example 5.4.4. Taking the numbers from Example 5.1.5, we compute $\theta_{\alpha,\beta} \approx 0.1\%$, thus

$$\frac{\Delta x_1^{(a)}}{x_1^{(a)}} \approx 0.1\%, \quad \forall a \in (0,1).$$

The price of the product of the domestic company and its competitiveness is almost not affected by a modification of the tax system. The reason is that the insurance ratio of companies is typically small. In addition, the rate of input of insurance companies is not very large.

Example 5.4.5. Insurance contracts are an input to the production of goods. Their contribution varies across different industry sectors and so does the effect of a modification of the tax system on production costs and product prices. Suppose that the input ratio is set to α =5.2% as in the previous example. Insurance ratios for different US industry sectors are based on a survey of MarketStance and were obtained from sigma (2012) (p. 17). The data are displayed in Table 5.3. Again, $\theta_{\alpha,\beta}$ and $\frac{\Delta x_1^{(a)}}{x_1^{(a)}}$ are computed according to our model. In all cases, the effects are very small.

Industrial Sector	**Premium/Business Vol.** β in %	**Saving/Business Vol.** $\theta_{\alpha,\beta}$ in %	**Shift in Demand** $\Delta x_1^{(a)}/x_1^{(a)}$ in %
Mining	0.80	0.16	0.16
Construction	1.31	0.26	0.26
Manufacturing	0.31	0.06	0.06
Transport, utilities, communication	1.21	0.24	0.24
Retail trade	0.36	0.07	0.07
Wholesale trade	0.14	0.03	0.03
Financial	0.38	0.08	0.08
Services	0.70	0.14	0.14

Table 5.3: Shift in product demand related to industrial sectors.

5.4.2 | Constant Elasticity of Substitution Utility Function

We recall the definition of a utility function with *constant elasticity of substitution (CES)*:

$$u_{a,b}(x_1, x_2) := \left(a x_1^b + (1-a) x_2^b\right)^{\frac{1}{b}},$$

where $a \in (0,1)$ and $b \neq 0$. The latter quantity is called the *parameter of substitution.* Again, we denote the budget of the consumer by w. The consumer's optimal demand is

$$x_1^{a,b} = \frac{w\ (p_1/a)^{-\eta}}{a^\eta p_1^{1-\eta} + (1-a)^\eta p_2^{1-\eta}}, \qquad x_2^{a,b} = \frac{w\ (p_2/(1-a))^{-\eta}}{a^\eta p_1^{1-\eta} + (1-a)^\eta p_2^{1-\eta}},$$

where $\eta := \frac{1}{1-b}$ denotes the elasticity of substitution.

Lemma 5.4.6 (Shift in Demand)**.** *Let p_1 be the original price of product 1 and $x_1^{a,b}$ the corresponding demand. Suppose that $\tilde{p}_1$ is the price of product 1 after modifying the tax system, see Lemma 5.4.2, and $\tilde{x}_1^{a,b}$ the corresponding demand. Setting $\Delta x_1^{a,b} = \tilde{x}_1^{a,b} - x_1^{a,b}$, we obtain*

$$\frac{\Delta x_1^{a,b}}{x_1^{a,b}} = (1-\theta_{\alpha,\beta})^{-\eta} \frac{a^\eta\ p_1^{1-\eta} + (1-a)^\eta\ p_2^{1-\eta}}{(1-\theta_{\alpha,\beta})^{1-\eta}\ a^\eta\ p_1^{1-\eta} + (1-a)^\eta\ p_2^{1-\eta}} - 1.$$

Proof. The proof is given in Section 5.8. □

In contrast to a Cobb-Douglas utility, the relative demand shift depends on the parameters of the utility and the price level of the products in the case of CES-utility. The impact of these inputs on demand is illustrated in Figure 5.5 for the parameter values given in Table 5.4.

Saving:	$\theta_{\alpha,\beta} = 0.1\%$
Share Parameter:	$a \in (0,1)$
Price of Product 1:	$p_1 = 1$
Price of Product 2:	$p_2 \in [0,2]$

Table 5.4: Parameters for case studies with CES utility function.

In particular, we consider different parameters of substitution b and elasticity of substitution $\eta = 1/(1-b)$. For $\eta > 1$, the products are gross substitutes, for $\eta < 1$, they are gross complements. We fix $p_1 = 1$ and vary η, a and p_2. The resulting relative demand shifts are displayed in Figure 5.5. In the case of gross substitutes ($\eta > 1$), the increase in demand caused by the price change is, of course, higher than in the case of gross complements. The effect is, however, in all cases very small.

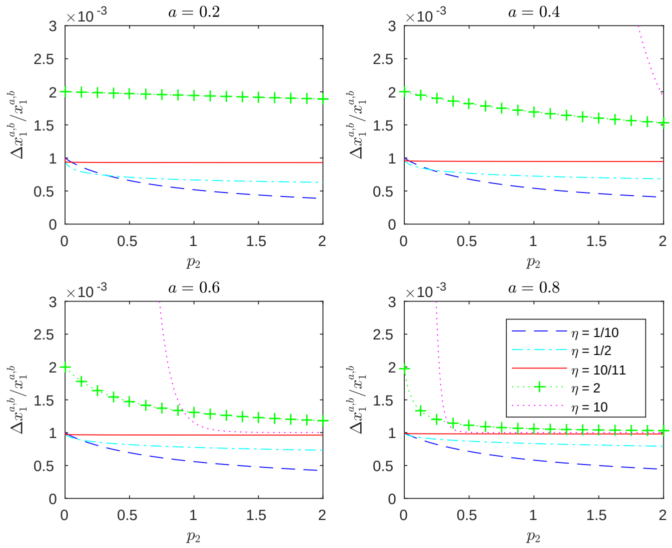


Figure 5.5: Relative shift in demand with CES utility functions.

All case studies clearly indicate that the international competitiveness (in terms of product pricing) of corporate policyholders is almost not affected by the difference of premium tax and VAT.

5.5 | Impact on Ruin Probability

In this section, we investigate the impact of the premium tax on the ruin probability of insurance companies. For this purpose, we extend the classical Cramér-Lundberg model by including taxes and compare ruin probabilities for different systems of taxation. A concise introduction to ruin theory is Mikosch (2009). A comprehensive presentation can be found in Asmussen & Albrecher (2010).

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We consider a family of *risk processes* of insurance companies $(R_t^w)_{t\geq 0}$ enumerated by the initial wealth $R_0^w = w \in \mathbb{R}$. The ruin probability of these companies is a function of initial wealth:

$$\psi^w(\pi) := P\left(\inf_{t\geq 0} R_t^w < 0\right).$$

For later reference, we also emphasize the dependence on the premium rate.

5.5.1 | The Cramér-Lundberg Model

In the current section, we recall the classical Cramér-Lundberg model and its basic definition. On the basis of Mikosch (2009) and Asmussen & Albrecher (2010), we collect the results that will be needed for an analysis of the impact of insurance tax on ruin.

Model 5.5.1 (Cramér-Lundberg)**.** Denote the initial capital of the insurance company by $w \in \mathbb{R}$ and its premium rate by $\pi \in \mathbb{R}$. Insurance losses are modeled by a compound Poisson process $\left(\sum_{k=1}^{N_t} X_k\right)_{t\geq 0}$ where individual losses $(X_k)_{k\in\mathbb{N}}$ are strictly positive, integrable, identically distributed with law B, jointly independent and independent of the Poisson process $(N_t)_{t\geq 0}$ with intensity $\vartheta > 0$. The risk process in the Cramér-Lundberg model is given by

$$R_t^w = w + \pi t - \sum_{k=1}^{N_t} X_k.$$

Wald's equation and the strong law of large numbers imply that $\frac{1}{t}\sum_{k=1}^{N_t} X_k \xrightarrow[t\to\infty]{} \vartheta\,\mathbb{E}[X_1] =: r$ almost surely. It is well-known that ruin occurs with probability 1, unless the net profit condition (NPC) holds, i.e., $\pi > r$. This is equivalent to premium payments being larger than the expected value of the losses for any time horizon t, i.e.,

$$\pi t \ = \ (1+\rho)\,\mathbb{E}\left[\sum_{k=1}^{N_t} X_k\right]$$

with safety loading $\rho = \frac{\pi - r}{r} > 0$. If the NPC holds, the asymptotic behaviour of the ruin probability ψ^w for $w \to \infty$ can be characterized; large w corresponds to high initial capital. We recall the key results for light-tailed and heavy-tailed losses in the Cramér-Lundberg model.

Notation 5.5.2. If $\lim_{w\uparrow\infty} \frac{\psi^w(\pi)}{\varphi^w(\pi)} = 1$, we write $\psi^w(\pi) \sim \varphi^w(\pi)$.

The classical result of ruin theory considers the case of light-tailed distributions and involves the Cramér-Lundberg coefficient. Assume that the moment-generating function of X_1, i.e., $\hat{B}(h) = \int e^{hz} dB(z) = \mathbb{E}\left[e^{hX_1}\right]$, exists for all $h \in (-h_0, h_0)$ for some $h_0 > 0$. The Cramér-Lundberg coefficient $l > 0$, if it exists, is the unique solution of the equation

$$\hat{B}(l) = 1 + \frac{\pi l}{\vartheta}$$

Theorem 5.5.3 (Cramér-Lundberg Approximation)**.** *Assume that the NPC holds and that the distribution of X_1 has a density and a moment-generating function in some neighborhood of 0. In addition, we suppose that the Cramér-Lundberg coefficient $l > 0$ exists. Setting $C = \frac{\pi - r}{\vartheta \hat{B}'(l) - \pi}$ the asymptotic behaviour of the ruin probability can be characterized as follows:*

$$\lim_{w\to\infty} e^{lw}\psi^w(\pi) = C,$$

i.e., $\psi^w(\pi) \sim Ce^{-lw}$ as $w \to \infty$.

Example 5.5.4. For independent, exponentially distributed losses $X_1, X_2, \ldots$ with parameter $\iota > 0$, i.e., $X_k \sim \mathrm{Exp}(\iota)$, $k \in \mathbb{N}$, the Cramér-Lundberg coefficient is $l = \iota - \frac{\vartheta}{\pi}$, and the ruin probability equals the asymptotic approximation of Theorem 5.5.3:

$$\psi^w(\pi) = \frac{\vartheta}{\iota\pi} e^{-\left(\iota - \frac{\vartheta}{\pi}\right)w} \tag{5.4}$$

So far, we considered light-tailed losses. In the case of heavy-tailed losses, the occurrence of ruin is qualitatively different from the light-tailed case. For light-tailed loss distributions, ruin happens if a large number of sufficiently large claims accumulate. For heavy-tailed loss distributions, ruin can occur spontaneously and is typically due to a large single claim. Quantitatively, this is related to the integrated tail distribution. The corresponding theorem of Embrechts & Veraverbeke (1982) requires the notion of subexponential distributions.

Remark 5.5.5. (i) If the positive random variable X has distribution function F, then the function $F_{X,I} : \mathbb{R} \to [0,1]$ with

$$F_{X,I}(x) = \left(\frac{1}{\mathbb{E}[X]} \int_0^x (1 - F(y))\, dy \right) \cdot \mathbb{1}_{(0,\infty)}(x)$$

is the *integrated tail distribution function* of X. The function $F_{X,I}$ is a distribution function of a probability measure on the positive half line.

(ii) Subexponential distributions provide a natural definition of being heavy-tailed. They formalize that the tail of the sum $S_n = X_1 + \cdots + X_n$ is essentially determined by the tail of the maximum $M_n = \max_{k=1,\ldots,n} X_k$ for independent copies of the distribution of X_1. A formal definition is

$$\forall n \geq 2 : \quad \lim_{x\to\infty} \frac{P(S_n > x)}{P(X_1 > x)} = n.$$

Theorem 5.5.6. *Assume that the NPC holds. In addition, suppose that the losses X_k have a density and that $F_{X_1,I}$, the integrated tail distribution function of X_1, is subexponential. Then,*

$$\lim_{w\to\infty} \frac{\psi^w(\pi)}{1 - F_{X_1,I}(w)} = \frac{1}{\rho} = \frac{r}{\pi - r},$$

i.e., $\psi^w(\pi) \sim \frac{1-F_{X_1,I}(w)}{\rho}$ as $w \to \infty$.

Example 5.5.7. Examples of parametric distributions that satisfy the conditions of this theorem can be found in Table 3.2.19 in Mikosch (2009). These include the log-normal and the Pareto distribution.

5.5.2 | The Cramér-Lundberg Model with Taxes

We extend the model and add taxes.

Model 5.5.8 (Cramér-Lundberg Model with Taxes)**.** Let $\tau \in [0,1]$ be a constant tax rate that is charged on the gross premium income $\pi \in \mathbb{R}$. We denote taxed premiums by $\bar{\pi} = (1+\tau)\cdot\pi$. We assume that the *realized tax* charged on the insurer's input costs $L \in \mathbb{R}$ is given by the tax rate $\varepsilon\tau$ which we represent as a fraction $\varepsilon \in [0,1]$ of the premium tax rate $\tau \in [0,1]$. The term *realized tax* refers to the tax on inputs minus deductions that are allowed. The costs L do not include insurance payments due to losses. The *after-tax risk process* is

$$R_t^{w,\tau,\varepsilon} = w + \left[\frac{\bar{\pi}}{1+\tau} - (1+\varepsilon\tau)L \right] t - \sum_{k=1}^{N_t} X_k. \tag{5.5}$$

The effective insurance premium (after subtracting all expenses and taxes) is

$$\pi^{\tau,\varepsilon} := \frac{\bar{\pi}}{1+\tau} - (1+\varepsilon\tau)L. \tag{5.6}$$

The model allows to mimic different tax systems.

(i) For $\tau = \tau_{PT} = \tau_{VAT}$ and $\varepsilon = 1$, we obtain a tax system with premium tax in which VAT is applied to inputs but cannot be deducted from the premium tax payments. This captures the current German tax system, if we choose $\tau = 19\%$.

(ii) For $\tau = \tau_{VAT}$ and $\varepsilon = 0$, we obtain a counterfactual tax system in which premium tax is replaced by VAT. In this case, VAT paid on inputs is fully deductable from VAT paid on insurance premiums.

(iii) Suppose that we are given a tax system with premium tax and VAT as in (i), i.e., $\tau = \tau_{PT} = \tau_{VAT}$ and $\varepsilon = 1$. As in Example 5.1.2, we consider a counterfactual tax system (ii) in which premium tax is replaced by VAT. We assume that VAT paid on value added of the insurance firm in the new tax system is equal to premium tax revenues in the original tax system. Moreover, we hold π constant. We denote the modified quantities with a tilde. In this case, $\tilde{\tau} = \widetilde{\tau_{VAT}} = \tau_{PT} \cdot \frac{\pi}{\widetilde{W}}$ and $\tilde{\varepsilon} = 0$ where $\widetilde{W}$ is the value that the insurance company adds to its inputs by producing the insurance contract, see Section 5.1. The modified taxed premium rate is given by $\bar{\pi} = (1+\tilde{\tau}) \cdot \pi > (1+\tau_{PT}) \cdot \pi$. We implicitly assumed that the higher tax on the premium is paid by the policyholder. The financial situation of the insurance company is thus improved in this case, since it can take advantage of tax deductions.

(iv) Suppose that we are in a counterfactual tax system (ii). We can extend the arguments in (iii) to construct a corresponding tax system with premium tax. We assume that insurance tax revenues in the new tax system are equal to VAT paid on value added of the insurance firm in tax system (ii). Moreover, we hold π constant. In this case, the premium tax rate needs to be adjusted, i.e., $\tilde{\tau} = \widetilde{\tau_{PT}} = \tau_{VAT} \cdot \frac{\widetilde{W}}{\pi}$ and $\tilde{\varepsilon} = 1$. The modified taxed premium rate is given by $\bar{\pi} = (1+\tilde{\tau}) \cdot \pi < (1+\tau_{VAT}) \cdot \pi$. This describes the opposite scenario to the situation in (iii). The modification of the tax system leads to a higher tax burden of the insurance company, since tax deductions are no longer possible. The benefits are in this case transferred to the policyholders.

The risk process $R^{w,\tau,\varepsilon}$ defined in Eq. (5.5) is a function of the tax parameters τ and ε. We investigate how ruin probabilities depend on the tax system. In contrast to the examples (iii) and (iv) above, we now keep $\bar{\pi}$ fixed instead of π. Tax expenses and tax savings are not transferred to the buyer of insurance, but are fully absorbed by the insurance firm. This implies, in particular, that the design of the tax system and a modified tax rate on premiums alter the financial resources and ruin probability of the insurance company.

The effective insurance premium is computed according to Eq. (5.6). For both the light-tailed and the heavy-tailed case, we consider the dependence of the ratio $\psi^w(\pi^{\tau,\varepsilon})/\psi^w(\pi^{0,0})$ on the tax rate τ and compare the cases $\varepsilon = 1$ and $\varepsilon = 0$ that correspond to a system with premium tax and VAT, respectively.

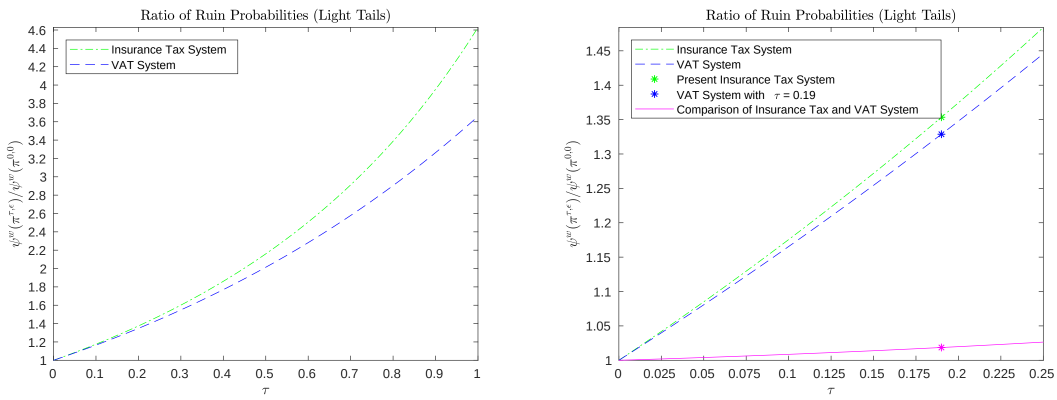
Example 5.5.9. First, we consider light-tailed loss distributions. If the conditions of Theorem 5.5.3 hold, then clearly $\frac{\psi^w(\pi^{\tau,\varepsilon})}{\psi^w(\pi^{0,0})} \sim \frac{C(\pi^{\tau,\varepsilon})e^{-l(\pi^{\tau,\varepsilon})w}}{C(\pi^{0,0})e^{-l(\pi^{0,0})w}}$ with $C(\pi) = \frac{\pi - r}{\vartheta \hat{B}'(l(\pi)) - \pi}$ and Cramér-Lundberg coefficient $l(\pi)$ for any effective insurance premium π. In the case of exponential losses, the approximation equals the exact ruin probability. Let $X_1, X_2, \ldots$ be exponentially distributed with parameter $\iota > 0$. Then,

$$\frac{\psi^w(\pi^{\tau,\varepsilon})}{\psi^w(\pi^{0,0})} = \frac{\pi^{0,0}}{\pi^{\tau,\varepsilon}} e^{\vartheta w \left(\frac{1}{\pi^{\tau,\varepsilon}} - \frac{1}{\pi^{0,0}}\right)} \tag{5.7}$$

In the numerical example, we choose $\iota = 1$ and $\vartheta = 1$, thus $r = 1$. Since the NPC should be satisfied, we assume $\pi^{0,0} = 2$. We set[9] $w = 1 \approx 0.87 = 43.5\% \cdot \pi^{0,0}$ and choose $L = \alpha\pi^{0,0} = 2\alpha$. Observe that $\bar{\pi} = \pi^{0,0} + L$. Setting the mean rate of input $\alpha = 5.2\%$ as in Example 5.1.1, we obtain $L = 0.104$ and deduce from formula (5.6):

$$\pi^{\tau,\varepsilon} := \frac{2 + L}{1 + \tau} - (1 + \varepsilon\tau)L = \frac{2.104}{1 + \tau} - 0.104 \cdot (1 + \varepsilon\tau) \approx \frac{2.1}{1 + \tau} - \frac{1 + \varepsilon\tau}{10}$$

Plugging this result into Eq. (5.7), we compute the ratio of ruin probabilities as a function of τ. This is displayed in Figure 5.6.



(a) (b)

Figure 5.6: Ratio of ruin probabilites for exponentially distributed losses.

Clearly, the higher the tax rate τ the higher the probability of ruin compared to a system without taxes. As expected, the increase of ruin probabilities is stronger in a tax system with premium tax. A change of the tax system from type (i) to type (ii) would thus reduce the probability of ruin. We observe that $\psi^w(\pi^{0.19,1})/\psi^w(\pi^{0,0}) = 1.353$, $\psi^w(\pi^{0.19,0})/\psi^w(\pi^{0,0}) = 1.329$, thus $\psi^w(\pi^{0.19,1})/\psi^w(\pi^{0.19,0}) = 1.018$. In Germany, at the

[9] According to BaFin (2011-2015), Issue 2015, p. 158, Table 520, equity capital of non-life insurance firms in Germany in 2015 was 43.5% of gross premium income.

prevailing rate of 19%, the ruin probability in a tax system with premium tax is only about 2% larger than the ruin probability in a tax system with VAT.

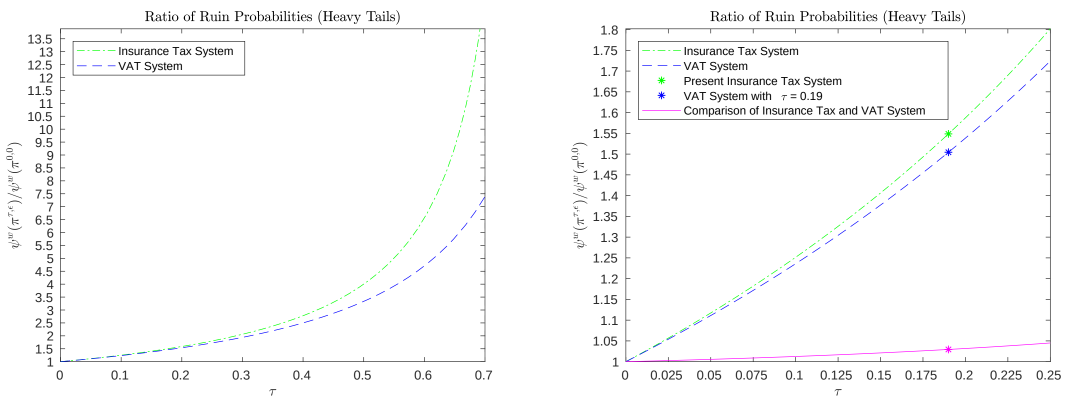
Example 5.5.10. Second, we consider the heavy-tailed loss distributions. As in the previous example, we assume that

$$\pi^{\tau,\varepsilon} \approx \frac{2.1}{1+\tau} - \frac{1+\varepsilon\tau}{10},$$

and choose $\vartheta = 1$. Let Z be log-normally distributed with parameters $(0,1)$, thus $\mathbb{E}[Z] = e^{1/2}$. We assume that the independent losses $X_1, X_2, \ldots$ have the same distribution as $e^{-1/2} \cdot Z$, i.e., X_k is log-normally distributed with parameters $\left(-\frac{1}{2}, 1\right)$. This implies $\mathbb{E}[X_1] = 1$, thus $r = 1$ as in the example of light-tailed losses. We compute:

$$\frac{\psi^w(\pi^{\tau,\varepsilon})}{\psi^w(\pi^{0,0})} \sim \frac{\pi^{0,0}-1}{\pi^{\tau,\varepsilon}-1} = \left(\frac{2.1}{1+\tau} - \frac{11+\varepsilon\tau}{10}\right)^{-1}$$

This is displayed in Figure 5.7.



(a) (b)

Figure 5.7: Ratio of ruin probabilites for heavy-tailed loss distributions.

In particular, we obtain $\psi^w(\pi^{0.19,1})/\psi^w(\pi^{0,0}) = 1.549$, $\psi^w(\pi^{0.19,0})/\psi^w(\pi^{0,0}) = 1.504$, thus $\psi^w(\pi^{0.19,1})/\psi^w(\pi^{0.19,0}) = 1.03$. In Germany, at the prevailing rate of 19%, the ruin probability in a tax system with premium tax is only about 3% larger than the ruin probability in a tax system with VAT. Let us finally stress that the results do not depend on the loss distributions being log-normal; only the condition $\mathbb{E}[X_1] = 1$ was used in the derivation.

The quantities derived in Examples 5.5.9 & 5.5.10 and displayed in Figures 5.6 & 5.7 are all ratios of ruin probabilities over an infinite time horizon. In absolute terms, annual ruin probabilities of real insurance companies are limited by regulatory standards. Solvency II, for example, restricts annual ruin probabilities to at most 0.5%. Otherwise, companies face serious interventions of the regulator. Our analysis thus indicates that absolute changes of annual ruin probabilities due to a modified tax system would be extremely moderate – on the order of less than 10 basis points. However, even small absolute changes in ruin probabilities might be costly in terms of solvency capital, if regulatory constraints are tight and binding. This issue is discussed in the next section.

Remark 5.5.11. Our focus is on premium tax and VAT, and we deduced the implications of tax systems on ruin probabilities from standard results in the literature on ruin theory. The key assumption was that Eq. (5.6) describes the tax impact on the risk process. In the context of premium tax and VAT, Eq. (5.6) is a reasonable hypothesis. However, the functional dependence of risk processes on other types of taxes might be more complicated than assumed in this chapter. We briefly summarize some previous key contributions. Albrecher & Hipp (2007) analyze the effect of tax payments under a loss-carry forward system in the Cramér-Lundberg model. They suppose that taxes are only paid when the company is in a profitable situation, meaning that the risk process is at its running maximum. The authors study ruin probabilities with and without taxes in their model and find that the survival probability with tax is a power of the survival probability without tax, i.e., $1 - \psi_\gamma(w) = (1 - \psi_0(w))^{\frac{1}{1-\gamma}}$, where $0 < \gamma < 1$ is the constant tax rate. Moreover, they compute the optimal surplus level at which taxation should start in order to maximize the expected discounted tax payments before ruin. Albrecher, Badescu & Landriault (2008) conduct a similar analysis in the dual risk model. The description of the ruin probability with taxes in terms of the ruin probability without taxes becomes more complicated. A considerable generalization of the results is derived by Albrecher, Renaud & Zhou (2008) who embed the model by Albrecher & Hipp (2007) into a general Lévy framework. The relation between ruin probabilities is recovered, and also the structure of many other results is preserved in the Lévy setup. Kyprianou & Zhou (2009) and Albrecher, Borst, Boxma & Resing (2009) introduce a surplus-dependent tax rate.

5.6 | Impact on Solvency Capital Requirement

As explained in the previous sections, premium tax generates more tax revenues than VAT, if both tax rates are equal. We now compare these two alternative tax systems from the point of view of solvency capital requirements. For this purpose, we assume that insurance firms keep their taxed insurance premiums constant, but retain the tax savings that accrue when premium tax is replaced by VAT. Obviously, the solvency capital requirement is then decreased by this amount, and insurance companies can distribute all tax savings to their shareholders. If this occurs, risk will be back at its original level. In the current section, we review the notion of solvency capital requirements in the context of internal models[10] and explain in detail why the dividend payments to shareholders may be increased.

To this end, we review the basic definition of distribution-based monetary risk mea-

[10] Another approach is the standard formula of Solvency II, a modular construction for the computation of the solvency capital requirement, see Appendx A, Corollary A.0.8. While an internal model attempts to describe and evaluate the stochastic evolution of the balance sheet of the insurance firm, the standard formula is an auxiliary construction that facilitates the computation of a solvency capital requirement. Aggregation of risk modules is based on correlations and a square-root formula. It is well-known and easily demonstrated that the modular construction cannot be interpreted as an approximation of a capital requirement that limits the probability of ruin to less than 0.5% as requested by Directive 2009/138/EC, see, e.g., Pfeifer (2016). In this chapter, we focus exclusively on internal models.

sures. These include all risk measures that are typically used in practice. For a short introduction to monetary risk measures, we refer to Appendix A; for a detailed exposition on the subject, see Artzner et al. (1999), Föllmer & Schied (2011), and Föllmer & Weber (2015).

Definition 5.6.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{X}$ a vector space of random variables on Ω that contains the constants. We identify random variables that are P-almost surely equal. A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is called a *monetary risk measure* on $\mathcal{X}$, if $\rho(X) = \rho(Y)$ for $X = Y$ P-almost surely and if ρ satisfies the following properties:

(i) *Monotonicity*: If $X \geq Y$ P-almost surely, then $\rho(X) \leq \rho(Y)$.
(Better payoff profiles are less risky.)

(ii) *Cash-invariance:* If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.
(Adding a fixed amount m to the risky position decreases the risk exactly by this amount.)

The risk measure ρ is called *distribution-based*, if $\rho(X) = \rho(Y)$ whenever X and Y have the same distribution under P.

Example 5.6.2. Examples of distribution-based monetary risk measures are value at risk (V@R) and average value at risk (AV@R), also called expected shortfall, conditional value at risk, tail value at risk, or worst conditional expectation. V@R and AV@R are the basis of the definition of solvency capital requirements in Solvency II and in the Swiss Solvency Test, respectively.

(i) Value at risk at level $y \in (0, 1)$ is defined as a quantile:

$$\mathrm{V@R}_y(X) := \inf\{m \in \mathbb{R} \,|\, P(X + m < 0) \leq y\}.$$

It is equal to the smallest monetary amount m that needs to be added to the financial position X such that the probability of a loss does not exceed the level y.

(ii) Average value at risk at level $y \in (0, 1)$ is the average of the V@Rs below y, i.e.,

$$\mathrm{AV@R}_y(X) := \frac{1}{y} \int_0^y \mathrm{V@R}_c(X)\, dc.$$

Under technical conditions, e.g., if X has a continuous distribution, it is equal to the conditional expectation of a loss beyond the $\mathrm{V@R}_y(X)$.

We will now explain – in a stylized way – how solvency capital requirements are defined in an internal model; see Chapters 3 & 4 for further details. The evolution of assets, liabilities and capital of an insurance firm can be captured by solvency balance sheets at time horizons that are specified by regulators. The time horizon of Solvency II and the Swiss Solvency Test is one year. Table 5.5 displays the balance sheet of a company at time $t = 0$ and $t = 1$.

$t = 0$

Assets	Liabilities
A_0	$E_0 = A_0 - L_0$
	L_0

$t = 1$

Assets	Liabilities
A_1	$E_1 = A_1 - L_1$
	L_1

Table 5.5: Balance sheet of an insurance company for different points in time.

The assets are denoted by A_t, the liabilities by L_t, $t = 0, 1$. The quantities at time $t = 0$ are known, the quantities at time $t = 1$ are random variables. The difference between assets and liabilities $E_t = A_t - L_t$, $t = 0, 1$, is the net asset value (NAV) of the firm. We set $X = E_1 - E_0$ for the change of the NAV over the considered time horizon.

The solvency capital requirement (SCR) for Solvency II is defined in the Directive 2009/138/EC of the European Parliament and of the Council on the taking-up and pursuit of the business of Insurance and Reinsurance – Solvency II (see European Commission (2009)):

> The Solvency Capital Requirement should be determined as the economic capital to be held by insurance and reinsurance undertakings in order to ensure that ruin occurs no more often than once in every 200 cases or, alternatively, that those undertakings will still be in a position, with a probability of at least 99.5 %, to meet their obligations to policy holders and beneficiaries over the following 12 months. That economic capital should be calculated on the basis of the true risk profile of those undertakings, taking account of the impact of possible risk-mitigation techniques, as well as diversification effects.

This definition is specified in terms of condition on the acceptability of E_1. An equivalent formulation[11] provides the definition of the SCR under Solvency II:

$$P(E_1 < 0) \leq y \quad \Leftrightarrow \quad \mathrm{V@R}_y(E_1) \leq 0 \quad \Leftrightarrow \quad \mathrm{V@R}_y(E_1 - E_0) \leq E_0 \quad \Leftrightarrow \quad \mathrm{V@R}_y(X) \leq E_0.$$

Setting $\mathrm{SCR} := \mathrm{V@R}_y(X)$, the solvency condition of the company becomes

$$\mathrm{SCR} \leq E_0.$$

An analogous argument holds, if V@R is replaced by any other risk measure ρ.[12] The acceptance set of ρ is the family of positions with non-positive risk, i.e.,

$$\mathcal{A}_\rho = \{X \in \mathcal{X} : \ \rho(X) \leq 0\}.$$

[11] For simplicity, we assume in this chapter that interest rates over the one-year horizon are approximately zero. For adjustments on the definition of the SCR if interest rates are non-zero, see Christiansen & Niemeyer (2014).

[12] V@R has been criticized in the context of capital regulation, since it neglects losses beyond the V@R and – due to its lack of coherence – it might mislead investment decisions and asset-liability management. In addition, in corporate networks it is possible "to sweep the downside risk under the carpet", see Chapter 4 and Weber (2018).

If we assume again for simplicity that interest rates over a one-year horizon are zero, setting SCR := $\rho(X)$, we obtain the following solvency condition:

$$E_1 \in \mathcal{A}_\rho \quad \Leftrightarrow \quad \rho(E_1) \leq 0 \quad \Leftrightarrow \quad \text{SCR} \leq E_0.$$

The Swiss Solvency Test chooses AV@R as the basis for the definition of solvency.

Let us now return to the original question regarding the impact of the tax system on solvency capital. In Eq. (5.1), we computed the tax savings that would accrue if a deduction of VAT paid on inputs was permitted. We assume that these savings of $\bar{L} - L = \tau_{VAT}\alpha\pi$ are retained by the insurance company. While initial capital E_0 remains unchanged, capital E_1 at the solvency time horizon is increased by this amount. This leads to a reduction of the SCR.

Lemma 5.6.3. *We denote by SCR the solvency capital requirement in the original tax system with premium tax. Assume that the tax system is modified such that a deduction of VAT paid on inputs is permitted. In this case, the solvency capital requirement is reduced to*

$$SCR - \tau_{VAT}\alpha\pi.$$

Proof. The proof is given in Section 5.8. □

Considering the situation in Example 5.1.1, the reduction of the solvency capital by $\tau_{VAT}\alpha\pi$ would amount to 0.99% of gross premium income. This is due to decreased government revenues. The company could increase the dividend payments to its shareholders by this amount. If this is done, the NAV at time 1 will return to its original level E_1. The solvency situation of the insurance company, i.e., $E_1 \in \mathcal{A}_\rho$, will then be the same as before. Conversely, if the NAV of an insurance firm at time 0, i.e., E_0, is close to the SCR in a tax system with VAT, the firm would need a capital injection of 0.99% of gross premium income from its shareholders to satisfy the same solvency capital constraint in a tax system with premium tax.

5.7 | Conclusion

We analyzed the impact of premium tax on total tax revenues, insurance demand, the competitiveness of corporate buyers of insurance, the ruin probability of insurance firms and their solvency capital requirement. We find that the competitiveness of corporate buyers of insurance, the ruin probability of insurance firms and their solvency capital are hardly affected. In contrast, the tax system (i.e., premium tax vs. VAT) has a significant influence on the cost of insurance, insurance demand, government revenues and the profitability of insurance firms. The increased cost of insurance in tax systems with premium tax in contrast to VAT might promote alternative risk transfer mechanisms such as off-shore captive insurance, derivatives, or preventative measures that are not subject to premium tax. These instruments might provide more cost-efficient solutions to the risk management needs of corporations. On the one hand, some tax-efficient products might offer new business opportunities for insurance firms. On the other hand, alternative risk transfer mechanisms might also cannibalize their traditional business. The design of such instruments and their implications for corporate risk management, insurance companies and government revenues are interesting topics for further research.

5.8 | Appendix: Proofs

In this section, we provide the proofs of the results presented in this chapter.

Proof of Lemma 5.1.4.

Proof. The savings are given by Eq. (5.1) plus the input tax reduction of the company for insurance products, i.e., $E = \tau_{VAT}\ \alpha V + \tau_{PT}\ V$. □

Proof of Lemma 5.2.1.

Proof. The result follows immediately from the condition $\widetilde{\tau_{VAT}}\widetilde{W} \ = \ S$. □

Proof of Theorem 5.3.4.

Proof. Here, we provide a condensed proof of the theorem. Detailed computations are given in Appendix D, Section D.2.1.

(i) CARA-utility:
We first consider the case $X \sim \text{Ber}(\hat{x}, p)$. We compute

$$\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right] \ = \ 1 - e^{-\kappa(w-\nu\pi)}\left(e^{\kappa(1-\nu)\hat{x}}p + 1 - p\right).$$

This implies

$$\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right] \ = \ \kappa e^{\kappa(\pi\nu-\hat{x}\nu+\hat{x}-w)}\left(\pi(p-1)e^{\kappa\hat{x}\nu-\kappa\hat{x}} + p(\hat{x}-\pi)\right).$$

At the boundary $\nu = 0$, we obtain

$$\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right]_{|\nu=0} \ = \ \kappa e^{\kappa(\hat{x}-w)}\left(\pi(p-1)e^{-\kappa\hat{x}} + p(\hat{x}-\pi)\right).$$

Thus, $\nu(\pi) = 0$ is the optimal solution, if and only if

$$\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right]_{|\nu=0} \ \leq \ 0 \quad \Longleftrightarrow \quad \pi \ \geq \ \frac{p\hat{x}e^{\kappa\hat{x}}}{1-p+pe^{\kappa\hat{x}}}$$

At the boundary $\nu = 1$, we obtain $\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right]_{|\nu=1} = \kappa e^{\kappa(\pi-w)}\left(p\hat{x}-\pi\right)$. Thus, the optimal solution is $\nu(\pi) = 1$, iff $\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right]_{|\nu=1} \geq 0$, i.e., $\pi \leq p\hat{x}$. In all other cases, we need to solve $\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right] = 0$, leading to the stated solution. The first-order conditions are sufficient due to the strict concavity.
Second, we derive the optimal contract for $X \sim \Gamma(\xi, \mu)$. In this case,

$$\begin{aligned}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right] \ &= \ 1 - e^{-\kappa(w-\nu\pi)}\ \left(\frac{\mu}{\mu-\kappa(1-\nu)}\right)^{\xi},\\ \frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right] \ &= \ e^{-\kappa(w-\nu\pi)}\left(\frac{\mu}{\mu-\kappa(1-\nu)}\right)^{\xi+1}\left(-\frac{\kappa}{\mu}\right)\left(\pi(\mu-\kappa(1-\nu))-\xi\right).\end{aligned}$$

The solution can now be derived by analogous arguments as before.

(ii) HARA-utility: Using the same steps as above, the solution is computed, observing

$$\begin{aligned}\mathbb{E}\left[u_2^\lambda\left(X_\nu\right)\right] &= \frac{1}{\lambda}\cdot\Big(((1-\nu)(w-\hat{x})+\nu(w-\pi))^\lambda\cdot p \\ &\qquad +((1-\nu)w+\nu(w-\pi))^\lambda\cdot(1-p)\Big),\\ \frac{\partial}{\partial\nu}\mathbb{E}\left[u_2^\lambda\left(X_\nu\right)\right] &= (w-\nu\pi+\hat{x}(\nu-1))^{\lambda-1}p(\hat{x}-\pi)+(-\pi)(1-p)(w-\nu\pi)^{\lambda-1}.\end{aligned}$$

(iii) Logarithmic utility: Again, the solution is derived by analogous arguments, noting

$$\begin{aligned}\mathbb{E}\left[u_2^0\left(X_\nu\right)\right] &= \log((1-\nu)(w-\hat{x})+\nu(w-\pi))\cdot p \\ &\qquad +\log((1-\nu)w+\nu(w-\pi))\cdot(1-p),\\ \frac{\partial}{\partial\nu}\mathbb{E}\left[u_2^0\left(X_\nu\right)\right] &= \frac{1}{w-\nu\pi+\hat{x}\nu-\hat{x}}p(\hat{x}-\pi)+(-\pi)(1-p)\frac{1}{w-\nu\pi}\end{aligned}$$

□

Proof of Lemma 5.3.6.

Proof. The result follows from $\bar{\pi}_{VAT}=\bar{\pi}_{PT}-E$ where tax savings E are computed according to Eq. (5.3) with $\bar{\pi}_{PT}=(1+\tau_{PT})\beta U$. Hence,

$$\begin{aligned}\bar{\pi}_{VAT}&=\bar{\pi}_{PT}-E=(1+\tau_{PT})\beta U-(\tau_{VAT}\alpha+\tau_{PT})\beta U=(1+\tau_{PT})\beta U\left(1-\frac{\tau_{VAT}\alpha+\tau_{PT}}{1+\tau_{PT}}\right)\\ &=\bar{\pi}_{PT}\left(1-\frac{\tau_{VAT}\alpha+\tau_{PT}}{1+\tau_{PT}}\right).\end{aligned}$$

□

Proof of Lemma 5.4.2.

Proof. We have $\theta_{\alpha,\beta}=\frac{E}{U}$ where U is the revenue or business volume of the company. Now, the result follows from Eq. (5.3). □

Proof of Lemma 5.4.3.

Proof. This is an application of Lemma 5.4.2 to the solution of the optimization problem, thus

$$\frac{\Delta x_1^{(a)}}{x_1^{(a)}}=\frac{\frac{aw}{\tilde{p}_1}-\frac{aw}{p_1}}{\frac{aw}{p_1}}=\frac{\frac{aw}{(1-\theta_{\alpha,\beta})p_1}-\frac{aw(1-\theta_{\alpha,\beta})}{(1-\theta_{\alpha,\beta})p_1}}{\frac{aw(1-\theta_{\alpha,\beta})}{(1-\theta_{\alpha,\beta})p_1}}=\frac{1}{1-\theta_{\alpha,\beta}}-1=\frac{\theta_{\alpha,\beta}}{1-\theta_{\alpha,\beta}}$$

□

Proof of Lemma 5.4.6.

Proof. This is an application of Lemma 5.4.2 to the solution of the optimization problem, thus

$$\frac{\Delta x_1^{a,b}}{x_1^{a,b}} = \frac{\frac{w(\tilde{p}_1/a)^{-\eta}}{a^\eta \tilde{p}_1^{1-\eta}+(1-a)^\eta p_2^{1-\eta}} - \frac{w(p_1/a)^{-\eta}}{a^\eta p_1^{1-\eta}+(1-a)^\eta p_2^{1-\eta}}}{\frac{w(p_1/a)^{-\eta}}{a^\eta p_1^{1-\eta}+(1-a)^\eta p_2^{1-\eta}}}$$

$$= \frac{\left(\frac{(1-\theta_{\alpha,\beta})p_1}{a}\right)^{-\eta} \cdot \left(a^\eta p_1^{1-\eta} + (1-a)^\eta p_2^{1-\eta}\right)}{\left(\frac{p_1}{a}\right)^{-\eta} \cdot \left(a^\eta((1-\theta_{\alpha,\beta})p_1)^{1-\eta} + (1-a)^\eta p_2^{1-\eta}\right)} - 1$$

$$= (1-\theta_{\alpha,\beta})^{-\eta} \cdot \frac{a^\eta p_1^{1-\eta} + (1-a)^\eta p_2^{1-\eta}}{a^\eta (1-\theta_{\alpha,\beta})^{1-\eta} p_1^{1-\eta} + (1-a)^\eta p_2^{1-\eta}} - 1$$

□

Proof of Lemma 5.6.3.

Proof. Adjusted quantities are labeled with a tilde. The random economic capital at time $t = 1$ becomes $\widetilde{E}_1 = E_1 + \tau_{VAT}\alpha\pi$. Due to cash-invariance of ρ, we compute

$$\widetilde{\mathrm{SCR}} = \rho(X + \tau_{VAT}\alpha\pi) = \rho(X) - \tau_{VAT}\alpha\pi = \mathrm{SCR} - \tau_{VAT}\alpha\pi.$$

□

5.9 | Appendix: Computations of Section 5.1

Rate of Input

	2015	2014	2013	2012	2011
Gross premiums earned π	75,008,740	71,216,091	69,298,052	66,922,556	63,514,681
Capital income	7,431,575	7,246,143	7,207,242	7,451,052	6,988,566
Investment expenses	1,269,941	923,197	1,010,400	1,098,403	1,740,899
Total losses	56,243,800	52,078,719	55,722,781	50,255,508	48,929,622
Acquisition costs & adminis. expenses	18,921,252	18,083,843	17,594,251	17,113,492	16,486,877
Taxes	1,462,200	1,479,200	963,100	1,483,700	1,147,200
Input L	4,543,122	5,897,275	1,214,762	4,422,505	2,198,649

Table 5.6: Computation of input costs in thousands of euros (TEUR).
These values are given in Table 540 of the corresponding annual report of BaFin (2011-2015). (Negative) Taxes are disclosed in Table 79 of the same reports.

	2015	2014	2013	2012	2011	Mean
Gross premiums earned	100.00	100.00	100.00	100.00	100.00	
Capital income	9.91	10.17	10.40	11.13	11.00	
Investment expenses	1.69	1.30	1.46	1.64	2.74	
Total losses	75.00	73.10	80.40	75.10	77.00	
Acquisition costs & administrative expenses	25.20	25.40	25.40	25.60	26.00	
Taxes	1.95	2.08	1.39	2.22	1.81	
Rate of Input α	6.07	8.30	1.75	6.58	3.46	5.23

Table 5.7: Computation of rate of input α as ratio of earned gross premium π.
Total losses as well as acquisition costs and administrative expenses can be adopted from Table 540 in the corresponding annual report of BaFin (2011-2015). Other quantities are calculated using Table 5.6.

Value Added

	2015	2014	2013	2012	2011
Acquisition costs & adminis. expenses	18,921,252	18,083,843	17,594,251	17,113,492	16,486,877
Profits before taxes	2,548,300	2,587,900	2,135,800	2,472,600	1,986,700
Changes in equilization provisions	295,400	684,500	-180,700	858,400	-368,700
Gross technical result	4,859,078	5,076,151	-165,970	3,442,647	1,812,497
Net technical result	2,931,694	2,908,918	288,506	1,573,496	377,868
Value added $\widetilde{W}$	23,692,336	23,523,476	19,094,875	22,313,643	19,539,506

Table 5.8: Computation of value added in TEUR.
Profits before taxes and (negative) changes in equilization provisions are given in Table 79 of the corresponding annual report of BaFin (2011-2015). Gross and net technical results can be found in Table 540 of the same reports.

	2015	2014	2013	2012	2011	Mean
Acquisition costs & administrative expenses	25.20	25.40	25.40	25.60	26.00	
Profits before taxes	3.40	3.63	3.08	3.69	3.13	
Changes in equilization provisions	0.39	0.96	-0.26	1.28	-0.58	
Gross technical result	6.50	7.10	-0.20	5.10	2.90	
Net technical result	3.91	4.08	0.42	2.35	0.59	
Value added $\frac{\widetilde{W}}{\pi}$	31.58	33.01	27.60	33.33	30.85	31.28

Table 5.9: Computation of value added as ratio of earned gross premium π.
Acquisition costs and administrative expenses as well as gross technical result can be adopted from Table 540 in the corresponding annual report of BaFin (2011-2015). Other quantities are calculated using Table 5.8 and π in Table 5.6.

Value added for different lines of insurance

Accident

	2015	2014	2013	2012	2011
Total gross premium earned $\underline{\pi}$	69,448,394	66,146,203	64,535,515	62,102,602	59,310,517
Gross premium earned accident $\underline{\pi}^*$	6,388,854	6,440,961	6,416,895	6,500,627	6,383,714
Net premium earned accident $\hat{\underline{\pi}}^*$	5,487,886	5,545,237	5,707,299	5,638,822	5,587,643
Total profits before taxes	2,548,300	2,587,900	2,135,800	2,472,600	1,986,700
Total changes in equilization provisions	295,400	684,500	-180,700	858,400	-368,700
Net technical result accident	24.70	16.60	17.10	17.20	17.30

Table 5.10: Needed data for computation of value added accident.
Total gross premium earned (direct business), gross premium earned accident (direct business), net premium earned accident (direct business) and net technical result accident are given in Table 541 of the corresponding annual report of BaFin (2011-2015). As above, total profits before taxes and total (negative) changes in equilization provisions can be adopted from Table 79 in the same reports. Positions 1 to 5 are specified in TEUR, position 6 as ratio of $\hat{\underline{\pi}}^*$.

	2015	2014	2013	2012	2011
Acquisition costs and administrative expenses	1,993,322	2,003,139	2,002,071	2,054,198	2,055,556
Profits before taxes	234,429	251,996	212,367	258,821	213,833
Changes in equilization provisions	27,175	66,653	-17,967	89,854	-39,684
Gross technical result	1,252,215	1,210,901	1,270,545	1,280,624	1,238,441
Net technical result	1,355,508	920,509	975,948	969,877	966,662
Value added accident	2,151,634	2,612,179	2,491,068	2,713,619	2,501,483

Table 5.11: Computation of value added accident in TEUR.
All quantities except net technical result are computed using Table 5.12 and $\underline{\pi}^*$ given in Table 5.10. Net technical result is calculated by multiplying the corresponding quantities in Table 5.10.

	2015	2014	2013	2012	2011	Mean
Acquisition costs and administrative expenses	31.20	31.10	31.20	31.60	32.20	
Profits before taxes	3.67	3.91	3.31	3.98	3.35	
Changes in equilization provisions	0.43	1.03	-0.28	1.38	-0.62	
Gross technical result	19.60	18.80	19.80	19.70	19.40	
Net technical result	21.22	14.29	15.21	14.92	15.14	
Value added accident	33.68	40.56	38.82	41.74	39.19	38.80

Table 5.12: Computation of value added accident as ratio of earned gross premium $\underline{\pi}^*$.
Acquisition costs and administrative expenses as well as gross technical results are given in Table 541 of the corresponding annual report of BaFin (2011-2015). Profits before taxes and changes in equilization provisions are calculated by using total values in Table 5.10. Net technical results are based on the absolute values in Table 5.11 and $\underline{\pi}^*$ given in Table 5.10.

Public Liability (publicL)

	2015	2014	2013	2012	2011
Total gross premium earned $\underline{\pi}$	69,448,394	66,146,203	64,535,515	62,102,602	59,310,517
Gross premium earned publicL $\underline{\pi}^*$	9,246,435	8,837,457	8,360,776	8,023,858	7,706,079
Net premium earned publicL $\hat{\underline{\pi}}^*$	6,714,540	6,536,798	6,670,263	6,437,008	5,979,624
Total profits before taxes	2,548,300	2,587,900	2,135,800	2,472,600	1,986,700
Total changes in equilization provisions	295,400	684,500	-180,700	858,400	-368,700
Net technical result publicL	7.50	9.00	7.20	9.80	9.80

Table 5.13: Needed data for computation of value added public liability.
Total gross premium earned (direct business), gross premium earned public liability (direct business), net premium earned public liability (direct business) and net technical result public liability are given in Table 541 of the corresponding annual report of BaFin (2011-2015). As above, total profits before taxes and total (negative) changes in equilization provisions can be adopted from Table 79 in the same reports. Positions 1 to 5 are specified in TEUR, position 6 as ratio of $\hat{\underline{\pi}}^*$.

	2015	2014	2013	2012	2011
Acquisition costs and administrative expenses	2,810,916	2,695,424	2,575,119	2,527,515	2,450,533
Profits before taxes	339,283	345,756	276,700	319,468	258,127
Changes in equilization provisions	39,330	91,453	-23,410	110,908	-47,904
Gross technical result	674,990	821,884	627,058	866,577	608,780
Net technical result	503,591	588,312	480,259	630,827	586,003
Value added public liability	3,360,929	3,366,205	2,975,208	3,193,641	2,683,533

Table 5.14: Computation of value added public liability in TEUR.
All quantities except net technical result are computed using Table 5.15 and $\underline{\pi}^*$ given in Table 5.13. Net technical result is calculated by multiplying the corresponding quantities in Table 5.13.

	2015	2014	2013	2012	2011	Mean
Acquisition costs and administrative expenses	30.40	30.50	30.80	31.50	31.80	
Profits before taxes	3.67	3.91	3.31	3.98	3.35	
Changes in equilization provisions	0.43	1.03	-0.28	1.38	-0.62	
Gross technical result	7.30	9.30	7.50	10.80	7.90	
Net technical result	5.45	6.66	5.74	7.86	7.60	
Value added public liability	36.35	38.09	35.59	39.80	34.82	36.93

Table 5.15: Computation of value added public liability as ratio of earned gross premium $\underline{\pi}^*$.
Acquisition costs and administrative expenses as well as gross technical results are given in Table 541 of the corresponding annual report of BaFin (2011-2015). Profits before taxes and changes in equilization provisions are calculated by using total values in Table 5.13. Net technical results are based on the absolute values in Table 5.14 and $\underline{\pi}^*$ given in Table 5.13.

Car Total

	2015	2014	2013	2012	2011
Total gross premium earned $\underline{\pi}$	69,448,394	66,146,203	64,535,515	62,102,602	59,310,517
Gross premium earned car $\underline{\pi}^*$	24,601,179	23,637,844	22,503,977	21,234,566	20,113,638
Net premium earned car $\hat{\underline{\pi}}^*$	19,146,675	18,312,796	18,352,347	17,317,716	16,402,766
Total profits before taxes	2,548,300	2,587,900	2,135,800	2,472,600	1,986,700
Total changes in equilization provisions	295,400	684,500	-180,700	858,400	-368,700
Net technical result car total	2.00	3.70	-2.80	-3.30	-8.10

Table 5.16: Needed data for computation of value added car total.
Total gross premium earned (direct business), gross premium earned car total (direct business), net premium earned car total (direct business) and net technical result car total are given in Table 541 of the corresponding annual report of BaFin (2011-2015). As above, total profits before taxes and total (negative) changes in equilization provisions can be adopted from Table 79 in the same reports. Positions 1 to 5 are specified in TEUR, position 6 as ratio of $\hat{\underline{\pi}}^*$.

	2015	2014	2013	2012	2011
Acquisition costs and administrative expenses	4,206,802	4,089,347	3,960,700	3,822,222	3,640,568
Profits before taxes	902,702	924,806	744,768	845,449	673,738
Changes in equilization provisions	104,642	244,611	-63,011	293,510	-125,035
Gross technical result	615,029	827,325	-1,012,679	-509,630	-1,528,636
Net technical result	382,934	677,573	-513,866	-571,485	-1,328,624
Value added car total	5,446,241	5,408,515	4,143,643	5,023,036	3,989,259

Table 5.17: Computation of value added car total in TEUR.
All quantities except net technical result are computed using Table 5.18 and $\underline{\pi}^*$ given in Table 5.16. Net technical result is calculated by multiplying the corresponding quantities in Table 5.16.

	2015	2014	2013	2012	2011	Mean
Acquisition costs and administrative expenses	17.10	17.30	17.60	18.00	18.10	
Profits before taxes	3.67	3.91	3.31	3.98	3.35	
Changes in equilization provisions	0.43	1.03	-0.28	1.38	-0.62	
Gross technical result	2.50	3.50	-4.50	-2.40	-7.60	
Net technical result	1.56	2.87	-2.28	-2.69	-6.61	
Value added car total	22.14	22.88	18.41	23.65	19.83	21.38

Table 5.18: Computation of value added car total as ratio of earned gross premium $\underline{\pi}^*$.
Acquisition costs and administrative expenses as well as gross technical results are given in Table 541 of the corresponding annual report of BaFin (2011-2015). Profits before taxes and changes in equilization provisions are calculated by using total values in Table 5.16. Net technical results are based on the absolute values in Table 5.17 and $\underline{\pi}^*$ given in Table 5.16.

Defense

	2015	2014	2013	2012	2011
Total gross premium earned $\underline{\pi}$	69,448,394	66,146,203	64,535,515	62,102,602	59,310,517
Gross premium earned defense $\underline{\pi}^*$	3,949,994	3,824,287	3,756,450	3,695,395	3,401,014
Net premium earned defense $\hat{\underline{\pi}}^*$	3,440,597	3,317,429	3,367,084	3,306,620	3,048,240
Total profits before taxes	2,548,300	2,587,900	2,135,800	2,472,600	1,986,700
Total changes in equilization provisions	295,400	684,500	-180,700	858,400	-368,700
Net technical result defense	0.50	-0.40	0.50	3.50	3.40

Table 5.19: Needed data for computation of value added defense.
Total gross premium earned (direct business), gross premium earned defense (direct business), net premium earned defense (direct business) and net technical result defense are given in Table 541 of the corresponding annual report of BaFin (2011-2015). As above, total profits before taxes and total (negative) changes in equilization provisions can be adopted from Table 79 in the same reports. Positions 1 to 5 are specified in TEUR, position 6 as ratio of $\hat{\underline{\pi}}^*$.

	2015	2014	2013	2012	2011
Acquisition costs and administrative expenses	1,319,298	1,269,663	1,228,359	1,249,044	1,091,725
Profits before taxes	144,939	149,621	124,320	147,131	113,922
Changes in equilization provisions	16,801	39,575	-10,518	51,079	-21,142
Gross technical result	31,600	-22,946	7,513	136,730	112,233
Net technical result	17,203	-13,270	16,835	115,732	103,640
Value added defense	1,495,435	1,449,183	1,332,838	1,468,251	1,193,099

Table 5.20: Computation of value added defense in TEUR.
All quantities except net technical result are computed using Table 5.21 and $\underline{\pi}^*$ given in Table 5.19. Net technical result is calculated by multiplying the corresponding quantities in Table 5.19.

	2015	2014	2013	2012	2011	Mean
Acquisition costs and administrative expenses	33.40	33.20	32.70	33.80	32.10	
Profits before taxes	3.67	3.91	3.31	3.98	3.35	
Changes in equilization provisions	0.43	1.03	-0.28	1.38	-0.62	
Gross technical result	0.80	-0.60	0.20	3.70	3.30	
Net technical result	0.44	-0.35	0.45	3.13	3.05	
Value added defense	37.86	37.89	35.48	39.73	35.08	37.21

Table 5.21: Computation of value added defense as ratio of earned gross premium $\underline{\pi}^*$.
Acquisition costs and administrative expenses as well as gross technical results are given in Table 541 of the corresponding annual report of BaFin (2011-2015). Profits before taxes and changes in equilization provisions are calculated by using total values in Table 5.19. Net technical results are based on the absolute values in Table 5.20 and $\underline{\pi}^*$ given in Table 5.19.

Fire

	2015	2014	2013	2012	2011
Total gross premium earned $\underline{\pi}$	69,448,394	66,146,203	64,535,515	62,102,602	59,310,517
Gross premium earned fire $\underline{\pi}^*$	2,150,739	1,888,463	1,840,158	1,736,250	1,763,792
Net premium earned fire $\hat{\underline{\pi}}^*$	1,131,264	1,091,608	1,050,167	1,048,086	1,064,826
Total profits before taxes	2,548,300	2,587,900	2,135,800	2,472,600	1,986,700
Total changes in equilization provisions	295,400	684,500	-180,700	858,400	-368,700
Net technical result fire	-13.30	-7.50	-4.20	-10.20	-7.50

Table 5.22: Needed data for computation of value added fire.
Total gross premium earned (direct business), gross premium earned fire (direct business), net premium earned fire (direct business) and net technical result fire are given in Table 541 of the corresponding annual report of BaFin (2011-2015). As above, total profits before taxes and total (negative) changes in equilization provisions can be adopted from Table 79 in the same reports. Positions 1 to 5 are specified in TEUR, position 6 as ratio of $\hat{\underline{\pi}}^*$.

	2015	2014	2013	2012	2011
Acquisition costs and administrative expenses	615,111	523,104	506,043	503,513	502,681
Profits before taxes	78,918	73,884	60,900	69,128	59,081
Changes in equilization provisions	9,148	19,542	-5,152	23,999	-10,964
Gross technical result	-204,320	-18,885	58,885	-83,340	-15,874
Net technical result	-150,458	-81,871	-44,107	-106,905	-79,862
Value added fire	649,315	679,517	664,783	620,205	614,785

Table 5.23: Computation of value added fire in TEUR.
All quantities except net technical result are computed using Table 5.24 and $\underline{\pi}^*$ given in Table 5.22. Net technical result is calculated by multiplying the corresponding quantities in Table 5.22.

	2015	2014	2013	2012	2011	Mean
Acquisition costs and administrative expenses	28.60	27.70	27.50	29.00	28.50	
Profits before taxes	3.67	3.91	3.31	3.98	3.35	
Changes in equilization provisions	0.43	1.03	-0.28	1.38	-0.62	
Gross technical result	-9.50	-1.00	3.20	-4.80	-0.90	
Net technical result	-7.00	-4.34	-2.40	-6.16	-4.53	
Value added fire	30.19	35.98	36.13	35.72	34.86	34.58

Table 5.24: Computation of value added fire as ratio of earned gross premium $\underline{\pi}^*$.
Acquisition costs and administrative expenses as well as gross technical results are given in Table 541 of the corresponding annual report of BaFin (2011-2015). Profits before taxes and changes in equilization provisions are calculated by using total values in Table 5.22. Net technical results are based on the absolute values in Table 5.23 and $\underline{\pi}^*$ given in Table 5.22.

Household

	2015	2014	2013	2012	2011
Total gross premium earned $\underline{\pi}$	69,448,394	66,146,203	64,535,515	62,102,602	59,310,517
Gross premium earned househ. $\underline{\pi}^*$	2,814,327	2,742,306	2,683,368	2,622,915	2,578,722
Net premium earned househ. $\hat{\underline{\pi}}^*$	2,426,927	2,370,466	2,434,656	2,375,543	2,330,813
Total profits before taxes	2,548,300	2,587,900	2,135,800	2,472,600	1,986,700
Total changes in equilization provisions	295,400	684,500	-180,700	858,400	-368,700
Net technical result household	15.30	12.30	14.40	14.70	16.30

Table 5.25: Needed data for computation of value added household.
Total gross premium earned (direct business), gross premium earned household (direct business), net premium earned household (direct business) and net technical result household are given in Table 541 of the corresponding annual report of BaFin (2011-2015). As above, total profits before taxes and total (negative) changes in equilization provisions can be adopted from Table 79 in the same reports. Positions 1 to 5 are specified in TEUR, position 6 as ratio of $\hat{\underline{\pi}}^*$.

	2015	2014	2013	2012	2011
Acquisition costs and administrative expenses	982,200	959,807	936,495	925,889	902,553
Profits before taxes	103,267	107,290	88,806	104,431	86,378
Changes in equilization provisions	11,971	28,378	-7,513	36,255	-16,030
Gross technical result	484,064	394,892	402,505	445,896	477,064
Net technical result	371,320	291,567	350,590	349,205	379,923
Value added household	1,210,183	1,198,800	1,069,703	1,163265	1,070,042

Table 5.26: Computation of value added household in TEUR.
All quantities except net technical result are computed using Table 5.27 and $\underline{\pi}^*$ given in Table 5.25. Net technical result is calculated by multiplying the corresponding quantities in Table 5.25.

	2015	2014	2013	2012	2011	Mean
Acquisition costs and administrative expenses	34.90	35.00	34.90	35.30	35.00	
Profits before taxes	3.67	3.91	3.31	3.98	3.35	
Changes in equilization provisions	0.43	1.03	-0.28	1.38	-0.62	
Gross technical result	17.20	14.40	15.00	17.00	18.50	
Net technical result	13.19	10.63	13.07	13.31	14.73	
Value added household	43.00	43.72	39.86	44.35	41.50	42.49

Table 5.27: Computation of value added household as ratio of earned gross premium $\underline{\pi}^*$.
Acquisition costs and administrative expenses as well as gross technical results are given in Table 541 of the corresponding annual report of BaFin (2011-2015). Profits before taxes and changes in equilization provisions are calculated by using total values in Table 5.25. Net technical results are based on the absolute values in Table 5.26 and $\underline{\pi}^*$ given in Table 5.25.

Residential Building (ResBui)

	2015	2014	2013	2012	2011
Total gross premium earned $\underline{\pi}$	69,448,394	66,146,203	64,535,515	62,102,602	59,310,517
Gross premium earned ResBui $\underline{\pi}^*$	6,144,732	5,782,479	5,388,303	5,033,876	4,764,973
Net premium earned ResBui $\underline{\hat{\pi}}^*$	4,702,787	4,425,298	4,329,537	4,064,797	3,834,273
Total profits before taxes	2,548,300	2,587,900	2,135,800	2,472,600	1,986,700
Total changes in equilization provisions	295,400	684,500	-180,700	858,400	-368,700
Net technical result residential building	-9.40	-8.30	-22.20	-12.50	-14.40

Table 5.28: Needed data for computation of value added residential building.
Total gross premium earned (direct business), gross premium earned residential building (direct business), net premium earned residential building (direct business) and net technical result residential building are given in Table 541 of the corresponding annual report of BaFin (2011-2015). As above, total profits before taxes and total (negative) changes in equilization provisions can be adopted from Table 79 in the same reports. Positions 1 to 5 are specified in TEUR, position 6 as ratio of $\underline{\hat{\pi}}^*$.

	2015	2014	2013	2012	2011
Acquisition costs and administrative expenses	1,720,525	1,624,877	1,530,278	1,424,587	1,362,782
Profits before taxes	225,471	226,233	178,326	200,423	159,610
Changes in equilization provisions	26,137	59,839	-15,087	69,580	-29,621
Gross technical result	-147,474	-138,779	-1,929,012	-241,626	-385,963
Net technical result	-442,062	-367,300	-961,157	-508,100	-552,135
Value added residential building	2,266,721	2,139,469	725,661	1,961,063	1,658,944

Table 5.29: Computation of value added residential building in TEUR.
All quantities except net technical result are computed using Table 5.30 and $\underline{\pi}^*$ given in Table 5.28. Net technical result is calculated by multiplying the corresponding quantities in Table 5.28.

	2015	2014	2013	2012	2011	Mean
Acquisition costs and administrative expenses	28.00	28.10	28.40	28.30	28.60	
Profits before taxes	3.67	3.91	3.31	3.98	3.35	
Changes in equilization provisions	0.43	1.03	-0.28	1.38	-0.62	
Gross technical result	-2.40	-2.40	-35.80	-4.80	-8.10	
Net technical result	-7.19	-6.35	-17.84	-10.09	-11.59	
Value added residential building	36.89	37.00	13.47	38.96	34.82	32.23

Table 5.30: Computation of value added residential building as ratio of earned gross premium $\underline{\pi}^*$.
Acquisition costs and administrative expenses as well as gross technical results are given in Table 541 of the corresponding annual report of BaFin (2011-2015). Profits before taxes and changes in equilization provisions are calculated by using total values in Table 5.28. Net technical results are based on the absolute values in Table 5.29 and $\underline{\pi}^*$ given in Table 5.28.

Credit and Guarantee (CreGua)

	2015	2014	2013	2012	2011
Total gross premium earned $\underline{\pi}$	69,448,394	66,146,203	64,535,515	62,102,602	59,310,517
Gross premium earned CreGua $\underline{\pi}^*$	450,905	415,173	988,984	958,490	1,235,832
Net premium earned CreGua $\hat{\underline{\pi}}^*$	429,550	403,996	585,225	560,561	845,929
Total profits before taxes	2,548,300	2,587,900	2,135,800	2,472,600	1,986,700
Total changes in equilization provisions	295,400	684,500	-180,700	858,400	-368,700
Net technical result credit and guarantee	32.40	29.10	20.20	23.00	29.20

Table 5.31: Needed data for computation of value added credit and guarantee.
Total gross premium earned (direct business), gross premium earned credit and guarantee (direct business), net premium earned credit and guarantee (direct business) and net technical result credit and guarantee are given in Table 541 of the corresponding annual report of BaFin (2011-2015). As above, total profits before taxes and total (negative) changes in equilization provisions can be adopted from Table 79 in the same reports. Positions 1 to 5 are specified in TEUR, position 6 as ratio of $\hat{\underline{\pi}}^*$.

	2015	2014	2013	2012	2011
Acquisition costs and administrative expenses	132,115	119,155	273,949	272,211	359,627
Profits before taxes	16,545	16,243	32,730	38,162	41,396
Changes in equilization provisions	1,918	4,296	-2,769	13,249	-7,682
Gross technical result	124,450	126,213	206,698	150,483	411,532
Net technical result	139,174	117,563	118,215	128,929	247,011
Value added credit and guarantee	135,854	148,344	392,392	345,176	557,862

Table 5.32: Computation of value added credit and guarantee in TEUR.
All quantities except net technical result are computed using Table 5.33 and $\underline{\pi}^*$ given in Table 5.31. Net technical result is calculated by multiplying the corresponding quantities in Table 5.31.

	2015	2014	2013	2012	2011	Mean
Acquisition costs and administrative expenses	29.30	28.70	27.70	28.40	29.10	
Profits before taxes	3.67	3.91	3.31	3.98	3.35	
Changes in equilization provisions	0.43	1.03	-0.28	1.38	-0.62	
Gross technical result	27.60	30.40	20.90	15.70	33.30	
Net technical result	30.87	28.32	11.95	13.45	19.99	
Value added credit and guarantee	30.13	35.73	39.68	36.01	45.14	37.34

Table 5.33: Computation of value added credit and guarantee as ratio of earned gross premium $\underline{\pi}^*$.
Acquisition costs and administrative expenses as well as gross technical results are given in Table 541 of the corresponding annual report of BaFin (2011-2015). Profits before taxes and changes in equilization provisions are calculated by using total values in Table 5.31. Net technical results are based on the absolute values in Table 5.32 and $\underline{\pi}^*$ given in Table 5.31.

Note: When data were available from different BaFin-reports, we have always chosen the most current data source. Whenever possible, we used data from BaFin (2011-2015), Issue 2015, for the years 2013-2015, BaFin (2011-2015), Issue 2014, for the year 2012 and BaFin (2011-2015), Issue 2013, for the year 2011. This refers to Table 540 in the corresponding issues. For the year 2014, we had to rely on Table 80 instead of Table 79 in BaFin (2011-2015), Issue 2014.

6 | Market Consistent Embedded Value

A German version of the introductory part, Sections 6.1 & 6.2 was previously published in *Der Aktuar*, p. 4-8, 2014, see Becker, Cottin, Fahrenwaldt, Hamm, Nörtemann & Weber (2014).

The main objective of the market consistent embedded value (MCEV) is to define a methodologically sound, meaningful, comparable and consistent measurement of the profitability of the total business in-force of an insurance company. The CFO Forum (2009*b*), Principle 1, states that the "MCEV is a measure of the consolidated value of shareholders' interests in the *covered business.* [...]". This value serves as an important indicator not only for the management of these insurance firms, but also for analysts, investors and shareholders of the undertakings.

The measurement of the intrinsic value of an insurance company is not a trivial task. Contracts are associated with expenses and gains at different points in time that are usually stochastic. A measure of value attempts to summarize the inherent value in a single number. Traditional accounting measures are not able to capture how the two dimensions randomness and time are intertwined. As a consequence, they are largely useless for the company's management, investors and capital providers.

The MCEV was suggested in 2009, see CFO Forum (2009*b*). Its computation is based on stochastic balance sheet projections. The random evolutions of the balance sheet positions depend on the random evolution of financial and insurance risk as well as the random behavior of policyholders and on the management policies of the company. The computation of the MCEV requires an asset-liability management model that probabilistically models both the asset and the liability side. Actuarial and financial valuation are both required for the analysis. In case of perfect pooling, the value of insurance liabilities could be computed by a law of large numbers under the statistical measure. But pooling is never perfect and stochastic fluctuations remain. At the same time, systematic risk such as financial risk or stochastic mortality influences both the liability and the asset side of the company's balance sheet. Two complementary methodologies need to be applied to value these risks. The hedgeable part can be priced under a martingale measure using risk-neutral valuation. The non-hedgeable part contributes to the capital requirement of the firm. The capital requirement is computed on the basis of a monetary risk measure that encodes risk tolerance governing the firm, typically under the statistical measure. This risk measure is either specified by a regulatory authority or by policies of the company. Providing capital is associated with a cost, the *cost of capital*, that is charged for non-hedgeable risks.

Literature. A preliminary measure of value in the industry is the Embedded Value, now known as *classical* or *traditional Embedded Value* (TEV), that is, for example, described in DAV (2005). The TEV is not yet capable of fully integrating the dimensions time and value, as it is based on deterministic computations. A *Present Value of Future Profits* (PVFP) is calculated by means of a single deterministic projection of future cash flows. Discounting is done according to the risk discount rate, which is given by the sum of a risk-free interest rate and a company specific risk loading. The TEV has multiple obvious drawbacks: First, stochastic fluctuations of future cash flows, e.g., caused by financial markets, systematic insurance risk or the policyholders' behavior, are not included. Risk analysis is clumsy and naïve. Second, risk loadings are company specific, not standardized, thus arbitrary to a large extent. Third, universal reporting standards cannot easily be developed on the basis of a deficient methodology such as TEV. Reported values for different companies can hardly be compared to each other in a meaningful manner.

Realizing these challenges, the CFO Forum – a discussion group of Chief Financial Officers of large European insurance companies – proposed the *European Embedded Value* (EEV) to facilitate the transparent and comparable valuation of insurance companies, see CFO Forum (2004*b*) and CFO Forum (2004*a*). These ideas were further developed and led to the definition of the *Market Consistent Embedded Value* (MCEV). The framework for the computation of the MCEV in life insurance companies is organized in 17 principles and 91 guidances in the document "Market Consistent Embedded Value Principles", see CFO Forum (2009*b*), which was published in October 2009. These basic principles are listed in Table 6.1. The additional document "Market Consistent Embedded Value – Basis for Conclusions", see CFO Forum (2009*a*), explains the calculation in detail and consists of 198 comments. Actuarial societies published more detailed methodologies, see, e.g., the following publications by *Deutsche Aktuarvereinigung (DAV)*: "Stochastischer Embedded Value", DAV (2006), "Best Estimate in der Lebensversicherung", DAV (2010), and "Market Consistent Embedded Value", DAV (2011).

No.	Heading	Contents
P1	Introduction	Definition MCEV, Group MCEV, MCEV-Methodology
P2	Coverage	Area of Application, Definition of Covered Business
P3	MCEV Definitions	Components of MCEV
P4	Free Surplus	Definition of Free Surplus
P5	Required Capital	Definition of Required Capital
P6	Value of In-Force Covered Business	Components of Value of In-Force Covered Business, Comment on Present Value of Future Profits
P7	Financial Options and Guarantees	Computation of the Time Value of Financial Options and Guarantees
P8	Frictional Costs of Required Capital	Definition of Frictional Costs of Required Capital
P9	Cost of Residual Non-	Definition and Computation of Costs of Residual

	Hedgeable Risks	Non-Hedgeable Risks
P10	New Business and Renewals	Definition of New Business, Allowance of Renewals, Additional Value to Shareholders
P11	Assessment of Appropriate Non-Economic Projection Assumptions	Recommended Practice on Appropriate Assumptions for Future Experience; Demographic Assumptions, Expenses, Taxation and Legislation
P12	Economic Assumptions	Recommended Practice on Appropriate Economic Assumptions; Comment on Inflation and Smoothing
P13	Investment Returns and Discount Rates	Recommended Practice on Capital Market Consistent Discount Rates
P14	Reference Rates	Recommended Practice on Reference Rates as Proxy for the Risk-Free Rate
P15	Stochastic Models	Recommended Practice on Stochastic Models and their Parameters
P16	Participating Business	Recommended Practice on Participating Business
P17	Disclosure	Reporting Requirements

Table 6.1: Summary of MCEV-Principles.

There are several previous scientific contributions on the MCEV that focus on various important aspects of this valuation methodology. Diers, Eling, Kraus & Reuß (2012) generalize the concept of the MCEV and apply it to non-life insurance business in order to achieve comparability in performance measurement. Kraus (2013) shows that the concepts of economic value added and MCEV can be linked under the residual income valuation theory which facilitates the consistent application of both at a corporate level. Graf, Pricking, Schmidt & Zwiesler (2012) utilize Monte Carlo simulations in order to analyze the time value of financial options and guarantees (TVOG) in private health insurance business. They find that the impact of TVOG is significantly smaller than in the life insurance business. Sector-specific issues as well as their impact on the MCEV-computation are qualitatively discussed in Schmidt (2012). Schmidt (2014) applies an adjusted version of the asset-liability management (ALM) model introduced by Gerstner, Griebel, Holtz, Goschnik & Haep (2008) to conduct a quantitative study of the MCEV in German private health insurance. The focus is on the TVOG, and the author finds that its impact on the MCEV is rather small. Bauer, Reuß & Singer (2012) present a technique for the numerical implementation of the MCEV which is based on nested simulations. Reuß, Ruß & Wieland (2015) focus on the impact of product design on the insurer's capital efficiency in a market consistent valuation model. They introduce alternative products to the traditional participating life insurance contracts and illustrate how these can reduce the risk and increase the profitability of the insurer. The cost of capital approach for the allowance of non-hedgeable risk is considered in Waszink (2013). In particular, the author focuses on the discount rate suitable for determining the risk margin.

This chapter extends a preliminary analysis of the MCEV in Becker et al. (2014). The key contribution is to explain how actuarial and financial valuation are intertwined in the computation of the MCEV and how non-hedgeable risks are valued on the basis of monetary risk measures. We explicitly analyze randomness on both sides of the balance sheet, i.e., the asset and the liability side. Our case studies include the impact of several factors (e.g., interest rates, stochastic mortality, management decisions etc.) on the MCEV. In order to keep our ALM-analysis tractable, we use a very stylized model. Extensions could be developed along the lines of Gerstner et al. (2008) and Schmidt (2014).

Outline. The chapter is organized as follows: In Section 6.1, we define the MCEV and provide an overview of its components. In Section 6.2, we discuss in detail the components of the MCEV. We explain how actuarial and financial valuation methods are intertwined. The computation of the MCEV is presented in Section 6.3. While Section 6.3.1 introduces our asset-liability management model, Section 6.3.2 illustrates the methodology in several case studies. Section 6.4 concludes with a summary and short discussion.

6.1 | The MCEV and its Components

We begin with a brief exposition of the MCEV and its components.[1] According to Principle 3 in CFO Forum (2009*b*), the "MCEV represents the present value of shareholders' interests in the *earnings distributable* from assets allocated to the *covered business* after sufficient allowance for the aggregate risks in the covered business. The allowance for risk should be calibrated to match the market price for risk where reliably observable. The MCEV consists of the following components:

- Free surplus allocated to the covered business,
- Required capital, and
- Value of in-force covered business.

The value of future new business is excluded from the MCEV."

Following this definition, the MCEV is decomposed as the sum of three components: *Free Surplus* (FS), *Required Capital* (ReC) and *Value of In-Force Covered Business* (VIF):

$$\text{MCEV} = \text{FS} + \text{ReC} + \text{VIF}. \tag{6.1}$$

The sum of FS and ReC is called *Shareholder Net Worth*, cf. DAV (2011), or *Net Asset Value* (NAV), i.e.,

$$\text{NAV} = \text{FS} + \text{ReC}. \tag{6.2}$$

The NAV is part of the balance sheet of an insurance company and corresponds to the market value at valuation date of those assets that are not required to cover the technical insurance obligations. The FS is that part of the NAV "[...] allocated to, but not required to support, the in-force *covered business* at the valuation date", see Principle 4 in CFO Forum (2009*b*). Hence, the FS can be distributed to shareholders without restrictions. The ReC, however, is that part of the NAV, whose distribution to shareholders is restricted. There

[1] Similar introductions to the MCEV can also be found in Becker et al. (2014) and Schmidt (2012).

are several reasons for companies to hold additional capital not backing the liabilities: regulatory requirements such as solvency requirements according to Solvency II, rating purposes, reputation, etc. Consequently, the ReC is given by (see DAV (2011), p. 5)

$$\text{ReC} = \max\left\{\text{regulatory capital requirement}, \text{in-house capital requirement}\right\}.$$

The NAV can be computed from the balance sheet as a residual quantity. The asset side of the balance sheet can be evaluated by financial methods. Its total value equals the total liabilities on the balance sheet. The NAV is the difference between total liabilities and the liabilities that cover the technical insurance obligations. Those could, for example, consist of a reserve and a bonus account that are calculated according to prudent actuarial valuation methods.

Although the NAV corresponds to the market value of those assets that are *not* required to cover the technical insurance obligations, it is not equal to the MCEV. The reason is that the estimates of the reserves are prudent. In many cases, they are larger than the insurance payments that will be realized in the future. This leads to additional future profits of the insurance companies. The value of these profits is the VIF which is not yet included in NAV.[2] As a consequence, the MCEV equals the sum of NAV and VIF, see Equations (6.1) & (6.2).

How is the VIF computed? As explained, the full actuarial reserves will not be needed in all scenarios. In order to gauge these random gains of the shareholders, a stochastic balance sheet projection is needed. Over any fixed time period, gross profits are reflected as the incremental net asset value. Using a suitable martingale measure, a value of the stream of these profits can be computed. This quantity is called the *Present Value of Future Profits* (PVFP(stoch)). In order to obtain the VIF, two positions are subtracted from the PVFP(stoch), the *Frictional Costs of Required Capital* (FC) and the *Cost of Residual Non-Hedgeable Risks* (CRNHR). FC refers, for example, to taxation and investment costs on the assets backing the ReC. CRNHR are discussed in Principle 9 in CFO Forum (2009*b*): "An allowance should be made for the cost of *non-hedgeable risks* not already allowed for in [...] the PVFP. This allowance should include the impact of *non-hedgeable non-financial risks* and *non-hedgeable financial risks.* [...]." Such risks are, for example, mortality risk, operational risk, lapse risk, etc.

The cited definition of the CRNHR is vague, and one key insight of this chapter is how it can be made precise. Our key idea is to compute the PVFP(stoch) under a specific pricing measure. The CRNHR is a value adjustment that reflects the particular structure of the pricing measure. The chosen pricing measure prices replicable risks correctly. But we assume on purpose that it does not add any risk premium to non-hedgeable risks, see Section 6.2. Non-hedgeable risks thus require an additional value adjustment which is reflected by the CRNHR. In this chapter, we suggest and explain an approach how

[2]In Section 6.2 we explain the difference of financial and actuarial valuation in detail. Section 6.3 includes a detailed illustration of a company's balance sheet and the valuation of its positions. Section 6.2 focuses on the computation of the VIF.

CRNHR could be computed. Finally, we obtain the following equation for the VIF:

$$\mathrm{VIF} = \mathrm{PVFP(stoch)} - \mathrm{FC} - \mathrm{CRNHR}. \tag{6.3}$$

The computation of PVFP(stoch) is based on stochastic balance sheet projections. This method needs Monte Carlo simulations which are computationally expensive. Historically, deterministic methods were used in actuarial mathematics. In this spirit, a simpler, but crude version of PVFP is PVFP(CE). This quantity is obtained from a single deterministic scenario: the "certainty equivalent" (CE). The CE-scenario is constructed on the basis of best estimate assumptions. Stochastic fluctuations are not included – implying that PVFP(CE) and PVFP(stoch) typically do not coincide. Their difference is called the *Time Value of Financial Options and Guarantees* (TVOG)

$$\mathrm{TVOG} = \mathrm{PVFP(CE)} - \mathrm{PVFP(stoch)}. \tag{6.4}$$

It can be interpreted as a quantitative measure of the stochastic fluctuations that are neglected by the PVFP(CE). Random fluctuations of insurance liabilities are, for example, due to imperfect pooling, mortality risk, surrenders, embedded options, dynamic agreements, etc. This leads to an alternative representation of VIF:

$$\mathrm{VIF} = \mathrm{PVFP(CE)} - \mathrm{TVOG} - \mathrm{FC} - \mathrm{CRNHR}.$$

Figure 6.1 illustrates the identities between the various quantities that were introduced.

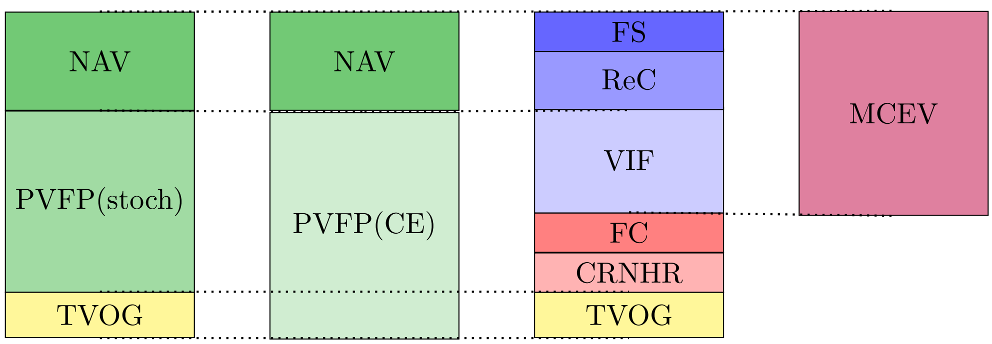


Figure 6.1: MCEV and its components.

6.2 | Valuation Methods

In this section, we explain the computation of the MCEV and its components in more detail. In particular, we discuss the interplay of actuarial and financial valuation. We suggest a unified framework that shows how a martingale measure and the statistical measure are combined – the former being used for the valuation of replicable risks, and the latter for the computation of the cost of capital of non-hedgeable risks. Our definition of the cost of capital builds on monetary risk measures.

The MCEV is the sum of NAV and VIF. The NAV is easily computed as the difference of assets and reserves. Reserves are prudently calculated and embed potential for future cash flows to the shareholders. The key task in the MCEV-computation consists of assigning a value to these cash flows, the VIF. The valuation of cash flows intertwines risk pooling and arbitrage-free pricing.

We will work within the following abstract setting, see Wüthrich, Bühlmann & Furrer (2010), Section 2. Consider a discrete time model on a probability space $(\Omega, \mathcal{H}, P)$ with dates $t = 0, 1, \ldots, T$ and final time horizon $T \in \mathbb{N}$. The information flow is captured by the filtration $\mathbb{H} = (\mathcal{H}_t)_{t=0,1,\ldots,T}$, i.e.,

$$\{\emptyset, \Omega\} = \mathcal{H}_0 \subset \mathcal{H}_1 \subset \ldots \subset \mathcal{H}_T = \mathcal{H}.$$

The primary products available in the financial market are adapted stochastic processes $\left(S^0, S^1, \ldots, S^N\right)$, $N \in \mathbb{N}$. The product $S^0_t = (1 + r)^t$, $t = 0, 1, \ldots, T$, is a deterministic savings account and will be chosen as numéraire.
We assume that the filtered probability space possesses a product structure, i.e.,

$$\Omega = \Omega^1 \times \Omega^2, \quad \mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2, \quad P = P^1 \otimes P^2.$$

The probability measures $P^1 : \mathcal{H}^1 \to [0, 1]$ and $P^2 : \mathcal{H}^2 \to [0, 1]$ denote the statistical measures on the σ-algebras $\mathcal{H}^1$ and $\mathcal{H}^2$, respectively. The first component captures replicable, the second non-replicable risks. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,\ldots,T}$ the filtration of replicable risks and by $\mathbb{T} = (\mathcal{T}_t)_{t=0,1,\ldots,T}$ the filtration of non-replicable risks. The σ-algebra $\mathcal{H}_t = \sigma(\mathcal{F}_t \cup \mathcal{T}_t)$ contains all information at time t. This includes, e.g., information on the financial market, demographic evolution, economic events, policyholder behavior, etc. We assume that up to nullsets

$$\mathcal{F}_t = \sigma\left(\left(\mathcal{H}^1 \otimes \{\emptyset, \Omega^2\}\right) \cap \mathcal{H}_t\right), \quad \mathcal{T}_t = \sigma\left(\left(\{\emptyset, \Omega^1\} \otimes \mathcal{H}^2\right) \cap \mathcal{H}_t\right) \qquad t = 0, 1, \ldots, T.$$

In particular, $\left(S^0, S^1, \ldots, S^N\right)$ are adapted to $\mathbb{F}$.

We suggest the following valuation approach: The value of replicable risks is the cost of their perfect replication. In a first step, non-replicable risks are simply priced according to best estimates. This leads to a pricing measure with the following structure

$$Q = Q^1 \otimes P^2,$$

where Q^1 can be interpreted as a martingale measure on the first component. To be more precise, the processes $\frac{S^1}{S^0}, \frac{S^2}{S^0}, \ldots, \frac{S^N}{S^0}$ are Q-martingales. In a second step, a value adjustment for the non-replicable risk is computed using monetary risk measures.

Present Value of Future Profits. The present value of future profits, PVFP(stoch)_t, computes the value of future profits at time t with respect to the martingale measure Q, i.e.,

$$\mathrm{PVFP(stoch)}_t = \mathbb{E}_Q\left[\sum_{s=t+1}^{T} \frac{1}{(1+r)^{s-t}} \cdot \mathrm{Profit}_s \;\middle|\; \mathcal{H}_t\right]. \tag{6.5}$$

Profits are defined as the one-year increments of NAV adjusted for interest:

$$\mathrm{Profit}_t = \mathrm{NAV}_t - (1+r)\cdot \mathrm{NAV}_{t-1}.$$

Here, r denotes the interest rate. Due to our choice of Q, the valuation of replicable risks is market-consistent, while best estimates are taken for all residual risks. Residual risks that lead to future annual profits are thus priced too optimistically, without including a proper risk premium. A valuation adjustment is necessary, called the costs of residual non-hedgeable risks CRNHR.

Costs of Residual Non-Hedgeable Risks. Risks that cannot fully be hedged include, for example, fluctuations due to imperfect pooling, mortality risk, surrenders, embedded options, dynamic agreements, etc. As long as these risks can be quantitatively modeled and embedded into an economic scenario generator, the following methodology is applicable to compute a valuation adjustment, the CRNHR:

1. Determine the non-hedgeable fluctuations around PVFP(stoch), i.e.,

   $$Z_t - \mathrm{PVFP(stoch)}_t$$

   with

   $$Z_t = \mathbb{E}_Q\left[\sum_{s=t+1}^{T} \frac{1}{(1+r)^{s-t}} \cdot \mathrm{Profit}_s \;\middle|\; \mathcal{H}_t, \mathcal{T}_T\right]. \tag{6.6}$$

2. Compute a capital buffer for the downside risk. For this purpose, an $\mathcal{H}_t$-conditional risk measure ρ_t is applied to the fluctuations:

   $$\rho_t\left(Z_t - \mathrm{PVFP(stoch)}_t\right).$$

   Then, the risk $\rho_t\left(Z_t - \mathrm{PVFP(stoch)}_t\right)$ is an $\mathcal{H}_t$-measurable random variable. For a brief introduction to risk measures and their extension to conditional, resp. dynamic, risk measures, we refer to Section 6.5.

3. Finally, the CRNHR_t is computed by multiplying the capital buffer with a cost of capital rate $\xi \in [0,1]$, thus

   $$\mathrm{CRNHR}_t = \xi \cdot \rho_t\left(Z_t - \mathrm{PVFP(stoch)}_t\right). \tag{6.7}$$

   This quantity can be considered as risk premium for non-hedgeable risks.

For positively homogeneous risk measures, we obtain the following representation of CRNHR.

Proposition 6.2.1. *Let ρ_t be a positively homogeneous conditional risk measure with $\rho_t(0) = 0$, $\xi \in [0,1]$, then*

$$CRNHR_t = \xi \cdot \frac{1}{(1+r)^{T-t}} \cdot \rho_t \left(\mathbb{E}_Q \left[NAV_T \,\middle|\, \mathcal{H}_t, \mathcal{T}_T \right] - \mathbb{E}_Q \left[NAV_T \,\middle|\, \mathcal{H}_t \right] \right),$$

for all $0 \leq t \leq T-1$ and $CRNHR_T = 0$.

Proof. By Equations (6.5) & (6.6), we obtain

$$\text{PVFP(stoch)}_t = \frac{1}{(1+r)^{T-t}} \cdot \mathbb{E}_Q \left[\text{NAV}_T \,\middle|\, \mathcal{H}_t \right] - \text{NAV}_t$$

and

$$Z_t = \frac{1}{(1+r)^{T-t}} \cdot \mathbb{E}_Q \left[\text{NAV}_T \,\middle|\, \mathcal{H}_t, \mathcal{T}_T \right] - \text{NAV}_t,$$

for $0 \leq t \leq T-1$ and $\text{PVFP(stoch)}_T = Z_T = 0$. The positive homogeneity of ρ_t leads to

$$\begin{aligned} \text{CRNHR}_t &= \xi \cdot \rho_t \left(Z_t - \text{PVFP(stoch)}_t \right) \\ &= \xi \cdot \frac{1}{(1+r)^{T-t}} \cdot \rho_t \left(\mathbb{E}_Q \left[\text{NAV}_T \,\middle|\, \mathcal{H}_t, \mathcal{T}_T \right] - \mathbb{E}_Q \left[\text{NAV}_T \,\middle|\, \mathcal{H}_t \right] \right). \end{aligned}$$

□

Frictional Costs of Required Capital. As mentioned in Section 6.1, FC corresponds to those expenses resulting from ReC. These are, for example, costs of capital investments or taxes on profits due to assets backing the ReC. The FC can be interpreted as opportunity costs resulting from the investment of equity capital into the insurance company rather than a direct investment in the capital market, see DAV (2011) and Christiansen & Niemeyer (2014) for further details.

Required Capital and Free Surplus. The segmentation of NAV into ReC and FS, see Eq. (6.2), allows for different approaches. In-house capital requirements can be based on requirements provided by regulators, rating agencies, investors or the management of the company. For example, the *risk capital* (RC) can be computed by applying a conditional risk measure to the random change in the MCEV over a one year time horizon with respect to the available information, i.e.,

$$\text{RC}_t = \rho_t \left(\text{MCEV}_{t+1} - \text{MCEV}_t \right), \quad t = 0, 1, \ldots, T-1.$$

Since VIF represents the value of the business in force from the perspective of equity investors, it can be subsidized with the RC. Hence, ReC is given by

$$\text{ReC}_t = \max \left\{ \text{SCR}_t, \max \left\{ 0, \text{RC}_t - \text{VIF}_t \right\} \right\},$$

where SCR denotes the *solvency capital requirement* according to regulatory requirements, see, e.g., Chapters 3 & 4 for further details on the SCR. Afterwards, FS can be computed as residual, i.e.,

$$\text{FS}_t = \text{NAV}_t - \text{ReC}_t.$$

6.3 | An Application to Asset-Liability Management

In this section, we illustrate the methodology in the context of a stylized asset-liability management model. We analyze the impact of different investment strategies, stochastic mortality and interest rates on the evolution of the stochastic balance sheet and the MCEV.

6.3.1 | The Model

Asset-liability management controls insurance firms on the basis of the random future evolution of assets and liabilities, see Zwiesler (2004). Our model of the stochastic dynamics of the balance sheet of an insurance firm is described in this section. A snapshot of a simplified balance sheet is displayed in Table 6.2. The sum of the assets equals the sum of the liabilities: $\tilde{S}_t + \tilde{B}_t = \text{NAV}_t + \bar{V}_t + \text{BA}_t$. Setting $A_t := \tilde{S}_t + \tilde{B}_t$ and $L_t := \bar{V}_t + \text{BA}_t$, we obtain $\text{NAV}_t = A_t - L_t$.

ASSETS		LIABILITIES	
Stocks	$\tilde{S}_t$	Net Asset Value	NAV_t
Bonds	$\tilde{B}_t$	Actuarial Reserve	$\bar{V}_t$
		Bonus Account	BA_t

Table 6.2: Simplified balance sheet of an insurance firm.

Asset Model. We suppose that there are two primary products available in the financial market, i.e., $N = 1$ and we set $(B_t, S_t) = (S_t^0, S_t^1), t = 0, 1, \ldots, T$. The product B is a risk-free bond and S is a risky stock. In the time period $(t-1, t]$, the insurance company holds $n_{B_{t-1}}$ bonds and $n_{S_{t-1}}$ shares of the stock. At time t, the value of the bond position is $\tilde{B}_t = n_{B_{t-1}} \cdot B_t$ and the value of the stock position is $\tilde{S}_t = n_{S_{t-1}} \cdot S_t$. Gains and losses on the asset side are due to price changes of the bond and the stock and denoted by

$$\Delta\tilde{B}_t = \tilde{B}_t - \tilde{B}_{t-1} \quad \text{and} \quad \Delta\tilde{S}_t = \tilde{S}_t - \tilde{S}_{t-1},$$

respectively. For simplicity, we consider a constant interest rate r, i.e., $B_t = (1+r) \cdot B_{t-1}$, $t = 1, 2, \ldots, T$, with $B_0 = 1$. A model for the stock price is described in Section 6.3.2.

Liability Model. We consider an insurance company selling pure endowments. The computation of the actuarial reserve is based on prudent actuarial assumptions on the interest rate and survival probabilities. These quantities will be denoted by r^* and p^*, respectively. We assume that survival probabilities may deviate from the actuarial assumptions. This may lead to gains and losses.

We suppose that the technical assumptions are fixed and do not change when new information becomes available. We use the actuarial standard notation, i.e., ${}_tp_x^*$ for the technical probability of a person of age x to survive additional t years. Realized empirical survival probabilities typically deviate from these values and may not be stationary. We denote the time t-realized fractions of survival of persons of age x at time 0 by the $\mathcal{H}_t$-measurable random variable ${}_tp_x$. A stochastic mortality model is described in Section 6.3.2.

The actuarial reserve is determined as follows: We consider a pure endowment with maturity T and single premium payment π in $t = 0$. Let X denote the sum insured, then the single premium is given by

$$\pi = \frac{1}{(1+r^*)^T}\ {}_Tp_x^* \cdot X.$$

The first-order actuarial reserve at any time t, V_t, is determined by charging interest to the value of the actuarial reserve at time $t-1$:

$$V_t = (1+r^*) \cdot V_{t-1}, \quad V_0 = \pi.$$

However, the actual actuarial reserve at time t must be adjusted due to biometrical corrections:

$$\bar{V}_t = \frac{{}_tp_x}{{}_tp_x^*} \cdot V_t = \frac{1}{(1+r^*)^{T-t}}\ {}_tp_x\ {}_{T-t}p_{x+t}^*\ X, \quad \bar{V}_0 = V_0.$$

This is due to the fact that the realized probability of survival up to time t typically deviates from the technical assumptions.

Like the asset side, the liability side generates gains and losses at each point in time. These gains and losses are due to the evolution of the financial market or stochastic mortality and are denoted by

$$\Delta\bar{V}_t = \bar{V}_{t-1} - \bar{V}_t = \underbrace{(\bar{V}_{t-1} - (1+r^*)\bar{V}_{t-1})}_{\text{interest rate losses}} + \underbrace{((1+r^*)\bar{V}_{t-1} - \bar{V}_t)}_{\text{mortality gains/losses}}.$$

We denote interest rate losses by $\Delta\bar{V}_t^r := \bar{V}_{t-1} - (1+r^*)\bar{V}_{t-1}$. These should be fully captured by gains on the asset side. Observe that at this point ongoing low interest rates in the financial market cause a crucial problem. Mortality gains/losses are denoted by $\Delta\bar{V}_t^m := (1+r^*)\bar{V}_{t-1} - \bar{V}_t$. They occur if the actuarial reserve at time t is not given by simply paying interest on the reserve at time $t-1$, since there have been deviations in the in fact realized survival probabilities from the assumed actuarial survival probabilities. If there are no deviations, $\Delta\bar{V}_t^m = 0$ and hence, there are neither gains nor losses.

Management Model. The evolution of the balance sheet depends on the action of the management through time that we describe by management rules. We focus on investment strategies and surplus distribution.

Investment Strategies:
The asset side of the balance sheet is the sum of the bond and stock holdings. The fraction

invested in stocks at time t is denoted by $\alpha_t \in [0,1]$. We consider two alternative investment strategies.

1. **Strategy A: Buy and hold.**
   Stocks and bonds are bought at time 0 and then held throughout. The number of stocks thus equals
   $$n_{S_t} = \frac{\alpha_0(\text{NAV}_0 + V_0)}{S_0},$$
   the number of bonds
   $$n_{B_t} = \frac{(1-\alpha_0)(\text{NAV}_0 + V_0)}{B_0} = (1-\alpha_0)(\text{NAV}_0 + V_0),$$
   noting $B_0 = 1$.

2. **Strategy B: Rearranging the portfolio according to a reserve rate.**
   The reserve rate at time t is defined by
   $$\varepsilon_t = \frac{\text{NAV}_t}{\bar{V}_t + \text{BA}_t} = \frac{\text{NAV}_t}{L_t}$$
   If ε_t differs from a given target reserve rate of the insurance company, $\varepsilon^{\text{target}} \in (0,1)$, investments are adjusted; see Gerstner et al. (2008) for a similar approach. In addition, stock investments are bounded, i.e., $\alpha_t \le \alpha^{\max}$. With $\alpha_0 \le \alpha^{\max}$ the investment rule is given by
   $$\alpha_t = \min\left\{\max\left\{\alpha_0 + \left(\varepsilon_t - \varepsilon^{\text{target}}\right), 0\right\}, \alpha^{\max}\right\}, \quad t \ge 1.$$
   The number of stocks at time t is $n_{S_t} = \frac{\alpha_t A_t}{S_t}$, the number of bonds $n_{B_t} = \frac{(1-\alpha_t)A_t}{B_t}$.

Surplus Distribution Strategy:
Investment and biometric gains have to be split between the insurance company and the policyholders. We assume that in each period a fraction $\beta \in [0,1]$ of investment gains and a fraction $\gamma \in [0,1]$ of mortality gains is allocated to the shareholders, while the rest is allocated to the bonus account of the policyholders. If the insurance company experiences investment or biometric losses, the net asset value, i.e., the shareholders' account, is affected only.

The value of the bonus account at time t is denoted by BA_t. The bonus account is not adjusted for biometrical corrections; rather, it is charged interest by the actuarial interest rate. Consequently, the costs of the insurer for the bonus account are given by

$$\Delta b_t = \text{BA}_{t-1} - (1+r^*)\text{BA}_{t-1}.$$

The evolution of the bonus account is provided in Section 6.3.2.

6.3.2 | Numerical Case Studies

In the current section, we compute the quantities introduced in Sections 6.1 & 6.2 in the model defined in Section 6.3.1.

We consider a two period model, i.e., $t = 0, 1, 2$ and $T = 2$. The buyer of insurance has to pay a single premium π, where we set the sum insured to $X = 1,000$. Moreover, we assume ${}_1p_x^* = 1$, ${}_1p_{x+1}^* = 0.9$, i.e., ${}_Tp_x^* = 0.9$ and death can occur in the second period only. The capital buffer for CRNHR is computed by the risk measures value at risk and average value at risk at level $\lambda = 0.005$, respectively, see Appendix A, Example A.0.3. According to Allianz (2016), the cost of capital rate is set to $\xi = 0.06$.

At the beginning, 10% of the available capital is invested in stocks, i.e., $\alpha_0 = 0.1$, the remaining 90% are invested in bonds. In our case studies, we analyze the two investment strategies introduced in Section 6.3.1. To this end, we set $\alpha^{\max} = 0.15$ and $\varepsilon^{\text{target}} = 0.12$ in Strategy B. Moreover, we simulate both mortality gains and losses. In our setting, 90% of positive capital market gains and 75% of positive biometric gains are credited to the policyholder, whereas $\beta = 10\%$ and $\gamma = 25\%$ of these gains are invested in the net asset value of the company. Losses are fully carried by the insurance firm. Hence, the evolution of the net asset value is given by

$$
\begin{aligned}
\text{NAV}_t = \text{NAV}_{t-1} & \\
&+ \mathbb{1}_{\{\Delta\bar{V}_t>0\}} \\
&\Bigg[\mathbb{1}_{\{\Delta\tilde{S}_t+\Delta\tilde{B}_t+\Delta b_t>0\}}\Bigg(\frac{\text{NAV}_{t-1}}{\text{NAV}_{t-1}+\bar{V}_{t-1}}\cdot(\Delta\tilde{S}_t+\Delta\tilde{B}_t+\Delta b_t) \\
&\qquad + \beta\cdot\frac{\bar{V}_{t-1}}{\text{NAV}_{t-1}+\bar{V}_{t-1}}\cdot(\Delta\tilde{S}_t+\Delta\tilde{B}_t+\Delta b_t)+\gamma\cdot\Delta\bar{V}_t\Bigg) \\
&\quad + \mathbb{1}_{\{\Delta\tilde{S}_t+\Delta\tilde{B}_t+\Delta b_t\leq 0\}}\Big((\Delta\tilde{S}_t+\Delta\tilde{B}_t+\Delta b_t)\ +\gamma\cdot\Delta\bar{V}_t\Big)\Bigg] \\
&+ \mathbb{1}_{\{\Delta\bar{V}_t\leq 0\}} \\
&\Bigg[\mathbb{1}_{\{\Delta\tilde{S}_t+\Delta\tilde{B}_t+\Delta\bar{V}_t^r+\Delta b_t>0\}}\Bigg(\frac{\text{NAV}_{t-1}}{\text{NAV}_{t-1}+\bar{V}_{t-1}}\cdot(\Delta\tilde{S}_t+\Delta\tilde{B}_t+\Delta\bar{V}_t^r+\Delta b_t) \\
&\qquad + \beta\cdot\frac{\bar{V}_{t-1}}{\text{NAV}_{t-1}+\bar{V}_{t-1}}\cdot(\Delta\tilde{S}_t+\Delta\tilde{B}_t+\Delta\bar{V}_t^r+\Delta b_t)+\Delta\bar{V}_t^m\Bigg) \\
&\quad + \mathbb{1}_{\{\Delta\tilde{S}_t+\Delta\tilde{B}_t+\Delta\bar{V}_t^r+\Delta b_t\leq 0\}}\Big(\Delta\tilde{S}_t+\Delta\tilde{B}_t+\Delta\bar{V}_t+\Delta b_t\Big)\Bigg]
\end{aligned}
$$

Here, the indicator function $\mathbb{1}_{\{\Delta\bar{V}_t>0\}}$ decides whether there is a mortality gain or not. If there is a mortality gain, the second indicator function $\mathbb{1}_{\{\Delta\tilde{S}_t+\Delta\tilde{B}_t+\Delta b_t>0\}}$ decides whether there are capital market gains or not. If there are additional capital market gains, the fraction of the gains which results from the net asset value of the company as well as the fraction β which results from the investment of the reserve remain within the company and are added to the net asset value. Moreover, the fraction γ of the mortality gains is added. If there are mortality gains only, the company has to pay for the capital market losses.

If there are mortality losses, the insurer has to compensate for those by reducing the net asset value. Again, capital market gains are split as above, losses are completely taken by the firm.
Accordingly, the evolution of the bonus account is given by

$$
\begin{aligned}
\mathrm{BA}_t &= (1+r^*) \cdot \mathrm{BA}_{t-1} \\
&\quad + \mathbb{1}_{\{\Delta\bar{V}_t > 0\}} \\
&\quad \Bigg[\mathbb{1}_{\{\Delta\tilde{S}_t + \Delta\tilde{B}_t + \Delta b_t > 0\}} \left((1-\beta) \cdot \frac{\bar{V}_{t-1}}{\mathrm{NAV}_{t-1} + \bar{V}_{t-1}} \cdot (\Delta\tilde{S}_t + \Delta\tilde{B}_t + \Delta b_t) + (1-\gamma) \cdot \Delta\bar{V}_t \right) \\
&\quad + \mathbb{1}_{\{\Delta\tilde{S}_t + \Delta\tilde{B}_t + \Delta b_t \le 0\}} \left((1-\gamma) \cdot \Delta\bar{V}_t \right) \Bigg] \\
&\quad + \mathbb{1}_{\{\Delta\bar{V}_t \le 0\}} \\
&\quad \Bigg[\mathbb{1}_{\{\Delta\tilde{S}_t + \Delta\tilde{B}_t + \Delta\bar{V}_t^r + \Delta b_t > 0\}} \left((1-\beta) \cdot \frac{\bar{V}_{t-1}}{\mathrm{NAV}_{t-1} + \bar{V}_{t-1}} \cdot (\Delta\tilde{S}_t + \Delta\tilde{B}_t + \Delta\bar{V}_t^r + \Delta b_t) \right) \\
&\quad + \mathbb{1}_{\{\Delta\tilde{S}_t + \Delta\tilde{B}_t + \Delta\bar{V}_t^r + \Delta b_t \le 0\}} \cdot 0 \Bigg]
\end{aligned}
$$

with $\mathrm{BA}_0 = 0$.

As presented in Section 6.2, we denote by $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,\dots,T}$ the filtration of replicable (financial) risks and by $\mathbb{T} = (\mathcal{T}_t)_{t=0,1,\dots,T}$ the filtration of non-replicable risks (stochastic mortality). The complete information flow is captured by the filtration $\mathbb{H} = (\mathcal{H}_t)_{t=0,1,\dots,T}$, i.e.,

$$\{\emptyset, \Omega\} = \mathcal{H}_0 \subset \mathcal{H}_1 \subset \ldots \subset \mathcal{H}_T = \mathcal{H}.$$

Thus, the σ-algebra $\mathcal{H}_t = \sigma(\mathcal{F}_t \cup \mathcal{T}_t)$ contains all information at time t. With respect to Section 6.2, a complete scenario has the form $\omega = (\omega^1, \omega^2) \in \Omega^1 \times \Omega^2 = \Omega$, where Ω^1 represents the sample space of replicable financial risk of the stock evolution and $\Omega^2 = [0,1]$ is the sample space covering non-replicable risk of stochastic frequency of mortality. Our filtered probability space is given by

$$\Omega = \Omega^1 \times [0,1], \quad \mathcal{H} = \mathcal{P}\Omega^1 \otimes \mathcal{B}_{[0,1]}, \quad P = P^1 \otimes P^2,$$

where $\mathcal{P}\Omega^1$ is the power set of Ω^1 and $\mathcal{B}_{[0,1]}$ denotes the Borel σ-algebra on the interval $[0,1]$.

The evolution of the stock in our two period model is given by a possible upwards movement u or downwards movement d per period. Hence, the associated sample space adds up to

$$\Omega^1 = \{uu,\ du,\ ud,\ dd\}.$$

The random evolution of the stock is illustrated in Figure 6.2.

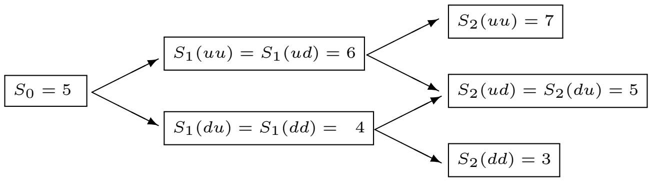


Figure 6.2: Stock evolution.

For the risk of stochastic mortality, the actual realized survival probability ${}_1p_{x+1}$ is modeled by a random variable on the interval $[0,1]$. For simplicity, we assume that the random variable is Beta-distributed. For a Beta distribution with parameters $a, b \geq 1$ and density

$$f_{a,b}(x) = \frac{1}{B(a,b)}\, x^{a-1}(1-x)^{b-1}\, \mathbb{1}_{[0,1]}(x), \quad x \in \mathbb{R},$$

we use the notation $\text{Beta}(a,b)$. Note that $B(\cdot,\cdot)$ denotes the Beta function given by $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$. For further simplicity, we assume that individuals can die only a short time before $t = 2$, i.e., ${}_1p_x = 1 = {}_1p_x^*$. However, we point out that any stochastic mortality model can be included within our model framework.

In the following case studies, we compute PVFP(stoch)_t according to Eq. (6.5). The particular pricing measure is $Q = Q^1 \otimes P^2$, where Q^1 represents the martingale measure in the pure financial market and P^2 represents the statistical measure in the pure stochastic mortality model. In our case studies, Q^1 varies with respect to the interest rate in the financial market and P^2 is given by a Beta distribution, i.e., ${}_1p_{x+1} \sim \text{Beta}(a,b)$. However, in the computation of PVFP(stoch)_t, ${}_1p_{x+1}$ is not random, but fixed to a best estimate assumption by averaging. We set

$$\bar{p} := \mathbb{E}_{P^2}\left[{}_1p_{x+1}\right].$$

Then, we compute

$$\text{CRNHR}_t = \xi \cdot \rho_t\left(Z_t - \text{PVFP(stoch)}_t\right)$$

according to Eq. (6.7), where by Eq. (6.6) it is

$$Z_t = \mathbb{E}_Q\left[\sum_{s=t+1}^{T} \frac{1}{(1+r)^{s-t}} \cdot \left(\text{NAV}_s - (1+r)\text{NAV}_{s-1}\right) \middle| \mathcal{H}_t, \mathcal{T}_T\right]$$

the PVFP(stoch) without averaging over the non-hedgeable risks, i.e., ${}_1p_{x+1} \sim \text{Beta}(a,b)$ is random.

Based on these quantities, VIF and MCEV are easily computed by

$$\text{VIF}_t = \text{PVFP(stoch)}_t - \text{CRNHR}_t - \text{FC}_t,$$

where we assume $\mathrm{FC}_t = 0$ for all $0 \le t \le T$, and

$$\mathrm{MCEV}_t = \mathrm{NAV}_t + \mathrm{VIF}_t.$$

As illustrated in Section 6.2, we compute ReC_t by

$$\mathrm{ReC}_t = \max\left\{\mathrm{SCR}_t, \max\left\{\mathrm{RC}_t - \mathrm{VIF}_t, 0\right\}\right\},$$

where $\mathrm{SCR}_t = \rho_t\left(\mathrm{NAV}_{t+1} - (1+r)\mathrm{NAV}_t\right)$ and $\mathrm{RC}_t = \rho_t\left(\mathrm{MCEV}_{t+1} - \mathrm{MCEV}_t\right)$. Finally, FS_t is given by

$$\mathrm{FS}_t = \mathrm{NAV}_t - \mathrm{ReC}_t.$$

Our case studies are structured as follows: We consider three different scenarios in the financial market:

- Zero interest rate scenario: The interest rate in the financial market as well as the actuarial interest rate is zero, i.e., $r = r^* = 0$.
- Low interest rate scenario: The interest rate in the financial market is lower than the actuarial interest rate. We set $r = 0.01$ and $r^* = 0.02$.
- High interest rate scenario: The interest rate in the financial market is higher than the actuarial interest rate. We set $r = 0.03$ and $r^* = 0.02$.

In each interest rate scenario, we compare the two investment strategies *buy and hold* (Strategy A) vs. *rearranging the portfolio* (Strategy B). At the same time, we simulate three mortality scenarios:

- Scenario 1: ${}_1p_{x+1} \sim \mathrm{Beta}(92.1842, 10.2427)$, i.e., $\bar{p} = \mathbb{E}_{P^2}\left[{}_1p_{x+1}\right] = 0.9 = {}_1p^*_{x+1}$
- Scenario 2: ${}_1p_{x+1} \sim \mathrm{Beta}(138.2685, 30.3516)$, i.e., $\bar{p} = \mathbb{E}_{P^2}\left[{}_1p_{x+1}\right] = 0.82 < {}_1p^*_{x+1}$ (mortality gains)
- Scenario 3: ${}_1p_{x+1} \sim \mathrm{Beta}(50.9071, 2.6793)$, i.e., $\bar{p} = \mathbb{E}_{P^2}\left[{}_1p_{x+1}\right] = 0.95 > {}_1p^*_{x+1}$ (mortality losses)

The distributions are chosen such that the variance in all three cases is the same and equal to $8.7017 \cdot 10^{-4}$. The corresponding probability density functions are displayed in Figure 6.3.

Moreover, we compare two regulatory systems. The first requires value at risk for risk measurement whereas the second uses average value at risk for the computation of capital requirements.

Zero Interest Rate Scenario. In the following example, we compute the stochastic balance sheet as well as the economic quantities introduced in the previous sections in the zero interest rate scenario for different strategies of the insurance company by additionally considering the three mortality scenarios.

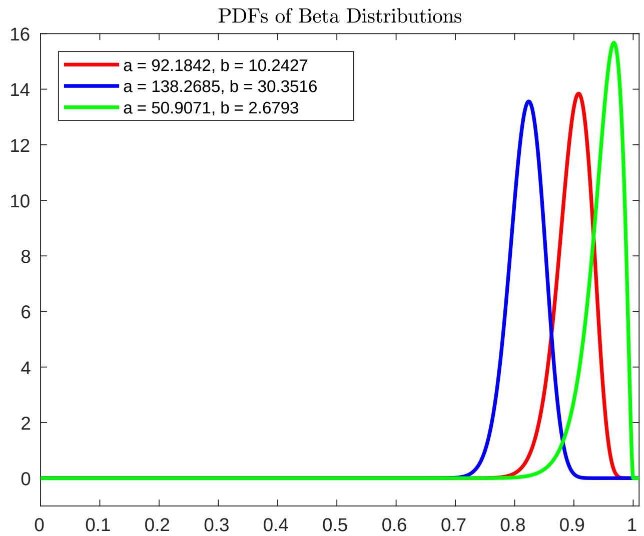


Figure 6.3: Probability density functions of the considered Beta distributions.

Example 6.3.1. We assume $\text{NAV}_0 = 50$ and compute $V_0 = \pi = 900$. Since $\alpha_0 = 0.1$, we obtain the following balance sheet:

$\boldsymbol{t = 0}$:

ASSETS							**LIABILITIES**						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	95			95			NAV	50			50		
Bond	855			855			V	900			900		
							BA	0			0		
Σ	950						Σ	950					

Table 6.3: Stochastic balance sheet for $t = 0$.

In the first period, the stock can move upwards or downwards as given in Figure 6.2 and the company realizes gains or losses depending on the evolution of the capital market. Due to our model framework (i.e., ${}_1p_x = 1$), there are no mortality gains or losses in the first period. According to the 10/90 rule, positive capital market gains are split between the insurer (NAV) and the policyholder (bonus account). Losses are fully carried by the company and consequently, there is no capital on the bonus account.

$\boldsymbol{t = 1, \omega = (u\cdot, {}_1p_x)}$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	114			114			NAV	52.8			52.8		
Bond	855			855			$\bar{V}$	900.0			900.0		
							BA	16.2			16.2		
Σ	969						Σ	969					

Table 6.4: Stochastic balance sheet for $t = 1$ when the stock goes up.

$\boldsymbol{t = 1, \omega = (d\cdot, {}_1p_x)}$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	76			76			NAV	31			31		
Bond	855			855			$\bar{V}$	900			900		
							BA	0			0		
Σ	931						Σ	931					

Table 6.5: Stochastic balance sheet for $t = 1$ when the stock goes down.

Note that until now, there is no difference in Strategy A and B, since the initial stock investment for both strategies is given by $\alpha_0 = 0.1$. In Strategy A, this investment strategy is maintained. But in Strategy B, the stock investment in the second period depends on the reserve rate ε_1. In both scenarios $\omega = (u\cdot, {}_1p_x)$ and $\omega = (d\cdot, {}_1p_x)$, it is $\varepsilon_1 < \varepsilon^{\text{target}} = 0.12$ and hence the stock investment is reduced to $\alpha_1 = 0.0376$ in the upwards-scenario and to $\alpha_1 = 0.0144$ in the downwards-scenario. Consequently, the balance sheet in $t = 2$ depends on the investment strategy, the mortality scenario and the stock evolution.

Let us first consider the case where the stock moves up again. For both Strategies A and B, Scenario 1 results in further capital market gains and neither gains nor losses from biometric risks. Thus, the capital market gains are split as in $t = 1$ and the NAV as well as the bonus account increase. In Strategy B, the increase is less strong since the capital gains are lower due to the reduced stock investment. In Scenario 2, additional to capital market gains, mortality gains are realized and thus the NAV as well as the bonus account increase stronger compared to Scenario 1. This is true for both investment strategies. In the last Scenario 3, there are mortality losses which have to be fully carried by the company. Since mortality losses dominate capital market gains, the NAV is reduced. The bonus account is not affected by these mortality losses and hence, it is increased by the capital market gains and therefore coincides with the bonus account in Scenario 1. Again, this argumentation is true for Strategy A and B.

$\boldsymbol{t = 2, \omega = (uu, \bar{p})}$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	133			42.54			NAV	55.65	75.65	5.65	53.71	73.71	3.71
Bond	855			932.54			$\bar{V}$	900.00	820.00	950.00	900.00	820.00	950.00
							BA	32.35	92.35	32.35	21.37	81.37	21.37
Σ	988			975.08			Σ	988			975.08		

Table 6.6: Stochastic balance sheet for $t = 2$ when the stock goes up again.

In the second case, the stock moves downwards in $t = 2$ after it moved upwards in $t = 1$. Hence, the company realizes losses in the capital market. These losses are fully carried by the insurer and thus, in Scenario 1, the NAV decreases and the bonus account remains the same due to zero interest rates. Now, in Strategy B, the NAV is reduced less because of smaller capital losses due to the lower stock investment. In Scenario 2, mortality gains dominate capital market losses and hence, both the NAV and the bonus account increase. Scenario 3 leads to mortality losses and now the NAV becomes negative due to losses in the capital market as well as in biometric risk. Again, the bonus account is not affected.

$\boldsymbol{t = 2, \omega = (ud, \bar{p})}$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	95			30.38			NAV	33.8	53.8	-16.2	46.72	66.72	-3.28
Bond	855			932.54			$\bar{V}$	900.0	820.0	950.0	900.00	820.00	950.00
							BA	16.2	76.2	16.2	16.20	76.20	16.20
Σ	950			962.92			Σ	950			962.92		

Table 6.7: Stochastic balance sheet for $t = 2$ when the stock goes down.

In the third case, the stock moves upwards in $t = 2$ after it moved downwards in $t = 1$. Hence, the company realizes gains in the capital market. The interpretation of the results is along the lines of the interpretation given for Table 6.6.

$\boldsymbol{t = 2, \omega = (du, \bar{p})}$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	95			16.81			NAV	33.47	53.47	-16.53	31.44	51.44	-18.56
Bond	855			917.55			$\bar{V}$	900.00	820.00	950.00	900.00	820.00	950.00
							BA	16.53	76.53	16.53	2.92	62.92	2.92
Σ	950			934.36			Σ	950			934.36		

Table 6.8: Stochastic balance sheet for $t = 2$ when the stock goes up.

In the fourth case, the stock moves down again. Hence, the company realizes losses in the capital market. The interpretation of the results is along the lines of the interpretation given for Table 6.7.

$\boldsymbol{t = 2, \omega = (dd, \bar{p})}$:

ASSETS							**LIABILITIES**						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	57			10.09			NAV	12	32	-38	27.64	47.64	-22.36
Bond	855			917.55			$\bar{V}$	900	820	950	900.00	820.00	950.00
							BA	0	60	0	0	60.00	0
Σ	912			927.64			Σ	912			927.64		

Table 6.9: Stochastic balance sheet for $t = 2$ when the stock goes down again.

Now, let us compute the economic quantities decribed above in our model framework. The results for Strategy A and B under Scenario 1, 2 and 3 are given in Table 6.10. Results in brackets belong to an insurance company measuring risk by average value at risk; otherwise value at risk is applied for the computation of capital requirements. Whenever results for both risk measures are the same in all considered cases, we only state them once.

	Strategy/Scenario	$t = 0$	$t = 1$, $(u\cdot, \cdot)$	$t = 1$, $(d\cdot, \cdot)$
PVFP	A1	-16.2708	-8.0762	-8.2653
	A2	3.7292	11.9238	11.7347
	A3	-66.2708	-58.0762	-58.2653
	B1	-10.1228	-2.5832	-1.4625
	B2	9.8772	17.4168	18.5375
	B3	-60.1228	-52.5832	-51.4625
CRNHR	A1	3.6679 (3.9441)	3.6679 (3.9441)	3.6679 (3.9441)
	A2	1.0401 (1.2080)	1.0401 (1.2080)	1.0401 (1.2080)
	A3	2.7094 (2.7895)	2.7094 (2.7895)	2.7094 (2.7895)
	B1	3.5833 (3.9219)	3.5833 (3.9219)	3.5833 (3.9219)
	B2	1.0187 (1.1484)	1.0187 (1.1484)	1.0187 (1.1484)
	B3	2.6837 (2.7617)	2.6837 (2.7617)	2.6837 (2.7617)
VIF	A1	-19.9386 (-20.2149)	-11.7441 (-12.0203)	-11.9332 (-12.2094)
	A2	2.6891 (2.5212)	10.8837 (10.7158)	10.6946 (10.5267)
	A3	-68.9802 (-69.0603)	-60.7856 (-60.8657)	-60.9747 (-61.0548)
	B1	-13.7061 (-14.0447)	-6.1665 (-6.5051)	-5.0458 (-5.3844)
	B2	8.8585 (8.7288)	16.3981 (16.2684)	17.5188 (17.3891)
	B3	-62.8065 (-62.8845)	-55.2669 (-55.3449)	-54.1462 (-54.2242)
MCEV	A1	30.0613 (29.7851)	41.0559 (40.7797)	19.0668 (18.7906)
	A2	52.6891 (52.5212)	63.6837 (63.5158)	41.6946 (41.5267)
	A3	-18.9802 (-19.0603)	-7.9856 (-8.0657)	-29.9747 (-30.0548)
	B1	36.2939 (35.9553)	46.6335 (46.2949)	25.9542 (25.6156)
	B2	58.8585 (58.7288)	69.1981 (69.0684)	48.5188 (48.3891)
	B3	-12.8065 (-12.8845)	-2.4669 (-2.5449)	-23.1462 (-23.2242)
SCR	A1	19.00	19.00	19.00
	A2	19.00	-1.00	-1.00
	A3	19.00	69.00	69.00

	B1	19.00	6.0771	3.3619
	B2	19.00	-13.9229	-16.6381
	B3	19.00	56.0771	53.3619
RC	A1	10.9945	7.2559 (6.9797)	7.0668 (6.7906)
	A2	10.9945	9.8837 (9.7158)	9.6945 (9.5267)
	A3	10.9945	8.2144 (8.1343)	8.0253 (7.9452)
	B1	10.3397	-0.0893 (-0.4279)	-1.6839 (-2.0225)
	B2	10.3397	2.4753 (2.3455)	0.8808 (0.7510)
	B3	10.3397	0.8103 (0.7322)	-0.7842 (-0.8623)
ReC	A1	30.9332 (31.2094)	19.00	19.00
	A2	19.00 (19.00)	0	0
	A3	79.9747 (80.0548)	69.00	69.00
	B1	24.0458 (24.3844)	6.0772	3.3619
	B2	19.00 (19.00)	0	0
	B3	73.1462 (73.2242)	56.0772	53.3620
FS	A1	19.0668 (18.7906)	33.80	12.00
	A2	31.00 (31.00)	52.80	31.00
	A3	-29.9747 (-30.0548)	-16.20	-38.00
	B1	25.9542 (25.6156)	46.7228	27.6381
	B2	31.00 (31.00)	52.80	31.00
	B3	-23.1462 (-23.2242)	-3.2772	-22.3620

Table 6.10: Computation of economic quantities in a zero interest rate scenario.

The computation of PVFP(stoch) (in Table 6.10 only denoted by PVFP) does not depend on any risk measurement procedure at all and thus single values are given. Of course, mortality gains lead to a higher PVFP(stoch) than mortality losses. An active asset management modeled by Strategy B increases PVFP(stoch), since losses can be reduced efficiently due to the reduced stock investment. In Strategy A, PVFP(stoch) is higher if the stock moves upwards in $t = 1$ than in the case where the stock moves downwards, whereas in Strategy B it is the other way around, because the stock investment is even stronger reduced when the stock moves down in $t = 1$. The CRNHR can be reduced as well by applying Strategy B. Of course, CRNHR is higher if average value at risk is applied. In all settings, CRNHR is constant over time, since interest rates are zero and the non-hedgeable risk – stochastic mortality – is realized in the last period, i.e., from $t = 1$ to $t = 2$. The results for VIF and MCEV are direct consequences of the computations of NAV, PVFP(stoch) and CRNHR. In order to calculate ReC, we need to quantify SCR and RC first. Since possible balance sheets in $t = 1$ are equal for all strategies and scenarios, the SCR in $t = 0$ coincides in all cases. In $t = 1$, the SCR is reduced in Scenario 2, especially in Strategy B, because the mortality gains increase the NAV. In the case of mortality losses, i.e., Scenario 3, the NAV decreases and thus the SCR increases due to the downside risk. In Scenario 1, the SCR remains 19 in Strategy A whereas it is reduced in Strategy B as a result of active asset management. While applying Strategy A, the SCR coincides in both scenarios of the financial market (up or down in $t = 1$). Since losses in the NAV are the same if the stock moves downwards in $t = 2$, independent of the stock evolution in $t = 1$, the downside risk in both scenarios is equal. This is not true for Strategy B. In Strategy B, the stock investment is higher if the stock moved upwards in $t = 1$ than in the case where the stock moved downwards. The higher stock investment leads to higher losses if the stock moves downwards in $t = 2$. Thus, the downside risk is increased which leads to a higher SCR compared to the SCR we obtain if the stock moves downwards first. In $t = 0$, the RC is equal in all scenarios but differs slightly among the strategies. Again, in $t = 1$, the RC can be reduced efficiently by applying Strategy B. As we have seen before, the reduction is even more efficient when the stock moved downwards in $t = 1$, because of the reduction of the stock investment to $\alpha_1 = 0.0144$. The results for ReC are direct consequences of the computations of SCR, RC and VIF. In turn, FS follows from NAV and ReC. Observe, since ReC is equal to zero in Scenario 2 for both strategies, that the FS is equal to the NAV in this case.

Example 6.3.2. In Example 6.3.1, the computation of the TVOG was not needed in order to compute the MCEV, since we directly computed PVFP(stoch). By computing PVFP(CE), we obtain TVOG as residual, see Eq. (6.4). In the CE-scenario, the interest paid on all assets is the same; in our example this interest rate is $r = 0$. The best estimate for survival probabilities is $\bar{p} = \mathbb{E}_{P^2}[{}_1p_{x+1}] = 0.9$ in Scenario 1, $\bar{p} = \mathbb{E}_{P^2}[{}_1p_{x+1}] = 0.82$ in Scenario 2 and $\bar{p} = \mathbb{E}_{P^2}[{}_1p_{x+1}] = 0.95$ in Scenario 3. Hence, the balance sheet in $t = 2$ is given by Table 6.11.

ASSETS							**LIABILITIES**						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	95			33.78			NAV	50	70	0	50	70	0
Bond	855			916.22			V	900	820	950	900	820	950
							BA	0	60	0	0	60	0
Σ	950						Σ	950					

Table 6.11: Stochastic balance sheet for $t = 2$ in the CE-scenario.

Due to the CE-scenario, there are no gains from the investment in the capital market for both strategies. Observe that in Strategy B, the invesment is just shifted from the stock to the bond due to the investment rule, we defined. Thus, only mortality has an impact on the evolution of the NAV and the bonus account. In Scenario 2, there is a mortality gain in the amount of 80. This gain is split according to the 25/75 rule between the insurer and the policyholder. In Scenario 3, there are mortality losses of 50 that are fully captured by the company. In Scenario 1, there are neither gains nor losses.

Table 6.12 displays the resulting PVFP(CE) and TVOG. While PVFP(CE) varies for different scenarios but not in time, TVOG varies in time but is not affected by different mortality scenarios.

	Strategy/Scenario	$t = 0$	$t = 1$, $(u\cdot, \cdot)$	$t = 1$, $(d\cdot, \cdot)$
PVFP(CE)	A1	0	0	0
	A2	20	20	20
	A3	-50	-50	-50
	B1	0	0	0
	B2	20	20	20
	B3	-50	-50	-50
TVOG	A1	16.2708	8.0762	8.2653
	A2	16.2708	8.0762	8.2653
	A3	16.2708	8.0762	8.2653
	B1	10.1228	2.5832	1.4625
	B2	10.1228	2.5832	1.4625
	B3	10.1228	2.5832	1.4625

Table 6.12: Computation of TVOG.

Low Interest Rate Scenario. In the second interest rate scenario, we assume $r = 0.01$ and $r^* = 0.02$. In Example 6.3.3 below, we analyze the economic quantities compared to Example 6.3.1 only. The stochastic balance sheet projections are given in Section 6.6.

Example 6.3.3. We assume $\mathrm{NAV}_0 = 50$ and compute $V_0 = \pi \approx 865.05$. The interpretation of the results in Table 6.13 is along the lines of the interpretation of the values in the previous example. However, values such as PVFP(stoch), VIF and MCEV are lower whereas costs such as SCR and RC are higher in nearly all cases. This is due to the interest rate losses coming from the low interest rates in the financial market and which have to be covered by the NAV of the company. Only in the case of mortality gains, i.e., Scenario 2, these gains still dominate interest rate losses, at least when the stock moved upwards (compare the SCRs). Moreover, we emphasize that the martingale measure in this example differs from the martingale measure in the previous example, where the upwards and the downwards movement of the stock had the same probability, due to a non-zero interest rate. Here, an upwards movement of the stock is more likely. Furthermore, the interest rate of $r = 0.01$ leads to non-constant CRNHR due to discounting. Besides, RC is not equal in all scenarios in $t = 0$, but depends on mortality gains/losses.

	Strategy/Scenario	$t = 0$	$t = 1$, $(u\cdot, \cdot)$	$t = 1$, $(d\cdot, \cdot)$
PVFP	A1	-25.2905	-12.7479	-12.7894
	A2	-0.5534	12.2827	12.1443
	A3	-74.3053	-62.2528	-62.2944
	B1	-21.3395	-8.8140	-8.7362
	B2	3.4907	16.2959	16.3075
	B3	-70.3543	-58.3189	-58.2411
CRNHR	A1	3.5208 (3.8468)	3.5560 (3.8853)	3.5560 (3.8853)
	A2	0.9037 (1.2377)	0.9142 (1.2525)	0.9121 (1.2475)
	A3	2.6726 (2.7563)	2.6993 (2.7839)	2.6993 (2.7839)
	B1	3.5184 (3.8405)	3.5536 (3.8789)	3.5536 (3.8789)
	B2	0.9065 (1.2425)	0.9165 (1.2566)	0.9146 (1.2531)
	B3	2.6696 (2.7405)	2.6963 (2.7679)	2.6963 (2.7679)

VIF	A1	-28.8113 (-29.1373)	-16.3039 (-16.6332)	-16.3454 (-16.6747)
	A2	-1.4571 (-1.7911)	11.3685 (11.0302)	11.2322 (10.8968)
	A3	-76.9779 (-77.0616)	-64.9521 (-65.0367)	-64.9937 (-65.0783)
	B1	-24.8579 (-25.1800)	-12.3676 (-12.6929)	-12.2898 (-12.6151)
	B2	2.5842 (2.2482)	15.3794 (15.0393)	15.3929 (15.0544)
	B3	-73.0239 (-73.0948)	-61.0152 (-61.0868)	-60.9374 (-61.0090)
MCEV	A1	21.1887 (20.8627)	35.0738 (34.7445)	6.2880 (5.9587)
	A2	48.5429 (48.2089)	62.7462 (62.4079)	33.8656 (33.5302)
	A3	-26.9779 (-27.0616)	-13.5744 (-13.6590)	-42.3603 (-42.4449)
	B1	25.1421 (24.8200)	39.0101 (38.6848)	10.3436 (10.0183)
	B2	52.5842 (52.2482)	66.7571 (66.4170)	38.0263 (37.6878)
	B3	-23.0239 (-23.0948)	-9.6375 (-9.7091)	-38.3040 (-38.3756)
SCR	A1	27.8666	28.3012	27.8566
	A2	27.8666	-4.9341	-5.3787
	A3	27.8666	78.3012	77.8566
	B1	27.8666	15.1757	10.1532
	B2	27.8666	-15.5208	-16.3077
	B3	27.8666	65.1757	60.1532
RC	A1	15.4479 (15.5079)	12.0362 (11.7643)	11.8375 (11.5655)
	A2	14.8161 (14.8385)	13.9576 (13.6948)	13.7609 (13.5010)
	A3	15.8127 (15.8315)	13.2699 (13.2035)	13.0712 (13.0048)
	B1	15.3425 (15.4033)	2.8439 (2.5767)	-1.8134 (-2.0806)
	B2	14.6975 (14.7245)	4.7636 (4.4991)	0.1083 (-0.1547)
	B3	15.7070 (15.7270)	4.0779 (4.0257)	-0.5794 (-0.6316)
ReC	A1	44.2592 (44.6452)	28.3401 (28.3975)	28.1829 (28.2402)
	A2	27.8666 (27.8666)	2.5891 (2.6646)	2.5287 (2.6042)
	A3	92.7906 (92.8931)	78.3012 (78.3012)	78.0649 (78.0831)
	B1	40.2004 (40.5833)	15.2115 (15.2696)	10.4764 (10.5345)
	B2	27.8666 (27.8666)	0 (0)	0 (0)
	B3	88.7309 (88.8218)	65.1757 (65.1757)	60.3580 (60.3744)
FS	A1	5.7408 (5.3548)	23.0376 (22.9802)	-5.5495 (-5.6068)
	A2	22.1334 (22.1334)	48.7886 (48.7131)	20.1047 (20.0292)
	A3	-42.7906 (-42.8931)	-26.9235 (-26.9235)	-55.4315 (-55.4497)
	B1	9.7996 (9.4167)	36.1662 (36.1081)	12.1570 (12.0989)
	B2	22.1334 (22.1334)	51.3777 (51.3777)	22.6334 (22.6334)
	B3	-38.7309 (-38.8218)	-13.7980 (-13.7980)	-37.7246 (-37.7440)

Table 6.13: Computation of economic quantities in a low interest rate scenario.

High Interest Rate Scenario. In the third interest rate scenario, we assume $r = 0.03$ and $r^* = 0.02$. Again, in Example 6.3.4 below, we analyze the economic quantities compared to Examples 6.3.1 & 6.3.3 only. The stochastic balance sheet projections are given in Section 6.6.

Example 6.3.4. We assume $\mathrm{NAV}_0 = 50$ and compute $V_0 = \pi \approx 865.05$. The interpretation of the results in Table 6.14 is along the lines of the interpretation of the values in the previous examples. Values such as PVFP(stoch), VIF and MCEV are higher whereas costs such as SCR and RC are lower in most of the cases. This is due to the interest rate gains coming from the high interest rates in the financial market. Again, we point out that the martingale measure in this example differs from the martingale measures in the two examples before. Besides, we note the dependence of the results on the underlying Beta distributions.

	Strategy/Scenario	$t = 0$	$t = 1, (u\cdot, \cdot)$	$t = 1, (d\cdot, \cdot)$
PVFP	A1	-7.3835	-3.6910	-3.6635
	A2	13.3952	17.7364	17.7042
	A3	-54.5133	-52.2347	-52.2072
	B1	-3.7266	0.0022	0.2023
	B2	13.3952	17.7364	17.7042
	B3	-50.8564	-48.5415	-48.3414
CRNHR	A1	3.4494 (3.7444)	3.5529 (3.8567)	3.5529 (3.8567)
	A2	0.8580 (1.0250)	0.8838 (1.0565)	0.8834 (1.0549)
	A3	2.5423 (2.6263)	2.6186 (2.7051)	2.6186 (2.7051)
	B1	3.4369 (3.7157)	3.5400 (3.8272)	3.5400 (3.8272)
	B2	0.8458 (0.9745)	0.8716 (1.0058)	0.8712 (1.0011)
	B3	2.5328 (2.6046)	2.6088 (2.6828)	2.6088 (2.6828)
VIF	A1	-10.8329 (-11.1279)	-7.2439 (-7.5477)	-7.2164 (-7.5202)
	A2	12.5372 (12.3702)	16.8526 (16.6799)	16.8208 (16.6493)
	A3	-57.0556 (-57.1396)	-54.8533 (-54.9398)	-54.8258 (-54.9123)
	B1	-7.1635 (-7.4423)	-3.5378 (-3.8250)	-3.3377 (-3.6249)
	B2	12.5494 (12.4207)	16.8648 (16.7306)	16.8330 (16.7031)
	B3	-53.3892 (-53.4610)	-51.1503 (-51.2243)	-50.9502 (-51.0242)
MCEV	A1	39.1671 (38.8721)	46.5945 (46.2907)	31.8879 (31.5841)
	A2	62.5372 (62.3702)	70.6910 (70.5183)	55.9251 (55.7536)
	A3	-7.0556 (-7.1396)	-1.0149 (-1.1014)	-15.7215 (-15.8080)
	B1	42.8365 (42.5577)	50.3006 (50.0134)	35.7666 (35.4794)
	B2	62.5494 (62.4207)	70.7032 (70.5690)	55.9373 (55.8074)
	B3	-3.3892 (-3.4610)	2.6881 (2.6141)	-11.8459 (-11.9199)
SCR	A1	12.3957	12.5567	11.6736
	A2	12.3957	-14.9876	-15.4028
	A3	12.3957	62.5567	61.6736
	B1	12.3957	1.1320	0.6587
	B2	12.3957	-17.1343	-17.3683
	B3	12.3957	51.1320	50.6587
RC	A1	8.8703 (9.0541)	5.3329 (5.2093)	4.9230 (4.7994)
	A2	7.0195 (7.0746)	5.8654 (5.8924)	5.4558 (5.4841)
	A3	9.8998 (9.9503)	7.3556 (7.3185)	6.9457 (6.9086)
	B1	8.6628 (8.8308)	-2.3839 (-2.5067)	-2.2114 (-2.3342)
	B2	7.0141 (7.0447)	-1.4714 (-1.5580)	-1.2915 (-1.3752)
	B3	9.6839 (9.7335)	-0.3730 (-0.3982)	-0.2005 (-0.2257)
ReC	A1	19.7032 (20.1820)	12.5768 (12.7570)	12.1394 (12.3196)
	A2	12.3957 (12.3957)	0 (0)	0 (0)
	A3	66.9554 (67.0899)	62.5567 (62.5567)	61.7715 (61.8209)
	B1	15.8263 (16.2731)	1.1539 (1.3183)	1.1263 (1.2907)

	B2	12.3957 (12.3957)	0 (0)	0 (0)
	B3	63.0731 (63.1945)	51.1320 (51.1320)	50.7497 (50.7985)
FS	A1	30.2968 (29.8180)	41.2616 (41.0814)	26.9649 (26.7847)
	A2	37.6043 (37.6043)	53.8384 (53.8384)	39.1043 (39.1043)
	A3	-16.9554 (-17.0899)	-8.7183 (-8.7183)	-22.6672 (-22.7166)
	B1	34.1737 (33.7269)	52.6845 (52.5201)	37.9780 (37.8136)
	B2	37.6043 (37.6043)	53.8384 (53.8384)	39.1043 (39.1043)
	B3	-13.0731 (-13.1945)	2.7064 (2.7064)	-11.6454 (-11.6942)

Table 6.14: Computation of economic quantities in a high interest rate scenario.

Comparative Statics. Finally, our main results concerning the MCEV are clearly shown in Figures 6.4 - 6.7. Figure 6.4 displays the impact of different interest rate scenarios on the MCEV in $t = 0$. In particular, the prevailing situation of a low interest rate environment is illustrated. Obviously, low interest rates in the financial market reduce the MCEV since the higher actuarial interest rate requires equivalent gains in the financial market in order to avoid losses. Moreover, we find that the active investment strategy of rearranging the portfolio (Strategy B) dominates the passive buy & hold strategy (Strategy A), i.e., the MCEV can be increased when the asset portfolio is rearranged according to the considered investment rule. As an example, Figure 6.4 shows the results in the case of neither mortality gains nor losses (Scenario 1).

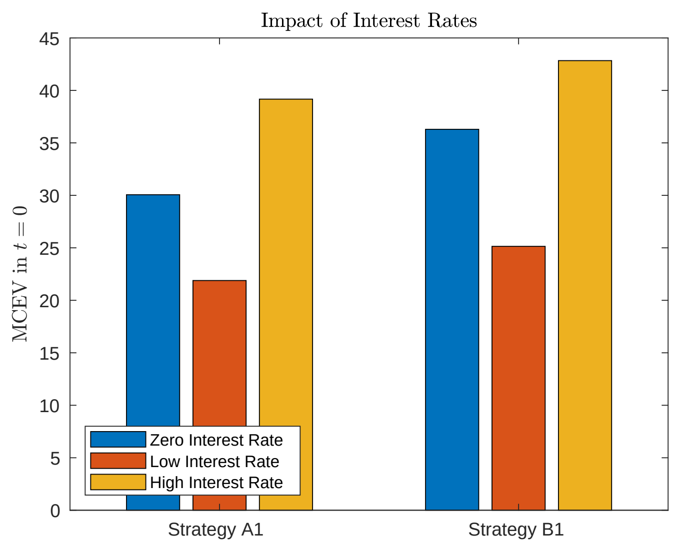


Figure 6.4: Comparison of MCEV in $t = 0$ for different interest rate scenarios in Strategy A1 and Strategy B1.

Figure 6.5 presents the impact of different investment strategies on the MCEV in $t = 0$ in detail. Again, we see that the active investment strategy (Strategy B) leads to a higher MCEV than the buy & hold strategy (Strategy A). This is true for each interest rate and each mortality scenario.

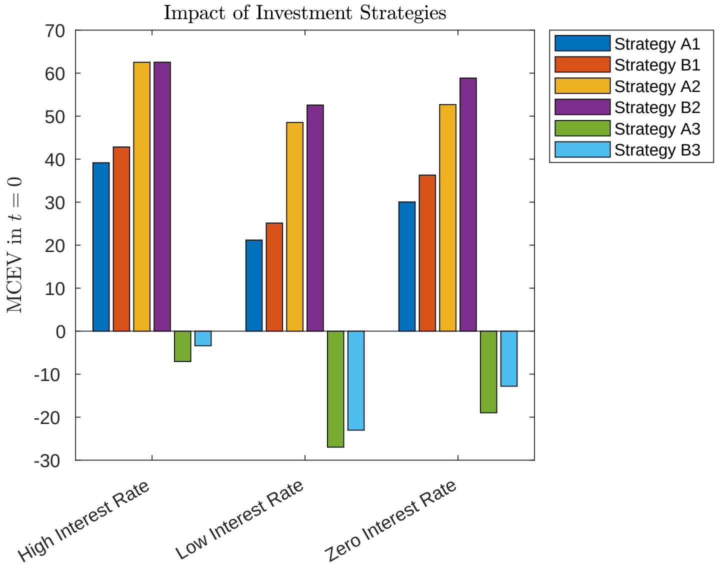


Figure 6.5: Comparison of MCEV in $t = 0$ for Strategies A & B in three interest rate scenarios.

The impact of stochastic mortality on the MCEV in $t = 0$ is shown is Figure 6.6. We choose the results of Strategy A as an example. Equivalent results are obtained for Strategy B. As expected, mortality gains increase the MCEV and losses lead to reductions. In particular, we observe that including stochastic mortality gains and losses varies the MCEV significantly in each interest rate scenario.

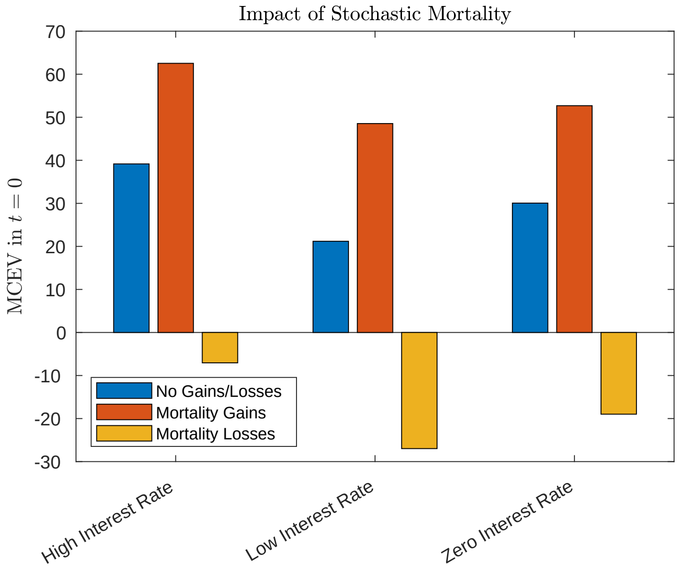


Figure 6.6: Comparison of MCEV in $t = 0$ for different stochastic mortality scenarios in three interest rate scenarios.

Figure 6.7 displays the composition of the MCEV in $t = 0$. As an example, we choose the results obtained for Strategy B in Scenario 2, i.e., the insurance company rearranges its asset portfolio and mortality gains are assumed. Low interest rates lead to a higher required capital whereas in the high interest rate scenario the free surplus can be increased.

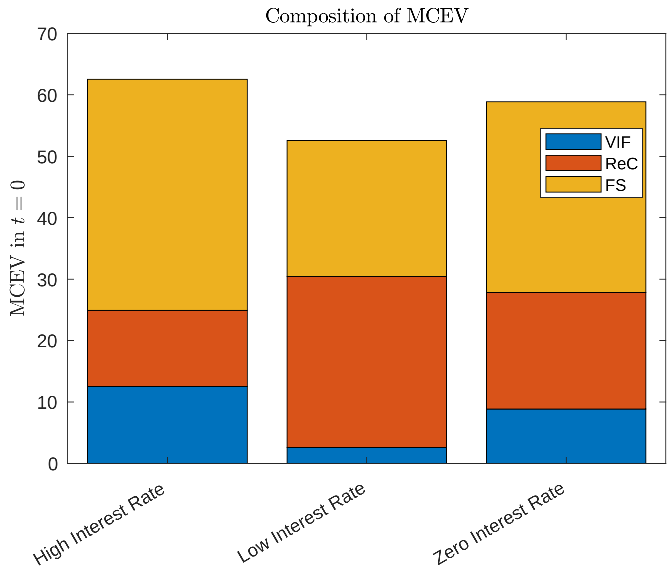


Figure 6.7: Comparison of the composition of the MCEV in $t = 0$ in VIF, ReC and FS in three interest rate scenarios.

6.4 | Conclusion

We provided a comprehensive overview of the MCEV and the valuation of its components. These valuations are based on the interplay of actuarial and financial approaches which is explicitly illustrated in this chapter. While presenting the computation of the MCEV, we provided a complete asset-liability management model that includes, first, an interest rate and a stock model for the asset model. Second, a product and a stochastic mortality model for the liability model are incorporated. Third, we suggested investment and surplus distribution strategies in order to investigate the influence of a management model. Moreover, we compared different regulatory systems by applying value at risk as well as average value at risk for risk measurement and the determination of capital requirements. Our numerical results, in particular, indicate that active asset-liability management dominates passive asset-liability management in the sense that capital requirements can be reduced and, in turn, the free surplus is increased. Future research needs to adjust each component of our model in order to become more realistic. We emphasize that the complexity will increase tremendously.

6.5 | Appendix: Conditional Risk Measures

In this section, we provide a brief introduction to risk measures and their extension to conditional risk measures. To this end, we consider a discrete time model on a probability space $(\Omega, \mathcal{H}, P)$ with dates $t = 0, 1, \ldots, T$ and final time horizon $T \in \mathbb{N}$. The information flow is captured by the filtration $\mathbb{H} = (\mathcal{H}_t)_{t=0,1,\ldots,T}$, i.e.,

$$\{\emptyset, \Omega\} = \mathcal{H}_0 \subset \mathcal{H}_1 \subset \ldots \subset \mathcal{H}_T = \mathcal{H}.$$

We denote by L^0, resp. L^∞, the space of real-valued random variables, resp. bounded random variables, on the given probability space.

Definition 6.5.1. A mapping $\rho : L^\infty \to \mathbb{R}$ is called a *monetary risk measure* if the following properties are satisfied:

(i) *Monotonicity*: If $X \geq Y$ P-almost surely, then $\rho(X) \leq \rho(Y)$.
(Better payoff profiles are less risky.)

(ii) *Cash-invariance:* If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.
(Adding a fixed amount m to the risky position decreases the risk exactly by this amount.)

Examples of risk measures are provided in Appendix A, e.g, in Example A.0.3.

The definition of monetary risk measures is static. A dynamic extension to conditional, resp. dynamic, risk measures incorporates the evolution of time. The assessment of risk is conditional on available information up to time t.
In the subsequent exposition, we follow Detlefsen & Scandolo (2005). We define the subspaces

$$L^0_{\mathcal{H}_t} = \left\{ X \in L^0 \;\middle|\; X \text{ is } \mathcal{H}_t\text{-measurable} \right\}$$

and

$$L^\infty_{\mathcal{H}_t} = L^\infty \cap L^0_{\mathcal{H}_t},$$

where $\mathcal{H}_t \subseteq \mathcal{H}, t \in \{0, 1, \ldots, T\}$.

Definition 6.5.2. (i) A mapping $\rho_t : L^\infty \to L^0_{\mathcal{H}_t}$, $t \in \{0, 1, \ldots, T\}$, is called an $\mathcal{H}_t$-*conditional risk measure.*

(ii) A *dynamic risk measure* is a family $(\rho_t)_{t=0}^T$ such that ρ_t is an $\mathcal{H}_t$-conditional risk measure.

Remark 6.5.3. (i) For $t = 0$, $\rho_0 : L^\infty \to L^0_{\mathcal{H}_0} = \mathbb{R}$ corresponds to the classical scalar risk measure in Definition 6.5.1, i.e., the unconditional case.

(ii) The conditional risk $\rho_t(X)$ is a random variable in $L^0_{\mathcal{H}_t}$. When the scenario $\omega \in \Omega$ occurs, we assess the risk $\rho_t(X)(\omega)$ to the random position X.

6.6 | Appendix: Computations of Section 6.3.2

In this section, we provide the stochastic balance sheet projections of Examples 6.3.3 & 6.3.4. We start with Example 6.3.3:

$\boldsymbol{t=0}$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	91.50			91.50			NAV	50.00			50.00		
Bond	823.55			823.55			V	865.05			865.05		
							BA	0			0		
Σ	915.05						Σ	915.05					

Table 6.15: Stochastic balance sheet for $t = 0$ (Low interest rate).

$\boldsymbol{t=1, \omega=(u\cdot, {}_1p_x)}$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	109.81			109.81			NAV	51.38			51.38		
Bond	831.78			831.78			$\bar{V}$	882.35			882.35		
							BA	7.86			7.86		
Σ	941.59						Σ	941.59					

Table 6.16: Stochastic balance sheet for $t = 1$ when the stock goes up (Low interest rate).

$\boldsymbol{t=1, \omega=(d\cdot, {}_1p_x)}$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	73.20			73.20			NAV	22.63			22.63		
Bond	831.78			831.78			$\bar{V}$	882.35			882.35		
							BA	0			0		
Σ	904.98						Σ	904.98					

Table 6.17: Stochastic balance sheet for $t = 1$ when the stock goes down (Low interest rate).

$\boldsymbol{t=2, \omega=(uu, \bar{p})}$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	128.11			41.43			NAV	52.70	70.92	2.70	48.56	69.18	-1.44
Bond	840.10			915.14			$\bar{V}$	900.00	820.00	950.00	900.00	820.00	950.00
							BA	15.51	77.29	15.51	8.01	67.39	8.01
Σ	968.21			956.57			Σ	968.21			956.57		

Table 6.18: Stochastic balance sheet for $t = 2$ when the stock goes up again (Low interest rate).

$t = 2, \omega = (ud, \bar{p})$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	91.5			29.59			NAV	23.59	56.82	-26.41	36.72	67.41	-13.28
Bond	840.1			915.14			$\bar{V}$	900.00	820.00	950.00	900.00	820.00	950.00
							BA	8.01	54.78	8.01	8.01	57.32	8.01
Σ	931.6			944.73			Σ	931.6			944.73		

Table 6.19: Stochastic balance sheet for $t = 2$ when the stock goes down (Low interest rate).

$t = 2, \omega = (du, \bar{p})$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	91.5			6.39			NAV	23.73	41.48	-26.27	15.26	39.48	-34.74
Bond	840.1			908.87			$\bar{V}$	900.00	820.00	950.00	900.00	820.00	950.00
							BA	7.87	70.12	7.87	0	55.78	0
Σ	931.6			915.26			Σ	931.6			915.26		

Table 6.20: Stochastic balance sheet for $t = 2$ when the stock goes up (Low interest rate).

$t = 2, \omega = (dd, \bar{p})$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	54.9			3.84			NAV	-5	28.24	-55	12.71	39.17	-37.29
Bond	840.1			908.87			$\bar{V}$	900	820.00	950	900.00	820.00	950.00
							BA	0	46.76	0	0	53.54	0
Σ	895			912.71			Σ	895			912.71		

Table 6.21: Stochastic balance sheet for $t = 2$ when the stock goes down again (Low interest rate).

For Example 6.3.4, we obtain:

$t = 0$:

ASSETS							LIABILITIES						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	91.50			91.50			NAV	50.00			50.00		
Bond	823.55			823.55			V	865.05			865.05		
							BA	0			0		
Σ	915.05						Σ	915.05					

Table 6.22: Stochastic balance sheet for $t = 0$ (High interest rate).

$t = 1, \omega = (u\cdot, {}_1p_x)$:

ASSETS			LIABILITIES		
Strategy	A	B		A	B
Scenario	1 2 3	1 2 3		1 2 3	1 2 3
Stock	109.81	109.81	NAV	53.84	53.84
Bond	848.25	848.25	$\bar{V}$	882.35	882.35
			BA	21.87	21.87
Σ	958.06		Σ	958.06	

Table 6.23: Stochastic balance sheet for $t = 1$ when the stock goes up (High interest rate).

$t = 1, \omega = (d\cdot, {}_1p_x)$:

ASSETS			LIABILITIES		
Strategy	A	B		A	B
Scenario	1 2 3	1 2 3		1 2 3	1 2 3
Stock	73.20	73.20	NAV	39.10	39.10
Bond	848.25	848.25	$\bar{V}$	882.35	882.35
			BA	0	0
Σ	921.45		Σ	921.45	

Table 6.24: Stochastic balance sheet for $t = 1$ when the stock goes down (High interest rate).

$t = 2, \omega = (uu, \bar{p})$:

ASSETS			LIABILITIES						
Strategy	A	B		A			B		
Scenario	1 2 3	1 2 3		1	2	3	1	2	3
Stock	128.11	44.19	NAV	57.73	76.0	7.73	56.24	74.50	6.24
Bond	873.70	947.79	$\bar{V}$	900.00	820.0	950.00	900.00	820.00	950.00
			BA	44.08	105.8	44.08	35.74	97.48	35.74
Σ	1001.8	991.98	Σ	1001.8			991.98		

Table 6.25: Stochastic balance sheet for $t = 2$ when the stock goes up again (High interest rate).

$t = 2, \omega = (ud, \bar{p})$:

ASSETS			LIABILITIES						
Strategy	A	B		A			B		
Scenario	1 2 3	1 2 3		1	2	3	1	2	3
Stock	91.5	31.56	NAV	42.9	70.44	-7.1	54.32	72.59	4.32
Bond	873.7	947.79	$\bar{V}$	900.0	820.00	950.0	900.00	820.00	950.00
			BA	22.3	74.76	22.3	25.03	86.76	25.03
Σ	965.2	979.35	Σ	965.2			979.35		

Table 6.26: Stochastic balance sheet for $t = 2$ when the stock goes down (High interest rate).

$t = 2, \omega = (du, \bar{p})$:

ASSETS			LIABILITIES						
Strategy	A	B		A			B		
Scenario	1 2 3	1 2 3		1	2	3	1	2	3
Stock	91.5	28.01	NAV	42.71	60.74	-7.29	41.17	59.19	-8.83
Bond	873.7	926.02	$\bar{V}$	900.0	820.00	950.00	900.00	820.00	950.00
			BA	22.49	84.46	22.49	12.86	74.84	12.86
Σ	965.2	954.03	Σ	965.2			954.03		

Table 6.27: Stochastic balance sheet for $t = 2$ when the stock goes up (High interest rate).

$t = 2, \omega = (dd, \bar{p})$:

ASSETS							**LIABILITIES**						
Strategy	A			B				A			B		
Scenario	1	2	3	1	2	3		1	2	3	1	2	3
Stock	54.9			16.81			NAV	28.6	55.68	-21.4	39.62	57.65	-10.38
Bond	873.7			926.02			$\bar{V}$	900.0	820.00	950.0	900.00	820.00	950.00
							BA	0	52.92	0	3.21	65.18	3.21
Σ	928.6			942.83			Σ	928.6			942.83		

Table 6.28: Stochastic balance sheet for $t = 2$ when the stock goes down again (High interest rate).

A | A Short Introduction to Risk Measures

For the convenience of the reader, we review the basic definitions of scalar-valued risk measures. For detailed information on risk measures and their properties, we refer, e.g., to Artzner et al. (1999), Föllmer & Schied (2011), McNeil et al. (2015) and Föllmer & Weber (2015).

We denote by $\mathcal{X}(\mathbb{R})$ a vector space of measurable, real-valued functions on a measurable space $(\Omega, \mathcal{F})$ that contains the constants. If P is a probability measure on $(\Omega, \mathcal{F})$, typical examples of $\mathcal{X}(\mathbb{R})$ are L^p-spaces, $p \in [1, \infty]$, where P-almost sure equal functions are identified with each other.

Definition A.0.1. A mapping $\rho : \mathcal{X}(\mathbb{R}) \to \mathbb{R}$ is called *monetary risk measure* if it satisfies

(i) *Monotonicity*: If $X(\omega) \geq Y(\omega), \forall \omega \in \Omega$, then

$$\rho(X) \leq \rho(Y).$$

(Better payoff profiles are less risky.)

(ii) *Cash-invariance:* If $m \in \mathbb{R}$, then

$$\rho(X + m) = \rho(X) - m.$$

(Adding a fixed cash amount m to the risky position decreases the risk exactly by this amount.)

Further properties are:

(iii) *Convexity:* If $\lambda \in [0, 1]$ and $X, Y \in \mathcal{X}(\mathbb{R})$, then

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y).$$

(Diversification does not increase risk.)
For monetary risk measures convexity is equivalent to *Quasi-convexity:*

$$\rho(\lambda X + (1 - \lambda)Y) \leq \max\{\rho(X), \rho(Y)\}.$$

(iv) *Positive homogeneity:* If $\lambda \geq 0$, then

$$\rho(\lambda X) = \lambda\rho(X).$$

(Risk increases in a linear way.)

(v) *Subadditivity:*

$$\rho(X+Y) \leq \rho(X) + \rho(Y).$$

(Risk of the aggregate position is bounded by the sum of the individual risks.)

Any two of the properties convexity, positive homogeneity and subadditivity imply the third. A monetary risk measure satisfying additionally property (iii) is called *convex risk measure*. A convex risk measure satisfying additionally property (iv) is called *coherent risk measure*. In general, a mapping $\rho : \mathcal{X}(\mathbb{R}) \to \mathbb{R}, X \mapsto \rho(X)$ is called *risk measure*.

In particular, any monetary risk measure corresponds to its *acceptance set*, $\mathcal{A} = \{X \in \mathcal{X} \mid \rho(X) \leq 0\}$, from which it can be recovered via

$$\rho(X) = \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}.$$

Thus, a monetary risk measure can be viewed as a capital requirement: $\rho(X)$ is the minimal capital that has to be added to the position X to make it acceptable.

For later references, we define (lower and upper) quantiles.

Definition A.0.2. The upper and lower λ-quantile, $\lambda \in (0,1)$, of a random variable X on $(\Omega, \mathcal{F}, P)$ are defined by

$$q_X^+(\lambda) := \inf\{x \in \mathbb{R} \mid P(X \leq x) > \lambda\} = \sup\{x \in \mathbb{R} \mid P(X < x) \leq \lambda\}$$

and

$$q_X^-(\lambda) := \sup\{x \in \mathbb{R} \mid P(X < x) < \lambda\} = \inf\{x \in \mathbb{R} \mid P(X \leq x) \geq \lambda\},$$

respectively. A λ-quantile is any $q \in \mathbb{R}$ with

$$P(X \leq q) \geq \lambda \quad \text{and} \quad P(X < q) \leq \lambda.$$

The set of all λ-quantiles of X is an interval $\left[q_X^-(\lambda), q_X^+(\lambda)\right]$.

We recall well-known risk measures with different properties.

Example A.0.3. (i) Standard deviation risk measure:

$$\rho(X) := -\mathbb{E}[X] + \gamma\sqrt{\mathrm{Var(X)}}, \quad \gamma > 0.$$

Since the standard deviation is not monotone, it is not a monetary risk measure in general.

(ii) Value at risk: The most commonly used risk measure in practice - and in particular the prescribed risk measure for *Solvency II* purposes - is *value at risk* (V@R). For a given level $\lambda \in (0,1)$, we denote by $\mathrm{V@R}_\lambda$ the monetary risk measure defined by the acceptance set

$$\mathcal{A}_{\mathrm{V@R}_\lambda} = \{X \in \mathcal{X} \mid P(X < 0) \leq \lambda\}. \tag{A.1}$$

For a financial position X, the value $\mathrm{V@R}_\lambda(X)$ specifies the smallest monetary amount that needs to be added to X such that the probability of a loss becomes smaller than λ:

$$\mathrm{V@R}_\lambda(X) := \inf\{m \in \mathbb{R} \mid P(X + m < 0) \leq \lambda\} = -q_X^+(\lambda).$$

$\mathrm{V@R}_\lambda$ has two main deficiencies: Firstly, value at risk is not a convex risk measure and may thus penalize diversification beyond the setting of Gaussian or more generally elliptic financial positions. Secondly, $\mathrm{V@R}_\lambda$ neglects extreme losses that occur with small probability. These deficiencies of value at risk were a major reason to develop a systematic theory of coherent and convex risk measures, as initiated by Artzner et al. (1999) and Föllmer & Schied (2002). Further, $\mathrm{V@R}_\lambda$ is positively homogeneous, but not subadditive in general.

(iii) Average value at risk: Another basic example is the *average value at risk* (AV@R), also known as *conditional value at risk*, *tail value at risk*, or *expected shortfall*, which plays a prominent role in the *Swiss Solvency Test*. The average value at risk at level $\lambda \in (0,1]$ is defined by

$$\mathrm{AV@R}_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda \mathrm{V@R}_\alpha(X)\ d\alpha.$$

$\mathrm{AV@R}_\lambda(X)$ corresponds, under weak technical conditions, to the conditional expectation of a loss beyond the $\mathrm{V@R}_\lambda(X)$. In contrast to value at risk, $\mathrm{AV@R}_\lambda$ accounts for extreme losses per definition, and it provides incentives for diversification. More precisely, $\mathrm{AV@R}_\lambda$ is a coherent measure of risk.

(iv) Range value at risk: Cont, Deguest & Scandolo (2010) suggest an alternative to V@R and AV@R, called *range value at risk* (RV@R). Letting $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$, they define

$$\mathrm{RV@R}_{\alpha,\beta}(X) = \tfrac{1}{\beta}\int_\alpha^{\alpha+\beta} \mathrm{V@R}_\gamma(X)\, d\gamma.$$

Note that the limiting cases of $\mathrm{RV@R}_{\alpha,\beta}$ correspond to $\mathrm{V@R}_\alpha$ for $\beta \to 0$ and $\mathrm{AV@R}_\beta$ for $\alpha \to 0$. Like V@R, RV@R is a non convex risk measure, and it may thus penalize diversification.

The following lemma shows V@R as the lower quantile of the random variable $-X$.

Lemma A.0.4. *The value at risk at level $\lambda \in (0,1)$ is given by*

$$V@R_\lambda(X) = q_{-X}^-(1-\lambda).$$

Proof. It is

$$
\begin{aligned}
\mathrm{V@R}_\lambda(X) &= -q_X^+(\lambda) = -\sup\{x \in \mathbb{R} \mid P(X < x) \le \lambda\} = \inf\{y \in \mathbb{R} \mid P(X < -y) \le \lambda\} \\
&= \inf\{y \in \mathbb{R} \mid P(-X > y) \le \lambda\} = \inf\{y \in \mathbb{R} \mid P(-X \le y) \ge 1 - \lambda\} \\
&= q_{-X}^-(1-\lambda)
\end{aligned}
$$

□

For normally distributed random variables, we derive the risk for the risk measures V@R and AV@R explicitly.

Lemma A.0.5. *Let X be a normally distributed random variable, i.e., $X \sim \mathcal{N}(\mu, \sigma^2)$.*

(i) $V@R_\lambda$ is given explicitly by

$$V@R_\lambda(X) = -\mathbb{E}[X] + \Phi^{-1}(1-\lambda)\sqrt{\mathrm{Var}(X)},$$

where Φ^{-1} is the quantile function of the standard normal distribution.

(ii) $AV@R_\lambda$ is given explicitly by

$$AV@R_\lambda(X) = -\mathbb{E}[X] + \frac{\phi(\Phi^{-1}(1-\lambda))}{\lambda}\sqrt{\mathrm{Var}(X)},$$

where Φ^{-1} is the quantile function of the standard normal distribution and ϕ the corresponding density.

Proof. (i) It is

$$
\begin{aligned}
\mathrm{V@R}_\lambda(X) &= \inf\{m \in \mathbb{R} \mid P(X + m < 0) \le \lambda\} \\
&= \inf\left\{m \in \mathbb{R} \,\middle|\, P\left(\frac{X - \mathbb{E}[X]}{\sqrt{\mathrm{Var}(X)}} < \frac{-m - \mathbb{E}[X]}{\sqrt{\mathrm{Var}(X)}}\right) \le \lambda\right\} \\
&= \inf\left\{m \in \mathbb{R} \,\middle|\, \Phi\left(\frac{-m - \mathbb{E}[X]}{\sqrt{\mathrm{Var}(X)}}\right) \le \lambda\right\} \\
&= \inf\left\{m \in \mathbb{R} \,\middle|\, m \ge -\mathbb{E}[X] + \Phi^{-1}(1-\lambda)\sqrt{\mathrm{Var}(X)}\right\} \\
&= -\mathbb{E}[X] + \Phi^{-1}(1-\lambda)\sqrt{\mathrm{Var}(X)}
\end{aligned}
$$

(ii) For a continuous distribution function, $\mathrm{AV@R}_\lambda$ can be represented by

$$\mathrm{AV@R}_\lambda(X) = \mathbb{E}\left[-X \mid -X \ge \mathrm{V@R}_\lambda(X)\right].$$

If $\tilde{X}$ is standard normally distributed, i.e., $\tilde{X} \sim \mathcal{N}(0,1)$, then

$$
\begin{aligned}
\mathrm{AV@R}_\lambda(\tilde{X}) &= \mathbb{E}\left[-\tilde{X} \mid -\tilde{X} \ge \mathrm{V@R}_\lambda(\tilde{X})\right] = \mathbb{E}\left[-\tilde{X} \mid -\tilde{X} \ge -\Phi^{-1}(\lambda)\right] \\
&= \frac{1}{P\left(\tilde{X} \le \Phi^{-1}(\lambda)\right)}\,\mathbb{E}\left[-\tilde{X} \cdot \mathbb{1}_{\{\tilde{X} \le \Phi^{-1}(\lambda)\}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{P\left(\tilde{X} \leq \Phi^{-1}(\lambda)\right)} \int_{-\infty}^{\Phi^{-1}(\lambda)} -y\,\phi(y)\,dy \\
&= \frac{1}{\Phi\left(\Phi^{-1}(\lambda)\right)} \int_{\Phi^{-1}(1-\lambda)}^{\infty} y\,\phi(y)\,dy \\
&= \frac{1}{\lambda} \int_{\Phi^{-1}(1-\lambda)}^{\infty} y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy = \lim_{T\to\infty} \frac{1}{\lambda} \left[-\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right]_{\Phi^{-1}(1-\lambda)}^{T} \\
&= \lim_{T\to\infty} \frac{1}{\lambda} \left[-\phi(y)\right]_{\Phi^{-1}(1-\lambda)}^{T} = \lim_{T\to\infty} \frac{1}{\lambda} \left(-\phi(T) + \phi\left(\Phi^{-1}(1-\lambda)\right)\right) \\
&= \frac{\phi\left(\Phi^{-1}(1-\lambda)\right)}{\lambda}
\end{aligned}
$$

Since $\mathrm{AV@R}_\lambda$ is cash-invariant and positively homogeneous, we obtain for $X \sim \mathcal{N}(\mu, \sigma^2)$

$$
\begin{aligned}
\mathrm{AV@R}_\lambda(X) &= \mathrm{AV@R}_\lambda\left(\mathbb{E}[X] + \tilde{X}\sqrt{\mathrm{Var}(X)}\right) = -\mathbb{E}[X] + \mathrm{AV@R}_\lambda(\tilde{X})\sqrt{\mathrm{Var}(X)} \\
&= -\mathbb{E}[X] + \frac{\phi(\Phi^{-1}(1-\lambda))}{\lambda}\sqrt{\mathrm{Var}(X)}
\end{aligned}
$$

Since the standard normal distribution is continuous and symmetric, we have that the upper- and lower quantile coincide and that X and $-X$ are both standard normally distributed. Hence, $-\Phi^{-1}(\lambda) = \Phi^{-1}(1-\lambda)$ and the results above are true. □

Remark A.0.6. The representations given in Lemma A.0.5 show that for normally distributed random variables, V@R and AV@R are standard deviation risk measures. To be more precise: For $\lambda \leq 0.5$, it is $\Phi^{-1}(1-\lambda) > 0$, and hence, $\mathrm{V@R}_\lambda$ is a standard deviation risk measure. Thus, the subadditivity property is fulfilled for normally distributed random variables in that case. $\mathrm{AV@R}_\lambda$ is a standard deviation risk measure since $\frac{\phi(\Phi^{-1}(1-\lambda))}{\lambda} > 0$ for all $\lambda \in (0,1)$.

The following proposition and subsequent corollary state aggregation results. Since this thesis is also concerned with solvency capital requirements, regulation and risk aggregation, we present the results for the sake of completeness.

Proposition A.0.7. *Let $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ be normally distributed. If $X = \sum_{i=1}^n X_i$, then*

$$
V@R_\lambda(X) = -\sum_{i=1}^{n} \mathbb{E}[X_i] + \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\; V@R_\lambda(X_i - \mathbb{E}[X_i])\; V@R_\lambda(X_j - \mathbb{E}[X_j])}, \tag{A.2}
$$

where ρ_{ij} is the correlation coefficient between X_i and X_j.

Proof. According to Lemma A.0.5 (i), it is

$$
\mathrm{V@R}_\lambda(X_i) = -\mathbb{E}[X_i] + \Phi^{-1}(1-\lambda)\sqrt{\mathrm{Var}(X_i)} \;\Leftrightarrow\; \sqrt{\mathrm{Var}(X_i)} = \frac{\mathrm{V@R}_\lambda(X_i) + \mathbb{E}[X_i]}{\Phi^{-1}(1-\lambda)}
$$

This leads to

$$
\begin{aligned}
\mathrm{V@R}_\lambda(X) &= -\mathbb{E}[X] + \Phi^{-1}(1-\lambda)\sqrt{\mathrm{Var}(X)} \\
&= -\mathbb{E}[X] + \Phi^{-1}(1-\lambda)\sqrt{\sum_{i=1}^n \sum_{j=1}^n \mathrm{Cov}(X_i, X_j)} \\
&= -\mathbb{E}[X] + \Phi^{-1}(1-\lambda)\sqrt{\sum_{i=1}^n \sum_{j=1}^n \rho_{ij}\sqrt{\mathrm{Var}(X_i)}\sqrt{\mathrm{Var}(X_j)}} \\
&= -\mathbb{E}[X] + \Phi^{-1}(1-\lambda)\sqrt{\sum_{i=1}^n \sum_{j=1}^n \rho_{ij}\,\frac{\mathrm{V@R}_\lambda(X_i)+\mathbb{E}[X_i]}{\Phi^{-1}(1-\lambda)}\,\frac{\mathrm{V@R}_\lambda(X_j)+\mathbb{E}[X_j]}{\Phi^{-1}(1-\lambda)}} \\
&= -\sum_{i=1}^n \mathbb{E}[X_i] + \sqrt{\sum_{i=1}^n \sum_{j=1}^n \rho_{ij}\,\mathrm{V@R}_\lambda(X_i-\mathbb{E}[X_i])\,\mathrm{V@R}_\lambda(X_j-\mathbb{E}[X_j])}
\end{aligned}
$$

□

Corollary A.0.8. *Let $X_i \sim \mathcal{N}(0,\sigma_i^2)$ be normally distributed with $\mathbb{E}[X_i]=0\ \forall i=1,\dots,n$. If $X=\sum_{i=1}^n X_i$, then*

$$
V@R_\lambda(X) = \sqrt{\sum_{i=1}^n \sum_{j=1}^n \rho_{ij}\ V@R_\lambda(X_i)\ V@R_\lambda(X_j)}, \tag{A.3}
$$

where ρ_{ij} is the correlation coefficient between X_i and X_j.

Equation (A.3) is well known as the *square root formula* or *standard formula*. It is proposed in the guidelines of Solvency II in order to compute the solvency capital requirement of insurance firms. To this end, the aggregate risk, e.g., due to different lines of business, market risk etc., is quantified by prescribed correlation matrices for individual risks. Corollary A.0.8 shows that aggregated risk can be computed by the standard formula when risks are centered jointly normally distributed. Hence, solvency capital calculations are based on very strong assumptions that are usually not satisfied in real world applications.

Observe that the results in Proposition A.0.7 and Corollary A.0.8 are also true for AV@R.

The following example presents further monetary risk measures.

Example A.0.9. (i) Utility-based shortfall risk: Letting $l:\mathbb{R}\to\mathbb{R}$ be an increasing, non-constant, convex loss function and λ the threshold level. The utility-based shortfall risk ($\mathrm{UBSR}_{l,\lambda}$) is a convex risk measure defined by

$$
\mathrm{UBSR}_{l,\lambda}(X) := \inf\{m \in \mathbb{R} \,|\, \mathbb{E}[l(-(X+m))] \le \lambda\}.
$$

Thus, $\mathrm{UBSR}_{l,\lambda}$ of a financial position X equals the smallest monetary amount m that needs to be added to X in order to avoid that the expected utility $-\mathbb{E}[l(-(X+m))]$ is less than the threshold level $-\lambda$. The risk measure is normalized when $\lambda = l(0)$.

Setting $l(x) := -u(-x)$, where u is a utility function (i.e., non-decreasing and strictly concave), then l is a strictly convex and increasing function, and the maximization of expected utility is equivalent to minimizing the expected loss, i.e.,

$$\mathrm{UBSR}_{l,\lambda}(X) = \inf\{m \in \mathbb{R} \,|\, \mathbb{E}[l(-(X+m))] \leq \lambda\} = \inf\{m \in \mathbb{R} \,|\, \mathbb{E}[u(X+m)] \geq -\lambda\},$$

see, e.g., Föllmer & Schied (2011).

(ii) Optimized certainty equivalent: Letting $u : \mathbb{R} \to [-\infty, \infty)$ be a concave and non-decreasing utility function satisfying $u(0) = 0, u(x) \geq 0\, \forall x \geq 0$ and $u(x) < x\, \forall x$. The optimized certainty equivalent (OCE_u) is defined by the map $\mathrm{OCE}_u : L^\infty \to \mathbb{R}$ with

$$\mathrm{OCE}_u(X) = \sup_{\eta \in \mathbb{R}} \{\eta + \mathbb{E}[u(X - \eta)]\}.$$

The OCE_u is the present value of an optimal split of the uncertain future income X into a certain amount η and an uncertain future amount $X - \eta$, see Föllmer & Weber (2015). As shown by Ben-Tal & Teboulle (2007), the negative of an OCE defines a convex risk measure, i.e.,

$$\rho_{\mathrm{OCE}_u}(X) := -\mathrm{OCE}_u(X), \quad X \in L^\infty.$$

The subsequent lemma summarizes representations of $\mathrm{UBSR}_{l,\lambda}$ and OCE_u for some explicit choices of loss and utility functions, respectively.

Lemma A.0.10. *(i) $UBSR_{l,\lambda}$: For an exponential loss function $l(x) = e^{\beta x}$, we obtain*

$$UBSR_{l,\lambda}(X) = \inf\left\{m \in \mathbb{R} \,\middle|\, \mathbb{E}\left[e^{-\beta(X+m)}\right] \leq \lambda\right\} = \frac{1}{\beta}\left(\log \mathbb{E}\left[e^{-\beta X}\right] - \log(\lambda)\right).$$

For the threshold level $\lambda = 1$, this risk measure coincides with the entropic risk measure, i.e., $UBSR_{l,1}(X) = \rho_\beta(X)$, cf. Appendix C.1, Example C.1.2 (i) & (iv).

(ii) OCE_u: For an exponential utility function $u(t) = 1 - e^{-\beta t}$, we obtain

$$\rho_{OCE_u}(X) = UBSR_{l,\frac{1}{\beta}}(X) - \frac{\beta - 1}{\beta}$$

For $\beta = 1$, this risk measure coincides with the entropic risk measure with parameter 1, i.e., $\rho_{OCE_u}(X) = \rho_\beta(X) = \rho_1(X)$.

(iii) OCE_u: For the utility function $u(t) = \min\{0, \alpha\, t\}$, $\alpha > 1$, we obtain

$$\rho_{OCE_u}(X) = AV@R_{\frac{1}{\alpha}}(X).$$

See as well Appendix C.1, Example C.1.2 (v).

Proof. (i) This follows by the fact that $\mathrm{UBSR}_{l,\lambda}(X) = s^*$ is the unique solution of the equation $\mathbb{E}[l(-X - s)] = \lambda$, see, e.g., Föllmer & Schied (2011), Dunkel & Weber

(2010) for details. Hence, computing the root of the function $g_\lambda(s) := \mathbb{E}[l(-X-s)]-\lambda$ leads to

$$\begin{aligned}
\mathbb{E}[l(-X-s)] - \lambda \overset{!}{=} 0 \quad &\Leftrightarrow \quad \mathbb{E}\left[e^{-\beta(X+s)}\right] = \lambda \\
&\Leftrightarrow \quad -\beta s + \log\left(\mathbb{E}\left[e^{-\beta X}\right]\right) = \log(\lambda) \\
&\Leftrightarrow \quad s^* = \frac{1}{\beta}\left(\log\left(\mathbb{E}\left[e^{-\beta X}\right]\right) - \log(\lambda)\right)
\end{aligned}$$

(ii) Since $u'(t) = \beta\, e^{-\beta t}$, the first-order condition (1.2) in Chapter 1 yields

$$\begin{aligned}
\mathbb{E}\left[\beta e^{-\beta(X-\eta^*)}\right] = 1 \quad &\Leftrightarrow \quad \log(\beta) + \beta\eta^* = -\log\mathbb{E}\left[e^{-\beta X}\right] \\
&\Leftrightarrow \quad \eta^* = \frac{1}{\beta}\left(-\log\mathbb{E}\left[e^{-\beta X}\right] + \log\left(\frac{1}{\beta}\right)\right) \\
&\Leftrightarrow \quad \eta^* = -\mathrm{UBSR}_{l,\frac{1}{\beta}}(X)
\end{aligned}$$

Thus, the OCE_u is given by

$$\begin{aligned}
\mathrm{OCE}_u(X) &= \eta^* + \mathbb{E}[u(X-\eta^*)] \\
&= -\mathrm{UBSR}_{l,\frac{1}{\beta}}(X) + \mathbb{E}\left[1 - e^{-\beta\left(X - \frac{1}{\beta}\left(-\log\mathbb{E}\left[e^{-\beta X}\right] - \log(\beta)\right)\right)}\right] \\
&= -\mathrm{UBSR}_{l,\frac{1}{\beta}}(X) + 1 - \mathbb{E}\left[e^{-\beta X}\right]\,\mathbb{E}\left[\mathbb{E}\left[e^{-\beta X}\right]^{-1}\,\beta^{-1}\right] \\
&= -\mathrm{UBSR}_{l,\frac{1}{\beta}}(X) + \frac{\beta-1}{\beta}
\end{aligned}$$

Hence, we obtain the corresponding risk measure

$$\rho(X) = \mathrm{UBSR}_{l,\frac{1}{\beta}}(X) - \frac{\beta-1}{\beta}$$

(iii) Let X be a continuous random variable with continuous distribution function F_X and density f_X. Then,

$$\begin{aligned}
\mathrm{OCE}_u(X) &= \sup_{\eta\in\mathbb{R}}\ \{\eta + \mathbb{E}[u(X-\eta)]\} = \sup_{\eta\in\mathbb{R}}\ \{\eta + \alpha\,\mathbb{E}[\min\{0, X-\eta\}]\} \\
&= \sup_{\eta\in\mathbb{R}}\ \left\{\eta + \alpha\int_{-\infty}^{\eta}(x-\eta)f_X(x)\,dx\right\} \\
&=: \sup_{\eta\in\mathbb{R}}\ \{f(\eta)\}
\end{aligned}$$

We obtain

$$\begin{aligned}
f'(\eta) \overset{!}{=} 0 \quad &\Leftrightarrow \quad 1 + \alpha\frac{d}{d\eta}\left(\int_{-\infty}^{\eta} x f_X(x)\,dx - \eta\int_{-\infty}^{\eta} f_X(x)\,dx\right) = 0 \\
&\Leftrightarrow \quad 1 + \alpha\left(\eta f(\eta) - \left(1\cdot\int_{-\infty}^{\eta} f_X(x)\,dx + \eta f_X(\eta)\right)\right) = 0 \\
&\Leftrightarrow \quad 1 + \alpha\left(-(F_X(\eta) - 0)\right) = 0
\end{aligned}$$

$$\Leftrightarrow \quad \eta^* = F_X^{-1}\left(\frac{1}{\alpha}\right),$$

where $F_X^{-1}(\lambda) = q_X(\lambda)$ is the λ-quantile of the random variable X. Hence, according to Example A.0.3 (ii), it is $\eta^* = -\mathrm{V@R}_{\frac{1}{\alpha}}(X)$. Now, it follows by Föllmer & Schied (2011), Lemma 4.51,

$$\begin{aligned}
\mathrm{OCE}_u &= \sup_{\eta\in\mathbb{R}} \{\eta + \mathbb{E}[u(X-\eta)]\} = \sup_{\eta\in\mathbb{R}} \{\eta + \alpha\,\mathbb{E}[\min\{0, X-\eta\}]\} \\
&= \sup_{\eta\in\mathbb{R}} \{\eta - \alpha\,\mathbb{E}[\max\{0, -(X-\eta)\}]\} \\
&= q_X\left(\frac{1}{\alpha}\right) - \alpha\,\mathbb{E}\left[\max\left\{0, q_X\left(\frac{1}{\alpha}\right) - X\right\}\right] \\
&= -\mathrm{AV@R}_{\frac{1}{\alpha}}(X)
\end{aligned}$$

Hence, the corresponding risk measure is given by

$$\rho(X) = \mathrm{AV@R}_{\frac{1}{\alpha}}(X).$$

□

B | Redistribution Risk Measures

In this additional chapter, we introduce an extension of the classical monetary risk measurement theory. The key contribution of this extension is the integration of a so-called redistribution function. This enables the consideration of any possible reallocation of terminal wealth within the risk measurement procedure. Special cases of redistribution risk measures are discussed within this thesis, see, e.g., Chapters 2 & 3.

In the seminal paper by Artzner et al. (1999), the classical theory of monetary risk measures is provided. Here, the risk of a random variable is measured by the following procedure: Let the random variable X denote the stochastic evolution of a risky position. For instance, X might be any financial product. Then, the risk of X is quantified by applying a risk measure ρ that determines the minimal amount of capital k which needs to be added to X in order to make the position acceptable with respect to the corresponding acceptance set $\mathcal{A}$, i.e.,

$$\rho(X) = \inf\{k \in \mathbb{R} \,|\, k + X \in \mathcal{A}\}. \tag{B.1}$$

We extend the risk measure given in Eq. (B.1) to redistribution risk measures, see Eq. (B.2) below. This is motivated by the following observation: Often the *terminal wealth* of a company or an investor does not only depend on X, but also on a second redistribution mechanism which reallocates wealth by the end of a fixed time horizon. The outcome of this redistribution depends on both the stochastic evolution of X and the initial capital $c \in \mathbb{R}$ the company or the investor is endowed with. For example, the portfolio of an investor could need to be adjusted due to given trading constraints or a network of financial institutions might transfer capital between the entities. These redistribution channels need to be taken into account when overall risk is measured. Given initial capital c, we formalize any redistribution by a random field f that captures the stochastic terminal wealth, and consider the resulting redistribution risk measure

$$\tilde{\rho}(f; c) := \inf\{k \in \mathbb{R} \,|\, f(k + c) \in \mathcal{A}\}. \tag{B.2}$$

By means of the random field f, any possible redistribution channel which determines the terminal financial wealth of a company or an investor can be modeled. Thus, $\tilde{\rho}(f; c)$ in Eq. (B.2) quantifies the capital requirement that is needed in order to make the future random outcome of a function depending on the evolution of X and the initial capital c, i.e., $f(k + c)$, acceptable. In contrast, $\rho(X)$ in Eq. (B.1) focuses on the acceptability of X only.

Furthermore, we emphasize that computing $\tilde{\rho}(f;c)$ as given in Eq. (B.2) is not the same as calculating

$$\bar{\rho}(f;c) := \inf\{k \in \mathbb{R} \mid k + f(c) \in \mathcal{A}\}. \tag{B.3}$$

In Eq. (B.3), risk is measured by applying a classical risk measure to the redistributed value $f(c)$, i.e., $\bar{\rho}(f;c) = \rho(f(c))$. This approach does not capture the impact of the additional capital k on the redistribution mechanism itself since k is added to the terminal redistributed value instead of the initial position. However, risk measures as given in Eq. (B.3) are often proposed in the literature, see, e.g., Acerbi & Scandolo (2008), Chen, Iyengar & Moallemi (2013), Hoffmann, Meyer-Brandis & Svindland (2016), Brunnermeier & Cheridito (2014) and Haier et al. (2016). There are two significant differences in the interpretation of the two approaches given in (B.2) and (B.3):

1. Our redistribution risk measure in Eq. (B.2) determines the amount of capital that needs to be added today, i.e., in $t = 0$, in order to make the random outcome tomorrow, i.e., in $t = 1$, acceptable. The risk measure in Eq. (B.3) attempts to add an amount of capital after the actual redistribution is already done. This might be too late. See Chapter 2, Section 2.3, for an illustrative example.

2. In general, the risk measure given in Eq. (B.3) is not cash-invariant anymore in the sense of adding an amount of cash to the eligible asset. Instead, the capital k is added to the redistributed value which does not correspond to the natural interpretation of risk as a capital requirement. In contrast, our redistribution risk measure in Eq. (B.2) maintains the property of cash-invariance and therefore its natural interpretation as a capital requirement.

Literature. In this chapter, we analyze redistribution risk measures in both the scalar, see Eq. (B.2), and the set-valued setting. For classical scalar-valued risk measures, a huge literature exists. In recent years, the excellent book by Föllmer & Schied (2011) became a standard reference for risk measures and their properties. We refer the interested reader to references therein for further information. For set-valued risk measures and its applications to finance and economics, we refer, e.g., to Feinstein et al. (2017), Molchanov & Cascos (2016), Haier et al. (2016) and the references therein. Haier et al. (2016) suggest a set-valued risk measure that incorporates intragroup transfer strategies. They apply the framework of set-valued risk measures based on selections introduced in Molchanov & Cascos (2016). A set-valued network risk measure that can be embedded within our framework of redistribution risk measures is proposed in Hamm et al. (2019*a*). The concept of redistribution risk measures previously appeared in the literature under several notations, see, e.g., Weber et al. (2013) for scalar-valued liquidity-adjusted risk measures or Feinstein et al. (2017) for set-valued systemic risk measures. However, until now a unifying framework did not exist. We provide this missing general setup in this chapter.

Outline. This additional chapter is structured as follows: In Section B.1, we give precise definitions of our comprehensive redistribution risk measure in both the scalar and the set-valued setting. Section B.2 illustrates several examples of practical applications within the unifying framework. In Section B.3, we analyze the properties of the proposed risk measure. Section B.4 concludes with a summary and suggestions for future research.

B.1 | Definition of Redistribution Risk Measures

Throughout this chapter, the set of financial positions at time 1 whose risk needs to be assessed is a vector space of random variables on a probability space $(\Omega, \mathcal{F}, P)$ that contains the constants. For example, $\mathcal{X}(\mathbb{R}) = L^0(\mathbb{R})$ is the family of random variables, $\mathcal{X}(\mathbb{R}) = L^\infty(\mathbb{R})$ is the subspace of bounded random variables, or $\mathcal{X}(\mathbb{R}) = L^p(\mathbb{R})$ is the subspace of p-integrable random variables. Analogously, we use the notation $\mathcal{X}(\mathbb{R}^n)$, where $\mathcal{X}(\mathbb{R}^n) \subseteq L^0(\mathbb{R}^n)$ is a subspace of the family of n-dimensional random vectors.

All risky positions X which are acceptable with respect to the monetary risk measure ρ (see Appendix A, Definition A.0.1) are collected in the corresponding acceptance set

$$\mathcal{A} = \{X \in \mathcal{X}(\mathbb{R}) \,|\, \rho(X) \leq 0\} \subseteq \mathcal{X}(\mathbb{R}).$$

Hence, we call X *acceptable* if no additional capital is needed for acceptance.

Remark B.1.1. The defining properties of a general acceptance set $\mathcal{A}$ are given by (see, e.g., Föllmer & Schied (2011), Section 4.1, and Feinstein et al. (2017), Section 2.2):

(i) $\inf\{m \in \mathbb{R} \,|\, m \in \mathcal{A}\} > -\infty$.
(Not any deterministic monetary amount is acceptable.)

(ii) If $X \in \mathcal{A}, Y \in \mathcal{X}(\mathbb{R})$ and $Y(\omega) \geq X(\omega)$ $\forall \omega \in \Omega$, then $Y \in \mathcal{A}$.
(Positions that dominate acceptable positions are again acceptable.)

(iii) $\mathcal{A}$ is closed in $\mathcal{X}(\mathbb{R})$.

The scalar risk measure corresponding to $\mathcal{A}$ is given by

$$\rho(X) = \inf\{k \in \mathbb{R} \,|\, k + X \in \mathcal{A}\}, \quad X \in \mathcal{X}(\mathbb{R}). \tag{B.4}$$

Thus, the risk of X is quantified by the minimal amount of capital k which has to be added to the risky position X in order to make it acceptable. Hence, there is no distinction between risk and capital requirements.

We extend the classical static risk measure given in Eq. (B.4) to our redistribution risk measure as follows. We assume that the risky position X is reallocated, e.g., within a portfolio or among agents. Here, we suppose that this reallocation depends on the initial capital c and a capital injection k. This redistribution of terminal wealth is modeled by a random field f. Using a redistribution risk measure, the risk of X is quantified by the minimal amount of capital k which has to be added to the initial capital c in order to make the terminal wealth $f(k+c)$ acceptable with respect to a classical monetary risk measure.

In particular, our redistribution risk measure takes the capital injection k – which in turn influences the redistribution – into account for the risk measurement procedure itself.

For the remaining part of the chapter, let us denote the sets $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$ for $m, n \in \mathbb{N}$.

Definition B.1.2. We call the random field

$$f : \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^m) \quad \text{with} \quad f(c) = (f_1(c), f_2(c), \ldots, f_m(c)),$$

$f_j : \mathbb{R}^n \to \mathcal{X}(\mathbb{R})$, $\forall j \in M$, $c \in \mathbb{R}^n$, a *redistribution function*. We denote by $\mathcal{Y}$ the space of random fields, i.e., $f, f_j \in \mathcal{Y}$, $\forall j \in M$, $c \in \mathbb{R}^n$.

In Section B.2, several examples of redistribution functions are illustrated.

Now, we define our redistribution risk measures based on redistribution functions in both the scalar and set-valued setting. These risk measures either map to a single value or to a set of vectors and capture any redistribution of the risky position X.

Definition B.1.3. Let $f : \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^m)$ be a redistribution function. Given that component $j \in M$ needs to be acceptable and given eligible assets $i \in N$, we define redistribution risk measures as follows:

(i) For $k \in \mathbb{R}$, we call the map

$$\tilde{\rho} : \mathcal{Y} \times \mathbb{R}^n \to \mathbb{R}$$

a *redistribution risk measure*, if for some acceptance set $\mathcal{A}_j \subseteq \mathcal{X}(\mathbb{R})$ of a static monetary risk measure

$$\tilde{\rho}(f; c) = \inf \left\{ k \in \mathbb{R} \,\middle|\, f_j(k + c) \in \mathcal{A}_j \right\}, \quad c \in \mathbb{R}^n,\, j \in M,$$

where f_j is the component of f that needs to be acceptable with respect to $\mathcal{A}_j$ and

$$k + c = (c_1, \ldots, k + c_i, \ldots, c_n), \quad i \in N.$$

Here, i is the eligible asset in which the capital k is invested.

(ii) For $k \in \mathbb{R}^n$, we call the map

$$\tilde{\rho} : \mathcal{Y} \times \mathbb{R}^n \to \mathcal{P} \subseteq \mathbb{R}^n$$

a *set-valued redistribution risk measure*, if for some acceptance set $\mathcal{A}_j \subseteq \mathcal{X}(\mathbb{R})$ of a static scalar monetary risk measure

$$\tilde{\rho}(f; c) = \left\{ k \in \mathbb{R}^n \,\middle|\, f_j(k + c) \in \mathcal{A}_j \right\}, \quad c \in \mathbb{R}^n,\, j \in M.$$

Again, f_j is the component of f that needs to be acceptable with respect to $\mathcal{A}_j$ and

$$k + c = (k_1 + c_1, \ldots, k_n + c_n).$$

Thus, in each asset $i \in N$ capital k_i is invested.

Remark B.1.4. (i) Let $c \in \mathbb{R}$, $C \in \mathcal{X}(\mathbb{R})$ and let $X = C - c$ denote the random change in wealth of the financial position. If $n = m = 1$ and $f(c) = X + c$, we obtain the classical scalar risk measure as provided in Eq. (B.4).

(ii) The acceptability conditions given in Definition B.1.3 can easily be extended to the condition of simultaneous acceptance of several positions $J \subseteq M$ by setting

$$\tilde{\rho}(f;c) = \inf\left\{k \in \mathbb{R} \,\middle|\, f_j(k+c) \in \mathcal{A}_j \;\forall j \in J\right\}, \quad c \in \mathbb{R}^n,$$

and

$$\tilde{\rho}(f;c) = \left\{k \in \mathbb{R}^n \,\middle|\, f_j(k+c) \in \mathcal{A}_j \;\forall j \in J\right\}, \quad c \in \mathbb{R}^n,$$

respectively. We refer to Example B.2.3 as well as Chapter 3 for an application with $J = M$.

B.2 | Examples of Application

In this section, we suggest several examples for practical applications which illustrate the quality of a unifying general framework as provided in Section B.1. The main objective is to specify the redistribution mechanism f which captures the random terminal wealth.

Example B.2.1 (Insolvency Risk)**.** Let us consider an insurance company that models its internal wealth by the market consistent embedded value (MCEV). The MCEV can be calculated as the sum of the net asset value and the present value of future profits reduced by the sum of the costs of residual non-hedgeable risks and any further costs. An overview with respect to the valuation of the MCEV and its components is provided in Chapter 6.

Based on our introduced framework, the capital $k \in \mathbb{R}$ that the insurance firm has to add to its initial economic capital $c \in \mathbb{R}$ in order to make the future random value of the company, modeled by the MCEV, acceptable can be determined by applying a redistribution risk measure. With $m = n = 1$ and

$$f : \mathbb{R} \to \mathcal{X}(\mathbb{R}), \quad f(c) = \mathrm{MCEV}(c),$$

the insolvency risk of the company can be measured by

$$\tilde{\rho}(f;c) = \inf\left\{k \in \mathbb{R} \,\middle|\, \mathrm{MCEV}(k+c) \in \mathcal{A}\right\}.$$

Example B.2.2 (Liquidity Risk)**.** Let us consider an agent who attempts to make the future random liquidity-adjusted portfolio value of his portfolio $c \in \mathbb{R}^{n+1}$ acceptable by trading. Here, the liquidity-adjusted portfolio value is given in terms of an optimization problem subject to liquidity and portfolio constraints. As analyzed in Chapter 2, the liquidity-adjusted portfolio value $V(c)$ can be determined by

$$V(c) = \sup\left\{U(l,z) \,\middle|\, l \in \mathcal{L}(c,z,\phi,a) \cap (\mathbb{R} \times \mathcal{K})\right\},$$

where $z = \{z_0, \ldots, z_n\} \in \mathcal{M}^{n+1}$ is the spot market of n risky assets consisting of $n+1$ random marginal supply-demand curves, $U(l,z)$ is the maximal mark-to-market value of a portfolio l and $\mathcal{L}(c,z,\phi,a) \cap (\mathbb{R} \times \mathcal{K})$ formalizes the liquidity and portfolio constraints obligatory for the trader.

Based on our introduced framework, the capital $k \in \mathbb{R}$ the agent has to invest in the eligible asset c_i of his portfolio $c \in \mathbb{R}^{n+1}$ in order to make the future random liquidity-adjusted portfolio value acceptable can be determined by applying a redistribution risk measure. With $m = 1$ and

$$f : \mathbb{R}^{n+1} \to \mathcal{X}(\mathbb{R}), \quad f(c) = V(c),$$

the liquidity risk of the investor can be measured by

$$\tilde{\rho}(f;c) = \inf\left\{k \in \mathbb{R} \,\middle|\, V(k+c) \in \mathcal{A}\right\}.$$

We refer to Chapter 2 for the precise notations and a detailed analysis of liquidity risk and liquidity-adjusted risk measures.

Example B.2.3 (Network Risk)**.** Let us consider a financial network of n entities such that $i \in N$ represents firm i of the network. The objective of the system's regulator is to quantify the risk that arises due to the characteristics of the system itself. Thus, acceptability of the entities within the network is required. The regulatory system might be based on a legal entity or consolidated approach:

(i) The legal entity approach requires that the terminal wealth of each entity within the network is acceptable with respect to its specific regulatory guideline. Based on our introduced framework and Remark B.1.4 (ii), the capital $k_i \in \mathbb{R}$ that each entity $i \in N$ of the network needs to add to its initial economic capital $c_i \in \mathbb{R}$ in order to make its future random wealth acceptable can be determined by applying a set-valued redistribution risk measure. With $m = n$ and

$$f : \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^n), \quad f(c) = C(c),$$

such that $C(c) \in \mathcal{X}(\mathbb{R}^n)$ models the random terminal wealth for each entity by taking capital and risk transfer strategies into account, the network risk can be measured by

$$\tilde{\rho}(f;c) = \left\{k \in \mathbb{R}^n \,\middle|\, C_i(k+c) \in \mathcal{A}_i \, \forall i \in N\right\}.$$

(ii) The consolidated approach requires that the aggregated terminal wealth of the total network is acceptable for one regulator. Based on our introduced framework, the capital $k_i \in \mathbb{R}$ that each entity $i \in N$ of the network needs to add to its initial economic capital $c_i \in \mathbb{R}$ in order to make the total future random wealth of the network acceptable can be determined by applying a set-valued redistribution risk measure. In this case, with $m = 1$ and

$$f : \mathbb{R}^n \to \mathcal{X}(\mathbb{R}), \quad f(c) = C^{\text{network}}(c),$$

such that $C^{\text{network}}(c)$ models the random terminal wealth of the total financial network, the network risk can be measured by

$$\tilde{\rho}(f; c) = \left\{ k \in \mathbb{R}^n \,\middle|\, C^{\text{network}}(k + c) \in \mathcal{A} \right\}.$$

For a detailed analysis on regulatory systems, network risk, network risk measures and risk sharing within networks, we refer to Chapters 3 & 4.

B.3 | Properties of Redistribution Risk Measures

In this section, we analyze the properties of the suggested redistribution risk measures in the scalar and set-valued setting.

Scalar-valued setting. Throughout this paragraph, let us consider the scalar-valued redistribution risk measure $\tilde{\rho}$ as provided in Definition B.1.3 (i). The following proposition specifies conditions for $\tilde{\rho}$ being a monetary convex risk measure.

Proposition B.3.1. *Let $f : \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^m)$ be a redistribution function and $\tilde{\rho}$ the redistribution risk measure provided in Definition B.1.3 (i). The following properties are satisfied:*

(i) Cash-invariance:

$$\tilde{\rho}(f; c + l) = \tilde{\rho}(f; c) - l, \ \forall l \in \mathbb{R}.$$

(If initial capital is increased by l, the required additional capital is reduced by l.)

(ii) Monotonicity in the initial capital: If f is non-decreasing in its j-th component, $j \in M$, then

$$\tilde{\rho}(f; c) \leq \tilde{\rho}(f; z)$$

whenever $c_i \geq z_i, \forall i = 1, \dots, n$.
(Larger initial capital leads to less additional capital.)

(iii) Convexity: If f is concave in its j-th component and $\mathcal{A}_j$ is convex, $j \in M$, then

$$\alpha\tilde{\rho}(f; c) + (1 - \alpha)\tilde{\rho}(f; z) \geq \tilde{\rho}(f; \alpha c + (1 - \alpha)z), \quad \alpha \in [0, 1].$$

(Diversification in the initial endowment does not increase risk.)

Proof. (i) Letting $l \in \mathbb{R}$, $j \in M$. It is $\tilde{\rho}(f; c+l) = \inf\{k \in \mathbb{R} \mid f_j(c+l+k) \in \mathcal{A}_j\}$. Setting $b = l + k$ leads to

$$\tilde{\rho}(f; c+l) = \inf\{b-l \in \mathbb{R} \mid f_j(c+b) \in \mathcal{A}_j\} = \inf\{b \in \mathbb{R} \mid f_j(c+b) \in \mathcal{A}_j\} - l = \tilde{\rho}(f; c) - l.$$

(ii) Let $c_i \geq z_i, \forall i = 1, \dots, n$, $j \in M$. Since f_j is non-decreasing, it is $f_j(k+z) \leq f_j(k+c)$ for any $k \in \mathbb{R}$. Hence, if $f_j(k+z) \in \mathcal{A}_j$ then also $f_j(k+c) \in \mathcal{A}_j$, because of Remark B.1.1 (ii). Thus, $\tilde{\rho}(f; c) \leq \tilde{\rho}(f; z)$.

(iii) Let $\alpha \in [0, 1]$ be fixed, $j \in M$. Since $\mathcal{A}_j$ is convex, it is $\alpha f_j(k+c) + (1-\alpha) f_j(l+z) \in \mathcal{A}_j$, for all $k, l \in \mathbb{R}$ such that $f_j(k+c), f_j(l+z) \in \mathcal{A}_j$. The concavity of f_j yields

$$\alpha f_j(k+c) + (1-\alpha) f_j(l+z) \leq f_j(\alpha(k+c) + (1-\alpha)(l+z))$$

which leads to $f_j(\alpha(k+c) + (1-\alpha)(l+z)) = f_j(\alpha k + (1-\alpha) l + \alpha c + (1-\alpha) z) \in \mathcal{A}_j$. We obtain the upper bound

$$\tilde{\rho}(f; \alpha c + (1-\alpha) z) = \inf\{y \in \mathbb{R} \mid f_j(\alpha c + (1-\alpha) z + y) \in \mathcal{A}_j\} \leq \alpha k + (1-\alpha) l.$$

Taking the limits $k \downarrow \tilde{\rho}(f; c)$ and $l \downarrow \tilde{\rho}(f; z)$ implies convexity of $\tilde{\rho}$:

$$\tilde{\rho}(f; \alpha c + (1-\alpha) z) \leq \alpha \tilde{\rho}(f; c) + (1-\alpha) \tilde{\rho}(f; z).$$

□

The following proposition provides properties of $\tilde{\rho}$ in the random field.

Proposition B.3.2. *Let $f : \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^m)$ and $t : \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^m)$ be redistribution functions and $\tilde{\rho}$ the redistribution risk measure provided in Definition B.1.3 (i). The following properties are satisfied:*

(i) Monotonicity in the random field: If $f_j(c) \geq t_j(c), \forall c \in \mathbb{R}^n$ and $j \in M$, then

$$\tilde{\rho}(f; c) \leq \tilde{\rho}(t; c).$$

(Larger terminal wealth is less risky.)

(ii) Quasi-convexity: If $\mathcal{A}_j$, $j \in M$, is convex, then

$$\tilde{\rho}(\alpha f + (1-\alpha) t; c) \leq \max\, \{\tilde{\rho}(f; c), \tilde{\rho}(t; c)\}.$$

(Diversification in the terminal wealth does not increase risk.)

Proof. (i) Let $f_j(c) \geq t_j(c), \forall c \in \mathbb{R}^n, j \in M$. Hence, if $t_j(k+c) \in \mathcal{A}_j$ then also $f_j(k+c) \in \mathcal{A}_j$, because of Remark B.1.1 (ii). Thus, $\tilde{\rho}(f; c) \leq \tilde{\rho}(t; c)$.

(ii) Let $f_j(c) \geq t_j(c), \forall c \in \mathbb{R}^n, j \in M$, and $\tilde{\rho}(t;c) = l$. Hence, it is $t_j(c+l) \in \mathcal{A}_j$ which implies $f_j(c+l) \in \mathcal{A}_j$. Convexity of $\mathcal{A}_j$ yields $\alpha f_j(l+c) + (1-\alpha)t_j(l+c) \in \mathcal{A}_j$. This leads to

$$\tilde{\rho}(\alpha f + (1-\alpha)t;c) = \inf\{z \in \mathbb{R} \mid \alpha f_j(z+c) + (1-\alpha)t_j(z+c) \in \mathcal{A}_j\} \leq l = \tilde{\rho}(t;c).$$

If $f_j(c) \leq t_j(c), \forall c \in \mathbb{R}^n, j \in M$, then

$$\tilde{\rho}(\alpha f + (1-\alpha)t;c) \leq \tilde{\rho}(f;c).$$

This proofs the claim due to (i).

□

We compare the risk measures $\tilde{\rho}$ and $\bar{\rho}$, see Eq.(B.3), in the subsequent proposition. In particular, we find that for translation-supervariant redistribution functions, capital requirements can be reduced when these are positive and $\tilde{\rho}$ is applied. When capital requirements are negative, $\tilde{\rho}$ is more prudent than $\bar{\rho}$.

Proposition B.3.3. *Let $f : \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^m)$ be a redistribution function, $\tilde{\rho}$ the redistribution risk measure provided in Definition B.1.3 (i) and $\bar{\rho}(f;c) = \inf\{k \in \mathbb{R} \mid f_j(c)+k \in \mathcal{A}_j\}, j \in M$. The following properties are satisfied:*

(i) If f is translation-invariant in its j-th component, $j \in M$, then $\tilde{\rho}(f;c) = \bar{\rho}(f;c)$.

(ii) If f is translation-supervariant in its j-th component, $j \in M$, then

$$|\tilde{\rho}(f;c)| \leq |\bar{\rho}(f;c)|$$

and $\tilde{\rho}(f;c)$ and $\bar{\rho}(f;c)$ have the same sign, if $\tilde{\rho}(f;c) \neq 0$.

Proof. (i) Let $j \in M$. Since $f_j(c+k) = f_j(c) + k$, this property is obvious.

(ii) Let $j \in M$. The property is shown by case-by-case analysis. First, let us consider $\tilde{\rho}(f;c) > 0$. Then, $f_j(c+k) \notin \mathcal{A}_j$ for any fixed $k \in (0, \tilde{\rho}(f;c))$. Since f_j is translation-supervariant, it is $f_j(c+k) \geq f_j(c) + k$ which implies $f_j(c) + k \notin \mathcal{A}_j$, because of Remark B.1.1 (ii). This leads to $\bar{\rho}(f;c) \geq k > 0$. Letting k increase to $\tilde{\rho}(f;c)$, we obtain $\tilde{\rho}(f;c) \leq \bar{\rho}(f;c)$ for $\tilde{\rho}(f;c) > 0$.
Now, we consider $\tilde{\rho}(f;c) < 0$. In this case, $f_j(c+k) \in \mathcal{A}_j$ for any fixed $k \in (\tilde{\rho}(f;c), 0)$. Since f_j is translation-supervariant, it is $f_j(c+k) \leq f_j(c)+k$ which implies $f_j(c)+k \in \mathcal{A}_j$, because of Remark B.1.1 (ii). This leads to $\bar{\rho}(f;c) \leq k < 0$. Taking the limit $k \downarrow \tilde{\rho}(f;c)$, we obtain $\tilde{\rho}(f;c) \geq \bar{\rho}(f;c)$ for $\tilde{\rho}(f;c) < 0$.

□

Set-valued setting. Throughout this paragraph, let us consider the set-valued redistribution risk measure as provided in Definition B.1.3 (ii). The following proposition summarizes its properties.

Proposition B.3.4. *Let $f : \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^m)$ be a redistribution function and $\tilde{\rho}$ the set-valued redistribution risk measure provided in Definition B.1.3 (ii). The following properties are satisfied:*

(i) *Cash-invariance:*

$$\tilde{\rho}(f; c + l) = \tilde{\rho}(f; c) - l, \ \forall l \in \mathbb{R}^n.$$

(ii) *Monotonicity in the initial capital: If f is non-decreasing in its j-th component, $j \in M$, then*

$$\tilde{\rho}(f; z) \subseteq \tilde{\rho}(f; c)$$

whenever $z_i \le c_i$ $\forall i = 1, \dots, n$.

(iii) *Monotonicity in the random field: Let $t \in \mathcal{Y}$ be another redistribution function such that $f_j(c) \ge t_j(c), \ \forall c \in \mathbb{R}^n$ and $j \in M$. Then,*

$$\tilde{\rho}(t; c) \subseteq \tilde{\rho}(f; c).$$

(iv) *Convexity: If f is concave in its j-th component and $\mathcal{A}_j$ is convex, $j \in M$, then $\tilde{\rho}$ is a convex subset of $\mathbb{R}^n$, i.e., if $k, l \in \tilde{\rho}(f; c)$, then*

$$\alpha\, k + (1 - \alpha)\, l \in \tilde{\rho}(f; c), \quad \alpha \in [0, 1].$$

Proof. (i) Letting $l \in \mathbb{R}^n$, $j \in M$. It is $\tilde{\rho}(f; c+l) = \{k \in \mathbb{R}^n \mid f_j(c+l+k) \in \mathcal{A}_j\}$. Setting $b = l + k$ leads to

$$\tilde{\rho}(f; c+l) = \{b - l \in \mathbb{R}^n \mid f_j(c+b) \in \mathcal{A}_j\} = \{b \in \mathbb{R}^n \mid f_j(c+b) \in \mathcal{A}_j\} - l = \tilde{\rho}(f; c) - l.$$

(ii) Let $z_i \le c_i$ $\forall i = 1, \dots, n$, $j \in M$. Since f_j is non-decreasing, it is $f_j(k+z) \le f_j(k+c)$. Hence, if $f_j(k+z) \in \mathcal{A}_j$ then also $f_j(k+c) \in \mathcal{A}_j$, because of Remark B.1.1 (ii). Therefore, $k \in \tilde{\rho}(f; c)$ whenever $k \in \tilde{\rho}(f; z)$. Thus, $\tilde{\rho}(f; z) \subseteq \tilde{\rho}(f; c)$.

(iii) Let $f_j(c) \ge t_j(c), \forall c \in \mathbb{R}^n, j \in M$. Hence, if $t_j(k+c) \in \mathcal{A}_j$ then also $f_j(k+c) \in \mathcal{A}_j$, because of Remark B.1.1 (ii). Thus, $\tilde{\rho}(t; c) \subseteq \tilde{\rho}(f; c)$.

(iv) Let $\alpha \in [0, 1]$ be fixed, $j \in M$. Since $k, l \in \tilde{\rho}(f; c)$, it is $f_j(k+c), f_j(l+c) \in \mathcal{A}_j$. Convexity of $\mathcal{A}_j$ leads to $\alpha f_j(k+c) + (1-\alpha) f_j(l+c) \in \mathcal{A}_j$. The concavity of f_j yields

$$\alpha f_j(k+c) + (1-\alpha) f_j(l+c) \le f_j(\alpha(k+c) + (1-\alpha)(l+c))$$

which implies $f_j(\alpha(k+c) + (1-\alpha)(l+c)) = f_j(\alpha\, k + (1-\alpha)\, l + c) \in \mathcal{A}_j$. Due to the definition of $\tilde{\rho}$, we obtain

$$\alpha k + (1-\alpha) l \in \tilde{\rho}(f; c).$$

□

The properties of the set-valued redistribution risk measure in Proposition B.3.4 can be interpreted analogously to the corresponding properties of the classical scalar risk measure: Characteristic (i) again describes that adding a fixed capital vector to the financial positions reduces the risk exactly by this amount. Monotonicity formalizes that multivariate capital requirements are stricter if initial (ii) resp. future (iii) random wealth levels decrease. Property (iv) states that convex combinations of acceptable capital allocations are again acceptable.

Remark B.3.5. In Proposition B.3.4 (ii), the redistribution risk measure can be specified by

$$\tilde{\rho} : \mathcal{Y} \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n; \mathbb{R}^n_+) := \{A \subseteq \mathbb{R}^n \mid A = A + \mathbb{R}^n_+\}.$$

Hence, $\tilde{\rho}$ maps into the collection of upper sets with ordering cone $\mathbb{R}^n_+$. For non-decreasing redistribution functions f_j, the properties of the acceptance set $\mathcal{A}_j$, $j \in M$, imply $l + z \in \tilde{\rho}(f; c)$ whenever $l \in \tilde{\rho}(f; c)$ and $z \in \mathbb{R}^n_+$.

We compare the risk measures $\tilde{\rho}$ and $\bar{\rho}$ in the set-valued setting. The findings are consistent with the scalar-valued framework.

Proposition B.3.6. *Let $f : \mathbb{R}^n \to \mathcal{X}(\mathbb{R}^m)$ be a redistribution function, $\tilde{\rho}$ the redistribution risk measure provided in Definition B.1.3 (ii) and $\bar{\rho}(f; c) = \{k \in \mathbb{R}^n \mid f_j(c) + k \in \mathcal{A}_j\}, j \in M$. The following properties are satisfied:*

(i) If f is translation-invariant in its j-th component, $j \in M$, then $\tilde{\rho}(f; c) = \bar{\rho}(f; c)$, i.e., the elements of both sets coincide.

(ii) Let f be translation-supervariant in its j-th component, $j \in M$.

(a) Let $k \in \mathbb{R}^n_+$. If $k \in \bar{\rho}(f; c)$, then $k \in \tilde{\rho}(f; c)$.

(b) Let $k \in \mathbb{R}^n_-$. If $k \in \tilde{\rho}(f; c)$, then $k \in \bar{\rho}(f; c)$.

Proof. (i) Let $j \in M$. Since $f_j(c + k) = f_j(c) + k$, this property is obvious.

(ii) (a) Let $k^* \in \mathbb{R}^n_+$ and $k^* \in \bar{\rho}(f; c) = \{k \in \mathbb{R}^n \mid f_j(c) + k \in \mathcal{A}_j\}$, $j \in M$. Since f_j is translation-supervariant, i.e., $f_j(c) + k^* \leq f_j(c + k^*)$, and $\mathcal{A}_j$ is the acceptance set of a static scalar monetary risk measure, it is $f_j(c + k^*) \in \mathcal{A}_j$. This implies $k^* \in \{k \in \mathbb{R}^n \mid f_j(c + k) \in \mathcal{A}_j\} = \tilde{\rho}(f; c)$.

(b) Let $k^* \in \mathbb{R}^n_-$ and $k^* \in \tilde{\rho}(f; c) = \{k \in \mathbb{R}^n \mid f_j(c + k) \in \mathcal{A}_j\}$, $j \in M$. Since f_j is translation-supervariant, i.e., $f_j(c + k^*) \leq f_j(c) + k^*$, and $\mathcal{A}_j$ is the acceptance set of a static scalar monetary risk measure, it is $f_j(c) + k^* \in \mathcal{A}_j$. This implies $k^* \in \{k \in \mathbb{R}^n \mid f_j(c) + k \in \mathcal{A}_j\} = \bar{\rho}(f; c)$.

□

Remark B.3.7. In contrast to scalar-valued redistribution risk measures, set-valued redistribution risk measures do not quantify a minimum amount of additional capital, but a whole set of additional capital endowments that all generate acceptable outcomes. In Feinstein et al. (2017), the definition of orthant risk measures is suggested in order to provide a methodology of choosing an adequate capital allocation out of the set of acceptable allocations. Orthant risk measures specify a minimum vector of capital endowments which is unique if the lower boundary of the set of possible positions is not a line segment. A similar approach is proposed in Chapter 3, Section 3.2.1, in order to determine optimal set-valued capital allocations in financial networks.

B.4 | Conclusion

We introduced a concept of meaningful risk measurement when a redistribution channel is present. We pointed out the advantages – in particular, in terms of interpretation – and analyzed the properties of the suggested redistribution risk measure in the scalar and set-valued setting. Examples demonstrating the quality of this unifying concept for practical applications were discussed. For detailed analyses of liquidity-adjusted and network risk measures, we refer to Chapters 2 & 3, respectively. A comprehensive introduction to the MCEV is provided in Chapter 6.

The investigation of the statistical properties of the suggested redistribution risk measure is an interesting field for future research. Moreover, the development of efficient algorithms for the computation of the provided risk measures is an important objective, because the inclusion of optimization problems or network models into the risk measurement procedure significantly increases the effort of calculating capital requirements.

C | Appendix of Chapter 3

In this additional appendix, we introduce the subgradient allocation principle that relies on the robust representation of convex risk measures. First, we provide a short review on the robust representation of convex risk measures in Section C.1. In Section C.2, we define our subgradient allocation principle and provide explicit formulae for its computation.

C.1 | Robust Representation of Convex Risk Measures

Under mild conditions, convex risk measures admit a robust representation, cf., e. g., Föllmer & Schied (2011), Chapter 4.2, for an overview. To recall some basis facts, let $\mathcal{X}(\mathbb{R}) = L^\infty$ be the space of all bounded measurable functions on some measurable space $(\Omega, \mathcal{F})$. The dual space $\mathcal{X}'$ can be identified with the space $ba := ba(\Omega, \mathcal{F})$ of finitely additive set functions with finite total variation, see Definition A.50 in Föllmer & Schied (2011). Let $\mathcal{M}_1$ denote the set of all probability measures on $(\Omega, \mathcal{F})$, and let $\mathcal{M}_{1,f}$ be the set of all finitely additive set functions $Q : \mathcal{F} \to [0, 1]$ with $Q(\Omega) = 1$.

Definition C.1.1. Let ρ be a convex risk measure. We call ρ *proper* if the domain of ρ is non-emtpy, i. e., $\text{dom}\rho = \{X \in \mathcal{X}(\mathbb{R}) \mid \rho(X) < \infty\} \neq \emptyset$.

The robust representation is derived from convex duality theory. The *convex conjugate* (Fenchel-Legendre transform) of a proper convex risk measure ρ is given by $\rho^*(l_Q) = \alpha^{\min}(Q)$, where the functions are defined as follows:

$$\rho^* : \mathcal{X}' \to \mathbb{R} \cup \{+\infty\}, \quad \rho^*(l) = \sup_{X \in \mathcal{X}(\mathbb{R})} \{l(X) - \rho(X)\},$$

where $l_Q \in \mathcal{X}'$ is given by $l_Q(X) = \mathbb{E}_Q[-X]$ for $Q \in \mathcal{M}_{1,f}$ and thus

$$\alpha^{\min} : \mathcal{M}_{1,f} \to \mathbb{R} \cup \{+\infty\}, \quad \alpha^{\min}(Q) = \sup_{X \in \mathcal{X}(\mathbb{R})} \{\mathbb{E}_Q[-X] - \rho(X)\},$$

see Remark 4.18 in Föllmer & Schied (2011). The functional $\alpha^{\min}$ is called *minimal penalty function.* Since

$$l_Q(X) \leq \rho(X) + \rho^*(l_Q) \quad \text{and} \quad \mathbb{E}_Q[-X] \leq \rho(X) + \alpha^{\min}(Q),$$

respectively, we have the dual - also called robust - representation of convex risk measures

$$\rho(X) = \sup_{l_Q \in \text{ba}} \{l_Q(X) - \rho^*(l_Q)\} \quad \text{and} \quad \rho(X) = \sup_{Q \in \mathcal{M}_{1,f}} \{\mathbb{E}_Q[-X] - \alpha^{\min}(Q)\},$$

respectively.

Let us now fix a probability measure P on $(\Omega, \mathcal{F})$, and let us denote by $\mathcal{M}_1(P)$ the class of all probability measures $Q \in \mathcal{M}_1$ which are absolutely continuous with respect to P. We write $Q \ll P$, whenever Q is absolutely continuous with respect to P, and use the notation $\varphi = dQ/dP$ for the Radon-Nikodym density of Q with respect to P. Let ρ be a convex risk measure on $L^\infty(\Omega, \mathcal{F})$ which respects the null sets of P, i. e., $\rho(X) = \rho(Y)$ whenever the equivalence relation $X = Y$ P-a. s. holds. Then, ρ can be regarded as a convex risk measure on the Banach space $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, P)$ of equivalence classes. In this case, the robust representation

$$\rho(X) = \sup_{Q \in \mathcal{M}_1(P)} \{\mathbb{E}_Q[-X] - \alpha(Q)\}$$

holds iff ρ is *continuous from above*, i. e., $\rho(X_n) \nearrow \rho(X)$ whenever $X_n \searrow X$ P-a. s.. This representation clarifies that the risk with respect to any convex risk measure is computed as worst-case of expected loss among a class of probabilistic models, taken more or less seriously according to a so-called penalty function.
The following example recalls dual representations of well-known risk measures.

Example C.1.2. (i) Entropic risk measure (cf. Föllmer & Schied (2011), Example 4.34):

$$\rho_\beta(X) = \sup_{Q \in \mathcal{M}_1(P)} \left\{\mathbb{E}_Q[-X] - \tfrac{1}{\beta} H(Q|P)\right\}, \quad \beta > 0,$$

where the minimal penalty function

$$\alpha : \mathcal{M}_1(P) \to (0, \infty], Q \mapsto \alpha(Q) := \tfrac{1}{\beta} H(Q|P)$$

is given in terms of the *relative entropy* $H(Q|P) := \mathbb{E}_Q[\log \frac{dQ}{dP}]$ of $Q \in \mathcal{M}_1(P)$ with respect to P. Note that the supremum is attained by the measure with density $\varphi = e^{-\beta X}/\mathbb{E}[e^{-\beta X}]$, and this yields the explicit formula $\rho_\beta(X) = \frac{1}{\beta} \log \mathbb{E}[e^{-\beta X}]$. The risk measure ρ_β is a special case of the utility-based shortfall risk measure in (iv).

(ii) Divergence risk measure (cf. Föllmer & Schied (2011), Example 4.36):

$$\rho_g(X) := \sup_{Q \ll P} \{\mathbb{E}_Q[-X] - I_g(Q|P)\},$$

where $g : [0, \infty[\to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous convex function satisfying $g(1) < \infty$ and the superlinear growth condition $\frac{g(x)}{x} \to +\infty$ as $x \uparrow \infty$. The minimal penalty function $\alpha_g : \mathcal{M}_1(P) \to \mathbb{R} \cup \{+\infty\}$ is given by the *g-divergence*, i.e.,

$$\alpha_g(Q) := I_g(Q|P) := \mathbb{E}\left[g\left(\tfrac{dQ}{dP}\right)\right], \quad Q \in \mathcal{M}_1(P).$$

For $g(x) = \frac{1}{\beta} x \log x$, ρ_g coincides with the entropic risk measure ρ_β in (i). Let $g^*(y) = \sup_{x>0}\{xy - g(x)\}$ be the convex conjugate of g, then (cf. Föllmer & Schied

(2011), Theorem 4.122)

$$\rho_g(X) = \inf_{z\in\mathbb{R}} \{\mathbb{E}[g^*(z - X)] - z\}, \quad X \in L^\infty.$$

(iii) Average value at risk (cf. Föllmer & Schied (2011), Theorem 4.52):

$$\mathrm{AV@R}_\lambda(X) = \max_{Q\in\mathcal{Q}_\lambda} \mathbb{E}_Q[-X], \quad \lambda \in (0, 1],$$

where $\mathcal{Q}_\lambda := \left\{Q \in \mathcal{M}_1(P) \mid \frac{dQ}{dP} \le \frac{1}{\lambda}\ P\text{-a.s.}\right\}$. For $\lambda \in (0, 1)$, the maximum is attained by the measure $Q_0 \in \mathcal{Q}_\lambda$, whose density is given by

$$\frac{dQ_0}{dP} = \frac{1}{\lambda}\left(\mathbb{1}_{\{X<q\}} + \kappa\mathbb{1}_{\{X=q\}}\right),$$

where q is a $\lambda-$quantile of X and κ is defined by

$$\kappa := \begin{cases} 0, & \text{if } P[X = q] = 0, \\ \dfrac{\lambda - P[X < q]}{P[X = q]}, & \text{otherwise.} \end{cases}$$

With $g(x) := 0$ for $x \le \frac{1}{\lambda}$ and $g(x) := +\infty$ for $x > \frac{1}{\lambda}$ in (ii), $\mathrm{AV@R}_\lambda$ belongs to the class of divergence risk measures.

(iv) Utility-based shortfall risk (cf. Föllmer & Schied (2011), Theorem 4.115): Letting $l : \mathbb{R} \to \mathbb{R}$ be an increasing, non-constant, convex loss function, and let λ denote a threshold level. The utility-based shortfall risk satisfies the robust representation

$$\mathrm{UBSR}_{l,\lambda}(X) = \max_{Q\in\mathcal{M}_1(P)} \left\{\mathbb{E}_Q[-X] - \inf_{z>0} \frac{1}{z}\left(\lambda + \mathbb{E}\left[l^*\left(z\,\tfrac{dQ}{dP}\right)\right]\right)\right\}, \quad X \in L^\infty,$$

in terms of the convex conjugate $l^*(y) := \sup_{x\in\mathbb{R}}\{yx - l(x)\}$ of the convex loss function l and the minimal penalty function

$$\alpha^{\min}(Q) = \inf_{z>0} \frac{1}{z}\left(\lambda + \mathbb{E}\left[l^*\left(z\,\tfrac{dQ}{dP}\right)\right]\right), \quad Q \in \mathcal{M}_1(P).$$

For an exponential loss function and the threshold level $\lambda = 1$, this risk measure coincides with the entropic risk measure in (i), cf. Appendix A, Lemma A.0.10 (i).

(v) Optimized certainty equivalents: For an utility function u, the *optimized certainty equivalent* of a financial position $X \in \mathcal{X}$ is defined as

$$\mathrm{OCE}_u(X) := \sup_{\eta\in\mathbb{R}}\{\eta + E_P[u(X - \eta)]\}.$$

This can be interpreted as the present value of an optimal allocation of the uncertain future income X between a certain present amount η and the uncertain future amount $X - \eta$.

Up to sign, the OCE coincides with a divergence risk measure. Indeed, let g be a divergence function, and let g^* denote its convex conjugate. Since (see (ii))

$$\inf_{Q\in\mathcal{M}_1(P)}\{\mathbb{E}_Q[X]+I_g(Q|P)\}=\sup_{\eta\in\mathbb{R}}\{\eta-\mathbb{E}[g^*(\eta-X)]\},$$

we have $\mathrm{OCE}_u(X)=-\rho_g(X)$, for $u(z)\equiv -g^*(-z)$, i. e., the divergence function g is given by $g(y)=\sup_{x\in\mathbb{R}}\{xy-l(x)\}$, see, e. g., Föllmer & Weber (2015). This is the convex conjugate function of the loss function l associated to u via $l(x)=-u(-x)$. Hence, $g(y)=l^*(y)$ and $u(x)=-g^*(-x)$, and we obtain

$$\mathrm{OCE}_{-g^*(-x)}(X)=-\rho_{g(x)}(X)\quad\text{and}\quad\mathrm{OCE}_{u(x)}(X)=-\rho_{l^*(x)}(X).$$

For an exponential utility function $u(t)=1-e^{-\beta t}$, we obtain

$$\rho_{\mathrm{OCE}_u}(X)=\mathrm{UBSR}_{l,\frac{1}{\beta}}(X)-\tfrac{\beta-1}{\beta}$$

For $\beta=1$, this risk measure coincides with the entropic risk measure $\rho_\beta(X)=\rho_1(X)$ in (i). For the utility function $u(t)=\min\{0,\alpha\, t\}$, $\alpha>1$, we have

$$\rho_{\mathrm{OCE}_u}(X)=\mathrm{AV@R}_{\frac{1}{\alpha}}(X),$$

cf. Appendix A, Lemma A.0.10 (ii) & (iii).

C.2 | Special Case of Subgradient Capital Allocation

The Euler allocation principle requires a positively homogeneous risk measure. To extend the Euler allocation to convex, but not necessarily positively homogeneous risk measures, we introduce the subgradient allocation principle and derive explicit formulae for its computation. To the best of our knowledge, these precise capital allocation principles with respect to certain risk measures are new to the literature. In contrast to Kromer & Overbeck (2014), our definition of the subgradient capital allocation principle is based on the robust representation of convex risk measures and, in particular, involves the minimal penalty function. The subgradient allocation principle can be embedded into our set-valued framework similar to the Euler allocation principle. In particular, the subgradient allocation principle corresponds to the consolidated approach (insensitive) together with the management's objective function to minimize the total additional capital.

To outline the mathematical details, consider the setting of Section C.1. We define the subdifferential following Kromer & Overbeck (2014), Section 2. For further reading on the weak*-subdifferential, we refer to Delbaen (2000).

Definition C.2.1. Let ρ be a convex risk measure on L^p.

(i) The subdifferential of ρ at $X\in\mathrm{dom}\rho, 1\le p<\infty$, is the set

$$\partial\rho(X)=\{\varphi\in L^q\,|\,\rho(X+Y)\ge\rho(X)+\mathbb{E}[\varphi Y]\ \forall Y\in L^p\},$$

where L^q is the dual space of L^p, i. e., q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) For $p = \infty$ consider the weak*-subdifferential

$$\partial\rho(X) = \{\varphi \in L^1 \mid \rho(X+Y) \geq \rho(X) + \mathbb{E}[\varphi Y] \; \forall Y \in L^\infty\}.$$

(iii) If $\partial\rho(X) \neq \emptyset$, then ρ is called *subdifferentiable* at X.

If ρ is a convex risk measure, then ρ is continuous and subdifferentiable on the interior of its domain, see Kromer & Overbeck (2014).

Proposition C.2.2 and Corollary C.2.3 characterize the elements of the subdifferential defined in Definition C.2.1. They are the key prerequisites for the concrete computation of the subgradient allocation principle. We refer to Appendix C.1 for details, in particular, for the equality of minimal penalty functions and convex conjugates in our setting.

Proposition C.2.2. *Let ρ be a proper convex risk measure. Then, $\varphi \in \partial\rho(X)$ if and only if*

$$\rho(X) = \mathbb{E}[-\varphi X] - \alpha^{\min}(Q),$$

where $\alpha^{\min}$ is the minimal penalty function in the robust representation of the risk measure ρ.

Proof. The proof is given in Ekeland & Témam (1999), Prop. 5.1 in Chapter I. Remember that $\varphi = \frac{dQ}{dP}$ and thus $\rho(X) = \mathbb{E}_Q[-X] - \alpha^{\min}(Q) = \mathbb{E}[-\varphi X] - \alpha^{\min}(Q)$. □

Corollary C.2.3. *Let ρ be a proper convex risk measure. Then, $\varphi \in \partial\rho(X)$ if and only if*

$$\varphi \in \arg\max_{\tilde{\varphi} \in \mathcal{X}'} \{\mathbb{E}[-\tilde{\varphi} X] - \alpha^{\min}(\tilde{Q})\},$$

where $\alpha^{\min}$ is the minimal penalty function in the robust representation of the risk measure ρ.

Proof. By Föllmer & Schied (2011), Theorem 4.16, any convex risk measure ρ on $\mathcal{X}(\mathbb{R})$ is of the form

$$\rho(X) = \max_{Q \in \mathcal{M}_{1,f}} \{\mathbb{E}_Q[-X] - \alpha^{\min}(Q)\}, \quad X \in \mathcal{X}(\mathbb{R}).$$

Together with Prop. C.2.2, we thus see that

$$\begin{aligned} \varphi = \frac{dQ}{dP} \in \partial\rho(X) \;&\Leftrightarrow\; Q \in \arg\max_{\tilde{Q} \in \mathcal{M}_{1,f}} \rho(X) = \arg\max_{\tilde{Q} \in \mathcal{M}_{1,f}} \{\mathbb{E}_{\tilde{Q}}[-X] - \alpha^{\min}(\tilde{Q})\} \\ &\Leftrightarrow\; \varphi \in \arg\max_{\tilde{\varphi} \in \mathcal{X}'} \{\mathbb{E}[-\tilde{\varphi} X] - \alpha^{\min}(\tilde{Q})\}. \end{aligned}$$

□

We are now ready to define our subgradient allocation principle based on the robust representation of convex risk measures and the corresponding minimal penalty function $\alpha^{\min}$. Recall that $\Lambda^w(E(e, \mathbf{0})) = \sum_{i=1}^n w_i E_i(e, \mathbf{0}) \in \mathcal{X}(\mathbb{R})$ and $E^{\mathbf{0}}(e,k) = E(e, \mathbf{0}) \in \mathcal{X}(\mathbb{R}^n)$.

Definition C.2.4. Let $E^{\mathbf{0}} \in \mathcal{Y}$ be the random field given in Chapter 3, (3.5). Let $\rho : L^p \to \mathbb{R}, 1 \leq p \leq \infty$, be a convex risk measure which is subdifferentiable at $\Lambda^w\left(E(e, \mathbf{0})\right) \in \mathrm{dom}\rho$. The *subgradient allocation principle* is given by

$$A_\partial : \mathcal{Y} \times W \to \mathbb{R}^n, \quad (E^{\mathbf{0}}, w) \mapsto \begin{pmatrix} A_{\partial,1}\left(E^{\mathbf{0}}, w\right) \\ \vdots \\ A_{\partial,n}\left(E^{\mathbf{0}}, w\right) \end{pmatrix}$$

with

$$A_{\partial,i}(E^{\mathbf{0}}, w) = w_i \, \mathbb{E}\left[-\varphi E_i(e, \mathbf{0})\right] - \gamma_i \alpha^{\min}(Q), \quad \varphi \in \partial\rho\left(\Lambda^w\left(E(e, \mathbf{0})\right)\right),$$

where $\sum_{i=1}^n \gamma_i = 1$ and $\alpha^{\min}$ is the minimal penalty function in the robust representation of ρ.

For $w = \mathbf{1}$, Definition C.2.4 leads to

$$A_{\partial,i}(E^{\mathbf{0}}, \mathbf{1}) = \mathbb{E}\left[-\varphi E_i(e, \mathbf{0})\right] - \gamma_i \alpha^{\min}(Q), \quad \varphi \in \partial\rho(\Lambda^{\mathbf{1}}\left(E(e, \mathbf{0})\right)).$$

Since $\varphi = \frac{dQ}{dP} \in \partial\rho\left(\Lambda^{\mathbf{1}}\left(E(e, \mathbf{0})\right)\right)$, it follows that (see Proposition C.2.2)

$$\begin{aligned} \sum_{i=1}^n A_{\partial,i}(E^{\mathbf{0}}, \mathbf{1}) &= \mathbb{E}\left[-\varphi \sum_{i=1}^n E_i(e, \mathbf{0})\right] - \alpha^{\min}(Q) \sum_{i=1}^n \gamma_i = \mathbb{E}\left[-\varphi \Lambda^{\mathbf{1}}\left(E(e, \mathbf{0})\right)\right] - \alpha^{\min}(Q) \\ &= \rho(\Lambda^{\mathbf{1}}\left(E(e, \mathbf{0})\right)), \end{aligned}$$

i. e., the full allocation property is satisfied in analogy to Eq. (3.6) in Chapter 3. The embedding of the subgradient principle into our set-valued framework of capital allocation presented in Chapter 3 is along the lines of the embedding of the Euler allocation principle, since both allocation principles are based on a regulatory system that sums up individual positions (consolidated approach (insensitive)) in order to allocate risk capital, and both approaches minimize the total risk capital at the same time, see Corollary 3.2.6 in Chapter 3.

Remark C.2.5. Recall that a convex risk measure is coherent if and only if its penalty function satisfies $\alpha^{\min} \in \{0, \infty\}$. Thus, for coherent risk measures the subgradient allocation principle reduces to

$$A_{\partial,i}(E^{\mathbf{0}}, w) = w_i \, \mathbb{E}\left[-\varphi E_i(e, \mathbf{0})\right], \quad \varphi \in \partial\rho\left(\Lambda^w\left(E(e, \mathbf{0})\right)\right), i \in N.$$

In the following corollaries, we derive explicit results for the subgradient allocation for several risk measures. We begin with an alternative formulation of the Euler capital allocation for average value at risk resulting from the subgradient allocation principle, cf. Chapter 3, Section 3.6 (iii). Both allocations are equivalent.

Corollary C.2.6. *Let ρ be the average value at risk at level $\lambda \in (0, 1)$. The subgradient* *allocation is given by*

$$A_{\partial,i}^{\mathrm{AV@R}_\lambda}(E^{\mathbf{0}}, w) = w_i \mathbb{E}\left[-\varphi E_i(e, \mathbf{0})\right], \quad i = 1, \dots, n,$$

where $\varphi = \frac{1}{\lambda}\left(\mathbb{1}_{\{\Lambda^w(E(e,\mathbf{0}))<-\mathrm{V@R}_\lambda(\Lambda^w(E(e,\mathbf{0})))\}} + \kappa \mathbb{1}_{\{\Lambda^w(E(e,\mathbf{0}))=-\mathrm{V@R}_\lambda(\Lambda^w(E(e,\mathbf{0})))\}}\right)$ *and* κ *is defined by*

$$\kappa := \begin{cases} 0, & \text{if } P\left[\Lambda^w\left(E(e,\mathbf{0})\right) = -\mathrm{V@R}_\lambda\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right)\right] = 0, \\ \frac{\lambda - P[\Lambda^w(E(e,\mathbf{0}))<-\mathrm{V@R}_\lambda(\Lambda^w(E(e,\mathbf{0})))]}{P[\Lambda^w(E(e,\mathbf{0}))=-\mathrm{V@R}_\lambda(\Lambda^w(E(e,\mathbf{0})))]}, & \text{otherwise.} \end{cases}$$

Proof. By Remark C.2.5, the subgradient capital allocation for the coherent risk measure AV@R is given by

$$A_{\partial,i}^{\mathrm{AV@R}_\lambda}(E^{\mathbf{0}}, w) = w_i \, \mathbb{E}\left[-\varphi E_i(e, \mathbf{0})\right], \quad i = 1, \dots, n,$$

where $\varphi \in \partial \mathrm{AV@R}_\lambda\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right)$.
From Corollary C.2.3, we obtain that $\varphi \in \partial \mathrm{AV@R}_\lambda\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right)$ if and only if φ maximizes the robust representation of AV@R. Now, we know from Example C.1.2 (iii) that the robust representation is given by

$$\mathrm{AV@R}_\lambda\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right) = \max_{Q\in\mathcal{Q}_\lambda} \mathbb{E}_Q\left[-\Lambda^w\left(E(e,\mathbf{0})\right)\right],$$

where for $\lambda \in (0,1)$ the maximum is attained by the measure with density

$$\varphi = \frac{1}{\lambda}\left(\mathbb{1}_{\{\Lambda^w(E(e,\mathbf{0}))<-\mathrm{V@R}_\lambda(\Lambda^w(E(e,\mathbf{0})))\}} + \kappa \mathbb{1}_{\{\Lambda^w(E(e,\mathbf{0}))=-\mathrm{V@R}_\lambda(\Lambda^w(E(e,\mathbf{0})))\}}\right)$$

and κ is defined such as given in the corollary. □

In contrast to average value at risk, the entropic risk measure, utility-based shortfall risk as well as optimized certainty equivalent risk measures are not positively homogeneous in general. Hence, the Euler allocation cannot be applied, while alternative capital allocation principles such as the subgradient allocation can be implemented.

Corollary C.2.7. *Let* ρ *be the utility-based shortfall risk measure with exponential loss function* $l(x) = e^{\beta x}$*. The subgradient allocation is given by*

$$A_{\partial,i}^{UBSR_{l,\lambda}}(E^{\mathbf{0}}, w) = w_i \mathbb{E}\left[-\varphi E_i(e, \mathbf{0})\right] - \tfrac{\gamma_i}{\beta}\left(\mathbb{E}\left[\varphi \log(\varphi)\right] + \log(\lambda)\right), \quad i = 1, \dots, n,$$

where $\varphi = e^{-\beta\Lambda^w(E(e,\mathbf{0}))}/\mathbb{E}[e^{-\beta\Lambda^w(E(e,\mathbf{0}))}]$ *is chosen such that* $\sum_{i=1}^n \gamma_i = 1$*. Setting* $\lambda = 1$*, the subgradient allocation for the entropic risk measure follows.*

Proof. For the convex risk measure UBSR, Definition C.2.4 provides the subgradient capital allocation

$$A_{\partial,i}^{\mathrm{UBSR}_{l,\lambda}}(E^{\mathbf{0}}, w) = w_i \mathbb{E}\left[-\varphi E_i(e, \mathbf{0})\right] - \gamma_i \, \alpha^{\min}(Q), \quad i = 1, \dots, n,$$

where $\varphi \in \partial \mathrm{UBSR}_{l,\lambda}\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right)$ and $\sum_{i=1}^n \gamma_i = 1$. From Corollary C.2.3, we obtain that $\varphi \in \partial \mathrm{UBSR}_{l,\lambda}\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right)$ if and only if φ maximizes the robust representation of $\mathrm{UBSR}_{l,\lambda}$. In the case of an exponential loss function, the robust representation of UBSR is given by (see Lemma A.0.10 and Example C.1.2 (iv))

$$\begin{aligned}
\mathrm{UBSR}_{l,\lambda}\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right) &= \rho_\beta\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right) - \tfrac{1}{\beta}\log(\lambda) \\
&= \max_{Q\in\mathcal{M}_1(P)} \left\{\mathbb{E}_Q\left[-\Lambda^w\left(E(e,\mathbf{0})\right)\right] - \tfrac{1}{\beta}H(Q|P)\right\} - \tfrac{1}{\beta}\log(\lambda) \\
&= \tfrac{1}{\beta}\log\mathbb{E}[e^{-\beta\Lambda^w(E(e,\mathbf{0}))}] - \tfrac{1}{\beta}\log(\lambda), \quad \beta > 0,
\end{aligned}$$

where the maximum is attained by the measure with density $\varphi = \frac{e^{-\beta\Lambda^w(E(e,\mathbf{0}))}}{\mathbb{E}[e^{-\beta\Lambda^w(E(e,\mathbf{0}))}]}$. The minimal penalty function is given by

$$\begin{aligned}
\alpha^{\min}(Q) &= \sup_{\Lambda^w(E(e,\mathbf{0}))\in L^\infty} \left\{\mathbb{E}_Q\left[-\Lambda^w\left(E(e,\mathbf{0})\right)\right] - \mathrm{UBSR}_{l,\lambda}\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right)\right\} \\
&= \sup_{\Lambda^w(E(e,\mathbf{0}))\in L^\infty} \left\{\mathbb{E}_Q\left[-\Lambda^w\left(E(e,\mathbf{0})\right)\right] - \tfrac{1}{\beta}(\log\mathbb{E}[e^{-\beta\Lambda^w(E(e,\mathbf{0}))}] - \log(\lambda))\right\} \\
&= \sup_{\Lambda^w(E(e,\mathbf{0}))\in L^\infty} \left\{\mathbb{E}_Q[-\Lambda^w\left(E(e,\mathbf{0})\right)] - \tfrac{1}{\beta}\log\mathbb{E}[e^{-\beta\Lambda^w(E(e,\mathbf{0}))}]\right\} + \tfrac{1}{\beta}\log(\lambda) \\
&= \tfrac{1}{\beta}H(Q|P) + \tfrac{1}{\beta}\log(\lambda) = \tfrac{1}{\beta}\left(\mathbb{E}\left[\varphi\log(\varphi)\right] + \log(\lambda)\right).
\end{aligned}$$

□

Corollary C.2.8. *Let ρ_{OCE_u} be the optimized certainty equivalent risk measure with exponential utility function $u(x) = 1 - e^{-\beta x}$. The subgradient allocation is given by*

$$A_{\partial,i}^{\rho_{OCE_u}}(E^{\mathbf{0}}, w) = w_i\mathbb{E}\left[-\varphi E_i(e,\mathbf{0})\right] - \tfrac{\gamma_i}{\beta}\left(\mathbb{E}\left[\varphi\log(\varphi)\right] + \log\left(\tfrac{1}{\beta}\right) + (\beta-1)\right), \quad i = 1,\dots,n,$$

where $\varphi = e^{-\beta\Lambda^w(E(e,\mathbf{0}))}/\mathbb{E}[e^{-\beta\Lambda^w(E(e,\mathbf{0}))}]$ and γ_i is chosen such that $\sum_{i=1}^n \gamma_i = 1$.

Proof. Since ρ_{OCE_u} is a convex risk measure, the subgradient capital allocation is given by

$$A_{\partial,i}^{\rho_{\mathrm{OCE}_u}}(E^{\mathbf{0}}, w) = w_i\mathbb{E}\left[-\varphi E_i(e,\mathbf{0})\right] - \gamma_i\,\alpha^{\min}(Q), \quad i = 1,\dots,n,$$

where $\varphi \in \partial\rho_{\mathrm{OCE}_u}\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right)$ and $\sum_{i=1}^n \gamma_i = 1$ (cf. Definition C.2.4). Corollary C.2.3 yields that $\varphi \in \partial\rho_{\mathrm{OCE}_u}\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right)$ if and only if φ maximizes the robust representation of ρ_{OCE_u}.

In the case of an exponential utility function, the robust representation of OCE_u is given by (see Example C.1.2 (v))

$$\begin{aligned}
\rho_{\mathrm{OCE}_u}\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right) &= \mathrm{UBSR}_{l,\frac{1}{\beta}}\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right) - \tfrac{\beta-1}{\beta} \\
&= \max_{Q\in\mathcal{M}_1(P)} \left\{\mathbb{E}_Q\left[-\Lambda^w\left(E(e,\mathbf{0})\right)\right] - \tfrac{1}{\beta}\left(H(Q|P) + \log\left(\tfrac{1}{\beta}\right)\right)\right\} - \tfrac{\beta-1}{\beta} \\
&= \tfrac{1}{\beta}\left(\log\mathbb{E}[e^{-\beta\Lambda^w(E(e,\mathbf{0}))}] - \log\left(\tfrac{1}{\beta}\right)\right) - \tfrac{\beta-1}{\beta} \\
&= \tfrac{1}{\beta}\left(\log\mathbb{E}[e^{-\beta\Lambda^w(E(e,\mathbf{0}))}] - \log\left(\tfrac{1}{\beta}\right) - (\beta-1)\right), \quad \beta > 0,
\end{aligned}$$

where the maximum is attained by the measure with density $\varphi = \frac{e^{-\beta\Lambda^w(E(e,\mathbf{0}))}}{\mathbb{E}[e^{-\beta\Lambda^w(E(e,\mathbf{0}))}]}$. The minimal penalty function for the OCE-risk measure is given by

$$\begin{aligned}
\alpha^{\min}(Q) &= \sup_{\Lambda^w(E(e,\mathbf{0}))\in L^\infty} \left\{\mathbb{E}_Q\left[-\Lambda^w\left(E(e,\mathbf{0})\right)\right] - \rho_{\mathrm{OCE}_u}\left(\Lambda^w\left(E(e,\mathbf{0})\right)\right)\right\} \\
&= \sup_{\Lambda^w(E(e,\mathbf{0}))\in L^\infty} \left\{\mathbb{E}_Q\left[-\Lambda^w\left(E(e,\mathbf{0})\right)\right]\right. \\
&\qquad\qquad \left. -\frac{1}{\beta}\left(\log\mathbb{E}[e^{-\beta\Lambda^w(E(e,\mathbf{0}))}] - \log\left(\frac{1}{\beta}\right) - (\beta-1)\right)\right\} \\
&= \sup_{\Lambda^w(E(e,\mathbf{0}))\in L^\infty} \left\{\mathbb{E}_Q\left[-\Lambda^w\left(E(e,\mathbf{0})\right)\right] - \frac{1}{\beta}\log\mathbb{E}[e^{-\beta\Lambda^w(E(e,\mathbf{0}))}]\right\} \\
&\qquad\qquad +\frac{1}{\beta}\left(\log\left(\frac{1}{\beta}\right) + (\beta-1)\right) \\
&= \frac{1}{\beta}H(Q|P) + \frac{1}{\beta}\left(\log\left(\frac{1}{\beta}\right) + (\beta-1)\right).
\end{aligned}$$

□

D | Further Computations

In this section, we provide detailed computations of results presented in the thesis.

D.1 | Computations of Section 1.4.3

In this section, we provide detailed computations of results presented in Chapter 1, Section 1.4.3.

Assume that

$$u(t) := \begin{cases} 1 - (t-1)^4, & \text{if } t \leq 1, \\ 1, & \text{otherwise.} \end{cases}$$

We compare two different distributions for the financial position X. In the first case, we assume that X is normally distributed with variance $\sigma^2 = \frac{5}{3}$; in the second case, we suppose that X has a Student's t-distribution with $\nu = 5$ degrees of freedom. Both random variables have the same mean and variance, but different tail behavior. In both cases, the expression $\eta + \mathbb{E}\left[u\left(X - \eta\right)\right]$ can be written as a closed-form expression of η:

$$\begin{aligned}
\mathrm{OCE}_u(X) &= \sup_{\eta \in \mathbb{R}} \; \{\eta + \mathbb{E}[u(X-\eta)]\} \\
&= \sup_{\eta \in \mathbb{R}} \; \left\{\eta + \mathbb{E}[\mathbb{1}_{\{X-\eta>1\}}] + \mathbb{E}[(1-(X-\eta-1)^4)\cdot \mathbb{1}_{\{X-\eta\leq 1\}}]\right\} \\
&= \sup_{\eta \in \mathbb{R}} \; \left\{\eta + \int_{1+\eta}^{\infty} f_X(x)\,dx + \int_{-\infty}^{1+\eta} f_X(x)\,dx - \int_{-\infty}^{1+\eta} (x-\eta-1)^4 \cdot f_X(x)\,dx\right\} \\
&= \sup_{\eta \in \mathbb{R}} \; \left\{\eta + 1 - \int_{-\infty}^{1+\eta} (x-\eta-1)^4 \cdot f_X(x)\,dx\right\} \\
&=: \sup_{\eta \in \mathbb{R}} \; \{f(\eta)\},
\end{aligned}$$

where f_X is the density of the random variable X. Computing the derivative yields

$$\begin{aligned}
f'(\eta) = 1 - \frac{d}{d\eta}\Bigg(&\int_{-\infty}^{\eta+1} x^4 \cdot f_X(x)\,dx - 4(\eta+1)\int_{-\infty}^{\eta+1} x^3 \cdot f_X(x)\,dx \\
&+ 6(\eta+1)^2 \int_{-\infty}^{\eta+1} x^2 \cdot f_X(x)\,dx - 4(\eta+1)^3 \int_{-\infty}^{\eta+1} x \cdot f_X(x)\,dx \\
&+(\eta+1)^4 \int_{-\infty}^{\eta+1} f_X(x)\,dx\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \Bigg((\eta+1)^4 \cdot f_X(\eta+1) - \Bigg(4\int_{-\infty}^{\eta+1} x^3 \cdot f_X(x)\,dx + 4(\eta+1)^4 \cdot f_X(\eta+1) \Bigg) \\
&\qquad + \Bigg(12(\eta+1)\int_{-\infty}^{\eta+1} x^2 \cdot f_X(x)\,dx + 6(\eta+1)^4 \cdot f_X(\eta+1) \Bigg) \\
&\qquad - \Bigg(12(\eta+1)^2\int_{-\infty}^{\eta+1} x \cdot f_X(x)\,dx + 4(\eta+1)^4 \cdot f_X(\eta+1) \Bigg) \\
&\qquad + \Bigg(4(\eta+1)^3\int_{-\infty}^{\eta+1} f_X(x)\,dx + (\eta+1)^4 \cdot f_X(\eta+1) \Bigg) \Bigg) \\
&= 1 - \Bigg(-4\int_{-\infty}^{\eta+1} x^3 \cdot f_X(x)\,dx + 12(\eta+1)\int_{-\infty}^{\eta+1} x^2 \cdot f_X(x)\,dx \\
&\qquad -12(\eta+1)^2\int_{-\infty}^{\eta+1} x \cdot f_X(x)\,dx + 4(\eta+1)^3\int_{-\infty}^{\eta+1} f_X(x)\,dx \Bigg)
\end{aligned}
$$

Using, e.g., Mathematica, we obtain

$$
f'(\eta) = 0 \;\Leftrightarrow\; \eta^* = -2.16359 \quad \text{and} \quad f'(\eta) = 0 \;\Leftrightarrow\; \eta^* = -3.73624
$$

respectively. Now, the OCE follows by the subsequent calculation: If X is normally distributed, we obtain

$$
\int_{-\infty}^{-2.16359+1} (x+2.16359-1)^4 \frac{1}{\sqrt{2\pi\frac{5}{3}}} e^{-\frac{1}{2}\frac{x^2}{\frac{5}{3}}}\,dx = 0.48751,
$$

which implies $\mathrm{OCE}_u(X) = -2.16359 + 1 - 0.48751 = -1.6511$. If X has a Student's t-distribution, it is

$$
\begin{aligned}
&\int_{-\infty}^{-3.73624+1} (x+3.73624-1)^4 \frac{\Gamma(3)}{\sqrt{5\pi}\,\Gamma\left(\frac{5}{2}\right)} \left(1+\frac{x^2}{5}\right)^{-3} dx \\
&= \int_{-\infty}^{-3.73624+1} (x+3.73624-1)^4 \frac{2}{\sqrt{5\pi}\frac{3\sqrt{\pi}}{4}} \left(1+\frac{x^2}{5}\right)^{-3} dx = 3.19451,
\end{aligned}
$$

which implies $\mathrm{OCE}_u(X) = -3.73624 + 1 - 3.19451 = -5.93075$.

D.2 | Computations of Section 5.3

In this section, we provide detailed computations of results presented in Chapter 5, Section 5.3.

D.2.1 | Computations of Theorem 5.3.4

We provide a detailed proof of Theorem 5.3.4.

(i) CARA-utility:

We first consider the case $X \sim \text{Ber}(\hat{x}, p)$. We compute

$$\begin{aligned}
\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right] &= 1-\mathbb{E}\left[e^{-\kappa[(1-\nu)(w-X)+\nu(w-\pi)]}\right] \\
&= 1-\mathbb{E}\left[e^{-\kappa(1-\nu)w}e^{\kappa(1-\nu)X}e^{-\kappa\nu(w-\pi)}\right] \\
&= 1-e^{-\kappa[(1-\nu)w+\nu(w-\pi)]}\mathbb{E}\left[e^{\kappa(1-\nu)X}\right] \\
&= 1-e^{-\kappa(w-\nu\pi)}\ \left(e^{\kappa(1-\nu)\hat{x}}p+1-p\right)
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right] &= \frac{\partial}{\partial\nu}\left(1-e^{-\kappa w+\kappa\nu\pi}\cdot e^{\kappa\hat{x}-\kappa\nu\hat{x}}\cdot p-e^{-\kappa w+\kappa\nu\pi}+p\cdot e^{-\kappa w+\kappa\nu\pi}\right) \\
&= -\kappa\pi e^{-\kappa w+\kappa\nu\pi}\cdot e^{\kappa\hat{x}-\kappa\nu\hat{x}}\cdot p+e^{-\kappa w+\kappa\nu\pi}\cdot\kappa\hat{x}e^{\kappa\hat{x}-\kappa\nu\hat{x}}\cdot p \\
&\quad -\kappa\pi e^{-\kappa w+\kappa\nu\pi}+p\kappa\pi e^{-\kappa w+\kappa\nu\pi} \\
&= \kappa e^{-\kappa w+\kappa\nu\pi}e^{\kappa\hat{x}-\kappa\nu\hat{x}}\left(-p\pi+\hat{x}p-\pi e^{-\kappa\hat{x}+\kappa\nu\hat{x}}+p\pi e^{-\kappa\hat{x}+\kappa\nu\hat{x}}\right) \\
&= \kappa e^{-\kappa w+\kappa\nu\pi+\kappa\hat{x}-\kappa\nu\hat{x}}\left(\pi\left(-p-e^{-\kappa\hat{x}+\kappa\nu\hat{x}}+pe^{-\kappa\hat{x}+\kappa\nu\hat{x}}\right)+\hat{x}p\right) \\
&= \kappa e^{\kappa(-w+\nu\pi+\hat{x}-\nu\hat{x})}\left(\pi e^{\kappa(\nu\hat{x}-\hat{x})}(-1+p)-\pi p+\hat{x}p\right) \\
&= \kappa e^{\kappa(\pi\nu-\nu\hat{x}-w+\hat{x})}\left(\pi(p-1)e^{\kappa(\nu-1)\hat{x}}+p(\hat{x}-\pi)\right)
\end{aligned}$$

At the boundary $\nu = 0$, we obtain

$$\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right]_{|\nu=0} \;=\; \kappa e^{\kappa(\hat{x}-w)}\left(\pi(p-1)e^{-\kappa\hat{x}}+p(\hat{x}-\pi)\right).$$

Thus, $\nu(\pi) = 0$ is the optimal solution, if and only if

$$\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right]_{|\nu=0} \;\le\; 0 \quad\Longleftrightarrow\quad \pi \;\ge\; \frac{p\hat{x}e^{\kappa\hat{x}}}{1-p+pe^{\kappa\hat{x}}}$$

since

$$\begin{aligned}
& \kappa e^{\kappa(\hat{x}-w)}\left(\pi(p-1)e^{-\kappa\hat{x}}+p(\hat{x}-\pi)\right) \;\le\; 0 \\
\Leftrightarrow\quad & \pi(p-1)e^{-\kappa\hat{x}}+p\hat{x}-p\pi \;\le\; 0 \\
\Leftrightarrow\quad & \pi\cdot\left((p-1)e^{-\kappa\hat{x}}-p\right) \;\le\; -p\hat{x} \\
\Leftrightarrow\quad & \pi \;\ge\; \frac{p\hat{x}}{(1-p)e^{-\kappa\hat{x}}+p} = \frac{p\hat{x}e^{\kappa\hat{x}}}{1-p+pe^{\kappa\hat{x}}}
\end{aligned}$$

At the boundary $\nu = 1$, we obtain $\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right]_{|\nu=1} = \kappa e^{\kappa(\pi-w)}(p\hat{x}-\pi)$. Thus, the optimal solution is $\nu(\pi) = 1$, iff $\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right]_{|\nu=1} \ge 0$, i.e., $p\hat{x}-\pi \ge 0 \;\Leftrightarrow\; \pi \le p\hat{x}$. In all other cases, we need to solve $\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right] = 0$, leading to the stated solution, since:

$$\begin{aligned}
& \kappa e^{\kappa(\pi\nu-\nu\hat{x}-w+\hat{x})}(\pi(p-1)e^{\kappa(\nu-1)\hat{x}}+p(\hat{x}-\pi)) \stackrel{!}{=} 0 \\
\Leftrightarrow \quad & \pi(p-1)e^{\kappa(\nu-1)\hat{x}}+p(\hat{x}-\pi) = 0 \\
\Leftrightarrow \quad & \frac{-p\hat{x}+p\pi}{\pi(p-1)} = e^{\kappa(\nu-1)\hat{x}} \\
\Leftrightarrow \quad & \ln\left(\frac{-p\hat{x}+p\pi}{\pi(p-1)}\right) = \kappa(\nu-1)\hat{x} \\
\Leftrightarrow \quad & 1+\frac{1}{\kappa\hat{x}}\cdot\ln\left(\frac{-p\hat{x}+p\pi}{\pi(p-1)}\right) = \nu \\
\Leftrightarrow \quad & \nu = 1+\left(-\frac{1}{\kappa\hat{x}}\cdot\ln\left(\frac{\pi p-\pi}{-p\hat{x}+p\pi}\right)\right) \\
\Leftrightarrow \quad & \nu = 1-\frac{1}{\kappa\hat{x}}\cdot\ln\left(\frac{\pi(1-p)}{p(\hat{x}-\pi)}\right) = 1-\frac{1}{\kappa\hat{x}}\cdot\ln\left(\frac{\frac{1-p}{p}}{\frac{\hat{x}-\pi}{\pi}}\right) = 1-\frac{1}{\kappa\hat{x}}\cdot\ln\left(\frac{\frac{1}{p}-1}{\frac{\hat{x}}{\pi}-1}\right)
\end{aligned}$$

The first-order conditions are sufficient due to the strict concavity.
Second, we derive the optimal contract for $X \sim \Gamma(\xi, \mu)$. In this case, we compute

$$\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right] = 1-e^{-\kappa[(1-\nu)w+\nu(w-\pi)]}\mathbb{E}\left[e^{\kappa(1-\nu)X}\right] = 1-e^{-\kappa(w-\nu\pi)}\left(\frac{\mu}{\mu-\kappa(1-\nu)}\right)^{\xi}$$

This implies

$$\begin{aligned}
\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right] &= \frac{\partial}{\partial\nu}\left(1-e^{-\kappa w+\kappa\nu\pi}\cdot\mu^{\xi}\cdot(\mu-\kappa+\kappa\nu)^{-\xi}\right) \\
&= \mu^{\xi}(-\kappa\pi)e^{-\kappa w+\kappa\nu\pi}(\mu-\kappa+\kappa\nu)^{-\xi} \\
&\quad - \mu^{\xi}e^{-\kappa w+\kappa\nu\pi}(-\xi)(\mu-\kappa+\kappa\nu)^{-\xi-1}\kappa \\
&= e^{-\kappa(w-\nu\pi)}\left(-\kappa\pi\left(\frac{\mu}{\mu-\kappa(1-\nu)}\right)^{\xi}+\xi\kappa\mu^{\xi}\frac{1}{(\mu-\kappa(1-\nu))^{\xi+1}}\right) \\
&= e^{-\kappa(w-\nu\pi)}\cdot\left(\frac{\mu}{\mu-\kappa(1-\nu)}\right)^{\xi+1}\left(-\kappa\pi\frac{\mu-\kappa(1-\nu)}{\mu}+\xi\frac{\kappa}{\mu}\right) \\
&= e^{-\kappa(w-\nu\pi)}\cdot\left(\frac{\mu}{\mu-\kappa(1-\nu)}\right)^{\xi+1}\left(-\frac{\kappa}{\mu}\right)(\pi(\mu-\kappa(1-\nu))-\xi)
\end{aligned}$$

At the boundary $\nu = 0$, we obtain

$$\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right]_{|\nu=0} = e^{-\kappa w}\left(\frac{\mu}{\mu-\kappa}\right)^{\xi+1}\left(-\frac{\kappa}{\mu}\right)(\pi(\mu-\kappa)-\xi).$$

Thus, $\nu(\pi) = 0$ is the optimal solution, if and only if

$$\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right]_{|\nu=0} \le 0 \quad \Longleftrightarrow \quad \pi \ge \frac{\xi}{\mu-\kappa}$$

since

$$e^{-\kappa w}\left(\frac{\mu}{\mu-\kappa}\right)^{\xi+1}\left(-\frac{\kappa}{\mu}\right)(\pi(\mu-\kappa)-\xi) \ \leq\ 0 \quad \Leftrightarrow \quad \left(-\frac{\kappa}{\mu}\right)(\pi(\mu-\kappa)-\xi) \ \leq\ 0$$
$$\Leftrightarrow \quad \pi((\mu-\kappa)-\xi) \ \geq\ 0$$
$$\Leftrightarrow \quad \pi \ \geq\ \frac{\xi}{\mu-\kappa}$$

At the boundary $\nu = 1$, we obtain $\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right]_{|\nu=1} = e^{-\kappa(w-\pi)}\left(-\frac{\kappa}{\mu}\right)(\pi\mu-\xi)$. Thus, the optimal solution is $\nu(\pi) = 1$, iff $\frac{\partial}{\partial\nu}\mathbb{E}\left[u_1^\kappa\left(X_\nu\right)\right]_{|\nu=1} \geq 0$, i.e., $\left(-\frac{\kappa}{\mu}\right)(\pi\mu-\xi) \geq 0 \ \Leftrightarrow\ \pi \leq \frac{\xi}{\mu}$.
In all other cases, we need to solve $\frac{\partial}{\partial\nu}\mathbb{E}[u_1^\kappa(X_\nu)] = 0$, leading to the stated solution, since:

$$e^{-\kappa(w-\nu\pi)}\cdot\left(\frac{\mu}{\mu-\kappa(1-\nu)}\right)^{\xi+1}\left(-\frac{\kappa}{\mu}\right)(\pi(\mu-\kappa(1-\nu))-\xi) \ \stackrel{!}{=}\ 0$$
$$\Leftrightarrow \quad \pi(\mu-\kappa(1-\nu))-\xi \ =\ 0$$
$$\Leftrightarrow \quad \pi\mu-\kappa\pi+\pi\kappa\nu-\xi \ =\ 0$$
$$\Leftrightarrow \quad \nu \ =\ 1+\frac{\xi}{\pi\kappa}-\frac{\mu}{\kappa}$$

The first-order conditions are sufficient due to the strict concavity.

(ii) HARA-utility: We compute

$$\begin{aligned}\mathbb{E}\left[u_2^\lambda\left(X_\nu\right)\right] &= \mathbb{E}\left[\frac{1}{\lambda}\cdot((1-\nu)(w-X)+\nu(w-\pi))^\lambda\right]\\ &= \frac{1}{\lambda}\cdot\Big(((1-\nu)(w-\hat{x})+\nu(w-\pi))^\lambda\cdot p\\ &\qquad +((1-\nu)w+\nu(w-\pi))^\lambda\cdot(1-p)\Big)\end{aligned}$$

This implies

$$\begin{aligned}\frac{\partial}{\partial\nu}\mathbb{E}\left[u_2^\lambda\left(X_\nu\right)\right] &= \frac{\partial}{\partial\nu}\left(\frac{1}{\lambda}\cdot\Big((w-\hat{x}-\nu w+\nu\hat{x}+\nu w-\nu\pi)^\lambda\cdot p\right.\\ &\qquad \left.+(w-\nu w+\nu w-\nu\pi)^\lambda\cdot(1-p)\Big)\right)\\ &= \frac{1}{\lambda}\cdot\Big(\lambda\cdot(w-\hat{x}+\nu\hat{x}-\nu\pi)^{\lambda-1}\cdot p\cdot(\hat{x}-\pi)\\ &\qquad +\lambda\cdot(w-\nu\pi)^{\lambda-1}\cdot(1-p)\cdot(-\pi)\Big)\\ &= p(\hat{x}-\pi)(\hat{x}(\nu-1)+w-\nu\pi)^{\lambda-1}+(-\pi)(1-p)(w-\nu\pi)^{\lambda-1}\end{aligned}$$

At the boundary $\nu = 0$, we obtain

$$\frac{\partial}{\partial\nu}\mathbb{E}\left[u_2^\lambda\left(X_\nu\right)\right]_{|\nu=0} \ =\ p(\hat{x}-\pi)(-\hat{x}+w)^{\lambda-1}+(-\pi)(1-p)w^{\lambda-1}.$$

Thus, $\nu(\pi) = 0$ is the optimal solution, if and only if

$$\frac{\partial}{\partial \nu}\mathbb{E}\left[u_2^\lambda(X_\nu)\right]_{|\nu=0} \leq 0 \quad \Longleftrightarrow \quad \pi \geq \frac{p\hat{x}w^{1-\lambda}}{pw^{1-\lambda} + (1-p)(w-\hat{x})^{1-\lambda}}$$

since

$$\begin{aligned} & p(\hat{x}-\pi)(-\hat{x}+w)^{\lambda-1} + (-\pi)(1-p)w^{\lambda-1} \leq 0 \\ \Leftrightarrow \quad & p\hat{x}(w-\hat{x})^{\lambda-1} \leq \pi\left((1-p)w^{\lambda-1} + p(w-\hat{x})^{\lambda-1}\right) \\ \Leftrightarrow \quad & \pi \geq \frac{p\hat{x}(w-\hat{x})^{\lambda-1}}{(1-p)w^{\lambda-1} + p(w-\hat{x})^{\lambda-1}} = \frac{p\hat{x}w^{1-\lambda}}{pw^{1-\lambda} + (1-p)(w-\hat{x})^{1-\lambda}} \end{aligned}$$

compare Appendix D.2.2, part (ii), for the last computation. At the boundary $\nu = 1$, we obtain $\frac{\partial}{\partial \nu}\mathbb{E}\left[u_2^\lambda(X_\nu)\right]_{|\nu=1} = p(\hat{x}-\pi)(w-\pi)^{\lambda-1} + (-\pi)(1-p)(w-\pi)^{\lambda-1}$. Thus, the optimal solution is $\nu(\pi) = 1$, iff $\frac{\partial}{\partial \nu}\mathbb{E}\left[u_2^\lambda(X_\nu)\right]_{|\nu=1} \geq 0 \Leftrightarrow \pi \leq p\hat{x}$, since

$$\begin{aligned} & p(\hat{x}-\pi)(w-\pi)^{\lambda-1} + (-\pi)(1-p)(w-\pi)^{\lambda-1} \geq 0 \\ \Leftrightarrow \quad & p\hat{x}(w-\pi)^{\lambda-1} \geq \pi\left((1-p)(w-\pi)^{\lambda-1} + p(w-\pi)^{\lambda-1}\right) \\ \Leftrightarrow \quad & \pi \leq \frac{p\hat{x}(w-\pi)^{\lambda-1}}{(1-p)(w-\pi)^{\lambda-1} + p(w-\pi)^{\lambda-1}} = p\hat{x} \end{aligned}$$

In all other cases, we need to solve $\frac{\partial}{\partial \nu}\mathbb{E}\left[u_2^\lambda(X_\nu)\right] = 0$, leading to the stated solution, since:

$$\begin{aligned} & p(\hat{x}-\pi)(\hat{x}(\nu-1) + w - \nu\pi)^{\lambda-1} + (-\pi)(1-p)(w-\nu\pi)^{\lambda-1} \overset{!}{=} 0 \\ \Leftrightarrow \quad & p(\hat{x}-\pi)(w-\nu\pi)^{1-\lambda} - \pi(1-p)((\nu-1)\hat{x} - \nu\pi + w)^{1-\lambda} = 0 \\ \Leftrightarrow \quad & (1-p)\pi((\nu-1)\hat{x} - \nu\pi + w)^{1-\lambda} = p(\hat{x}-\pi)(w-\nu\pi)^{1-\lambda} \\ \Leftrightarrow \quad & \pi^\zeta(1-p)^\zeta((\nu-1)\hat{x} - \nu\pi + w) = p^\zeta(\hat{x}-\pi)^\zeta(w-\nu\pi), \qquad \text{where } \zeta = \frac{1}{1-\lambda} \\ \Leftrightarrow \quad & \pi^\zeta(1-p)^\zeta(\nu-1)\hat{x} - (1-p)^\zeta\pi^\zeta\nu\pi + (1-p)^\zeta\pi^\zeta w \\ & = p^\zeta(\hat{x}-\pi)^\zeta w - p^\zeta(\hat{x}-\pi)^\zeta\nu\pi \\ \Leftrightarrow \quad & (1-p)^\zeta\pi^\zeta\nu\hat{x} - (1-p)^\zeta\pi^{\zeta+1}\nu + p^\zeta(\hat{x}-\pi)^\zeta\nu\pi \\ & = (1-p)^\zeta\pi^\zeta\hat{x} - (1-p)^\zeta\pi^\zeta w + p^\zeta(\hat{x}-\pi)^\zeta w \\ \Leftrightarrow \quad & \nu = \frac{\pi^\zeta\left((1-p)^\zeta\hat{x} - (1-p)^\zeta w\right) + p^\zeta(\hat{x}-\pi)^\zeta w}{\pi^\zeta(1-p)^\zeta(\hat{x}-\pi) + p^\zeta(\hat{x}-\pi)^\zeta\pi} \end{aligned}$$

The first-order conditions are sufficient due to the strict concavity.

(iii) Logarithmic utility: We compute

$$\begin{aligned} \mathbb{E}\left[u_2^0(X_\nu)\right] &= \mathbb{E}\left[\log((1-\nu)(w-X) + \nu(w-\pi))\right] \\ &= \log((1-\nu)(w-\hat{x}) + \nu(w-\pi)) \cdot p + \log((1-\nu)w + \nu(w-\pi)) \cdot (1-p) \end{aligned}$$

This implies

$$\begin{aligned}\frac{\partial}{\partial\nu}\mathbb{E}\left[u_2^0\left(X_\nu\right)\right] &= \frac{\partial}{\partial\nu}\left(\log(w-\hat{x}+\nu\hat{x}-\nu\pi)\cdot p+\log(w-\nu\pi)\cdot(1-p)\right)\\ &= \frac{1}{w-\hat{x}+\nu\hat{x}-\nu\pi}\cdot p\cdot(\hat{x}-\pi)+\frac{1}{w-\nu\pi}\cdot(1-p)\cdot(-\pi)\end{aligned}$$

At the boundary $\nu = 0$, we obtain

$$\frac{\partial}{\partial\nu}\mathbb{E}\left[u_2^0\left(X_\nu\right)\right]_{|\nu=0} = \frac{1}{w-\hat{x}}p(\hat{x}-\pi)+(-\pi)(1-p)\frac{1}{w}$$

Thus, $\nu(\pi) = 0$ is the optimal solution, if and only if

$$\frac{\partial}{\partial\nu}\mathbb{E}\left[u_2^0\left(X_\nu\right)\right]_{|\nu=0} \leq 0 \quad\Longleftrightarrow\quad \pi \geq \frac{p\hat{x}w}{w+\hat{x}(p-1)}$$

since

$$\begin{aligned}& \frac{1}{w-\hat{x}}p(\hat{x}-\pi)+(-\pi)(1-p)\frac{1}{w} \leq 0\\ \Leftrightarrow\quad & \frac{1}{w-\hat{x}}p\hat{x} \leq \pi\left(\frac{1-p}{w}+\frac{p}{w-\hat{x}}\right)\\ \Leftrightarrow\quad & \pi \geq \frac{p\hat{x}(w-\hat{x})^{-1}}{(1-p)w^{-1}+p(w-\hat{x})^{-1}} = \frac{p\hat{x}w}{w+\hat{x}(p-1)}\end{aligned}$$

The last equation is obtained by

$$\begin{aligned}\frac{p\hat{x}(w-\hat{x})^{-1}}{(1-p)w^{-1}+p(w-\hat{x})^{-1}} &= \frac{\frac{p\hat{x}}{w-\hat{x}}}{\frac{(1-p)(w-\hat{x})}{w(w-\hat{x})}+\frac{pw}{w(w-\hat{x})}} = \frac{\frac{p\hat{x}}{w-\hat{x}}}{\frac{(1-p)(w-\hat{x})+pw}{w(w-\hat{x})}}\\ &= \frac{p\hat{x}}{w-\hat{x}}\cdot\frac{w(w-\hat{x})}{(1-p)(w-\hat{x})+pw} = \frac{p\hat{x}w}{(1-p)(w-\hat{x})+pw}\\ &= \frac{p\hat{x}w}{w-\hat{x}+p\hat{x}} = \frac{p\hat{x}w}{w+\hat{x}(p-1)}\end{aligned}$$

At the boundary $\nu = 1$, we obtain $\frac{\partial}{\partial\nu}\mathbb{E}\left[u_2^0\left(X_\nu\right)\right]_{|\nu=1} = \frac{1}{w-\pi}p(\hat{x}-\pi)+(-\pi)(1-p)\frac{1}{w-\pi}$. Thus, the optimal solution is $\nu(\pi) = 1$, iff $\frac{\partial}{\partial\nu}\mathbb{E}\left[u_2^0\left(X_\nu\right)\right]_{|\nu=1} \geq 0$, i.e., $p(\hat{x}-\pi) \geq \pi(1-p) \Leftrightarrow \pi \leq p\hat{x}$.
In all other cases, we need to solve $\frac{\partial}{\partial\nu}\mathbb{E}\left[u_2^0\left(X_\nu\right)\right] = 0$, leading to the stated solution, since:

$$\begin{aligned}& \frac{1}{w-\hat{x}+\nu\hat{x}-\nu\pi}\cdot p\cdot(\hat{x}-\pi)+\frac{1}{w-\nu\pi}\cdot(1-p)\cdot(-\pi) \overset{!}{=} 0\\ \Leftrightarrow\quad & p(\hat{x}-\pi)\frac{1}{w-\hat{x}+\nu\hat{x}-\nu\pi} = \pi(1-p)\frac{1}{w-\nu\pi}\\ \Leftrightarrow\quad & p(w-\nu\pi)(\hat{x}-\pi) = \pi(1-p)(w-\hat{x}+\nu\hat{x}-\nu\pi)\\ \Leftrightarrow\quad & -p(\hat{x}-\pi)\nu\pi-\pi(1-p)\nu\hat{x}+\pi(1-p)\nu\pi\\ & = \pi(1-p)w-\pi(1-p)\hat{x}-p(\hat{x}-\pi)w\\ \Leftrightarrow\quad & \nu(-p(\hat{x}-\pi)\pi-\pi(1-p)\hat{x}+\pi^2(1-p)) = \pi(1-p)(w-\hat{x})-pw(\hat{x}-\pi)\end{aligned}$$

$$\Leftrightarrow \qquad \nu \;=\; \frac{\pi(1-p)(w-\hat{x}) - pw(\hat{x}-\pi)}{\pi(1-p)(\pi-\hat{x}) - p\pi(\hat{x}-\pi)} = \frac{\pi(w-\hat{x}) - p\hat{x}(w-\pi)}{\pi(\pi-\hat{x})}$$

The first-order conditions are sufficient due to the strict concavity.

D.2.2 | Computations of Remark 5.3.5

First, observe that

$$w - \frac{\mathbb{E}\left[(w-X)u'(w-X)\right]}{\mathbb{E}\left[u'(w-X)\right]} = \frac{\mathbb{E}\left[Xu'(w-X)\right]}{\mathbb{E}\left[u'(w-X)\right]}$$

(i) CARA-utility:
For $X \sim \mathrm{Ber}(\hat{x}, p)$, it is $\mathbb{E}[X] = p\hat{x}$ and $\frac{\mathbb{E}[Xu'(w-X)]}{\mathbb{E}[u'(w-X)]} = \frac{\mathbb{E}\left[Xe^{\kappa X}\right]}{\mathbb{E}[e^{\kappa X}]} = \frac{p\hat{x}e^{\kappa\hat{x}}}{pe^{\kappa\hat{x}}+(1-p)}$.
For $X \sim \Gamma(\xi, \mu)$, it is $\mathbb{E}[X] = \frac{\xi}{\mu}$ and $\frac{\mathbb{E}[Xu'(w-X)]}{\mathbb{E}[u'(w-X)]} = \frac{\mathbb{E}\left[Xe^{\kappa X}\right]}{\mathbb{E}[e^{\kappa X}]}$. Since X is Gamma-distributed, we have

$$\frac{1}{\mathbb{E}\left[e^{\kappa X}\right]} = \frac{1}{\left(\frac{\mu}{\mu-\kappa}\right)^{\xi}} = \frac{(\mu-\kappa)^{\xi}}{\mu^{\xi}}$$

and

$$\mathbb{E}\left[X\, e^{\kappa X}\right] = \int_{\mathbb{R}} x\, e^{\kappa x}\, f_{\xi,\mu}(x)\; dx = \frac{\mu^{\xi}}{\Gamma(\xi)} \int_0^{\infty} x^{\xi}\, e^{(\kappa-\mu)x}\; dx.$$

The gamma function satisfies $\Gamma(\xi+1) = \xi\,\Gamma(\xi)$. By substituting $y = -(\kappa-\mu)x$, we compute

$$\begin{aligned}\mathbb{E}\left[X\, e^{\kappa X}\right] &= \frac{\mu^{\xi}}{\Gamma(\xi)}\,\frac{1}{(\mu-\kappa)^{\xi+1}} \int_0^{\infty} y^{\xi}\, e^{-y}\; dy = \frac{\mu^{\xi}}{\Gamma(\xi)}\,\frac{1}{(\mu-\kappa)^{\xi+1}}\,\Gamma(\xi+1)\\ &= \frac{\mu^{\xi}}{\Gamma(\xi)}\,\frac{1}{(\mu-\kappa)^{\xi+1}}\,\xi\,\Gamma(\xi) = \frac{\mu^{\xi}\,\xi}{(\mu-\kappa)^{\xi+1}}\end{aligned}$$

Finally,

$$\frac{\mathbb{E}\left[Xe^{\kappa X}\right]}{\mathbb{E}\left[e^{\kappa X}\right]} = \frac{(\mu-\kappa)^{\xi}}{\mu^{\xi}} \cdot \frac{\mu^{\xi}\,\xi}{(\mu-\kappa)^{\xi+1}} = \frac{\xi}{\mu-\kappa}$$

(ii) HARA-utility: It is $\mathbb{E}[X] = p\hat{x}$ and for $\lambda \in (0,1)$

$$\begin{aligned}\frac{\mathbb{E}\left[Xu'(w-X)\right]}{\mathbb{E}\left[u'(w-X)\right]} &= \frac{p\hat{x}(w-\hat{x})^{\lambda-1}}{p(w-\hat{x})^{\lambda-1} + (1-p)w^{\lambda-1}} = \frac{\frac{p\hat{x}}{(w-\hat{x})^{1-\lambda}}}{\frac{p}{(w-\hat{x})^{1-\lambda}} + \frac{1-p}{w^{1-\lambda}}}\\ &= \frac{\frac{p\hat{x}}{(w-\hat{x})^{1-\lambda}}}{\frac{pw^{1-\lambda}+(1-p)(w-\hat{x})^{1-\lambda}}{(w-\hat{x})^{1-\lambda}w^{1-\lambda}}} = \frac{p\hat{x}(w-\hat{x})^{1-\lambda}w^{1-\lambda}}{(w-\hat{x})^{1-\lambda}(pw^{1-\lambda} + (1-p)(w-\hat{x})^{1-\lambda})}\\ &= \frac{p\hat{x}w^{1-\lambda}}{pw^{1-\lambda} + (1-p)(w-\hat{x})^{1-\lambda}}\end{aligned}$$

(iii) Logarithmic utility: It is $\mathbb{E}[X] = p\hat{x}$ and $\frac{\mathbb{E}[Xu'(w-X)]}{\mathbb{E}[u'(w-X)]} = \frac{p\hat{x}w}{w+\hat{x}(p-1)}$

D.2.3 | Computations of Premium Principles in Table 5.2

We consider the following premium principles:

1. Expected Value Principle: $\pi = \mathbb{E}[X] + \delta \cdot \mathbb{E}[X]$,
2. Variance Principle: $\pi = \mathbb{E}[X] + \delta \cdot \text{Var}(X)$,
3. Semi-Variance Principle: $\pi = \mathbb{E}[X] + \delta \cdot \mathbb{E}\left[\left((X - \mathbb{E}[X])^+\right)^2\right]$,
4. Standard Deviation Principle: $\pi = \mathbb{E}[X] + \delta \cdot \sqrt{\text{Var}(X)}$,
5. Semi-Standard Deviation Principle: $\pi = \mathbb{E}[X] + \delta \cdot \sqrt{\mathbb{E}\left[\left((X - \mathbb{E}[X])^+\right)^2\right]}$.

Note that in addition to Section 5.3, we also compute the variance and semi-variance principle.
For $X \sim \text{Ber}(\hat{x}, p)$, we calculate:

1. Expected Value Principle: $\pi = p\hat{x} + \delta p\hat{x} = p\hat{x} \cdot (1 + \delta)$
2. Variance Principle:

$$\begin{aligned}\pi &= p\hat{x} + \delta \cdot \left(\mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2\right) = p\hat{x} + \delta \cdot (p\hat{x}^2 - p^2\hat{x}^2) = p\hat{x} + \delta p\hat{x}^2 \cdot (1 - p) \\ &= p\hat{x} \cdot (1 + \delta(1 - p)\hat{x})\end{aligned}$$

3. Semi-Variance Principle:

$$\begin{aligned}\pi &= p\hat{x} + \delta \cdot (p(\hat{x} - p\hat{x})^2) = p\hat{x} + \delta \cdot (p(\hat{x}^2 - 2p\hat{x}^2 + p^2\hat{x}^2) \\ &= p\hat{x} + \delta \cdot (p\hat{x}^2(1 - 2p + p^2)) = p\hat{x} + \delta \cdot (p\hat{x}^2(1 - p)^2) = p\hat{x} \cdot (1 + \delta(1 - p)^2\hat{x})\end{aligned}$$

4. Standard Deviation Principle:

$$\begin{aligned}\pi &= p\hat{x} + \delta \cdot \sqrt{p\hat{x}^2 - p^2\hat{x}^2} = p\hat{x} + \delta\hat{x}\sqrt{p(1 - p)} \\ &= p\hat{x} \cdot \left(1 + \delta\frac{\sqrt{p}\sqrt{1 - p}}{p}\right) = p\hat{x} \cdot \left(1 + \delta\sqrt{\frac{1 - p}{p}}\right)\end{aligned}$$

5. Semi-Standard Deviation Principle:

$$\pi = p\hat{x} + \delta \cdot \sqrt{p\hat{x}^2(1 - p)^2} = p\hat{x} + \delta\hat{x}(1 - p)\sqrt{p} = p\hat{x} \cdot \left(1 + \delta\frac{1 - p}{\sqrt{p}}\right)$$

For $X \sim \Gamma(\xi, \mu)$, we calculate:

1. Expected Value Principle: $\pi = \frac{\xi}{\mu} + \delta \cdot \frac{\xi}{\mu} = \frac{\xi}{\mu} \cdot (1 + \delta)$

2. Variance Principle: $\pi = \frac{\xi}{\mu} + \delta \cdot \frac{\xi}{\mu^2} = \frac{\xi}{\mu}\left(1 + \delta\frac{1}{\mu}\right)$

3. Semi-Variance Principle: We have to compute $\mathbb{E}\left[\left((X - \mathbb{E}[X])^+\right)^2\right]$. To this end, define the upper incomplete gamma function

$$\underline{\Gamma}(\alpha, t) := \int_t^\infty x^{\alpha-1}e^{-x}\,dx$$

and note that,

$$\underline{\Gamma}(\alpha+1, t) = e^{-t}t^\alpha + \alpha\underline{\Gamma}(\alpha, t) \quad \text{and} \quad \underline{\Gamma}(\alpha, t) = \lambda^\alpha \int_{\frac{t}{\lambda}}^\infty x^{\alpha-1}e^{-\lambda x}\,dx,$$

cf. Remark D.2.1. Hence, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left((X - \mathbb{E}[X])^+\right)^2\right] \\
&= \int_{\frac{\xi}{\mu}}^\infty \left(x - \frac{\xi}{\mu}\right)^2 \frac{\mu^\xi}{\Gamma(\xi)} x^{\xi-1}e^{-\mu x}\,dx = \int_{\frac{\xi}{\mu}}^\infty \left(x^2 - 2\frac{\xi}{\mu}x + \frac{\xi^2}{\mu^2}\right)\frac{\mu^\xi}{\Gamma(\xi)}x^{\xi-1}e^{-\mu x}\,dx \\
&= \frac{\mu^\xi}{\Gamma(\xi)}\left(\int_{\frac{\xi}{\mu}}^\infty x^{\xi+1}e^{-\mu x}\,dx - 2\frac{\xi}{\mu}\int_{\frac{\xi}{\mu}}^\infty x^\xi e^{-\mu x}\,dx + \frac{\xi^2}{\mu^2}\int_{\frac{\xi}{\mu}}^\infty x^{\xi-1}e^{-\mu x}\,dx\right) \\
&= \frac{\mu^\xi}{\Gamma(\xi)}\left(\frac{1}{\mu^{\xi+2}}\underline{\Gamma}(\xi+2,\xi) - 2\frac{\xi}{\mu}\frac{1}{\mu^{\xi+1}}\underline{\Gamma}(\xi+1,\xi) + \frac{\xi^2}{\mu^2}\frac{1}{\mu^\xi}\underline{\Gamma}(\xi,\xi)\right) \\
&= \frac{\mu^\xi}{\Gamma(\xi)}\left(\frac{1}{\mu^{\xi+2}}\left(e^{-\xi}\xi^{\xi+1} + (\xi+1)\underline{\Gamma}(\xi+1,\xi) - 2\xi\left(e^{-\xi}\xi^\xi + \xi\underline{\Gamma}(\xi,\xi)\right) + \xi^2\underline{\Gamma}(\xi,\xi)\right)\right) \\
&= \frac{1}{\Gamma(\xi)}\frac{1}{\mu^2}\left(e^{-\xi}\xi^{\xi+1} + (\xi+1)\left(e^{-\xi}\xi^\xi + \xi\underline{\Gamma}(\xi,\xi)\right)\right. \\
&\qquad \left. -2e^{-\xi}\xi^{\xi+1} - 2\xi^2\underline{\Gamma}(\xi,\xi) + \xi^2\underline{\Gamma}(\xi,\xi)\right) \\
&= \frac{1}{\mu^2\Gamma(\xi)}\left(\xi^{\xi+1}e^{-\xi} + \xi^{\xi+1}e^{-\xi} + \xi^\xi e^{-\xi} + \xi^2\underline{\Gamma}(\xi,\xi) + \xi\underline{\Gamma}(\xi,\xi)\right. \\
&\qquad \left. -2\xi^{\xi+1}e^{-\xi} - 2\xi^2\underline{\Gamma}(\xi,\xi) + \xi^2\underline{\Gamma}(\xi,\xi)\right) \\
&= \frac{1}{\mu^2\Gamma(\xi)}\left(\xi^\xi e^{-\xi} + \xi\underline{\Gamma}(\xi,\xi)\right)
\end{aligned}$$

Thus,

$$\pi = \frac{\xi}{\mu} + \delta \cdot \frac{1}{\mu^2\Gamma(\xi)}\left(\xi^\xi e^{-\xi} + \xi\underline{\Gamma}(\xi,\xi)\right) = \frac{\xi}{\mu}\cdot\left(1 + \delta\frac{1}{\mu\Gamma(\xi)}\left(\xi^{\xi-1}e^{-\xi} + \underline{\Gamma}(\xi,\xi)\right)\right).$$

For $\xi = 1$, this leads to

$$\pi = \frac{1}{\mu}\cdot\left(1 + \delta\frac{2}{\mu}e^{-1}\right).$$

4. Standard Deviation Principle: $\pi = \frac{\xi}{\mu} + \delta \cdot \sqrt{\frac{\xi}{\mu^2}} = \frac{\xi}{\mu}\cdot\left(1 + \delta\frac{1}{\sqrt{\xi}}\right)$

5. Semi-Standard Deviation Principle:

$$\pi = \frac{\xi}{\mu} + \delta \cdot \sqrt{\frac{1}{\mu^2 \Gamma(\xi)} \left(\xi^\xi e^{-\xi} + \xi \underline{\Gamma}(\xi, \xi)\right)} = \frac{\xi}{\mu} \cdot \left(1 + \delta \frac{1}{\xi} \sqrt{\frac{1}{\Gamma(\xi)} \left(\xi^\xi e^{-\xi} + \xi \underline{\Gamma}(\xi, \xi)\right)}\right)$$

For $\xi = 1$, this leads to

$$\pi = \frac{1}{\mu} \cdot \left(1 + \delta \sqrt{2e^{-1}}\right).$$

Remark D.2.1. It is

(i) $\underline{\Gamma}(\alpha + 1, t) = e^{-t} t^\alpha + \alpha \underline{\Gamma}(\alpha, t)$,

(ii) $\underline{\Gamma}(\alpha, t) = \lambda^\alpha \int_{\frac{t}{\lambda}}^\infty x^{\alpha-1} e^{-\lambda x}\, dx$.

Proof. (i) By partial integration, we obtain

$$\begin{aligned}
\underline{\Gamma}(\alpha + 1, t) &= \int_t^\infty x^\alpha e^{-x}\, dx = \lim_{T \to \infty} \int_t^T x^\alpha e^{-x}\, dx \\
&= \lim_{T \to \infty} \left(\left[-e^{-x} x^\alpha\right]_t^T + \int_t^T e^{-x} \alpha x^{\alpha-1}, dx \right) \\
&= \lim_{T \to \infty} \left(-e^{-T} T^\alpha + e^{-t} t^\alpha + \alpha \int_t^T x^{\alpha-1} e^{-x}\, dx \right) \\
&= \lim_{T \to \infty} -e^{-T} T^\alpha + e^{-t} t^\alpha + \alpha \int_t^\infty x^{\alpha-1} e^{-x}\, dx \\
&= e^{-t} t^\alpha + \alpha \underline{\Gamma}(\alpha, t)
\end{aligned}$$

(ii) By substituting $y = \lambda x$, we obtain

$$\int_{\frac{t}{\lambda}}^\infty x^{\alpha-1} e^{-\lambda x}\, dx = \frac{1}{\lambda^{\alpha-1}} \int_t^\infty y^{\alpha-1} e^{-y} \frac{1}{\lambda}\, dy = \frac{1}{\lambda^\alpha} \int_t^\infty y^{\alpha-1} e^{-y}\, dy = \frac{1}{\lambda^\alpha} \underline{\Gamma}(\alpha, t).$$

Hence,

$$\underline{\Gamma}(\alpha, t) = \lambda^\alpha \int_{\frac{t}{\lambda}}^\infty x^{\alpha-1} e^{-\lambda x}\, dx.$$

□

Remark D.2.2. Note that

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = 0 + e^{-0} = 1$$

and

$$\underline{\Gamma}(1, 1) = \int_1^\infty e^{-x}\, dx = \left[-e^{-x}\, dx\right]_1^\infty = 0 + e^{-1} = e^{-1}.$$

Bibliography

Acciaio, B. (2005), Two Problems Related to Utility Theory Under Unusual Assumptions, PhD thesis, University of Perugia.

Acciaio, B. (2007), 'Optimal Risk Sharing with Non-Monotone Monetary Functions', *Finance & Stochastics* **11**(2), 267–289.

Acerbi, C. & G. Scandolo (2008), 'Liquidity Risk Theory and Coherent Risk Measures', *Quantitative Finance* **8**(7), 681–692.

Albrecher, H., A. Badescu & D. Landriault (2008), 'On the Dual Risk Model with Tax Payments', *Insurance: Mathematics and Economics* **42**, 1086–1094.

Albrecher, H. & C. Hipp (2007), 'Lundberg's Risk Process with Tax', *Blätter der DGVFM* **28**, 13–28.

Albrecher, H., J.-F. Renaud & X. Zhou (2008), 'A Lévy Insurance Risk Process with Tax', *Journal of Applied Probability* **45**, 363–375.

Albrecher, H., S. Borst, O. Boxma & J. Resing (2009), 'The Tax Identity in Risk Theory – A Simple Proof and an Extension', *Insurance: Mathematics and Economics* **44**, 304–306.

Allianz (2016), Market Consistent Embedded Value Report 2016, Technical report, Allianz SE - Group Actuarial.

Andradóttir, S. (1998), 'A Review of Simulation Optimization Techniques', *Proceedings of the 1998 Winter Simulation Conference,* pp. 151–158.

Artzner, P. (1999), 'Application of Coherent Risk Measures to Capital Requirements in Insurance', *Northern American Actuarial Journal* **3**(2), 11–25.

Artzner, P., F. Delbaen, J.-M. Eber & D. Heath (1999), 'Coherent Measures of Risk', *Mathematical Finance* **9**(3), 203–228.

Artzner, P., F. Delbaen & P. Koch Medina (2009), 'Risk Measures and Efficient Use of Capital', *ASTIN Bulletin* **39**(1), 101–116.

Asmussen, S. & H. Albrecher (2010), *Ruin Probabilities*, Vol. 14 of *Advanced Series on Statistical Science & Applied Probability*, 2 edn, World Scientific, Singapore.

Astic, F. & N. Touzi (2007), 'No Arbitrage Conditions and Liquidity', *Journal of Mathematical Economics* **43**(6), 692–708.

BaFin (2011-2015), Statistik der BaFin - Erstversicherungsunternehmen, Technical report, Bundesanstalt für Finanzdienstleistungsaufsicht, Bonn und Frankfurt am Main.

Barrieu, P. & N. El Karoui (2005), 'Inf-Convolution of Risk Measures and Optimal Risk Transfer', *Finance and Stochastics* **9**, 269–298.

Barrieu, P. & N. El Karoui (2008), Pricing, Hedging and Optimally Designing Derivatives via Minimization of Risk Measures, *in* R.Carmona, ed., 'Indifference Pricing: Theory and Applications', Princeton University Press, Princeton, USA.

Bauer, D., A. Reuß & D. Singer (2012), 'On the Calculation of the Solvency Capital Requirement Based on Nested Simulations', *ASTIN Bulletin* **42**(2), 453–499.

Bauer, H. (2001), *Measure and Integration Theory*, Walter de Gruyter, Berlin - New York.

Becker, T., C. Cottin, M. Fahrenwaldt, A.-M. Hamm, S. Nörtemann & S. Weber (2014), 'Market Consistent Embedded Value - eine praxisorientierte Einführung', *Der Aktuar* **1**, 4–8.

Ben-Tal, A. & M. Teboulle (2007), 'An old-new Concept of Convex Risk Measures: The Optimized Certainty Equivalent', *Mathematical Finance* **17**(3), 449–476.

Brunnermeier, M.K. & P. Cheridito (2014), Measuring and Allocating Systemic Risk. Submitted.

Buch, A. & G. Dorfleitner (2008), 'Coherent Risk Measures, Coherent Capital Allocations and the Gradient Allocation Principle', *Insurance: Mathematics and Economics* **42**(1), 235–242.

Bundesfinanzministerium (2016), 'Kassenmäßige Steuereinnahmen nach Steuerarten in den Kalenderjahren 2010 - 2015'. [Online; accessed 04-April-2017].

Bundesministerium der Justiz und für Verbraucherschutz (2017), 'Insurance Tax Act', `https://www.gesetze-im-internet.de/englisch_versstg/englisch_versstg.html`. [Online; accessed 04-April-2017].

Cetin, U. & L.C.G. Rogers (2007), 'Modeling Liquidity Effects in Discrete Time', *Mathematical Finance* **17**(1), 15–29.

Cetin, U., R.A. Jarrow & P.E. Protter (2004), 'Liquidity Risk and Arbitrage Pricing Theory', *Finance and Stochastics* **8**(3), 311–341.

CFO Forum (2004*a*), Basis for Conclusions – European Embedded Value Principles, Technical report, CFO Forum.

CFO Forum (2004*b*), European Embedded Value Principles, Technical report, CFO Forum.

CFO Forum (2009*a*), Market Consistent Embedded Value – Basis for Conclusions, Technical report, CFO Forum.

CFO Forum (2009*b*), Market Consistent Embedded Value Principles, Technical report, CFO Forum.

Chen, C., G. Iyengar & C.C. Moallemi (2013), 'An Axiomatic Approach to Systemic Risk', *Management Science* **59**(6), 1373–1388.

Christiansen, M. & A. Niemeyer (2014), 'Fundamental Definition of the Solvency Capital Requirement in Solvency II', *ASTIN Bulletin* **44**(3), 501–533.

Cont, R., R. Deguest & G. Scandolo (2010), 'Robustness and Sensitivity Analysis of Risk Measurement Procedures', *Quantitative Finance* **10**(6), 593–606.

DAV, Deutsche Aktuarvereinigung (2005), Embedded Value, Hinweis, DAV, Köln.

DAV, Deutsche Aktuarvereinigung (2006), Stochastischer Embedded Value, Hinweis, DAV, Köln.

DAV, Deutsche Aktuarvereinigung (2010), Best Estimate in der Lebensversicherung, Technical report, DAV-Arbeitsgruppe Best Estimate in der Lebensversicherung.

DAV, Deutsche Aktuarvereinigung (2011), Market Consistent Embedded Value, Hinweis, DAV, Köln.

Degelmann, M., A.-M. Hamm & S. Weber (2018), 'The Impact of Insurance Premium Taxation', *European Actuarial Journal* **8**(1), 127–167.

Delbaen, F. (2000), Coherent Risk Measures, Lecture Notes, Cattedra Galileiana, Scuola Normale di Pisa.

Denault, M. (2001), 'Coherent Allocation of Risk Capital', *Journal of Risk* **4**(1), 1–33.

Detlefsen, K. & G. Scandolo (2005), 'Conditional and Dynamic Convex Risk Measures', *Finance and Stochastics* **9**(4), 539–561.

Diers, D., M. Eling, C. Kraus & A. Reuß (2012), 'Market Consistent Embedded Value in Non-Life Insurance: How to Measure it and Why', *Journal of Risk Finance* **13**(4), 320–346.

Dunkel, J. & S. Weber (2007), 'Efficient Monte Carlo Methods for Convex Risk Measures in Portfolio Credit Risk Models', *Proceedings of the 2007 Winter Simulation Conference,* pp. 958–966.

Dunkel, J. & S. Weber (2010), 'Stochastic Root Finding and Efficient Estimation of Convex Risk Measures', *Operations Research* **58**(5), 1505–1521.

Ekeland, I. & R. Témam (1999), *Convex Analysis and Variational Problems*, Vol. 28 of *Classics in Applied Mathematics*, SIAM, Philadelphia.

Embrechts, P., H. Liu & R. Wang (2018), 'Quantile-Based Risk Sharing', *Operations Research* **66**(4), 936–949.

Embrechts, P. & N. Veraverbeke (1982), 'Estimates for the Probability of Ruin with Special Emphasis on the Possibility of Large Claims', *Insurance: Mathematics and Economics* **1**, 55 – 72.

European Commission (2009), Directive 2009/138/EC of the European Parliament and of the Council of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II), Directives, Official Journal of the European Commission.

European Commission (2017), VAT Rates Applied in the Member States of the European Union, Technical report, European Commission.

Feinstein, Z., B. Rudloff & S. Weber (2017), 'Measures of Systemic Risk', *SIAM Journal on Financial Mathematics* **8**(1), 672–708.

Filipović, D. (2008), 'Optimal Numéraires for Risk Measures', *Mathematical Finance* **18**(2), 333–336.

Filipović, D. & A. Kunz (2008), 'Realizable Group Diversification Effects', *Life & Pension,* pp. 33–40.

Filipović, D. & G. Svindland (2008), 'Optimal Capital and Risk Allocations for Law- and Cash-Invariant Convex Functions', *Finance and Stochastics* **12**, 423–439.

Filipović, D. & M. Kupper (2007), 'On the Group Level Swiss Solvency Test', *Bulletin of the Swiss Association of Actuaries* **1**, 97–115.

Filipović, D. & M. Kupper (2008*a*), 'Equilibrium Prices for Monetary Utility Functions', *International Journal of Theoretical and Applied Finance* **11**, 325–343.

Filipović, D. & M. Kupper (2008*b*), 'Optimal Capital and Risk Transfers for Group Diversification', *Mathematical Finance* **18**(1), 55–76.

Föllmer, H. & A. Schied (2002), 'Convex Measures of Risk and Trading Constraints', *Finance and Stochastics* **6**(4), 429–447.

Föllmer, H. & A. Schied (2004), *Stochastic Finance - An Introduction in Discrete Time, 2nd Edition*, Walter de Gruyter, Berlin.

Föllmer, H. & A. Schied (2011), *Stochastic Finance - An Introduction in Discrete Time*, 3rd extended edn, Walter de Gruyter & Co., Berlin.

Föllmer, H. & S. Weber (2015), 'The Axiomatic Approach to Risk Measurement for Capital Determination', *Annual Review of Financial Economics* **7**.

Freund, R.M. (2004), Issues in Non-Convex Optimization, Manuscript, Massachusetts Institute of Technology. With assistance from B.W. Anthony.

Fu, M.C. (2006), Gradient Estimation, *in* S.Henderson & B.Nelson, eds, 'Simulation', Vol. 13 of *Handbooks in Operations Research and Management Science*, Elsevier, Amsterdam, The Netherlands, pp. 575–616. Chapter 19.

Fu, M.C. (2008), 'What Should You Know About Simulation and Derivatives', *Naval Research Logistics (NRL)* **55**(8), 723–736.

Gerstner, T., M. Griebel, M. Holtz, R. Goschnik & M. Haep (2008), 'A General Asset-Liability-Management Model for the Efficient Simulation of Portfolios of Life Insurance Policies', *Insurance: Mathematics and Economics* **42**(2), 704–716.

Giesecke, K., T. Schmidt & S. Weber (2008), 'Measuring the Risk of Extreme Events', *Journal of Investment Management* **6**(4), 1–15.

Giorgi, G. & T.H. Kjeldsen, eds (2014), *Traces and Emergence of Nonlinear Programming*, Springer, Basel.

Graf, S., J. Pricking, J.-P. Schmidt & H.-J. Zwiesler (2012), 'Marktkonsistente Bewertung in der Privaten Krankenversicherung', *Versicherungswirtschaft* **67**(14), 1064–1066.

Haier, A., I. Molchanov & M. Schmutz (2016), 'Intragroup Transfers, Intragroup Diversification and Their Risk Assessment', *Annals of Finance* **12**(3-4), 363–392.

Hamel, A. H. & F. Heyde (2010), 'Duality for Set-Valued Measures of Risk', *SIAM Journal on Financial Mathematics* **1**(1), 66–95.

Hamel, A. H., F. Heyde & B. Rudloff (2011), 'Set-Valued Risk Measures for Conical Market Models', *Mathematics and Financial Economics* **5**(1), 1–28.

Hamm, A.-M. (2012), Stochastische Nullstellensuche für Optimized Certainty Equivalents, Master's thesis, Leibniz Universität Hannover.

Hamm, A.-M., T. Knispel & S. Weber (2019*a*), Network Risk, Network Regulation, and Network Optimization. Working Paper.

Hamm, A.-M., T. Knispel & S. Weber (2019*b*), Optimal Risk Sharing in Insurance Networks: An Application to Asset-Liability Management. Accepted and soon to be published in *European Actuarial Journal*, Springer.

Hamm, A.-M., T. Salfeld & S. Weber (2013), 'Stochastic Root Finding for Optimized Certainty Equivalents', *Proceedings of the 2013 Winter Simulation Conference,* pp. 922–932.

Henderson, S.G. & B.L. Nelson, eds (2006), *Simulation*, Vol. 13 of *Handbooks in Operations Research and Management Science*, Elsevier, Amsterdam, The Netherlands.

Hildebrandt, M. (2013), Auswirkungen der Versicherungsteuer auf Unternehmen und Märkte, Master's thesis, Leibniz Universität Hannover.

Hoffmann, H., T. Meyer-Brandis & G. Svindland (2016), 'Risk-Consistent Conditional Systemic Risk Measures', *Stochastic Processes and their Applications* **126**(7), 2014–2037.

Holzheu, T. (1997), 'Die Belastung von Versicherungsdienstleistungen mit Verkehrsteuern', Verlag Versicherungswirtschaft e.V., Karlsruhe; Veröffentlichungen der Hamburger Gesellschaft zur Förderung des Versicherungswesens mbH, Hamburg; Heft 19.

Holzheu, T. (2000), 'Die Einbeziehung der Schaden-/Unfallversicherung in das Umsatzsteuersystem', Verlag Versicherungswirtschaft Gmbh, Karlsruhe; Veröffentlichungen der Hamburger Gesellschaft zur Förderung des Versicherungswesens mbH, Hamburg; Heft 23.

Hong, L.J. & B.L. Nelson (2009), 'A Brief Introduction to Optimization via Simulation', *Proceedings of the 2009 Winter Simulation Conference,* pp. 75–85.

Insurance Europe (2016), Indirect Taxation on Insurance Contracts in Europe, Technical report, Insurance Europe (European Insurance and Reinsurance Federation), Brussels.

Jarrow, R.A. & P.E. Protter (2005), 'Liquidity Risk and Risk Measure Computation', *Review of Future Markets* **11**(1), 27–39.

Jouini, E., M. Meddeb & N. Touzi (2004), 'Vector-Valued Coherent Risk Measures', *Finance and Stochastics* **8**(4), 531–552.

Jouini, E., W. Schachermayer & N. Touzi (2008), 'Optimal Risk Sharing for Law Invariant Monetary Utility Functions', *Mathematical Finance* **18**(2), 269–292.

Kalkbrener, M. (2005), 'An Axiomatic Approach to Capital Allocation', *Mathematical Finance* **15**(3), 425–437.

Keller, P. (2007), 'Group Diversification', *The Geneva Papers on Risk and Insurance, Issues and Practice,* **32**, 382–392.

Kim, S. (2006), 'Gradient-Based Simulation Optimization', *Proceedings of the 2006 Winter Simulation Conference,* pp. 159–167.

Kraus, C. (2013), 'EVA/RAROC vs. MCEV Earnings: A Unification Approach', *The Geneva Papers on Risk and Insurance – Issues and Practice* **38**(1), 113–136.

Kromer, E. & L. Overbeck (2014), 'Suitability of Capital Allocations for Performance Measurement', *Journal of Risk* **16**(6).

Kushner, H.-J. & G.-G. Yin (2003), *Stochastic Approximation and Recursive Algorithms and Applications*, 2nd edn, Springer, New York.

Kyprianou, A.E. & X. Zhou (2009), 'General Tax Structures and the Lévy Insurance Risk Model', *Journal of Applied Probability* **46**, 1146–1156.

Luder, T. (2007), 'Modelling of Risks in Insurance Groups for the Swiss Solvency Test', *Bulletin of the Swiss Association of Actuaries* **1**, 85–97.

McNeil, A.J., R. Frey & P. Embrechts (2015), *Quantitative Risk Management - Concepts, Techniques, Tools*, revised edn, Princeton Series in Finance, Princeton University Press, Princeton and Oxford.

Mikosch, Thomas (2009), *Non-Life Insurance Mathematics – An Introduction with the Poisson Process*, Universitext, 2 edn, Springer-Verlag, Berlin Heidelberg.

Molchanov, I. & I. Cascos (2016), 'Multivariate Risk Measures: A Constructive Approach Based on Selections', *Mathematical Finance* **26**(4), 867–900.

Pennanen, T. & I. Penner (2010), 'Hedging of Claims with Physical Delivery under Convex Transaction Costs', *SIAM Journal on Financial Mathematics* **1**, 158–178.

Pfeifer, D. (2016), Hält das Standardmodell unter Solvency II, was es verspricht?, *in* Koch, Weber & Winter, eds, 'Der Forschung – der Lehre – der Bildung. 100 Jahre Hamburger Seminar für Versicherungswissenschaft', Verlag Versicherungswirtschaft, Karlsruhe, pp. 767–788.

Reuß, A., J. Ruß & J. Wieland (2015), Participating Life Insurance Contracts under Risk Based Solvency Framworks: How to Increase Capital Efficiency by Product Design, *in* K.Glau, M.Scherer & R.Zagst, eds, 'Innovations in Quantitative Risk Management', Vol. 99 of *Springer Proceedings in Mathematics & Statistics*, Springer, Cham, pp. 185–208.

Rockafellar, R.T. (1970), *Convex Analysis*, Princeton Landmarks in Mathematics and Physics, Princeton University Press, Princeton.

Schied, A. & A. Slynko (2011), Some Mathematical Aspects of Market Impact Modeling, *in* 'Surveys in Stochastic Processes. Proceedings of the 33rd SPA', EMS Series of Congress Reports, ed. by J. Blath, P. Imkeller, S. Roelly.

Schmidt, J.-P. (2012), 'Market Consistent Embedded Value in der Privaten Krankenversicherung', *Zeitschrift für die gesamte Versicherungswissenschaft* **101**(2), 223–253.

Schmidt, J.-P. (2014), 'Market-Consistent Valuation of Long-Term Insurance Contracts - Valuation Framework and Application to German Private Health Insurance', *European Actuarial Journal* **4**(1), 125–153.

Schmidt, K.D. (2009), *Versicherungsmathematik*, 3 edn, Springer, Berlin Heidelberg.

Schrinner, A., ed. (1997), *Versicherungsteuer als Standortproblem*, Institut für "Finanzen und Steuern" e.V., Bonn. IFSt-Schrift Nr. 352.

sigma (2012), Insuring Ever-Evolving Commercial Risks, Study 5, Swiss Re, Zurich. Authors: T. Holzheu, O. Futterknecht, R, Lechner; *sigma* co-editor: J. Villat Cordova; Managing editor responsible for *sigma* series: K. Karl.

Straubhaar, T. (2006), Auswirkungen einer Erhöhung der Versicherungsteuer auf die deutsche Wirtschaft, die Verbraucher, die öffentlichen Haushalte und den Versicherungsstandort Deutschland, Gutachten des HWWI, Hamburgisches WeltWirtschaftsInstitut.

Tasche, D. (2000), Risk Contributions and Performance Measurement. Working Paper, Technische Universität München.

Tasche, D. (2004), Allocating Portfolio Economic Capital to Sub-Portfolios, *in* A.Dev, ed., 'Economic Capital', Risk Books, pp. 275–302.

Tasche, D. (2008), Capital Allocation to Business Units and Sub-Portfolios: The Euler Principle, *in* A.Resti, ed., 'Pillar II in the New Basel Accord: The Challenge of Economic Capital', Risk Books, pp. 423–453.

Waszink, H. (2013), 'Considerations on the Discount Rate in the Cost of Capital Method for the Risk Margin'. Preprint.

Weber, S. (2006), 'Distribution-Invariant Risk Measures, Information, and Dynamic Consistency', *Mathematical Finance* **16**(2), 419–442.

Weber, S. (2018), 'Solvency II, or How to Sweep the Downside Risk Under the Carpet', *Insurance: Mathematics and Economics* **82**, 191–200.

Weber, S., W. Anderson, A.-M. Hamm, T. Knispel, M. Liese & T. Salfeld (2013), 'Liquidity-Adjusted Risk Measures', *Mathematics and Financial Economics* **7**(1), 69–91.

Wüthrich, M. V., H. Bühlmann & H. Furrer (2010), *Market-Consistent Actuarial Valuation*, EAA, 2 edn, Springer-Verlag, Berlin Heidelberg.

Zwiesler, H.-J. (2004), Asset-Liability-Management – die Versicherung auf dem Weg von der Planungsrechnung zum Risikomanagement, *in* K.Spremann, ed., 'Versicherung im Umbruch', Springer, Heidelberg, pp. 117–131.

List of Figures

List of Tables

List of Publications and Scientific Talks

Publications:

- **Hamm, A.-M.**, T. Salfeld & S. Weber: "Stochastic Root Finding for Optimized Certainty Equivalents", *Proceedings of the 2013 Winter Simulation Conference*, IEEE, pp. 922-932, 2013.
- Weber, S., W. Anderson, **A.-M. Hamm**, T. Knispel, M. Liese & T. Salfeld: "Liquidity-Adjusted Risk Measures", *Mathematics and Financial Economics* **7**(1), pp. 69-91, 2013.
- Becker, T., C. Cottin, M. Fahrenwaldt, **A.-M. Hamm**, S. Nörtemann & S. Weber: "Market Consistent Emebedded Value – eine praxisorientierte Einführung", *Der Aktuar* **1**, pp. 4-8, 2014.
- Degelmann, M., **A.-M. Hamm** & S. Weber: "The Impact of Insurance Premium Taxation", *European Actuarial Journal* **8**(1), pp. 127-167, 2018.

Scientific Talks:

- 7th Conference in Actuarial Science & Finance on Samos, Greece, June 2012.
- 16th International Congress on Insurance: Mathematics and Economics, Hong Kong, June 2012.
- 2013 ASTIN Colloquium, The Hague, Netherlands, May 2013.
- Joint PhD Seminar in Statistics, Financial and Actuarial Mathematics, Hannover, Germany, November 2013.
- Quantitative Methods in Finance, Sydney, Australia, December 2015.
- Joint PhD Seminar in Statistics, Financial and Actuarial Mathematics, Oldenburg, Germany, February 2016.
- 31st International Congress of Actuaries, Berlin, Germany, June 2018.
- DAV/DGVFM-Jahrestagung 2019, Düsseldorf, Germany, April 2019.

Acknowledgements

I wish to thank my supervisor Prof. Dr. Stefan Weber for giving me the opportunity to write my PhD thesis at the Institute of Probability and Statistics, Leibniz Universität Hannover. I am thankful for a lot of inspiring discussions focusing on mathematics as well as economics. His support and professional advise improved my research with regard to both contents and presentation. Moreover, I am deeply grateful for his enthusiastic engagement for establishing an international and interdisciplinary research environment.

Further, I like to thank Prof. Dr. Ralf Korn and Prof. Dr. Ludger Overbeck for being second reviewers of my thesis, as well as Prof. Dr. Marc Steinbach for taking the chair of the examination board.

Additionally, I want to thank Prof. Dr. Thomas Knispel for fruitful discussions and collaborations that lead to jointly-authored publications and preprints enriching my thesis, see Hamm et al. (2019*a*), Hamm et al. (2019*b*), and Weber et al. (2013). I am also thankful to Dr. Thomas Salfeld, Moritz Degelmann, Dr. William Anderson & Maren Liese, as well as Prof. Dr. Fahrenwaldt, Prof. Dr. Torsten Becker, Prof. Dr. Claudia Cottin & Dr. Stefan Nörtemann for succesful collaborative research, see Hamm et al. (2013), Degelmann et al. (2018), Weber et al. (2013), and Becker et al. (2014).

Furthermore, I would like to thank all my colleagues for their companionship; especially, Kerstin Awiszus for reading my thesis, sharing thoughts as well as the office, and becoming a friend. Also, I thank Marcel Kleiber for giving helpful comments on my thesis. Special thanks go to Oliver Janke for the best friendship I ever had.

Finally, I wish to thank my parents for their love and support, and, in particular, for giving me the best sisters in the world. Last but not least, I thank Daniel for coming into my life and making me happy!

The numerical results presented in this thesis were carried out on the cluster system at Leibniz Universität Hannover, Germany.