Topics in Singular Analysis with Applications to Representation Theory and to Numerical Analysis

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Zusammenfassung

Diese Arbeit befasst sich mit partiellen Differentialoperatoren und Funktionenräumen auf nichtkompakten oder Lipschitz–regulären Mannigfaltigkeiten, insbesondere in Bezug auf Anwendungen in der theoretischen numerischen Analysis und der Darstellungstheorie. Der erste Teil analysiert die numerische Approximation gewisser vollständig nichtlinearer Transmissions– und Kontaktprobleme mit Hilfe von finiten Elementen und Randelementmethoden. Im zweiten Teil werden analytische Darstellungen reeller Liegruppen untersucht und ein Analogon des Faktorisierungssatzes von Dixmier und Malliavin für den Raum der analytischen Vektoren gezeigt.

Die ersten zwei Kapitel zur numerischen Analysis führen eine numerisch effiziente variationelle Formulierung für gewisse nichtlineare Transmissionsprobleme aus der Elastizität und der Theorie der Phasenübergänge ein. Dabei wird ein stark nichtlinearer Operator in einem beschränkten Lipschitzgebiet durch Transmissions– oder Kontaktbedingungen an ein homogenes linear–elliptisches Problem auf dem unbeschränkten Komplement im $\mathbb{R}^n$ gekoppelt. Ein Hauptergebnis zeigt, dass in gewissen gemischten $L^p - L^2$–Sobolevräumen eine natürliche a priori and a posteriori Analyse des numerischen Fehlers durchgeführt werden kann. In einem ersten Schritt wird die Konvergenz eines schnellen adaptiven Finite Element/Randelementverfahrens für entartete Operatoren wie den p–Laplaceoperator mit Reibungskontakt gezeigt:

Satz. a) Das Kontaktproblem besitzt eine eindeutige und stetig von den Daten abhängende Lösung in $W^{1,p}(\Omega) \times H^{1/2}(\partial\Omega)$.
b) Die numerischen Lösungen konvergieren in der Norm gegen die exakte Lösung.
c) Der Fehler wird durch einen gradient–recovery–Schätzer für den p–Laplace und eine residuale a posteriori Abschätzung auf $\partial\Omega$ dominiert.

Anschließend werden die Betrachtungen auf ein nichtkonvexes Double–Well–Problem mit Kontakt erweitert. Obwohl dieses Problem nicht einmal eine schwache Lösung besitzt, sind gewisse makroskopische Eigenschaften von Näherungslösungen eindeutig bestimmt und lassen sich numerisch effizient berechnen.

Der darstellungstheoretische Teil führt eine allgemeine Sprache zur Untersuchung der analytischen Darstellungen einer reellen Liegruppe G ein. Insbesondere wird ein Analogon des Faktorisierungssatzes von Dixmier–Malliavin für den Raum der analytischen Vektoren E^ω einer Darstellung (π, E) von G gezeigt: Unter gewissen Wachstumsvoraussetzungen, die z.B. im Fall eines Banachraums E erfüllt sind, induziert π eine Wirkung einer natürlichen Algebra superexponentiell fallender analytischer Functionen $\mathcal{A}(G)$, so dass gilt:

Satz. $\mathcal{A}(G) = \mathcal{A}(G) * \mathcal{A}(G)$ und $E^\omega = \mathcal{A}(G) * E^\omega$.

Im letzten Teil der Arbeit werden schließlich Kategorien $\mathcal{A}(G)$–temperierter und nicht–temperierter analytischer Darstellungen eingeführt und ihre grundlegenden topologischen und darstellungstheoretischen Eigenschaften untersucht.

Schlagworte: analytische Darstellungen, nichtlineares Transmissionsproblem, Randelementmethode

Abstract

This thesis considers partial differential operators and function spaces on non-compact or Lipschitz–regular manifolds in the context of theoretical numerical analysis and representation theory. The first part analyzes the approximate solution of certain fully nonlinear transmission and contact problems using coupled finite and boundary elements. In the second part, analytic representations of real Lie groups are studied and, in particular, an analytic analogue of Dixmier–Malliavin's factorization theorem is obtained.

The first two chapters on theoretical numerical analysis present numerically efficient variational formulations for fully nonlinear transmission problems arising e.g. in elasticity and phase transitions. The basic set–up consists of an "unpleasant" operator in a bounded Lipschitz domain, which is coupled via transmission or contact conditions to a homogeneous linear elliptic system in the complement in $\mathbb{R}^n$ with radiation conditions at infinity. As the main result, we show how certain mixed $L^p - L^2$–Sobolev spaces provide the proper formulation for an a priori and a posteriori analysis of the error committed in the numerical approximation. In a first step, convergence of a fast adaptive finite element/boundary element algorithm for degenerate operators such as the p–Laplacian with friction conditions is established:

Theorem. a) The contact problem admits a unique solution in $W^{1,p}(\Omega) \times H^{1/2}(\partial\Omega)$, which depends continuously on the data.
b) The numerical approximations are norm–convergent to the solution.
c) The error is estimated by a gradient recovery for the p–Laplacian and a residual a posteriori estimate on $\partial\Omega$.

The analysis is then extended to a nonconvex double–well problem with contact. Although this problem does not even admit a weak solution, approximate solutions share certain "macroscopic" features captured in a Young measure and may be computed efficiently.

The representation–theoretical part introduces a general framework to study analytic representations of a real Lie group G. It presents an analogue of the

Dixmier–Malliavin factorization theorem for the space of analytic vectors E^ω associated to a representation (π, E) of G: Under suitable growth assumptions, satisfied e.g. whenever E is a Banach space, π gives rise to an action of a natural algebra of superexponentially decaying analytic functions $\mathcal{A}(G)$ such that:

Theorem. $\mathcal{A}(G) = \mathcal{A}(G) * \mathcal{A}(G)$ and $E^\omega = \mathcal{A}(G) * E^\omega$.

A final part formally introduces categories of $\mathcal{A}(G)$–tempered and non–tempered analytic representations and analyzes their fundamental topological and representation theoretical properties.

key words: analytic representations, nonlinear transmission problems, boundary element method

Contents

Chapter 1

Introduction

This thesis considers partial differential operators and function spaces on non-compact or Lipschitz–regular manifolds in the context of theoretical numerical analysis and representation theory. The first part analyzes the approximate solution of certain fully nonlinear transmission and contact problems using coupled finite and boundary elements. In the second part, analytic representations of real Lie groups are studied and, in particular, an analytic analogue of Dixmier–Malliavin's factorization theorem is obtained.

The unifying theme behind the work in two seemingly distant subjects is the quantitative study of partial differential equations in the sense of estimates for the solutions or derived quantities and the investigation of the corresponding function spaces.
The basic set–up for the transmission problems consists of a nonlinear and degenerate operator in a Lipschitz domain $\Omega \subset \mathbb{R}^n$, which is coupled via contact conditions to a linear elliptic system in $\mathbb{R}^n \setminus \overline{\Omega}$ with radiation conditions at infinity. Typically, the exterior problem is formally solved and reduced to $\partial\Omega$ with the help of the Dirichlet–Neumann (or Poincaré–Steklov) operator. This leads to a boundary contact problem with generalized pseudodifferential boundary conditions. Key results of our work are a priori estimates showing a certain well–posedness of the equations and the norm convergence of numerical approximations in $L^p(\Omega) - L^2(\partial\Omega)$–Sobolev spaces.
The factorization problem for Lie group representations, on the other hand, essentially amounts to representing an arbitrary rapidly decaying analytic function on a Lie group G as a sum of convolutions of such functions. For compactly supported C^∞–functions, the analogous problem was posed by Ehrenpreis when studying the solvability of convolution equations on $\mathbb{R}^n$ [23]. To obtain a factorization, we endow G with a left–invariant Riemannian metric and use the functional calculus for the Laplace–Beltrami operator to decompose the identity operator into smoothing and unbounded convolution operators. Practically, thus, we use the basic properties of the wave equation to verify Cauchy inequalities for

functions of the Laplacian and thereby deduce the analyticity of their kernels. Apart from factorization, some topological properties of spaces of analytic functions associated to Lie group representations are discussed. In the particular case of our algebra of rapidly decaying analytic functions this crucially involves the maximum principle for a suitable complex Laplacian.

Large parts of this work are available as preprint and have been submitted for publication. The various coauthors will be mentioned below. We now outline the content of the thesis, but refer to the individual chapters for more technical information.

The first two chapters on theoretical numerical analysis present numerically efficient variational formulations for nonlinear transmission problems arising e.g. in elasticity and phase transitions. While the numerical approximation of uniformly elliptic linear and nonlinear problems has been well–understood for many years, the correct framework to couple degenerate or not even monotone nonlinear operators best studied in L^p–spaces to an elliptic exterior L^2–problem remained unclear. For a homogeneous linear elliptic boundary problem in a domain Ω, the idea of the boundary element method is to compute the solution by discretizing and solving the equivalent integral equation for the layer potentials on $\partial\Omega$. This approach has been well established for many years and leads to rapidly convergent algorithms [51]. While for smooth, or at least $C^{1,\alpha}$, boundaries the analysis of the resulting equations is amenable to pseudodifferential techniques [16], the practically relevant case of polygonal domains or their complements relies on hard results by Coifman, McIntosh and Meyer [14] for integral operators on Lipschitz boundaries. Adapting their work to the needs of numerical analysis, Costabel [15] in particular proved the relevant continuity and coercivity estimates to justify realistic boundary element procedures.
The analysis of the contact problems discussed in this thesis couples these results for a homogeneous linear exterior problem to a fully nonlinear, possibly nonmonotone equation such as the p–Laplacian or a double–well potential in Ω. As the main result, we introduce a convenient framework based on mixed $L^p - L^2$–Sobolev spaces to discuss the a priori and a posteriori analysis of the error committed in the numerical approximation with finite and boundary elements.

In Chapter 2, monotone p–Laplacian–like operators ($p \geq 2$) are considered with transmission and Coulomb friction contact to the homogeneous Laplace equation. They arise, for example, as toy models of quasi–Newtonian fluids or nonlinearly elastic Hencky materials. Finite element approximations of local boundary problems for the p–Laplacian have been extensively studied since the first analysis by Barrett and Liu [1], and efficient a posteriori estimates of gradient recovery type have been legitimized only recently [7]. We obtain:

Theorem.
a) The contact problem admits a unique solution in $W^{1,p}(\Omega) \times H^{1/2}(\partial\Omega)$, which depends continuously on the data.
b) The numerical approximations are norm–convergent to the solution.
c) The gradient recovery of [7] can be combined with a residual a posteriori estimate on $\partial\Omega$.

Our specific estimates in Chapter 2 generalize the results from the elliptic case. However, without elliptic regularity one can only expect local $C^{1,\alpha}$–smoothness of the solution (cf. [18]), and the convergence of the algorithm might be arbitrarily slow. Numerical experiments against exact solutions in the frictionless case nevertheless exhibit a positive rate of convergence and show that the error is efficiently controlled by the a posteriori estimate.

Chapter 3 extends the analysis to a double–well problem with Signiorini contact, a prototypical example for a nonmonotone operator with explicitly known convex relaxation [8]. In the case of Dirichlet boundary conditions, the double–well problem has served as a benchmark to test numerical algorithms. Physically, it describes the generic behavior of a material passing the critical point of a phase transition into a finely textured mixture of locally energetically equivalent configurations of lower symmetry, the so–called microstructure. Although this problem does not even admit a weak solution, approximate solutions share certain "macroscopic" features, like the stress or exterior boundary value. It is for these quantities that an analogue of the above theorem is obtained.

The approach to contact problems readily generalizes beyond the scalar toy models studied in this work and does not depend on the details of the numerical method.

Chapters 2 and 3 are joint preprints with Matthias Maischak, Elmar Schrohe and Ernst P. Stephan. I have written most of the technical parts, leaving the numerical simulations and their discussion to Matthias Maischak.

The representation–theoretic part of this work presents a general framework to study analytic representations of a real Lie group G. Let (π, E) be a representation of G on a locally convex space E. Smoothness or analyticity of a vector $v \in E$ is defined in terms of the orbit map $x \mapsto \pi(x)v$, and the spaces of smooth E^∞ or analytic E^ω vectors are endowed with the corresponding topology of vector–valued functions on G. The representation (π, E) will be called analytic, if $E = E^\omega$ as topological vector spaces.
For a Fréchet space E, a fundamental result by Dixmier–Malliavin allows to factorize E^∞: If Π denotes the induced action of the convolution algebra $C_c^\infty(G)$

on E^∞,

$$\Pi(f)v = \int_G f(g)\pi(g)v \; dg \quad (f \in \mathrm{C}_c^\infty(G), v \in E),$$

then

$$E^\infty = \Pi(\mathrm{C}_c^\infty(G))E^\infty := \mathrm{span}\{\Pi(f)v : f \in C_c^\infty(G),\ v \in E^\infty\}.$$

It generalizes Cohen's factorization theorem for Banach algebras with an approximate unit [13], in particular $L^1(G) = L^1(G)*L^1(G)$, to $C_c^\infty(G) = C_c^\infty(G)*C_c^\infty(G)$ and answers the above–mentioned problem by Ehrenpreis. Choosing an approximate identity in $C_c^\infty(G)$, one recovers Gårding's result that E^∞ is dense in E [24]. Outside representation theory, the result is frequently applied in noncommutative geometry and abstract studies of distributions to show surjectivity.

Chapter 4 presents an analogue of Dixmier–Malliavin's theorem in the analytic category, with the algebra of test functions replaced by analytic functions $\mathcal{A}(G)$ of superexponential decay with respect to a left–invariant Riemannian metric $\mathbf{g}$. Under suitable growth assumptions, satisfied e.g. whenever E is a Banach space, the main theorem says:

Theorem.
$\mathcal{A}(G) = \mathcal{A}(G) * \mathcal{A}(G)$ and $E^\omega = \Pi(\mathcal{A}(G))E^\omega$.

As hinted at above, the proof relies on the functional calculus for the Laplace–Beltrami operator Δ associated to $\mathbf{g}$ and, in particular, on Cheeger, Gromov and Taylor's representation

$$f(\sqrt{\Delta}) = \int_{\mathbb{R}} \hat{f}(\lambda)\ \cos(\lambda\sqrt{\Delta})\ d\lambda\,, \qquad f \in \mathcal{S}(\mathbb{R}) \text{ even},$$

in terms of the solution operator $\cos(\lambda\sqrt{\Delta})$ for the wave equation.

Corollary.
a) E^ω is dense in E [46].
b) $v \in E^\omega$ if and only if there exists $\varepsilon > 0$ such that for every continuous seminorm p on E: $\sum_{j=0}^\infty \frac{\varepsilon^j}{(2j)!}\, p(\Delta^j v) < \infty$.
c) $E^\infty = \bigcap_{j=0}^\infty \mathrm{Dom}(\Delta^j)$.

b) and c) generalize the elliptic regularity results for partial differential equations to an algebraic characterization in terms of the enveloping Lie algebra.

Chapter 5 formally introduces categories of $\mathcal{A}(G)$–tempered and non–tempered analytic representations and analyzes their fundamental topological and representation theoretical properties. This includes a further discussion of a similar algebra $\mathcal{A}(G)$ and, using the factorization theorem, of the analytic globalization of Harish–Chandra modules. Because of the inductive limit structure of E^ω, analytic representations tend to carry complicated, non-Fréchet topologies, and we may not impose completeness assumptions if the quotient of an analytic representation by a closed invariant subspace is to be analytic. However, important special cases like the analytic vectors associated to a Banach representation, the algebra $\mathcal{A}(G)$, or the analytic globalization of a Harish–Chandra module are better behaved.

As an application, analytic representations might provide a convenient language to study weakly holomorphic modular forms and the factorization of the associated matrix coefficients.

Chapters 4 and 5 have been compounded from joint preprints with Bernhard Krötz, Christoph Lienau and Henrik Schlichtkrull. I have written a substantial portion of the technical parts on the factorization theorem and, in Chapter 5, contributed to the topological properties, especially those of $\mathcal{A}(G)$.

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Chapter 2

Adaptive FE–BE coupling for the p–Laplacian

Adaptive finite element / boundary element procedures provide an efficient and extensively investigated tool for the numerical solution of uniformly elliptic transmission or contact problems. However, models of strongly nonlinear materials often lead to nonelliptic partial differential equations, where the standard Hilbert space techniques are no longer appropriate to analyze the computational methods. In this chapter, we consider the numerical approximation of the following degenerate transmission and contact problem on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$:

$$\begin{aligned} -\text{div}\,(\varrho(|\nabla u_1|)\nabla u_1) = f \quad \text{in } \Omega, - \Delta u_2 =& 0 \quad \text{in } \Omega^c, \\ \varrho(|\nabla u_1|)\partial_\nu u_1 - \partial_\nu u_2 = t_0 \quad \text{on } \partial\Omega, u_1 - u_2 =& u_0 \quad \text{on } \Gamma_t, \\ -\varrho(|\nabla u_1|)\partial_\nu u_1(u_0 + u_2 - u_1) + g|(u_0 + u_2 - u_1)| =& 0, \\ |\varrho(|\nabla u_1|)\partial_\nu u_1| \leq g \quad \text{on } \Gamma_s. \\ u_2(x) = \begin{cases} a + o(1) & , n = 2 \\ \mathcal{O}(|x|^{2-n}), & n > 2 \end{cases} . \end{aligned} \tag{0.1}$$

Here $\varrho(t)$ denotes a function $\varrho(x,t) \in C(\overline{\Omega} \times (0,\infty))$ satisfying

$$0 \leq \varrho(t) \leq \varrho^*[t^\delta(1+t)^{1-\delta}]^{p-2},$$

$$|\varrho(t)t - \varrho(s)s| \leq \varrho^*[(t+s)^\delta(1+t+s)^{1-\delta}]^{p-2}|t-s|$$

and

$$\varrho(t)t - \varrho(s)s \geq \varrho_*[(t+s)^\delta(1+t+s)^{1-\delta}]^{p-2}(t-s)$$

for all $t \geq s > 0$ uniformly in $x \in \Omega$ ($\delta \in [0,1]$, $\varrho_*, \varrho^* > 0$). The interface $\partial\Omega = \overline{\Gamma_s \cup \Gamma_t}$ is divided into the disjoint components Γ_s and $\Gamma_t \neq \emptyset$, and the data belong to the following spaces:

$$f \in L^{p'}(\Omega),\ u_0 \in W^{\frac{1}{2},2}(\partial\Omega),\ t_0 \in W^{-\frac{1}{2},2}(\partial\Omega),\ g \in L^\infty(\Gamma_s),\ a \in \mathbb{R}.$$

As usual, the normal derivatives are understood in terms of a Green's formula, and it is convenient to set $a = 0$ for $n > 2$. In two dimensions one further condition is required to enforce uniqueness:

$$\int_\Omega f + \langle t_0, 1\rangle = 0. \tag{0.2}$$

We are looking for weak solutions $(u_1, u_2) \in W^{1,p}(\Omega) \times W^{1,2}_{loc}(\Omega^c)$ when $p \geq 2$. A typical example is given by $\varrho(t) = [t^\delta(1+t)^{1-\delta}]^{p-2}$, $\delta \in [0,1]$, with the p–Laplacian corresponding to the maximally degenerate case $\delta = 1$.
We use layer potentials for the Laplace equation on Ω^c to reduce the system to a uniquely solvable variational problem on $W^{1,p}(\Omega) \times W_0^{\frac{1}{2},2}(\Gamma_s)$. The main idea of our theoretical analysis is simple: Because the traces of $W^{1,p}(\Omega)$–functions are continuously embedded into $W^{\frac{1}{2},2}(\partial\Omega)$ for $p \geq 2$, the quadratic form $\langle Su, u\rangle$ associated to the Steklov–Poincaré operator is accessible to Hilbert space methods whenever it is defined. In this slightly weaker setting, Friedrichs' inequality (Prop. 2.1.3) allows to recover control over the L^p–norms in the interior, and as a consequence the full variational functional associated to the above equations is coercive in $W^{1,p}(\Omega)$.
The numerical part of this chapter, contributed by Matthias Maischak, investigates a model problem, which shows singularities resulting from the given boundary data, as well as from the change of boundary conditions, leading to a suboptimal convergence rate for uniform mesh refinements. We also present a Uzawa solver to deal with the variational inequality.
With the help of a Korn inequality (Prop. 2.1.6), our method easily carries over to transmission problems in nonlinear elasticity, e.g. Hencky materials in Ω coupled to the Lamé equation in Ω^c. A generalization to a certain nonconvex energy functional will be discussed in Chapter 3.

2.1 Preliminaries

Let Ω be an open subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Set $p' = \frac{p}{p-1}$ whenever $p \in (1, \infty)$.

Definition 2.1.1. The Sobolev spaces $W^{k,p}_{(0)}(\Omega)$, $k \in \mathbb{N}_0$, are the completion of $C^\infty_{(c)}(\Omega)$ with respect to the norm $\|u\|_{W^{k,p}(\Omega)} = \|u\|_{k,p} = \|u\|_p + \sum_{|\gamma|=k} \|\partial^\gamma u\|_p$. The second term in the norm will be denoted by $|u|_{W^{1,p}(\Omega)} = |u|_{k,p}$. Let $W_0^{-k,p'}(\Omega) = \left(W^{k,p}(\Omega)\right)'$ and $W^{-k,p'}(\Omega) = \left(W_0^{k,p}(\Omega)\right)'$. $W^{1-\frac{1}{p},p}(\partial\Omega)$ denotes the space of traces of $W^{1,p}(\Omega)$–functions on the boundary. It coincides with the Besov space $B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)$ as obtained by real interpolation of Sobolev spaces [52], and one may define $W^{s,p}(\partial\Omega) = B^s_{p,p}(\partial\Omega)$ for $s \in (-1, 1)$.

Remark 2.1.2. We are going to need the following properties for bounded $\partial\Omega$ [52]:
a) All the above spaces are reflexive and $(W^{s,p}(\partial\Omega))' = W^{-s,p'}(\partial\Omega)$.
b) For $p = 2$ they coincide with the Sobolev spaces H^s.
c) $W^{1-\frac{1}{p},p}(\partial\Omega) \hookrightarrow W^{\frac{1}{2},2}(\partial\Omega)$ for $p \geq 2$.
d) If $\partial\Omega$ is smooth, pseudodifferential operators of order m with symbol in the Hörmander class $S^m_{1,0}(\partial\Omega)$ map $W^{s,p}(\partial\Omega)$ continuously to $W^{s-m,p}(\partial\Omega)$. For Lipschitz $\partial\Omega$, at least the first–order Steklov–Poincaré operator S of the Laplacian on Ω^c is continuous between $W^{\frac{1}{2},2}(\partial\Omega)$ and $W^{-\frac{1}{2},2}(\partial\Omega)$ [15].
e) Points a) to d) imply that the quadratic form $\langle Su, u\rangle$ associated to S is well-defined on $W^{1-\frac{1}{p},p}(\partial\Omega)$ if $p \geq 2$. S being elliptic, the form cannot be defined for $p < 2$ even if $\partial\Omega$ is smooth.

Uniform monotony will be shown using a variant of Friedrichs' inequality.

Proposition 2.1.3. *Assume Ω is bounded and that $\Gamma \subset \partial\Omega$ has positive $(n-1)$–dimensional measure. Then there is a $C > 0$ such that*

$$\|u\|_p \leq C(\|\nabla u\|_p + \|u|_\Gamma\|_{L^1(\Gamma)}) \quad \textit{for all } u \in W^{1,p}(\Omega).$$

Proof. We apply an interpolation argument to the well-known Friedrichs' inequality

$$\|u - u_\Omega\|_p \leq C\|\nabla u\|_p, \qquad u_\Omega = \frac{1}{|\Omega|}\int_\Omega u,$$

on $W^{1,p}(\Omega)$ (see e.g. [45]). Let $L : W^{1,p}(\Omega) \to L^p(\Omega)$ be the rank–1 operator $Lu = \frac{1}{|\Gamma|}\int_\Gamma u|_\Gamma$ and I the inclusion of $W^{1,p}(\Omega)$ into $L^p(\Omega)$. Then $I-L : W^{1,p}(\Omega) \to L^p(\Omega)$ is bounded and

$$\|u - Lu\|_p = \|(I - L)(u - u_\Omega)\|_p \leq \|I - L\|\|u - u_\Omega\|_{1,p} \leq C\|\nabla u\|_p$$

for all $u \in W^{1,p}(\Omega)$. The assertion follows. □

Let $\omega(x, y) = (|x| + |y|)^\delta(1 + |x| + |y|)^{1-\delta}$, $0 \leq \delta \leq 1$. In addition to the above norms, the following family of quasi–norms will prove useful:

Definition 2.1.4. For $v, w \in W^{1,p}(\Omega)$ and $k \in \mathbb{N}_0$, define

$$|v|_{(k,w,p)} = \left(\int_\Omega \omega(\nabla w, D^k v)^{p-2}|D^k v|^2\right)^{\frac{1}{2}},$$

where $|D^k v|^2 = \sum_{|\gamma|=k} |\partial^\gamma v|^2$.

Remark 2.1.5. a) If $p \geq 2$, the $(1, w, p)$–quasi–norm can be estimated from above and below by suitable powers of the $W^{1,p}$–seminorm [22]:

$$|v|^p_{1,p} \leq |v|^2_{(1,w,p)} \leq C(|v|_{1,p}, |w|_{1,p})|v|^2_{1,p}.$$

b) In the nondegenerate case $\delta = 0$, we have $|v|_{1,2}^2 \leq |v|_{(1,w,p)}^2$.
c) The following inequality is useful for computations with quasi–norms:

$$\lambda\mu \leq \max\{\varepsilon^{-1}, \varepsilon^{1/(1-p)}\}(a^{p-1} + \lambda)^{p'-2}\lambda^2 + \varepsilon(a+\mu)^{p-2}\mu^2$$

for $\lambda, \mu, a \geq 0$ and $\varepsilon > 0$.

The results of this chapter easily generalize to the systems of equations describing certain inelastic materials. In this case, Lemma 2.1.3 has to be replaced by the following Korn inequality:

Proposition 2.1.6. *Assume $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $\Gamma \subset \partial\Omega$ has positive $(n-1)$–dimensional measure. Then there is a $C > 0$ such that*

$$\|u\|_{1,p} \leq C(\|\varepsilon(u)\|_p + \|u|_\Gamma\|_{L^1(\Gamma)}) \quad \text{for all } u \in (W^{1,p}(\Omega))^n.$$

Proof. The L^p–version $\|u\|_{1,p} \leq C(\|\varepsilon(u)\|_p + \|u\|_p)$ of Korn's inequality is well-known (see e.g. [19]). Assume the assertion was false. Then

$$\|\varepsilon(u_n)\|_p + \|u_n|_\Gamma\|_{L^1(\Gamma)} \leq \frac{1}{n}$$

for some sequence in $W^{1,p}(\Omega)$ normalized to $\|u_n\|_{1,p} = 1$. By the compactness of $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, we may assume u_n to converge in $L^p(\Omega)$. The cited variant of Korn's inequality shows that u_n is even Cauchy in $W^{1,p}(\Omega)$, hence converges to some u_0 with $\|\varepsilon(u_0)\|_p = \|u_0|_\Gamma\|_{L^1(\Gamma)} = 0$. The kernel of ε consists of skew–symmetric affine transformations $Ax + b$, $A = -A^T$. As $\dim\ker A \equiv n \bmod 2$, u_0 cannot vanish on all of the ($n-1$–dimensional) Γ unless $u_0 = 0$. Contradiction to $\|u_0\|_{1,p} = 1$. □

2.2 Variational formulation and reduction to $\partial\Omega$

We continue to use the notation from the Introduction and mainly follow [43]. Fix some $p \geq 2$ and, for $q(t) = \int_0^t s\varrho(s)\,\mathrm{d}s$, let $G(u) = \int_\Omega q(|\nabla u|)$ with derivative

$$DG(u,v) = \langle G'u, v\rangle = \int_\Omega \varrho(|\nabla u|)\nabla u \nabla v \qquad (u, v \in W^{1,p}(\Omega))$$

and $j(v) = \int_{\Gamma_s} g|v|$, $v \in L^1(\Gamma_s)$. G is known to be strictly convex and $G' : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ bounded and uniformly monotone, hence coercive, with respect to the seminorm $|\cdot|_{1,p}$: There is some $\alpha_G > 0$ such that for all $u, v \in W^{1,p}(\Omega)$

$$\langle G'u - G'v, u - v\rangle \geq \alpha_G |u - v|_{1,p}^p \quad \text{and} \quad \lim_{|u|_{1,p}\to\infty} \frac{\langle G'u, u\rangle}{|u|_{1,p}} = \infty.$$

The naive variational formulation of the transmission problem (0.1) minimizes the functional

$$\Phi(u_1, u_2) = G(u_1) + \frac{1}{2}\int_{\Omega^c} |\nabla u_2|^2 - \int_\Omega f u_1 - \langle t_0, u_2|_{\partial\Omega}\rangle + j((u_2 - u_1 + u_0)|_{\Gamma_s})$$

over a suitable convex set.

Lemma 2.2.1. *Minimizing Φ over the nonempty, closed and convex subset*

$$C = \{(u_1, u_2) \in W^{1,p}(\Omega) \times W^{1,2}_{loc}(\Omega^c) : (u_1 - u_2)|_{\Gamma_t} = u_0,\ u_2 \in \mathcal{L}_2\},$$

$$\mathcal{L}_2 = \{v \in W^{1,2}_{loc}(\Omega^c) : \Delta v = 0 \text{ in } W^{-1,2}(\Omega^c) + \text{radiation condition at } \infty\},$$

is equivalent to the system (0.1) in the sense of distributions if $\varrho \in C^1(\overline{\Omega}\times(0,\infty))$.

Proof. C is apparently convex. A similar argument as in Remarks 2 and 4 of [6] shows that C is closed and nonempty. The proof there almost exclusively involves the exterior problem in $\mathcal{L}_2$ and only requires basic measure theoretic properties of $W^{1,2}(\Omega)$, which also hold for $W^{1,p}(\Omega)$. Finally, repeat the computations of [43] to obtain equivalence with (0.1). □

To reduce the exterior problem to the boundary, we are going to need the layer potentials

$$\begin{aligned}
\mathcal{V}\phi(x) &= -\frac{1}{\pi}\int_{\partial\Omega} \phi(x')\ \log|x - x'|\ dx',\\
\mathcal{K}\phi(x) &= -\frac{1}{\pi}\int_{\partial\Omega} \phi(x')\ \partial_{\nu_{x'}} \log|x - x'|\ dx',\\
\mathcal{K}'\phi(x) &= -\frac{1}{\pi}\int_{\partial\Omega} \phi(x')\ \partial_{\nu_x} \log|x - x'|\ dx',\\
\mathcal{W}\phi(x) &= \frac{1}{\pi}\ \partial_{\nu_x}\int_{\partial\Omega} \phi(x')\ \partial_{\nu_{x'}} \log|x - x'|\ dx'
\end{aligned}$$

associated to the Laplace equation on Ω^c. They extend from $C^\infty(\partial\Omega)$ to a bounded map $\begin{pmatrix} -\mathcal{K} & \mathcal{V} \\ \mathcal{W} & \mathcal{K}' \end{pmatrix}$ on the Sobolev space $W^{\frac{1}{2},2}(\partial\Omega) \times W^{-\frac{1}{2},2}(\partial\Omega)$. If the capacity of $\partial\Omega$ is less than 1, which can always be achieved by scaling, $\mathcal{V}$ and $\mathcal{W}$ considered as operators on $W^{-\frac{1}{2},2}(\partial\Omega)$ are selfadjoint, $\mathcal{V}$ is positive and $\mathcal{W}$ non-negative. Similarly, the Steklov-Poincaré operator

$$S = \mathcal{W} + (1 - \mathcal{K}')\mathcal{V}^{-1}(1 - \mathcal{K}) : W^{\frac{1}{2},2}(\partial\Omega) \subset W^{-\frac{1}{2},2}(\partial\Omega) \to W^{-\frac{1}{2},2}(\partial\Omega)$$

defines a positive and selfadjoint operator (pseudodifferential of order 1, if $\partial\Omega$ is smooth) with the main property

$$\partial_\nu u_2|_{\partial\Omega} = -S(u_2|_{\partial\Omega} - a)$$

for solutions $u_2 \in \mathcal{L}_2$ of the Laplace equation on Ω^c. By Remark 2.1.2 e), S gives rise to a coercive and symmetric bilinear form $\langle Su, u\rangle$ on $W^{\frac{1}{2},2}(\partial\Omega)$ and, in particular, a pairing on the traces of $W^{1,p}(\Omega)$ if and only if $p \geq 2$.

Using the weak definition of $\partial_\nu|_{\partial\Omega}$, S reduces the integral over Ω^c in Φ to the boundary:

$$\int_{\Omega^c} |\nabla u_2|^2 = -\langle \partial_\nu u_2|_{\partial\Omega}, u_2|_{\partial\Omega}\rangle = \langle S(u_2|_{\partial\Omega} - a), u_2|_{\partial\Omega}\rangle \quad \text{for } u_2 \in \mathcal{L}_2.$$

Easy manipulations allow to substitute u_2 by a function v on Γ_s (cf. [43]): Let

$$\widetilde{W}^{\frac{1}{2},2}(\Gamma_s) = \{u \in W^{\frac{1}{2},2}(\partial\Omega) : \text{supp } u \subset \bar{\Gamma}_s\}, \quad X^p = W^{1,p}(\Omega) \times \widetilde{W}^{\frac{1}{2},2}(\Gamma_s)$$

and $(u, v) = (u_1 - c, u_0 + u_2|_{\partial\Omega} - u_1|_{\partial\Omega}) \in X^p$ for a suitable $c \in \mathbb{R}$. Collecting the data–dependent terms in

$$\lambda(u, v) = \langle t_0 + Su_0, u|_{\partial\Omega} + v\rangle + \int_\Omega fu$$

leads to

$$\Phi(u_1, u_2) = G(u) + \frac{1}{2}\langle S(u|_{\partial\Omega} + v), u|_{\partial\Omega} + v\rangle - \lambda(u, v) + j(v) + \frac{1}{2}\langle Su_0, u_0\rangle + \langle t_0, u_0\rangle.$$

The first three terms on the right hand side will be called $J(u, v)$.

Lemma 2.2.2. *Minimizing Φ over C is equivalent to minimizing $J + j$ over the nonempty closed convex set $D = \{(u, v) \in X^p : \langle S(u|_{\partial\Omega} + v - u_0), 1\rangle = 0$ if $n = 2\}$*

Proof. As in [43]. The main additional observation here is that the substitution $v = u_0 + u_2|_{\partial\Omega} - u_1|_{\partial\Omega}$ indeed defines an element of $\widetilde{W}^{\frac{1}{2},2}(\Gamma_s)$, because $u_0, u_2|_{\partial\Omega} \in W^{\frac{1}{2},2}(\partial\Omega)$, $u_1|_{\partial\Omega} \in W^{1-\frac{1}{p},p}(\partial\Omega) \subset W^{\frac{1}{2},2}(\partial\Omega)$ by Remark 2.1.2 and $v|_{\Gamma_t} = 0$, if $(u_1, u_2) \in C$. □

2.3 Existence and uniqueness

Minimization of $J + j$ over D translates into the following variational inequality: Find $(\hat{u}, \hat{v}) \in X^p$ such that

$$\langle G'\hat{u}, u - \hat{u}\rangle + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u - \hat{u})|_{\partial\Omega} + v - \hat{v}\rangle + j(v) - j(\hat{v}) \geq \lambda(u - \hat{u}, v - \hat{v})$$

for all $(u, v) \in X^p$. Note that D has been replaced by X^p.

We now prove the crucial monotony estimate:

Lemma 2.3.1. *The operator in the variational inequality is uniformly monotone on X^p. There exists an $\alpha = \alpha(C) > 0$ such that for all $\|u, v\|_X, \|\hat{u}, \hat{v}\|_X < C$*

$$\alpha(\|u - \hat{u}\|^p_{W^{1,p}(\Omega)} + \|v - \hat{v}\|^p_{\widetilde{W}^{\frac{1}{2},2}(\Gamma_s)}) \leq \langle G'\hat{u} - G'u, \hat{u} - u\rangle + \langle S((\hat{u} - u)|_{\partial\Omega} + \hat{v} - v), (\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\rangle.$$

Proof. Recall the monotony estimate for G' from Section 2.2:

$$\langle G'\hat{u} - G'u, \hat{u} - u\rangle \geq \alpha_G |\hat{u} - u|^p_{1,p}.$$

The triangle inequality and convexity of x^p imply

$$\begin{aligned} \|\hat{v} - v\|^p_{\widetilde{W}^{\frac{1}{2},2}(\Gamma_s)} &\leq (\|(\hat{u} - u)|_{\Gamma_s} + \hat{v} - v\|_{W^{\frac{1}{2},2}(\Gamma_s)} + \|(\hat{u} - u)|_{\Gamma_s}\|_{W^{\frac{1}{2},2}(\Gamma_s)})^p \\ &\leq 2^{p-1}\ (\|(\hat{u} - u)|_{\Gamma_s} + \hat{v} - v\|^p_{W^{\frac{1}{2},2}(\Gamma_s)} + \|(\hat{u} - u)|_{\Gamma_s}\|^p_{W^{\frac{1}{2},2}(\Gamma_s)}). \end{aligned}$$

Using $W^{1-\frac{1}{p},p}(\Gamma_s) \hookrightarrow W^{\frac{1}{2},2}(\Gamma_s)$ as well as the boundedness of the trace operator,

$$2^{1-p}\ \|\hat{v} - v\|^p_{\widetilde{W}^{\frac{1}{2},2}(\Gamma_s)} - \beta\ \|\hat{u} - u\|^p_{W^{1,p}(\Omega)} \leq \|(\hat{u} - u)|_{\Gamma_s} + \hat{v} - v\|^p_{W^{\frac{1}{2},2}(\Gamma_s)}$$

follows for some $\beta \geq 1$. Let

$$K = \{(u, v, \hat{u}, \hat{v}) \in X^p \times X^p : \|(\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\|_{W^{\frac{1}{2},2}(\partial\Omega)} < 2\beta C\}$$

and $0 < \varepsilon < \beta^{-1}$. Since S is positive definite on $W^{\frac{1}{2},2}(\partial\Omega)$, we obtain from Friedrichs' inequality for $(u, v, \hat{u}, \hat{v}) \in K$ or, in particular, if $\|u, v\|_X, \|\hat{u}, \hat{v}\|_X < C$:

$$\begin{aligned} &\langle G'\hat{u} - G'u, \hat{u} - u\rangle + \langle S((\hat{u} - u)|_{\partial\Omega} + \hat{v} - v), (\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\rangle \\ &\gtrsim |\hat{u} - u|^p_{1,p} + \|(\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} \\ &\gtrsim |\hat{u} - u|^p_{1,p} + \|(\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\|^p_{W^{\frac{1}{2},2}(\partial\Omega)} \\ &\gtrsim |\hat{u} - u|^p_{1,p} + \varepsilon\ \|(\hat{u} - u)|_{\Gamma_s} + \hat{v} - v\|^p_{W^{\frac{1}{2},2}(\Gamma_s)} + \|(\hat{u} - u)|_{\Gamma_t}\|^p_{W^{\frac{1}{2},2}(\Gamma_t)} \\ &\gtrsim \|\hat{u} - u\|^p_{W^{1,p}(\Omega)} + \varepsilon\ \|(\hat{u} - u)|_{\Gamma_s} + \hat{v} - v\|^p_{W^{\frac{1}{2},2}(\Gamma_s)} \\ &\gtrsim (1 - \varepsilon\beta)\ \|\hat{u} - u\|^p_{W^{1,p}(\Omega)} + 2^{1-p}\varepsilon\ \|\hat{v} - v\|^p_{\widetilde{W}^{\frac{1}{2},2}(\Gamma_s)}. \end{aligned}$$

Uniform monotony on all of X^p is shown similarly, but on the unbounded complement $(X^p \times X^p) \setminus K$ the exponents p on the left hand side have to be replaced by 2. □

Theorem 2.3.2. *The variational inequality is equivalent to the transmission problem (0.1) and has a unique solution.*

Proof. We repeat the computations in [43] to get the equivalence with the minimization of $J + j$ over D, and hence with (0.1). Existence and uniqueness follow from Lemma 2.3.1, e.g. by applying [57], Proposition 32.36. □

2.4 Discretization and error analysis

In order to avoid using $S = \mathcal{W} + (1-\mathcal{K}')\mathcal{V}^{-1}(1-\mathcal{K})$ explicitly, the numerical implementation involves a variant of the variational inequality

$$\langle G'\hat{u}, u-\hat{u}\rangle + \langle S(\hat{u}|_{\partial\Omega}+\hat{v}), (u-\hat{u})|_{\partial\Omega}+v-\hat{v}\rangle + j(v)-j(\hat{v}) \geq \lambda(u-\hat{u}, v-\hat{v})$$

in terms of the layer potentials. Our a posteriori analysis is therefore based on the following equivalent problem: Find $(\hat{u},\hat{v},\hat{\phi}) \in X^p \times W^{-\frac{1}{2},2}(\partial\Omega) =: Y^p$, such that

$$\begin{aligned} &\langle G'\hat{u}, u-\hat{u}\rangle + \langle \mathcal{W}(\hat{u}|_{\partial\Omega}+\hat{v}) + (\mathcal{K}'-1)\hat{\phi}, (u-\hat{u})|_{\partial\Omega}+v-\hat{v}\rangle \\ &\qquad + j(v)-j(\hat{v}) \;\geq\; \langle t_0+\mathcal{W}u_0, (u-\hat{u})|_{\partial\Omega}+v-\hat{v}\rangle + \int_\Omega f(u-\hat{u}), \\ &\langle \phi, \mathcal{V}\hat{\phi} + (1-\mathcal{K})(\hat{u}|_{\partial\Omega}+\hat{v})\rangle = \langle \phi, (1-\mathcal{K})u_0\rangle \end{aligned}$$

for all $(u,v,\phi)\in Y^p$. More concisely,

$$B(\hat{u},\hat{v},\hat{\phi}; u-\hat{u}, v-\hat{v}, \phi-\hat{\phi}) + j(v)-j(\hat{v}) \geq \Lambda(u-\hat{u}, v-\hat{v}, \phi-\hat{\phi})$$

with

$$\begin{aligned} B(u,v,\phi;\bar{u},\bar{v},\bar{\phi}) &= \langle G'u,\bar{u}\rangle + \langle \mathcal{W}(u|_{\partial\Omega}+v) + (\mathcal{K}'-1)\phi, \bar{u}|_{\partial\Omega}+\bar{v}\rangle \\ &\quad + \langle \bar{\phi}, \mathcal{V}\phi + (1-\mathcal{K})(u|_{\partial\Omega}+v)\rangle, \\ \Lambda(u,v,\phi) &= \langle t_0+\mathcal{W}u_0, u|_{\partial\Omega}+v\rangle + \int_\Omega fu + \langle \phi, (1-\mathcal{K})u_0\rangle. \end{aligned}$$

The more detailed a priori and a posteriori error analysis requires a few basic properties of the quasi–norms [22].

Remark 2.4.1. a) The continuity and coercivity estimates can be sharpened: For all $u,v\in W^{1,p}(\Omega)$

$$\langle G'u-G'v, u-v\rangle \lesssim |u-v|^2_{(1,u,p)} \lesssim \langle G'u-G'v, u-v\rangle.$$

b) There is $\theta>0$ such that for all $\varepsilon\in(0,\infty)$ and all $u,v,w\in W^{1,p}(\Omega)$

$$|\langle G'u-G'v, w\rangle| \lesssim \varepsilon|u-v|^2_{(1,u,p)} + \varepsilon^{-\theta}|w|^2_{(1,u,p)}.$$

Lemma 2.4.2. *For all* $(\hat{u},\hat{v},\hat{\phi}),(u,v,\phi)\in Y^p$ *we have*

$$\begin{aligned} &|\hat{u}-u|^2_{(1,\hat{u},p)} + \|(\hat{u}-u)|_{\partial\Omega}+\hat{v}-v\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\eta\|^2_{W^{-\frac{1}{2},2}(\partial\Omega)} \\ &\lesssim |\hat{u}-u|^2_{(1,\hat{u},p)} + \|(\hat{u}-u)|_{\partial\Omega}+\hat{v}-v\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\hat{\phi}-\phi\|^2_{W^{-\frac{1}{2},2}(\partial\Omega)} \\ &\lesssim B(\hat{u},\hat{v},\hat{\phi};\hat{u}-u,\hat{v}-v,\eta) - B(u,v,\phi;\hat{u}-u,\hat{v}-v,\eta), \end{aligned}$$

where $2\eta = \hat{\phi}-\phi+V^{-1}(1-K)((\hat{u}-u)|_{\partial\Omega}+\hat{v}-v)$.

Proof. The right hand side of the identity

$$\begin{aligned} &B(\hat{u},\hat{v},\hat{\phi};\hat{u}-u,\hat{v}-v,\eta)-B(u,v,\phi;\hat{u}-u,\hat{v}-v,\eta)\\ &=\langle G'\hat{u}-G'u,\hat{u}-u\rangle+\tfrac{1}{2}\langle\mathcal{W}((\hat{u}-u)|_{\partial\Omega}+\hat{v}-v),(\hat{u}-u)|_{\partial\Omega}+\hat{v}-v)\rangle\\ &\quad+\tfrac{1}{2}\langle S((\hat{u}-u)|_{\partial\Omega}+\hat{v}-v),(\hat{u}-u)|_{\partial\Omega}+\hat{v}-v)\rangle+\tfrac{1}{2}\langle\mathcal{V}(\hat{\phi}-\phi),\hat{\phi}-\phi\rangle. \end{aligned}$$

is, up to a constant, larger than $\|\hat{u}-u,\hat{v}-v,\hat{\phi}-\phi\|^2_{(\hat{u},Y^p)}$. Furthermore,

$$\|\eta\|_{W^{-\frac{1}{2},2}(\partial\Omega)}\lesssim\|\hat{\phi}-\phi\|_{W^{-\frac{1}{2},2}(\partial\Omega)}+\|(\hat{u}-u)|_{\partial\Omega}+\hat{v}-v\|_{W^{\frac{1}{2},2}(\partial\Omega)}.$$

□

Let $\{\mathcal{T}_h\}_{h\in I}$ a regular triangulation of Ω into disjoint open regular triangles K, so that $\overline{\Omega}=\bigcup_{K\in\mathcal{T}_h}K$. Each element has at most one edge on $\partial\Omega$, and the closures of any two of them share at most a single vertex or edge. Let h_K denote the diameter of $K\in\mathcal{T}_h$ and ρ_K the diameter of the largest inscribed ball. We assume that $1\leq\max_{K\in\mathcal{T}_h}\frac{h_K}{\rho_K}\leq R$ independent of h and that $h=\max_{K\in\mathcal{T}_h}h_K$. $\mathcal{E}_h$ is going to be the set of all edges of the triangles in $\mathcal{T}_h$, D the set of nodes. Associated to $\mathcal{T}_h$ is the space $W_h^{1,p}(\Omega)\subset W^{1,p}(\Omega)$ of functions whose restrictions to any $K\in\mathcal{T}_h$ are linear.
$\partial\Omega$ is triangulated by $\{l\in\mathcal{E}_h : l\subset\partial\Omega\}$. $W_h^{\frac{1}{2},2}(\partial\Omega)$ denotes the corresponding space of piecewise linear functions, and $\widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)$ the subspace of those supported on Γ_s. Finally, $W_h^{-\frac{1}{2},2}(\partial\Omega)\subset W^{-\frac{1}{2},2}(\partial\Omega)$.
We denote by $i_h : W_h^{1,p}(\Omega)\hookrightarrow W^{1,p}(\Omega)$, $j_h:\widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)\hookrightarrow\widetilde{W}^{\frac{1}{2},2}(\Gamma_s)$ and $k_h : W_h^{-\frac{1}{2},2}(\partial\Omega)\hookrightarrow W^{-\frac{1}{2},2}(\partial\Omega)$ the canonical inclusion maps. Set $X_h^p=W_h^{1,p}(\Omega)\times\widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)$, We denote by $i_h : W_h^{1,p}(\Omega)\hookrightarrow W^{1,p}(\Omega)$, $j_h:\widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)\hookrightarrow\widetilde{W}^{\frac{1}{2},2}(\Gamma_s)$ and $k_h : W_h^{-\frac{1}{2},2}(\partial\Omega)\hookrightarrow W^{-\frac{1}{2},2}(\partial\Omega)$ the canonical inclusion maps. Set $X_h^p=W_h^{1,p}(\Omega)\times\widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)$,

$$S_h=\frac{1}{2}(W+(I-K')k_h(k_h^*Vk_h)^{-1}k_h^*(I-K))$$

and

$$\lambda_h(u_h,v_h)=\langle t_0+S_hu_0,u|_{\partial\Omega}+v\rangle+\int_\Omega fu_h.$$

As is well–known, there exists $h_0>0$ such that the approximate Steklov–Poincaré operator S_h is coercive uniformly in $h<h_0$, i.e. $\langle S_hu_h,u_h\rangle\geq\alpha_S\|u_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)}$ with α_S independent of h.
The discretized variational inequality reads as follows: Find $(\hat{u}_h,\hat{v}_h,\hat{\phi}_h)\in Y_h^p$ such that

$$B(\hat{u}_h,\hat{v}_h,\hat{\phi}_h;u_h-\hat{u}_h,v_h-\hat{v}_h,\phi_h-\hat{\phi}_h)+j(v_h)-j(\hat{v}_h)\geq\Lambda(u_h-\hat{u}_h,v_h-\hat{v}_h,\phi_h-\hat{\phi}_h)$$

for all $(u_h, v_h, \phi_h) \in Y_h^p$. Repeating the arguments from the previous section, one obtains a unique solution to the discretized variational inequality.

Theorem 2.4.3. *Let $(\hat{u}, \hat{v}, \hat{\phi}) \in Y^p$, $(\hat{u}_h, \hat{v}_h, \hat{\phi}_h) \in Y_h^p$ be the solutions of the continuous resp. discretized variational problem. The following a priori bound for the error holds uniformly in $h < h_0$:*

$$
\begin{aligned}
&\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \hat{\phi} - \hat{\phi}_h\|_{Y^p}^p \\
&\lesssim |\hat{u} - \hat{u}_h|_{(1,\hat{u},p)}^2 + \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 + \|\hat{\phi} - \hat{\phi}_h\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^2 \\
&\lesssim \inf_{(u_h,v_h,\phi_h)\in Y_h^p} \|\hat{u} - u_h, \hat{v} - v_h, \hat{\phi} - \phi_h\|_{Y^p}^2 + \|\hat{v} - v_h\|_{L^2(\Gamma_s)}.
\end{aligned}
$$

Proof. Let $(u, v, \phi) \in Y^p$, $(u_h, v_h, \phi_h) \in Y_h^p$. Lemma 2.4.2 and the variational inequality imply

$$
\begin{aligned}
&|\hat{u} - \hat{u}_h|_{(1,\hat{u},p)}^2 + \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 + \|\hat{\phi} - \hat{\phi}_h\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^2 \\
&\lesssim B(\hat{u}, \hat{v}, \hat{\phi}; \hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \hat{\phi} - \hat{\phi}_h) - B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; \hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \hat{\phi} - \hat{\phi}_h) \\
&\lesssim B(\hat{u}, \hat{v}, \hat{\phi}; u, v, \phi) - \Lambda(u - \hat{u}, v - \hat{v}, \phi - \hat{\phi}) + j(v) - j(\hat{v}) \\
&\quad + B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; u_h, v_h, \phi_h) - \Lambda(u_h - \hat{u}_h, v_h - \hat{v}_h, \phi_h - \hat{\phi}_h) + j(v_h) - j(\hat{v}_h) \\
&\quad - B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; \hat{u}, \hat{v}, \hat{\phi}) - B(\hat{u}, \hat{v}, \hat{\phi}; \hat{u}_h, \hat{v}_h, \hat{\phi}_h)
\end{aligned}
$$

Setting $(u, v, \phi) = (\hat{u}_h, \hat{v}_h, \hat{\phi}_h)$ and adding 0, the right hand side turns into

$$
\begin{aligned}
&B(\hat{u}, \hat{v}, \hat{\phi}; u_h - \hat{u}, v_h - \hat{v}, \phi_h - \hat{\phi}) - \Lambda(u_h - \hat{u}, v_h - \hat{v}, \phi_h - \hat{\phi}) + j(v_h) - j(\hat{v}) \\
&\quad + B(\hat{u}, \hat{v}, \hat{\phi}; \hat{u} - u_h, \hat{v} - v_h, \hat{\phi} - \phi_h) - B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; \hat{u} - u_h, \hat{v} - v_h, \hat{\phi} - \phi_h).
\end{aligned}
$$

We first consider the friction terms:

$$
j(v_h) - j(\hat{v}) = \int_{\Gamma_s} g(|v_h| - |\hat{v}|) \leq \int_{\Gamma_s} g(|v_h - \hat{v}|) \leq \|g\|_{L^2(\Gamma_s)} \|v_h - \hat{v}\|_{L^2(\Gamma_s)}.
$$

The last two terms are bounded using Remark 2.4.1b and Cauchy-Schwarz:

$$
\begin{aligned}
\langle G'\hat{u} - G'\hat{u}_h, \hat{u} - u_h \rangle &\lesssim \varepsilon |\hat{u} - \hat{u}_h|_{(1,\hat{u},p)}^2 + \varepsilon^{-\theta} |\hat{u} - u_h|_{(1,\hat{u},p)}^2, \\
&\lesssim \varepsilon |\hat{u}_h - \hat{u}|_{(1,\hat{u},p)}^2 + \varepsilon^{-\theta} C(|\hat{u}|_{1,p}, |u_h|_{1,p}) |u_h - \hat{u}|_{1,p}^2
\end{aligned}
$$

for sufficiently small $\varepsilon > 0$. We may replace $C(|\hat{u}|_{1,p}, |u_h|_{1,p})$ by an honest constant noting that the coercivity of our functional gives an a priori bound on $\|\hat{u}\|_{W^{1,p}(\Omega)}$ and that we can restrict to those u_h satisfying $\|u_h\|_{W^{1,p}(\Omega)} \leq 2\|\hat{u}\|_{W^{1,p}(\Omega)}$. Moreover,

$$
\begin{aligned}
&\langle \mathcal{W}((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h) + (1 - \mathcal{K}')(\hat{\phi} - \hat{\phi}_h), (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h \rangle \\
&\lesssim \varepsilon \|(\hat{u} - \hat{u}_h|_{\partial\Omega} + \hat{v} - \hat{v}_h\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 + \varepsilon \|\hat{\phi} - \hat{\phi}_h\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^2 \\
&\quad + \varepsilon^{-1} \|\hat{u} - u_h)\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 + \varepsilon^{-1} \|\hat{v} - v_h\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2,
\end{aligned}
$$

and

$$\langle \hat{\phi} - \phi_h, \mathcal{V}(\hat{\phi} - \hat{\phi}_h) + (1 - \mathcal{K})((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h)\rangle$$
$$\lesssim \varepsilon^{-1}\|\hat{\phi} - \phi_h\|^2_{W^{-\frac{1}{2},2}(\partial\Omega)} + \varepsilon\|\hat{\phi} - \hat{\phi}_h\|^2_{W^{-\frac{1}{2},2}(\partial\Omega)} + \varepsilon\|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)}.$$

Substituting $(u, v, \phi) = (u_h, \hat{v}, 0)$ and $(u, v, \phi) = (2\hat{u} - u_h, \hat{v}, 0)$ into the variational inequality on Y^p and using that also the ϕ part is really an equality, the remaining two terms reduce to

$$\begin{aligned}
&\langle -t_0 - \mathcal{W}u_0 + \mathcal{W}(\hat{u}|_{\partial\Omega} + \hat{v}) + (\mathcal{K}' - 1)\hat{\phi}, v_h - \hat{v}\rangle \\
&= -\langle t_0 - S(\hat{u}|_{\partial\Omega} + \hat{v} - u_0), v_h - \hat{v}\rangle \\
&= -\langle \varrho(|\nabla u|)\partial_\nu u, v_h - \hat{v}\rangle \;\leq\; \|g\|_{L^2(\Gamma_s)}\|v_h - \hat{v}\|_{L^2(\Gamma_s)}.
\end{aligned}$$

Applying these various estimates to the terms of the right hand side, the assertion follows from

$$\|\hat{u}-\hat{u}_h, \hat{v}-\hat{v}_h, \hat{\phi}-\hat{\phi}_h\|^p_{Y^p} \lesssim |\hat{u}-\hat{u}_h|^2_{(1,\hat{u},p)} + \|(\hat{u}-\hat{u}_h)|_{\partial\Omega} + \hat{v}-\hat{v}_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\hat{\phi}-\hat{\phi}_h\|^2_{W^{-\frac{1}{2},2}(\partial\Omega)}$$

as in Lemma 2.3.1. □

In the nondegenerate case $\delta = 0$, we essentially recover the estimates for uniformly elliptic operators from [6, 43].

Corollary 2.4.4. *For $\delta = 0$, we obtain*

$$\|\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \hat{\phi} - \hat{\phi}_h\|^2_{Y^2} \;\lesssim\; \inf_{(u_h, v_h, \phi_h) \in Y_h^p} \|\hat{u} - u_h, \hat{v} - v_h, \hat{\phi} - \phi_h\|^2_{Y^p} + \|\hat{v} - v_h\|_{L^2(\Gamma_s)}$$

uniformly in $h < h_0$

Proof. Use 2.1.5b) to estimate $|\hat{u}_h - \hat{u}|_{(1,\hat{u},p)}$ in Theorem 3.2.1 from below. □

2.5 Adaptive grid refinement

Denote by

$$(e, \tilde{e}, \epsilon) = (\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \hat{\phi} - \hat{\phi}_h) \in Y^p$$

the error of the Galerkin approximation, and let $2\nu = \epsilon + \mathcal{V}^{-1}(1 - \mathcal{K})(e|_{\partial\Omega} + \tilde{e})$. Our basic a posteriori estimate is the following.

Lemma 2.5.1. *For all* $(e_h, \tilde{e}_h, \nu_h) \in Y_h^p$

$$\begin{aligned}
&|e|^2_{(1,\hat{u},p)} + \|e|_{\partial\Omega} + \tilde{e}\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\epsilon\|^2_{W^{-\frac{1}{2},2}(\partial\Omega)} \\
&\lesssim \Lambda(e - e_h, \tilde{e} - \tilde{e}_h, \nu - \nu_h) + j(\tilde{e}_h + \hat{v}_h) - j(\hat{v}) \\
&\quad - B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; e - e_h, \tilde{e} - \tilde{e}_h, \nu - \nu_h) \\
&= \int_\Omega f(e - e_h) - \langle G'\hat{u}_h, e - e_h\rangle + \int_{\Gamma_s} g(|\tilde{e}_h + \hat{v}_h| - |\tilde{e} + \hat{v}_h|) \\
&\quad - \langle \nu - \nu_h, \mathcal{V}\hat{\phi}_h + (1 - \mathcal{K})(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0)\rangle \\
&\quad + \langle t_0 - \mathcal{W}(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) - (\mathcal{K}' - 1)\hat{\phi}_h, (e - e_h)|_{\partial\Omega} + \tilde{e} - \tilde{e}_h\rangle.
\end{aligned}$$

Proof. Lemma 2.4.2, the continuous and the discretized variational inequality imply

$$\begin{aligned}
&|\hat{u} - u|^2_{(1,\hat{u},p)} + \|(\hat{u} - u)|_{\partial\Omega} + \hat{v} - v\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\hat{\phi} - \phi\|^2_{W^{-\frac{1}{2},2}(\partial\Omega)} \\
&\lesssim B(\hat{u}, \hat{v}, \hat{\phi}; \hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \nu) - B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; \hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \nu) \\
&\lesssim \Lambda(\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \nu) + j(\hat{v}_h) - j(\hat{v}) - B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; \hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h, \nu) \\
&\lesssim \Lambda(\hat{u} - \hat{u}_h - (u_h - \hat{u}_h), \hat{v} - \hat{v}_h - (v_h - \hat{v}_h), \nu - \nu_h) + j(v_h) - j(\hat{v}) \\
&\quad - B(\hat{u}_h, \hat{v}_h, \hat{\phi}_h; \hat{u} - \hat{u}_h - (u_h - \hat{u}_h), \hat{v} - \hat{v}_h - (v_h - \hat{v}_h), \nu - \nu_h).
\end{aligned}$$

Note that the variational inequalities are identities when restricted to the ϕ-variable. The claim follows by setting $e_h = u_h - \hat{u}_h$ and $\tilde{e}_h = v_h - \hat{v}_h$. □

Simplifying the right hand side along the lines of [7] leads to a gradient recovery scheme in the interior with a residual type estimator on the boundary. With a straight forward modification of [42], also a method purely based on residual type estimates could be justified.
For $1 < p < \infty$ and $0 \leq \delta \leq 1$, define

$$G_{p,\delta}(x, y) = |y|^2 \omega(x, y)^{p-2} = |y|^2[(|x| + |y|)^\delta (1 + |x| + |y|)^{1-\delta}]^{p-2}$$

whenever $|x| + |y| > 0$ and 0 otherwise. As in [7], our analysis will be based on the following consequences of the monotony and convexity properties of $G_{p,\delta}$.

Lemma 2.5.2. *Assume that* Ω *is connected. Let* q *be a continuous linear form on* $W^{1,p}(\Omega)$ *with* $\mathbb{R} \cap \ker q = \{0\}$*, where* $\mathbb{R}$ *is identified with the space of constant functions on* Ω*. Then for any* $1 < p < \infty$ *there exists* $C_P = C_P(p, q, \Omega) > 0$ *such that for all* $a \geq 0$ *and* $u \in W^{1,p}(\Omega)$*,*

$$\int_\Omega G_{p,\delta}(a, u) \leq C_P \left(G_{p,\delta}(a, q(u)) + \int_\Omega G_{p,\delta}(a, |\nabla u|) \right).$$

Proof. Cf. [7], Lemma 4.1 and its generalization in Remark 4.3. □

Lemma 2.5.3. *For any $d, k \in \mathbb{N}$ there is $C_\Sigma = C_\Sigma(p, d, k) > 0$ such that for all $a_1, a_2, \dots, a_k \in \mathbb{R}^d$*

$$\sum_{j=1}^{k}\sum_{l=1}^{j-1} G_{p,\delta}(a_j, a_j - a_l) \lesssim C_\Sigma \sum_{j=1}^{k-1} \min_{1\le m\le k} G_{p,\delta}(a_m, a_{j+1} - a_j).$$

Proof. Cf. [7], Lemma 4.2 and its generalization in Remark 4.3. □

Even though Lemma 2.5.5 and Lemma 2.5.6 hold for any $1 < p < \infty$ with minor modifications of the proofs (see [7] for a similar discussion), we will from now on concentrate on the range $2 \le p < \infty$ relevant to our transmission problem.

Definition 2.5.4. Let $z \in D$ be a node of the triangulation $\mathcal{T}_h$ and $\varphi_z \in W_h^{1,p}(\Omega)$ the associated nodal basis function. Let $\omega_z = \{x \in \Omega : \varphi_z(x) > 0\}$ be the interior of the support of φ_z. The interpolation operator $\pi : W^{1,p}(\Omega) \to W_h^{1,p}(\Omega)$ is defined as

$$\pi u = \sum_{z\in D} u_z \varphi_z, \qquad u_z = \int_\Omega \varphi_z u \Big/ \int_\Omega \varphi_z.$$

Lemma 2.5.5. *Let $\mathcal{E}_h^z = \{l \in \mathcal{E}_h : l = \bar{K}_i \cap \bar{K}_j$ for some $K_i, K_j \subset \omega_z\}$. Given $u_h \in W_h^{1,p}(\Omega)$, let $[\partial_{\nu_\mathcal{E}} u_h]_l$ denote the jump of the normal derivative across the inner edge l of the triangulation. Then, if $v \in W^{1,p}(\Omega)$ and $K \in \mathcal{T}_h$, the following estimate holds:*

$$\int_K G_{p,\delta}(\nabla u_h, h_K^{-1}(v - \pi v)) + \int_K G_{p,\delta}(\nabla u_h, \nabla(v - \pi v))$$
$$\lesssim \sum_{z\in D\cap\bar{K}} \left(\int_{\omega_z} G_{p,\delta}(\nabla u_h, \nabla v) + \sum_{l\in\mathcal{E}_h^z} \min_{\bar{K}'\cap l\neq\emptyset} \int_{\omega_z} G_{p,\delta}(\nabla u_h|_{K'}, [\partial_{\nu_\mathcal{E}} u_h]_l) \right).$$

Proof. The proof is a modification of [7], Lemma 4.3. Concerning the first term on the left hand side, the convexity of $G_{p,\delta}$ in its second argument (a "triangle inequality") and enlarging the domain of integration leads to

$$\begin{aligned}
\int_K G_{p,\delta}(\nabla u_h, h_K^{-1}(v - \pi v)) &= \int_K G_{p,\delta}(\nabla u_h, \sum_{z\in D\cap\bar{K}} h_K^{-1}(v - v_z)\varphi_z) \\
&\lesssim \sum_{z\in D\cap\bar{K}} \int_K G_{p,\delta}(\nabla u_h, h_K^{-1}(v - v_z)\varphi_z) \\
&\le \sum_{z\in D\cap\bar{K}} \int_{\omega_z} G_{p,\delta}(\nabla u_h|_K, h_K^{-1}(v - v_z)\varphi_z).
\end{aligned}$$

As $G_{p,\delta}(\nabla u_h|_K,\cdot)$ is increasing and $|\varphi_z|\le 1$, Lemma 2.5.2 with $q(u)=\int_{\omega_z}\varphi_z u$ implies

$$\begin{aligned}\int_{\omega_z} G_{p,\delta}(\nabla u_h|_K, h_K^{-1}(v-v_z)\varphi_z) &\le \int_{\omega_z} G_{p,\delta}(\nabla u_h|_K, h_K^{-1}(v-v_z))\\ &\le C_P\int_{\omega_z} G_{p,\delta}(\nabla u_h|_K,\nabla(v-v_z)) \qquad (5.1)\\ &= C_P\int_{\omega_z} G_{p,\delta}(\nabla u_h|_K,\nabla v)\end{aligned}$$

for every term in the sum over $z\in D\cap\bar K$. To replace the constant $\nabla u_h|_K$ by ∇u_h, we repeatedly apply the usual triangle inequality and the convexity of $G_{p,\delta}$ to obtain

$$\begin{aligned}&G_{p,\delta}(\nabla u_h|_K,\nabla v)\\ &\le G_{p,\delta}(\nabla u_h|_K, |\nabla v|+|\nabla u_h|_K-\nabla u_h|)\\ &= (|\nabla v|+|\nabla u_h|_K-\nabla u_h|)^2(|\nabla u_h|_K|+|\nabla v|+|\nabla u_h|_K-\nabla u_h|)^{\delta(p-2)}\\ &\quad\times(1+|\nabla u_h|_K|+|\nabla v|+|\nabla u_h|_K-\nabla u_h|)^{(1-\delta)(p-2)}\\ &\le (|\nabla v|+|\nabla u_h|_K-\nabla u_h|)^2(|\nabla v|+2(|\nabla u_h|+|\nabla u_h|_K-\nabla u_h|))^{\delta(p-2)}\\ &\quad\times(1+|\nabla v|+2(|\nabla u_h|+|\nabla u_h|_K-\nabla u_h|))^{(1-\delta)(p-2)}\\ &\lesssim G_{p,\delta}(\nabla u_h, |\nabla v|+|\nabla u_h|_K-\nabla u_h|)\\ &\lesssim G_{p,\delta}(\nabla u_h,\nabla v)+G_{p,\delta}(\nabla u_h,\nabla u_h|_K-\nabla u_h).\end{aligned}$$

Altogether

$$\int_K G_{p,\delta}(\nabla u_h|_K, h_K^{-1}(v-\pi v))\lesssim \sum_{z\in D\cap\bar K}\int_{\omega_z}\{G_{p,\delta}(\nabla u_h,\nabla v)+G_{p,\delta}(\nabla u_h,\nabla u_h|_K-\nabla u_h)\}\,.$$

Let $\overline{\omega}_z=\bar K_1\cup\dots\cup\bar K_k$. Applying Lemma 2.5.3 with $a_j=\nabla u_h|_{K_j}$, $1\le j\le k$, leads to the asserted bound for the first term. For the proof, note that the conormal derivatives of the piecewise linear function u_h are determined by its boundary values on the corresponding edge. But $u_h\in W_h^{1,p}(\Omega)\subset W^{1,p}(\Omega)$, so the restrictions from both sides have to coincide, and the conormal derivative does not jump: $a_j-a_{j-1}=[\partial_{\nu_\mathcal{E}}u_h|_{\bar K_j\cap\bar K_{j-1}}]$.
As for the second term, let $c=\frac{1}{|K|}\int_K v$. Because

$$\int_K G_{p,\delta}(\nabla u_h,\nabla(v-\pi v))\lesssim\int_K G_{p,\delta}(\nabla u_h,\nabla v)+\int_K G_{p,\delta}(\nabla u_h,\nabla(\pi v-c))$$

by convexity and the triangle inequality, it only remains to consider the second term $\int_K G_{p,\delta}(\nabla u_h,\nabla(\pi v-c))$. The inverse estimate

$$|\nabla(\pi v-c)|\lesssim\frac{1}{|K|}\int_K h_K^{-1}|\pi v-c|$$

for the affine function $\pi v - c$ and Jensen's inequality show

$$\begin{aligned}\int_K G_{p,\delta}(\nabla u_h, \nabla(\pi v - c)) &\lesssim \int_K \frac{1}{|K|}\int_K G_{p,\delta}(\nabla u_h, h_K^{-1}(\pi v - c))\\ &= \int_K G_{p,\delta}(\nabla u_h, h_K^{-1}(\pi v - c)).\end{aligned}$$

However, as before

$$\int_K G_{p,\delta}(\nabla u_h, h_K^{-1}(\pi v - c)) \lesssim \int_K G_{p,\delta}(\nabla u_h, h_K^{-1}(v - \pi v)) + \int_K G_{p,\delta}(\nabla u_h, h_K^{-1}(v - c)),$$

and the first term has been considered in the first step of the proof. Lemma 2.5.2 with $q(u) = \int_K u$ also bounds the final term by $\int_K G_{p,\delta}(\nabla u_h, \nabla v)$. □

Lemma 2.5.6. *For any $\varepsilon > 0$, $u_h \in W_h^{1,p}(\Omega)$, $v \in W^{1,p}(\Omega)$ and $f \in L^{p'}(\Omega)$,*

$$\begin{aligned}\int_\Omega f(v - \pi v) \;\leq\; & C\varepsilon \int_\Omega G_{p,\delta}(\nabla u_h, \nabla v)\\ &+C(\varepsilon)\sum_{z\in D}\sum_{K\subset\overline{\omega}_z}\int_K G_{p',1}(|\nabla u_h|^{p-1}, h_K(f - f_K))\\ &+C\varepsilon\sum_{z\in D}\sum_{l\in\mathcal{E}_h^z}\min_{\bar{K}'\cap l\neq\emptyset}\int_{\omega_z} G_{p,\delta}(\nabla u_h|_{K'}, [\partial_{\nu_\mathcal{E}} u_h]_l).\end{aligned}$$

Here, $f_K = \frac{1}{|K|}\int_K f$. If $f \in W^{1,p'}(\Omega)$, the second term may be replaced by

$$C(\varepsilon)\sum_{z\in D}\sum_{K\subset\overline{\omega}_z}\int_K G_{p',1}(|\nabla u_h|^{p-1}, h_K^2 \nabla f).$$

Proof. We adapt the proof of [7], Lemma 4.4. Let $\tilde{K} \subset \overline{\omega}_z$ such that $|\nabla u_h|_{\tilde{K}}| = \max_{K'\subset\overline{\omega}_z} |\nabla u_h|_{K'}|$. Applying the inequality from Remark 2.1.5c) for some $\varepsilon > 0$ and $C(\varepsilon) = C_P \max\{\varepsilon^{-1}, \varepsilon^{1/(1-p)}\}$,

$$\begin{aligned}\int_\Omega f(v - \pi v) \;=\;& \sum_{z\in D}\sum_{K\subset\overline{\omega}_z}\int_K h_K(f - f_K)h_K^{-1}(v - v_z)\varphi_z\\ \leq\;& C_P^{-1}\, C(\varepsilon)\sum_{z\in D}\sum_{K\subset\overline{\omega}_z}\int_K (|\nabla u_h|_{\tilde{K}}|^{p-1} + h_K|f - f_K|)^{p'-2} h_K^2 |f - f_K|^2\\ &+\varepsilon\sum_{z\in D}\sum_{K\subset\overline{\omega}_z}\int_K (|\nabla u_h|_{\tilde{K}}| + h_K^{-1}|v - v_z|\varphi_z)^{p-2} h_K^{-2}|v - v_z|^2\varphi_z^2\\ \leq\;& C_P^{-1}\, C(\varepsilon)\sum_{z\in D}\sum_{K\subset\overline{\omega}_z}\int_K G_{p',1}(|\nabla u_h|_{\tilde{K}}|^{p-1}, h_K(f - f_K))\\ &+\varepsilon\sum_{z\in D}\sum_{K\subset\overline{\omega}_z}\int_K G_{p,\delta}(\nabla u_h|_{\tilde{K}}, h_K^{-1}(v - v_z)\varphi_z),\end{aligned}$$

because $\sum_{K\subset\overline{\omega}_z}\int_K f_K(v-v_z)\varphi_z = 0$. However, by our choice of $\tilde{K}$ and because $p' \le 2$,

$$\int_K G_{p',1}(|\nabla u_h|_{\tilde{K}}|^{p-1}, h_K(f-f_K)) \le \int_K G_{p',1}(|\nabla u_h|^{p-1}, h_K(f-f_K)).$$

If $f \in W^{1,p'}(\Omega)$, Lemma 2.5.2 with $q(u) = \int_K u$ gives:

$$\int_K G_{p',1}(|\nabla u_h|^{p-1}, h_K(f-f_K)) \le C_P \int_K G_{p',1}(|\nabla u_h|^{p-1}, h_K^2 \nabla f).$$

Concerning the $G_{p,\delta}$–term, equation (5.1) in the proof of Lemma 2.5.5 shows that it is dominated by $\varepsilon\int_{\omega_z} G_{p,\delta}(\nabla u_h|_{\tilde{K}}, \nabla v)$, which in turn was bounded by

$$\varepsilon \int_{\omega_z} G_{p,\delta}(\nabla u_h, \nabla v) + \varepsilon \sum_{l\in\mathcal{E}_h^z} \min_{\bar{K}'\cap l\neq\emptyset} \int_{\omega_z} G_{p,\delta}(\nabla u_h|_{K'}, [\partial_{\nu_\mathcal{E}} u_h]_l).$$

□

In order to define the a posteriori estimator, we still need to introduce some notation. For any $z \in D$, denote by $K_{j,z} \in \mathcal{T}_h$, $1 \le j \le N_z$, the triangles neighboring z in the sense that $\overline{\omega}_z = \bigcup_{j=1}^{N_z} \bar{K}_{j,z}$. To each $K_{j,z}$ we associate a weight factor $\alpha_{j,z} \ge 0$ normalized to $\sum_{j=1}^{N_z} \alpha_{j,z} = 1$.

Definition 2.5.7. Given $u_h \in W_h^{1,p}(\Omega)$, define the gradient recovery

$$G_h u_h = \sum_{z\in D} (G_h v_h)(z)\ \varphi_z, \quad (G_h v_h)(z) = \sum_{j=1}^{N_z} \alpha_{j,z} \nabla u_h|_{K_{j,z}}.$$

The following theorem states our reliable, but presumably not efficient a posteriori estimate.

Theorem 2.5.8. *Let $f \in L^{p'}(\Omega)$ and denote by $(e, \tilde{e}, \epsilon)$ the error between the Galerkin solution $(\hat{u}_h, \hat{v}_h, \hat{\phi}_h) \in Y_h^p$ and the true solution $(\hat{u}, \hat{v}, \hat{\phi}) \in Y^p$. If $\Gamma_s \neq \emptyset$, assume that $\nabla\hat{u}|_{\Gamma_s} \in L^p(\Gamma_s)$. Then*

$$\begin{aligned} \|e, \tilde{e}, \epsilon\|_{Y^p}^p &\lesssim |e|^2_{(1,\hat{u},p)} + \|e|_{\partial\Omega} + \tilde{e}\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\epsilon\|^2_{W^{-\frac{1}{2},2}(\partial\Omega)} \\ &\lesssim \eta_{gr}^2 + \eta_f^2 + \eta_S^2 + \eta_\partial^2 + \eta_g^2, \end{aligned}$$

where

$$\eta_{gr}^2 = \sum_{K\in\mathcal{T}_h} \int_K G_{p,\delta}(\nabla\hat{u}_h, \nabla\hat{u}_h - G_h\hat{u}_h),$$

$$\eta_f^2 = \sum_{K\in\mathcal{T}_h} \int_K G_{p',1}(|\nabla\hat{u}_h|^{p-1}, h_K(f - f_K)),$$

$$\eta_S^2 = \sum_{l\subset\partial\Omega} l\ \|\partial_s\{\mathcal{V}\hat{\phi}_h + (1-\mathcal{K})(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0)\}\|_{L^2(l)}^2$$

$$\eta_\partial^2 = \sum_{l\subset\partial\Omega} l\ \| - \varrho(\nabla\hat{u}_h)\ \partial_\nu\hat{u}_h + t_0 - \mathcal{W}(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) - (\mathcal{K}'-1)\hat{\phi}_h\|_{L^2(l)}^2$$

$$\eta_g^2 = \sum_{l\subset\Gamma_s} l\|\varrho(\nabla\hat{u}_h)\ \partial_\nu\hat{u}_h|_{\Gamma_s}\|_{L^2(l)}^2 + \|g\|_{W^{-\frac{1}{2},2}(\Gamma_s)}^2$$

If $f \in W^{1,p'}(\Omega)$, we may replace η_f^2 by $\sum_{K\in\mathcal{T}_h}\int_K G_{p',1}(|\nabla\hat{u}_h|^{p-1}, h_K^2\nabla f)$.

Proof. From Lemma 2.5.1 we know that for all $(e_h, \tilde{e}_h, \nu_h) \in Y_h^p$

$$\begin{aligned}
\|e,\tilde{e},\epsilon\|_{Y^p}^p &\lesssim |e|_{(1,\hat{u},p)}^2 + \|e|_{\partial\Omega} + \tilde{e}\|_{W^{\frac{1}{2},2}(\partial\Omega)}^2 + \|\epsilon\|_{W^{-\frac{1}{2},2}(\partial\Omega)}^2 \\
&\lesssim \int_\Omega f(e-e_h) - \sum_{K\in\mathcal{T}_h}\int_{\partial K} \varrho(\nabla\hat{u}_h)\ \partial_\nu\hat{u}_h|_{\partial K}\ (e-e_h) \\
&\quad + \int_{\Gamma_s} g(|\tilde{e}_h + \hat{v}_h| - |\hat{v}|) - \langle \nu - \nu_h, \mathcal{V}\hat{\phi}_h + (1-\mathcal{K})(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0)\rangle \\
&\quad + \langle t_0 - \mathcal{W}(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) - (\mathcal{K}'-1)\hat{\phi}_h, (e-e_h)|_{\partial\Omega} + \tilde{e} - \tilde{e}_h\rangle,
\end{aligned}$$

with $2\nu = \epsilon + \mathcal{V}^{-1}(1-\mathcal{K})(e|_{\partial\Omega} + \tilde{e})$. The first two terms are mainly going to give the gradient recovery in the interior, the fourth term the error η_S of constructing the Steklov-Poincaré operator, while the remaining terms add up to η_∂.
Concerning the first term:

$$\begin{aligned}
\int_\Omega f(e-e_h) \quad &\lesssim \quad \varepsilon\sum_{K\in\mathcal{T}_h}\int_K G_{p,\delta}(\nabla\hat{u}_h, \nabla e) \\
&\quad + C(\varepsilon)\sum_{K\in\mathcal{T}_h}\int_K G_{p',1}(|\nabla\hat{u}_h|^{p-1}, h_z(f-f_z)) \\
&\quad + \varepsilon\sum_{z\in D}\sum_{l\in\mathcal{E}_h^z}\min_{\bar{K}'\cap l\neq\emptyset}\int_{\omega_z} G_{p,\delta}(\nabla\hat{u}_h|_{K'}, [\partial_{\nu_\mathcal{E}} u_h]_l) \\
&\lesssim \quad \varepsilon|e|_{(1,\hat{u},p)}^2 + C(\varepsilon)\ \eta_f^2 + \varepsilon\sum_{z\in D}\sum_{l\in\mathcal{E}_h^z}\min_{\bar{K}'\cap l\neq\emptyset}\int_{\omega_z} G_{p,\delta}(\nabla\hat{u}_h|_{K'}, [\partial_{\nu_\mathcal{E}} u_h]_l).
\end{aligned}$$

$G_h\hat{u}_h$ is continuous across any interior edge l, so that $[\partial_\nu\hat{u}_h]_l = [\partial_\nu\hat{u}_h - G_h\hat{u}_h]_l$ and

$$\min_{\bar{K}'\cap l\neq\emptyset}\int_{\omega_z} G_{p,\delta}(\nabla\hat{u}_h|_{K'}, [\partial_\nu\hat{u}_h - G_h u_h]_l) \lesssim \int_{\omega_z} G_{p,\delta}(\nabla\hat{u}_h, \nabla\hat{u}_h - G_h\hat{u}_h).$$

Therefore,

$$\begin{aligned}\int_\Omega f(e-e_h) &\lesssim \varepsilon|e|^2_{(1,\hat{u},p)} + C(\varepsilon)\eta_f^2 + \varepsilon\sum_{z\in D}\sum_{l\in\mathcal{E}_h^z}\int_{\omega_z} G_{p,\delta}(\nabla\hat{u}_h,[\partial_\nu\hat{u}_h - G_h\hat{u}_h]_l)\\ &\lesssim \varepsilon|e|^2_{(1,\hat{u},p)} + C(\varepsilon)\eta_f^2 + \varepsilon\sum_{K\in\mathcal{T}_h}\int_K G_{p,\delta}(\nabla\hat{u}_h,\nabla\hat{u}_h - G_h\hat{u}_h)\\ &= \varepsilon|e|^2_{(1,\hat{u},p)} + C(\varepsilon)\eta_f^2 + \varepsilon\eta_{gr}^2.\end{aligned}$$

Concerning the second term, let

$$A_l = \varrho(\nabla\hat{u}_h|_{K_{l,1}})\ \partial_\nu\hat{u}_h|_{K_{l,1}} - \varrho(\nabla\hat{u}_h|_{K_{l,2}})\ \partial_\nu\hat{u}_h|_{K_{l,2}},$$

where again $l\subset\bar{K}_{l,1}\cap\bar{K}_{l,2}$, and the unit normal ν points outward of $K_{l,1}$. Therefore

$$\begin{aligned}-\langle G'\hat{u}_h, e-\pi e\rangle &= -\sum_{K\in\mathcal{T}_h}\int_{\partial K}\varrho(\nabla\hat{u}_h)\ \partial_\nu\hat{u}_h|_{\partial K}\ (e-\pi e)\\ &= -\sum_{l\not\subset\partial\Omega}\int_l A_l(e-\pi e) - \sum_{l\subset\partial\Omega}\int_l\varrho(\nabla\hat{u}_h)\ \partial_\nu\hat{u}_h|_l\ (e-\pi e).\end{aligned}$$

Repeating the analysis of [7], Theorem 5.1, with the help of Lemma 2.5.5 gives

$$-\sum_{l\not\subset\partial\Omega}\int_l A_l(e-\pi e)\lesssim \eta_{gr}^2 + \varepsilon(|e|^2_{(1,\hat{u}_h,p)} + \eta_{gr}^2).$$

Thus

$$\begin{aligned}\|e,\tilde{e},\epsilon\|^p_{Y^p} &\lesssim |e|^2_{(1,\hat{u},p)} + \|e|_{\partial\Omega}+\tilde{e}\|^2_{W^{\frac12,2}(\partial\Omega)} + \|\epsilon\|^2_{W^{-\frac12,2}(\partial\Omega)}\\ &\lesssim \eta_f^2 + \varepsilon(\eta_{gr}^2 + |e|^2_{(1,\hat{u},p)}) + \eta_{gr}^2 + \varepsilon(|e|^2_{(1,\hat{u}_h,p)} + \eta_{gr}^2)\\ &\quad + \int_{\Gamma_s}\{-\varrho(\nabla\hat{u}_h)\ \partial_\nu\hat{u}_h|_{\Gamma_s}(\tilde{e}_h-\tilde{e}) + g(|\tilde{e}_h+\hat{v}_h| - |\tilde{e}+\hat{v}_h|)\}\\ &\quad - \int_{\partial\Omega}\varrho(\nabla\hat{u}_h)\ \partial_\nu\hat{u}_h|_{\partial\Omega}\ ((e-\pi e)|_{\partial\Omega}+\tilde{e}-\tilde{e}_h)\\ &\quad + \langle t_0 - \mathcal{W}(\hat{u}_h|_{\partial\Omega}+\hat{v}_h-u_0) - (\mathcal{K}'-1)\hat{\phi}_h, (e-\pi e)|_{\partial\Omega}+\tilde{e}-\tilde{e}_h\rangle\\ &\quad - \langle\nu-\nu_h, \mathcal{V}\hat{\phi}_h + (1-\mathcal{K})(\hat{u}_h|_{\partial\Omega}+\hat{v}_h-u_0)\rangle.\end{aligned}$$

We bound the second, third + fourth as well as the final line individually. Cauchy-Schwarz and Young's inequality allow to estimate the last term by

$$\varepsilon\|e|_{\partial\Omega}+\tilde{e}\|^2_{W^{\frac12,2}(\partial\Omega)} + \varepsilon\|\epsilon\|^2_{W^{-\frac12,2}(\partial\Omega)} + \varepsilon^{-1}\ \|\mathcal{V}\hat{\phi}_h + (1-\mathcal{K})(\hat{u}_h|_{\partial\Omega}+\hat{v}_h-u_0)\|^2_{W^{\frac12,2}(\partial\Omega)},$$

and the latter by η_S^2 (cf. [10]). The third and fourth lines are estimated by (cf. [10])

$$\|-\varrho(\nabla\hat{u}_h)\,\partial_\nu\hat{u}_h+t_0-\mathcal{W}(\hat{u}_h|_{\partial\Omega}+\hat{v}_h-u_0)-(\mathcal{K}'-1)\hat{\phi}_h\|_{W^{-\frac{1}{2},2}(\partial\Omega)}\|(e-\pi e)|_{\partial\Omega}+\tilde{e}\|_{W^{\frac{1}{2},2}(\partial\Omega)}$$

which lead to η_∂, where we have choosen $\tilde{e}_h = 0$, i.e. $v_h = \hat{v}_h$. Finally, using the triangle inequality, the second line is simplified as follows:

$$\begin{aligned}
&\int_{\Gamma_s}\{-\varrho(\nabla\hat{u}_h)\,\partial_\nu\hat{u}_h|_{\Gamma_s}(\tilde{e}_h-\tilde{e})+g(|\tilde{e}_h+\hat{v}_h|-|\tilde{e}+\hat{v}_h|)\}\\
&\le\int_{\Gamma_s}\{-\varrho(\nabla\hat{u}_h)\,\partial_\nu\hat{u}_h|_{\Gamma_s}(\tilde{e}_h-\tilde{e})+g|\tilde{e}_h-\tilde{e}|\}\\
&=\int_{\Gamma_s}\{\varrho(\nabla\hat{u}_h)\,\partial_\nu\hat{u}_h|_{\Gamma_s}\tilde{e}+g|\tilde{e}|\}\\
&\le\|\varrho(\nabla\hat{u}_h)\,\partial_\nu\hat{u}_h|_{\Gamma_s}\|_{W^{-\frac{1}{2},2}(\Gamma_s)}\|\tilde{e}\|_{W^{\frac{1}{2},2}(\Gamma_s)}+\|g\|_{W^{-\frac{1}{2},2}(\Gamma_s)}\|\tilde{e}\|_{W^{\frac{1}{2},2}(\Gamma_s)}.
\end{aligned}$$

We may use the Cauchy-Schwartz inequality and the inverse inequality, leading to η_g.

□

2.6 Numerical results

With the subset Λ_h of $\widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)$ given by

$$\Lambda_h = \{\sigma_h \in \widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)\,:\,|\sigma_h(x)|\le 1 \text{ a.e. on } \Gamma_s\},$$

we can define an Uzawa algorithm for solving the variational inequality analogously to [43]. In order to introduce this algorithm, let P_Λ be the projection of $\widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)$ onto Λ_h, i.e. for every nodal point of the mesh $\mathcal{T}_h|_{\Gamma_s}$ holds $\delta \mapsto P_\Lambda(\delta) = \sup\{-1, \inf(1,\delta)\}$.

Algorithm 1 (Uzawa).

(i) Choose $\sigma_h^0 \in \Lambda_h$.

(ii) For $n = 0, 1, 2, \dots$ find $(u_h^n, v_h^n) \in X_h^p$ such that

$$\langle G'u_h^n, u_h\rangle + \langle S_h(u_h^n|_{\partial\Omega} + v_h^n), u_h|_{\partial\Omega} + v_h\rangle + \int_{\Gamma_s} g\sigma_h^n v_h\,ds = \lambda_h(u_h, v_h)$$

for all $(u_h, v_h) \in X_h^p$.

(iii) Set
$$\sigma_h^{n+1} = P_\Lambda(\sigma_h^n + \rho g v_h^n),$$
where $\rho > 0$ is a sufficiently small parameter that will be specified later.

(iv) Repeat with 2. until a convergence criterion is satisfied.

In our first example the model problem is defined on the L-shape with $\Omega = [-\frac{1}{4}, \frac{1}{4}]^2 \backslash [0, \frac{1}{4}]^2$, $\Omega^c = \mathbb{R}^2 \backslash \Omega$. The friction part of the interface is $\Gamma_s = \overline{(-\frac{1}{4}, -\frac{1}{4})(\frac{1}{4}, -\frac{1}{4})} \cup \overline{(-\frac{1}{4}, -\frac{1}{4})(-\frac{1}{4}, \frac{1}{4})}$, see Figure 2.1.
In this example we choose $\varrho(t) = (\varepsilon + t)^{p-2}$, with $p = 3$ and $\varepsilon = 0.00001$. Our volume and boundary data are given by $f = 0$ and $u_0 = r^{2/3} \sin \frac{2}{3}(\varphi - \frac{\pi}{2})$, $t_0 = \partial_\nu u_0|_{\partial\Omega}$. The friction parameter is $g = 0.5$, leading to slip conditions on the interface. We have applied the Uzawa algorithm as introduced above with the damping parameter $\rho = 25$ to solve the variational inequality. The nonlinear variational problem in the Uzawa algorithm is then solved by Newton's method in every Uzawa-iteration step.
In Table 2.1 we give the degrees of freedom, the value $J_h(\hat{u}_h, \hat{v}_h)$ and the error measured with the help of J, i.e. $\delta J = J_h(\hat{u}_h, \hat{v}_h) - J(\hat{u}, \hat{v})$, where we have obtained the value $J(\hat{u}, \hat{v})$ by extrapolation of $J_h(\hat{u}_h, \hat{v}_h)$. Due to the slip condition, we need only a few Uzawa steps. But as a consequence of the degeneration of the system matrix, due to the nonlinearity, the iteration numbers for the MINRES solver, applied to the linearized system, are very high, leading to large computation times. The convergence rate α_J is suboptimal, due to the presence of singularities, in the boundary data as well, as due to the change of boundary conditions.

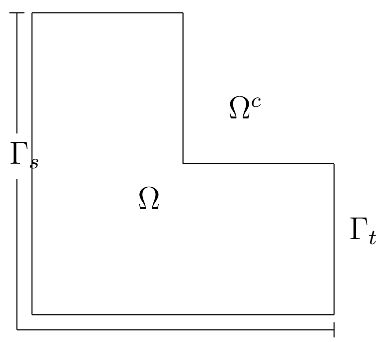

Figure 2.1: Geometry and interface of the model problem

In our second example we use the same model geometry as before (see Fig. 2.1). Here we choose the friction boundary $\Gamma_s = \emptyset$. Therefore our model problem reduces to a non-linear p-Laplacian FEM-BEM coupling problem, where we can prescribe the solution.

DOF	$J_h(\hat{u}_h, \hat{v}_h)$	δJ	α_J	It_{Uzawa}	$\tau(s)$
28	-0.511609	0.017249	—	2	0.190
80	-0.517938	0.010920	-0.435	2	0.640
256	-0.521857	0.007001	-0.382	2	2.440
896	-0.524293	0.004566	-0.341	2	11.05
3328	-0.525841	0.003017	-0.316	2	61.85
12800	-0.526865	0.001993	-0.308	2	437.5
50176	-0.527571	0.001287	-0.320	2	4218.

Table 2.1: Convergence rates and Uzawa steps for uniform meshes (Example 1)

In this example we choose $\varrho(t) = (\varepsilon + t)^{p-2}$, with $p = 3$ and $\varepsilon = 0.00001$. We prescribe the solution by $u_1 = r^{2/3} \sin \frac{2}{3}(\varphi - \frac{\pi}{2})$ and $u_2 = 0$. Then the boundary data u_0, t_0 and volume data f are given by $u_0 = u_1|_\Gamma$, $t_0 = \varrho(|\nabla u_1|)\partial_\nu u_1$ and $f = -\operatorname{div}(\varrho(|\nabla u_1|)\nabla u_1)$.
In the following we give errors in the $\|\cdot\|_{W^{1,p}(\Omega)}$ norm and in the quasinorm $|u - u_h|_Q = \|u - u_h\|_{(1,u_h,p)}$.
In Tab. 2.2 we give the errors, convergence rates, number of Newton iterations It_{Newton} and the computing time for the uniform h-version with rectangles. We observe that the convergence rate in the quasi-norm $|\cdot|_Q$ is better than in the $\|\cdot\|_{W^{1,3}(\Omega)}$-norm. The number of Newton iterations appears to be bounded.
In Tab. 2.3 for the uniform h-version with triangles, we give the errors, convergence rates, error estimator η, efficiency indices δ_u/η for the $\|\cdot\|_{W^{1,3}(\Omega)}$-norm and δ_q/η for the $|\cdot|_Q$-norm, number of Newton iterations and the computing time. Again, here we observe that the convergence rate in the quasi-norm $|\cdot|_Q$ is better than in the $\|\cdot\|_{W^{1,3}(\Omega)}$-norm and the number of Newton iterations is bounded. The efficiency index δ_u/η appears to be constant, whereas the efficiency index δ_q/η appears to be decreasing.
Tab. 2.4 gives the corresponding numbers for the adaptive version, using a blue-green refining strategy for triangles and refining the 10% elements with the largest indicators. Here we observe that the convergence rates for both norms are very similar and that both efficiency indices are bounded.
Figure 2.2 give the errors for all methods in the $\|\cdot\|_{W^{1,3}(\Omega)}$-norm and the $|\cdot|_Q$ quasi-norm together with the error indicators for the uniform and adaptive methods.
Figure 2.3 presents the sequence of meshes generated by the adaptive refinement strategy. We clearly observe the refinement towards the reentrant corner with the singularity of the solution.

DOF	$\|u-u_h\|_{1,3}$	α	$\|u-u_h\|_Q$	α	It_{Newton}	$\tau(s)$
21	0.1711499	—	0.1293512	—	22	0.224
65	0.1308635	-0.238	0.0860870	-0.360	22	0.424
225	0.1039326	-0.186	0.0612225	-0.274	23	1.668
833	0.0826578	-0.175	0.0438478	-0.255	23	6.804
3201	0.0657091	-0.170	0.0314280	-0.247	23	27.28
12545	0.0522196	-0.168	0.0225589	-0.243	24	120.8
49665	0.0414910	-0.167	0.0162319	-0.239	24	560.1
197633	0.0329617	-0.167	0.0117169	-0.236	24	2678.

Table 2.2: Errors, convergence rates (Example 2, uniform mesh with rectangles)

DOF	$\|u-u_h\|_{1,3}$	α	$\|u-u_h\|_Q$	α	η	δ_u/η	δ_q/η	It_{New}	$\tau(s)$
21	0.1945908	—	0.1510064	—	1.027	0.190	0.147	22	0.620
65	0.1535874	-0.209	0.1081632	-0.295	0.690	0.223	0.157	22	2.212
225	0.1219287	-0.186	0.0774765	-0.269	0.516	0.236	0.150	22	8.617
833	0.0969249	-0.175	0.0555005	-0.255	0.394	0.246	0.141	23	36.00
3201	0.0770270	-0.171	0.0396882	-0.249	0.304	0.253	0.131	23	144.2
12545	0.0611994	-0.168	0.0283778	-0.246	0.236	0.260	0.120	24	608.7
49665	0.0486160	-0.167	0.0203130	-0.243	0.184	0.265	0.111	24	2530.
197633	0.0386151	-0.167	0.0145686	-0.241	0.144	0.269	0.102	24	11000

Table 2.3: Errors, onvergence rates, estimator η, reliability δ_u/η and δ_q/η (Example 2, uniform mesh with triangles)

DOF	$\|u-u_h\|_{1,3}$	α	$\|u-u_h\|_Q$	α	η	δ_u/η	δ_q/η	It_{New}	$\tau(s)$
21	0.1945908	—	0.1510064	—	1.027	0.190	0.147	22	0.196
32	0.1602214	-0.461	0.1205155	-0.535	0.804	0.199	0.150	22	0.332
54	0.1275298	-0.436	0.0918131	-0.520	0.603	0.212	0.152	22	0.648
93	0.1019990	-0.411	0.0699054	-0.501	0.442	0.231	0.158	22	1.132
152	0.0821754	-0.440	0.0540462	-0.524	0.325	0.253	0.166	23	2.000
249	0.0679251	-0.386	0.0449420	-0.374	0.246	0.276	0.183	23	3.352
400	0.0558447	-0.413	0.0369614	-0.412	0.190	0.294	0.194	23	5.700
625	0.0439784	-0.535	0.0277857	-0.639	0.148	0.297	0.188	24	9.896
986	0.0352491	-0.485	0.0217361	-0.539	0.116	0.305	0.188	24	17.45
1528	0.0279287	-0.531	0.0167409	-0.596	0.091	0.308	0.184	25	31.16
2322	0.0222760	-0.540	0.0129489	-0.614	0.071	0.312	0.181	25	53.98
3620	0.0177640	-0.510	0.0102552	-0.525	0.056	0.316	0.182	25	106.7
5544	0.0142059	-0.524	0.0080233	-0.576	0.044	0.320	0.181	25	205.3
8449	0.0112965	-0.544	0.0063426	-0.558	0.035	0.322	0.181	26	422.4
12810	0.0090396	-0.536	0.0050706	-0.538	0.028	0.325	0.183	26	1060.
19222	0.0072288	-0.551	0.0040370	-0.562	0.022	0.329	0.184	26	2400.
29006	0.0057984	-0.536	0.0032478	-0.529	0.018	0.333	0.186	27	5460.
43593	0.0046615	-0.536	0.0026230	-0.524	0.014	0.337	0.190	27	13000

Table 2.4: p-Laplacian (adaptive), convergence rates, estimator η, reliability δ_u/η and δ_q/η

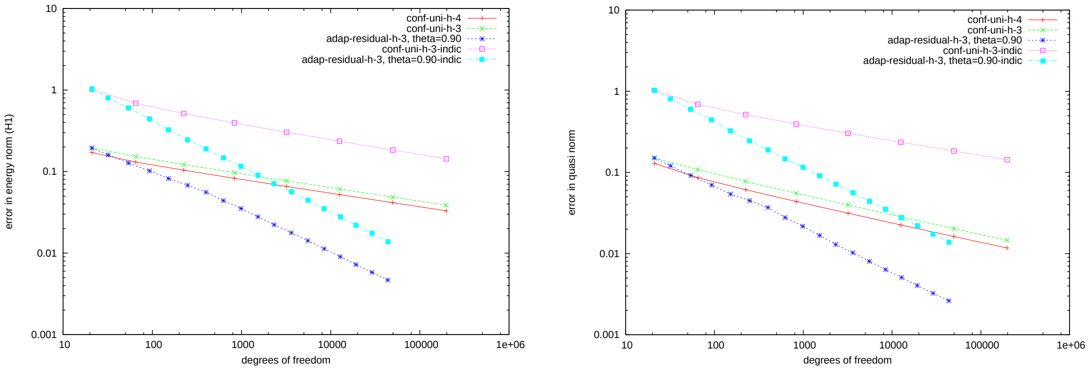

Figure 2.2: $\|u - u_n\|_{W^{1,3}(\Omega)}$ (left) and $|u - u_n|_Q$ (right).

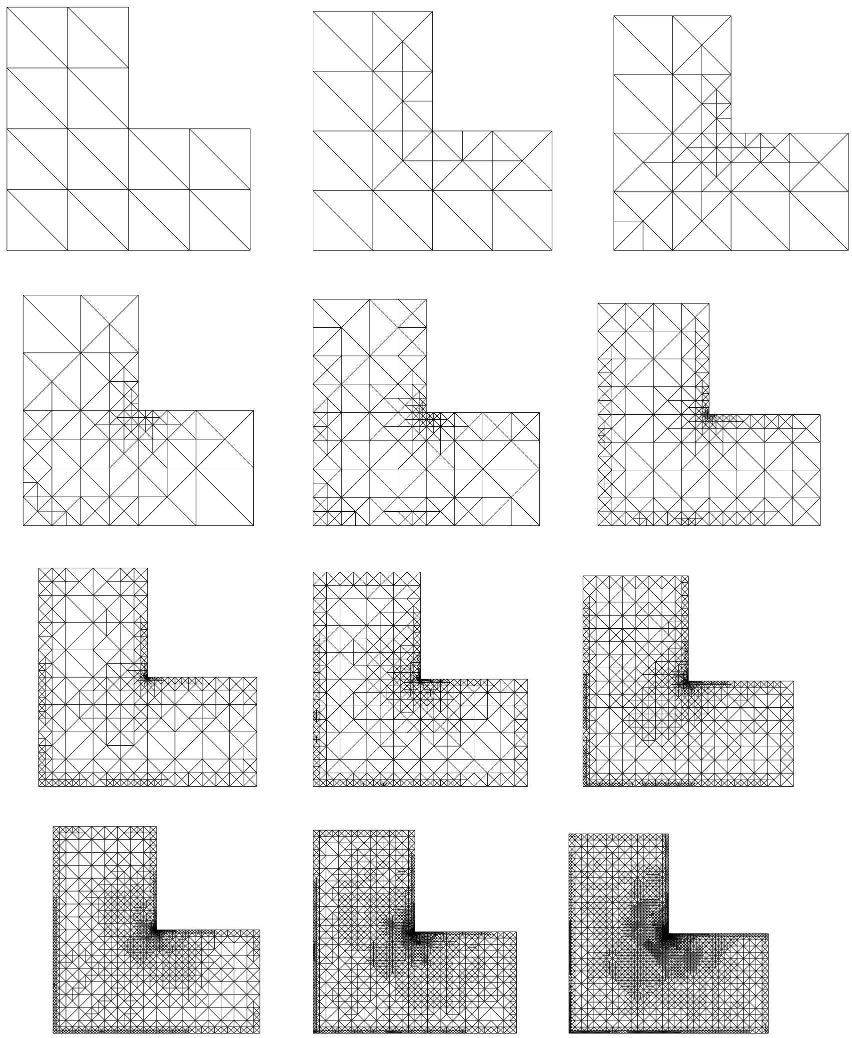

Figure 2.3: The first 12 meshes generated by the adaptive refinement algorithm

Chapter 3

Contact problems involving microstructure

In the previous chapter we showed that certain mixed $L^2 - L^p$–Sobolev spaces provide a convenient setting to study the numerical approximation of contact problems for monotone operators like the p–Laplacian. This section extends the approach to nonconvex functionals, discussing the prototypical model case of a double–well potential in Signiorini and transmission contact with the linear Laplace equation. As a proof of principle, it intends to clarify the mathematical basis – including wellposedness, convergence, a priori and a simple a posteriori estimate – of adaptive finite element / boundary element methods in this highly degenerate nonlinear setting. The methods readily extend to certain systems of equations from nonlinear elasticity, frictional contact or more elaborate a posteriori estimates.
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\partial\Omega = \overline{\Gamma_t \cup \Gamma_s}$ a decomposition of its boundary into disjoint open subsets, $\Gamma_t \neq \emptyset$. We consider the problem of minimizing the functional

$$\Phi(u_1, u_2) = \int_\Omega W(\nabla u_1) + \frac{1}{2}\int_{\Omega^c} |\nabla u_2|^2 - \int_\Omega f u_1 - \langle t_0, u_2|_{\partial\Omega}\rangle$$

with nonconvex energy density $W(F) = |F - F_1|^2\ |F - F_2|^2$ ($F_1 \neq F_2 \in \mathbb{R}^n$) over the closed convex set

$$\{(u_1, u_2) \in W^{1,4}(\Omega) \times W^{1,2}_{loc}(\Omega^c) : (u_1 - u_2)|_{\Gamma_t} = u_0,\ (u_1 - u_2)|_{\Gamma_s} \leq u_0,\ u_2 \in \mathcal{L}_2\},$$

$$\mathcal{L}_2 = \left\{\{v \in W^{1,2}_{loc}(\Omega^c) : \Delta v = 0 \text{ in } W^{-1,2}(\Omega^c),\ v = \left\{\begin{array}{ll} o(1) & ,n = 2 \\ \mathcal{O}(|x|^{2-n}) & ,n > 2\end{array}\right\}\right\}.$$

The data $f \in L^{4/3}(\Omega)$, $t_0 \in W^{-\frac{1}{2},2}(\partial\Omega)$ and $u_0 \in W^{\frac{1}{2},2}(\partial\Omega)$ are taken from the appropriate spaces.

Classical exact minimizers of Φ satisfy the equations

$$\begin{aligned} -\text{div } DW(\nabla u_1) = f \quad \text{in } \Omega, \quad \Delta u_2 =& 0 \quad \text{in } \Omega^c, \\ \nu \cdot DW(\nabla u_1) - \partial_\nu u_2 = t_0 \quad \text{on } \partial\Omega,\ u_1 - u_2 =& u_0 \text{ on } \Gamma_t, \\ u_1 - u_2 \le u_0,\ \nu \cdot DW(\nabla u_1) \le 0,\ \nu \cdot DW(\nabla u_1)(u_1 - u_2 - u_0) =& 0 \quad \text{on } \Gamma_s, \\ + \text{ radiation condition for } u_2 \text{ at } \infty.& \end{aligned}$$

Therefore, the minimization problem for Φ is a variational formulation of a contact problem between the double–well potential W and the Laplace equation, with transmission (Γ_t) and Signiorini (Γ_s) contact at the interface.
Nonconvex minimization problems of this type arise naturally when a material in Ω passes the critical point of a phase transition into a finely textured mixture of locally energetically equivalent configurations of lower symmetry, the so–called microstructure. Lacking convexity in Ω, the minimum of Φ is usually not attained. Nevertheless, it is possible and of practical interest to extract average physical properties of the sequences minimizing Φ. Examples of such quantities include the displacement in the exterior, stresses, the region, where minimizing sequences develop microstructure, or also the gradient of the displacement away from the microstructure. Crucially for the use of boundary elements, the exterior boundary value on the interface is not affected by the presence of microstructure.
The increasingly fine length scale of the microstructure often prevents the direct numerical minimization, and starting with works of Carstensen and Plecháv**c** [8, 9] computational approaches based on *relaxed* formulations have been considered. Relaxation amounts to replacing the nonconvex functional by its quasi–convex envelope, in our setting the degenerate functional

$$\Phi^{**}(u_1, u_2) = \int_\Omega W^{**}(\nabla u_1) + \frac{1}{2}\int_{\Omega^c} |\nabla u_2|^2 - \int_\Omega f u_1 - \langle t_0, u_2|_{\partial\Omega}\rangle.$$

If $A = \frac{1}{2}(F_2 - F_1)$ and $B = \frac{1}{2}(F_1 + F_2)$, the convex integrand W^{**} is given by the formula (cf. [8])

$$W^{**}(F) = \big(\max\{0, |F-B|^2 - |A|^2\}\big)^2 + 4|A|^2|F-B|^2 - 4(A(F-B))^2.$$

The theory of relaxation for nonconvex integrands shows that the weak limit of any Φ–minimizing sequence minimizes Φ^{**}. Macroscopic quantities like the stress DW^{**} on Ω defined by this weak limit coincide with the averages such as the average stress $\int DW(u)\ \mathrm{d}\nu(u)$ defined by the Young measure ν associated to the minimizing sequence. To extract the average physical properties of sequences minimizing Φ, it is thus sufficient to understand the minimizers of the degenerately convex functional Φ^{**}.
We are thus going to analyze a finite element / boundary element scheme which numerically minimizes Φ^{**} and thereby approximates certain macroscopic quantities independent of the particular minimizer. Our approach is based on previous

works by Carstensen / Plecháv**c** [8, 9] and Bartels [2] for double–well potentials with Dirichlet or Neumann boundary conditions.

For later reference, we recall from [8] the following estimates for the relaxed double–well potential ($E, F \in \mathbb{R}^n$):

$$\max\{C_1|F|^4 - C_2, 0\} \le W^{**}(F) \le C_3 + C_4|F|^4, \tag{0.1}$$

$$|DW^{**}(F)| \le C_5(1 + |F|^3), \tag{0.2}$$

$$|DW^{**}(F) - DW^{**}(E)|^2 \le C_6(1 + |F|^2 + |E|^2)(DW^{**}(F) - DW^{**}(E))(F - E), \tag{0.3}$$

$$8|A|^2|\mathbb{P}F - \mathbb{P}E|^2 + 2\frac{Q(F) + Q(E)}{|A|}|A(F - E)|^2 + 2(Q(F) - Q(E))^2 \le (DW^{**}(F) - DW^{**}(E))(F - E) \tag{0.4}$$

where $Q(F) = \max\{0, |F - B|^2 - |A|^2\}$ and $\mathbb{P}$ is the orthogonal projection onto the subspace of vectors orthogonal to A.

3.1 Analysis of the relaxed problem

We first outline how the minimization problem for Φ^{**} can be reduced to a boundary–domain variational inequality. As it involves the exterior problem, it is not affected by the nonconvex part of the functional. See the previous chapter for a more detailed exposition.
As before, we are going to need the Steklov–Poincaré operator

$$S : W^{\frac{1}{2},2}(\partial\Omega) \to W^{-\frac{1}{2},2}(\partial\Omega)$$

with defining property

$$\partial_\nu u_2|_{\partial\Omega} = -S(u_2|_{\partial\Omega})$$

for solutions $u_2 \in \mathcal{L}_2$ of the Laplace equation on Ω^c. Let

$$\widetilde{W}^{\frac{1}{2},2}(\Gamma_s) = \{v \in W^{\frac{1}{2},2}(\partial\Omega) : \text{supp } v \subset \bar{\Gamma}_s\}, \quad X = W^{1,4}(\Omega) \times \widetilde{W}^{\frac{1}{2},2}(\Gamma_s).$$

Using S and the affine change of variables,

$$(u_1, u_2) \mapsto (u, v) = (u_1 - c, u_0 + u_2|_{\partial\Omega} - u_1|_{\partial\Omega}) \in X,$$

for a suitable $c \in \mathbb{R}$ reduces the exterior part of Φ^{**} to Γ_s:

$$\Phi^{**}(u_1, u_2) = \int_\Omega W^{**}(\nabla u) + \frac{1}{2}\langle S(u|_{\partial\Omega} + v), u|_{\partial\Omega} + v\rangle - \lambda(u, v) + C \equiv J(u, v) + C,$$

where

$$\lambda(u,v) = \langle t_0 + Su_0, u|_{\partial\Omega} + v\rangle + \int_\Omega fu$$

and $C = C(u_0, t_0)$ is a constant independent of u, v. Therefore, instead of Φ^{**} one may equivalently minimize J over

$$\mathcal{A} = \{(u,v) \in X : v \geq 0 \text{ and } \langle S(u|_{\partial\Omega} + v - u_0), 1\rangle = 0 \text{ if } n = 2\}.$$

A reformulation as a variational inequality reads as follows: Find $(\hat{u}, \hat{v}) \in \mathcal{A}$ such that

$$\int_\Omega DW^{**}(\nabla\hat{u})\nabla(u-\hat{u}) + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (u-\hat{u})|_{\partial\Omega} + v - \hat{v}\rangle \geq \lambda(u-\hat{u}, v-\hat{v}) \quad (1.1)$$

for all $(u,v) \in \mathcal{A}$.
Convexity and the closedness of $\mathcal{A}$ assure that the relaxed functional J assumes its minimum. Due to the lack of coercivity, the minimizer may fail to be unique, though certain macroscopic quantities are uniquely determined.

Lemma 3.1.1. *The set of minimizers is nonempty and bounded in X. The stress $DW^{**}(\hat{u})$, the projected gradient $\mathbb{P}\nabla\hat{u}$, the region of microstructure $\{x \in \Omega : Q(\nabla\hat{u}) = 0\}$ and the boundary value $\hat{u}|_{\partial\Omega} + \hat{v}$ are independent of the minimizer $(\hat{u}, \hat{v}) \in \mathcal{A}$ of J (up to sets of measure 0).*

For the proof, we recall the variant

$$\|\hat{u}\|_{W^{1,4}(\Omega)} \lesssim \|\nabla\hat{u}\|_{L^4(\Omega)} + \|\hat{u}\|_{W^{\frac{1}{2},2}(\Gamma_t)} \quad (1.2)$$

of Friedrichs' inequality from Chapter 2.

Proof of Lemma 3.1.1. By (0.1) and the coercivity of S, we have

$$\begin{aligned} |J(\hat{u},\hat{v})| \geq\ & C_1\|\nabla\hat{u}\|^4_{L^4(\Omega)} - C_2 \text{vol}\,\Omega + \frac{1}{2}C_S\|\hat{u}|_{\Gamma_s} + v\|^2_{W^{\frac{1}{2},2}(\Gamma_s)} \\ & + \frac{1}{2}C_S\|\hat{u}\|^2_{W^{\frac{1}{2},2}(\Gamma_t)} - \|f\|_{L^{4/3}(\Omega)}\|\hat{u}\|_{W^{1,4}(\Omega)} \\ & - \|t_0 + Su_0\|_{W^{-\frac{1}{2},2}(\partial\Omega)}\|\hat{u}|_{\Gamma_s} + \hat{v}\|_{W^{\frac{1}{2},2}(\Gamma_s)} \\ & - \|t_0 + Su_0\|_{W^{-\frac{1}{2},2}(\partial\Omega)}\|\hat{u}\|_{W^{\frac{1}{2},2}(\Gamma_t)} \end{aligned}$$

for any minimizer $(\hat{u}, \hat{v}) \in \mathcal{A}$ of J. Consequently

$$\|\nabla\hat{u}\|^4_{L^4(\Omega)} + \|\hat{u}|_{\Gamma_s} + \hat{v}\|^2_{W^{\frac{1}{2},2}(\Gamma_s)} + \|\hat{u}\|^2_{W^{\frac{1}{2},2}(\Gamma_t)} - C\|\hat{u}\|_{W^{1,4}(\Omega)}$$

is bounded for some $C > 0$. The inequality (1.2) easily yields the boundedness of $\|(\hat{u}, \hat{v})\|_X$.

If $(\hat{u}_1, \hat{v}_1), (\hat{u}_2, \hat{v}_2) \in \mathcal{A}$ are two minimizers, J is constant on $\{(\hat{u}_1, \hat{v}_1) + s(\hat{u}_2 - \hat{u}_1, \hat{v}_2 - \hat{v}_1) : s \in [0,1]\}$: If not, the restriction of J to this set would have a maximum $> J(\hat{u}_1, \hat{v}_1) = J(\hat{u}_2, \hat{v}_2)$ for some $0 < s < 1$, contradicting the convexity of J. Therefore

$$\langle J'(\hat{u}_2, \hat{v}_2) - J'(\hat{u}_1, \hat{v}_1), (\hat{u}_2 - \hat{u}_1, \hat{v}_2 - \hat{v}_1)\rangle = 0,$$

or for our particular J

$$\begin{aligned} 0 &= \int_\Omega (DW^{**}(\nabla \hat{u}_2) - DW^{**}(\nabla \hat{u}_1))\nabla(\hat{u}_2 - \hat{u}_1) \\ &\quad + \langle S((\hat{u}_2 - \hat{u}_1)|_{\partial\Omega} + \hat{v}_2 - \hat{v}_1), (\hat{u}_2 - \hat{u}_1)|_{\partial\Omega} + \hat{v}_2 - \hat{v}_1\rangle. \end{aligned}$$

Both of the terms on the right hand side are non–negative, because S is coercive and W^{**} convex, and hence

$$\hat{u}_1|_{\partial\Omega} + \hat{v}_1 = \hat{u}_2|_{\partial\Omega} + \hat{v}_2 \quad \text{and} \quad (DW^{**}(\nabla \hat{u}_2) - DW^{**}(\nabla \hat{u}_1))\nabla(\hat{u}_2 - \hat{u}_1) = 0$$

almost everywhere. The inequality (0.3),

$$\begin{aligned} &|DW^{**}(\nabla \hat{u}_2) - DW^{**}(\nabla \hat{u}_1)|^2 \\ &\qquad \lesssim (1 + |\nabla \hat{u}_2|^2 + |\nabla \hat{u}_1|^2)(DW^{**}(\nabla \hat{u}_2) - DW^{**}(\nabla \hat{u}_1))\nabla(\hat{u}_2 - \hat{u}_1), \end{aligned}$$

implies $DW^{**}(\nabla \hat{u}_1) = DW^{**}(\nabla \hat{u}_2)$ almost everywhere. The assertions about the projected gradients and the region of microstructure are immediate consequences of inequality (0.4),

$$\begin{aligned} &|\mathbb{P}\nabla \hat{u}_2 - \mathbb{P}\nabla \hat{u}_1|^2 + (Q(\nabla \hat{u}_2) - Q(\nabla \hat{u}_1))^2 \\ &\qquad \lesssim (DW^{**}(\nabla \hat{u}_2) - DW^{**}(\nabla \hat{u}_1))\nabla(\hat{u}_2 - \hat{u}_1). \end{aligned}$$

□

In particular, the displacement $\hat{u}_2$ on Ω^c is uniquely determined and may be computed from $\hat{u}|_{\partial\Omega} + \hat{v}$ with the help of layer potentials. Due to the lack of convexity of W, neither $\hat{u}$ nor $\nabla \hat{u}$ needs to be unique. However, Lemma 3.1.1 allows to identify subsets of Ω, on which these quantities are well–defined.

Corollary 3.1.2. *a) Let $\Omega_{t,A}$ be the set of points $x \in \Omega$ for which the component of a hyperplane perpendicular to A through x intersects Γ_t. Then the displacement $u|_{\Omega_{t,A}}$ is independent of the minimizer.*
b) The same holds for the gradient $\nabla \hat{u}$ outside the region of microstructure.

Proof. The proof closely follows the arguments of [8], Theorem 3.
a) Let $(\hat{u}_1, \hat{v}_1)$, $(\hat{u}_2, \hat{v}_2) \in \mathcal{A}$ be two minimizers, and consider $w = \hat{u}_2 - \hat{u}_1$. Because $\mathbb{P}\nabla \hat{u}_1 = \mathbb{P}\nabla \hat{u}_2$, ∇w is parallel to A almost everywhere. It is easy to see

that, therefore, w may be modified on a set of measure zero to yield an absolutely continuous function which is locally constant along the hyperplanes perpendicular to A. With $w|_{\Gamma_t}$ being 0 by Lemma 3.1.1, w also has to vanish on almost every hyperplane hitting Γ_t.
b) is a consequence of $\mathbb{P}\nabla\hat{u}_1 = \mathbb{P}\nabla\hat{u}_2$, $DW^{**}(\nabla\hat{u}_1) = DW^{**}(\nabla\hat{u}_2)$ and (0.4):

$$(Q(\nabla\hat{u}_2) + Q(\nabla\hat{u}_1))|A\nabla(\hat{u}_2 - \hat{u}_1)|^2 \lesssim (DW^{**}(\nabla\hat{u}_2) - DW^{**}(\nabla\hat{u}_1))\nabla(\hat{u}_2 - \hat{u}_1).$$

$\square$

3.2 Discretization and a priori estimates

We are now going to analyze which quantities can be computed numerically with a Galerkin method.
Let $\{\mathcal{T}_h\}_{h\in I}$ a regular triangulation of $\Omega \subset \mathbb{R}^2$ into disjoint open regular triangles K, so that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$. Each element has at most one edge on $\partial\Omega$, and the closures of any two of them share at most a single vertex or edge. Let h_K denote the diameter of $K \in \mathcal{T}_h$ and ρ_K the diameter of the largest inscribed ball. We assume that $1 \leq \max_{K\in\mathcal{T}_h} \frac{h_K}{\rho_K} \leq R$ independent of h and that $h = \max_{K\in\mathcal{T}_h} h_K$. $\mathcal{E}_h$ is going to be the set of all edges of the triangles in $\mathcal{T}_h$, D the set of nodes. Associated to $\mathcal{T}_h$ is the space $W_h^{1,4}(\Omega) \subset \mathcal{W}$ of functions whose restrictions to any $K \in \mathcal{T}_h$ are linear.
$\partial\Omega$ is triangulated by $\{l \in \mathcal{E}_h : l \subset \partial\Omega\}$. $W_h^{\frac{1}{2},2}(\partial\Omega)$ denotes the corresponding space of piecewise linear functions, and $\widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)$ the subspace of those supported on Γ_s. Finally, $W_h^{-\frac{1}{2},2}(\partial\Omega) \subset W^{-\frac{1}{2},2}(\partial\Omega)$,

$$\mathcal{A}_h = \mathcal{A} \cap (W_h^{1,4}(\Omega) \times W_h^{\frac{1}{2},2}(\partial\Omega))$$

and $X_h^4 = W_h^{1,4}(\Omega) \times \widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s)$.
We denote by $i_h : W_h^{1,4}(\Omega) \hookrightarrow \mathcal{W}$, $j_h : \widetilde{W}_h^{\frac{1}{2},2}(\Gamma_s) \hookrightarrow \widetilde{W}^{\frac{1}{2},2}(\Gamma_s)$ and $k_h : W_h^{-\frac{1}{2},2}(\partial\Omega) \hookrightarrow W^{-\frac{1}{2},2}(\partial\Omega)$ the canonical inclusion maps. A discretization of the Steklov–Poincaré operator is defined as

$$S_h = \frac{1}{2}(W + (I - K')k_h(k_h^* V k_h)^{-1}k_h^*(I - K))$$

from the single resp. double layer potentials V and K and the hypersingular integral operator W of the exterior problem. S_h is well–known to be uniformly coercive for small h in the sense that there exists $h_0 > 0$ and an h–independent $\alpha_S > 0$ such that for all $0 < h < h_0$

$$\langle S_h u_h, u_h\rangle \geq \alpha_S \|u_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)}.$$

Furthermore, in this case

$$\|(S_h - S)u\|_{W^{-\frac{1}{2},2}(\partial\Omega)} \leq C_S \operatorname{dist}_{W^{-\frac{1}{2},2}(\partial\Omega)}(V^{-1}(1-K)u, W_h^{-\frac{1}{2},2}(\partial\Omega)) \qquad (2.1)$$

for all $u \in W^{\frac{1}{2},2}(\partial\Omega)$ and all $0 < h < h_0$.
As before, $(\hat{u}, \hat{v})$ denotes a minimizer of J over $\mathcal{A}$, while $(\hat{u}_h, \hat{v}_h)$ minimizes the approximate functional

$$J_h(u_h, v_h) = \int_\Omega W^{**}(\nabla u_h) + \frac{1}{2}\langle S_h(u_h|_{\partial\Omega} + v_h), u_h|_{\partial\Omega} + v_h\rangle - \lambda_h(u_h, v_h),$$

$$\lambda_h(u_h, v_h) = \langle t_0 + S_h u_0, u_h|_{\partial\Omega} + v_h\rangle + \int_\Omega f u_h,$$

over $\mathcal{A}_h$. For simplicity, abbreviate the stress $DW^{**}(\nabla\hat{u})$ by σ and the indicator $Q(\nabla\hat{u})$ for microstructure by ξ. Similarly, write σ_h and ξ_h for the corresponding quantities associated to $\hat{u}_h$. The following a priori estimate holds.

Theorem 3.2.1. *The Galerkin approximations of the stress σ, exterior boundary values $u|_{\partial\Omega} + v$ and the other quantities in Lemma 3.1.1 converge for $h \to 0$.*
a) There is an h–independent $C > 0$ such that

$$\begin{aligned}
&\|\sigma - \sigma_h\|^2_{L^{\frac{4}{3}}(\Omega)} + \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} \\
&\quad + \|\mathbb{P}\nabla\hat{u} - \mathbb{P}\nabla\hat{u}_h\|^2_{L^2(\Omega)} + \|(\xi + \xi_h)^{1/2} A\nabla(\hat{u} - \hat{u}_h)\|^2_{L^2(\Omega)} + \|\xi - \xi_h\|^2_{L^2(\Omega)} \\
&\leq C \inf_{(U_h, V_h) \in \mathcal{A}_h} \left\{\|u - U_h\|_{W^{1,4}(\Omega)} + \|(u - U_h)|_{\partial\Omega} + v - V_h)\|_{W^{\frac{1}{2},2}(\partial\Omega)}\right\} \\
&\qquad + \operatorname{dist}_{W^{-\frac{1}{2},2}(\partial\Omega)}(V^{-1}(1-K)(\hat{u} + \hat{v} - u_0), W_h^{-\frac{1}{2},2}(\partial\Omega))^2.
\end{aligned}$$

b) For pure transmission conditions, $\Gamma_t = \partial\Omega$, the slightly better estimate

$$\begin{aligned}
&\|\sigma - \sigma_h\|^2_{L^{\frac{4}{3}}(\Omega)} + \|\hat{u} - \hat{u}_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\mathbb{P}\nabla\hat{u} - \mathbb{P}\nabla\hat{u}_h\|^2_{L^2(\Omega)} \\
&\quad + \|(\xi + \xi_h)^{1/2} A\nabla(\hat{u} - \hat{u}_h)\|^2_{L^2(\Omega)} + \|\xi - \xi_h\|^2_{L^2(\Omega)} \\
&\leq C \inf_{U_h \in \mathcal{A}_h} \left\{\|\nabla\hat{u} - \nabla U_h\|^2_{L^4(\Omega)} + \|\hat{u} - U_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)}\right\} \\
&\qquad + \operatorname{dist}_{W^{-\frac{1}{2},2}(\partial\Omega)}(V^{-1}(1-K)(\hat{u} - u_0), W_h^{-\frac{1}{2},2}(\partial\Omega))^2
\end{aligned}$$

holds.

Proof. We integrate (0.3) and use Hölder's inequality as well as the uniform bound on the norm of minimizers (the first assertion in Lemma 3.1.1) to obtain

$$\|\sigma - \sigma_h\|^2_{L^{\frac{4}{3}}(\Omega)} \lesssim \int_\Omega (\sigma - \sigma_h)\nabla(\hat{u} - \hat{u}_h). \qquad (2.2)$$

Most of the remaining terms on the left hand side are similarly bounded with the help of (0.4):

$$\|\mathbb{P}\nabla\hat{u} - \mathbb{P}\nabla\hat{u}_h\|^2_{L^2(\Omega)} + \|(\xi+\xi_h)^{1/2}A\nabla(\hat{u}-\hat{u}_h)\|^2_{L^2(\Omega)} + \|\xi-\xi_h\|^2_{L^2(\Omega)} \lesssim \int_\Omega (\sigma-\sigma_h)\nabla(\hat{u}-\hat{u}_h). \quad (2.3)$$

Adding the inequalities,

$$\begin{aligned}
LHS^2 \;&:=\; \|\sigma-\sigma_h\|^2_{L^{\frac{4}{3}}(\Omega)} + \|(\hat{u}-\hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} \\
&\quad + \|\mathbb{P}\nabla\hat{u} - \mathbb{P}\nabla\hat{u}_h\|^2_{L^2(\Omega)} + \|(\xi+\xi_h)^{1/2}A\nabla(\hat{u}-\hat{u}_h)\|^2_{L^2(\Omega)} + \|\xi-\xi_h\|^2_{L^2(\Omega)} \\
&\lesssim\; \textstyle\int_\Omega (\sigma-\sigma_h)(\nabla\hat{u}-\nabla\hat{u}_h) + \langle S(\hat{u}-\hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h, (\hat{u}-\hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle \\
&=\; -\int_\Omega \sigma\nabla\hat{u}_h - \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), \hat{u}_h|_{\partial\Omega} + \hat{v}_h\rangle \\
&\qquad -\int_\Omega \sigma_h\nabla\hat{u} - \langle S(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), \hat{u}|_{\partial\Omega} + \hat{v}\rangle \\
&\qquad +\int_\Omega \sigma\nabla\hat{u} + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), \hat{u}|_{\partial\Omega} + \hat{v}\rangle \\
&\qquad +\int_\Omega \sigma_h\nabla\hat{u}_h + \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), \hat{u}_h|_{\partial\Omega} + \hat{v}_h\rangle \\
&\qquad +\langle (S-S_h)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), \hat{u}_h|_{\partial\Omega} + \hat{v}_h\rangle.
\end{aligned}$$

Let $(U,V) \in \mathcal{A}$, $(U_h, V_h) \in \mathcal{A}_h$. Applying the variational inequality (1.1) to the third and fourth line and rearranging terms leads to

$$\begin{aligned}
LHS^2 \;&\lesssim\; \int_\Omega \sigma\nabla(U-\hat{u}_h) + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), U|_{\partial\Omega} + V - \hat{u}_h|_{\partial\Omega} - \hat{v}_h\rangle \\
&\qquad + \int_\Omega \sigma_h\nabla(U_h-\hat{u}) + \langle S(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), U_h|_{\partial\Omega} + V_h - \hat{u}|_{\partial\Omega} - \hat{v}\rangle \\
&\qquad + \lambda(\hat{u}-U, \hat{v}-V) + \lambda(\hat{u}_h - U_h, \hat{v}_h - V_h) \\
&\qquad + \langle (S-S_h)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0), (\hat{u}_h - U_h)|_{\partial\Omega} + \hat{v}_h - V_h\rangle \\
&=\; \int_\Omega \sigma\nabla(U-\hat{u}_h) + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (U-\hat{u}_h)|_{\partial\Omega} + V - \hat{v}_h\rangle \\
&\qquad + \int_\Omega \sigma\nabla(U_h-\hat{u}) + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (U_h-\hat{u})|_{\partial\Omega} + V_h - \hat{v}\rangle \\
&\qquad + \int_\Omega (\sigma_h - \sigma)\nabla(U_h - \hat{u}) \\
&\qquad + \langle S((\hat{u}_h - \hat{u})|_{\partial\Omega} + \hat{v}_h - \hat{v}), (U_h-\hat{u})|_{\partial\Omega} + V_h - \hat{v}\rangle \\
&\qquad + \lambda(\hat{u}-U, \hat{v}-V) + \lambda(\hat{u}_h - U_h, \hat{v}_h - V_h) \\
&\qquad + \langle (S-S_h)(\hat{u}|_{\partial\Omega} + \hat{v} - u_0), (\hat{u}_h - U_h)|_{\partial\Omega} + \hat{v}_h - V_h\rangle \\
&\qquad + \langle (S-S_h)((\hat{u}_h - \hat{u})|_{\partial\Omega} + \hat{v}_h - \hat{v}), (\hat{u}_h - U_h)|_{\partial\Omega} + \hat{v}_h - V_h\rangle.
\end{aligned}$$

Hölder's inequality tells us that

$$\int_\Omega (\sigma_h - \sigma)\nabla(U_h - \hat{u}) \leq \|\sigma_h - \sigma\|_{L^{\frac{4}{3}}(\Omega)} \|\nabla(U_h - \hat{u})\|_{L^4(\Omega)},$$

and the continuity of S allows to bound

$$\langle S((\hat{u}_h - \hat{u})|_{\partial\Omega} + \hat{v}_h - \hat{v}), (U_h - \hat{u})|_{\partial\Omega} + V_h - \hat{v}\rangle$$

by a multiple of

$$\begin{aligned} &\|(\hat{u}_h - \hat{u})|_{\partial\Omega} + \hat{v}_h - \hat{v}\|_{W^{\frac{1}{2},2}(\partial\Omega)} \|(U_h - \hat{u})|_{\partial\Omega} + V_h - \hat{v}\|_{W^{\frac{1}{2},2}(\partial\Omega)} \\ \lesssim\ & \varepsilon \|(\hat{u}_h - \hat{u})|_{\partial\Omega} + \hat{v}_h - \hat{v}\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \frac{1}{\varepsilon}\|(U_h - \hat{u})|_{\partial\Omega} + V_h - \hat{v}\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} \end{aligned}$$

for small $\varepsilon > 0$. Similarly, the last two lines are, up to prefactors, bounded by

$$\begin{aligned} &\varepsilon \|(\hat{u}_h - \hat{u})|_{\partial\Omega} + \hat{v}_h - \hat{v}\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + (1 + \frac{1}{\varepsilon})\|(U_h - \hat{u})|_{\partial\Omega} + V_h - \hat{v}\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} \\ &+ \|(S - S_h)(\hat{u}|_{\partial\Omega} + \hat{v} - u_0)\|^2_{W^{\frac{1}{2},2}(\partial\Omega)}. \end{aligned}$$

Thus, choosing $(U, V) = (\hat{u}_h, \hat{v}_h)$,

$$\begin{aligned} LHS^2 \ \lesssim\ & \|\sigma_h - \sigma\|_{L^{\frac{4}{3}}(\Omega)} \|\nabla(U_h - \hat{u})\|_{L^4(\Omega)} + \|(U_h - \hat{u})|_{\partial\Omega} + V_h - \hat{v}\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} \\ & + \|(S - S_h)(\hat{u}|_{\partial\Omega} + \hat{v} - u_0)\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \int_\Omega \sigma \nabla(U_h - \hat{u}) \\ & + \langle S(\hat{u}|_{\partial\Omega} + \hat{v}), (U_h - \hat{u})|_{\partial\Omega} + V_h - \hat{v}\rangle - \lambda(U_h - \hat{u}, V_h - \hat{v}). \end{aligned}$$

If $\Gamma_t = \partial\Omega$, the variational inequality (1.1) becomes an equality, the last line vanishes and b) follows. In the general case, we estimate the last line by

$$\begin{aligned} & \|\sigma\|_{L^{\frac{4}{3}}(\Omega)} \|\nabla(U_h - \hat{u})\|_{L^4(\Omega)} + \|f\|_{L^{\frac{4}{3}}(\Omega)} \|U_h - \hat{u}\|_{L^4(\Omega)} \\ + & \|S(\hat{u}|_{\partial\Omega} + \hat{v} - u_0) - t_0\|_{W^{-\frac{1}{2},2}(\partial\Omega)} \|(U_h - \hat{u})|_{\partial\Omega} + V_h - \hat{v}\|_{W^{\frac{1}{2},2}(\partial\Omega)}, \end{aligned}$$

recalling that

$$\lambda(U_h - \hat{u}, V_h - \hat{v}) = \langle t_0 + S u_0, (U_h - \hat{u})|_{\partial\Omega} + V_h - \hat{v}\rangle + \int_\Omega f(U_h - \hat{u}).$$

□

In particular, we can stably compute the approximate solutions in the exterior domain from $\hat{u}_h|_{\partial\Omega} + \hat{v}_h$.

3.3 An a posteriori error estimate

In order to set up an adaptive algorithm, we now establish an a posteriori estimate of residual type. It allows to localize the approximation error and leads to an adaptive mesh refinement strategy. A related and somewhat more involved estimate for the linear Laplace operator with unilateral Signiorini contact has been considered in [35].
Let $(\hat{u}, \hat{v}) \in \mathcal{A}$, $(\hat{u}, \hat{v}_h) \in \mathcal{A}_h$ solutions of the continuous resp. discretized variational inequality. We define a simple approximation $(\pi_h \hat{u}, \pi_h \hat{v}) \in \mathcal{A}_h$ of $(\hat{u}, \hat{v})$ as follows: $\pi_h \hat{u}$ is going to be the Clement interpolant of $\hat{u}$, and $\pi_h \hat{v} = \hat{v}_h$.
The next Lemma collects the crucial properties of Clement interpolation (see e.g. [5]).

Lemma 3.3.1. *Let $K \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$. Then with $\omega_K = \bigcup_{\overline{K'} \cap \overline{K} \neq \emptyset} K'$ and $\omega_E = \bigcup_{\overline{E'} \cap E \neq \emptyset} E'$ we have:*

$$\begin{aligned} \|\hat{u} - \pi_h \hat{u}\|_{L^4(K)} &\lesssim h_K \|\hat{u}\|_{W^{1,4}(\omega_K)} , \\ \|\hat{u} - \pi_h \hat{u}\|_{L^2(E)} &\lesssim h_E^{1/2} \|\hat{u}\|_{W^{\frac{1}{2},2}(\omega_E)} . \end{aligned}$$

We are going to prove the following a posteriori estimate:

Theorem 3.3.2.

$$\begin{aligned} &\|\sigma - \sigma_h\|^2_{L^{\frac{4}{3}}(\Omega)} + \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\mathbb{P}\nabla\hat{u} - \mathbb{P}\nabla\hat{u}_h\|^2_{L^2(\Omega)} \\ &+ \|(\xi + \xi_h)^{1/2} A \nabla(\hat{u} - \hat{u}_h)\|^2_{L^2(\Omega)} + \|\xi - \xi_h\|^2_{L^2(\Omega)} \\ &\quad \lesssim \eta_\Omega + \eta_C + \eta_S + \operatorname{dist}_{W^{-\frac{1}{2},2}(\partial\Omega)}(V^{-1}(1-K)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0), W_h^{-\frac{1}{2},2}(\partial\Omega))^2 , \end{aligned}$$

where

$$\begin{aligned} \eta_\Omega &= \sum_K h_K \|f\|_{L^{4/3}(K)} + \sum_{E \cap \partial\Omega = \emptyset} h_E \|[\nu_E \cdot \sigma_h]\|_{L^2(E)} , \\ \eta_C &= \eta_{C,1} + \eta_{C,2} = \sum_{E \subset \Gamma_s} \|(\nu_E \cdot \sigma_h)_+\|_{L^2(E)} + \sum_{E \subset \Gamma_s} \int_E (\nu_E \cdot \sigma_h)_- \ \hat{v}_h , \\ \eta_S &= \sum_{E \subset \partial\Omega} h_E^{1/2} \ \|S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) + (\nu_{\partial\Omega} \cdot \sigma_h) - t_0\|_{L^2(E)} . \end{aligned}$$

Remark 3.3.3. a) Also the constant prefactors, suppressed in our notation $\lesssim$, are explicitly known.
b) The main point of this estimate is to show that the a posteriori estimates for the contact part ([35]) and the double–well term ([8]) are compatible. More sophisticated bounds related to a different choice of π_h generalize to our setting

in a similar way. As a simple case, a more considerate (sign–preserving) choice of $\pi_h \hat{v}$ with $\|\hat{v} - \pi_h \hat{v}\|_{L^2(\Gamma_s)} \lesssim h^\alpha \|\hat{v}\|_{L^2(\Gamma_s)}$ could be used to gain an h^α in $\eta_{C,1}$ at the expense of modifying

$$\eta_{C,2} = \sum_{E \subset \Gamma_s} \int_E (\nu_E \cdot \sigma_h)_- \ \pi_h^1 \hat{v}_h \ ,$$

as long as we only assure that $\int_E (\pi_h \hat{v}_h - \hat{v}) \leq \int_E \pi_h^1 \hat{v}_h$ for some auxiliary interpolation operator π_h^1 (see e.g. [35]).

c) As in Chapter 2, it is straight forward to introduce an additional variable on the boundary to obtain estimates that do not involve the incomputable difference $S_h - S$. Similarly, we might also use the formulation of Bartels [2] with explicit Young measures in the interior part.

Proof of Theorem 3.3.2. As in the proof of Theorem 3.2.1, we start with the inequality

$$\begin{aligned} LHS^2 &:= \|\sigma - \sigma_h\|^2_{L^{\frac{4}{3}}(\Omega)} + \|(\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\|^2_{W^{\frac{1}{2},2}(\partial\Omega)} \\ &\quad + \|\mathbb{P}\nabla\hat{u} - \mathbb{P}\nabla\hat{u}_h\|^2_{L^2(\Omega)} + \|(\xi + \xi_h)^{1/2} A \nabla(\hat{u} - \hat{u}_h)\|^2_{L^2(\Omega)} + \|\xi - \xi_h\|^2_{L^2(\Omega)} \\ &\lesssim \int_\Omega (\sigma - \sigma_h)\nabla(\hat{u} - \hat{u}_h) + \langle S((\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle. \end{aligned}$$

Using the variational inequality and its discretized variant results in

$$\begin{aligned} LHS^2 &\lesssim \lambda(\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h) - \int_\Omega \sigma_h \nabla(\hat{u} - \hat{u}_h) \\ &\quad - \langle S(\hat{u}_h|_{\partial\Omega} + \hat{v}_h, (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle \\ &= \lambda(\hat{u} - \hat{u}_h, \hat{v} - \hat{v}_h) - \int_\Omega \sigma_h \nabla(\hat{u} - \hat{u}_h) \\ &\quad - \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h, (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle \\ &\quad + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h, (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle \\ &\leq \lambda(\hat{u} - u_h, \hat{v} - v_h) - \int_\Omega \sigma_h \nabla(\hat{u} - u_h) \\ &\quad - \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h, (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\rangle \\ &\quad + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle \\ &= \int_\Omega f(\hat{u} - u_h) - \sum_{E \cap \partial\Omega = \emptyset} \int_E [\nu_E \cdot \sigma_h](\hat{u} - u_h) \\ &\quad - \langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) + (\nu_{\partial\Omega} \cdot \sigma_h) - t_0), (\hat{u} - u_h)|_{\partial\Omega} + \hat{v} - v_h\rangle \\ &\quad + \int_{\Gamma_s} (\nu_{\partial\Omega} \cdot \sigma_h) \ (\hat{v} - v_h) \\ &\quad + \langle (S_h - S)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0), (\hat{u} - \hat{u}_h)|_{\partial\Omega} + \hat{v} - \hat{v}_h\rangle \end{aligned}$$

for all $(u_h, v_h) \in \mathcal{A}_h$. Here, ν_E and $\nu_{\partial\Omega}$ denote the outward–pointing unit normal vector to an edge $E \subset \overline{K}$, resp. to $\partial\Omega$, and $[\nu_E \cdot \sigma_h]$ is the jump of the discretized normal stress across E. According to estimate (2.1) for $S_h - S$ and Young's inequality, the last term contributes the not explicitly computable $\mathrm{dist}_{W^{-\frac{1}{2},2}(\partial\Omega)}(V^{-1}(1-K)(\hat{u}_h|_{\partial\Omega} + \hat{v}_h), W_h^{-\frac{1}{2},2}(\partial\Omega))^2$.
We are going to choose $(u_h, v_h) = (\pi_h \hat{u}, \pi_h \hat{v})$. Then, the first three terms on the right hand side can be estimated with the help of Lemma 3.3.1 and Hölder's inequality:

$$\int_\Omega f(\hat{u} - \pi_h \hat{u}) \leq \|\hat{u}\|_{\mathcal{W}} \Big(\sum_K h_K^{4/3} \int_K |f|^{4/3} \Big)^{3/4} ,$$

$$\Big| \sum_{E \cap \partial\Omega = \emptyset} \int_E [\nu_E \cdot \sigma_h](\hat{u} - \pi_h \hat{u}) \Big| \leq \|\hat{u}\|_{W^{\frac{1}{2},2}(\partial\Omega)} \Big(\sum_{E \cap \partial\Omega = \emptyset} h_E \int_E |[\nu_E \cdot \sigma_h]|^2 \Big)^{1/2}$$

and

$$\begin{aligned} &|\langle S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) + (\nu_{\partial\Omega} \cdot \sigma_h) - t_0), (\hat{u} - \pi_h \hat{u})|_{\partial\Omega} + \hat{v} - \pi_h \hat{v} \rangle| \\ &\qquad \leq (\|\hat{u}\|_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\hat{v}\|_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\hat{v}_h\|_{W^{\frac{1}{2},2}(\partial\Omega)}) \\ &\qquad\qquad \times \|S_h(\hat{u}_h|_{\partial\Omega} + \hat{v}_h - u_0) + (\nu_{\partial\Omega} \cdot \sigma_h) - t_0\|_{W^{-\frac{1}{2},2}(\partial\Omega)} . \end{aligned}$$

Note that the boundedness of the set of minimizers, Lemma 3.1.1, provides an explicit uniform bound on both $\|\hat{u}, \hat{v}\|_X$ and $\|\hat{u}_h, \hat{v}_h\|_X$ in terms of the norms of the data. The $W^{-\frac{1}{2},2}(\partial\Omega)$–norm leads to η_S ([10]).
The remaining term requires a slightly more precise analysis. Decompose

$$(\nu_{\partial\Omega} \cdot \sigma_h) = (\nu_{\partial\Omega} \cdot \sigma_h)_+ - (\nu_{\partial\Omega} \cdot \sigma_h)_-$$

into its positive and negative parts. For a classical exact solution, the Signiorini condition requires $(\nu_{\partial\Omega} \cdot \sigma)_+ = 0$, and we estimate the corresponding term as above:

$$\Big| \int_{\Gamma_s} (\nu_{\partial\Omega} \cdot \sigma_h)_+ \ (\hat{v} - \pi_h \hat{v}) \Big| \lesssim (\|\hat{v}\|_{W^{\frac{1}{2},2}(\partial\Omega)} + \|\hat{v}_h\|_{W^{\frac{1}{2},2}(\partial\Omega)}) \Big(\int_{\Gamma_s} |(\nu_E \cdot \sigma_h)_+|^2 \Big)^{1/2} .$$

For the negative part, we may drop the unknown term:

$$\begin{aligned} -\int_{\Gamma_s} (\nu_{\partial\Omega} \cdot \sigma_h)_- \ (\hat{v} - v_h) &= \sum_{E \subset \Gamma_s} (\nu_E \cdot \sigma_h)_- \int_E (v_h - \hat{v}) \\ &\leq \sum_{E \subset \Gamma_s} (\nu_E \cdot \sigma_h)_- \int_E \hat{v}_h . \end{aligned}$$

The a posteriori estimate follows. □

As in Chapter 2, the a posteriori estimate allows to localize the error and gives rise to a strategy for adaptive mesh refinements.

Chapter 4

Analytic factorization of Lie group representations

Consider a category $\mathcal{C}$ of modules over a nonunital algebra $\mathcal{A}$. We say that $\mathcal{C}$ has the *factorization property* if for all $\mathcal{M} \in \mathcal{C}$,

$$\mathcal{M} \;=\; \mathcal{A} \cdot \mathcal{M} \;:=\; \operatorname{span}\{a \cdot m \;\mid\; a \in \mathcal{A},\; m \in \mathcal{M}\}\,.$$

In particular, if $\mathcal{A} \in \mathcal{C}$ this implies $\mathcal{A} = \mathcal{A} \cdot \mathcal{A}$.
Let (π, E) be a representation of a real Lie group G on a Fréchet space E. Then the corresponding space of smooth vectors E^∞ is again a Fréchet space. The representation (π, E) induces a continuous action Π of the algebra $C_c^\infty(G)$ of test functions on E given by

$$\Pi(f)v = \int_G f(g)\pi(g)v \; dg \quad (f \in \mathrm{C}_c^\infty(G), v \in E),$$

which restricts to a continuous action on E^∞. Hence the smooth vectors associated to such representations are a $C_c^\infty(G)$–module, and a result by Dixmier and Malliavin [21] states that this category has the factorization property.
In this chapter we prove an analogous result for the category of analytic vectors. For simplicity, we outline our approach for a Banach representation (π, E). In this case, the space E^ω of analytic vectors is endowed with a natural inductive limit topology, and gives rise to a representation (π, E^ω). To define an appropriate algebra acting on E^ω, we fix a left–invariant Riemannian metric on G and let d be the associated distance function. The continuous functions $\mathcal{R}(G)$ on G which decay faster than $e^{-nd(g,\mathbf{1})}$ for all $n \in \mathbb{N}$ form a $G \times G$–module under the left–right regular representation. We define $\mathcal{A}_{LR}(G)$ to be the space of analytic vectors of this action. Both $\mathcal{R}(G)$ and $\mathcal{A}_{LR}(G)$ form an algebra under convolution, and the action Π of $C_c^\infty(G)$ extends to give E^ω the structure of an $\mathcal{A}_{LR}(G)$–module.
In this setting, our main theorem says that the category of analytic vectors for Banach representations of G has the factorization property. More generally, we obtain a result for F–representations:

Theorem 4.0.4. *Let G be a real Lie group and (π, E) an F–representation of G. Then*

$$\mathcal{A}_{LR}(G) \;=\; \mathcal{A}_{LR}(G) * \mathcal{A}_{LR}(G)$$

and

$$E^\omega \;=\; \Pi(\mathcal{A}_{LR}(G))\; E^\omega \;=\; \Pi(\mathcal{A}_{LR}(G))\; E.$$

Let us remark that the special case of bounded Banach representations of $(\mathbb{R}, +)$ has been proved by one of the authors in [41]. The above factorization theorem is a crucial tool to understand the minimal analytic globalization of Harish–Chandra modules in Chapter 5.

As a corollary of Theorem 4.0.4 we obtain that a vector is analytic if and only if it is analytic for the Laplace–Beltrami operator, which generalizes a result of Goodman [29] for unitary representations.

In particular, the theorem extends Nelson's result that $\Pi(\mathcal{A}_{LR}(G))\; E^\omega$ is dense in E^ω [46]. Gårding had obtained an analogous theorem for the smooth vectors [24]. However, while Nelson's proof is based on approximate units constructed from the fundamental solution $\varrho_t \in \mathcal{A}_{LR}(G)$ of the heat equation on G by letting $t \to 0^+$, our strategy relies on some more sophisticated functions of the Laplacian.

To prove Theorem 4.0.4, we first consider the case $G = (\mathbb{R}, +)$. Here the proof is based on the key identity

$$\alpha_\varepsilon(z)\cosh(\varepsilon z) + \beta_\varepsilon(z) = 1,$$

for the entire functions $\alpha_\varepsilon(z) = 2e^{-\varepsilon z\,\mathrm{erf}(z)}$ and $\beta_\varepsilon(z) = 1 - \alpha_\varepsilon(z)\cosh(\varepsilon z)$ on the complex plane [1]. We consider this as an identity for the symbols of the Fourier multiplication operators $\alpha_\varepsilon(i\partial)$, $\beta_\varepsilon(i\partial)$ and $\cosh(i\varepsilon\partial)$. The functions α_ε and β_ε are easily seen to belong to the Fourier image of $\mathcal{A}(\mathbb{R})$, so that $\alpha_\varepsilon(i\partial)$ and $\beta_\varepsilon(i\partial)$ are given by convolution with some $\kappa_\alpha^\varepsilon, \kappa_\beta^\varepsilon \in \mathcal{A}(\mathbb{R})$. For every $v \in E^\omega$ and sufficiently small $\varepsilon > 0$, we may also apply $\cosh(i\varepsilon\partial)$ to the orbit map $\gamma_v(g) = \pi(g)v$ and conclude that

$$(\cosh(i\varepsilon\partial)\;\gamma_v) * \kappa_\alpha^\varepsilon + \gamma_v * \kappa_\beta^\varepsilon = \gamma_v.$$

The theorem follows by evaluating in 0.

Unlike in the work of Dixmier and Malliavin, the rigid nature of analytic functions requires a global geometric approach in the general case. The idea is to refine the functional calculus of Cheeger, Gromov and Taylor [12] for the Laplace-Beltrami operator in the special case of a Lie group. Using this tool, the general proof then closely mirrors the argument for $(\mathbb{R}, +)$. See Remark 4.4.4 for the heuristics relating the functional calculus to the Fourier transform on $\mathbb{R}$.

The chapter concludes by showing in Section 4.6 how our strategy may be adapted to solve some related factorization problems.

[1] Some basic properties of these functions and the Gaussian error function erf are collected in Appendix A.

4.1 Banach representations and F-representations

For a Hausdorff, locally convex and sequentially complete topological vector space E we denote by $GL(E)$ the associated group of isomorphisms. Let G be a real Lie group. By a *representation* of G we shall understand a continuous action

$$G \times E \to E, \quad (g, v) \mapsto g \cdot v \ ,$$

on some topological vector space E. Each representation gives rise to a group homomorphism

$$\pi : G \to E, \quad g \mapsto \pi(g), \quad \pi(g)v := g \cdot v \quad (v \in E) \, ,$$

and it is custom to denote the representation by the symbol (π, E).
A representation (π, E) is called a *Banach representation* if E is a Banach space. We say that (π, E) is an F-representation, if E is a Fréchet space for which there exists a defining family of seminorms $(p_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ the action

$$G \times (E, p_n) \to (E, p_n)$$

is continuous. Here (E, p_n) refers to the vector space E endowed with the locally convex structure induced by p_n.

Remark 4.1.1. (a) Every Banach representation is an F-representation.
(b) Let (π, E) be an F-representation. For each $n \in \mathbb{N}$ let us denote by $\hat{E}_n$ the Banach completion of (E, p_n), i.e. the completion of the normed space $E/\{p_n = 0\}$. The action of G on (E, p_n) factors to a continuous action on the normed space $E/\{p_n = 0\}$ and thus induces a Banach representation of G on $\hat{E}_n$.
(c) The left regular action of G on the Fréchet space $C(G)$ defines a representation, but in general not an F-representation.

Let E^∞ denote the space of smooth vectors in E, that is, the vectors $v \in E$ for which the orbit map $g \mapsto \pi(g)v$ is smooth into E. Then $E^\infty \subset E$ is an invariant subspace, and it is dense if E is complete. The orbit map provides an injection of E^∞ into $C^\infty(G, E)$, from which E^∞ inherits a topological vector space structure. Then (π, E^∞) is a representation. Furthermore, E^∞ is a Fréchet space if E is a Fréchet space, and (π, E^∞) is an F-representation if (π, E) is an F-representation. By definition, a *smooth representation* is a representation for which $E^\infty = E$ as topological vector spaces.

Growth of representations

We call a function $w : G \to \mathbb{R}^+$ a *weight* if

- w is locally bounded,

- w is sub-multiplicative, i.e. if

$$w(gh) \leq w(g)w(h)$$

for all $g, h \in G$.

To every Banach representation (π, E) we associate the function

$$w_\pi(g) := \|\pi(g)\| \qquad (g \in G)\,,$$

where $\|\cdot\|$ denotes the standard operator norm. It follows from the uniform boundedness principle that w_π is locally bounded. Hence w_π is a weight.
Sub-multiplicative functions can be dominated in a geometric way. For that let us fix a left invariant Riemannian metric $\mathbf{g}$ on G. Associated to $\mathbf{g}$ we obtain the Riemannian distance function $\mathbf{d} : G \times G \to \mathbb{R}_{\geq 0}$. The distance function is left G-invariant and hence is recovered as $\mathbf{d}(g, h) = d(g^{-1}h)$ from the function

$$d(g) := \mathbf{d}(g, \mathbf{1}) \qquad (g \in G),$$

where $\mathbf{1} \in G$ is the neutral element. Notice that it follows from the elementary properties of the metric that d is compatible with the group structure in the sense that

$$d(g^{-1}) = d(g) \quad \text{and} \quad d(gh) \leq d(g) + d(h) \tag{1.1}$$

for all $g, h \in G$. In particular, $g \mapsto e^{d(g)}$ is a weight. Note also that the metric balls $\{g \in G | \, d(g) \leq R\}$ in G are compact ([25], p. 74).
Here is a key property of d, see [24]:

Lemma 4.1.2. *If $w : G \to \mathbb{R}_+$ is locally bounded and submultiplicative (i.e. $w(gh) \leq w(g)w(h)$), then there exist $c, C > 0$ such that*

$$w(g) \leq Ce^{cd(g)} \qquad (g \in G).$$

Remark 4.1.3. In particular, it follows that a Banach representation has at most exponential growth

$$\|\pi(g)\| \leq Ce^{c\,d(g)}.$$

By applying Remark 4.1.1(b) we obtain for an F-representation (π, E) with defining seminorms $(p_n)_{n\in\mathbb{N}}$ that for each n there exist constants c_n, C_n such that

$$p_n(\pi(g)v) \leq C_n e^{c_n d(g)} p_n(v) \qquad (g \in G, v \in E). \tag{1.2}$$

Finally, notice that it follows from Remark 4.1.3 that if $d_1(g) = d_{\mathbf{g}_1}(g, \mathbf{1})$ is the function associated with a different choice of a G-invariant metric, then d_1 is compatible with d, in the sense that there exist constants $c, C > 0$ such that

$$d_1(g) \leq cd(g) + C \qquad (g \in G)$$

(and vice–versa with d, d_1 interchanged).

Remark 4.1.4. Suppose that G is a real reductive group and $\|\cdot\|$ is a norm of G (see [53], Sect 2.A.2). Then $\|\cdot\|$ is a weight and hence there exist constants $c_1, C_1 > 0$ such that

$$\log \|g\| \leq c_1 d(g) + C_1 \qquad (g \in G).$$

Conversely, by following the proof of [53], Lemma 2.A.2.2, one finds constants $c_2, C_2 > 0$ such that

$$d(g) \leq c_2 \log \|g\| + C_2 \qquad (g \in G).$$

4.2 The space of analytic vectors

Let us denote by $\mathfrak{g}$ the Lie algebra of G. To simplify the exposition we will assume that $G \subset G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is a complex group with Lie-algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} =: \mathfrak{g}_{\mathbb{C}}$. We stress, however, that this assumption is not necessary, since the use of $G_{\mathbb{C}}$ essentially only takes place locally in neighborhoods G.
We extend the left invariant metric $\mathbf{g}$ to a left $G_{\mathbb{C}}$-invariant metric on $G_{\mathbb{C}}$ and denote the associated distance function as before by d. For every $n \in \mathbb{N}$ we set

$$V_n := \{g \in G_{\mathbb{C}} \mid d(g) < \frac{1}{n}\} \qquad \text{and} \qquad U_n := V_n \cap G\,.$$

It is clear that the V_n's, resp. U_n's, form a base of the neighborhood filter of $\mathbf{1}$ in $G_{\mathbb{C}}$, resp. G. Note that V_n is symmetric, and that $xy \in V_n$ for all $x, y \in V_{2n}$.
Let (π, E) be a representation of G. For each $v \in E$ we denote by

$$\gamma_v : G \to E, \quad x \mapsto \pi(x)v\ ,$$

the associated continuous orbit map. We call v an *analytic* vector if γ_v extends to a holomorphic E-valued function (see Section 4.8) on some open neighborhood of G in $G_{\mathbb{C}}$.
If v is analytic, then γ_v is a real analytic map $G \to E$. The converse statement, that real analyticity of the orbit map implies the analyticity of v, holds under the assumption that E is sequentially complete. Hence our definition agrees with the standard notion of analytic vectors for Banach representations, see for example [46], [25], [28].

Remark 4.2.1. If E is a Banach space or more generally a complete *DF*-space (see [44], Ch. 25), then it follows from [47] Thm. 1, that v is an analytic vector already if the orbit map is *weakly analytic*, that is, $\lambda \circ \gamma_v : G \to \mathbb{C}$ is real analytic for all $\lambda \in E'$. Here E' denotes the dual space of continuous linear forms.

The space of analytic vectors is denoted by E^ω. A theorem of Nelson ([46] p. 599) asserts that E^ω is dense in E if E is a Banach space. More precisely, Nelson's theorem asserts the following. Let $h_t \in C^\infty(G)$ denote the *heat kernel* on G,

where $t > 0$, then $\Pi(h_t)v \in E^\omega$ and $\Pi(h_t)v \to v$ for $t \to 0$, for all $v \in E$. In fact, the proof of Nelson's theorem is valid more generally if E is sequentially complete and π has exponential growth (that is, for every $v \in E$ and for every continuous seminorm p there exist constants c, C such that $p(\pi(x)v) \leq Ce^{cd(x)}$ for all $x \in G$). In particular, this is the case for F-representations (see (1.2)). The density is false in general, as easy examples such as the left regular representation of $\mathbb{R}$ on $C_c(\mathbb{R})$ show.
We wish to emphasize that E^ω is a G-invariant vector subspace of E. This follows immediately from the identity $\gamma_{\pi(g)v}(x) = \gamma_v(xg)$. We also note that E^ω is a $\mathfrak{g}$-invariant subset of the space E^∞ of smooth vectors.
It is convenient to introduce the following notation. For every $n \in \mathbb{N}$ we define the subspace of E^ω,

$$E_n = \{v \in E \mid \gamma_v \text{ extends to a holomorphic map } GV_n \to E\}.$$

Since G is totally real in $G_\mathbb{C}$ and GV_n is connected, the holomorphic extension of γ_v is unique if it exists. Let us denote the extension by $\gamma_{v,n} \in \mathcal{O}(GV_n, E)$. For each $z \in GV_n$ the operator

$$\pi_n(z) : E_n \to E, \quad \pi_n(z)v := \gamma_{v,n}(z),$$

is linear. In particular, uniqueness implies

$$\pi_n(gz) = \pi(g)\pi_n(z)$$

for all $g \in G$, $z \in GV_n$. It is easily seen that if $m < n$, then $E_m \subset E_n$ and $\pi_m(z)v = \pi_n(z)v$ for $z \in GV_n$, $v \in E_m$. We shall omit the subscript n from the operator $\pi_n(z)$ if no confusion is possible.
A closely related space is

$$\tilde{E}_n = \{v \in E \mid \gamma_v|_{U_n} \text{extends holomorphically to } V_n\}.$$

Lemma 4.2.2. *The space of analytic vectors is given by the increasing unions*

$$E^\omega = \bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} \tilde{E}_n.$$

Furthermore,

$$E_n \subset \tilde{E}_n \subset E_{4n} \tag{2.1}$$

for all $n \in \mathbb{N}$.

Proof. The inclusions

$$\bigcup_{n\in\mathbb{N}} E_n \subset E^\omega \subset \bigcup_{n\in\mathbb{N}} \tilde{E}_n,$$

as well as the first inclusion in (2.1), are clear. Hence it suffices to prove the second inclusion in (2.1). Let $v \in V_n$ and let us denote the extension of γ_v by $f : V_n \to E$. For $g \in G$ and $z \in V_{4n}$ we define

$$F(gz) := \pi(g)f(z) \in E.$$

We need to show that the expression is well-defined. Assume $gz = g'z'$ with $g, g' \in G$ and $z, z' \in V_{4n}$. Then $g^{-1}g' = zz'^{-1} \in V_{2n}$, and hence $g^{-1}g'x \in V_n$ for all $x \in V_{2n}$. Since $\pi(g)\pi(g^{-1}g'x)v = \pi(g')\pi(x)v$ for $x \in G$, analytic continuation from U_{2n} implies $\pi(g)f(g^{-1}g'x) = \pi(g')f(x)$ for $x \in V_{2n}$. In particular, with $x = z'$ we obtain $\pi(g)f(z) = \pi(g')f(z')$, showing that F is well defined. As F is clearly holomorphic, we conclude that $v \in E_{4n}$. □

Next we want to topologize E^ω. For that we notice that the holomorphic extensions provide injections of E_n and $\tilde{E}_n$ into $\mathcal{O}(GV_n, E)$ and $\mathcal{O}(V_n, E)$, respectively. We topologize E_n and $\tilde{E}_n$ by means of these maps and the standard compact open topologies. It is easily seen that the inclusion maps $E_n \to E_{n+1} \to E$ and $\tilde{E}_n \to \tilde{E}_{n+1} \to E$ are all continuous. Furthermore:

Lemma 4.2.3. *The inclusion maps in (2.1) are continuous for all $n \in \mathbb{N}$.*

Proof. Identifying E_n and $\tilde{E}_n$ with the corresponding spaces of holomorphic functions, we obtain the following neighborhood bases of 0. In E_n, the members are all sets

$$W_{K,Z} := \{f \in E_n \mid f(K) \subset Z\},$$

where $K \subset GV_n$ is compact and $Z \subset E$ a zero neighborhood. Similarly in $\tilde{E}_n$, members are

$$\tilde{W}_{K,Z} := \{f \in \tilde{E}_n \mid f(K) \subset Z\},$$

where $K \subset V_n$ is compact and $Z \subset E$ a zero neighborhood. The continuity of the first inclusion is then obvious.
With the mentioned identifications, the second inclusion is given by the map $f \to F$ described in the previous proof. Let a neighborhood $W = W_{K,O} \subset E_{4n}$ be given. Let $K' \subset V_{4n}$ be an arbitrary compact neighborhood of 0. By compactness of $K \subset GV_{4n}$ we obtain a finite union $K \subset \cup g_i K' \subset GV_{4n}$. Let $O' = \cap \pi(g_i)^{-1}(O)$, then $\tilde{W} = \tilde{W}_{K',O'}$ is an open neighborhood of 0 in $\tilde{E}_n$, and $f \in \tilde{W} \Rightarrow F \in W$. □

We endow E^ω with the inductive limit topology of the ascending unions in Lemma 4.2.2. The Hausdorff property follows, since E is assumed to be Hausdorff. It follows from Lemma 4.2.3, that the two unions give rise to the same topology. In symbols:

$$E^\omega = \lim_{n\to\infty} E_n = \lim_{n\to\infty} \tilde{E}_n \subset E \tag{2.2}$$

with continuous inclusion into E. Since the restriction $\mathcal{O}(GV_n, E) \to C^\infty(G, E)$ is continuous for all $n \in \mathbb{N}$, we have $E^\omega \subset E^\infty$ with continuous inclusion.

Observe that an intertwining operator $T\colon E \to F$ between two representations $(\pi, E), (\rho, F)$ carries E^ω continuously into F^ω. In fact, if $v \in E_n$ with the holomorphically extended orbit map $z \mapsto \pi(z)v$, then $Tv \in F_n$ since $z \mapsto T\pi(z)v$ is a holomorphic extension of the orbit map $g \mapsto \rho(g)Tv = T\pi(g)v$. It follows that T maps E_n continuously into F_n for each n.
Notice that if we define a continuous action of G on $\mathcal{O}(GV_n, E)$ by

$$(g \cdot f)(z) := \pi(g)f(g^{-1}z) \qquad (g \in G, z \in GV_n),$$

then the image of $v \mapsto \pi_n(\cdot)v$ is the subspace $\mathcal{O}(GV_n, E)^G$ of G-invariant functions, with inverse map given by evaluation at $\mathbf{1}$. Thus E_n is identified with a *closed* subspace of $\mathcal{O}(GV_n, E)$. In particular, it follows (see [36], p. 365) that E_n is complete/Fréchet if E has this property.
Let us briefly recall the structure of the open neighborhoods of zero in the limit E^ω. If A is a subset of some vector space, then we write $\Gamma(A)$ for the convex hull of A. Now given for each n an open 0-neighborhood W_n in E_n (or $\tilde{E}_n$), the set

$$W := \Gamma(\bigcup_{n \in \mathbb{N}} W_n) \tag{2.3}$$

is an open convex neighborhood of 0 in E^ω. The set of neighborhoods W thus obtained form a filter base of the 0-neighborhoods in E^ω.

Proposition 4.2.4. *Let (π, E) be a representation of a Lie group on a topological vector space E. Then the following assertions hold:*

(i) The action $G \times E^\omega \to E^\omega$ is continuous, hence defines a representation (π, E^ω) of G.

(ii) Each $v \in E^\omega$ is an analytic vector for (π, E^ω) and

$$(E^\omega)^\omega = E^\omega$$

as topological vector spaces.

Proof. In (i) it suffices to prove continuity at $(\mathbf{1}, v)$ for each $v \in E^\omega$. We first prove the separate continuity of $g \mapsto \pi(g)v \in E^\omega$. Let $v \in E_n$, and consider the E-valued holomorphic extension of $g \mapsto \pi(g)v$. Since multiplication in $G_\mathbb{C}$ is holomorphic and $V_{2n} \cdot V_{2n} \subset V_n$, it follows that for each $z_1 \in V_{2n}$ the element $\pi_{2n}(z_1)v$ belongs to E_{2n}, with the holomorphic extension

$$z_2 \mapsto \pi_{2n}(z_2)\pi_{2n}(z_1)v := \pi_n(z_2 z_1)v \qquad (z_1, z_2 \in GV_{2n},\ v \in E_n) \tag{2.4}$$

of the orbit map. In particular, (2.4) holds for $z_1 = g \in U_{2n}$. The element $\pi(z_2 g)v \in E$ depends continuously on g, locally uniformly with respect to z_2. It follows that $g \mapsto \pi(g)v$ is continuous $U_{2n} \to E_{2n}$, hence into E^ω.

In order to conclude the full continuity of (i) it now suffices to establish the following:
(*) *For all compact subsets* $B \subset G$ *the operators* $\{\pi(g) \mid g \in B\}$ *form an equicontinuous subset of* $\mathrm{End}(E^\omega)$.
Before proving this, we note that for every compact subset $B \subset G$ and every $m \in \mathbb{N}$ there exists $n > m$ such that

$$b^{-1} V_n b \subset V_m \quad (b \in B).$$

This follows from the continuity of the adjoint action. Then $zb \in GV_m$ for all $z \in GV_n$, and hence $\pi(b)v \in E_n$ for all $b \in B$, $v \in E_m$ with

$$\pi_n(z)\pi(b)v = \pi_m(zb)v. \tag{2.5}$$

In order to prove (*) we fix a compact set $B \subset G$. Given $m \in N$ we choose $n > m$ as above. We are going to prove equicontinuity $B \times E_m \to E_n$. An open neighborhood of 0 in E_n can be assumed of the form

$$(K, Z) := \{f \in E_n \mid f(K) \subset Z\},$$

where $K \subset GV_n$ is compact and $Z \subset E$ a zero neighborhood. Then with $K' = \cup_{b \in B} b^{-1}Kb$ and $Z' = \cap_{b \in B} \pi(b)^{-1}(Z)$ we obtain

$$f(K') \subset Z' \Rightarrow \pi(b)f(b^{-1}Kb) \subset Z$$

for all $b \in B$ and all functions $f : GV_m \to E$. If in addition f is G-invariant, then the conclusion is $f(Kb) \subset Z$, and we have shown that the right translation by b maps the zero neighborhood (K', Z') in E_m into the zero neighborhood (K, Z) in E_n.
The equicontinuity $B \times E^\omega \to E^\omega$ is an easy consequence given the description (2.3) of the neighborhoods in the inductive limit. This completes the proof of (i).
For the proof of (ii), let $v \in E_n$. In the first part of the proof we saw that $\pi(z_1)v \in E_{2n}$ for each $z_1 \in V_{2n}$, with the holomorphically extended orbit map given by (2.4). It then follows from Lemma 4.8.1, applied to $V_{2n} \times GV_{2n}$ and the map $(z_1, z_2) \mapsto \pi(z_2 z_1)v$, that $z_1 \mapsto \pi(\cdot)\pi(z_1)v$ is holomorphic $V_{2n} \to \mathcal{O}(GV_{2n}, E)$. Hence $z_1 \mapsto \pi(z_1)v$ is holomorphic into E_{2n}, hence also into E^ω. Thus $g \mapsto \pi(g)v$ extends to a holomorphic E^ω-valued map on V_{2n}, and hence $v \in (E^\omega)^\omega$ by the second description in (2.2).
For the topological statement in (ii), we need to show that the identity map is continuous $E^\omega \to (E^\omega)^\omega$. We just saw that the identity map takes

$$E_n \to \widetilde{(E^\omega)}_{2n},$$

hence it suffices to show continuity of this map for each n. The proof given above reduces to the statement that the map mentioned below (8.1) is continuous. $\square$

Corollary 4.2.5. $(E^\infty)^\omega = (E^\omega)^\infty = E^\omega$ *as topological vector spaces.*

Proof. The continuous inclusions $E^\omega \subset E^\infty \subset E$ induce continuous inclusions $E^\omega = (E^\omega)^\omega \subset (E^\infty)^\omega \subset E^\omega$. With E replaced by E^ω, the same inclusions also imply $(E^\omega)^\omega \subset (E^\omega)^\infty \subset E^\omega$. □

We are interested in the functorial properties of the construction.

Lemma 4.2.6. *Let (π, E) be a representation, and let $F \subset E$ be a closed invariant subspace. Then*

(i) $F^\omega = E^\omega \cap F$ as a topological space,

(ii) $E^\omega/F^\omega \subset (E/F)^\omega$ continuously.

Proof. (i) Obviously $F_n \subset E_n$ for all n. Conversely, if $v \in E_n \cap F$ with holomorphically extended orbit map $z \mapsto \pi(z)v \in E$, then $\pi(g)v \in F$ for all $g \in G$ implies $\pi(z)v \in F$ for all $z \in GV_n$. Hence $v \in F_n$. The topological statement follows easily.
(ii) The quotient map induces a continuous map $E^\omega \to (E/F)^\omega$, which in view of (i) induces the mentioned continuous inclusion. □

Notice also that if E_1, E_2 are representations, then the product representations satisfy $E_1^\omega \times E_2^\omega \simeq (E_1 \times E_2)^\omega$.

4.3 Algebras of superexponentially decaying functions

We define a convolution algebra of analytic functions with fast decay. The purpose is to obtain an algebra which acts on representations of restricted growth, such as F-representations.
Let us denote by dg the Riemannian measure on G associated to the metric $\mathbf{g}$ and note that dg is a left Haar measure. It is of some relevance below that there is a constant $c > 0$ such that

$$\int_G e^{-cd(g)}\, dg < \infty \tag{3.1}$$

(see [25], p. 75, Lemme 2).
We define the space of superdecaying continuous function on G by

$$\mathcal{R}(G) := \{f \in C(G) \mid \forall N \in \mathbb{N} : \sup_{g \in G} |f(g)| e^{Nd(g)} < \infty\}$$

and equip it with the corresponding family of seminorms. Note that $\mathcal{R}(G)$ is independent of the choice of the left-invariant metric, and that it has the following properties:

Proposition 4.3.1.

(i) $\mathcal{R}(G)$ *is a Fréchet space and the natural action of* $G \times G$ *by left-right displacements defines an F-representation,*

(ii) $\mathcal{R}(G)$ *becomes a Fréchet algebra under convolution:*

$$f * h(x) = \int_G f(y)h(y^{-1}x)\ dy$$

for $f, h \in \mathcal{R}(G)$ *and* $x \in G$*,*

(iii) Every F-representation (π, E) *of* G *gives rise to a continuous algebra representation of* $\mathcal{R}(G)$*,*

$$\mathcal{R}(G) \times E \to E, \quad (f, v) \mapsto \Pi(f)v,$$

where

$$\Pi(f)v := \int_G f(g)\pi(g)v\ dg \qquad (f \in \mathcal{R}(G), v \in E)$$

as an E-valued integral.

Proof. Easy. Use (1.1), (1.2) and (3.1). □

Our concern is now with the analytic vectors of $(L \otimes R, \mathcal{R}(G))$. We set $\mathcal{A}_{LR}(G) := \mathcal{R}(G)^\omega$ and record that

$$\mathcal{A}_{LR}(G) = \lim_{U \to \{\mathbf{1}\}} \mathcal{R}(G)_U,$$

where

$$\mathcal{R}(G)_U = \left\{ \varphi \in \mathcal{O}(UGU) \mid \forall Q \Subset U\ \forall n \in \mathbb{N} : \sup_{g \in G} \sup_{q_1, q_2 \in Q} |\varphi(q_1 g q_2)|\ e^{nd(g)} < \infty \right\}.$$

It is clear that $\mathcal{A}_{LR}(G)$ is a subalgebra of $\mathcal{R}(G)$ and that

$$\Pi(\mathcal{A}_{LR}(G))\ E \subset E^\omega$$

whenever (π, E) is an F-representation.
In the next chapter, these and further properties will be verified for the very similar space of analytic vectors for $(L, \mathcal{R}(G))$.

4.4 Some geometric analysis on Lie groups

Let us denote by $\mathcal{V}(G)$ the space of left-invariant vector fields on G. It is common to identify γ with $\mathcal{V}(G)$ where $X \in \gamma$ corresponds to the vector field $\widetilde{X}$ given by

$$(\widetilde{X} f)(g) = \frac{d}{dt}\Big|_{t=0} f(g \exp(tX)) \qquad (g \in G, f \in C^\infty(G)) .$$

We note that the adjoint of $\widetilde{X}$ on the Hilbert space $L^2(G)$ is given by

$$\widetilde{X}^* = -\widetilde{X} - \operatorname{tr}(\operatorname{ad} X) .$$

Note that $\widetilde{X}^* = -\widetilde{X}$ in case γ is unimodular. Let us fix an an orthonormal basis $X_1, \ldots, X_n$ of γ with respect to $\mathbf{g}$. Then the Laplace–Beltrami operator $\Delta = d^* d$ associated to $\mathbf{g}$ is given explicitly by

$$\Delta = \sum_{j=1}^{n} (-\widetilde{X_j} - \operatorname{tr}(\operatorname{ad} X_j)) \; \widetilde{X_j} .$$

As $(G, \mathbf{g})$ is complete, Δ is essentially selfadjoint. We denote by

$$\sqrt{\Delta} = \int \lambda \; dP(\lambda)$$

the corresponding spectral resolution. It provides us with a measurable functional calculus, which allows to define

$$f(\sqrt{\Delta}) = \int f(\lambda) \; dP(\lambda)$$

as an unbounded operator $f(\sqrt{\Delta})$ on $L^2(G)$ with domain

$$D(f(\sqrt{\Delta})) = \left\{ \varphi \in L^2(G) \mid \int |f(\lambda)|^2 \; d\langle P(\lambda)\varphi, \varphi\rangle < \infty \right\} .$$

Let $c, \vartheta > 0$. We are going to apply the above calculus to functions in the space

$$\mathcal{F}_{c,\vartheta} = \left\{ \varphi \in \mathcal{O}(\mathbb{C}) \mid \forall N \in \mathbb{N} : \sup_{z \in \mathcal{W}_{N,\vartheta}} |\varphi(z)| \; e^{c|z|} < \infty \right\} ,$$

$$\mathcal{W}_{N,\vartheta} = \{ z \in \mathbb{C} \mid |\operatorname{Im} z| < N \} \cup \{ z \in \mathbb{C} \mid |\operatorname{Im} z| < \vartheta |\operatorname{Re} z| \} .$$

The resulting operators are bounded on $L^2(G)$ and given by a symmetric and left invariant integral kernel $K_f \in C^\infty(G \times G)$. Hence there exists a convolution kernel $\kappa_f \in C^\infty(G)$ with $\kappa_f(x) = \overline{\kappa_f(x^{-1})}$ such that $K_f(x, y) = \kappa_f(x^{-1}y)$, and for all $x \in G$:

$$f(\sqrt{\Delta}) \; \varphi = \int_G K_f(x, y) \; \varphi(y) \; dy = \int_G \kappa_f(y^{-1}x) \; \varphi(y) \; dy = (\varphi * \kappa_f)(x).$$

In the special case $G = (\mathbb{R}, +)$ we have for an even function f:

$$f(\sqrt{\Delta})\ \varphi = f(i\partial_x)\ \varphi = (f\ \hat{\varphi})\check{} = \varphi * \check{f},$$

where $\hat{}$ and $\check{}$ denote the Fourier transform resp. its inverse, so that $\kappa_f = \check{f}$ in this case.
The results of Cheeger, Gromov and Taylor [12] show that certain estimates satisfied by the Fourier transform carry over to κ_f for arbitrary Lie groups:

Theorem 4.4.1. *Let $c, \vartheta > 0$ and $f \in \mathcal{F}_{c,\vartheta}$ even. Then $\kappa_f \in \mathcal{R}(G)$.*

We are going to need an analytic variant of their result

Theorem 4.4.2. *Under the assumptions of the previous theorem: $\kappa_f \in \mathcal{A}_{LR}(G)$.*

The proof relies on some basic properties of the wave equation

$$\partial_t^2\varphi + \Delta\varphi = 0, \quad \varphi(0) = \varphi_0, \quad \partial_t|_{t=0}\varphi(0) = 0$$

in $L^2(G)$. The equation is formally solved in terms of the functional calculus by $\cos(t\sqrt{\Delta})\varphi_0$, and two key facts about the solution are the boundedness of $\cos(t\sqrt{\Delta})$ as an operator on $L^2(G)$ with norm ≤ 1 and the finite speed of propagation. The former follows from the functional calculus, but also Stone's theorem stating that unitary semigroups are precisely the exponentials of skew–adjoint operators can be employed:

$$\|\cos(t\sqrt{\Delta})\|_{\mathcal{L}(L^2(G))} = \frac{1}{2}\|e^{it\sqrt{\Delta}} + e^{-it\sqrt{\Delta}}\|_{\mathcal{L}(L^2(G))} \leq 1\ .$$

Finite speed of propagation, on the other hand, is a local variant of the conservation of energy and assures that the support of $\cos(t\sqrt{\Delta})\varphi_0$ will be contained in

$$\{x \in G \mid d(x, \operatorname{supp}\ \varphi_0) \leq t\}\ .$$

Therefore, the Schwartz kernel of $\cos(t\sqrt{\Delta})$ is supported in a strip of width $2t$ around the diagonal in $G \times G$. For example,

$$\cos(t\sqrt{\Delta})\varphi_0(x) = \frac{1}{2}(\varphi_0(x+t) + \varphi_0(x-t))$$

when $G = (\mathbb{R}, +)$.

Proof. We only have to establish local regularity, as the decay at infinity is already contained in [12].
The Fourier inversion formula allows to express κ_f as an integral of the wave kernel:

$$\kappa_f(\cdot) = K_f(\cdot, \mathbf{1}) = f(\sqrt{\Delta})\ \delta_{\mathbf{1}} = \int_{\mathbb{R}} \hat{f}(\lambda)\ \cos(\lambda\sqrt{\Delta})\ \delta_{\mathbf{1}}\ d\lambda.$$

As we would like to employ $\| \cos(\lambda\sqrt{\Delta})\|_{\mathcal{L}(L^2(G))} \leq 1$, we cut off a fundamental solution of Δ^k to write

$$\delta_{\mathbf{1}} = \Delta^k \varphi + \psi$$

for a fixed $k > \frac{1}{4}\dim(G)$ and some compactly supported $\varphi, \psi \in L^2$. Hence,

$$\begin{aligned} \Delta^l \kappa_f(\cdot) &= \int_{\mathbb{R}} \lambda^{2k+2l}\ \hat{f}(\lambda)\ \cos(\lambda\sqrt{\Delta})\ \varphi\ d\lambda + \int_{\mathbb{R}} \lambda^{2l}\ \hat{f}(\lambda)\ \cos(\lambda\sqrt{\Delta})\ \psi\ d\lambda \\ &= \int_{\mathbb{R}} \hat{f}^{(2k+2l)}(\lambda)\ \cos(\lambda\sqrt{\Delta})\ \varphi\ d\lambda + \int_{\mathbb{R}} \hat{f}^{(2l)}(\lambda)\ \cos(\lambda\sqrt{\Delta})\ \psi\ d\lambda. \end{aligned}$$

In Appendix A we show the following inequality for all $n \in \mathbb{N}$ and some constants $C_n, R > 0$:

$$|\hat{f}^{(l)}(\lambda)| \leq C_n\ l!\ R^l e^{-n|\lambda|}.$$

Using $\| \cos(\lambda\sqrt{\Delta})\|_{\mathcal{L}(L^2(G))} \leq 1$ and the Sobolev inequality, we obtain

$$|\Delta^l \kappa_f(\cdot)| \leq C_1\ (2l)!\ S^{2l}$$

for some $S > 0$. A classical result by Goodman [54] now implies that κ_f extends to holomorphic function on a complex neighborhood U of $\mathbf{1}$. By equivariance, $\kappa_f \in \mathcal{O}(GU)$. Left analyticity follows from $\kappa_f(x) = \overline{\kappa_f(x^{-1})}$, and Browder's theorem (Theorem 3.3.3 in [38]) then implies joint analyticity. The decay at infinity follows from [12]. □

Remark 4.4.3. Instead of citing the weaker results by Goodman, a slightly longer, possibly more self–contained proof could be obtained from standard theorems for analytic partial differential equations along the lines of [12].

Remark 4.4.4. We briefly sketch the analogy between κ_f and the Fourier transform $\hat{f}$ for more general functions, leading to a heuristic dictionary between the qualitative properties of f and κ_f. As above, the crucial idea is to express $f(\sqrt{\Delta})$ in terms of the wave kernel:

$$f(\sqrt{\Delta}) = \int_{\mathbb{R}} \hat{f}(\lambda)\ \cos(\lambda\sqrt{\Delta})\ d\lambda.$$

If f does not decay rapidly at infinity, κ_f will typically have a singularity at $\mathbf{1}$, as is the case for differential operators. In the integral representation, the singular behavior arises as the contribution of λ in a neighborhood of 0, or the short–time behavior of the wave kernel. One thus decomposes

$$f(\sqrt{\Delta}) = \int_{|\lambda|>\varepsilon} \hat{f}(\lambda)\ \cos(\lambda\sqrt{\Delta})\ d\lambda + \int_{|\lambda|<\varepsilon} \hat{f}(\lambda)\ \cos(\lambda\sqrt{\Delta})\ d\lambda$$

for small $\varepsilon > 0$. The finite speed of propagation for the wave equation assures that the second term has a properly supported integral kernel and does not contribute to the behavior of κ_f at infinity. The singular behavior at $\mathbf{1}$ can be further

analyzed if we replace $\cos(\lambda\sqrt{\Delta})$ by a short–time parametrix, which locally resembles the one from $\mathbb{R}^n$ [12]. On the other hand, the first term $\|\cos(\lambda\sqrt{\Delta})\|$ is estimated by 1 so that, assuming sufficient integrability, the decay of $\hat{f}$ carries over to κ_f.
The following heuristic picture emerges, mimicking what is known for $\hat{f}$: If f is smooth, κ_f is going to have superpolynomial decay at infinity. In the above case, when f is holomorphic on a strip, κ_f will decay exponentially corresponding to the width of the strip. And if f is a Paley–Wiener function, i.e. the Fourier transform of a compactly supported smooth function, also κ_f will have compact support. Conversely, the growth of f at infinity translates into local regularity: The polynomial bounds of pseudodifferential symbols translate into finite–order distributions with singular support at $\mathbf{1}$, and superpolynomial or exponential decay gives rise to smooth resp. holomorphic convolution kernels. The first of these cases will be used to obtain a strong factorization result for test functions in Section 4.6.

Regularized distance function

In the last part of this section we are going to discuss a holomorphic regularization of the distance function. Later on this will be used to construct certain holomorphic replacements for cut-off functions.
Consider the time–1 heat kernel $\varrho := \kappa_{e^{-\lambda^2}}$ and define $\tilde{d}$ on G by

$$\tilde{d}(g) := e^{-\Delta} d(g) = \int_G \varrho(x^{-1}g)\ d(x)\ dx.$$

Lemma 4.4.5. *There exist $U \in \mathcal{U}_{\mathbb{C}}$ and a constant $C_U > 0$ such that $\tilde{d} \in \mathcal{O}(GU)$ and for all $g \in G$ and all $u \in U$*

$$|\tilde{d}(gu) - d(g)| \leq C_U.$$

Proof. According to Theorem 4.4.2 the heat kernel ϱ admits an analytic continuation to a superexponentially decreasing function on GU for some bounded $U \in \mathcal{U}_{\mathbb{C}}$. This allows to extend $\tilde{d}$ to GU. To prove the inequality, we consider the integral

$$\bar{\varrho}(y) = \int_G \varrho(x^{-1}y)\ dx$$

as a holomorphic function of $y \in GU$. By the left invariance of the Haar measure and the normalization of the heat kernel, $\bar{\varrho} = 1$ on G, and hence on GU. Recall the triangle inequality on G: $|d(x) - d(g)| \leq d(x^{-1}g)$. This implies the uniform bound

$$\begin{aligned}\left|\tilde{d}(gu) - d(g)\right| &= \left|\int_G \varrho(x^{-1}gu)\ (d(x) - d(g))\ dx\right| \\ &\leq \int_G \left|\varrho(x^{-1}gu)\right|\ d(x^{-1}g)\ dx \\ &\leq \sup_{v \in U} \int_G \left|\varrho(x^{-1}v)\right|\ d(x^{-1})\ dx.\end{aligned}$$

□

4.5 Proof of the factorization theorem

Let (π, E) be a representation of G on a sequentially complete locally convex Hausdorff space and consider the Laplacian as an element

$$\Delta = \sum_{j=1}^{n} (-X_j - \mathrm{tr}(\mathrm{ad}\, X_j))\ X_j$$

of the universal enveloping algebra of γ. A vector $v \in E$ will be called Δ-*analytic*, if there exists $\varepsilon > 0$ such that for all continuous seminorms p on E one has

$$\sum_{j=0}^{\infty} \frac{\varepsilon^j}{(2j)!}\ p(\Delta^j v) < \infty\,.$$

Lemma 4.5.1. *Let E be a sequentially complete locally convex Hausdorff space and $\varphi \in \mathcal{O}(U, E)$ for some $U \in \mathcal{U}_{\mathbb{C}}$. Then there exists $R = R(U) > 0$ such that for all continuous semi-norms p on E there exists a constant C_p such that*

$$p\left(\left(\widetilde{X_{i_1}} \cdots \widetilde{X_{i_k}} \varphi\right)(\mathbf{1})\right) \leq C_p\ k!\ R^k$$

for all $(i_1, \ldots, i_k) \in \mathbb{N}^k$, $k \in \mathbb{N}$.

Proof. There exists a small neighborhood of 0 in γ in which the mapping

$$\Phi : \gamma \to E,\ X \mapsto \varphi(\exp(X)),$$

is analytic. Let $X = t_1 X_1 + \cdots + t_n X_n$. Because E is sequentially complete, Φ can be written for small X and t as

$$\Phi(X) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} \left(\widetilde{X_{\alpha_1}} \cdots \widetilde{X_{\alpha_k}} \varphi\right)(\mathbf{1})\, t^{\alpha}.$$

As this series is absolutely summable, there exists a $R > 0$ such that for every continuous semi-norm p on E there is a constant C_p with

$$p\left(\left(\widetilde{X_{i_1}} \cdots \widetilde{X_{i_k}} \varphi\right)(\mathbf{1})\right) \leq C_p \; k! \; R^k$$

for all $(i_1, \ldots, i_k) \in \mathbb{N}^k$, $k \in \mathbb{N}$. □

As a consequence we obtain:

Lemma 4.5.2. *Let (π, E) be a representation of G on some sequentially complete locally convex Hausdorff space E. Then analytic vectors are Δ–analytic.*

In Corollary 4.5.6 we will see that the converse holds for F-representations.
Let (π, E) be an F-representation of G. Then for each $n \in \mathbb{N}$ there exist $c_n, C_n > 0$ such that

$$\|\pi(g)\|_n \leq C_n \cdot e^{c_n d(g)} \qquad (g \in G),$$

where

$$\|\pi(g)\|_n := \sup_{\substack{p_n(v) \leq 1 \\ v \in E}} p_n(\pi(g)v) \, .$$

For $U \in \mathcal{U}_{\mathbb{C}}$ and $n \in \mathbb{N}$ we set

$$\mathcal{F}_{U,n} = \left\{ \varphi \in \mathcal{O}(GU, E) \mid \forall Q \Subset U \; \forall \varepsilon > 0 : \; \sup_{g \in G} \sup_{q \in Q} p_n(\varphi(gq)) \; e^{-(c_n + \varepsilon) d(g)} < \infty \right\}.$$

We are also going to need the subspace of superexponentially decaying functions in $\bigcap_n \mathcal{F}_{U,n}$:

$$\mathcal{R}(GU, E) = \left\{ \varphi \in \mathcal{O}(GU, E) \mid \forall Q \Subset U \; \forall n, N \in \mathbb{N} : \; \sup_{g \in G} \sup_{q \in Q} p_n(\varphi(gq)) \; e^{N d(g)} < \infty \right\}.$$

We record:

Lemma 4.5.3. *If $\kappa \in \mathcal{A}_{LR}(G)_V$, then right convolution with κ is a bounded operator from $\mathcal{F}_{U,n}$ to $\mathcal{F}_{V,n}$ for all $n \in \mathbb{N}$.*

We denote by $\mathcal{C}_\varepsilon$ the power series expansion $\sum_{j=0}^{\infty} \frac{\varepsilon^{2j}}{(2j)!} \Delta^j$ of $\cosh(\varepsilon \sqrt{\Delta})$. Note the following consequence of Lemma 4.5.1:

Lemma 4.5.4. *Let $U, V \in \mathcal{U}_{\mathbb{C}}$ such that $V \Subset U$. Then there exists $\varepsilon > 0$ such that $\mathcal{C}_\varepsilon$ is a bounded operator from $\mathcal{F}_{U,n}$ to $\mathcal{F}_{V,n}$ for all $n \in \mathbb{N}$.*

As in Appendix A, consider the functions $\alpha_\varepsilon(z) = 2e^{-\varepsilon z\,\mathrm{erf}(z)}$ and $\beta_\varepsilon(z) = 1 - \alpha_\varepsilon(z)\cosh(\varepsilon z)$, which belong to the space $\mathcal{F}_{2\varepsilon,\vartheta}$. We would like to substitute $\sqrt{\Delta}$ into our key identity (4.7.3)

$$\alpha_\varepsilon(z)\cosh(\varepsilon z) + \beta_\varepsilon(z) = 1$$

and replace the hyperbolic cosine by its Taylor expansion.

Lemma 4.5.5. *Let $U \in \mathcal{U}_\mathbb{C}$. Then there exist $\varepsilon > 0$ and $V \subset U$ such that for any $\varphi \in \mathcal{F}_{U,n}$, $n \in \mathbb{N}$,*

$$\mathcal{C}_\varepsilon(\varphi) * \kappa^\varepsilon_\alpha + \varphi * \kappa^\varepsilon_\beta = \varphi$$

holds as functions on GV.

Proof. Note that $\kappa^\varepsilon_\alpha, \kappa^\varepsilon_\beta \in \mathcal{A}_{LR}(G)$ according to Theorem 4.4.2. We first consider the case $E = \mathbb{C}$ and $\varphi \in L^2(G)$. With $|\alpha_\varepsilon(z)\cosh(\varepsilon z)|$ being bounded, $\cosh(\varepsilon\sqrt{\Delta})$ maps its domain into the domain of $\alpha_\varepsilon(\sqrt{\Delta})$, and the rules of the functional calculus ensure

$$\varphi - \beta_\varepsilon(\sqrt{\Delta})\varphi = (\alpha_\varepsilon(\cdot)\cosh(\varepsilon\cdot))(\sqrt{\Delta})\varphi = (\cosh(\varepsilon\sqrt{\Delta})\varphi) * \kappa^\varepsilon_\alpha$$

in $L^2(G)$ for all $\varphi \in D(\cosh(\varepsilon\sqrt{\Delta}))$. For such φ, the partial sums of $\mathcal{C}_\varepsilon\varphi$ converge to $\cosh(\varepsilon\sqrt{\Delta})\varphi$ in $L^2(G)$, and hence almost everywhere. Indeed,

$$\begin{aligned}
&\left\| \cosh(\varepsilon\sqrt{\Delta})\varphi - \sum_{j=0}^{N} \frac{\varepsilon^{2j}}{(2j)!}\Delta^j\varphi \right\|^2_{L^2(G)} \\
&= \int \left\langle dP(\lambda)\left(\cosh(\varepsilon\sqrt{\Delta})\varphi - \sum_{j=0}^{N}\frac{\varepsilon^{2j}}{(2j)!}\,\Delta^j\varphi\right), \cosh(\varepsilon\sqrt{\Delta})\varphi - \sum_{k=0}^{N}\frac{\varepsilon^{2k}}{(2k)!}\,\Delta^k\varphi \right\rangle \\
&= \int \left(\cosh(\varepsilon\lambda) - \sum_{k=0}^{N}\frac{(\varepsilon\lambda)^{2k}}{(2k)!}\right)^2 \langle dP(\lambda)\varphi,\varphi\rangle \\
&= \sum_{j,k=N+1}^{\infty} \int \frac{(\varepsilon\lambda)^{2j}}{(2j)!}\,\frac{(\varepsilon\lambda)^{2k}}{(2k)!}\,\langle dP(\lambda)\varphi,\varphi\rangle\ ,
\end{aligned}$$

and the right hand side tends to 0 for $N \to \infty$, because

$$\sum_{j,k=0}^{\infty} \int \frac{(\varepsilon\lambda)^{2j}}{(2j)!}\,\frac{(\varepsilon\lambda)^{2k}}{(2k)!}\,\langle dP(\lambda)\varphi,\varphi\rangle = \int \cosh(\varepsilon\lambda)^2\langle dP(\lambda)\varphi,\varphi\rangle < \infty\ .$$

In particular, given $\varphi \in \mathcal{R}(GU, E)$ and $\lambda \in E'$, we obtain $\mathcal{C}_\varepsilon\lambda(\varphi) = \cosh(\varepsilon\sqrt{\Delta})\lambda(\varphi)$ almost everywhere and

$$\mathcal{C}_\varepsilon(\lambda(\varphi)) * \kappa^\varepsilon_\alpha + \lambda(\varphi) * \kappa^\varepsilon_\beta = \lambda(\varphi)$$

as analytic functions on G for sufficiently small $\varepsilon > 0$.
Since the above identity holds for all $\lambda \in E'$, we obtain

$$\mathcal{C}_\varepsilon(\varphi) * \kappa_\alpha^\varepsilon + \varphi * \kappa_\beta^\varepsilon = \varphi$$

on any connected domain GV, $\mathbf{1} \in V \subset U$, on which the left hand side is holomorphic.
Recall the regularized distance function $\tilde{d}(g) = e^{-\Delta} d(g)$ from Lemma 4.4.5, and set $\chi_\delta(g) := e^{-\delta \tilde{d}(g)^2}$ $(\delta > 0)$. Given $\varphi \in \mathcal{F}_{U,n}$, $\chi_\delta \varphi \in \mathcal{R}(GU, E)$ and

$$\mathcal{C}_\varepsilon(\chi_\delta \varphi) * \kappa_\alpha^\varepsilon + (\chi_\delta \varphi) * \kappa_\beta^\varepsilon = \chi_\delta \varphi \,.$$

The limit $\chi_\delta \varphi \to \varphi$ in $\mathcal{F}_{U,n}$ as $\delta \to 0$ is easily verified. From Lemma 4.5.3 we also get $(\chi_\delta \varphi) * \kappa_\beta^\varepsilon \to \varphi * \kappa_\beta^\varepsilon$ as $\delta \to 0$. Finally Lemma 4.5.3 and Lemma 4.5.4 imply

$$\mathcal{C}_\varepsilon(\chi_\delta \varphi) * \kappa_\alpha^\varepsilon \to \ \mathcal{C}_\varepsilon(\varphi) * \kappa_\alpha^\varepsilon \quad (\delta \to 0).$$

The assertion follows. □

Proof of Theorem 4.0.4. Given $v \in E^\omega$, the orbit map γ_v belongs to $\bigcap_n \mathcal{F}_{U,n}$ for some $U \in \mathcal{U}_\mathbb{C}$. Applying Lemma 4.5.5 to the orbit map and evaluating at $\mathbf{1}$ we obtain the desired factorization

$$v = \gamma_v(\mathbf{1}) = \Pi(\kappa_\alpha^\varepsilon)\,(\mathcal{C}_\varepsilon(\gamma_v)(\mathbf{1})) + \Pi(\kappa_\beta^\varepsilon)\,(\gamma_v(\mathbf{1}))\,.$$

□

Note the following generalization of a theorem by Goodman for unitary representations [29, 54].

Corollary 4.5.6. *Let (π, E) be an F-representation. Then every Δ–analytic vector is analytic.*

Remark 4.5.7. a) A further consequence of our Theorem 4.0.4 is a simple proof of the fact that the space of analytic vectors for a Banach representation is complete.
b) We can also substitute $\sqrt{\Delta}$ into Dixmier's and Malliavin's presentation of the constant function 1 on the real line [21]. This invariant refinement of their argument shows that the smooth vectors for a Fréchet representation are precisely the vectors in the domain of Δ^k for all $k \in \mathbb{N}$.

4.6 Related problems

We conclude this chapter with a discussion of how our techniques can be modified to deal with a number of similar questions.
In the context of the introduction, given a nonunital algebra $\mathcal{A}$, a category $\mathcal{C}$ of $\mathcal{A}$–modules is said to have the *strong factorization property* if for all $\mathcal{M} \in \mathcal{C}$,

$$\mathcal{M} = \{am \ \mid \ a \in \mathcal{A}, \ m \in \mathcal{M}\}.$$

A strong factorization of test functions

Our methods may be applied to solve a related strong factorization problem for test functions. On $\mathbb{R}^n$ the Fourier transform allows to write a test function $\varphi \in C_c^\infty(\mathbb{R}^n)$ as the convolution $\psi * \Psi$ of two Schwartz functions, and [49] posed the natural problem whether one could demand $\psi, \Psi \in \mathcal{R}(\mathbb{R}^n)$. We are going to prove this in a more general setting.

Theorem 4.6.1. *For every real Lie group G*

$$C_c^\infty(G) \subset \{\psi * \Psi \ \mid \ \psi, \Psi \in \mathcal{R}(G)\}.$$

The proof is analogous to our main factorization theorem, and we only outline the argument. As above, we first regularize an appropriate distance function and set

$$\lambda(z) = \frac{1}{\sqrt{\pi}} e^{-z^2} * \log(1+|z|).$$

Lemma 4.6.2. *The function $\lambda(z)$ is entire and approximates* $\log(1+|z|)$ *in the sense that for all $N > 0$, $\vartheta \in (0,1)$ there exists a constant $C_{N,\vartheta}$ such that*

$$|\lambda(z) - \log(1+|z|)| \leq C_{N,\vartheta} \quad (z \in \mathcal{W}_{N,\vartheta}).$$

Let $m \in \mathbb{N}$. We would like to substitute the square root of the Laplacian associated to a left invariant metric G into a decomposition

$$1 = \widehat{\psi}_m(z)\ \widehat{\Psi}_m(z)$$

of the identity. In the current situation we use $\widehat{\psi}_m(z) = e^{-m\lambda(z)}$ and $\widehat{\Psi}_m(z) = e^{m\lambda(z)}$. Denote the convolution kernels of $\widehat{\psi}_m(\sqrt{\Delta})$ and $\widehat{\Psi}_m(\sqrt{\Delta})$ by ψ_m resp. Ψ_m. As mentioned in Remark 4.4.4, the ideas from the proof of Theorem 4.4.2 can be combined with the results of [12] to give:

Lemma 4.6.3. *Let $\chi \in C_c^\infty(G)$ with $\chi = 1$ in a neighborhood of* $\mathbf{1}$. *Then $\chi\Psi_m$ is a compactly supported distribution of order m and $(1-\chi)\Psi_m \in \mathcal{R}(G) \cap C^\infty(G)$. Given $k \in \mathbb{N}$, $\psi_m \in \mathcal{R}(G) \cap C^k(G)$ for sufficiently large m.*

Therefore $\widehat{\Psi}_m(\sqrt{\Delta})$ maps $C_c^\infty(G)$ to $\mathcal{R}(G)$. The functional calculus leads to a factorization

$$\mathrm{Id}_{C_c^\infty(G)} = \widehat{\psi}_m(\sqrt{\Delta})\ \widehat{\Psi}_m(\sqrt{\Delta})$$

of the identity, and in particular for any $\varphi \in C_c^\infty(G)$,

$$\varphi = (\widehat{\Psi}_m(\sqrt{\Delta})\ \varphi) * \psi_m \in \mathcal{R}(G) * \mathcal{R}(G).$$

Strong factorization of $\mathcal{A}_{LR}(G)$

It might be possible to strengthen Theorem 4.0.4 by showing that the analytic vectors have the strong factorization property.

Conjecture 4.6.4. *For any F-representation (π, E) of a real Lie group G,*

$$E^\omega = \{\Pi(\varphi)v \;\mid\; \varphi \in \mathcal{A}_{LR}(G),\ v \in E^\omega\}.$$

We provide some evidence in support of this conjecture and verify it for Banach representations of $(\mathbb{R}, +)$ using hyperfunction techniques.

Lemma 4.6.5. *The conjecture holds for every Banach representation of* $(\mathbb{R}, +)$.

Proof. Let (π, E) be a representation of $\mathbb{R}$ on a Banach space $(E, \|\cdot\|)$. Then there exist constants $c, C > 0$ such that $\|\pi(x)\| \leq Ce^{c|x|}$ for all $x \in \mathbb{R}$. If $v \in E^\omega$, there exists $R > 0$ such that the orbit map γ_v extends holomorphically to the strip $S_R = \{z \in \mathbb{C} \mid \operatorname{Im} z \in (-R, R)\}$. Let

$$\begin{aligned} \mathcal{F}_+(\gamma_v)(z) &= \int_{-\infty}^{0} \gamma_v(t)e^{-itz}\, dt, \quad \operatorname{Im} z > c, \\ -\mathcal{F}_-(\gamma_v)(z) &= \int_{0}^{\infty} \gamma_v(t)e^{-itz}\, dt, \quad \operatorname{Im} z < -c. \end{aligned}$$

Define the Fourier transform $\mathcal{F}(\gamma_v)$ of γ_v by

$$\mathcal{F}(\gamma_v)(x) = \mathcal{F}_+(\gamma_v)(x + 2ic) - \mathcal{F}_-(\gamma_v)(x - 2ic).$$

Note that $\|\mathcal{F}(\gamma_v)(x)\| e^{r|x|}$ is bounded for every $r < R$. Let $g(z) := \frac{Rz}{2}\operatorname{erf}(z)$ and write $\mathcal{F}(\gamma_v)$ as

$$\mathcal{F}(\gamma_v) = e^{-g}e^{g}\mathcal{F}(\gamma_v) \tag{6.1}$$

Define the inverse Fourier transform $\mathcal{F}^{-1}(\mathcal{F}(\gamma_v))$ for $x \in \mathbb{R}$ by

$$\mathcal{F}^{-1}(\mathcal{F}(\gamma_v))(x) = \int_{\operatorname{Im} t = 2c} \mathcal{F}_+(\gamma_v)(t)e^{itx}\, dt - \int_{\operatorname{Im} t = -2c} \mathcal{F}_-(\gamma_v)(t)e^{itx}\, dt.$$

Applying the inverse Fourier transform to both sides of (6.1) and evaluating at 0 yields

$$v = (2\pi)^{-1}\Pi\left(\mathcal{F}^{-1}\left(e^{-g}\right)\right)\left(\mathcal{F}^{-1}\left(e^{g}\mathcal{F}(\gamma_v)\right)(0)\right).$$

The assertion follows because $\mathcal{F}^{-1}\left(e^{-g}\right) \in \mathcal{A}(\mathbb{R})$. □

Strong factorization likewise holds for Banach representations of $(\mathbb{R}^n, +)$. Using the Iwasawa decomposition we are able to deduce from this the conjecture for $SL_2(\mathbb{R})$.
After the completion of this work, we verified that Cheeger, Gromov and Taylor's functional calculus could be extended to analyze $\alpha_\varepsilon^{-1}(\sqrt{\Delta})$ on suitable analytic functions. The main idea is to shift the contour in the Fourier inversion formula and express the solution operator to the wave equation at a complex time t in terms of elementary hyperbolic and trigonometric functions of $(\mathrm{Re}t)\sqrt{\Delta}$ and $(\mathrm{Im}t)\sqrt{\Delta}$. The strong factorization for an arbitrary Lie group then follows from the identity

$$\gamma_v = (\alpha_\varepsilon^{-1}(\sqrt{\Delta})\gamma_v) * \kappa_\alpha^\varepsilon$$

for $v \in E^\omega$ and sufficiently small $\varepsilon > 0$.

4.7 Appendix A. An identity of entire functions

Consider the following space of exponentially decaying holomorphic functions

$$\mathcal{F}_{c,\vartheta} = \left\{ \varphi \in \mathcal{O}(\mathbb{C}) \mid \forall N \in \mathbb{N} : \sup_{z \in \mathcal{W}_{N,\vartheta}} |\varphi(z)|\, e^{c|z|} < \infty \right\},$$
$$\mathcal{W}_{N,\vartheta} = \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| < N\} \cup \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| < \vartheta|\mathrm{Re}\, z|\}.$$

To understand the convolution kernel of a Fourier multiplication operator on $L^2(\mathbb{R})$ with symbol in $\mathcal{F}_{c,\vartheta}$, or more generally functions of $\sqrt{\Delta}$ on a manifold as in Section 4.4, we need some properties of the Fourier transformed functions.

Lemma 4.7.1. *Given $f \in \mathcal{F}_{c,\vartheta}$, there exist $C, R > 0$ such that*

$$|\hat{f}^{(k)}(z)| \leq C_N\ k!\ R^k e^{-N|z|}$$

for all $k, N \in \mathbb{N}$.

Proof. Note that if g is a holomorphic function on a strip $\{z \in \mathbb{C} \mid \mathrm{Im}\, z \leq N\}$ and $\int |g(x+iy)|\ dx < C$ for all $|y| \leq N$, then by shifting the contour of integration, the Fourier transform

$$\hat{g}(z) = \int_{\mathbb{R}} e^{-izx}\ g(x)\ dx = \int_{\mathbb{R}} e^{-iz(x+iy)}\ g(x+iy)\ dx$$

satisfies $|\hat{g}(z)|\ e^{N|z|} < C$ for all $z \in \mathbb{R}$. If $f \in \mathcal{F}_{c,\vartheta}$, then

$$\hat{f}(z) = \int e^{-izx}\ f(x)\ dx$$

extends to a holomorphic function on $\mathcal{W}_{c,\vartheta}$ by analytic continuation of the right hand side. The above argument implies $|\hat{f}(z)| \leq C_N e^{-N|z|}$ for all N even in $\mathcal{W}_{c',\vartheta}$,

$0 < c' < c$.
Estimates for higher derivatives are obtained from Cauchy's integral formula

$$\hat{f}^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{\hat{f}(\lambda)}{(\lambda - z)^{k+1}} \, d\lambda,$$

which leads to

$$|\hat{f}^{(k)}(z)| \leq C_N \; k! \; R^k e^{-N|z|}$$

for all $k, N \in \mathbb{N}$. □

Some important examples of functions in $\mathcal{F}_{c,\vartheta}$ may be constructed with the help of the Gaussian error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \; dt.$$

Interpreting the integral as an integral along a path from 0 to x, the error function extends to an entire function. It is odd, because the integrand is even, and

$$\operatorname{erf}(z) - 1 = O(z^{-1} e^{-z^2}) \tag{7.1}$$

as $z \to \infty$ in a sector $\{|\operatorname{Im} z| < \vartheta \operatorname{Re} z\}$ around $\mathbb{R}_+$ [56].

Remark 4.7.2. A straight–forward computation shows that

$$z \operatorname{erf}(z) = \frac{1}{\sqrt{\pi}} e^{-z^2} * |z| - \frac{1}{\sqrt{\pi}} e^{-z^2}.$$

It is therefore just a convenient regularization of the absolute value $|z|$, and the basic properties we need also hold for other similarly constructed functions. For example replace the heat kernel $\frac{1}{\sqrt{\pi}} e^{-z^2}$ by a suitable analytic probability density.

For any $\varepsilon > 0$, the asymptotic behavior of the error function from equation (7.1) implies that the even entire functions $\alpha_\varepsilon(z) = 2e^{-\varepsilon z \operatorname{erf}(z)}$ and $\beta_\varepsilon(z) = 1 - \alpha_\varepsilon(z) \cosh(\varepsilon z)$ decay exponentially as $z \to \infty$ in $\mathcal{W}_{N,\vartheta}$ for any $\vartheta < 1$. More precisely $\alpha_\varepsilon, \beta_\varepsilon \in \mathcal{F}_{2\varepsilon,\vartheta}$. Our later factorization hinges on a multiplicative decomposition of the constant function 1, which immediately follows from the definitions of α_ε and β_ε:

Lemma 4.7.3. *For all $\varepsilon > 0, \vartheta \in (0, 1)$, the functions $\alpha_\varepsilon, \beta_\varepsilon \in \mathcal{F}_{2\varepsilon,\vartheta}$ satisfy the identity*

$$\alpha_\varepsilon(z) \cosh(\varepsilon z) + \beta_\varepsilon(z) = 1.$$

4.8 Appendix B. Vector-valued holomorphy

Here we collect some results about analytic functions with values in a locally convex Hausdorff topological vector space E. Let $\Omega \subset \mathbb{C}^n$ be open.
It is a natural and common assumption that E is sequentially complete. Let us recall that under this assumption an E-valued function f on Ω is said to be holomorphic if it satisfies one of the following conditions, which are equivalent in this case:

(a) f is weakly holomorphic, that is, the scalar function $z \mapsto \zeta(f(z))$ is holomorphic for each continuous linear form $\zeta \in E'$;
(b) f is $\mathbb{C}$-differentiable in each variable at each $z \in \Omega$;
(c) f is infinitely often $\mathbb{C}$-differentiable at each $z \in \Omega$;
(d) f is continuous and is represented by a converging power series expansion with coefficients in E, in a neighborhood of each $z \in \Omega$.

In general, the conditions (c) and (d) are mutually equivalent and they imply (a) and (b). This follows by regarding f as a function into the completion $\bar{E}$ of E (see [27], Prop. 2.4). We shall call a function $f : \Omega \to E$ *holomorphic* if (c) or (d) is satisfied, or equivalently, if it is holomorphic into $\bar{E}$ with E-valued derivatives up to all orders.
Let M be an n-dimensional complex manifold. An E-valued function on M is called holomorphic if all its coordinate expressions are holomorphic. We denote by $\mathcal{O}(M, E)$ the space of E-valued holomorphic functions on M. Endowed with the compact open topology, it is a Hausdorff topological vector space, which is complete whenever E is complete.
The following isomorphism of topological vector spaces is useful.

Lemma 4.8.1. *Let M and N be complex manifolds, then*

$$\mathcal{O}(M \times N, E) \simeq \mathcal{O}(M, \mathcal{O}(N, E)) \tag{8.1}$$

under the natural map $f \mapsto (x \mapsto f(x, \cdot\,))$ from left to right.

Proof. Apart from the statement that $x \mapsto f(x, \cdot\,) \in \mathcal{O}(N, E)$ is holomorphic, this is straightforward from definitions. It is clear that $f(x, \cdot\,) \in \mathcal{O}(N, E)$. By regarding $\mathcal{O}(N, E)$ as a subspace of $\mathcal{O}(N, \bar{E})$ and noting that it carries the relative topology, we reduce to the case that E is complete, so that condition (b) applies. Assume for simplicity that $M = \mathbb{C}$. What needs to be established is then only that the complex differentiation

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{1}{h}[f(x + h, y) - f(x, y)] \in E$$

is valid locally uniformly with respect to $y \in N$. This follows from uniform continuity on compacta of the derivative. □

Chapter 5

Analytic representation theory of Lie groups

While analytic vectors are basic objects in the representation theory of real Lie groups, a coherent framework to study general analytic representations has been lacking so far. It is the aim of this chapter to introduce categories of tempered and non–tempered such representations and to analyze their fundamental properties. For a representation (π, E) of a Lie group G on a locally convex space E to be *analytic*, we are going to require that every vector in E be analytic and that the topology on the space of analytic vectors coincide with the topology of E. No completeness assumptions on E are imposed, so that the quotient of an analytic representation by a closed invariant subspace is again analytic.
Recall from the previous chapter that a vector $v \in E$ is called analytic provided that the orbit map $\gamma_v : x \mapsto \pi(x)v$ extends to a holomorphic E–valued function in a neighborhood of G within the complexification $G_{\mathbb{C}}$. The space E^ω of analytic vectors carries a natural inductive limit topology $E^\omega = \lim_{n\to\infty} E_n$,

$$E_n = \{v \in E \mid \gamma_v \text{ extends to a holomorphic map } GV_n \to E\} ,$$

indexed by a neighborhood basis $\{V_n\}_{n\in\mathbb{N}}$ of the identity in $G_{\mathbb{C}}$. We have already seen that the induced representation (π, E^ω) is continuous and indeed satisfies $E^\omega = (E^\omega)^\omega$ in the sense of topological vector spaces. Every analytic representation is obtained in this way. Due to the inductive limit structure of E^ω, interesting examples tend to involve complicated and possibly incomplete topologies. For instance, infinite dimensional Fréchet spaces do not carry any irreducible analytic representations of a reductive group. Still, in spite of examples by Grothendieck and others which show how incomplete spaces may naturally occur, important special cases are better behaved, like for instance the analytic vectors associated to a Banach representation, the algebra $\mathcal{A}(G)$ below, or the analytic globalization of a Harish–Chandra module.
Moderately growing analytic representations allow for an additional action by an algebra of superexponentially decaying functions. To be specific, consider a

Banach representation (π, E). Fix a left–invariant Riemannian metric on G and let $\mathbf{d}$ be the associated distance function. The continuous functions on G decaying faster than $e^{-n\mathbf{d}(\cdot,\mathbf{1})}$ for all $n \in \mathbb{N}$ form a convolution algebra $\mathcal{R}(G)$, which is a G–module under the left regular representation. If we denote the space of analytic vectors of $\mathcal{R}(G)$ by $\mathcal{A}(G)$, the map

$$\Pi : \mathcal{A}(G) \to \mathrm{End}(E^\omega),\ \Pi(f)v = \int_G f(x)\ \pi(x)v\ dx\ , \tag{0.1}$$

gives rise to a continuous algebra action on E^ω. More general representations will be called $\mathcal{A}(G)$*–tempered*, or of *moderate growth*, provided that the integral in (0.1) converges in the topology of E and defines a continuous action of $\mathcal{A}(G)$. Let us now specify to the case where G is a real reductive group, and let us recall that to each admissible G-representation E of finite length one can associate the Harish-Chandra module E_K of its K-finite vectors. Conversely, a *globalization* of a given Harish-Chandra module V is an admissible representation of G with $V = E_K$. The main result for this case is now as follows:

Theorem 5.0.2. *Let G be a real reductive group. Then every Harish-Chandra module V for G admits a unique $\mathcal{A}(G)$-tempered analytic globalization $V^{\min}$. Moreover, $V^{\min}$ has the property $V^{\min} = \Pi(\mathcal{A}(G))V$.*

It follows that $E^\omega \simeq V^{\min}$ for every $\mathcal{A}(G)$-tempered globalization E of V (in particular, for every Banach globalization). Let us mention the relationship to the announcements of Schmid and of Kashiwara-Schmid in [50] and [37], which (among others) assert that every Harish-Chandra module admits a unique *minimal globalization*, which is equivalent to E^ω for all Banach globalizations E. Our proof of Theorem 5.0.2 is independent of this theory. Instead, we rely on the corresponding theory by Casselman and Wallach for smooth globalizations, which is documented in [53] and more recently in [3].
The theorem features a worthwhile corollary, namely:

Corollary 5.0.3. *For an irreducible admissible Banach representation (π, E) of a real reductive group G, the space of analytic vectors E^ω is an algebraically simple $\mathcal{A}(G)$-module.*

5.1 Analytic representations

We refer to the previous chapter for the definition and basic properties of the space of analytic vectors E^ω of a representation (π, E) of G.

Completeness

As mentioned in the Introduction, the completeness of E does not ensure that E^ω is complete. For Banach representations this is the case as the following result shows.

Proposition 5.1.1. *Let (π, E) be a representation of G on a complete DF-space. Then E^ω is complete.*

Proof. Let (v_i) be a Cauchy net in E^ω. It is Cauchy in E, hence converges to some element $v \in E$. Moreover, the net of orbit maps (γ_{v_i}) converges pointwise on G to γ_v. We need to show that γ_v is real analytic, and using our assumptions on E it suffices to prove weak analyticity, see Remark 4.2.1.
Let $K \subset G$ be any compact set. We consider the space $A(K)$ of real analytic functions on K. These are germs of holomorphic functions defined on open neighborhoods V of K in $G_\mathbb{C}$, and $A(K)$ is equipped with the inductive topology. Since each $\mathcal{O}(V)$ has the Montel property, the limit is compact, so that $A(K)$ inherits completeness from $\mathcal{O}(V)$.
For every $\lambda \in E'$ we consider the mapping

$$E^\omega \to A(K), \quad E_n \ni v \mapsto \text{germ of } \lambda \circ \gamma_v.$$

It is clear that this is a continuous map. It follows that $\lambda \circ \gamma_{v_i}|_K$ converges in $A(K)$, so that $\lambda \circ \gamma_v$ is real analytic on K. □

Remark 5.1.2. Combining the proof above with [4], Theorem 3, leads to a more general result for representations on Fréchet spaces. In this case, E^ω is complete whenever there is a fundamental system of seminorms $\{p_n\}_{n\in\mathbb{N}}$ for the topology of E such that

$$\exists n \; \forall m \geq n \; \exists k \geq m \; \exists C > 0 \; \forall v \in E : \; p_m(v)^2 \leq C p_k(v) p_n(v).$$

Remark 5.1.3. An example by Grothendieck, [31], p. 95, may be adapted to give an example of an incomplete space of analytic vectors. Consider the regular representation of $G = S^1$ on the (complete) space $E = C(S^1, \mathbb{C}^\mathbb{N})$, where $\mathbb{C}^\mathbb{N}$ is endowed with the product topology. The analytic vectors for this action are sequences of functions, which extend holomorphically to a common annulus

$$\{z \in \mathbb{C} \mid 1 - \varepsilon < |z| < 1 + \varepsilon\}$$

for some $\varepsilon > 0$. Being a dense subspace of $(C(S^1)^\omega)^\mathbb{N}$, E^ω fails to be complete as well as sequentially complete.

Definition of analytic representations

Motivated by Proposition 4.2.4 we shall give the following definition.

Definition 5.1.4. A representation (π, E) is called *analytic* if $E = E^\omega$ holds as topological vector spaces.

Given a representation (π, E), Proposition 4.2.4 implies that (π, E^ω) is an analytic representation.

Lemma 5.1.5. *Let (π, E) be an analytic representation, and let $F \subset E$ be a closed invariant subspace. Then π induces analytic representations on both F and E/F.*

Proof. This follows from Lemma 4.2.6. From (i) in that lemma we infer immediately that $F^\omega = F$, and from (ii) we then conclude that $E/F = E^\omega/F^\omega \to (E/F)^\omega$ is continuous. The opposite inclusion is trivially valid and continuous. □

Example 5.1.6. We consider the Fréchet space $E := \mathcal{O}(G_\mathbb{C})$ with the right regular action of G,

$$\pi(g)f(z) = f(zg) \qquad (g \in G, z \in G_\mathbb{C}, f \in \mathcal{O}(G_\mathbb{C}))\,.$$

It is easy to see that (π, E) defines a representation. Given $v \in E$, it follows from (8.1) that the orbit map $\gamma_v : G \to E$ extends to a holomorphic mapping from $G_\mathbb{C}$ to E. The same equation implies easily that $E = E^\omega$ as topological spaces. Thus (π, E) is analytic.

Irreducible analytic representations

It is a natural question on which type of topological vector spaces E one can model irreducible analytic representations. The next result shows that this class is rather restrictive.

Theorem 5.1.7. *Let (π, E) be an irreducible representation of a reductive group on a Fréchet space E. If $E = E^\omega$ as vector spaces, then E is finite dimensional.*

Proof. By passing to a covering group if necessary, we may assume that $G_\mathbb{C}$ is simply connected. By assumption $E^\omega = \lim E_n$ identifies with E as vector spaces. The Grothendieck factorization theorem implies that $E = E_n$ for some n (see [32], Ch. 4, Sect. 5, Thm. 1). Hence the operator $\pi(x) := \pi_n(x)$ is defined on E, for all $x \in V_n$. We shall holomorphically extend to all $x \in G_\mathbb{C}$.
Let $v \in E$. By the monodromy theorem it suffices to extend $\pi(x)v$ along all simple smooth curves starting at $\mathbf{1}$. Let $\gamma : [0,1] \to G_\mathbb{C}$ be such a curve with $\gamma(0) = \mathbf{1}$. We select finitely many open sets $U_1, \ldots, U_k \subset G_\mathbb{C}$ which cover the curve $\gamma([0,1])$ and points

$$x_i = \gamma(t_i), \qquad 0 = t_1 < \cdots < t_k < 1,$$

such that $\mathbf{1} = x_1 \in U_1$ and $x_i \in U_i \cap U_{i-1}$ for $i > 1$. By choosing the sets U_i sufficiently small (and sufficiently many) we may assume that $U_i \subset V_{2n}x_i$ for each i and also that the only non–empty overlaps are among neighboring sets U_i and U_{i-1} (to attain these properties it may be useful from the outset to select the sets inside a tubular neighborhood around the curve).

In particular, $\pi(x)v$ is already defined for $x \in U_1 \subset V_{2n}$. On $U_2, \ldots, U_k$ we recursively define

$$\pi(x)v = \pi(z)\pi(x_i)v, \qquad x = zx_i \in U_i \subset V_{2n}x_i,$$

where $\pi(x_i)v$ is defined in the preceding step. Clearly this depends holomorphically on x. However, in order to obtain a proper extension of $x \mapsto \pi(x)v$, we need to verify that $\pi(x)v$ is well defined on overlaps between the U_i. What we need to show is that

$$\pi(z)\pi(x_i)v = \pi(zx_i)v, \qquad zx_i \in U_i \cap U_{i-1}.$$

Let $x_i = yx_{i-1}$ where $y \in V_{2n}$. By the recursive definition we have $\pi(x_i)v = \pi(y)\pi(x_{i-1})v$ and $\pi(zx_i)v = \pi(zy)\pi(x_{i-1})v$. Then the desired identity follows since $\pi(z)\pi(y) = \pi(zy)$ by (2.4).

Thus the representation extends to an irreducible holomorphic representation of $G_{\mathbb{C}}$ (also denoted by π) . If $U < G_C$ is a compact real form, then the Peter-Weyl theorem implies that $\pi|_U$ is irreducible and finite dimensional. $\square$

Remark 5.1.8. Non–reductive groups, on the other hand, may have irreducible analytic actions on a Fréchet space. As an example, consider the Schrödinger representation of the Heisenberg group $\mathbb{H}^n$ on the Fréchet space

$$E = \{f \in \mathcal{O}(\mathbb{C}^n) \mid \forall N, M \in \mathbb{N} : \sup_{x \in \mathbb{R}^n} \sup_{y \in (-N,N)^n} |f(x+iy)|\, e^{M|x|} < \infty\}.$$

It is irreducible as a restriction of the Schrödinger representation on $L^2(\mathbb{R}^n)$, and one readily verifies that $E = E^\omega$.

5.2 $\mathcal{A}(G)$–tempered representations

Analytic superdecaying functions revisited

As promised above, we now provide further details about the analytic vectors for the left–regular representation in $\mathcal{R}(G)$. We set $\mathcal{A}(G) := \mathcal{R}(G)^\omega$ and equip $\mathcal{A}(G)$ with the corresponding vector topology. This algebra is slightly larger than $\mathcal{A}_{LR}(G)$, but seems more natural in the context of algebra actions on representations, where right analyticity is generally lost under convolution.

With the notation from the preceding chapter we put $\mathcal{A}_n(G) := \mathcal{R}(G)_n$ for each $n \in \mathbb{N}$. Notice that $\mathcal{A}_n(G)$ is a Fréchet space for each n, since $\mathcal{R}(G)$ is Fréchet. Hence $\mathcal{A}(G)$ is an *LF*-space (inductive limit of Fréchet spaces). In the appendix we show that $\mathcal{A}(G)$ is complete and reflexive.

Proposition 5.2.1.

(i) $\mathcal{A}(G)$ *carries representations of* G *by left and right action,*

(ii) $\mathcal{A}(G)$ is a subalgebra of $\mathcal{R}(G)$ and convolution is continuous

$$\mathcal{A}(G) \times \mathcal{A}(G) \to \mathcal{A}(G).$$

Proof. (i) The statement about the left action is immediate from Proposition 4.2.4 (i). It is clear that $\mathcal{A}(G)$ is right invariant, since every right displacement $f \mapsto R_g f$ is an intertwining operator for the left regular representation. The continuity of the right action follows from Lemma 5.2.2 below, see Remark 5.2.3.
(ii) This follows from Proposition 5.2.5 (to be proved below) by taking $E = \mathcal{R}(G)$.
□

The next lemma gives us a concrete realization of $\mathcal{A}_n(G)$.

Lemma 5.2.2. *For all $n \in \mathbb{N}$, restriction to G provides a topological isomorphism of*

$$\left\{ f \in \mathcal{O}(V_n G) \;\middle|\; \begin{matrix} \forall N > 0, \forall \Omega \subset V_n \text{ compact}: \\ \sup_{g \in G, z \in \Omega} |f(zg)| e^{Nd(g)} < \infty \end{matrix} \right\}$$

onto $\mathcal{A}_n(G)$. Here the space above is topologized by the seminorms mentioned in its definition.

Proof. Let $f \in \mathcal{A}_n(G)$. Then $\gamma_f : G \to \mathcal{R}(G)$, $g \mapsto f(g^{-1}\cdot)$ extends to a holomorphic map $\gamma_{f,n} : GV_n \to \mathcal{R}(G)$. As point evaluations $\mathcal{R}(G) \to \mathbb{C}$ are continuous, it follows that $F(z) := \gamma_{f,n}(z^{-1})(\mathbf{1})$ defines a holomorphic extension of f to $V_n G$. Moreover, $F(zg) = \gamma_{f,n}(z^{-1})(g)$ for $z \in V_n, g \in G$. Let $N > 0$ and a compact set $\Omega \subset V_n$ be given, then

$$\sup_{g \in G, z \in \Omega} |F(zg)| e^{Nd(g)} = \sup_{z \in \Omega} p_N(\gamma_{f,n}(z^{-1})) < \infty$$

where $p_N(h) = \sup_{g \in G} |h(g)| e^{Nd(g)}$ is a defining seminorm of $\mathcal{R}(G)$. Hence F belongs to the space above. Moreover, we see that $f \mapsto F$ is an isomorphism onto its image.
Conversely, let F belong to the space above and put $f := F|_G$. Then it is clear that $f \in \mathcal{R}(G)$ (take $\Omega = \{\mathbf{1}\}$). We need to show that $f \in \mathcal{A}_n(G)$, i. e. that $\gamma_f : G \to \mathcal{R}(G)$ extends to a holomorphic map $GV_n \to \mathcal{R}(G)$. The extension is $z \mapsto F(z^{-1}\cdot)$, and we need to show that it is holomorphic.
We first show that $z \mapsto F(z^{-1}\cdot)$ is continuous into $\mathcal{R}(G)$. To see this, let $z_0 \in GV_n$ and $\epsilon, N > 0$ be given. We wish to find a neighborhood D of z_0 such that

$$p_N(F(z^{-1}\cdot) - F(z_0^{-1}\cdot)) < \epsilon \tag{2.1}$$

for all $z \in D$.
Let us fix a compact neighborhood D_0 of z_0 in GV_n. As

$$\sup_{g \in G, z \in D_0} |F(z^{-1}g)| e^{md(g)} < \infty$$

for all $m > N$ we find a compact subset $K \subset G$ such that

$$\sup_{g \in G \setminus K, z \in D_0} |F(z^{-1}g)| e^{Nd(g)} < \epsilon/2.$$

Shrinking D_0 to some possibly smaller neighborhood D we may request that

$$\sup_{g \in K, z \in D} |F(z^{-1}g) - F(z_0^{-1}g)| e^{Nd(g)} < \epsilon\,.$$

The required estimate 2.1 follows.
As continuity has been verified, holomorphicity follows provided $z \mapsto \lambda(F(z^{-1}\cdot))$ is holomorphic for λ ranging in a subset whose linear span is weakly dense in $\mathcal{R}(G)'$ (see [30], p. 39, Remarque 1). A convenient such subset is $\{\delta_g \mid g \in G\}$, and the proof is complete. $\square$

Remark 5.2.3. Let $q(f) := \sup_{g \in G, z \in \Omega} |f(zg)| e^{Nd(g)}$ be a seminorm on $\mathcal{A}_n(G)$ as above. Then (1.1) implies

$$q(R_x f) \leq e^{Nd(x)} q(f) \quad (f \in \mathcal{A}_n(G))\ ,$$

for $x \in G$, so $\mathcal{A}_n(G)$ is an F-representation for the right action.

Analytic vectors of F-representations

Let (π, E) be an F-representation of G, and let $v \in E$. The map $f \mapsto \Pi(f)v$ is intertwining from $\mathcal{R}(G)$ (with left action) to E. Hence $\Pi(f)v \in E_n$ for $f \in \mathcal{A}_n(G)$ and $\Pi(f)v \in E^\omega$ for $f \in \mathcal{A}(G)$. With the preceding characterization of $\mathcal{A}_n(G)$ we have

$$\pi(z)\Pi(f)v = \int_G f(z^{-1}g)\pi(g)v\, dg \tag{2.2}$$

for $f \in \mathcal{A}_n(G)$, $z \in GV_n$.

Remark 5.2.4. In particular

$$\Pi(\mathcal{A}(G))E^\omega \subset E^\omega$$

for F-representations. In fact, Chapter 4 shows that

$$\Pi(\mathcal{A}(G))E^\omega = E^\omega.$$

It is easily seen that the action of $\mathcal{A}(G)$ on E^ω is an algebra action. We shall now see that it is continuous.

Proposition 5.2.5. *Let (π, E) be an F-representation. The bilinear map $(f, v) \mapsto \Pi(f)v$ is continuous*

$$\mathcal{A}_n(G) \times E \to E_n,$$

for every $n \in \mathbb{N}$. Likewise, it is continuous

$$\mathcal{A}(G) \times E \to E^\omega.$$

Notice that since E^ω injects continuously in E, the last statement implies continuity of both

$$\mathcal{A}(G) \times E \to E \quad \text{and} \quad \mathcal{A}(G) \times E^\omega \to E^\omega.$$

Proof. Let $n \in \mathbb{N}$ be fixed and let $W \subset E_n$ be an open neighborhood of 0. We may assume

$$W = W_{K,p} := \{v \in E_n \mid p(\pi(K)v) < 1\}$$

with $K \subset GV_n$ compact and p a continuous seminorm on E such that

$$p(\pi(g)v) \leq Ce^{cd(g)}p(v) \qquad (g \in G, v \in E)$$

for some constants c, C (see 1.2).
Choose $N > 0$ so that (cf 3.1)

$$C_1 := \int_G e^{(c-N)d(g)}\, dg < \infty,$$

and let

$$O := \{f \in \mathcal{O}(V_nG) \mid \sup_{z\in K, g\in G} |f(z^{-1}g)|e^{Nd(g)} < \epsilon\} \subset \mathcal{A}_n(G)$$

(with ϵ to be specified below). According to Lemma 5.2.2, O is open.
For $f \in O$ and $z \in K$ we obtain by (2.2)

$$p(\pi(z)\Pi(f)v) \leq \int_G |f(z^{-1}g)|\, p(\pi(g)v))\; dg \leq \epsilon CC_1 p(v).$$

With $\epsilon < 1/(CC_1)$ we conclude that $\Pi(f)v \in W$ if $f \in O$ and $p(v) < 1$.
This proves the first statement. By taking inductive limits we infer continuity of $\lim(\mathcal{A}_n(G) \times E) \to E^\omega$. For the continuity of $\mathcal{A}(G) \times E \to E^\omega$ it now suffices to verify that $\lim(\mathcal{A}_n(G) \times E)$ and $\mathcal{A}(G) \times E = (\lim \mathcal{A}_n(G)) \times E$ are isomorphic. The map

$$\lim(\mathcal{A}_n(G) \times E) \to (\lim \mathcal{A}_n(G)) \times E$$

is clearly bijective and continuous. The left hand side is *LF*, and the right hand side is a product of ultrabornological spaces, hence itself ultrabornological. It follows that the open mapping theorem can be applied (see [44], Theorem 24.30 and Remarks 24.15, 24.36). □

For later use we note that $\mathcal{A}(G)$ contains a Dirac sequence.

Lemma 5.2.6. *The heat kernel h_t belongs to $\mathcal{A}(G)$ for each $t > 0$. Let E be an F-representation. Then $\Pi(h_t)v \to v$ in E for all $v \in E$.*

Proof. The convergence in E is Nelson's theorem (see Section 5.1). The heat kernel belongs to $\mathcal{A}(G)$ for all $t > 0$ by Thm. 4.4.2. □

Remark 5.2.7. It follows from the proof of Thm. 4.4.2, that there exists a common m such that $h_t \in \mathcal{A}_m(G)$ for all $t > 0$.

$\mathcal{A}(G)$-tempered representations

As we have seen that there is continuous algebra action of $\mathcal{A}(G)$ on the analytic vectors of F-representations, we shall make this property part of a definition.

Definition 5.2.8. A representation (π, E) is called $\mathcal{A}(G)$-tempered if for all $f \in \mathcal{A}(G)$ and $v \in E$ the vector valued integral

$$\Pi(f)v = \int_G f(g)\pi(g)v \, dg$$

converges in the topology of E, and $(f, v) \mapsto \Pi(f)v$ defines a continuous algebra action

$$\mathcal{A}(G) \times E \to E.$$

Example 5.2.9. (a) For every F-representation (π, E) both (π, E) itself and (π, E^ω) are $\mathcal{A}(G)$-tempered according to Proposition 5.2.5. In particular this holds for all Banach representations and also for $E = \mathcal{R}(G)$ with the left action (so that $E^\omega = \mathcal{A}(G)$).
(b) If (π, E) is an $\mathcal{A}(G)$-tempered representation and $F \subset E$ is a closed G-invariant subspace, then the induced representations on F and E/F are $\mathcal{A}(G)$-tempered.

5.3 Analytic globalizations of Harish-Chandra modules

In this section we will assume that G is a real reductive group. Let us fix a maximal compact subgroup $K < G$. We say that a complex vector space V is a $(\mathfrak{g}, K)$-module if V is endowed with a Lie algebra action of $\mathfrak{g}$ and a locally finite group action of K which are compatible in the sense that the derived and restricted actions of $\mathfrak{k}$ agree and, in addition,

$$k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v) \qquad (k \in K, X \in \mathfrak{g}, v \in V).$$

We call a $(\mathfrak{g}, K)$-module *admissible* if for any irreducible representation (σ, W) of K the multiplicity space $\mathrm{Hom}_K(W, V)$ is finite dimensional. Finally, an admissible $(\mathfrak{g}, K)$-module is called a *Harish-Chandra module* if V is finitely generated as a $\mathcal{U}(\mathfrak{g})$-module. Here, as usual, $\mathcal{U}(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$. By a *globalization* of a Harish-Chandra module V we understand a representation (π, E) of G such that the space of K-finite vectors

$$E_K := \{v \in E \mid \dim \mathrm{span}_{\mathbb{C}}\{\pi(K)v\} < \infty\}$$

is $(\mathfrak{g}, K)$-isomorphic to V and dense in E. Density of E_K is automatic whenever E is quasi-complete, see [34], Lemma 4. Each element $v \in E$ allows an expansion in K-types $v = \sum_{\tau \in \hat{K}} v_\tau$, where $v_\tau = \dim \tau \, \pi(\chi_\tau)v \in E_K$. Here, the integral over

K that defines $\pi(\chi_\tau)v$ may take place in the completion of E, but v_τ belongs to E_K by density and finite dimensionality of K-type spaces.
A *Banach (F-, analytic, $\mathcal{A}(G)$-tempered) globalization* is a globalization by a Banach (F-, analytic, $\mathcal{A}(G)$-tempered) representation. Note that according to Harish-Chandra [33], $E_K \subset E^\omega$ if E is a Banach globalization. In general, the orbit map of a vector $v \in E_K$ is weakly analytic (see Remark 4.2.1).
According to the subrepresentation theorem of Casselman (see [53] Thm. 3.8.3), V admits a Banach-globalization E. The space E^ω is then an analytic $\mathcal{A}(G)$-tempered globalization.
If V is a Harish-Chandra module, we denote by $\tilde{V}$ the Harish-Chandra module dual to V, i.e. the space of K-finite linear forms on V (see [53], p. 115). We note that if E is a globalization of V, then $\tilde{V}$ embeds into E' and identifies with the subspace of K-finite continuous linear forms (see [11], Prop. 2.2). Furthermore $\tilde{V}$ separates on E. Since the matrix coefficients $x \mapsto \xi(\pi(x)v)$ for $v \in V, \xi \in \tilde{V}$ are real analytic functions on G, they are determined by their germs at $\mathbf{1}$. It follows that these functions on G are independent of the globalization (see [11], p. 396).

Minimal analytic globalizations

Let V be a Harish-Chandra module and $\mathbf{v} = \{v_1, \dots, v_k\}$ be a set of $\mathcal{U}(\mathfrak{g})$-generators. We shall fix an arbitrary $\mathcal{A}$-tempered globalization (π, E) and regard V as a subspace in E.
On the product space $\mathcal{A}(G)^k = \mathcal{A}(G) \times \cdots \times \mathcal{A}(G)$ with diagonal G-action, we consider the G-equivariant map

$$\Phi_{\mathbf{v}} : \mathcal{A}(G)^k \to E, \quad \mathbf{f} = (f_1, \dots, f_k) \mapsto \sum_{j=1}^{k} \Pi(f_j)v_j \ ,$$

and write $I_{\mathbf{v}}$ for its kernel. This map is evidently continuous, and thus $I_{\mathbf{v}}$ is a closed G-invariant subspace of $\mathcal{A}(G)^k$. We note that $\mathbf{f} \in I_{\mathbf{v}}$ if and only if $\sum_j \int f_j(g)\xi(\pi(g)v_j)\, dg = 0$ for all $\xi \in \tilde{V}$. It follows that $I_{\mathbf{v}}$ is independent of the choice of globalization. Furthermore, the dependence on generators is easily described: If $\mathbf{v}'$ is another set of generators, say k' in number, there exists a $k \times k'$-matrix u of elements from $\mathcal{U}(\mathfrak{g})$ such that $\mathbf{f} \in I_{\mathbf{v}}$ if and only if $R_u\mathbf{f} \in I_{\mathbf{v}'}$.
Since $I_{\mathbf{v}}$ is closed and G-invariant, the quotient

$$V^{\min} := \mathcal{A}(G)^k / I_{\mathbf{v}}$$

carries a representation of G which we denote by $(\pi, V^{\min})$. It is independent of the choice of the globalization (π, E) and (up to equivalence) of the set $\mathbf{v}$ of generators.

Lemma 5.3.1. *Let V be a Harish-Chandra module. Then the following assertions hold:*

(i) $V^{\min}$ *is an analytic* $\mathcal{A}(G)$*-tempered globalization of* V*.*

(ii) $V^{\min} = \Pi(\mathcal{A}(G))V$*, that is,* $V^{\min}$ *is spanned by the vectors of form* $\Pi(f)v$*.*

(iii) If (λ, F) *is any* $\mathcal{A}$*-tempered globalization of* V*, then the identity mapping* $V \to F$ *lifts to a* G*-equivariant continuous injection* $V^{\min} \to F^{\omega}$*.*

Proof. (i) It follows from the definition that $V^{\min}$ is analytic (see Lemma 4.2.6) and $\mathcal{A}(G)$-tempered (see Example 5.2.9(b)). It remains to be seen that $(V^{\min})_K$ is $(\mathfrak{g}, K)$-isomorphic to V. By definition, $\Phi_{\mathbf{v}}$ induces a continuous G-equivariant injection $V^{\min} \to E$. In particular $(V^{\min})_K$ is isomorphic to a $(\mathfrak{g}, K)$-submodule of $V = E_K$. Moreover as $\mathcal{A}(G)$ contains a Dirac sequence by Lemma 5.2.6, and as we may assume E to be a Banach space, each generator v_j belongs to the E-closure of the image of $V^{\min}$. By admissibility and finite dimensionality of K-types, v_j belongs to $(V^{\min})_K$ for each j. Thus $(V^{\min})_K \simeq V$ and (i) follows. Assertions (ii) and (iii) are clear. □

Because of property (iii), we shall refer to $V^{\min}$ as the *minimal* $\mathcal{A}(G)$*-tempered globalization* of V. We record the following functorial properties of the construction.

Lemma 5.3.2. *Let* V, W *be Harish-Chandra modules.*

(i) Every $(\mathfrak{g}, K)$*-homomorphism* $T \colon W \to V$ *lifts to a unique intertwining operator* $T^{\min} \colon W^{\min} \to V^{\min}$ *with restriction* T *on* $W = (W^{\min})_K$ *and with closed image.*

(ii) Assume that $W \subset V$ *is a submodule. Then*

(a) $W^{\min}$ *is equivalent with a subrepresentation of* $V^{\min}$ *on a closed invariant subspace,*

(b) $(V/W)^{\min}$ *is equivalent with the quotient representation* $V^{\min}/W^{\min}$*.*

Proof. (i) Let $\tilde{T} \colon \tilde{V} \to \tilde{W}$ denote the dual map of T, and observe that

$$\tilde{T}\xi(\pi(g)w) = \xi(\pi(g)Tw)$$

for all $w \in W, \xi \in \tilde{V}$ and $g \in G$. Indeed, these are analytic functions of g whose power series at $\mathbf{1}$ agree because T is a $\mathfrak{g}$-homomorphism. It follows that if we choose generators $w_1, \ldots, w_l$ for W and $v_1, \ldots, v_k$ for V such that $v_j = Tw_j$ for $j = 1, \ldots, l$, then the inclusion map $\mathbf{f} \mapsto (\mathbf{f}, \mathbf{0})$ of $\mathcal{A}(G)^l$ into $\mathcal{A}(G)^k$ takes $I_{\mathbf{w}}$ into $I_{\mathbf{v}}$. Hence this inclusion map induces a map

$$T^{\min} \colon \mathcal{A}(G)^l / I_{\mathbf{w}} \to \mathcal{A}(G)^k / I_{\mathbf{v}}$$

which is continuous, intertwining and has closed image. Moreover, this map restricts to T on W, since it maps each generator w_j to $v_j = Tw_j$.

(ii) is obtained from (i) with T equal to (a) the inclusion map $W \to V$ or (b) the quotient map $V \to V/W$. $\square$

Our next concern will be to realize the analytic globalizations inside of Banach modules.

Proposition 5.3.3. *Let (π, E) be an analytic $\mathcal{A}(G)$-tempered globalization of a Harish-Chandra module V. Then there exists a Banach representation (σ, F) of G and a continuous G-equivariant injection $(\pi, E) \to (\sigma, F)$.*

Proof. We fix generators $\xi = \{\xi_1, \ldots, \xi_l\}$ of the dual Harish-Chandra module $\tilde{V} \subset E'$ and put $U := \{v \in E \mid \max_{1 \leq j \leq l} |\xi_j(v)| < 1\}$. Then U is an open neighborhood of 0 in E.
Fix $m \in \mathbb{N}$ such that $\mathcal{A}_m(G)$ contains a Dirac sequence (see Remark 5.2.7). As $\mathcal{A}_m(G) \times E \to E$ is continuous, we find an open neighborhood O of 0 in $\mathcal{A}_m(G)$ and an open neighborhood W of 0 in E such that $\Pi(O)W \subset U$. We may assume that O is of the type $O = \{f \in \mathcal{A}_m(G) \mid q(f) < 1\}$ where

$$q(f) = \sup_{\substack{g \in G \\ z \in \Omega}} |f(zg)| e^{N d(g)}$$

for some $N \in \mathbb{N}$ and $\Omega \subset V_m$ compact. Define the normed space $X := (\mathcal{A}_m(G), q)$. It follows from Remark 5.2.3 that the right regular action of G is a representation by bounded operators on X. Let $F := (X^*)^l$ be the topological dual of X^l and σ the corresponding dual diagonal action of G. Note that F is a Banach space, being the dual of a normed space, so that σ is a Banach representation. We claim that the map

$$\phi : E \to F, \quad v \mapsto \left(\mathbf{f} = (f_1, \ldots, f_l) \mapsto \sum_{j=1}^{l} \xi_j(\Pi(f_j)v) \right)$$

is G-equivariant, continuous and injective. Equivariance is clear, and in order to establish continuity we fix a closed convex neighborhood $\tilde{O}$ of 0 in F. We may assume that $\tilde{O}$ is a polar of the form $\tilde{O} = [B^l]^o$ where B is a bounded set $B \subset X$. Because B is bounded, there exists $\lambda > 0$ such that $B \subset \lambda O$. Choosing $\tilde{W} := \frac{1}{\lambda} W$ we have $\phi(\tilde{W}) \subset \tilde{O}$, as

$$\begin{aligned} \phi(\tilde{W})(B^l) &\subset \frac{1}{l} \phi(W)(O^l) \subset \frac{1}{l} \sum_{j=1}^{l} \xi_j(\Pi(O)W) \\ &\subset \frac{1}{l} \sum_{j=1}^{l} \xi_j(U) \subset \{z \in \mathbb{C} \mid |z| \leq 1\}. \end{aligned}$$

It remains to be shown that ϕ is injective. Suppose that $\phi(v) = 0$. Then $\phi(v_\tau) = 0$ for each element v_τ in the K-finite expansion of v, so that we may assume v is

K-finite. Then for all $f \in \mathcal{A}_m(G)$ and $\eta \in \tilde{V}$ one would have $\eta(\Pi(f)v) = 0$. Since K-finite matrix coefficients are independent of globalizations, we conclude by Lemma 5.2.6 that $\eta(v) = 0$ and hence $v = 0$. □

The minimal analytic globalization of a spherical principal series representation

Let $G = KAN$ be an Iwasawa decomposition of G and denote by M the centralizer of A in K, i.e. $M = Z_K(A)$. Then $P = MAN$ is a minimal parabolic subgroup. Let us denote by $\mathfrak{a}, \mathfrak{n}$ the Lie algebras of A and N and define $\rho \in \mathfrak{a}^*$ by $\rho(X) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} X|_{\mathfrak{n}})$, $X \in \mathfrak{a}$. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $a \in A$ we set $a^\lambda := e^{\lambda(\log a)}$.
The *smooth spherical principal series with parameter* $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ is defined by

$$V_\lambda^\infty := \{f \in C^\infty(G) \mid \forall man \in P\ \forall g \in G: \ f(mang) = a^{\rho+\lambda} f(g)\}.$$

The action of G on V_λ^∞ is by right displacements in the arguments, and in this way we obtain a smooth F-representation $(\pi_\lambda, V_\lambda^\infty)$ of G. We denote the Harish-Chandra module of V_λ^∞ by V_λ.
It is useful to observe that the restriction mapping to K,

$$\operatorname{Res}_K : V_\lambda^\infty \to C^\infty(M\backslash K),$$

is an K-equivariant isomorphism of Fréchet spaces, and henceforth we will identify V_λ^∞ with $C^\infty(M\backslash K)$. The space V_λ of K-finite vectors in V_λ^∞ is then identified as a K-module with the space $C(M\backslash K)_K$ of K-finite functions on $M\backslash K$.
Likewise, the Hilbert space $\mathcal{H}_\lambda := L^2(M\backslash K)$ is provided with the representation π_λ. The space of smooth vectors for this representation is $\mathcal{H}_\lambda^\infty = V_\lambda^\infty = C^\infty(M\backslash K)$, and the space of analytic vectors is the space $\mathcal{H}_\lambda^\omega = V_\lambda^\omega := C^\omega(M\backslash K)$ of analytic functions on $M\backslash K$ with its usual topology.

Theorem 5.3.4. *For every* $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ *one has*

$$\Pi_\lambda(\mathcal{A}(G))V_\lambda = C^\omega(M\backslash K).$$

In particular $V_\lambda^{\min} \simeq V_\lambda^\omega = C^\omega(M\backslash K)$ *as analytic representations.*

The proof of this theorem is similar to the corresponding result in the smooth case (see [3], Section 4). Note that from Lemma 5.3.1 we have

$$\Pi_\lambda(\mathcal{A}(G))V_\lambda = V_\lambda^{\min} \subset V_\lambda^\omega$$

with continuous inclusion. As the space $V_\lambda^{\min}$ admits a web (see [44], 24.8 and 24.28) and $C^\omega(M\backslash K)$ is ultrabornological (see [44], 24.16), we can apply the open mapping theorem ([44], 24.30) to obtain an identity of topological spaces from the set-theoretical identity. It thus suffices to prove that for each $v \in V_\lambda^\omega$ there exists $\xi \in V_\lambda$ and $F \in \mathcal{A}(G)$ such that $\Pi(F)\xi = v$.

We need some technical preparations. Let us denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$, and write θ for the corresponding Cartan involution. Let $(\cdot,\cdot)$ be a non–degenerate invariant bilinear form on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. Then $\langle\cdot,\cdot\rangle = -(\theta\cdot,\cdot)$ defines an inner product on $\mathfrak{g}$, which we use to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. We write $|\cdot|$ for the norms induced on $\mathfrak{g}$ and $\mathfrak{g}^*$.
Let $X_1, \ldots, X_s$ be an orthonormal basis of $\mathfrak{k}$ and $Y_1, \ldots, Y_l$ be an orthonormal basis of $\mathfrak{p}$. We define elements in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ by

$$\Delta = \sum_{j=1}^{s} X_j^2 + \sum_{i=1}^{l} Y_i^2\,, \qquad \Delta_K = \sum_{j=1}^{s} X_j^2 \quad \text{and} \quad \mathrm{C} := \Delta - 2\Delta_K\,.$$

Note that C is a Casimir element. In particular, it belongs to the center of $\mathcal{U}(\mathfrak{g})$.
Let $\mathfrak{t} \subset \mathfrak{k}$ be a maximal torus. We fix a positive system of the root system $\Sigma(\mathfrak{t}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ and identify the unitary dual $\hat{K}$ via their highest weights with a subset of $i\mathfrak{t}^*$. If (τ, W_τ) is an irreducible representation of K, then Δ_K acts as the scalarr multiple $|\tau + \rho_{\mathfrak{k}}|^2 - |\rho_{\mathfrak{k}}|^2$. For every $\tau \in \hat{K}$ we denote by $\chi_\tau \in C(K)$ the normalized character $\chi_\tau(k) = (\dim W_\tau)^{-1} \operatorname{tr} \tau(k)$. Note that $C(K)$ acts on $\mathcal{A}(G)$ by left convolution.
We denote the left regular representation of G on $\mathcal{A}(G)$ by L. The following proposition will be crucial in the proof of Theorem 5.3.4.

Proposition 5.3.5. *Let $(c_\tau)_{\tau\in\hat{K}}$ be a sequence of complex numbers and $(a_\tau)_{\tau\in\hat{K}}$ a sequence of elements in G. Assume that*

$$|c_\tau| \leq Ce^{-\epsilon|\tau|}, \qquad d(a_\tau) \leq c_1 \log(1 + |\tau|) + c_2$$

for some $C, \epsilon, c_1, c_2 > 0$. Let $f \in \mathcal{A}(G)$. Then

$$F := \sum_{\tau\in\hat{K}} c_\tau \chi_\tau * L(a_\tau) f \in \mathcal{A}(G)\,.$$

Proof. As $(L, \mathcal{R}(G))$ is an F-representation, it follows from Chapter 4 that $h \in \mathcal{R}(G)$ is belongs to $\mathcal{A}(G)$ if and only if there exists an $M > 0$ such that for all $N \in \mathbb{N}$ there exists a constant $C_N > 0$ with

$$\sup_{g\in G} e^{Nd(g)} |\Delta^k h(g)| \leq C_N M^{2k} (2k)! \tag{3.1}$$

for all $k \in \mathbb{N}$.
Observe that $\Delta = \mathrm{C} + 2\Delta_K$. For every $h \in \mathcal{R}(G)$ one has

$$\Delta_K(\chi_\tau * h) = (|\tau + \rho_{\mathfrak{k}}|^2 - |\rho_{\mathfrak{k}}|^2)\chi_\tau * h\,. \tag{3.2}$$

Moreover as C is central we obtain for every $g \in G$ and $h \in \mathcal{A}(G)$ that

$$\mathrm{C}(\chi_\tau * L(g)h) = \chi_\tau * L(g)(\mathrm{C}h)\,. \tag{3.3}$$

Let now $f \in \mathcal{A}(G)$. As f is an analytic vector for $\mathcal{R}(G)$, hence also for $L^2(G)$, we find (see [54], Cor. 4.4.6.4) a constant $M_1 > 0$ such that for all $N > 0$ there exists a constant $C_N > 0$ such that

$$\sup_{g\in G} e^{Nd(g)} |\mathrm{C}^k f(g)| \leq C_N M_1^{2k} (2k)!\,. \tag{3.4}$$

We first estimate $\Delta^k(\chi_\tau * L(a_\tau)f)$. For that we employ (3.2) and (3.3) in order to obtain that

$$\begin{aligned}\Delta^k(\chi_\tau * L(a_\tau)f) &= \sum_{j=0}^{k} \binom{k}{j} \mathrm{C}^j (2\Delta_K)^{k-j} (\chi_\tau * L(a_\tau)f) \\ &= \sum_{j=0}^{k} 2^{k-j} \binom{k}{j} (|\tau+\rho_{\mathfrak{k}}|^2 - |\rho_{\mathfrak{k}}|^2)^{k-j} (\chi_\tau * L(a_\tau)\mathrm{C}^j f)\,.\end{aligned}$$

For $N > 0$ we thus obtain using (3.4) that

$$\begin{aligned}&\sup_{g\in G} e^{Nd(g)} |\Delta^k(\chi_\tau * L(a_\tau)f)(g)| \\ &\quad\leq C_N 2^{2k} \sum_{j=0}^{k} (1+|\tau|)^{2(k-j)} \cdot \sup_{g\in G} e^{Nd(g)} |L(a_\tau)\mathrm{C}^j f(g)| \\ &\quad\leq C_N' 2^{2k} e^{Nd(a_\tau)} \sum_{j=0}^{k} M_1^{2j} (1+|\tau|)^{2(k-j)} (2j)! \\ &\quad\leq C_N'' M_2^{2k} \sum_{j=0}^{k} (1+|\tau|)^{2(k-j)+Nc_1} (2j)!\end{aligned}$$

for some $C_N, M_2 > 0$ independent of τ. Using these inequalities for F we arrive at

$$\sup_{g\in G} e^{Nd(g)} |\Delta^k F(g)| \leq C_N'' M_2^{2k} \sum_{\tau\in\hat{K}} \sum_{j=0}^{k} |c_\tau| (1+|\tau|)^{2(k-j)+c_1} (2j)!\,.$$

From the lemma below we obtain that

$$\sum_{\tau\in\hat{K}} |c_\tau| (1+|\tau|)^{2(k-j)+c_1} \leq C M^{2k-2j} (2k-2j)!$$

for some constants $C, M > 0$ independent of k, j. Since

$$\sum_{j=0}^{k} (2k-2j)!(2j)! \leq 2^{2k}(2k)!$$

we conclude that F satisfies the estimates (3.1). □

Lemma 5.3.6. *Let $\epsilon > 0$. There exist $C, M > 0$ such that*

$$\sum_{\tau \in \hat{K}} e^{-\epsilon|\tau|}(1+|\tau|)^n \leq C\, M^n\, n!$$

for all $n \in \mathbb{N}$.

Proof. We assume for simplicity that K is semisimple. The proof is easily adapted to the general case. The set $\hat{K}$ is parametrized by a semilattice in $i\mathfrak{t}^*$, say $\hat{K} = \{m_1\tau_1 + \cdots + m_l\tau_l \mid m_1, \ldots, m_l \in \mathbb{N}\}$. We shall perform the summation over $\hat{K}$ by summing over $m \in \mathbb{N}$, and over those elements τ for which the maximal m_j is m. There are $l\, m^{l-1}$ such elements, and they all satisfy $am \leq |\tau| \leq bm$ for some $a, b > 0$ independent of m. It follows that the sum above is dominated by

$$\sum_{m \in \mathbb{N}} l m^{l-1} e^{-\epsilon a m}(1+bm)^n.$$

The given estimate now follows easily. □

Before we give the proof of Theorem 5.3.4, we recall some harmonic analysis for the compact homogeneous space $M\backslash K$. We denote by $K^\wedge_M$ the M-spherical part of $\hat{K}$, that is, the equivalence classes of irreducible representations τ for which the space V_τ^M of M-fixed vectors is non–zero. Then

$$L^2(M\backslash K) = \hat{\oplus}_{\tau \in K^\wedge_M} \operatorname{Hom}(V_\tau^M, V_\tau) \tag{3.5}$$

by the Peter–Weyl theorem. We write $v = \sum_\tau v_\tau$ for the corresponding decomposition of a function v on $M\backslash K$ and note that with the right action of $k \in K$ on $L^2(M\backslash K)$ we have $[\pi(k)v]_\tau = \tau(k) \circ v_\tau$.
Furthermore,

$$C^\omega(M\backslash K) = \left\{ v = \sum_\tau v_\tau \,\middle|\, \exists \epsilon, C > 0\, \forall \tau : ||v_\tau|| \leq C e^{-\epsilon|\tau|} \right\},$$

where $\|v_\tau\|$ denotes the operator norm of v_τ.
Let $\tau \in K^\wedge_M$. The integral $\delta_\tau(k) = \dim(\tau)\int_M \chi_\tau(mk)\, dm$ of the character is bi-invariant under M. The components of δ_τ in the decomposition (3.5) are all 0 except the τ-component, which is the inclusion operator I_τ of V_τ^M into V_τ.

Proof. We can now finally give the proof of Theorem 5.3.4. Let $v = \sum_\tau v_\tau \in C^\omega(M\backslash K)$ be given, and let $\epsilon > 0$ be as above.
It follows from [3], Section 6, that there exists a K–finite function $\xi \in V_\lambda$, and for each $\tau \in K^\wedge_M$ elements $a_\tau \in A$ and $c_\tau \in \mathbb{C}$ such that

$$d(a_\tau) \leq c_1 \log(1+|\tau|) + c_2, \quad |c_\tau| \leq 2(1+|\tau|)^{c_3}$$

for some constants $c_1, c_2, c_3 > 0$ independent of τ, and such that

$$R_\tau := \delta_\tau - c_\tau[\pi_\lambda(a_\tau)\xi]_\tau$$

satisfies $\|R_\tau\| \leq 1/2$ for all τ. By integration of ξ and R_τ over M we can arrange that they are both M–biinvariant.
We now choose a function $f \in \mathcal{A}(G)$ such that $\Pi(f)\xi = \xi$. It exists because $\Pi(\sum_{\tau\in F}\chi_\tau * h_t)\xi$ converges to ξ for $t \to 0$ and some finite set F of K-types by Lemma 5.2.6, so that ξ belongs to the closure of a finite dimensional subspace of $\Pi(\mathcal{A}(G))\xi$. According to Proposition 5.3.5, the function

$$F = \sum_\tau c_\tau e^{-\frac{1}{2}\epsilon|\tau|}\chi_\tau * L(a_\tau)f$$

belongs to $\mathcal{A}(G)$. An easy calculation shows that

$$\Pi(F)\xi = \sum_\tau e^{-\frac{1}{2}\epsilon|\tau|}(\delta_\tau - R_\tau)\ .$$

Being of type τ and M–biinvariant, R_τ corresponds in (3.5) to an operator $R_\tau \in \mathrm{End}(V_\tau^M)$. Since $\|R_\tau\| \leq 1/2$, the operator $I_\tau - R_\tau \in \mathrm{End}(V_\tau^M)$ is invertible with $\|(I_\tau - R_\tau)^{-1}\| \leq 2$. Then $v_\tau(I_\tau - R_\tau)^{-1} \in \mathrm{Hom}(V_\tau^M, V_\tau)$ with $\|v_\tau(I_\tau - R_\tau)^{-1}\| \leq 2Ce^{-\epsilon|\tau|}$. It follows that the function on $M\backslash K$ with the expansion

$$\sum_\tau e^{\frac{1}{2}\epsilon|\tau|}v_\tau(I_\tau - R_\tau)^{-1}$$

belongs to $C^\omega(M\backslash K)$. We denote by $h(k^{-1})$ this function, so that h is a right M–invariant function on K. Another easy calculation now shows that

$$\Pi(h)\Pi(F)\xi = \sum_\tau v_\tau = v\,,$$

and hence $h * F \in \mathcal{A}(G)$ is the function we seek. □

Unique analytic globalization

The goal of this section is to prove the following version of Schmid's minimal globalization theorem ([37], Theorem 2.13).

Theorem 5.3.7. *Let V be a Harish-Chandra module. Every analytic $\mathcal{A}(G)$-tempered globalization of V is isomorphic to $V^{\min}$.*
In particular, if (π, E) is an arbitrary F-globalization of V, then

$$E^\omega \simeq V^{\min}.$$

Proof. We first treat the case of an irreducible Harish-Chandra module V.
We first claim that V admits a Hilbert globalization $\mathcal{H}$ such that $\mathcal{H}^\omega = \Pi(\mathcal{A}(G))V$, and hence in particular (see Lemma 5.3.1 (ii))

$$\mathcal{H}^\omega \simeq V^{\min}.$$

In case $V = V_\lambda$ we can take $\mathcal{H}_\lambda = L^2(M\backslash K)$ and the assertion follows from Theorem 5.3.4. If the Harish-Chandra module is of the type $V = V_\lambda \otimes W$ where W is a finite dimensional G-module, then $\mathcal{H} = \mathcal{H}_\lambda \otimes W$ is a Hilbert globalization with $\mathcal{H}^\omega = \mathcal{H}_\lambda^\omega \otimes W$. A straightforward generalization of [3], Lemma 5.4, yields that

$$(\Pi_\lambda \otimes \Sigma)(\mathcal{A}(G))V = \mathcal{H}^\omega .$$

Finally, every irreducible Harish-Chandra module is a subquotient of some $V_\lambda \otimes W$ (see for example [40], Thm. 4.10), and the claim follows by Lemma 5.3.2.
Let now (π, E) be an arbitrary analytic $\mathcal{A}(G)$-tempered globalization of V. We aim to prove $E \simeq V^{\min}$. From Lemma 5.3.1 we know that $V^{\min}$ injects G-equivariantly and continuously into $E = E^\omega$, hence it suffices to establish surjectivity of the injection.
We now fix the Hilbert globalization $\mathcal{H}$ of above. In view of Proposition 5.3.3 we can embed (π, E) into a Banach globalization F of V. As E is analytic, we obtain a continuous G-equivariant injection $E \to F^\omega$. In order to proceed we recall the Casselman-Wallach theorem (cf. [11], [53], or [3] for a more recent proof) which implies that F^∞ is equivalent to $\mathcal{H}^\infty$ as F-representation. It follows, see Corollary 4.2.5, that $F^\omega \simeq \mathcal{H}^\omega$. Collecting the established isomorphisms, we have

$$V^{\min} \to E \subset F^\omega \simeq \mathcal{H}^\omega \simeq V^{\min}.$$

The surjectivity follows from the completeness of $\mathcal{H}^\omega$ (see Proposition 5.1.1).
Finally, we prove the case of an arbitrary Harish-Chandra module. As Harish-Chandra modules have finite composition series, it suffices to prove the following statement: Let $0 \to V_1 \to V \to V_2 \to 0$ be an exact sequence of Harish-Chandra modules and suppose that both V_1 and V_2 have unique analytic $\mathcal{A}(G)$-tempered globalizations. Then so does V.
Let E be an analytic $\mathcal{A}(G)$-tempered globalization of V. Let E_1 be the closure of V_1 in E and $E_2 = E/E_1$. Then E_1 and E_2 are analytic $\mathcal{A}(G)$-tempered globalizations of V_1 and V_2. By assumption we get $E_1 = V_1^{\min}$ and $E_2 = V_2^{\min}$, and from Lemma 5.3.2 we infer $V_2^{\min} = V^{\min}/V_1^{\min}$. Observe that in an exact sequence of topological vector spaces $0 \to E_1 \to E \to E_2 \to 0$ the topology on E is uniquely determined by the topology of E_1 and E_2 (see [20], Lemma 1). We thus conclude that $E = V^{\min}$. $\square$

We conclude by summarizing the topological properties of $V^{\min}$. Recall that an inductive limit $E = \lim_{n\to\infty} E_n$ of Fréchet spaces is called regular if every bounded set is contained and bounded in one of the steps E_n.

Corollary 5.3.8. *The minimal globalization $V^{\min}$ is a nuclear, regular, reflexive and complete inductive limit of Fréchet–Montel spaces.*

Proof. Theorem 5.3.7 and Proposition 5.1.1 imply that $V^{\min}$ is complete. Furthermore, it then follows from [55] and [39] that $V^{\min}$ is regular and reflexive (see also the appendix). It is an inductive limit of Fréchet–Montel spaces, because $\mathcal{A}(G)$ is an inductive limit of Fréchet–Schwarz spaces and Hausdorff quotients of such spaces are Fréchet–Montel. Nuclearity is inherited from $C^\omega(M\backslash K)$, which is the strong dual of a nuclear Fréchet space and this property is preserved when passing to the quotient of a finite dimensional tensor product. □

5.4 Appendix. Topological properties of $\mathcal{A}(G)$

While the topology of a general inductive limit of Fréchet spaces may be complicated, $\mathcal{A}(G)$ inherits certain properties from the steps $\mathcal{A}(G)_n$.

Theorem 5.4.1. *The algebra $\mathcal{A}(G)$ is regular, complete and reflexive.*

A regular inductive limit of Fréchet–Montel spaces is known to be reflexive [39] and complete [55], so that we only have to show regularity. The following criterion from [55], Theorem 3.3, in terms of interpolation inequalities will be convenient:

Proposition 5.4.2. *An inductive limit $E = \lim_{n\to\infty} E_n$ of Fréchet-Montel spaces is regular if and only if for some fundamental system $\{p_{n,\nu}\}_{\nu\in\mathbb{N}}$ of seminorms on E_n: $\forall n\ \exists m > n\ \exists \nu\ \forall k > m\ \forall \mu\ \exists \kappa\ \exists C\ \forall f \in E_n$*

$$p_{m,\mu}(f) \ \leq\ C(p_{k,\kappa}(f) + p_{n,\nu}(f)) \ . \tag{4.1}$$

In the case of $\mathcal{A}(G)$, condition (4.1) should be thought of as a weighted geometric relative of Hadamard's Three Lines Theorem. To verify it, we need to introduce some notions from complex and Riemannian geometry, starting with the appropriate differential operators.

By common practice we identify the Lie algebra $\mathfrak{g}_\mathbb{C}$ with the space of right–invariant vector fields on $G_\mathbb{C}$, where $X \in \mathfrak{g}_\mathbb{C}$ corresponds to the differential operator

$$\widetilde{X}u(x) = \frac{d}{dt}\Big|_{t=0} u(\exp(-tX)x) \quad (x \in G_\mathbb{C},\ u \in C^\infty(G_\mathbb{C})).$$

If we denote the complex structure on the Lie algebra $\mathfrak{g}_\mathbb{C}$ by J, the Cauchy–Riemann operators $\overline{\partial}_Z$ and ∂_Z associated to $Z \in \mathfrak{g}_\mathbb{C}$ are given by $\overline{\partial}_Z := \widetilde{Z} + i\widetilde{JZ}$ resp. $\partial_Z := \widetilde{Z} - i\widetilde{JZ}$.

In this section it will be convenient to replace the left G-invariant metric $\mathbf{g}$ on G used in Section 4.1 by a right invariant one, which we shall denote by the same symbol. Note that the corresponding distance functions d on G are equivalent (see Remark 4.1.3). The function

$$K(\exp(JX)g) := \frac{1}{2}|X|^2 := \frac{1}{2}\mathbf{g_1}(X,X)$$

endows a sufficiently small complex neighborhood VG of G with a right G–invariant Kähler structure. To see this, choose an orthonormal basis $\{X_j\}_{j=1}^l$ of $\mathfrak{g}$ with respect to the metric. A straightforward computation results in

$$\partial_{X_i}\overline{\partial}_{X_j}K(\mathbf{1}) = \mathbf{g_1}(X_i, X_j),$$

so that the complex Hessian $(Z_1, Z_2) \mapsto \partial_{Z_1}\overline{\partial}_{Z_2}K(\mathbf{1})$ defines a positive definite Hermitian form on $\mathfrak{g}_{\mathbb{C}}$. By continuity and invariance, positivity extends to give a Kähler metric on a small neighborhood VG.
The complex Laplacian

$$\Delta_{\mathbb{C}} = \sum_{j=1}^{l} \partial_{X_j}\overline{\partial}_{X_j} = \sum_{j=1}^{l} \widetilde{X_j}^2 + \widetilde{JX_j}^2,$$

agrees with the Kähler Laplacian up to first–order terms and maps real–valued functions to real–valued functions. Therefore the following weak maximum principle holds:

Lemma 5.4.3. *If $u \in C^2(VG)$ is real–valued with a local maximum in $z \in VG$, then*

$$\Delta_{\mathbb{C}}u(z) \leq 0.$$

As $\Delta_{\mathbb{C}}$ is a trace of the complex Hessian, we may rely on well–known results about plurisubharmonic functions to conclude:

Lemma 5.4.4. *For $u \in \mathcal{O}(VG)$, $\Delta_{\mathbb{C}}u = 0$ and $\Delta_{\mathbb{C}} \log|u| \geq 0$.*

So while it may be less obvious how to control applications of $\Delta_{\mathbb{C}}$ to the Riemannian distance function d on G, $\Delta_{\mathbb{C}}$ annihilates the holomorphically regularized distance function $\tilde{d} := e^{-\Delta_{\mathfrak{g}}}d$ from Chapter 4. This is going to be useful in the proof of Theorem 5.4.1, and the following Lemma, whose proof is similar to the one of Lemma 4.4.5, collects the key properties of $\tilde{d}$.

Lemma 5.4.5. *a) The function $\tilde{d}$ extends to a function in $\mathcal{O}(UG)$ for some neighborhood U of $\mathbf{1} \in G_{\mathbb{C}}$.*
b) For all $U' \Subset U$, $\sup_{zg \in U'G} |\tilde{d}(zg) - d(g)| < \infty$ and $\widetilde{X_j}\ \tilde{d}$ as well as $\widetilde{JX_j}\ \tilde{d}$ are bounded on $U'G$ for all j.

Before finally coming to the proof of Theorem 5.4.1, we introduce an equivalent representation of $\mathcal{A}(G)$ based on geometrically more convenient neighborhoods. If we define for $n \in \mathbb{N}$, $\nu \in \mathbb{N}_0$, the neighborhoods

$$\widetilde{V}_n := \left\{ \exp(JX) \in G_{\mathbb{C}} \mid |X| < \frac{1}{n} \right\},$$

$$\Omega_n^\nu := \left\{ \exp(JX) \in G_{\mathbb{C}} \mid |X| < \frac{1}{n + (\nu+2)^{-1}} \right\}$$

and associated subspaces of $\mathcal{A}(G)$,

$$\widetilde{\mathcal{A}(G)}_n := \left\{ f \in \mathcal{O}(\widetilde{V}_n G) \mid \forall \nu \in \mathbb{N} : p_{n,\nu}(f) := \sup_{g \in G, z \in \Omega_n^\nu} |f(zg)| \, e^{\nu d(g)} < \infty \right\},$$

then $\mathcal{A}(G)$ is again an inductive limit $\lim_{n\to\infty} \widetilde{\mathcal{A}(G)}_n$ of Fréchet–Montel spaces. Condition (4.1) translates into

$$\sup_{zg \in \Omega_m^\mu G} |f(zg)| \, e^{\mu d(g)} \le C \, (\sup_{zg \in \Omega_k^\kappa G} |f(zg)| \, e^{\kappa d(g)} + \sup_{zg \in \Omega_n^\nu G} |f(zg)| \, e^{\nu d(g)}) \tag{4.2}$$

for $f \in \widetilde{\mathcal{A}(G)}_n$.

To show this, let n sufficiently large, $0 \not\equiv f \in \widetilde{\mathcal{A}(G)}_n$, $m = n+1$, $\nu = 0$, $k > m$ and $\mu \in \mathbb{N}$, and consider

$$u(z) = \log|f(z)| + N(z) \; D(z)$$

on $\widetilde{V}_n G \setminus \widetilde{V}_{k+1} G$, where we choose $N(\exp(JX)g) = N(\exp(JX)) = \bar{\nu}(|X|^{-2\alpha} - (n+\frac{1}{2})^{2\alpha})$ and $D(z) = D_0 + \operatorname{Re} \tilde{d}(z)$ for some $\bar{\nu}, \alpha, D_0 > 0$. First note that $\Delta_{\mathbb{C}} u > 0$ if D_0 and α are sufficiently large. Indeed, by Lemma 5.4.4 it is enough to show $\Delta_{\mathbb{C}}(N(z)D(z)) > 0$. But $\Delta_{\mathbb{C}} D = 0$, so that

$$\Delta_{\mathbb{C}}(N(z)D(z)) = (\Delta_{\mathbb{C}} N(z)) \; D(z) + \\ + 2 \sum_{j=1}^{l} \left\{ \widetilde{X}_j N(z) \widetilde{X}_j D(z) + \widetilde{JX}_j N(z) \widetilde{JX}_j D(z) \right\} .$$

With $D \ge 1$ on $\widetilde{V}_n G$ for large D_0 by Lemma 5.4.5, we only have to show that

$$\Delta_{\mathbb{C}} N(z) > \overline{D} \max_{j=1,\ldots,l} \{ |\widetilde{X}_j N(z)|, |\widetilde{JX}_j N(z)| \}$$

on $\widetilde{V}_n G$ for large n and $\overline{D} = 2 \; \sup\{|\widetilde{X}_j D|, \; |\widetilde{JX}_j D| \; : \; j = 1, \ldots, l\}$. By G–invariance, it is sufficient to do so in $z = \exp(\varepsilon JX)$ close to $\varepsilon = 0$. The Baker–Campbell–Hausdorff formula implies

$$\exp(tJX_j)\exp(\varepsilon JX) = \exp(\varepsilon JX + tJX_j + \mathcal{O}(\varepsilon t^2) + \mathcal{O}(\varepsilon^2 t)) \cdot \\ \cdot \exp(\frac{1}{2}\varepsilon t [JX_j, JX]) \; ,$$

so that

$$\begin{aligned}\widetilde{JX_j}\ N(\exp(\varepsilon JX)) &= \frac{d}{dt}\Big|_{t=0} N(\varepsilon JX + tJX_j + \mathcal{O}(\varepsilon t^2) + \mathcal{O}(\varepsilon^2 t)) \\ &= -2\alpha\bar{\nu}\ \varepsilon^{-1-2\alpha}\ \frac{\mathbf{g_1}(X_j, X)}{\mathbf{g_1}(X,X)^{\alpha+1}} + \mathcal{O}(\varepsilon^{-2\alpha}),\end{aligned}$$

Similarly,

$$\begin{aligned}&(\widetilde{JX_j})^2\ N(\exp(\varepsilon JX)) \\ &\quad = 2\alpha\bar{\nu}\ \varepsilon^{-2-2\alpha}\ \frac{2(\alpha+1)\mathbf{g_1}(X_j,X)^2 - \mathbf{g_1}(X_j,X_j)\mathbf{g_1}(X,X)}{\mathbf{g_1}(X,X)^{\alpha+2}}\end{aligned}$$

up to terms of order $\varepsilon^{-1-2\alpha}$. Summing over j establishes the assertion for large α and small ε, hence for large n.
For $\kappa \geq 0$, set $S_n := \sup_{\partial\Omega_n^0 G} u$ and $S_k^\kappa := \sup_{\partial\Omega_k^\kappa G} u$. Because $u(z)$ is bounded from above and $\leq \max\{S_k^\kappa, S_n\}$ on $\partial\Omega_k^\kappa G \cup \partial\Omega_n^0 G$, the maximum principle, Lemma 5.4.3, assures

$$u(z) \leq \max\{S_k^\kappa, S_n\}$$

in $\Omega_n^0 G \setminus \Omega_k^\kappa G$, or

$$\begin{aligned}|f(z)|\ e^{N(z)D(z)} &\leq e^{\max\{S_k^\kappa, S_n\}} \\ &\leq \sup_{w\in\partial\Omega_k^\kappa G} |f(w)|\ e^{N(w)D(w)} + \sup_{w\in\partial\Omega_n^0 G} |f(w)|\ e^{N(w)D(w)}.\end{aligned}$$

As $\widetilde{V}_m \Subset \Omega_n^0$, we may choose $\bar{\nu}$ such that $N|_{\widetilde{V}_m G\setminus\widetilde{V}_{k+1}G} \geq \mu$. Setting $\kappa := \sup_{\widetilde{V}_k G\setminus\widetilde{V}_{k+1}G} N \geq \mu$ we obtain

$$\begin{aligned}\sup_{z\in\Omega_m^\mu G} |f(z)|\ e^{\mu D(z)} &\leq \sup_{z\in\Omega_k^\kappa G} |f(z)|\ e^{\kappa D(z)} + \sup_{z\in\partial\Omega_n^0 G} |f(z)| \\ &\leq \sup_{z\in\Omega_k^\kappa G} |f(z)|\ e^{\kappa D(z)} + \sup_{z\in\Omega_n^0 G} |f(z)|.\end{aligned}$$

Lemma 5.4.5 implies $d(z) - C \leq D(z) \leq d(z) + C$ for some $C > 0$, and Theorem 5.4.1 follows.

Remark 5.4.6. It would be interesting to better understand the topology of $\mathcal{A}(G)^N/I$ for a stepwise closed, $\mathcal{A}(G)$–invariant subspace I. Because $\widetilde{\mathcal{A}(G)}_n$ is even Fréchet–Schwarz, the quotients $\widetilde{\mathcal{A}(G)}_n^N/(I \cap \widetilde{\mathcal{A}(G)}_n^N)$ are Fréchet–Montel and one might hope to verify condition (4.1) as before. However, adapting the above proof requires strong assumptions on I, and general Hausdorff quotients $\mathcal{A}(G)^N/I$ are likely to be incomplete: For a convex domain $\Omega \subset \mathbb{R}^n$, the space of test functions $\mathcal{D}(\Omega)$ is isomorphic to a similar weighted space of holomorphic functions by Paley–Wiener's theorem. However, given any non–surjective differential operator A on $\mathcal{D}'(\Omega)$, the quotient of $\mathcal{D}(\Omega)$ by the image of A^t will be incomplete.

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