Finite Element and Boundary Element Coupling for Fluid-Structure Interaction

Von der Fakultät Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover
zur Erlangung des Grades einer
Doktorin der Naturwissenschaften
Dr. rer. nat.

genehmigte Dissertation

von

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geboren am 09. Februar 1980 in Cali/Kolumbien

2010

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Tag der Promotion:	17. Dezember 2009

Abstract

This thesis deals with the coupling of finite elements and boundary elements to solve a fluid structure interaction problem. We consider a time-harmonic vibration and scattering problem for homogeneous, isotropic, elastic solids surrounded by a compressible, inviscid and homogeneous fluid.

We present a convergence analysis and implementation of the h-version of the FE/BE coupling methods that were introduced by Bielak et al. [6] for the two- and three-dimensional case. These methods combine integral equations for the exterior fluid and finite element methods for the elastic structure. The eigenvalues of the interior Helmholtz problem induce non-unique solutions of the integral equations. Therefore we focus on two stable variational formulations, a symmetric and a non-symmetric formulation. These formulations are stable in the sense that they now provide a unique solution. For both stable formulations we derive a posteriori error estimates, a residual error estimator and a hierarchical error estimator. We prove their reliability and efficiency. Numerical experiments underline our theoretical results. From the error estimators we compute local error indicators which allow us to develop an adaptive mesh refinement strategy. For the two-dimensional case we perform an adaptive algorithm using a blue-green refinement on triangles and for the three-dimensional case we use hanging nodes on hexahedrons.

Key words. Fluid structure interaction problem. FE/BE coupling method, Galerkin method, a posteriori error estimator, residual error estimator, two-level hierarchical error estimator, adaptive algorithm.

Zusammenfassung

Diese Arbeit behandelt die Kopplung von Finiten Elementen and Randelementen (FE/BE) zur Modellierung der Wechselwirkungen von Fluiden und Festkörpern. Wir betrachten ein zeitharmonisches Schwingungs- und Streuungsproblem für homogene, isotrope, elastische Festkörper, die von einem kompressiblen, reibungsfreien und homogenen Fluid umgeben sind.

Basierend auf der Methode von Bielak et al. [6] stellen wir unser Konvergenzanalyse und Implementierung der h-Version in zwei- und dreidimensionalen Fall vor. Diese Methoden verbinden Integralgleichungen für das Fluid und Finite Elemente für die elastische Struktur. Die Eigenwerte des inneren Helmholtz Problems führen zu nicht-eindeutigen Lösungen der Integralgleichungen. Daher konzentrieren wir uns auf zwei stabile Variationsformulierungen, eine symmetrische und eine nicht-symmetrische. Diese Formulierungen sind stabil in dem Sinne, dass sie eine eindeutige Lösung liefern.

Für beide stabilen Formulierungen leiten wir a-posteriori-Abschätzungen, einen residualen Fehlerschätzer und einen hierarchischen Fehlerschätzer her. Wir beweisen ihre Zuverlässigkeit und Effizienz. Numerische Experimente unterstreichen unsere theoretischen Ergebnisse. Mit Hilfe der Fehlerschätzer berechnen wir lokale Fehlerindikatoren, die es uns erlauben, eine adaptive Netzverfeinerungsstrategie zu entwickeln. Im zweidimensionalen Fall verwenden wir für den adaptiven Algorithmus eine Blau-Grün-Verfeinerung auf Dreiecken. Im dreidimensionalen Fall verwenden wir Hexaeder mit hängenden Knoten.

Schlagwörter. Fluid structure interaction problem. FE/BE-Kopplung, Galerkin-Verfahren, a posteriori Fehlerschätzer, residualer Fehlerschätzer, hierarchischer Fehlerschätzer, adaptive Verfahren.

Acknowledgements

Several people have been instrumental in allowing this project to be completed. I would like to thank my advisor Prof. E. P. Stephan, for giving me the opportunity to belong to his workgroup and for his encouragement and academic support during the realization of this project. I would also like to thank PD Matthias Maischak, for his constant encouragements, support, and help concerning the numerical analysis and numerical implementation of my investigations; his software package MaiProgs is the basis for the numerical experiments presented in this work. Also, my most sincere greetings to my co-reviewer, Prof. G. Hsiao for agreeing to review my thesis and for his corrections.

Many thanks to my colleagues at the Institute for Applied Mathematics of the Gottfried Wilhelm Leibniz Universität Hannover, especially to Elke Ostermann, Dr. Ricardo A. Prato, Dr. Florian Leydecker, Michael Andres and Leo Nesemann for advice, help and the numerous discussions, more or less related to this thesis, to German language and mostly with life. Many thanks to all of them for their friendship and camaraderie. They helped change my life positively and will be forever in my heart and my mind.

I wish to extend my thanks to the whole staff at the Institute for Applied Mathematics. Particularly to Mrs. Carmen Gatzen and Mrs. Ulla Fleischhauer for their kindness and dedication, and their essential support on technical issues and prototyping. Also, I would like to thank Mr. Dieter Janz.

I also warmly thank my Family for standing by me in good and bad times.

This project would not have been possible without the general support of the project *DFG Graduiertenkolleg 615* that provided me the PhD scholarships.

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Introduction

The problem of determining the manner in which an incoming acoustic wave is scattered by an elastic body immersed in a fluid is of central importance in detecting and identifying submersed objects. This is a special case of a fluid-structure interaction problem. One major approach for examining fluid-structure interaction phenomena are numerical methods. These transform the physical problem into a system of partial differential equations, which can be solved with different discretization methods. The Finite Element Method (FEM) is one of the most commonly used methods in this field due to its high flexibility and also its applicability to nonhomogeneous physical problems. However, for problems on the boundary, as well as exterior problems its accuracy and corresponding modifications are not efficient. In these situations, the *boundary integral equations* in combination with finite element analysis on the boundary have led to a theoretical and efficient computational tool, the Boundary Element Method (BEM) (see e.g. Hsiao and Wendland [47] or Stephan [75]). The main idea of this method is to eliminate the field equations in the domain and to reduce the boundary value problems to equivalent equations only on the boundary, requiring the knowledge of corresponding fundamental solutions.

Through the combination of both methods an *FE/BE coupling method* arises (see e.g. Costabel and Stephan [24]). Suppose one has equations $Lu = 0$ in a domain Ω and $L_+ p = 0$ in its complement Ω^+, with certain transmission conditions across $\Gamma = \partial\Omega$ between u and p. The coupling then consists in reducing the equations in the exterior domain Ω^+ to boundary integral equations by using an integral representation corresponding to the operator L_+ in Ω^+. One then uses the transmission conditions to obtain a relation between u and p on the boundary. Therefore the problem is reduced to one defined only over the finite region occupied by the solid Ω, with associated boundary conditions which will contain integral operators.

In our case, the problem under consideration consists of determining the dynamics in a fluid Ω^+ and displacements in an elastic body Ω due to a given excitation in the fluid Ω^+ using an FE/BE coupling method. Here, Ω is a bounded region in $\mathbb{R}^d$ $(d = 2, 3)$, with boundary Γ, and Ω^+ is the complement of $\bar{\Omega}$. We consider the scattering of time-harmonic acoustic waves by a bounded elastic obstacle, immersed in a compressible, inviscid, homogeneous fluid. For this type of problem, the displacement field $\mathbf{u}$ in the domain Ω and the pressure field p in the fluid are unknown. For a comprehensive survey of the subject, including descriptions of various physical applications, see Gaunard [35].

In Bielak et al. [5, 6, 83] and Hsiao et al. [44] an FE/BE coupling method for an elastic body is presented to solve the scattering problem, using standard integral representations in the infinite exterior region occupied by the fluid. These methods, however, suffer from the same common defect associated with the integral formulations for purely exterior regions; namely, there is a discrete set of frequencies for which the method fails. Two techniques have been used in applications to remedy this situation. One was developed by Burton and Miller [10], combining linearly the surface Helmholtz integral equation and its normal derivative, derived from Green's second theorem. This method always leads to unique solutions if a certain coupling constant α has a nonvanishing imaginary part. An earlier procedure, given by Brakhage and Werner [8], complementary to that of Burton and Miller, but used far less frequently, represents the solution in the exterior region as a linear combination of a single layer and a double layer potential, with the coupling constant α again required to have a nonvanishing imaginary part. Kress [50] investigated how to choose the parameter α in order to minimize the condition number of the discrete system derived from the integral equation, finding that the value $\alpha = 1/k$ is an optimal value, where k is the wavenumber of the acoustic waves in the fluid Ω^+.

The major drawback with those procedures has been that these formulations contain hypersingular integrals involving the second partial normal derivative of Green's function. However, special integration techniques have been developed to remedy this difficulty (see Nedelec [67], Meyer et al. [60], Hamdi [38], and Holm et al. [42]). Due to this, the majority of numerical work is focused on the case when the solid is a rigid body, see e.g. Meyer et al. [60] , Krishnasamy [51], Demkowicz [30], and Maischak et al. [57].

With respect to the numerical implementation of the FE/BE coupling for a fluid-structure interaction problem, there is the work of Bielak et al. [6], Chang and Demkowicz [18] and Gatica et al. [34]. Bielak et al. present numerical results for the two-dimensional case of a symmetric variational formulation, which is obtained using the procedures of Brakhage-Werner and Burton-Miller simultaneously. Chang and Demkowicz present the *hp*-numerical implementation of a variational formulation obtained by the procedure of Burton-Miller and an adaptive *hp*-method based on a residual error estimate that depends only on the pressure in the fluid for a scattering problem in a hollow sphere. While Gatica et al. present a mixed finite element method for a fluid-solid interaction problem posed in the plane (see Hamdi and Jean [37]). Here, a coupling of primal and dual mixed finite element methods is applied to compute both the pressure of the scattered wave in the linearized fluid and the elastic vibrations that take place in the elastic body.

The main objective of this thesis is the implementation and analysis of the h-version of the FE/BE coupling methods presented by Bielak et al. [6] for two- and three-dimensional cases. Throughout the work, we have focused on two *stable variational formulations*, the *symmetric formulation* (VP_1) and the *non-symmetric formulation* (VP_2). We call them stable formulations, because they lead to unique solutions if the coupling constant α has a nonvanishing imaginary part. The non-symmetric formulation stems from the procedure of Brakhage-Werner and the symmetric formulation from using the procedures of Brakhage-Werner and Burton-Miller simultaneously.

The sesquilinear forms corresponding to the variational formulations (VP_1) and (VP_2) are in general not positive definite but satisfy the *Gårding inequality*, since they are of the form $(\mathcal{D}+\mathcal{K})u = f$ where $\mathcal{D}$ is a positive definite and $\mathcal{K}$ is a compact sesquilinear form. This allows to apply abstract results of existence and uniqueness of a variational problem, as well as for the stability and convergence analysis of the FE/BE coupling method. The sesquilinear form $\mathcal{D}$ induces an energy norm of the problem.

We present reliability and efficiency of two new a-posteriori error estimates for the stable formulations, a *residual error estimate* and a *hierarchical error estimate*, respectively, which guarantee a quasi-optimal bound of the error in the energy norm induced by $\mathcal{D}$. Based on these a-posteriori error estimates, we define local indicators and present adaptive algorithms for the mesh refinement of the coupling procedure.

The *residual error estimate* is formulated in the L^2-norm using standard techniques for FE methods, see e.g. Johnson et al. [49], Stewart et al. [78] or Braess [7] and techniques for FE/BE coupling methods e.g. Carstensen and Stephan [17, 15, 16]. To prove its reliability we use arguments of duality, see e.g. Hsiao and Wendland [46] or Costabel and Stephan [25]. The efficiency proof is based on techniques used by Verfürth [80] and Leydecker [52] for the indicators of the FEM part. For indicators arising from with boundary integral operators we implement some ideas of Carstensen [11] and Chernov [19].

The hierarchical error estimate is an extension of the multilevel adaptive refinement strategies introduced by Yserentant [82] for 2D elliptic finite element problems. These strategies were extended to indefinite and nonlinear problems by Bank and Smith [4]. The first applications using hierarchical methods applied to boundary element methods were published by Stephan et al. [76], Mund and Stephan [64, 66], Mund et al. [63] and Maischak et al. [57]. In fact, our estimate is a direct combination of the work of Mund and Stephan [64] and Maischak et al. [57]. The first authors present an error estimator for an FE/BE coupling of a non linear equation based on the Laplace operator; the second derives an error estimator for an indefinite problem of the form $(\mathcal{D} + \mathcal{K})u = f$ stemming from a BE method for an acoustic scattering problem.

This thesis is organized as follows:

In **Chapter 1** we recall main definitions and concepts necessary in the forthcoming analysis. Here, we focus on the boundary integral equations of the Helmholtz equation and describe some basic properties of boundary integral operators. We remark the problem of uniqueness of solutions for the integral equation on Γ in an exterior problem. We present the representations given by Brakhage and Werner [8] and Burton and Miller [10] for solving the problem of uniqueness of integral equations concerning this problem. Also, we give some technical considerations for calculating the fundamental solutions for the Helmholtz equation.

Chapter 2 is devoted to the fluid-structure problem. The analysis described in this chapter is a compilation of the works of Bielak et al. [6], Bielak and MacCamy [5] and Hsiao et al. [45, 43, 44]. We present the *reduced* problems obtained by adapting the boundary integral equations for the exterior problem. Here, *reduced problem* means that the problem is reduced to finding the displacement $\mathbf{u}$ inside the domain Ω, and the pressure p only on the boundary Γ. Section 2.3 presents the variational formulations of the reduced problems and in Section 2.4 the discretization using the Galerkin method with linear test and trial functions is described. An a priori error analysis is carried out, where a convergence rate of the order $\mathcal{O}(h)$ in the energy norm is obtained.

In **Chapter 3** we derive an *a posteriori residual error estimate* for the solution of the non-symmetric and symmetric formulations. Here we present reliability and efficiency of the estimators for both formulations. The efficiency is proven for quasi-uniform meshes assuming regularity conditions on the solution. Additionally, an adaptive algorithm for the mesh refinement of the coupling procedure is given.

In **Chapter 4** we derive an *a posteriori hierarchical error estimate* for the solution of the non-symmetric and symmetric formulations. This estimator is based on the work of Mund and Stephan [65] and Maischak et al. [57]. The reliability and efficiency of the estimate for the case of quasi-uniform meshes, under the assumption of a saturation condition for the non-symmetric and symmetric formulations is proven. Also, we present an analysis for the relation of the hierarchical and residual error indicators based on the analysis of Carstensen et al. [14].

In **Chapter 5** and **Chapter 6** we present numerical experiments for the two- and three-dimensional case. The discretizations of the non-symmetric and symmetric formulations are implemented and solved in the scientific program package MaiProgs [58]. Here, we present the performance of non stable ($\alpha = 0$) and stable ($\alpha = i/k$) methods for different wave numbers k. In the following sections the convergence of h-uniform and adaptive refinements for each stable formulations using $\alpha = i/k$ is shown. Also, for the stable formulations the hierarchical error estimators from Chapter 4 and the residual error estimator from Chapter 3 are implemented to use adaptive refinements.

In the two-dimensional case, triangles are used for the discretization of the domain Ω and the green-refinement technique is applied in the adaptive algorithm. In the three-dimensional case, we use hexahedrons for the discretization of the domain Ω and the hanging nodes technique is applied for the adaptive refinement.

In the Appendix, we describe some characteristics of *hanging nodes* for the adaptive strategy in 3D.

Throughout this thesis, vector-valued functions are written in bold letters, scalar functions in normal typed letters. The symbol $\lesssim$ signifies "$\leq$ up to a multiplicative constant $c > 0$". The symbol $\simeq$ means "$\lesssim$ and $\gtrsim$".

Chapter 1

Notations and Definitions

We start this chapter with a brief introduction into the main concepts and definitions connected with the Sobolev spaces used and some standard notation for distributions (see e.g. Girault and Raviart [72], Hsiao and Wendland [47] and Lions and Magenes [53]).

In the sequel, we deal with complex valued functions and the symbol i is used for $\sqrt{-1}$. For $\alpha \in \mathbb{C}$ we write $\bar{\alpha}$ and $|\alpha|$ for the conjugate and the modulus of a complex number, respectively. The symbol $|\cdot|$ is used to denote the euclidean norm of vectors in $\mathbb{C}^d$, i.e.

$$|x| = x \cdot \bar{x} = \big(\sum_{i=1}^{d} x_i\, \bar{x}_i\big)^{1/2}.$$

We use boldface letters to denote vector-valued functions. In the following, let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open, bounded, simply connected domain with a closed smooth boundary $\Gamma := \partial\Omega$ and its exterior complement $\Omega^+ := \mathbb{R}^d \backslash \bar{\Omega}$.

We introduce the standard definition of the $L^2(\Omega)$-space as the set of all functions $u : \Omega \to \mathbb{C}$ which are square-integrable over Ω in the sense of Lebesgue. $L^2(\Omega)$ is a Hilbert space with scalar product

$$(u, v)_0 := (u, v)_{L^2} := \int_\Omega u(x)\, \bar{v}(x)\, dx$$

and the corresponding norm

$$\|u\|_0 = \sqrt{(u, u)_0}.$$

For $u \in L^2(\Omega)$, $\partial^\alpha u$ represents the *weak derivative* in $L^2(\Omega)$ given by

$$\partial^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_i^{\alpha_1} \cdots \partial x_d^{\alpha_d}},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_i \in \mathbb{N}_0$ is a multiindex and $|\alpha| := \alpha_1 + \cdots + \alpha_d$ and assuming that $\partial^\alpha u \in L^2(\Omega)$ and

$$(\phi, \partial^\alpha u)_0 = (-1)^{|\alpha|}(\partial^\alpha \phi, u)_0 \quad \forall \phi \in C_0^\infty(\Omega).$$

Here $C^\infty(\Omega)$ denotes the space of infinitely times differentiable functions, and $C_0^\infty(\Omega)$ denotes the set of $C^\infty(\Omega)$ functions with compact support in Ω.
We define the *Sobolev space* $H^m(\Omega)$ for a given integer $m \geq 0$ by

$$H^m(\Omega) = \{u \in L^2(\Omega) \,|\, \partial^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \leq m\}.$$

The scalar product on $H^m(\Omega)$ is defined by

$$(u, v)_m := \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v)_0,$$

the associated norm

$$\|u\|_m := (u, u)_m^{1/2} = \Big(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_0 \Big)^{1/2},$$

and the corresponding semi-norm

$$|u|_m := \Big(\sum_{|\alpha| = m} \|\partial^\alpha u\|_0^2 \Big)^{1/2}$$

We write $H^m(\Omega)$ instead of H^m, and $\|\cdot\|_{m,\Omega}$ instead of $\|\cdot\|_m$ whenever the corresponding domain is important to distinguish.
We now define the *Sobolev spaces on the boundary* Γ which are necessary to define the integral operators (for details see e.g. Dautray and Lions [27], Sauter and Schwab [73] or Hsiao and Wendland [47]). The L^2-space on Γ is defined similarly to the space $L^2(\Omega)$ and equipped with the norm

$$||u||_0 := \left(\int_\Gamma |u(x)|^2 \, ds_x \right)^{1/2}.$$

Here it is assumed, that there exists a piecewise parameterization of the boundary

$$\chi : \xi \mapsto x, \quad \xi = (\xi_1, \dots, \xi_{d-1}) \in \mathcal{G} \subset \mathbb{R}^{d-1}, \quad x \in \Gamma.$$

The definition of higher order Sobolev spaces on Γ requires the partial derivatives with respect to the parameter ξ

$$\partial^\alpha u(x) := \left(\frac{\partial}{\partial \xi_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial \xi_{d-1}}\right)^{\alpha_{d-1}} u(\xi_1, \dots, \xi_{d-1}), \qquad x \in \Gamma.$$

It should be noted, that the existence of the derivative $\partial^\alpha u(x)$ with $|\alpha| \leq l$ depends on the smoothness of Γ. In particular, $\Gamma \in C^{l-1,1}(\mathcal{G})$ provides the existence of $\partial^\alpha u(x)$ for $|\alpha| \leq l$. Now we can define the Sobolev spaces on the boundary of order $k \in \mathbb{N}_0$, $k \leq l$, as the closure of the space $\{u \in C^\infty(\Gamma) : ||u||_k < \infty\}$ with respect to the norm

$$||u||_k := \left(\sum_{|\alpha| \leq k} ||\partial^\alpha u||^2_{L_2(\Gamma)}\right)^{1/2}.$$

The generalization onto the case of the Sobolev spaces of real positive order $s = k + r$, where $k \in \mathbb{N}_0$, $r \in (0, 1)$ is realized by the corresponding Sobolev-Slobodeckii norm

$$||u||_s := \left(||u||_k^2 + |u|_r^2\right)^{1/2}$$

with the half-norm

$$|u|_r := \left(\sum_{|\alpha| = r} \int_\Gamma \int_\Gamma \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x - y|^{d-1+2r}} \, ds_x ds_y\right)^{1/2}.$$

Employing the dual product for $u \in H^{1/2}(\Gamma)$ and $v \in H^{-1/2}(\Gamma)$

$$\langle u, \bar{v} \rangle := \int_\Gamma u(x) \overline{v(x)} \, ds_x,$$

we introduce the Sobolev spaces $H^{-s}(\Gamma)$ of negative order for $s \in (0, l]$ as the dual spaces to $H^s(\Gamma)$

$$H^{-s}(\Gamma) = (H^s(\Gamma))', \qquad s > 0,$$

with the norm

$$||u||_{-s} := \sup_{0 \neq v \in H^s(\Gamma)} \frac{\langle u, v \rangle}{||v||_s}.$$

Throughout this thesis, we use subscripts in dual products, as $\langle \cdot, \cdot \rangle_\Gamma$ or $\langle \cdot, \cdot \rangle_s$ or $\langle \cdot, \cdot \rangle_{s,\Gamma}$, whenever it is important to distinguish the domain or the order of the space.

1.1 Boundary Integral Operators

Let $L_k \phi := \Delta \phi + k^2 \phi$ be the Helmholtz operator for $k \in \mathbb{R}$ and $k \neq 0$. For the definition of the boundary integral operators we need the *fundamental solution* of the Helmholtz equation $\gamma_k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$\gamma_k(x, y) := \begin{cases} \frac{i}{4} H_0^1(k|x - y|), & \text{for } d = 2, \\ \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, & \text{for } d = 3, \end{cases} \tag{1.1}$$

which satisfies $L_k \gamma_k(x, y) = 0$ for $x \neq y$, where H_0^1 is the Hankel function of the first kind and order zero (see (1.19) and Colton and Kress [21]).
ϕ satisfies the *Sommerfeld radiation conditions* (see e.g. Hsiao and Wendland [47]), if for $x \in \Omega^+$

$$\phi(x) = \mathcal{O}(|x|^{-(d-1)/2}), \qquad -ik\phi(x) + \frac{d\phi}{d|x|}(x) = o(|x|^{-(d-1)/2}), \quad |x| \to \infty. \tag{1.2}$$

Now, for $x \in \Omega$ or $x \in \Omega^+$, let the *single layer potential* be defined by

$$S(\phi)(x) = \int_\Gamma \phi(y) \gamma_k(x, y) \, ds_y, \tag{1.3}$$

and the *double layer potential* defined by

$$D(\phi)(x) = \int_\Gamma \phi(y) \frac{\partial \gamma_k(x, y)}{\partial n_y} \, ds_y, \tag{1.4}$$

where $\frac{\partial \phi}{\partial n} := \nabla\phi \cdot \mathbf{n}$ is the normal derivative of ϕ. On Γ we consider the following integral operators: For $x \in \Gamma$ and $\phi : \Gamma \to \mathbb{C}$ we define the *single layer operator* as

$$V(\phi)(x) := \int_\Gamma \phi(y)\gamma_k(x,y)ds_y,$$

the *double layer operator*

$$K(\phi)(x) := \int_\Gamma \phi(y)\frac{\partial\gamma_k(x,y)}{\partial n_y}ds_y,$$

the *adjoint double layer operator*

$$K'(\phi)(x) := \frac{\partial}{\partial n_x}\int_\Gamma \phi(y)\,\gamma_k(x,y)ds_y,$$

and the *hypersingular operator*

$$W(\phi)(x) := -\frac{\partial}{\partial n_x}\int_\Gamma \phi(y)\frac{\partial}{\partial n_y}\gamma_k(x,y)ds_y.$$

For more details about the above introduced integral operators for the Helmholtz case, see eg. Hsiao and Wendland [47].

We take the limit $x \to \Gamma$ of (1.3) and (1.4) and their normal derivatives, which yields the following *jump relations*:

$$\begin{aligned} S\phi^\pm &= V\phi, \\ D\phi^\pm &= \big(K \pm \frac{I}{2}\big)\phi, \\ \frac{\partial}{\partial n}S\phi^\pm &= (K' \mp \frac{I}{2})\phi, \\ \frac{\partial}{\partial n}D\phi^\pm &= -W\phi, \end{aligned} \tag{1.5}$$

where the upper-indices "−" or "+" indicate from which direction (interior or exterior) the limit is being taken.

For a solution of $L_k\phi = 0$ in Ω or Ω^+ with the Sommerfeld radiation condition, one has the following representation formulas:

$$\phi = S(\frac{\partial \phi^-}{\partial n}) - D(\phi^-) \quad \text{in } \Omega, \tag{1.6}$$

$$\phi = D(\phi^+) - S(\frac{\partial \phi^+}{\partial n}) \quad \text{in } \Omega^+. \tag{1.7}$$

Using the integral representations (1.6) and (1.7) and the jump relations (1.5), we obtain the following boundary integral equations:

Representation in Ω, for $x \in \Gamma$

$$(K + \frac{I}{2})\,\phi(x) = V\frac{\partial \phi(x)}{\partial n}, \tag{1.8a}$$

$$W\phi(x) = -(K' - \frac{I}{2})\frac{\partial \phi(x)}{\partial n}, \tag{1.8b}$$

and in Ω^+, for $x \in \Gamma$

$$(K - \frac{I}{2})\phi(x) = V\frac{\partial \phi(x)}{\partial n}, \tag{1.9a}$$

$$W\phi(x) = -(K' + \frac{I}{2})\frac{\partial \phi(x)}{\partial n}. \tag{1.9b}$$

The following Lemma collects a number of properties of integral operators needed in this thesis:

Lemma 1.1. *1. Let $\phi, \psi \in H^{-1/2}(\Gamma)$ and $\eta, \varphi \in H^{1/2}(\Gamma)$, then there holds*

$$\langle V\phi, \bar{\psi}\rangle = \langle \bar{\phi}, V\psi\rangle, \quad \langle K\eta, \bar{\phi}\rangle = \langle \bar{\eta}, K'\phi\rangle, \quad \langle W\varphi, \bar{\eta}\rangle = \langle \bar{\varphi}, W\eta\rangle. \tag{1.10}$$

2. The operators

$$\begin{aligned} V :& H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \\ K :& H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \\ K' :& H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma), \\ W :& H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \end{aligned} \tag{1.11}$$

are continuous.

Proof. See Costabel and Stephan [23, Lemma 3.9].

1.1.1 Existence and uniqueness of a solution for the Helmholtz problem

Here we give necessary and sufficient conditions for the existence and uniqueness of the solution of the Helmholtz problem. For more details about this problem consult e.g. Sauter and Schwab [73], Colton and Kress [21] and Dautray and Lions [27].
The interior problems of Dirichlet and of Neumann

$$\begin{aligned} L_k\phi = \Delta\phi + k^2\phi = 0 \text{ in } \Omega, \\ \phi = g_D \text{ on } \Gamma, \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} L_k\phi = \Delta\phi + k^2\phi = 0 \text{ in } \Omega, \\ \frac{\partial\phi}{\partial n} = g_N \text{ on } \Gamma, \end{aligned} \tag{1.13}$$

admit a unique solution in $\phi \in H^1(\Omega)$ for given $g_D \in H^{1/2}$ (resp. $g_N \in H^{-1/2}$) except for certain values of k^2 called *eigenvalues*. k^2 is an eingenvalue of Dirichlet (or Neumann) problem for the negative laplace operator $-\Delta$. The eigenvalues form a countable sequence of real numbers wich diverge to infinity. The eigenvalues of the interior Dirichlet problem (1.12) are in general distinct from those of the interior Neumann problem (1.13).
For the exterior domain we consider the following exterior Dirichlet and Neumann problems:

$$\begin{aligned} L_k\phi = \Delta\phi + k^2\phi = 0 \text{ in } \Omega^+, \\ \phi = g_D \text{ on } \Gamma, \\ \phi \text{ satisfies the Sommerfeld radiation condition (1.2)}, \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} L_k\phi = \Delta\phi + k^2\phi = 0 \text{ in } \Omega^+, \\ \frac{\partial\phi}{\partial n} = g_N \text{ on } \Gamma, \\ \phi \text{ satisfies the Sommerfeld radiation condition (1.2)}, \end{aligned} \tag{1.15}$$

which admit a unique solution $\phi \in H^1_{loc}(\Omega^+)$ for $g_D \in H^{1/2}(\Gamma)$ (resp. $g_N \in H^{-1/2}(\Gamma)$), see e.g. Dautray and Lions [27, Vol. 4 p. 143]. We also have the following results.

Theorem 1.2. *1. The single layer operator V is invertible if k^2 is not an eigenvalue of the interior Dirichlet problem* (1.12)*.*

2. The operator $K' - \frac{I}{2}$ is injective if k^2 is not an eigenvalue of the interior Dirichlet problem (1.12)*.*

3. The operators $K + \frac{I}{2}$ and W are invertible if k^2 is not an eigenvalue of the interior Neumann problem (1.13)*.*

Proof. See e.g. Dautray and Lions [27, Vol. 4] or Sauter and Schwab [73].

Remark 1.3. *These considerations show the following: Although the solutions of the exterior problems* (1.14) *and* (1.15) *are well defined for all $k \in \mathbb{R}$, $k \neq 0$, the boundary integral equations of the exterior problems, can not be solved for those values of k, that are eigenvalues of the interior Dirichlet* (1.12) *or Neumann* (1.13) *problems.*

1.1.2 Modified boundary integral equation

The occurrence of resonance frequencies for the interior problems is typical for coupling methods. However we can avoid this problem by taking the representation proposed by Brakhage and Werner in [9] used by Bielak et al. [6] and studied by Kress [21, 50]. For $\alpha \in \mathbb{C}$ with $\operatorname{Im} \alpha \neq 0$ we use a continuous function ψ to get

$$\phi(x) = S\psi(x) + \alpha D\psi(x) \quad \forall x \in \Omega^+. \tag{1.16}$$

Applying the jump relations (1.5) and taking the limit $x \to \Gamma$, we get for all $x \in \Gamma$

$$\phi(x) = V\psi(x) + \alpha(K + \frac{I}{2})\psi(x), \tag{1.17a}$$

$$\frac{\partial \phi(x)}{\partial n} = (K' - \frac{I}{2})\psi(x) - \alpha W\psi(x). \tag{1.17b}$$

Below we state a uniqueness theorem for the representation (1.16).

Theorem 1.4. *Let $\alpha \in \mathbb{C}$ with $\operatorname{Im} \alpha \neq 0$. If $S\psi + \alpha D\psi = 0$ in Ω^+ then $\psi = 0$.*

Proof. This proof is given by Bielak et al. [6] and Brakhage and Werner in [9]. Put $v = S\psi + \alpha D\psi$, then by the jumps relations (1.5) and the hypothesis one gets that $v^+ = V\psi + \alpha(K + \frac{I}{2})\psi = 0$ and $v^- = V\psi + \alpha(K - \frac{I}{2})\psi = -\alpha\psi$ on Γ, also $v_n^+ = (K' - \frac{I}{2})\psi - \alpha W\psi = 0$ and $v_n^- = (K' + \frac{I}{2})\psi - \alpha W\psi = -\psi$. By Green's Theorem and replacing v^- and v_n^-

$$\int_\Omega \big(|\nabla v|^2 - k^2|v|^2\big)dx - \int_\Gamma v^- v_n^- ds = \int_\Omega |\nabla v|^2 - k^2|v|^2 dx + \alpha \int_\Gamma |\psi|^2 \,\mathrm{d}s = 0,$$

thus $\psi = 0$ since $\operatorname{Im}\alpha \neq 0$. □

Remark 1.5. *Burton and Miller [10] present also the following boundary integral equation to remedy the problem of non-uniqueness*

$$-(K - \frac{I}{2})\phi + V\frac{\partial\phi}{\partial n} + \alpha\big((K' + \frac{I}{2})\frac{\partial\phi}{\partial n} + W\phi\big) = 0. \tag{1.18}$$

We will use this representation in combination with (1.17) *to establish a symmetric variational formulation of an FE/BE coupling method.*

Remark 1.6. *Kress [50] analyzed the condition numbers of the discrete system obtained by approximating the integral equation* (1.16) *in 2D and 3D for different values of* α*, concluding that the value* $\alpha = \frac{i}{k}$ *is the optimal value in the sense that it provides the lowest condition number.*

Remark 1.7. *Note that if* $\operatorname{Im}\alpha = 0$ *and* $\operatorname{Re}\alpha \neq 0$ *the non-invertibility of the operator* $V + \alpha(K + \frac{I}{2})$ *depends on whether* k^2 *is an eigenvalue of interior Dirichlet problem* (1.12) *or the interior Neumann problem* (1.13)*, contrary to the case when* $\alpha = 0$ *since this depends only on the eigenvalues of the interior Dirichlet problem* (1.12)*.*

1.1.3 Numerical implementation of the kernel function

For the numerical results for the models in 2D we want to mention some features on the practical implementation of the boundary integral operators. In order to compute the Galerkin matrices in the two-dimensional case with the program system MaiProgs [58] we have to evaluate the Hankel function H_0^1 (see Sec. 1.1). Therefore, we consider its power series, i.e., for $x \in \mathbb{R}$

$$H_0^1(x) = L(x) \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \left(\frac{x}{2}\right)^{2\nu} - \frac{2i}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{x}{2}\right)^{2r} \Phi(r) \tag{1.19}$$

where

$$L(x) = 1 + \frac{2i}{\pi}\left(C_{euler} + \log\left(\frac{x}{2}\right)\right) \qquad \Phi(r) = \sum_{s=1}^{r} \frac{1}{s}, \quad \Phi(0) = 0$$

with the Euler constant $C_{euler} \approx 0.57721$. To compute the values of this function numerically we truncate the series

$$\tilde{H}_0^1(x) = L(x) \sum_{\nu=0}^{m} \frac{(-1)^\nu}{(\nu!)^2} \left(\frac{x}{2}\right)^{2\nu} - \frac{2i}{\pi} \sum_{r=0}^{m} \frac{(-1)^r}{(r!)^2} \left(\frac{x}{2}\right)^{2r} \Phi(r)$$

and compute the remaining terms analytically. Typically we set $m = 80$. The analytical computation of integral operators using the above kernels is described in Maischak [55].

To compute the kernel function in the three-dimensional case, we use the truncated power series of the kernel

$$\frac{e^{ik|x-y|}}{|x-y|} = \frac{1}{|x-y|} \sum_{n=0}^{\infty} \frac{(ik|x-y|)^n}{n!}.$$

The analytical computation of the truncated power series in 3D is described in [42, 56].

1.1.4 Representation formula of hypersingular operator

In order to compute the Galerkin matrices of the hypersingular operator W in 3D we use the following representation. For more details see Nédélec [67], Meyer et al. [60, 61] and Hamdi [38].

$$\begin{aligned} \langle W\psi, \phi \rangle &= \int_\Gamma \psi(x) \frac{\partial}{\partial n_x} \int_\Gamma \phi(y) \frac{\partial \gamma_k(x,y)}{\partial n_y} ds_y \, ds_x \\ &= \int_\Gamma \int_\Gamma \psi(x)\, \phi(y) (n_x \cdot n_y) k^2 \gamma_k(x,y) \, ds_y \, ds_x \\ &- \int_\Gamma \int_\Gamma [n_x \times \nabla_x \psi(x)].[n_y \times \nabla_y \phi(y)] \gamma_k(x,y) \, ds_y \, ds_x \\ &= \langle V \nabla\phi, \nabla\psi \rangle_{0,\Gamma} + k^2 \langle V\phi, \psi \rangle_{0,\Gamma}. \end{aligned} \tag{1.20}$$

Chapter 2

A Fluid-Solid Interaction Problem

This chapter discusses the mathematical analysis for a fluid-solid interaction problem. In our case, the problem under consideration consists of determining the dynamics in a fluid Ω^+ and displacements in an elastic body Ω due to a given excitation in the fluid Ω^+ using an FE/BE coupling method. By using the integral equations we find that the uniqueness for the problem is not guaranteed for certain wave numbers $k \in \mathbb{R}$, however, the representation (1.16) proposed by Brakhage and Werner [9] provides the uniqueness for any k. We take different variational formulations based on this representation proposed by Bielak et al. [6]. The analysis described in this chapter is a compilation of the works of Bielak et al. [6], Bielak and MacCamy [5] and Hsiao et al. [44].

2.1 Interface scattering problem

An incident acoustic wave is totally reflected by a rigid obstacle. This phenomenon is called *rigid scattering of sound*. If the obstacle is elastic, a part of the incident energy is transmitted in the form of elastic vibrations. The acoustic pressure waves act as time-varying loads, causing elastic vibrations. In this case, we speak of *elastic scattering*. Conversely, if the acoustic medium picks up elastic vibrations of an embedded body in the form of acoustic waves, we say that sound is radiated from the body. In this section we will state the equations for all these effects, which are a special case of the general equations of fluid-structure interaction.

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded, simply connected domain with a closed smooth boundary $\partial\Omega = \Gamma$ and its exterior complement given by $\Omega^+ = \mathbb{R}^d \backslash \bar{\Omega}$, see Figure 2.1.

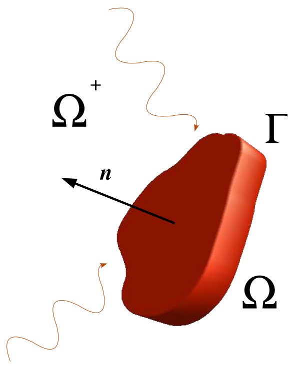


Fig. 2.1: Fluid-structure interaction (schematic plot).

We assume that all waves are steady-state (time harmonic) with angular frequency ω. If Ω is a linear elastic body, and/or the solid is subject to a time-harmonic driving force $\mathbf{F}(x,t) = \mathbf{f}(x)e^{-i\omega t}$, the displacement $\mathbf{u}$ is governed by the reduced elastodynamic equation

$$\operatorname{div}\sigma(\mathbf{u}) + \omega^2\rho\mathbf{u} = \mathbf{f}$$

where $\sigma(\mathbf{u}) := \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})$ is the stress tensor, λ and μ are the Lamé constants and ρ is the density of the body. Let the *traction operator* $\sigma(\mathbf{u})\mathbf{n}$ be defined by

$$\sigma(\mathbf{u})\mathbf{n} := 2\mu\frac{\partial\mathbf{u}}{\partial n} + \lambda\,\mathbf{n}\,\nabla\cdot\mathbf{u} + \mu\,\mathbf{n}\,\times\nabla\mathbf{u}.$$

σ_{n} denotes the normal component of $\sigma(\mathbf{u})\mathbf{n}$, i.e.

$$\sigma_{\mathrm{n}} = \mathbf{n}^T\,\sigma(\mathbf{u})\,\mathbf{n}. \tag{2.1}$$

Ω^+ represents an inviscid, compressible and homogeneous fluid with density ρ_0 and speed of sound c_0. The scalar pressure field in the fluid is denoted by $P(x,t)$. In the fluid Ω^+ an incident acoustic field $P^0(x,t) = p^0(x)e^{-ikt}$ is given. The objective is to determine the stationary acoustic field of the scattered pressure $p(x)$ for $x \in \Omega^+$, which satisfies the Helmholtz equation

$$\Delta p + k^2 p = 0,$$

where $k = \frac{\omega}{c_0}$ denotes the wave number, together with the *radiation condition*

$$p(x) = \mathcal{O}(|x|^{-(d-1)/2}), \qquad -ikp(x) + \frac{dp}{d|x|}(x) = o(|x|^{-(d-1)/2}), \quad |x| \to \infty. \tag{2.2}$$

The solution p is complex-valued. Then the term of physical interest is the real part of $P(x,t) = p(x)e^{-i\omega t}$, i.e.

$$\begin{aligned}\mathrm{Re}\, P &= \mathrm{Re}\,\big((\mathrm{Re}\,(p) + i\mathrm{Im}\, p)(\cos\omega t - i\sin\omega t)\big)\\ &= |p|\big(\frac{\mathrm{Re}\, p}{|p|}\cos(\omega t) + \frac{\mathrm{Im}\, p}{|p|}\sin(\omega t)\big)\\ &= |p|\sin(\theta + \omega t) \quad \text{ with } \quad \theta := \arctan\frac{\mathrm{Re}\, p}{\mathrm{Im}\, p}.\end{aligned}$$

Hence, the absolute value of the stationary solution is the amplitude of the physical solution, whereas θ is its phase. Moreover the pressure is in static equilibrium with the normal traction on the solid boundary:

$$\sigma(\mathbf{u})\mathbf{n} = -(p + p^0)\,\mathbf{n},$$

and the normal displacements of the solid and the fluid are equal on the surface. This leads to

$$\rho_0\omega^2\mathbf{u}\cdot\mathbf{n} = \frac{\partial p}{\partial n} + \frac{\partial p^0}{\partial n}.$$

For more details about the governing equations see Hsiao et al. [44], Ihlenburg [48], Feit [33] and Luke and Martin [54].

Finally, the fluid-solid interaction problem can be formulated as follows. For a given incident field $p^0 \in C^1$ which satisfies the equation $\Delta p^0 + k^2 p^0 = 0$ almost everywhere in Ω and Ω^+, find $\mathbf{u} \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma)$ and $p \in C^2(\Omega^+) \cap C^1(\Omega^+ \cup \Gamma)$ satisfying

$$\mathrm{div}\,\sigma(\mathbf{u}) + \rho\omega^2\mathbf{u} = \mathbf{f} \quad \text{ in } \Omega, \tag{2.3a}$$

$$\Delta p + k^2 p = 0 \quad \text{ in } \Omega^+, \tag{2.3b}$$

$$\sigma(\mathbf{u})\,\mathbf{n} = -(p + p^0)\mathbf{n} \quad \text{ on } \Gamma, \tag{2.3c}$$

$$\rho_0\omega^2\mathbf{u}\cdot\mathbf{n} = \frac{\partial p}{\partial n} + \frac{\partial p^0}{\partial n} \quad \text{ on } \Gamma, \tag{2.3d}$$

$$p \text{ satisfies the radiation condition (2.2) in } \Omega^+, \tag{2.3e}$$

where $\mathbf{n}$ is the normal on Γ exterior of Ω.

2.1.1 Existence and uniqueness of the fluid-solid interaction problem

There is a difficulty with the uniqueness of problem (2.3). To clarify this we consider the following theorem:

Theorem 2.1. *Assumming that* $(\mathbf{u}, p)$ *is a solution of* (2.3) *with* $\mathbf{f} = \mathbf{0}$, $p^0 = \frac{\partial p^0}{\partial n} = 0$, *then* $p = 0$ *in* Ω^+ *and*

$$\begin{aligned} \operatorname{div} \sigma(\mathbf{u}) + \rho\omega^2 \mathbf{u} = 0 \quad & \text{in } \Omega, \\ \sigma(\mathbf{u})\,\mathbf{n} = 0, \qquad \mathbf{u}\cdot\mathbf{n} = 0 \quad & \text{on } \Gamma. \end{aligned} \tag{2.4}$$

This does not necessarily imply that $\mathbf{u}$ vanishes in Ω. It is known that, for certain geometries and for certain frequencies, there are nontrivial solutions of this problem. We call these *Jones modes* and the associated frequencies *Jones frequencies*, as they were first discussed by Jones [31]. An elastic sphere could sustain torsional oscillations in which the radial component of the displacement is identically zero (see Eringen [32, Sec. 8.14] and Dallas [26]). Jones frequencies also exist for any axissymmetric body in which only the azimuthal component of the displacement is nonzero. However, intuitively, we do not expect Jones frequencies to exist for an "arbitrary body" which has been proved by Hargé [39]. In any event we will rule out this possibility by making the following assumption:

Assumption 2.1.1. (2.4) *has only the trivial solution.*

Now we can present the proof of Theorem 2.1, which appears in Luke and Martin [54].

Proof of Theorem **2.1*.*** Let $(\mathbf{u}, p)$ be the solution of (2.3) with $p^0 = \frac{\partial p^0}{\partial n} = 0$ and $\mathbf{f} = \mathbf{0}$. By application of the divergence theorem we obtain

$$\begin{aligned} \int_{\Gamma_a} p \frac{\partial \bar{p}}{\partial n}\, ds &= \int_{\Gamma} p \frac{\partial \bar{p}}{\partial n}\, ds + \int_{\Omega_a^+} \nabla p \cdot \nabla \bar{p} + p \Delta \bar{p}\, dx \\ &= -\rho_0 \omega^2 \int_{\Gamma} \bar{\mathbf{u}} \cdot \sigma(\mathbf{u})\mathbf{n}\, ds + \int_{\Omega_a^+} \nabla p \nabla \bar{p} - k^2 |p|^2\, dx, \end{aligned} \tag{2.5}$$

where Γ_a is the surface of a sphere B_a of radius a, which contains Ω and $\Omega_a^+ := B_a \setminus \bar{\Omega}$. Taking the imaginary part of (2.5) we get

$$\begin{aligned} \operatorname{Im} \int_{\Gamma_a} p \frac{\partial \bar{p}}{\partial n} ds &= -\rho_0 \omega^2 \operatorname{Im} \int_{\Gamma} \bar{\mathbf{u}} \cdot \sigma(\mathbf{u}) \cdot \mathbf{n} \, ds, \\ &= -\rho_0 \omega^2 \operatorname{Im} \Big(\int_{\Omega} \sigma(\mathbf{u}) : \nabla \bar{\mathbf{u}} - \rho \omega^2 \mathbf{u} \cdot \bar{\mathbf{u}} \, dx, \Big). \end{aligned}$$

The last step can be seen by applying the divergence theorem in Ω and (2.3a). Also, $\sigma(\mathbf{u}) : \nabla \bar{\mathbf{u}}$ and $\mathbf{u} \cdot \bar{\mathbf{u}}$ are real, then $\operatorname{Im} \int_{\Gamma_a} p \frac{\partial \bar{p}}{\partial n} ds = 0$. Furthermore p satisfies the radiation condition (2.2), it finally follows that

$$k \lim_{a \to \infty} \int_{\Gamma_a} |p|^2 \, ds = \lim_{a \to \infty} \operatorname{Im} \int_{\Gamma_a} p \frac{\partial \bar{p}}{\partial n} ds = 0,$$

and the Rellich's Lemma (see Colton and Kress [22, Lemma 3.11]) implies that $p = 0$ in Ω^+, thus $p = \frac{\partial p}{\partial n} = 0$ on Γ. Therefore we get with (2.3c) and (2.3d) that $\mathbf{u} \cdot \mathbf{n} = 0$ and $\sigma(\mathbf{u}) \, \mathbf{n} = 0$ on Γ and we get (2.4). □

2.2 Reduced Problems

In the following we consider different *reduced problems* of problem (2.3) presented in the works of Bielak et al. [6] and Bielak and MacCamy [5]. The problems are called reduced because they are limited to finding the displacement $\mathbf{u}$ inside the domain Ω but the pressure p only on the boundary Γ.

We present three problems, one non-stable problem (P_0) and two stable problems (P_1) and (P_2). (P_1) and (P_2) are called stable because they admit a unique solution for any wave number k in contrast to (P_0). First, we present the derivation of each problem and show their uniqueness.

Non-stable problem (P_0)***.*** *For given* $\mathbf{f}$, p^0 *and* $\frac{\partial p^0}{\partial n}$, *find* $\mathbf{u} \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma)$ *and* $p \in C^1(\Gamma)$ *such that*

$$\begin{aligned} \operatorname{div}\sigma(\mathbf{u}) + \rho\omega^2\mathbf{u} = \mathbf{f} \quad & \textit{in } \Omega, \\ \sigma(\mathbf{u})\,\mathbf{n} = -(p + p^0)\mathbf{n} \quad & \textit{on } \Gamma, \\ \rho_0\omega^2 V(\mathbf{u}\cdot\mathbf{n}) = (K - \frac{I}{2})p + V\frac{\partial p^0}{\partial n} \quad & \textit{on } \Gamma. \end{aligned} \tag{P_0}$$

Explanation. Suppose $(\mathbf{u}, p)$ is a solution of problem (2.3) then taking $p := Dp - S\frac{\partial p}{\partial n}$ in Ω^+ and by (1.9a) we have that

$$p = (K + \frac{I}{2})p - V\frac{\partial p}{\partial n} \quad \text{and} \quad \frac{\partial p}{\partial n} = -(K' - \frac{I}{2})\frac{\partial p}{\partial n} - Wp \qquad \text{on } \Gamma. \tag{2.6}$$

Applying the operator V in the boundary condition (2.3d) and using that $V\frac{\partial p}{\partial n} = (K - \frac{I}{2})p$, which comes from (2.6), we obtain the problem (P_0). □

Remark 2.2. *One easily verifies that if $(\mathbf{u}, p)$ is a solution of (P_0), and p is defined appropriately in Ω^+, $(\mathbf{u}, p)$ will be a solution of* (2.3)*.*

Lemma 2.3. *If Assumption 2.1.1 holds and k^2 is not an eigenvalue of the interior Neumann problem* (1.13)*, then for $p^0 = \frac{\partial p^0}{\partial n} = 0$ the problem (P_0) has only the trivial solution.*

Proof. In the following proof the plus and minus signs indicate limits from Ω^+ and Ω^-. Suppose $(\mathbf{u}, p^+)$ solves (P_0) with $p^0 = \frac{\partial p^0}{\partial n} = 0$. Defining $p = Dp^+ - S\frac{\partial p^+}{\partial n} = -D(\sigma_{\mathrm{n}}) - S(\rho_0\omega^2\mathbf{u}\cdot\mathbf{n})$ in Ω^+, where σ_{n} is the normal traction of $\mathbf{u}$ (see (2.1)) we have that $(\mathbf{u}, p)$ is a solution of (2.3). From Theorem 2.1 and Assumption 2.1.1 we have $\mathbf{u} = \mathbf{0}$ in Ω^+ and $\sigma_{\mathrm{n}} = 0$ on Γ, and from the first boundary condition of (P_0) we get $p = 0$ on Γ. Therefore the Problem (P_0) has only the trivial solution. □

Remark 2.4. *Note that for problem (P_0) the uniqueness of solutions for all wave numbers k is not guaranteed. Hence we say that the problem is non stable.*

The next problem is one of the main problems to consider in this work:

Stable problem (P_1)***.*** *For given $\mathbf{f}$, p^0 and $\frac{\partial p^0}{\partial n}$, find $\mathbf{u} \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma)$ and $\phi \in C^1(\Gamma)$ such that*

$$\begin{aligned} \operatorname{div}\sigma(\mathbf{u}) + \rho\omega^2\mathbf{u} = \mathbf{f} \quad & \textit{in } \Omega, \\ \sigma(\mathbf{u})\,\mathbf{n} = -(V\phi + \alpha(K + \frac{I}{2})\phi + p^0)\mathbf{n} \quad & \textit{on } \Gamma, \\ \rho_0\omega^2\mathbf{u}\cdot\mathbf{n} = (K' - \frac{I}{2})\phi - \alpha W\phi + \frac{\partial p^0}{\partial n} \quad & \textit{on } \Gamma. \end{aligned}$$

Explanation. We can represent p through equation (1.16) with $\operatorname{Im}\alpha \neq 0$ by

$$p(x) = S\phi(x) + \alpha D\phi(x), \qquad x \in \Omega^+, \tag{2.7}$$

then for $x \in \Gamma$

$$p(x) = V\phi(x) + \alpha(K + \frac{I}{2})\phi(x), \qquad \frac{\partial p(x)}{\partial n} = (K' - \frac{I}{2})\phi(x) - \alpha W\phi(x). \tag{2.8}$$

Replacing the above equalities in the boundary conditions (2.3c) and (2.3d), we obtain problem (P_1). □

For this case it can be verified that if $(\mathbf{u}, \phi)$ is a solution of (2.3), then for p defined by (2.7), $(\mathbf{u}, p)$ will be a solution of (2.3). Unlike the previous problem (P_0), this problem has a unique solution for all k.

Lemma 2.5. *If Assumption 2.1.1 holds and $\mathbf{f} = \mathbf{0}$ and $p^0 = \frac{\partial p^0}{\partial n} = 0$ then problem (P_1) has only the trivial solution.*

Proof. If $(\mathbf{u}, \phi)$ is solution of (P_1). Defining $p = S\phi + D\phi$ we have $(\mathbf{u}, p)$ is solution of (2.3). From Theorem 2.1 we have $(\mathbf{u}, p) = (\mathbf{0}, 0)$ and we obtain $p = S\phi + \alpha D\phi = 0$ hence by Theorem 1.4 $\phi = 0$. □

Remark 2.6. *If $\alpha = 0$, the problem (P_1) has a unique solution if k^2 is not an eigenvalue of the interior Dirichlet problem* (1.12). *If $\alpha \neq 0$ and $\operatorname{Im}\alpha = 0$ problem (P_1) has a unique solution if k^2 is not an eigenvalue of the interior problems of Dirichlet* (1.12) *or Neumann* (1.13).

Our last problem is derived from problem (P_1) and the integral equations (1.9) formulated by Burton and Miller [10].

Stable problem (P_2)***.*** *For given $\mathbf{f}$, p^0 and $\frac{\partial p^0}{\partial n}$, find $\mathbf{u} \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma)$, $\sigma_{\mathrm{n}} \in C(\Gamma)$ and $\phi \in C^1(\Gamma)$ such that*

$$
\begin{aligned}
\operatorname{div}\sigma(\mathbf{u}) + \rho\omega^2\mathbf{u} &= \mathbf{f} && \textit{in } \Omega,\\
\sigma(\mathbf{u})\,\mathbf{n} - \frac{1}{2}\,\sigma_{\mathrm{n}}\;\mathbf{n} + \frac{1}{2}\big(V\phi + \alpha(K + \frac{I}{2})\phi\big)\mathbf{n} &= -p^0\,\mathbf{n} && \textit{on } \Gamma,\\
-\frac{1}{2}\mathbf{u}\cdot\mathbf{n} + \frac{1}{2\rho_0\omega^2}\big((K' - \frac{I}{2})\phi - \alpha W\phi\big) &= -\frac{1}{2\rho_0\omega^2}\frac{\partial p^0}{\partial n} && \textit{on } \Gamma,\\
\frac{1}{2\rho_0\omega^2}\big((K - \frac{I}{2})\,\sigma_{\mathrm{n}} - \alpha W\,\sigma_{\mathrm{n}}\big) + \frac{1}{2}V(\mathbf{u})\cdot\mathbf{n} + \frac{\alpha}{2}(K' + \frac{I}{2})(\mathbf{u})\cdot\mathbf{n} &= \\
\frac{1}{2\rho_0\omega^2}\big(-(K - \frac{I}{2})p^0 + V\frac{\partial p^0}{\partial n} + \alpha(K' + \frac{I}{2})\frac{\partial p^0}{\partial n} + \alpha W p^0\big) & && \textit{on } \Gamma.
\end{aligned}
$$

Explanation. Using the Burton-Miller representation (1.18) for p on Γ, i.e.,

$$
-(K - \frac{I}{2}) + V\frac{\partial p}{\partial n} + \alpha\,\big((K' + \frac{I}{2})\frac{\partial p}{\partial n} + Wp\big) = 0 \tag{2.9}
$$

and considering σ_{n} as another unknown in the problem, we get together with the boundary transmission condition (2.3c)

$$
p = -\,\sigma_{\mathrm{n}} - p^0 \text{ on } \Gamma. \tag{2.10}
$$

Replacing (2.10) and (2.3d) in (2.9), we get

$$
\begin{aligned}
&(K - \frac{I}{2})\,\sigma_{\mathrm{n}} + (K - \frac{I}{2})p^0 + \rho_0\omega^2 V(\mathbf{u})\cdot\mathbf{n} - V\frac{\partial p^0}{\partial n} +\\
&\alpha\Big(\frac{\rho_0\omega^2}{2}\mathbf{u}\cdot\mathbf{n} - \frac{1}{2}\frac{\partial p^0}{\partial n} - W\,\sigma_{\mathrm{n}} - Wp^0 + \rho_0\omega^2 K'(\mathbf{u})\cdot\mathbf{n} - K'\frac{\partial p^0}{\partial n}\Big) = 0,
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{1}{\rho_0\omega^2}\big((K - \frac{I}{2})\,\sigma_{\mathrm{n}} - \alpha W\,\sigma_{\mathrm{n}}\big) &+ V(\mathbf{u})\cdot\mathbf{n} + \alpha(K' + \frac{I}{2})(\mathbf{u})\cdot\mathbf{n}\\
&= \frac{1}{\rho_0\omega^2}\Big(-(K - \frac{I}{2})p^0 + V\frac{\partial p^0}{\partial n} + \alpha(K' + \frac{I}{2})(\frac{\partial p^0}{\partial n} + \alpha W p^0\Big)\\
&= \frac{1}{\rho_0\omega^2}\Big(\frac{p^0}{2} - Kp^0 + V\frac{\partial p^0}{\partial n} + \alpha\frac{1}{2}\frac{\partial p^0}{\partial n} + \alpha(Wp^0 + K'\frac{\partial p^0}{\partial n})\Big)\\
&= \frac{1}{\rho_0\omega^2}\big(-(K - \frac{I}{2})p^0 + V\frac{\partial p^0}{\partial n} + \alpha(K' + \frac{I}{2})\frac{\partial p^0}{\partial n} + \alpha W p^0\big).
\end{aligned} \tag{2.11}
$$

Moreover, by definition of $\sigma(\mathbf{u})\mathbf{n} = \sigma_{\mathrm{n}}\,\mathbf{n}$ and from (2.3d) we obtain $2\sigma(\mathbf{u})\mathbf{n} = (\sigma_{\mathrm{n}} - p - p^0)\,\mathbf{n}$. Fynally using the boundary conditions of (P_1), (2.11) and dividing by 2, we obtain (P_2).

Remark 2.7. *If p^0 satisfies the Helmholtz equation for Ω and Ω^+, such as in the case of plane waves, the last term of equation* (2.11) *reads*

$$\frac{1}{2\rho_0\omega^2}\big((K-\frac{I}{2})\,\sigma_{\rm n} - \alpha W\,\sigma_{\rm n}\big) + \frac{1}{2}V(\mathbf{u})\cdot\mathbf{n} + \frac{\alpha}{2}(K'+\frac{I}{2})(\mathbf{u})\cdot\mathbf{n} = \frac{1}{2\rho_0\omega^2}\big(p^0 + \alpha\frac{\partial p^0}{\partial n}\big). \tag{2.12}$$

We obtain the following uniqueness result:

Lemma 2.8. *If Assumption 2.1.1 holds and $p^0 = \frac{\partial p^0}{\partial n} = 0$ then the problem (P_2) has only the trivial solution.*

Proof. The proof is analogue to the one of Lemma 2.5. If $(\mathbf{u}, \sigma_{\rm n}, \phi)$ is the solution of (P_2). Defining $p = S\phi + D\phi$ we have $(\mathbf{u}, p)$ is the solution of (2.3). From Theorem 2.1 we have $(\mathbf{u}, p) = (\mathbf{0}, 0)$ and we obtain $p = S\phi + \alpha D\phi = 0$ and $\sigma_{\rm n} = 0$. Finally, from Theorem 1.4 we get $\phi = 0$. □

Remark 2.9. *Unlike problem (P_0), the problems (P_1) and (P_2) have unique solutions for any value of $k \in \mathbb{C}$ with $\operatorname{Im}\alpha \neq 0$. For this reason we call these problems* stable.

2.3 Weak formulations

In the following we present the weak formulations for the reduced problems. Each one involves the weak formulation of the field equation: Seek $\mathbf{u} \in H^1(\Omega)$ such that

$$a_0(\mathbf{u},\mathbf{v}) - \rho\omega^2 a_1(\mathbf{u},\mathbf{v}) - \langle\sigma(\mathbf{u})\,\mathbf{n},\bar{\mathbf{v}}\rangle_\Gamma = -(\mathbf{f},\mathbf{v}), \qquad \forall\mathbf{v}\in H^1(\Omega)$$

where

$$a_0(\mathbf{u},\mathbf{v}) := \int_\Omega \sigma(\mathbf{u}) : \operatorname{grad}\bar{\mathbf{v}}\,\mathrm{d}x, \tag{2.13}$$

and

$$a_1(\mathbf{u},\mathbf{v}) := \int_\Omega \mathbf{u}\bar{\mathbf{v}}\,\mathrm{d}x, \tag{2.14}$$

which is obtained by multiplying the first differential equation in each problem, (P_0), (P_1) and (P_2) by a test function $\bar{\mathbf{v}}$, integrating over Ω and applying Green's theorem. Then we can use the boundary condition that involves the term $\sigma(\mathbf{u})\,\mathbf{n}$ to evaluate the boundary integral. We multiply the second boundary condition of (P_0) and (P_1) by a test function $\bar{q} \in L^2(\Gamma)$ and $\bar{\psi}$ and integrate over Γ.

Non-stable weak formulation (VP_0)***.*** *For given* $\mathbf{f} \in [H^1(\Omega)]^d$, $p^0, \frac{\partial p^0}{\partial n} \in L^2(\Gamma)$ *find* $(\mathbf{u}, p) \in \mathcal{H}_0 := [H^1(\Omega)]^d \times L^2(\Gamma)$ *such that*

$$\begin{aligned} a_0(\mathbf{u},\mathbf{v}) - \rho\omega^2 a_1(\mathbf{u},\mathbf{v}) + \langle p\,\mathbf{n}, \bar{\mathbf{v}}\rangle, &= -(\mathbf{f},\mathbf{v})_0 - \langle p^0\,\mathbf{n}, \bar{\mathbf{v}}\rangle \\ -\langle (K - \frac{I}{2})p, \bar{q}\rangle + \rho_0 w^2 \langle V(\mathbf{u}\cdot\mathbf{n}), \bar{q}\rangle &= \langle V\frac{\partial p^0}{\partial n}, \bar{q}\rangle \end{aligned} \qquad \forall (v,p) \in \mathcal{H}_0 \qquad (VP_0)$$

In short we write the formulation (VP_0) as: *Find* $(\mathbf{u}, p) \in \mathcal{H}_0$ *such that*

$$\mathcal{A}_0(\mathbf{u}, p; \mathbf{v}, q) = \mathcal{F}_0(\mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in \mathcal{H}_0,$$

where

$$\begin{aligned} \mathcal{A}_0(\mathbf{u}, p; \mathbf{v}, q) &:= a_0(\mathbf{u},\mathbf{v}) - \rho\omega^2 a_1(\mathbf{u},\mathbf{v}) + \langle p\,\mathbf{n}, \bar{\mathbf{v}}\rangle - \langle (K - \frac{I}{2})p, \bar{q}\rangle + \rho_0 w^2 \langle V(\mathbf{u}\cdot\mathbf{n}), \bar{q}\rangle, \\ \mathcal{F}_0(\mathbf{v}, q) &:= -(\mathbf{f},\mathbf{v})_0 - \langle p^0\,\mathbf{n}, \bar{\mathbf{v}}\rangle + \langle V(\frac{\partial p^0}{\partial n}), \bar{q}\rangle. \end{aligned}$$

For the following formulations we introduce the notation:

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &:= a_0(\mathbf{u},\mathbf{v}) - \rho\omega^2 a_1(\mathbf{u},\mathbf{v}), \\ b(\phi,\mathbf{v}) &:= \langle V(\phi)\mathbf{n}, \bar{\mathbf{v}}\rangle + \alpha\langle (K + \frac{I}{2})(\phi)\mathbf{n}, \bar{\mathbf{v}}\rangle, \\ b'(\mathbf{v},\phi) &:= \langle V(\mathbf{v})\cdot\mathbf{n}, \bar{\phi}\rangle + \alpha\langle (K' + \frac{I}{2})(\mathbf{v})\cdot\mathbf{n}, \bar{\phi}\rangle, \\ c(\phi,\mathbf{v}) &:= -\langle \phi\mathbf{n}, \bar{\mathbf{v}}\rangle, \\ c'(\mathbf{v},\phi) &:= -\langle \mathbf{v}\cdot\mathbf{n}, \bar{\phi}\rangle, \\ d(\phi,\psi) &:= \frac{1}{\rho_0\omega^2}\big(\langle (K' - \frac{I}{2})\phi, \bar{\psi}\rangle_\Gamma - \alpha\langle W\phi, \bar{\psi}\rangle\big). \end{aligned}$$

Remember that $\langle\cdot,\cdot\rangle$ indicates the dual product between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ (see p. 3). Then we get

Stable non-symmetric formulation (VP_1)***.*** *For given* $\mathbf{f} \in [H^1(\Omega)]^d$, $p^0, \frac{\partial p^0}{\partial n} \in H^{1/2}(\Gamma)$ *find* $(\mathbf{u}, \phi) \in \mathcal{H}_1 := [H^1(\Omega)]^d \times H^{1/2}(\Gamma)$ *such that*

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\phi,\mathbf{v}) &= -(\mathbf{f},\mathbf{v})_0 - \langle p^0\,\mathbf{n}, \bar{\mathbf{v}}\rangle, \\ c(\mathbf{u},\psi) + d(\phi,\psi) &= -\frac{1}{\rho_0\omega^2}\langle \frac{\partial p^0}{\partial n}, \bar{\psi}\rangle \end{aligned} \qquad \forall (v,\psi) \in \mathcal{H}_1 \qquad (VP_1)$$

In short: *Find* $(\mathbf{u},\phi) \in \mathcal{H}_1$ *such that*

$$\mathcal{A}_1(\mathbf{u},\phi;\mathbf{v},\psi) = \mathcal{F}_1(\mathbf{v},\psi) \quad \forall (\mathbf{v},\psi) \in \mathcal{H}_1 \qquad (VP_1)$$

where

$$\begin{aligned} \mathcal{A}_1(\mathbf{u},\phi;\mathbf{v},\psi) &:= a(\mathbf{u},\mathbf{v}) + b(\phi,\mathbf{v}) + c(\mathbf{u},\psi) + d(\phi,\psi), \\ \mathcal{F}_1(\mathbf{v},\psi) &:= -(\mathbf{f},\mathbf{v})_0 - \langle p^0\,\mathbf{n}, \bar{\mathbf{v}}\rangle - \frac{1}{\rho_0\omega^2}\langle \frac{\partial p^0}{\partial n}, \bar{q}\rangle. \end{aligned}$$

For problem (P_2) we multiply its second and third boundary conditions with test functions $\bar{\chi} \in H^{1/2}(\Gamma)$ and $\bar{\psi} \in H^{1/2}(\Gamma)$ and integrate over Γ, respectively. Note that $\overline{d(\sigma_{\mathrm{n}},\psi)} = d(\bar{\psi},\bar{\sigma_{\mathrm{n}}})$.

Stable symmetric formulation (VP_2)***.*** *For given* $\mathbf{f} \in [H^1(\Omega)]^d$, $p^0, \frac{\partial p^0}{\partial n} \in H^{1/2}(\Gamma)$ *find* $(\mathbf{u},\sigma_{\mathrm{n}},\phi) \in \mathcal{H}_2 := [H^1(\Omega)]^d \times H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$ *such that*

$$\begin{aligned} 2a(\mathbf{u},\mathbf{v}) + {} & c(\sigma_{\mathrm{n}},\mathbf{v}) + {} b(\phi,\mathbf{v}) = -2(\mathbf{f},\mathbf{v})_0 - \langle p^0\mathbf{n}, \bar{\mathbf{v}}\rangle, \\ c'(\mathbf{u},\chi) + {} & \qquad\qquad + d(\phi,\chi) = -\tfrac{1}{\rho_0\omega^2}\langle \tfrac{\partial p^0}{\partial n}, \bar{\chi}\rangle, \\ b'(\mathbf{u},\psi) + {} & d(\bar{\psi},\bar{\sigma_{\mathrm{n}}}) + \qquad\quad = \tfrac{1}{\rho_0\omega^2}\langle p^0 + \alpha\tfrac{\partial p^0}{\partial n}, \bar{\psi}\rangle \end{aligned} \qquad \forall (\mathbf{v},\chi,\psi) \in \mathcal{H}_2. \qquad (VP_2)$$

In short: *Find* $(\mathbf{u},\sigma_{\mathrm{n}},\phi) \in \mathcal{H}_2$ *such that*

$$\mathcal{A}_2(\mathbf{u},\sigma_{\mathrm{n}},\phi;\mathbf{v},\chi,\psi) = \mathcal{F}_2(\mathbf{v},\chi,\psi) \quad \forall (\mathbf{v},\chi,\psi) \in \mathcal{H}_2 \qquad (VP_2)$$

where

$$\begin{aligned}\mathcal{A}_2(\mathbf{u},\sigma_{\mathrm{n}},\phi;\mathbf{v},\chi,\psi) :=& 2a(\mathbf{u},\mathbf{v}) + b(\phi,\mathbf{v}) + b'(\mathbf{u},\psi),\\ &+ c(\sigma_{\mathrm{n}},\bar{\mathbf{v}}) + c'(\mathbf{u},\chi) + d(\phi,\chi) + d(\bar{\psi},\bar{\sigma_{\mathrm{n}}})\\ \mathcal{F}_2(\mathbf{v},\chi,\psi) :=& -2(\mathbf{f},\mathbf{v})_0 - \langle p^0\mathbf{n},\bar{\mathbf{v}}\rangle - \langle\frac{\partial p^0}{\partial n},\bar{\chi}\rangle + \tfrac{1}{\rho_0\omega^2}\langle p^0 + \alpha\frac{\partial p^0}{\partial n},\bar{\psi}\rangle.\end{aligned}$$

2.3.1 Existence and uniqueness of the weak formulations

This section is based on the work of Bielak et al. [6]. It is essential for this thesis and necessary to understand the type of problem and the involved theory. The variational formulations arise from the Helmholtz equation and the reduced elastodynamic equation, these are in general not positive definite. The existence and uniqueness can be concluded from a generalization of the positive definite case, this is the theory of forms that satisfy a Gårding inequality. This type of problems satisfies the Fredholm alternative: either the variational problem has a unique solution or there exists a nontrivial solution of the homogeneous problem. Hence the existence of the solution follows if one can show uniqueness. First we consider the following abstract problem which generalizes our variational problems:

Abstract problem. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A} : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ a bounded sesquilinear form and $\mathcal{F}$ an element of $\mathcal{H}'$, the dual of $\mathcal{H}$. Consider the variational problem: Find $\eta \in \mathcal{H}$ such that*

$$\mathcal{A}(\eta,\xi) = \mathcal{F}(\xi) \quad \forall\xi \in \mathcal{H}. \tag{VP_G}$$

Suppose that $\mathcal{A}$ can be decomposed into

$$\mathcal{A} = \mathcal{D} + \mathcal{K} \tag{2.15}$$

where $\mathcal{D}$ is a coercive sesquilinear form, i.e., there exist $\beta > 0$ such that

$$\mathcal{D}(\eta,\eta) \geq \beta\|\eta\|^2_{\mathcal{H}} \qquad \forall\eta \in \mathcal{H}, \tag{2.16}$$

and $\mathcal{K}$ is a compact sesquilinear form, i.e., for any bounded sequence $\{\eta_n\} \in \mathcal{H}$, there exists a subsequence η_{n_j} and $\eta \in \mathcal{H}$ such that $\mathcal{K}(\eta_{n_j},\xi) \to \mathcal{K}(\eta,\xi)$ for all $\xi \in \mathcal{H}$.

Theorem 2.10. *If $\mathcal{A}(\eta,\xi) = 0$ for all $\xi \in \mathcal{H}$ implies $\eta = 0$, then (VP_G) has a unique solution η and $\mathcal{F} \to \eta$ is a bounded map from $\mathcal{H}' \to \mathcal{H}$.*

Proof. The proof is based in the Fredholm alternative for variational equations. For more detail see e.g. Hsiao and Wendland [46], Sauter and Schwab [73]Satz 4.2.7 or Hackbusch [36]. □

Now, we show that the formulation (VP_1) satisfies the hypotheses of Theorem 2.10. A key is that a part of the sesquilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are coercive. For $a(\cdot,\cdot)$ this follows from Korn's second inequality which states that there are constants $\beta_1 > 0$, $\beta_0 > 0$ such that

$$\mathrm{Re}\int_{\Omega} \sigma(\mathbf{u}) : \nabla\bar{\mathbf{u}}\,dx \geq \beta_1\|\mathbf{u}\|_1^2 - \beta_0\|\mathbf{u}\|_0^2 \qquad \forall \mathbf{u} \in [H^1(\Omega)]^d. \tag{2.17}$$

We decompose the hypersingular operator W into two operators $W = W_0 + W_1$, where W_0 is a hypersingular operator obtained using the kernel γ_0, i.e, $\gamma_{k=0}(x,y) = -\frac{1}{4\pi}\frac{1}{|x-y|}$ and $W_1 = W - W_0$. It follows from Lemma 2.11 that W_0 is bounded and invertible.

Lemma 2.11. *There exists $m > 0$ such that*

$$\langle W_0\phi, \bar{\phi}\rangle \geq m\|\phi\|^2 \quad \forall\, \phi \in H^{1/2}(\Gamma).$$

Proof. See Costabel and Stephan [23] for the 2D- and 3D dimensional case. For the 3D case see also Sauter and Schwab [73]. □

Then, we make the following decomposition. We set

$$a(\cdot,\cdot) := a_d(\cdot,\cdot) + a_c(\cdot,\cdot), \qquad d(\cdot,\cdot) := \frac{\alpha}{\rho_0\omega^2}\langle W_0\cdot,\cdot\rangle + d_c(\cdot,\cdot) \tag{2.18}$$

where

$$\begin{aligned} a_d(u,v) &:= a_0(\mathbf{u},\mathbf{v}) + \beta_0\, a_0(\mathbf{u},\mathbf{v}) = (\tilde{A}_0(\mathbf{u}),\mathbf{v}),\\ a_c(u,v) &:= -(\rho\omega^2 + \beta_0)a_1(\mathbf{u},\mathbf{v}),\\ d_c(\phi,\chi) &:= \frac{1}{\rho_0\omega^2}\langle -(K' - \frac{I}{2})\phi + \alpha W_1\phi, \bar{\chi}\rangle. \end{aligned} \tag{2.19}$$

Next we define operators $\tilde{A}_c$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ by

$$a_c(\mathbf{u},\mathbf{v}) =: (\tilde{A}_c\mathbf{u},\mathbf{v}) \qquad b(\phi,\mathbf{v}) =: \langle \tilde{B}\phi,\bar{\mathbf{v}}\rangle$$

$$c(\mathbf{u},\psi) =: \langle \tilde{C}\mathbf{u},\bar{\psi}\rangle \qquad d_c(\phi,\chi) =: \langle \tilde{D}_c\phi,\bar{\chi}\rangle$$

The following Lemma is a result of Bielak [6], one can also find another version in Hsiao et al. [44] and Costabel and Stephan [23].

Lemma 2.12. *$\tilde{A}_c, \tilde{B}, \tilde{C}, \tilde{D}_c$ are compact operators.*

Thus we have shown that condition (2.15) is satisfied. With the next Lemma the hypothesis of Theorem 2.10 is satisfied.

Lemma 2.13. *If*

$$\mathcal{A}_1(\mathbf{u},\phi;\mathbf{v},\psi) = 0 \qquad \forall(\mathbf{v},\psi) \in \mathcal{H}_1 \qquad (VP_{1,hom})$$

then $(\mathbf{u},\phi) = \mathbf{0} \in \mathcal{H}_1$.

The following proof can be found in Bielak et al. [6].

Proof. Let $(\mathbf{u},\phi)$ be a solution of $VP_{1,hom}$, then $\mathbf{u}$ and $p = S\phi + \alpha D\phi$ will yield a solution of (2.3) for $p^0 = \frac{\partial p^0}{\partial n} = 0$, hence by Lemma 2.5 $(\mathbf{u},\phi) = 0$. To make this argument completely rigorous one needs to show that a solution of the homogeneous equation $\mathcal{A}_1(\mathbf{u},\phi;\mathbf{v},\psi) = 0$ has sufficient regularity such that Theorem 2.1 applies. Let us sketch slightly this argument. If $(\mathbf{u},0)$ is a solution of the homogeneous equation (VP_1) then we see that $\mathbf{u}$ is a generalized solution of $\operatorname{div}\sigma(\mathbf{u}) - k_0\mathbf{u} = -(k_0+\rho\omega^2)\mathbf{u} \equiv \chi$ with the boundary condition $\sigma(\mathbf{u})\,\mathbf{n} = -(V\phi + \alpha(K+\frac{I}{2})\phi + p^0)\mathbf{n} = \xi\mathbf{n}$. We have $\chi \in L^2(\Omega)$ and $\xi \in H^{1/2}(\Gamma)$. Standard elliptic theory then gives $\mathbf{u}_{|\Gamma} \in H^2(\Omega)$. Then $\mathbf{u}\cdot\mathbf{n} \in H^{3/2}(\Gamma)$ and the equation $(K' - \frac{I}{2})\phi + \alpha W\phi = \rho_0\omega^2\mathbf{u}\cdot\mathbf{n}$ will give $\phi \in H^{3/2}(\Gamma)$. One can now continue by a boot-strapping argument to show that $\mathbf{u}$ and ϕ have arbitrary degree of smoothness if Γ is smooth. □

Now, one can formulate a uniqueness and existence result for formulation (VP_1):

Theorem 2.14. *If $\alpha \neq 0$, $\alpha \in \mathbb{C}$ then (VP_1) has a unique solution $(\mathbf{u},\phi) \in \mathcal{H}_1$ for any k and any p^0, $\frac{\partial p^0}{\partial n}$.*

Proof. The result follows from Lemma 2.13 by applying Theorem 2.10. □

The analog result holds for the formulation (VP_2). In this case we use a similar decomposition of the sesquilinear forms.

Lemma 2.15. *Given* $\mathbf{f} \in [H^1(\Omega)]^d$, $p^0 \in H^{1/2}(\Omega)$, $\frac{\partial p^0}{\partial n} \in H^{-1/2}(\Omega)$, *if* $\alpha \in \mathbb{C}$ *and* $\alpha \neq 0$ *then* (VP_2) *has a unique solution* $(\mathbf{u}, \sigma_{\mathrm{n}}, \phi) \in \mathcal{H}_2$ *for all* k.

Proof. In this case we take

$$\mathcal{D}(\mathbf{u}, \sigma_{\mathrm{n}}, \phi; \mathbf{v}, \chi, \psi) := 2a_d(\mathbf{u}, \mathbf{v}) + \frac{\alpha}{\rho_0 \omega^2}\langle W_0 \phi, \chi \rangle + \frac{\alpha}{\rho_0 \omega^2}\langle W_0\, \sigma_{\mathrm{n}}, \psi \rangle$$

and

$$\mathcal{K}(\mathbf{u}, \sigma_{\mathrm{n}}, \phi; \mathbf{v}, \chi, \psi) := 2a_c(\mathbf{u}, \mathbf{v}) + b(\phi, \mathbf{v}) + b'(\mathbf{u}, \psi) \\ + c(\sigma_{\mathrm{n}}, \mathbf{v}) + c'(\mathbf{u}, \chi) + d_c(\phi, \chi) + d_c(\bar{\phi}, \bar{\sigma_{\mathrm{n}}})$$

Hence by (2.17), Lemmata 2.11 and 2.12 (VP_2) will have the form satisfy the condition (2.15) and the proof of the hypothesis is similar to the one given in Lemma 2.13. □

2.4 Galerkin Method

2.4.1 Discretization

Let $\mathcal{T}_h$ be a regular decomposition of $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, into non-overlapping elements τ of diameter h_τ, where $h := \max_{\tau \in \mathcal{T}_h} h_\tau$. We assume that $\mathcal{T}_h$ is quasi-uniform with mesh size $h > 0$ and shape-regular in the sense of Ciarlet [20], i.e., there exists a positive constant c_1 such that

$$\frac{h_\tau}{\rho_\tau} \le c_1 \quad \forall \tau \in \mathcal{T}_h$$

where $\rho_\tau := \max\{r : B_r \subseteq \mathcal{T}\}$, $B_r := \{x : x - x_0 < r, x_0 \in \mathcal{T}\}$.

Let $\mathcal{S}_{\Gamma,\tilde{h}}$ the set of faces s of elements $\tau \in \mathcal{T}_h$ which are contained in Γ of length $\tilde{h}_s$, where $\tilde{h} := \max_{s \in \mathcal{S}_{\Gamma,\tilde{h}}} \tilde{h}_s$. We also assume that $\mathcal{S}_{\Gamma,\tilde{h}}$ is a regular decomposition in the sense of Ciarlet. Now we define our approximation spaces.

2.4.2 Finite and boundary elements

Let $\mathcal{P}_t$ denote the set of *polynomials of degree* $\leq t$ in two or three variables. Let $\mathcal{W}_t^h \subset H^1(\Omega)$ be the space of continuous and piecewise polynomials with respect to a decomposition of Ω defined by

$$\mathcal{W}_t^h = \{\eta \in C^0(\Omega) : \eta_{|\tau} \in \mathcal{P}_t \text{ for every } \tau \in \mathcal{T}_h\}$$

and let $\mathcal{B}_t^h$ be the vector space of continuous and piecewise polynomials with respect to a decomposition of the boundary Γ defined by

$$\mathcal{B}_t^h := \{\eta \in C^0(\Gamma) : \eta_{|s} \in \mathcal{P}_t \text{ for every } s \in \mathcal{S}_{\Gamma,\tilde{h}}\}.$$

We write $\mathcal{W}^h$ instead of $\mathcal{W}_1^h$, and $\mathcal{B}^h$ instead of $\mathcal{B}_1^h$. Let us define the following spaces corresponding to the formulations (VP_0), (VP_1) and (VP_2),

$$\mathcal{H}_0^h = [\mathcal{W}^h]^d \times \mathcal{B}^h, \qquad \mathcal{H}_1^h = [\mathcal{W}^h]^d \times \mathcal{B}^h, \quad \mathcal{H}_2^h = [\mathcal{W}^h]^d \times \mathcal{B}^h \times \mathcal{B}^h.$$

Remark 2.16. *We need to approximate the unknows σ_{n} and ϕ by continuous linear functions, because the calculation of the Galerkin matrix $\langle W\phi, \bar{\psi}\rangle$ requires for the test basis functions at least class C^1, cf.* (1.20).

Remark 2.17. *The systems $\mathcal{H}_i^h$ $(i = 0, 1, 2)$ are regular $(2, r)$- systems of finite elements in the sense of Babuska and Aziz [2], where r describes the regularity of the solutions. In our cases W^h is a $(2, 1)$-system and S^h a $(2, 1/2)$-system.*

2.4.3 Discrete problems

Using the discretization spaces $\mathcal{W}^h$ and $\mathcal{B}^h$ we get the following discrete formulations of (VP_0), (VP_1) and (VP_2), respectively:

***Problem** (VP_0^h).* *Find $(\mathbf{u}^h, p^h) \in \mathcal{H}_0^h$ such that*

$$\mathcal{A}_1(\mathbf{u}^h, p^h; \mathbf{v}^h, q^h) = \mathcal{F}_1(\mathbf{v}^h, q^h) \qquad \forall (\mathbf{v}^h, q^h) \in \mathcal{H}_0^h. \qquad (VP_0^h)$$

Problem **(** VP_1^h **).** *Find* $(\mathbf{u}^h, \phi^h) \in \mathcal{H}_1^h$ *such that*

$$\mathcal{A}_1(\mathbf{u}^h, \phi^h; \mathbf{v}^h, \psi^h) = \mathcal{F}_1(\mathbf{v}^h, \psi^h) \qquad \forall (\mathbf{v}^h, \psi^h) \in \mathcal{H}_1^h. \tag{VP_1^h}$$

Problem **(** VP_2^h **).** *Find* $(\mathbf{u}^h, {\sigma_{\mathrm{n}}}^h, \phi^h) \in \mathcal{H}_2^h$ *such that*

$$\mathcal{A}_2(\mathbf{u}^h, {\sigma_{\mathrm{n}}}^h, \phi^h; \mathbf{v}^h, \chi^h, \psi^h) = \mathcal{F}_2(\mathbf{v}^h, \chi^h, \psi^h) \qquad (\mathbf{v}^h, \chi^h, \psi^h) \in \mathcal{H}_2^h. \tag{VP_2^h}$$

2.4.4 A priori estimate of the discretization error

In order to obtain an a priori error estimate for the solutions of problems (VP_0^h), (VP_1^h) and (VP_2^h) we need Theorem 2.21. It shows the uniqueness and convergence of the Galerkin method for a discrete variational problem for an indefinite problem of the form VP_G. We will show a sketch of the proof, which can be found in McCamy and Stephan [59], or Bielak and MacCamy [5] and apply it to prove the following quasi-optimal convergence results.

Theorem 2.18. *Let* $(\mathbf{u}, p)$ *be a solution of* (VP_0)*. Then there exist* $h_0 > 0$ *and* $c > 0$ *such that for any* $h < h_0$ (VP_0^h) *has a unique solution* $(\mathbf{u}^h, p^h) \in \mathcal{H}_0^h$ *and*

$$\|(\mathbf{u}, p) - (\mathbf{u}^h, p^h)\|_{\mathcal{H}_0} \leq c\Big(\inf_{v^h \in [W^h(\Omega)]^d} \|\mathbf{u} - \mathbf{v}^h\|_1^2 + \inf_{q^h \in S^h} \|p - q^h\|_0^2\Big)^{1/2}.$$

Theorem 2.19. *Let* $(\mathbf{u}, \phi)$ *solve* (VP_1)*. Then there exist* $h_0 > 0$ *and* $c > 0$ *such that for any* $h < h_0$ (VP_1^h) *has a unique solution* $(\mathbf{u}^h, \phi^h) \in \mathcal{H}_1^h$ *and*

$$\|(\mathbf{u}, \phi) - (\mathbf{u}^h, \phi^h)\|_{\mathcal{H}_1} \leq c\Big(\inf_{v^h \in [W^h(\Omega)]^d} \|\mathbf{u} - \mathbf{v}^h\|_1^2 + \inf_{\psi^h \in S^h} \|\phi - \psi^h\|_{1/2}^2\Big)^{1/2}$$

Theorem 2.20. *Let* $(\mathbf{u}, \sigma_{\mathrm{n}}, \phi)$ *solve* (VP_2)*. Then there exists* $h_0 > 0$ *and* $c > 0$ *such that for any* $h < h_0$ (VP_2^h) *has a unique solution* $(\mathbf{u}^h, {\sigma_{\mathrm{n}}}^h, \phi^h) \in \mathcal{H}_2^h$ *and*

$$\|(\mathbf{u}, \sigma_{\mathrm{n}}, \phi) - (\mathbf{u}^h, {\sigma_{\mathrm{n}}}^h, \phi^h)\|_{\mathcal{H}_2} \leq c\,\Big(\inf_{v^h \in [W^h(\Omega)]^d} \|\mathbf{u} - \mathbf{v}^h\|_1^2 + \inf_{\psi^h \in Sh} \|\,\sigma_{\mathrm{n}} - \psi^h\|_{1/2}^2 + \inf_{\chi^h \in Sh} \|\phi - \chi^h\|_{1/2}^2\Big)^{1/2}.$$

Now, to prove the above theorems we use the following abstract result, which is the convergence result of the approximation problem of the abstract problem (VP_G) (see p. 22). Let $\mathcal{H}_h \subset \mathcal{H}$ be a discrete dimensional approximation space of $\mathcal{H}$, hence we have the discrete problem:

Problem. *Find $\eta^h \in \mathcal{H}_h$ such that*

$$\mathcal{A}(\eta^h, \xi^h) = \mathcal{F}(\xi^h) \qquad \forall \xi^h \in \mathcal{H}^h. \tag{VP_G^h}$$

Theorem 2.21. *There exist $h_0 > 0$ and $c > 0$ such that for any $h < h_0$ VP_G^h has a unique solution η^h. Furthermore, there holds the following quasi-optimal convergence result*

$$\|\eta - \eta^h\|_{\mathcal{H}} \leq c\|\eta - \xi^h\|_{\mathcal{H}} \qquad \forall \xi^h \in \mathcal{H}^h. \tag{2.20}$$

In the following we give the proof of this theorem, which uses a known technique, for details see MacCamy and Stephan [59], Bielak and MacCamy [5]. This result follows from the coerciveness and compactness, used in Theorem 2.10.

Proof. By (2.15) we have that

$$\mathcal{A}(\eta, \xi) = \mathcal{D}(\eta, \xi) + \mathcal{K}(\eta, \xi) \qquad \forall \xi \in \mathcal{H}, \tag{2.21}$$

where $\mathcal{D}$ is a positive definite sesquilinear form and $\mathcal{K}$ is a sesquilinear form which generates a compact operator. First, we define a Galerkin operator $G_0^h : \mathcal{H} \to \mathcal{H}^h$ by $G_0^h \eta = \eta^h$ where η^h is the solution of

$$\mathcal{D}(\eta^h, \xi^h) = \mathcal{D}(\eta, \xi^h) \qquad \forall\, \xi^h \in \mathcal{H}^h. \tag{2.22}$$

If we expand η^h in terms of basis elements of $\mathcal{H}^h$, (2.22) becomes a system of linear algebraic equations and the coercivity of $\mathcal{D}$ guarantees that this system has a unique solution, so G_0^h is well defined. For any $w^h \in H^h$ we obtain from (2.22)

$$\mathcal{D}(\eta^h - w^h, \xi^h) = \mathcal{D}(\eta - w^h, \xi^h).$$

Hence, from (2.22) and the coercivity and continuity of $\mathcal{D}$, we have

$$\|\eta^h - w^h\| \leq \frac{M}{\beta}\|\eta - w^h\|$$

where β is the coercivity constant and M is the constant of continuity of $\mathcal{D}$. Thus there exists a constant $\beta_1 > 0$ such that

$$\|G_0^h\eta - \eta\| \leq \beta_1\|\eta - w^h\| \qquad \text{for any } \eta \in \mathcal{H}, w^h \in \mathcal{H}^h. \tag{2.23}$$

Hence, if the space $\mathcal{H}^h$ satisfies an approximation property to $\mathcal{H}$ this implies that G_0^h converges strongly to I as $h \to 0$, i.e.,

$$\|G_0^h\eta - \eta\| \to 0 \text{ as } h \to 0.$$

Moreover we can set $w^h = 0$ in (2.23) for any $\eta \in \mathcal{H}$. It follows that $\|G_0^h\|$ is uniformly bounded. Next we note that the sesquilinear form $\mathcal{K}$ generates a compact operator $\tilde{K}$ such that

$$\mathcal{K}(\eta, \xi) = \langle \tilde{K}\eta, \xi\rangle \qquad \forall(\eta, \xi) \in \mathcal{H}.$$

Theorem 2.10 states that the solution defines a bounded map $\tilde{\mathcal{A}}^{-1}$ from $\mathcal{H}'$ into $\mathcal{H}$ by the equation

$$\mathcal{A}(\tilde{\mathcal{A}}^{-1}F, \xi) = \mathcal{F}(\xi) \qquad \forall\, \xi \in \mathcal{H}. \tag{2.24}$$

Consider the map $\tilde{\mathcal{A}}^{-1}\tilde{K}(I - G_h^0)$. This is a map from $\mathcal{H}$ into itself and we assert that its norm tends to zero as $h \to 0$. If this in not true, then we find $\epsilon > 0$ and a sequence $h_n \to 0$ with $\eta_n \in \mathcal{H}$, $\|\eta_n\| = 1$ such that $\tilde{\mathcal{A}}^{-1}\tilde{K}(I - G_h^0)\eta_n \geq \epsilon$. Since $I - G_0^h$ is uniformly bounded $(I - G_0^h)\eta_n = w_n$ is a bounded sequence in $\mathcal{H}$ and by the definition of G_0^h one has

$$\mathcal{D}(w_n, \xi^h) = 0 \qquad \forall\xi^h \in \mathcal{H}^h. \tag{2.25}$$

Now $\{w_n\}$ is bounded in $\mathcal{H}$, hence there is a subsequence w_{n_k} which converges weakly to w in $\mathcal{H}$. From the approximation property and (2.25), we have that $\mathcal{D}(w, \xi) = 0$ for all $\xi \in \mathcal{H}$ and from coercivity of $\mathcal{D}$ we have $w = 0$. Thus w_n converges weakly to zero. But $\tilde{K}$ is compact,

so $\tilde{K}w_m$ converges strongly to zero and $\tilde{\mathcal{A}}^{-1}$ is bounded. It follows that $\tilde{\mathcal{A}}^{-1}\tilde{K}(I-G_h^0)$ converges strongly to zero which contradicts $\tilde{\mathcal{A}}^{-1}\tilde{K}(I-G_h^0)\eta_n \geq \epsilon$.
Then, since $\tilde{\mathcal{A}}^{-1}\tilde{K}(I-G_h^0)$ tends to zero as $h \to 0$, we can find a sufficiently small h_0 such that $(I-\tilde{\mathcal{A}}^{-1}\tilde{K}(I-G_h^0))^{-1}$ exists for $h < h_0$ and we can define

$$G^h := G_0^h(I-\tilde{\mathcal{A}}^{-1}\tilde{K}(I-G_h^0))^{-1}.$$

We assert that

$$\mathcal{A}(G^h\eta,\xi^h) = \mathcal{A}(\eta,\xi^h) \qquad \forall\, \xi^h \in \mathcal{H}^h, \tag{2.26}$$

i.e., G^h is the Galerkin operator for $\mathcal{A}$. To verify (2.26) we set

$$z := (I-\tilde{\mathcal{A}}^{-1}\tilde{K}(I-G_h^0))^{-1}\eta,$$

and get

$$z-\eta = \mathcal{A}^{-1}\tilde{K}(I-G_h^0)z. \tag{2.27}$$

Then (2.24), (2.27), (2.23) and the definition of $\mathcal{K}$ yields

$$\begin{aligned}\mathcal{A}(z,\xi) &= \mathcal{A}(\mathcal{A}^{-1}\tilde{K}(I-G_h^0)z,\xi) + \mathcal{A}(u,\xi)\\ &= \langle \xi, \tilde{K}(I-G_h^0)z\rangle + \mathcal{A}(u,\xi)\\ &= \mathcal{K}((I-G_h^0)z,\eta) + \mathcal{A}(u,v)\end{aligned}$$

or

$$\mathcal{D}(z,\xi) + \mathcal{K}(G_0^h z,\xi) = \mathcal{A}(\eta,\xi). \tag{2.28}$$

Inserting ξ^h in (2.28), then $\mathcal{D}(z,\xi^h) = \mathcal{D}(G_0^h z,\xi^h)$ and (2.28) yield

$$\mathcal{A}(G_0^h z,\xi^h) = \mathcal{A}(\eta,\xi^h) \qquad \forall \xi^h \in S^h$$

and this gives (2.26). Then for a solution η of (VP_G), equation (2.26) gives that the Galerkin solution is

$$\eta^h = G_0^h(I-\tilde{\mathcal{A}}^{-1}\tilde{K}(I-G_h^0))^{-1}$$

and this converges to η as $h \to 0$. Now, we proof the inequality (2.20). Let w^h be an arbitrary element in $\mathcal{H}^h$ and

$$\mathcal{A}(\eta - \eta^h, w^h) = 0.$$

We have that for any $\xi^h \in \mathcal{H}^h$ there exists a constant $\beta > 0$ such that

$$\begin{aligned}\beta\|\eta - \eta^h\| &\leq \mathcal{D}(\eta - \eta^h, \eta - \eta^h) + \mathcal{K}(\eta - \eta^h, \eta - \xi^h)\\ &= \mathcal{D}(\eta - \eta^h, \eta - \xi^h) + \mathcal{K}(\eta - \xi^h, \eta - \xi^h)\\ &\leq M\|\eta - \eta^h\|_{\mathcal{H}}\|\eta - \xi^h\|_{\mathcal{H}}\end{aligned}$$

where M is the maximum of the continuity constants of the forms $\mathcal{D}$ and $\mathcal{K}$, thus with $c = M/\beta$ the inequality (2.20) follows. □

2.4.5 Rate of convergence

For the convergence of the finite and boundary element methods we need the *approximation property* of continuous function spaces on Ω and Γ, respectively. We assume that the spaces $\mathcal{W}_t^h$ satisfy the approximation property:

Approximation property. Let $\tau \in \mathcal{T}_h$, $t \geq 1$ and $\eta \in H^m(\tau)$, then there exist $\eta^h \in \mathcal{W}_t^h(\tau)$ and a constant $c > 0$ which does not depend on h but on m, such that

$$\|\eta - \eta^h\|_{H^m(\tau)} \leq c h^{t+1-m}|\eta|_{t+1,\tau} \qquad \forall \eta \in H^{t+1}(\tau), \tag{2.29}$$

with $0 \leq m \leq t+1$.

Remark 2.22. *S^h satisfies the following approximation property in the spaces $H^q(\Gamma)$. The proof of this in the case $d = 3$ can be found in Babuška and Suri [3] and in the case $d = 2$ in Stephan and Suri [77].*

Approximation property on the boundary. *For each $s \in \mathcal{S}_{\Gamma,\tilde{h}}$ and $\eta \in H^m(s)$ there is $\eta^h \in \mathcal{B}_t^h$ and a constant $c > 0$ which does not depend on t and h, but depend of m, such that*

$$\|\eta - \eta^h\|_{H^q(s)} \leq c\frac{h_s^{\mu-q}}{t^{m-q}+1}\|\eta\|_{H^m(s)}$$

for all $0 \leq q \leq m$ *and* $\mu := \min\{t+1, m\}$.

Thus we obtain the following Lemma concerning the convergence of the discrete scheme (VP_1^h).

Lemma 2.23. *Let* $(\mathbf{u}, \phi) \in \mathcal{H}_1$ *be the solution of* (VP_1) *and let* $(\mathbf{u}^h, \phi^h) \in \mathcal{H}_1^h$ *be the Galerkin approximation of* $(\mathbf{u}, \phi)$*. Assuming regularity assumptions on* $(\mathbf{u}, \phi)$*, i.e.,* $(\mathbf{u}, \phi) \in [H^2(\Omega)]^d \times H^{3/2}(\Gamma))$*, there exists a constant* $c > 0$ *such that*

$$\|(\mathbf{u}, \phi) - (\mathbf{u}^h, \phi^h)\|_{\mathcal{H}_1} \leq c\, h \left(\|\mathbf{u}\|_2^2 + \|\phi\|_{\frac{3}{2}}^2\right)^{1/2}$$

Proof. The inequality follows from (2.20), (2.22), (2.29) and the above stated remarks. □

For our discrete scheme (VP_2^h) we obtain

Lemma 2.24. *Let* $(\mathbf{u}, \sigma_{\mathrm{n}}, \phi) \in \mathcal{H}_2$ *be the solution of* (VP_2) *and let* $(\mathbf{u}^h, {\sigma_{\mathrm{n}}}^h, \phi^h) \in \mathcal{H}_2^h$ *be the Galerkin approximation of* $(\mathbf{u}, \sigma_{\mathrm{n}}, \phi)$*. Assuming regularity assumptions on* $(\mathbf{u}, \sigma_{\mathrm{n}}, \phi)$*, i.e.,* $(\mathbf{u}, \sigma_{\mathrm{n}}, \phi) \in [H^2(\Omega)]^2 \times H^{3/2}(\Gamma) \times H^{3/2}(\Gamma)$*, there exists a constant* $c > 0$ *independent of* h *such that*

$$\|(\mathbf{u}, \sigma_{\mathrm{n}}, \phi) - (\mathbf{u}^h, {\sigma_{\mathrm{n}}}^h, \phi^h)\|_{\mathcal{H}_2} \leq ch\Big(\|\mathbf{u}\|_2^2 + \|\, \sigma_{\mathrm{n}}\, \|_{\frac{3}{2}}^2 + \|\phi\|_{\frac{3}{2}}^2\Big)^{1/2}.$$

Chapter 3

Residual Error Estimates

The purpose of this chapter is to present a posteriori error estimators and an h-adaptive strategy for the formulations (VP_1) and (VP_2). The *residual error estimate* is formulated in the L^2-norm using standard techniques for FE methods, see e.g. Johnson et al. [49], Stewart et al. [78] or Braess [7], and techniques for FE/BE coupling methods, see e.g. Carstensen et al. [12, 17, 15, 16].

In this chapter, we first present a residual error estimator together with its reliability for the formulation (VP_1) and then extend the results to the formulation (VP_2). To prove its reliability we use arguments of duality between $L^2(\Gamma)$ and $H^{1/2}(\Gamma)$, see e.g. Hsiao and Wendland [46] or Costabel and Stephan [25]. Based on these a posteriori error estimates we define local indicators and present adaptive algorithms for the mesh refinement. Subsequently, we present the efficiency of the estimators based on the techniques used by Verfürth [80] and Leydecker [52] for the indicators of the FE part. For the indicators with boundary integral operators we use some ideas of Carstensen [11] and Chernov [19].

Notation. Recall that $\mathcal{T}_h$ is a regular decomposition of Ω into non-overlapping elements τ of diameter h_τ, where $h := \max_{\tau\in\mathcal{T}_h} h_\tau$, and $\mathcal{S}_{\Gamma,\tilde{h}}$ is a regular decomposition of Γ into non-overlapping elements s of diameter h_s, where $\tilde{h} := \max_{s\in\mathcal{S}_{\Gamma,\tilde{h}}} h_s$. We recall that $\mathcal{S}_{\Gamma,\tilde{h}}$ is the set of faces s of elements $\tau \in \mathcal{T}_h$ which are contained in Γ.

Definition 3.1. Let $\mathcal{S}_i$ denote the set of faces of $\mathcal{T}_h$ which are not contained in Γ. Now, for $\tau \in \mathcal{T}_h$, we define the set of interior faces of τ by $\mathcal{S}_{\tau,i} = \{s_{\tau,i}\}$ and the boundary faces of τ by $\mathcal{S}_{\tau,\Gamma}$. Note that the set of faces of τ is given by $\mathcal{S}_\tau = \mathcal{S}_{\tau,i} \cup \mathcal{S}_{\tau,\Gamma}$.

3.1 An A Posteriori Error Estimator for the Coupling formulations $(\boldsymbol{VP_1})$ and $(\boldsymbol{VP_2})$. Reliability

First we present the analysis for the non symmetric formulation (VP_1) (see Section 2.3 p. 21). Then we present some results that are useful for the demonstration of the reliability of our error estimates.

Using the approximation properties and the inverse estimates given in Section 2.4.5 for an element $\tau \in \mathcal{T}_h$ and $s \in \mathcal{S}_{\Gamma,\tilde{h}}$, we obtain

Lemma 3.2. *There exist positive constants c_1, c_2 independent of $\tau \in \mathcal{T}_h$ and h, such that for every $\eta \in H^1(\tau)$ there exists $\eta^h \in \mathcal{W}^h(\tau)$ such that*

$$\|\eta - \eta^h\|_{0,\tau} \leq c_1 h |\eta|_{1,\tau}, \tag{3.1}$$

and

$$\|\eta - \eta^h\|_{0,\partial\tau} \leq c_2 h^{1/2} |\eta|_{1,\tau}. \tag{3.2}$$

There also exists a constant $c_3 > 0$ independent of $s \in \mathcal{S}_{\Gamma,\tilde{h}}$ and $\tilde{h}$, such that for every $\zeta \in H^{1/2}(s)$ there exists $\zeta^h \in \mathcal{B}$ such that

$$\|\zeta - \zeta^h\|_{0,s} \leq c_3 \tilde{h}^{1/2} \|\zeta\|_{\frac{1}{2},s}. \tag{3.3}$$

Remark 3.3. *From Section 2.3.1 we know that there exists a unique solution $(\mathbf{u}, \phi) \in \mathcal{H}$ of the variational problem (VP_1), given by*

$$\mathcal{A}_1(\mathbf{u}, \phi; \mathbf{v}, \psi) = \mathcal{F}_1(\mathbf{v}, \psi) \qquad \forall (\mathbf{v}, \psi) \in \mathcal{H}_1.$$

Since the sesquilinear form $\mathcal{A}_1$ is self-adjoint except in terms containing the double layer operator K, and their adjoint K', we can apply the same uniqueness and existence theory applied for the formulation $\mathcal{A}_1$ and obtain that there exists a unique solution $(\mathbf{u}, \phi) \in \mathcal{H}$ of the following adjoint variational problem

$$\mathcal{A}_1^*(\mathbf{v},\psi,\mathbf{u},\phi) := \mathcal{A}_1(\overline{\mathbf{v},\psi};\overline{\mathbf{u},\phi}) = \mathcal{F}_1(\mathbf{v},\psi) \qquad \forall(\mathbf{v},\psi)\in\mathcal{H}_1.$$

Let $\mathcal{H}_1' := H^{-1}(\Omega)\times H^{-1/2}(\Gamma)$. *From the above statements we obtain that there are continuous and invertible operators* $A_1 : \mathcal{H}_1 \to \mathcal{H}_1'$ *and* $A_1^* : \mathcal{H}_1 \to \mathcal{H}_1'$ *such that*

$$\mathcal{A}_1(\eta,\zeta) = \langle A_1\eta,\zeta\rangle_{\mathcal{H}_1'}, \quad \mathcal{A}_1^*(\zeta,\eta) = \langle A_1^*\zeta,\eta\rangle_{\mathcal{H}_1'},$$

for $(\eta,\zeta)\in\mathcal{H}_1\times\mathcal{H}_1$.

Theorem 3.4. *Let* $(\mathbf{u},\phi)\in\mathcal{H}_1$ *be the solution of problem* (VP_1) *and* $(\mathbf{u}^h,\phi^h)\in\mathcal{H}_1^h$ *be the solution of the discrete problem* (VP_1^h). *There exists a positive constant* c *independent of the meshsize* h *such that*

$$\|\mathbf{u}^h-\mathbf{u},\phi^h-\phi\|_{\mathcal{H}^1} \le c\big(R_1^h+R_2^h+R_3^h+R_4^h\big)^{1/2}, \tag{3.4}$$

where

$$\begin{aligned}
R_1^h &:= \sum_{\tau\in\mathcal{T}_h} h_\tau^2\|\operatorname{div}\sigma(\mathbf{u}^h)+\rho\omega^2\mathbf{u}^h-\mathbf{f}\|_{0,\tau}^2,\\
R_2^h &:= \sum_{s_i\in\mathcal{S}_i} h_{s_i}\|[\![\sigma(\mathbf{u}^h)\cdot\mathbf{n}]\!]\|_{0,s_i}^2,\\
R_3^h &:= \sum_{s\in\mathcal{S}_{\Gamma,\tilde h}} h_s\|\sigma(\mathbf{u}^h)\cdot\mathbf{n}+p^0\mathbf{n}+V\phi^h\,\mathbf{n}+\alpha(K+\frac{I}{2})\phi^h\,\mathbf{n}\|_{0,s}^2,\\
R_4^h &:= \sum_{s\in\mathcal{S}_{\Gamma,\tilde h}} h_s\|-\frac{1}{\rho_0\omega^2}\frac{\partial p^0}{\partial n}+\mathbf{u}^h\cdot\mathbf{n}-\frac{1}{\rho_0\omega^2}\big((K'-\frac{I}{2})-\alpha W\big)\phi^h\|_{0,s}^2.
\end{aligned}\tag{3.5}$$

Proof. We start by using a duality argument to bound $\|\mathbf{u}^h-\mathbf{u},\phi^h-\phi\|_{\mathcal{H}^1}$. Let $e := (\mathbf{u}^h-\mathbf{u},\phi^h-\phi)$. From Remark 3.3 it follows that the adjoint equation $A_1^*\delta=\eta$ is uniquely solvable for every $\eta\in\mathcal{H}_1'$. Moreover the continuity of $(A_1^*)^{-1} : \mathcal{H}_1'\to\mathcal{H}_1$ implies

$$\|\delta\|_{\mathcal{H}_1} \le c\|\eta\|_{\mathcal{H}_1'} \qquad \forall(\delta,\eta)\in\mathcal{H}_1\times\mathcal{H}_1'. \tag{3.6}$$

Since $\mathcal{H}_1$ and $\mathcal{H}_1'$ are pairing dual with respect to the scalar product of $\mathcal{L}^2 := [L^2(\Omega)]^d\times L^2(\Gamma)$, we have

$$\|e\|_{\mathcal{H}_1} \leq \sup_{\|\eta\|_{\mathcal{H}_1'} \leq 1} |(e,\eta)_{\mathcal{L}^2}| = \sup_{\|A_1^*\delta\|_{\mathcal{H}_1'} \leq 1} |(e, A_1^*\delta)_{\mathcal{L}^2}| = \sup_{\|A_1^*\delta\|_{\mathcal{H}_1'} \leq 1} |(A_1 e, \delta)_{\mathcal{L}^2}|. \tag{3.7}$$

We have that

$$\begin{aligned}(A_1 e, \delta) = \mathcal{A}_1(e;\delta) &= \mathcal{A}_1(\mathbf{u}^h, \phi^h; \delta) - \mathcal{A}_1(\mathbf{u}, \phi; \delta) \\ &= \mathcal{A}_1(\mathbf{u}^h, \phi^h; \delta) - \mathcal{F}_1(\delta).\end{aligned}$$

For $l \in \mathcal{H}_1^h$ there holds $\mathcal{A}_1(\mathbf{u}^h, \phi^h; l) - \mathcal{F}_1(l) = 0$. Then

$$\begin{aligned}\mathcal{A}_1(e;\delta) &= \mathcal{A}_1(\mathbf{u}^h, \phi^h; \delta) - \mathcal{F}_1(\delta) + \mathcal{F}_1(l) - \mathcal{A}_1(\mathbf{u}^h, \phi^h; l) \\ &= \mathcal{F}_1(l-\delta) - \mathcal{A}_1(\mathbf{u}^h, \phi^h; l-\delta).\end{aligned}$$

Taking l as the L^2-projection on $\mathcal{H}_1^h$ of $\delta \in \mathcal{H}_1$ and $\eta := (\eta_{\mathbf{v}}, \eta_\psi) = l - \delta \in \mathcal{H}_1$, we get

$$\mathcal{A}_1(e;\delta) = \mathcal{F}_1(\eta) - \mathcal{A}_1(\mathbf{u}^h, \phi^h; \eta). \tag{3.8}$$

Now we consider the bilinear form $a_0(\cdot,\cdot)$ in (2.13) on each element $\tau \in \mathcal{T}_h$. Integrating by parts over an $\tau \in \mathcal{T}_h$ and decomposing the integral over $\partial\tau$ into integrals on each interior face $s_{\tau,i} \in \mathcal{S}_{\tau,i}$ and each boundary face $s_{\tau,\Gamma} \in \mathcal{S}_{\tau,\Gamma}$, it follows

$$\begin{aligned}a_0(\mathbf{u}^h, \eta_{\mathbf{v}})_\tau =& (\sigma(\mathbf{u}^h) : \nabla \eta_{\mathbf{v}})_{0,\tau} = -(\operatorname{div}\sigma(\mathbf{u}^h), \eta_{\mathbf{v}})_{0,\tau} + \int_{\partial\tau} (\sigma(\mathbf{u}^h)\mathbf{n}) \cdot \bar{\eta}_{\mathbf{v}}\, \mathrm{d}s \\ =& -(\operatorname{div}\sigma(\mathbf{u}^h), \eta_{\mathbf{v}})_{0,\tau} \\ &+ \sum_{s_{\tau,i} \in \mathcal{S}_{\tau,i}} \langle \sigma(\mathbf{u}^h)\mathbf{n}, \bar{\eta}_{\mathbf{v}} \rangle_{0,s_{\tau,i}} + \sum_{s_{\tau,\Gamma} \in \mathcal{S}_{\tau,\Gamma}} \langle \sigma(\mathbf{u}^h)\mathbf{n}, \bar{\eta}_{\mathbf{v}} \rangle_{0,s_{\tau,\Gamma}}.\end{aligned}$$

Then, it follows from (3.8) that

$$
\begin{aligned}
\mathcal{A}_1(e;\delta) =& \mathcal{F}(\eta) - \mathcal{A}(\mathbf{u}^h, \phi^h; \eta) \\
=& \sum_{\tau\in\mathcal{T}_h} \Big((\operatorname{div}\sigma(\mathbf{u}^h), \eta_{\mathbf{v}})_{0,\tau} + \rho\omega^2(\mathbf{u}^h, \eta_{\mathbf{v}})_{0,\tau} - (\mathbf{f}, \eta_{\mathbf{v}})_{0,\tau} \Big) \\
& - \sum_{s_i\in\mathcal{S}_i} \langle [\![\sigma(\mathbf{u}^h)\mathbf{n}]\!], \bar{\eta}_{\mathbf{v}}\rangle_{0,s_i} - \langle \sigma(\mathbf{u}^h)\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_0 \\
& - \langle p^0\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_0 - \frac{1}{\rho_0\omega^2}\langle \frac{\partial p^0}{\partial n}, \bar{\eta}_\psi\rangle_0 - \langle V\phi^h\,\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_0 - \alpha\langle (K + \frac{I}{2})\phi^h\,\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_0 \\
& + \langle \mathbf{u}^h\cdot\mathbf{n}, \bar{\eta}_\psi\rangle_0 - \frac{1}{\rho_0\omega^2}\langle (\,(K' - \frac{I}{2}) - \alpha W)\phi^h, \bar{\eta}_\psi\rangle_0 \\
=& \sum_{\tau\in\mathcal{T}_h} (r_1^h, \eta_{\mathbf{v}})_\tau + (r_2^h, \bar{\eta})
\end{aligned}
\tag{3.9}
$$

where $[\![\cdot]\!]$ denotes the jump over an interior face, $r_1^h := \operatorname{div}\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f}$ is the residual defined on the interior elements, and

$$
\begin{aligned}
(r_2^h, \bar{\eta}) :=& - \sum_{s_i\in\mathcal{S}_i} \langle [\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!], \bar{\eta}_{\mathbf{v}}\rangle_{0,s_i} - \langle \sigma(\mathbf{u}^h)\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_0 \\
& - \langle p^0\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_0 - \frac{1}{\rho_0\omega^2}\langle \frac{\partial p^0}{\partial n}, \bar{\eta}_\psi\rangle_0 - \langle V\phi^h\,\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_0 - \alpha\langle (K + \frac{I}{2})\phi^h\,\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_0 \\
& + \langle \mathbf{u}^h\cdot\mathbf{n}, \bar{\eta}_\psi\rangle_0 - \frac{1}{\rho_0\omega^2}\langle (\,(K' - \frac{I}{2}) - \alpha W)\phi^h, \bar{\eta}_\psi\rangle_0 \\
=& - \sum_{s_i\in\mathcal{S}_i} \langle [\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!], \bar{\eta}_{\mathbf{v}}\rangle_{0,s_i} \\
& + \sum_{s\in\mathcal{S}_{\Gamma,\tilde{h}}} \Big(- \langle (\sigma(\mathbf{u}^h)\,\mathbf{n}\cdot\bar{\eta}_{\mathbf{v}}\rangle_{0,s} - \langle p^0\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_{0,s} - \frac{1}{\rho_0\omega^2}\langle \frac{\partial p^0}{\partial n}, \bar{\eta}_\psi\rangle_{0,s} \\
& - \langle V\phi^h\,\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_{0,s} - \alpha\langle (K + \frac{I}{2})\phi^h\,\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_{0,s} \\
& + \langle \mathbf{u}^h\cdot\mathbf{n}, \bar{\eta}_\psi\rangle_{0,s} - \frac{1}{\rho_0\omega^2}\langle (\,(K' - \frac{I}{2}) - \alpha W)\phi^h, \bar{\eta}_\psi\rangle_{0,s} \Big)
\end{aligned}
$$

is the residual defined on the boundary. Using the Cauchy-Schwarz inequality, Lemma 3.2 and the Hölder inequality we obtain

$$
\begin{aligned}
& \Big| \sum_{\tau\in\mathcal{T}_h} (r_1^h, \eta_{\mathbf{v}})_\tau + (r_2^h, \bar{\eta}) \Big| \\
\leq & \sum_{\tau\in\mathcal{T}_h} |(r_1^h, \eta_{\mathbf{v}})_\tau| + \Big| \sum_{s_i\in\mathcal{S}_i} \langle [\![\sigma(\mathbf{u}^h)\mathbf{n}]\!], \bar{\eta}_{\mathbf{v}}\rangle_{0,s_i} \Big| \\
& + \Big| \sum_{s\in\mathcal{S}_{\Gamma,\tilde{h}}} -\langle \sigma(\mathbf{u}^h)\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_{0,s} - \langle p^0\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_{0,s} - \langle V\phi^h\,\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_s - \alpha\langle (K + \frac{I}{2})\phi^h\,\mathbf{n}, \bar{\eta}_{\mathbf{v}}\rangle_s \Big|
\end{aligned}
$$

$$
\begin{aligned}
&+\Big| \sum_{s\in\mathcal{S}_{\Gamma,\tilde h}} -\frac{1}{\rho_0\omega^2}\langle \frac{\partial p^0}{\partial n}, \bar\eta_\psi\rangle_{0,s} + \langle \mathbf{u}^h\cdot\mathbf{n}, \eta_\psi\rangle_{0,s} - \frac{1}{\rho_0\omega^2}\langle ((K' - \frac{I}{2}) - \alpha W)\phi^h, \bar\eta_\psi\rangle_{0,s}\Big| \\
\le & \sum_{\tau\in\mathcal{T}_h} \|r_1^h\|_{0,\tau}\|\eta_{\mathbf{v}}\|_{0,\tau} + \sum_{s_i\in\mathcal{S}_i} \|[\![\sigma(\mathbf{u}^h)\,\mathbf{n})]\!]\|_{0,s_i}\,\|\eta_{\mathbf{v}}\|_{0,s_i} \\
&+ \sum_{s\in\mathcal{S}_{\Gamma,\tilde h}} \| -\sigma(\mathbf{u}^h)\,\mathbf{n} - p^0\mathbf{n} - V\phi^h\,\mathbf{n} - \alpha(K + \frac{I}{2})\phi^h\,\mathbf{n}\|_{0,s}\,\|\eta_{\mathbf{v}}\|_{0,s} \\
&+ \sum_{s\in\mathcal{S}_{\Gamma,\tilde h}} \| -\frac{1}{\rho_0\omega^2}\frac{\partial p^0}{\partial n} + \mathbf{u}^h\cdot\mathbf{n} - \frac{1}{\rho_0\omega^2}((K' - \frac{I}{2}) - \alpha W)\phi^h\|_{0,s}\,\|\eta_\psi\|_{0,s} \\
\le & c_1 \sum_{\tau\in\mathcal{T}_h} h_\tau\|r_1^h\|_{0,\tau}\,|\delta_{\mathbf{v}}|_{1,\tau} + c_2 \sum_{s_i\in\mathcal{S}_i} \|[\![\sigma(\mathbf{u}^h)\,\mathbf{n})]\!]\|_{0,s_i} h_{s_i}^{1/2}|\delta_{\mathbf{v}}|_{1/2,s_i} \\
&+ c_2 \sum_{s\in\mathcal{S}_{\Gamma,\tilde h}} \|\sigma(\mathbf{u}^h)\,\mathbf{n} + p^0\mathbf{n} + V\phi^h\mathbf{n} + \alpha(K + \frac{I}{2})\phi^h\mathbf{n}\|_{0,s} h_s^{1/2}|\delta_{\mathbf{v}}|_{1/2,s} \\
&+ c_3 \sum_{s\in\mathcal{S}_{\Gamma,\tilde h}} \| -\frac{1}{\rho_0\omega^2}\frac{\partial p^0}{\partial n} + \mathbf{u}^h\cdot\mathbf{n} - \frac{1}{\rho_0\omega^2}((K' - \frac{I}{2}) - \alpha W)\phi^h\|_{0,s} h_s^{1/2}\|\delta_\psi\|_{1/2,s} \\
\le & \max\{c_1,c_2,c_3\}\Big(\sum_{\tau\in\mathcal{T}_h} h_\tau^2\|r_1^h\|_{0,\tau}^2 + \sum_{s_i\in\mathcal{S}_i} h_{s_i}\|[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]\|_{0,s_i}^2 \\
&\qquad + \sum_{s\in\mathcal{S}_{\Gamma,\tilde h}} h_s\|\sigma(\mathbf{u}^h)\,\mathbf{n} + p^0\mathbf{n} + V\phi^h\,\mathbf{n} + \alpha(K + \frac{I}{2})\phi^h\,\mathbf{n}\|_{0,s}^2 \\
&\qquad + \sum_{s\in\mathcal{S}_{\Gamma,\tilde h}} h_s\| -\frac{1}{\rho_0\omega^2}\frac{\partial p^0}{\partial n} + \mathbf{u}^h\cdot\mathbf{n} - \frac{1}{\rho_0\omega^2}((K' - \frac{I}{2}) - \alpha W)\phi^h\|_{0,s}^2\Big)^{1/2} \\
&\quad \times \Big(\sum_{\tau\in\mathcal{T}_h} |\delta_{\mathbf{v}}|_{1,\tau}^2 + \sum_{\tau\in\mathcal{T}_h} |\delta_{\mathbf{v}}|_{1,s_i}^2 + \sum_{s\in\mathcal{S}_{\Gamma,\tilde h}} |\delta_{\mathbf{v}}|_{1,s}^2 + \sum_{s\in\mathcal{S}_{\Gamma,\tilde h}} \|\delta_\psi\|_{1/2,s}^2\Big)^{1/2} \\
\le & \max\{c_1,c_2,c_3\}(R_1^h + R_2^h + R_3^h + R_4^h)^{1/2} \\
&\times \Big(\sum_{\tau\in\mathcal{T}_h} \|\delta_{\mathbf{v}}\|_{1,\tau}^2 + \sum_{s\in\mathcal{S}_{\Gamma,\tilde h}} \|\delta_\psi\|_{1/2,s}^2\Big)^{1/2} \\
\le & \max\{c_1,c_2,c_3\}(R_1^h + R_2^h + R_3^h + R_4^h)^{1/2}\|\delta\|_{\mathcal{H}^1}.
\end{aligned}
$$

Finally, starting from (3.7), using (3.6), (3.9) and the above inequality, we get

$$
\begin{aligned}
\|\mathbf{u}^h - \mathbf{u}, \phi^h - \phi\|_{\mathcal{H}^1} &\le \max\{c_1,c_2,c_3\}(R_1^h + R_2^h + R_3^h + R_4^h)^{1/2}\|\delta\|_{\mathcal{H}^1} \\
&\le \max\{c_1,c_2,c_3\}(R_1^h + R_2^h + R_3^h + R_4^h)^{1/2}
\end{aligned}
$$

which is our result (3.4). □

Now we formulate the error estimator for the symmetric formulation (VP_2) (see Section 2.3, p. 21)

Theorem 3.5. *Let* $(\mathbf{u}, \sigma_{\rm n}, \phi) \in \mathcal{H}_2$ *be the solution of problem* (VP_2) *and let* $(\mathbf{u}^h, {\sigma_{\rm n}}^h, \phi^h) \in \mathcal{H}_2^h$ *be the solution of the discrete problem* (VP_2^h)*. There exists a positive constant c such that*

$$\|\mathbf{u}^h - \mathbf{u}, {\sigma_{\rm n}}^h - \sigma_{\rm n}, \phi^h - \phi\|_{\mathcal{H}_2} \le c\big(\tilde{R}_1^h + \tilde{R}_2^h + \tilde{R}_3^h + \tilde{R}_4^h + \tilde{R}_5^h\big)^{1/2} \tag{3.10}$$

where

$$\begin{aligned}
\tilde{R}_1^h &:= \sum_{\tau \in \mathcal{T}_h} 2h_\tau^2 \|\operatorname{div} \sigma(\mathbf{u}^h) + \rho\omega^2 \mathbf{u}^h - \mathbf{f}\|_{0,\tau}^2, \\
\tilde{R}_2^h &:= \sum_{s_i \in \mathcal{S}_i} 2h_{s_i} \| [\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!] \|_{s_i}^2, \\
\tilde{R}_3^h &:= \sum_{s \in \mathcal{S}_{\Gamma,\tilde{h}}} h_s \| -2\sigma(\mathbf{u}^h)\,\mathbf{n} - p^0 \mathbf{n} + {\sigma_{\rm n}}^h\,\mathbf{n} - (V + \alpha(K + \frac{I}{2})\,)\phi^h\,\mathbf{n}\|_s^2, \\
\tilde{R}_4^h &:= \sum_{s \in \mathcal{S}_{\Gamma,\tilde{h}}} h_s \Big\| -\frac{1}{\rho_0\omega^2}\frac{\partial p^0}{\partial n} + \mathbf{u}^h \cdot \mathbf{n} - \frac{1}{\rho_0\omega^2}(\,(K' - \frac{I}{2}) - \alpha W)\phi^h \Big\|_s^2, \\
\tilde{R}_5^h &:= \sum_{s \in \mathcal{S}_{\Gamma,\tilde{h}}} h_s \Big\| \frac{1}{\rho_0\omega^2}(p^0 + \alpha\frac{\partial p^0}{\partial n}) - (V + \alpha(K' + \frac{I}{2})\,)\mathbf{u}^h \cdot \mathbf{n} \\
&\qquad - \frac{1}{\rho_0\omega^2}(\,(K - \frac{I}{2}) - \alpha W\,)\,{\sigma_{\rm n}}^h \Big\|_s^2.
\end{aligned} \tag{3.11}$$

Proof. The proof is analogue to the one given in Theorem 3.4 and will therefore be left out.

3.2 Adaptive Strategy

Using the a posteriori error estimates (3.4) and (3.10), we can design an adaptive algorithm based on finding those elements with the biggest error. This process can be repeated until our estimate satisfies a tolerance. The constants c_1, c_2, c_3 depend on the approximation space, but do not depend on the particular problem. We are dividing the error estimator (3.4) by $\max\{c_1, c_2, c_3\}$, therefore our adaptive strategy is based on a calculation of a *scaled* error rather than an error estimator. To do this we calculate *local error indicators* $\eta_{R,1}^\tau$ for each element $\tau \in \mathcal{T}_h$, for the case (VP_1)

$$\eta_{R,1}^{\tau} := \left(R_1^{\tau} + R_2^{\tau} + R_3^{\tau} + R_4^{\tau}\right)^{1/2} \tag{3.12}$$

where

$$\begin{aligned}
R_1^{\tau} &:= h_{\tau}^2 \,\|\mathrm{div}\,\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f}\|_{0,\tau}^2,\\
R_2^{\tau} &:= \sum_{s_i \in \mathcal{S}_{i,\tau}} h_{s_i} \|[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]\|_{0,s_i}^2,\\
R_3^{\tau} &:= \sum_{s \in \mathcal{S}_{\Gamma,\tau}} h_s \|\sigma(\mathbf{u}^h)\,\mathbf{n} - p^0\mathbf{n} - (V + \alpha(K + \frac{I}{2})\,)\phi^h\,\mathbf{n}\|_{0,s}^2,\\
R_4^{\tau} &:= \sum_{s \in \mathcal{S}_{\Gamma,\tau}} h_s \big\| - \frac{1}{\rho_0\omega^2}\frac{\partial p^0}{\partial n} + \mathbf{u}^h\cdot\mathbf{n} - \frac{1}{\rho_0\omega^2}(\,(K' - \frac{I}{2}) - \alpha W)\phi^h\big\|_{0,s}^2,
\end{aligned}$$

and for the case (VP_2)

$$\eta_{R,2}^{\tau} := \left(\tilde{R}_1^{\tau} + \tilde{R}_2^{\tau} + \tilde{R}_3^{\tau} + \tilde{R}_4^{\tau} + \tilde{R}_5^{\tau}\right)^{1/2} \tag{3.13}$$

where

$$\begin{aligned}
\tilde{R}_1^{\tau} &:= 2h_{\tau}^2 \,\|\mathrm{div}\,\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f}\|_{0,\tau}^2,\\
\tilde{R}_2^{\tau} &:= \sum_{s_i \in \mathcal{S}_{i,\tau}} 2h_{s_{\tau,i}} \|[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]\|_{0,s_i}^2,\\
\tilde{R}_3^{\tau} &:= \sum_{s \in \mathcal{S}_{\Gamma,\tau}} h_s \| - 2\sigma(\mathbf{u}^h)\,\mathbf{n} - p^0\mathbf{n} + {\sigma_{\mathrm{n}}}^h\,\mathbf{n} - (V + \alpha(K + \frac{I}{2})\,)\phi^h\,\mathbf{n}\|_{s}^2,\\
\tilde{R}_4^{\tau} &:= \sum_{s \in \mathcal{S}_{\Gamma,\tau}} h_s \big\| - \frac{1}{\rho_0\omega^2}\frac{\partial p^0}{\partial n} + \rho_0\omega^2\mathbf{u}^h\cdot\mathbf{n} - \frac{1}{\rho_0\omega^2}(\,(K' - \frac{I}{2}) - \alpha W)\,\phi^h\big\|_{0,s}^2,\\
\tilde{R}_5^{\tau} &:= \sum_{s \in \mathcal{S}_{\Gamma,\tau}} h_s \big\|\frac{1}{\rho_0\omega^2}(p^0 + \alpha\frac{\partial p^0}{\partial n}) - (V + \alpha(K' + \frac{I}{2})\,)\,\mathbf{u}^h\cdot\mathbf{n}\\
&\qquad\qquad - \frac{1}{\rho_0\omega^2}(\,(K - \frac{I}{2}) + \alpha W)\ {\sigma_{\mathrm{n}}}^h\,\big\|_{0,s}^2.
\end{aligned}$$

We refine those elements of $\mathcal{T}_h$ where the local indicators are relatively large. In an independent manner, we can also compute the global estimates (3.4) and (3.10) for each formulation and verify its behavior during the adaptive method.

The global error estimator η_{R_i} in each formulation is given by

$$\eta_{R_i} := \Big(\sum_{k=1}^{n} (\eta_{R,i}^{\tau_k})^2 \Big)^{1/2} \qquad i = 1, 2,$$

where n is the number of elements number of $\mathcal{T}_h$. So, our adaptive refinement strategy is as follows:

Algorithm 1 Adaptive algorithm for (VP_1) and (VP_2)

Require: *TOL= error tolerance,* δ *= parameter of refinement*

for $i = 1, 2, \cdots$ **do**

1. Compute the Galerkin solution

$$(\mathbf{u}^h, \phi^h) \text{ for } (VP_1)$$
$$(\mathbf{u}^h, {\sigma_\mathrm{n}}^h, \phi^h) \text{ for } (VP_2)$$

of the fully-discrete system, respectively.

2. Compute for each $\tau \in \mathcal{T}_h$ the local error indicators given in (3.12) and (3.13)

$$\begin{aligned} \eta_{R,1}^{\tau} &:= R_1^\tau + R_2^\tau + R_3^\tau + R_4^\tau \qquad \text{for } (VP_1) \\ \eta_{R,2}^{\tau} &:= \tilde{R}_1^\tau + \tilde{R}_2^\tau + \tilde{R}_3^\tau + \tilde{R}_4^\tau + \tilde{R}_5^\tau \qquad \text{for } (VP_2) \end{aligned} \tag{3.14}$$

and set

$$\eta_{\max} := \max_{\tau \in \mathcal{T}} \eta_{R,i}^{\tau}$$

3. Refine any $\tau \in \mathcal{T}$ such that $\delta \cdot \eta_{\max} \leq \eta_{R,i}^{\tau}$.

4. Stop if $\big(\sum_{\tau \in \mathcal{T}} (\eta_{R,i}^{\tau})^2 \big)^{1/2} \leq \mathrm{TOL}$

3.3 Efficiency of the Residual Error Estimator of (VP_1)

In this section we prove the efficiency of the residual error estimator for the formulation (VP_1) on quasi-uniform meshes. The ideas of this proof can be found in Verfürth [80] and Leydecker [52] for the indicators of the FEM part. For the indicators with boundary integral operators, we use some ideas of Carstensen [11] and Chernov [19]. Again, In this chapter the symbol $\lesssim$ signifies "$\leq$ up a multiplicative constant $c > 0$"
We will bound the following error indicators:

$$
\begin{aligned}
R_1^h &:= \sum_{\tau\in\mathcal{T}_h} h_\tau^2 \|\operatorname{div}\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f}\|_{0,\tau}^2,\\
R_2^h &:= \sum_{s\in\mathcal{S}_i} h_s \|[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]\|_{0,s}^2,\\
R_3^h &:= \sum_{s\in\mathcal{S}_{\Gamma,\tilde{h}}} h_s \|\sigma(\mathbf{u}^h)\,\mathbf{n} - p^0\mathbf{n} - V\phi^h\,\mathbf{n} - \alpha\big(K + \frac{I}{2}\big)\phi^h\,\mathbf{n}\|_{0,s}^2,\\
R_4^h &:= \sum_{s\in\mathcal{S}_{\Gamma,\tilde{h}}} h_s \Big\| - \frac{1}{\rho_0\omega^2}\frac{\partial p^0}{\partial n} + \mathbf{u}^h\cdot\mathbf{n} - \frac{1}{\rho_0\omega^2}\big(\big((K' - \frac{I}{2}) - \alpha W\big)\phi^h\Big\|_{0,s}^2.
\end{aligned}
\tag{3.15}
$$

FEM-indicators in the interior

Initially, we present a local upper bound for R_1^h and R_2^h:

Lemma 3.6. *Let $\tau \in \mathcal{T}_h$ and $R_{1,\tau}^h := h_\tau^2\|\operatorname{div}\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f}\|_{0,\tau}^2$, then there holds*

$$
R_{1,\tau}^h \lesssim \|\sigma(\mathbf{u}^h - \mathbf{u})\|_{0,\tau}^2 + h_\tau^2\|\mathbf{u}^h - \mathbf{u}\|_{0,\tau}^2. \tag{3.16}
$$

Proof. For the case $d = 2$, we denote by $\lambda_{\tau,1}, \lambda_{\tau,2}, \lambda_{\tau,3}$ the barycentric coordinates of the triangle $\tau \in \mathcal{T}_h$. We define the *triangle-bubble function* b_τ by

$$
b_\tau := \begin{cases} 27\,\lambda_{\tau,1}\lambda_{\tau,2}\lambda_{\tau,3} & \text{on } \tau,\\ 0 & \text{on } \Omega\setminus\tau. \end{cases} \tag{3.17}
$$

For $d = 3$ we denote by $\lambda_{\tau,1}, \lambda_{\tau,2}, \lambda_{\tau,3}, \lambda_{\tau,4}$ the barycentric coordinates of a tetrahedron $\tau \in \mathcal{T}_h$. We define the *bubble function* b_τ by

$$b_\tau := \begin{cases} 256\,\lambda_{\tau,1}\lambda_{\tau,2}\lambda_{\tau,3}\lambda_{\tau,4} & \text{on } \tau, \\ 0 & \text{on } \Omega\setminus\tau. \end{cases} \tag{3.18}$$

In both cases ($d = 2,3$) the function b_τ has the following properties (see Verfürth [80, p. 10]) with a constant $c > 0$.

$$\begin{aligned} \text{supp } b_\tau \subset \tau, \quad 0 \le & b_\tau \le 1, \quad \max_{x\in\tau} b_\tau(x) = 1, \\ \|b_\tau^{1/2}\mathbf{u}\|_{0,\tau} \le \|\mathbf{u}\|_{0,\tau} & \le c\|b_\tau^{1/2}\mathbf{u}\|_{0,\tau}, \\ \|\nabla b_\tau^{1/2}\mathbf{u}\|_{0,\tau} & \le h_\tau^{-1}\|\mathbf{u}\|_{0,\tau}. \end{aligned} \tag{3.19}$$

We define $g_\tau := (\operatorname{div}\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f})b_\tau$. Then from (3.19)

$$\begin{aligned} &\|\operatorname{div}\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f}\|_{0,\tau}^2 \\ &\quad\lesssim \|(\operatorname{div}\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f})b_\tau^{1/2}\|_{0,\tau}^2 \\ &\quad= \int_\tau (\operatorname{div}\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f})g_\tau\,dx \\ &\quad= \int_\tau (\operatorname{div}\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h)g_\tau\,dx - \int_\tau (\operatorname{div}\sigma(\mathbf{u}) + \rho\omega^2\mathbf{u})g_\tau\,dx \\ &\quad= \int_\tau \operatorname{div}\sigma(\mathbf{u}^h - \mathbf{u})g_\tau\,dx + \int_\tau \rho\omega^2(\mathbf{u}^h - \mathbf{u})g_\tau\,dx \\ &\quad= \int_\tau \big(\,\sigma(\mathbf{u} - \mathbf{u}^h) : \nabla g_\tau + \rho\omega^2(\mathbf{u}^h - \mathbf{u})g_\tau\,\big)\,dx \\ &\quad\lesssim \|\sigma(\mathbf{u}^h - \mathbf{u})\|_{0,\tau}\|\nabla g_\tau\|_{0,\tau} + \|\mathbf{u}^h - \mathbf{u}\|_{0,\tau}\|g_\tau\|_{0,\tau} \\ &\quad\lesssim \|g_\tau\|_{0,\tau}\big(h_\tau^{-1}\|\sigma(\mathbf{u}^h - \mathbf{u})\|_{0,\tau} + \|\mathbf{u}^h - \mathbf{u}\|_{0,\tau}\big) \\ &\quad\lesssim \|\operatorname{div}\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f}\|_{0,\tau}\big(h_\tau^{-1}\|\sigma(\mathbf{u}^h - \mathbf{u})\|_{0,\tau} + \|\mathbf{u}^h - \mathbf{u}\|_{0,\tau}\big). \end{aligned}$$

Dividing the last inequality by $\|\operatorname{div}\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f}\|_{0,\tau}$ it follows that

$$\|\operatorname{div}\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f}\|_{0,\tau} \le c\big(h_\tau^{-1}\|\sigma(\mathbf{u}^h - \mathbf{u})\|_{0,\tau} + \|\mathbf{u}^h - \mathbf{u}\|_{0,\tau}\big).$$

Finally, raising powers to the square and multiplying by h_τ^2 we obtain (3.16). □

Next, we estimate the local indicator $R^h_{2,\tau}$ related to the jump on $s \in \mathcal{S}_{i,\tau}$.

Lemma 3.7. *Let $\tau \in \mathcal{T}_h$ and $R^h_{2,\tau} := \sum_{s_i\in\tau} h_s\|[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]\|_{0,s}^2$, then there holds*

$$R^h_{2,\tau} \lesssim \|\sigma(\mathbf{u}-\mathbf{u}^h)\|^2_{0,w_\tau} + h_s^2\|\mathbf{u}\|^2_{0,w_\tau}, \tag{3.20}$$

where w_τ are the element neighbours of τ.

Proof. We estimate the indicator R^h_2 related to the jump in $s \in \mathcal{S}_i$ with $s = \partial\tau_1 \cap \partial\tau_2$, where τ_1 and τ_2 are the elements that contain the face s. Let $w_s := \tau_1 \cup \tau_2$.

For this we need the following definitions: Considering the two-dimensional case, we define an *edge-bubble function* b_s (see Verfürth [80][p. 10]) by

$$b_s := \begin{cases} 4\lambda_{\tau_{i,1}}\lambda_{\tau_{i,2}} & \text{on } \tau_i, \quad i=1,2, \\ 0 & \text{on } \Omega\backslash w_s, \end{cases}$$

where $\lambda_{\tau_{i,1}}, \lambda_{\tau_{i,2}}$ are the barycentric coordinates of τ_i $(i = 1, 2)$ related to the edge s. For the three-dimensional case, we define a *face-bubble function* b_s with barycentric coordinates $\lambda_{\tau_{i,1}}, \lambda_{\tau_{i,2}}, \lambda_{\tau_{i,3}}$

$$b_s := \begin{cases} 27\lambda_{\tau_{i,1}}\lambda_{\tau_{i,2}}\lambda_{\tau_{i,3}} & \text{on } s \in \mathcal{S}_i, \\ 0 & \text{on } \Omega\backslash s. \end{cases}$$

In the case $d = 2$ the function b_s has the following properties:

$$\begin{aligned} \text{supp } b_s \subset w_s, \quad 0 \le & b_s \le 1, \quad \max_{x\in s} b_s(x) = 1, \\ \int_s b_s =& \frac{2}{3}h_s, \\ c_1h_s^2 \le \int_\tau b_s =& \frac{1}{3}|\tau| \le c_2h_s^2 \qquad \forall\tau \in w_s, \\ \|\nabla b_s\|_{0,\tau} \le & c_3h_s^{-1}\|b_s\|_{0,\tau} \qquad \forall\tau \in w_s. \end{aligned} \tag{3.21}$$

For the case $d = 3$, b_s is the same triangle-bubble function defined in (3.17), therefore b_s satisfies the properties (3.19).

We have to prove the following result:

$$\|[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]\|^2_{0,s} \lesssim h_s^{-1}\|\sigma(\mathbf{u}-\mathbf{u}^h)\|^2_{0,w_s} + h_s\|\mathbf{u}\|^2_{0,w_s}. \tag{3.22}$$

Using the same arguments given in Verfürth [80, p. 16], we define $g_s := [\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!] b_s$. Since we use continuous linear functions, the term $[\![\sigma(\mathbf{u}^h)\cdot\mathbf{n}]\!]$ is a complex number, then (3.21) implies that

$$\int_s [\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!] g_s = \frac{2}{3} h_s |[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]|^2 = \frac{2}{3} \|[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]\|_{0,s}^2. \tag{3.23}$$

Using Green's Theorem, (2.3a), the Cauchy-Schwarz inequality, properties (3.21), (3.23) and taking in account that $\operatorname{div}\sigma(\mathbf{u}^h) = \mathbf{0}$ we obtain

$$\begin{aligned}
\|[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!] b_s^{1/2}\|_{0,s}^2 &= \int_{w_s} \sigma(\mathbf{u}^h) : \nabla g_s\,dx + \int_{w_s} \operatorname{div}\sigma(\mathbf{u}^h)\cdot g_s\,dx - \int_{\partial w_s} \sigma(\mathbf{u}^h)\,\mathbf{n}\cdot g_s \\
&= \int_{w_s} \sigma(\mathbf{u}^h) : \nabla g_s\,dx \\
&\quad - \int_{w_s} \sigma(\mathbf{u}) : \nabla g_s\,dx - \int_{w_s} \operatorname{div}\sigma(\mathbf{u})\cdot g_s\,dx + \int_{\partial w_s} \sigma(\mathbf{u})\,\mathbf{n}\cdot g_s \\
&= \int_{w_s} \sigma(\mathbf{u}^h - \mathbf{u}) : \nabla g_s\,dx + \rho\omega^2 \int_{w_s} \mathbf{u}\cdot g_s\,dx \\
&\lesssim \|\sigma(\mathbf{u}^h - \mathbf{u})\|_{0,w_s} \|\nabla g_s\|_{0,w_s} + \|\mathbf{u}\|_{0,w_s} \|g_s\|_{0,w_s} \\
&\lesssim h_s^{-1} |[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]|\, \|b_s\|_{0,s}\, \|\sigma(\mathbf{u}^h - \mathbf{u})\|_{0,w_s} \\
&\qquad + |[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]|\, \|b_s\|_{0,s}\, \|\mathbf{u}\|_{0,w_s} \\
&\lesssim |[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]|\ h_s^{-1} \Big(\int_{w_s} b_s\Big)^{1/2} \|\sigma(\mathbf{u}^h - \mathbf{u})\|_{0,w_s} \\
&\quad + |[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]| \Big(\int_{w_s} b_s\Big)^{1/2} \|\mathbf{u}\|_{0,w_s} \\
&\lesssim \|[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]\|_{0,s} \Big(\int_{w_s} b_s\Big)^{1/2} \Big(h_s^{-3/2} \|\sigma(\mathbf{u}^h - \mathbf{u})\|_{0,w_s} + h_s^{-1/2} \|\mathbf{u}\|_{0,w_s}\Big) \\
&\lesssim \|[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]\|_{0,s} \big(h_s^{-1/2} \|\sigma(\mathbf{u}^h - \mathbf{u})\|_{0,w_s} + h_s^{1/2} \|\mathbf{u}\|_{0,w_s}\big)
\end{aligned}$$

Dividing the last inequality by $\|[\![\sigma(\mathbf{u}^h)\cdot\mathbf{n}]\!]\|_{0,s}$, raising powers to the square and multiplying by h_s yields (3.22). Finally, summing over each interior edge of τ result (3.20) follows. □

Indicators on the boundary

Next, we give upper local estimates for the estimators R_3^h and R_4^h. For is analysis we use some ideas of Carstensen [11] and Chernov [19].

Since we use $(2,r)$-regular boundary element $\mathcal{B}^h$ families in the sense of Babuška and Aziz [2], one can assume the *inverse assumption*. For more details see e.g. Hsiao and Wendland [46] or Wendland [81].

Assumption 3.3.1. *Inverse assumption: For $m \leq s \leq 2$, $|m|, |s| \leq r$ there exists a constant $c = c(m,s,r)$ for all $\eta^h \in \mathcal{B}^h$*

$$\|\eta^h\|_s \leq ch^{t-s}\|\eta^h\|_t \qquad \forall \eta \in \mathcal{B}^h.$$

Lemma 3.8. *Assuming $(\mathbf{u},\phi) \in [H^2(\Omega)]^d \times H^{3/2}(\Gamma)$ it follows*

$$R_3^h \lesssim h_{\max,\Gamma} \left(\|\sigma(\mathbf{u}-\mathbf{u}^h)\|^2_{0,\Gamma} + \|\phi-\phi^h\|^2_{0,\Gamma} + \|\phi-\phi^h\|^2_{-1,\Gamma}\right). \tag{3.24}$$

Proof. Noting that $p^0\mathbf{n} = -\sigma(\mathbf{u})\,\mathbf{n} + \big(V + \alpha\big(K + \frac{I}{2}\big)\big)(\phi)\,\mathbf{n}$, we obtain in each $s \in \mathcal{S}_{\Gamma,\tilde{h}}$

$$\begin{aligned}
&\|\sigma(\mathbf{u}^h)\,\mathbf{n} - p^0\mathbf{n} - V\phi^h\,\mathbf{n} - \alpha\big(K + \frac{I}{2}\big)\phi^h\,\mathbf{n}\|^2_{0,s} \\
&\qquad = \| -\sigma(\mathbf{u}-\mathbf{u}^h)\,\mathbf{n} + V(\phi-\phi^h)\,\mathbf{n} + \alpha\big(K + \frac{I}{2}\big)(\phi-\phi^h)\,\mathbf{n}\|^2_{0,s} \\
&\qquad \lesssim \|\sigma(\mathbf{u}-\mathbf{u}^h)\,\mathbf{n}\|^2_{0,s} + \|V(\phi-\phi^h)\,\mathbf{n}\|^2_{0,s} + \|\alpha\big(K + \frac{I}{2}\big)(\phi-\phi^h)\,\mathbf{n}\|^2_{0,s}.
\end{aligned} \tag{3.25}$$

Summing the estimate (3.25) over all elements $s \in \mathcal{S}_{\Gamma,\tilde{h}}$ and due to $\|\cdot\|^2_{0,\Gamma} = \sum_{s\in\mathcal{S}_{\Gamma,\tilde{h}}} \|\cdot\|^2_{0,s}$ we obtain

$$R_3^h \lesssim h_{\max,\Gamma}\big(\|\sigma(\mathbf{u}-\mathbf{u}^h)\,\mathbf{n}\|^2_{0,\Gamma} + \|V(\phi-\phi^h)\,\mathbf{n}\|^2_{0,\Gamma} + \|\alpha\big(K + \frac{I}{2}\big)(\phi-\phi^h)\,\mathbf{n}\|^2_{0,\Gamma}\big).$$

Since $V : H^{-1/2+\tilde{s}}(\Gamma) \to H^{1/2+\tilde{s}}(\Gamma)$ and $K : H^{1/2+\tilde{s}}(\Gamma) \to H^{1/2+\tilde{s}}(\Gamma)$ are continuous mappings for $\tilde{s} \in [-1/2, 1/2]$ we have that

$$\begin{aligned}
\|V(\phi-\phi^h)\,\mathbf{n}\|^2_{0,\Gamma} &\lesssim \|V(\phi-\phi^h)\|^2_{0,\Gamma} \lesssim \|\phi-\phi^h\|^2_{-1,\Gamma}, \\
\|\big(K + \frac{I}{2}\big)(\phi-\phi^h)\,\mathbf{n}\|^2_{0,\Gamma} &\lesssim \|(K + \frac{I}{2})(\phi-\phi^h)\|^2_{0,\Gamma} \lesssim \|\phi-\phi^h\|^2_{0,\Gamma}.
\end{aligned}$$

Thus

$$R_3^h \lesssim h_{\max,\Gamma} \left(\|\sigma(\mathbf{u}-\mathbf{u}^h)\|^2_{0,\Gamma} + \|\phi-\phi^h\|^2_{0,\Gamma} + \|\phi-\phi^h\|^2_{-1,\Gamma}\right).$$

$\square$

Lemma 3.9. *Let $\mathcal{I}_h : \mathbf{C}(\Gamma) \to S^h$ denote the Lagrange interpolation operator, see e.g. Press et al. [70], then there holds*

$$\begin{aligned} R_4^h \lesssim & h_{\max,\Gamma} \|\mathbf{u} - \mathbf{u}^h\|_{0,\Gamma}^2 + h_{\max,\Gamma} \|\phi - \phi^h\|_{1/2,\Gamma}^2 \\ & + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}} \|\mathcal{I}_h\phi - \phi\|_{1/2,\Gamma}^2 + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}} \|\phi - \phi^h\|_{1/2,\Gamma}^2. \end{aligned} \tag{3.26}$$

Proof. Noting that $\frac{1}{\rho_0\omega^2}\frac{\partial p^0}{\partial \mathbf{n}} = \mathbf{u}\cdot\mathbf{n} - \frac{1}{\rho_0\omega^2}(\,(K' - \frac{I}{2}) - \alpha W)\phi^h$ we obtain in each $s \in \mathcal{S}_{\Gamma,\tilde{h}}$

$$\begin{aligned} &\| - \frac{1}{\rho_0\omega^2}\frac{\partial p^0}{\partial n} + \mathbf{u}^h\cdot\mathbf{n} - \frac{1}{\rho_0\omega^2}(\,(K' - \frac{I}{2}) - \alpha W)\phi^h\|_{0,s}^2 \\ &\qquad \lesssim \|(\mathbf{u} - \mathbf{u}^h)\cdot\mathbf{n}\|_{0,s}^2 + \|\big(K' - \frac{I}{2}\big)(\phi - \phi^h)\|_{0,s}^2 + \|\alpha W(\phi - \phi^h)\|_{0,s}^2. \end{aligned} \tag{3.27}$$

Summing the estimate (3.27) over all elements $s \in \mathcal{S}_{\Gamma,\tilde{h}}$ we obtain

$$R_4^h \lesssim h_{\max,\Gamma}\Big(\|\rho_0\omega^2(\mathbf{u} - \mathbf{u}^h)\cdot\mathbf{n}\|_{0,\Gamma}^2 + \|\big(K' - \frac{I}{2}\big)(\phi - \phi^h)\|_{0,\Gamma}^2 + \|\alpha W(\phi - \phi^h)\|_{0,\Gamma}^2\Big),$$

Since $K' : H^{-1/2+\tilde{s}}(\Gamma) \to H^{-1/2+\tilde{s}}(\Gamma)$ and $W : H^{1/2+\tilde{s}}(\Gamma) \to H^{-1/2+\tilde{s}}(\Gamma)$ are continuous mappings for $\tilde{s} \in [-1/2, 1/2]$, we have that

$$\begin{aligned} \|\alpha W(\phi - \phi^h)\|_{0,\Gamma}^2 &\lesssim \|\phi - \phi^h\|_{1,\Gamma}^2, \\ \|\big(K' - \frac{I}{2}\big)(\phi - \phi^h)\|_{0,\Gamma}^2 &\lesssim \|\phi - \phi^h\|_{0,\Gamma}^2. \end{aligned} \tag{3.28}$$

The triangle inequality gives

$$\|\phi - \phi^h\|_{1,\Gamma}^2 \le \|\phi - \mathcal{I}_h\phi\|_{1,\Gamma}^2 + \|\mathcal{I}_h\phi - \phi^h\|_{1,\Gamma}^2,$$

since $(\mathcal{I}_h\phi - \phi^h) \in \mathcal{B}^h$ we can apply the inverse Assumption 3.3.1

$$\begin{aligned} \|\mathcal{I}_h\phi - \phi^h\|_{1,\Gamma}^2 &\le h_{\min,\Gamma}^{-1}\|\mathcal{I}_h\phi - \phi^h\|_{\frac{1}{2},\Gamma}^2 \\ &\le h_{\min,\Gamma}^{-1}(\|\mathcal{I}_h\phi - \phi\|_{\frac{1}{2},\Gamma}^2 + \|\phi - \phi^h\|_{\frac{1}{2},\Gamma}^2). \end{aligned} \tag{3.29}$$

Thus

$$\begin{aligned} R_4^h \lesssim & h_{\max,\Gamma}\big(\|(\mathbf{u}-\mathbf{u}^h)\cdot\mathbf{n}\|_{0,\Gamma}^2 + \|\big(K' - \frac{I}{2}\big)(\phi-\phi^h)\|_{0,\Gamma}^2 + \|\alpha W(\phi-\phi^h)\|_{0,\Gamma}^2\big) \\ \leq & h_{\max,\Gamma}\|\mathbf{u}-\mathbf{u}^h\|_{0,\Gamma}^2 + h_{\max,\Gamma}\|\phi-\phi^h\|_{0,\Gamma}^2 + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\mathcal{I}_h\phi-\phi^h\|_{1/2,\Gamma}^2 \\ & + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\phi-\phi^h\|_{1/2,\Gamma}^2. \end{aligned}$$

□

An upper bound for all indicators

Summing the local estimators from Lemma 3.6 and 3.7 we obtain the following result. Since $\|\cdot\|_{0,\Omega}^2 = \sum_{\tau\in\mathcal{T}_h}\|\cdot\|_{0,\tau}^2$ it follows

Lemma 3.10.

$$R_1^h \lesssim \|\sigma(\mathbf{u}^h-\mathbf{u})\|_{0,\Omega}^2 + h_{\max,\Omega}^2\|\mathbf{u}^h-\mathbf{u}\|_{0,\Omega}^2, \tag{3.30}$$

$$R_2^h \lesssim \|\sigma(\mathbf{u}-\mathbf{u}^h)\|_{0,\Omega}^2 + h_{\max,\Omega}^2\|\mathbf{u}\|_{0,\Omega}^2. \tag{3.31}$$

Now, we are ready to establish an upper bound for all indicators. Taking into account the results given in Lemmas 3.6 - 3.10 it follows the following theorem.

Theorem 3.11.

$$\begin{aligned} \big(R_1^h + R_2^h + R_3^h + R_4^h\big) \lesssim & \|\sigma(\mathbf{u}-\mathbf{u}^h)\|_{0,\Omega}^2 + h_{\max,\Omega}\|\sigma(\mathbf{u}-\mathbf{u}^h)\|_{0,\Gamma}^2 \\ & + h_{\max,\Omega}^2\|\mathbf{u}-\mathbf{u}^h\|_{0,\Omega}^2 + h_{\max,\Omega}^2\|\mathbf{u}\|_{0,\Omega}^2 + h_{\max,\Gamma}\|(\mathbf{u}-\mathbf{u}^h)\|_{0,\Gamma}^2 \\ & + h_{\max,\Gamma}\|\phi-\phi^h\|_{0,\Gamma}^2 + h_{\max,\Gamma}\|\phi-\phi^h\|_{-1,\Gamma}^2 \\ & + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\mathcal{I}_h\phi-\phi\|_{1/2,\Gamma}^2 + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\phi-\phi^h\|_{1/2,\Gamma}^2. \end{aligned}$$

Let us consider quasiuniform meshes on Ω and their boundary Γ, i.e., meshes for which there exist constants $c_1, c_2 > 0$, independent of the meshsize, such that

$$1 \leq \frac{h_{\max,\Omega}}{h_{\min,\Omega}} \leq c_1, \qquad 1 \leq \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}} \leq c_2. \tag{3.32}$$

Remark 3.12. *Using regularity assumptions on the solution, i.e.,* $(\mathbf{u}, \phi) \in [H^2(\Omega)]^d \times H^{3/2}(\Gamma)$ *and the approximation properties of the Lagrangian interpolation operator yield*

$$\|\phi - \mathcal{I}_h\phi\|_{1/2,\Gamma}^2 \lesssim h_{\max,\Gamma}^2 \|\phi\|_{3/2,\Gamma}^2,$$
$$h_{\max,\Gamma}\|\phi - \phi^h\|_{0,\Gamma}^2 \lesssim h_{\max,\Gamma}^4 \|\phi\|_{3/2,\Gamma}^2,$$
$$h_{\max,\Gamma}\|\phi - \phi^h\|_{-1,\Gamma}^2 \lesssim h_{\max,\Gamma}^6 \|\phi\|_{3/2,\Gamma}^2.$$

According to Carstensen [11, p. 318]) we expect that exists $h_0 > 0$ *such that for all* $h \leq h_0$

$$ch_{\max,\Gamma}^2 \leq \|\phi - \phi^h\|_{1/2,\Gamma}^2, \qquad ch_{\max,\Omega}^2 \leq \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2. \tag{3.33}$$

Thus,

$$\begin{aligned}
&\frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\mathcal{I}_h\phi - \phi\|_{1/2,\Gamma}^2 + h_{\max,\Gamma}\|\phi - \phi^h\|_{0,\Gamma}^2 + h_{\max,\Gamma}\|\phi - \phi^h\|_{-1,\Gamma}^2 \\
&\quad\lesssim h_{\max,\Gamma}^2\|\phi\|_{3/2,\Gamma}^2 + h_{\max,\Gamma}^4\|\phi\|_{3/2,\Gamma}^2 + h_{\max,\Gamma}^6\|\phi\|_{3/2,\Gamma}^2 \\
&\quad\lesssim h_{\max,\Gamma}^2\|\phi\|_{3/2,\Gamma}^2 \\
&\quad\lesssim \|\phi - \phi^h\|_{0,\Gamma}^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|\sigma(\mathbf{u} - \mathbf{u}^h)\|_{0,\Omega}^2 + h_{\max,\Omega}\|\sigma(\mathbf{u} - \mathbf{u}^h)\|_{0,\Gamma}^2 + h_{\max,\Omega}^2\|\mathbf{u} - \mathbf{u}^h\|_{0,\Omega}^2 \\
&\quad + h_{\max,\Gamma}\|(\mathbf{u} - \mathbf{u}^h)\|_{0,\Gamma}^2 + h_{\max,\Omega}^2\|\mathbf{u}\|_{0,\Omega}^2 \\
&\lesssim \|\sigma(\mathbf{u} - \mathbf{u}^h)\|_{0,\Omega}^2 + \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2 \\
&\lesssim \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2.
\end{aligned}$$

Remark 3.13. *Together with Theorem 3.11 and Remark 3.12 we get*

$$R_1^h + R_2^h + R_3^h + R_4^h \lesssim \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2 + \|\phi - \phi^h\|_{1/2,\Gamma}^2.$$

3.4 Efficiency of the Residual Error Estimator of (VP_2)

In this section we prove the efficiency of the residual error estimator for the formulation (VP_2) on quasi-uniform meshes. We will bound the following error indicators:

$$\begin{aligned}
\tilde{R}_1^h &:= \sum_{\tau\in\mathcal{T}_h} 2h_\tau^2 \|\mathrm{div}\,\sigma(\mathbf{u}^h) + \rho\omega^2\mathbf{u}^h - \mathbf{f}\|_{0,\tau}^2,\\
\tilde{R}_2^h &:= \sum_{s_i\in\mathcal{S}_i} 2h_{s_i} \|[\![\sigma(\mathbf{u}^h)\,\mathbf{n}]\!]\|_{s_{\tau,i}}^2,\\
\tilde{R}_3^h &:= \sum_{s\in\mathcal{S}_{\Gamma,\tilde{h}}} h_s\| -2\sigma(\mathbf{u}^h)\,\mathbf{n} - p^0\mathbf{n} + {\sigma_{\mathrm{n}}}^h\,\mathbf{n} - V\phi^h\,\mathbf{n} - \alpha(K+\frac{I}{2})\phi^h\,\mathbf{n}\|_s^2,\\
\tilde{R}_4^h &:= \sum_{s\in\mathcal{S}_{\Gamma,\tilde{h}}} h_s\Big\| -\frac{\partial p^0}{\partial n} + \rho_0\omega^2\mathbf{u}^h\cdot\mathbf{n} - (K'-\frac{I}{2})(\phi^h) + \alpha W(\phi^h)\Big\|_s^2,\\
\tilde{R}_5^h &:= + \sum_{s\in\mathcal{S}_{\Gamma,\tilde{h}}} h_s\Big\|\frac{1}{\rho_0\omega^2}(p^0+\alpha\frac{\partial p^0}{\partial n}) - V(\mathbf{u}^h)\cdot\mathbf{n} - \alpha(K'+\frac{I}{2})(\mathbf{u}^h)\cdot\mathbf{n}\\
&\qquad - \frac{1}{\rho_0\omega^2}(K-\frac{I}{2})\,{\sigma_{\mathrm{n}}}^h + \frac{\alpha}{\rho_0\omega^2}W\,{\sigma_{\mathrm{n}}}^h\,\Big\|_s^2.
\end{aligned} \tag{3.34}$$

Taking into account that the indicators tR_1^h, $\tilde{R}_2^h$ and $\tilde{R}_4^h$ are equal to those given for the formulation (VP_1) (except for a constant factor). For this formulation (VP_2) we are to establish upper bounds for the remaining estimators $\tilde{R}_3^h$, $\tilde{R}_5^h$. Note that these estimates contain the variable σ_{n} introduced exclusively for this formulation.

Lemma 3.14. *Assuming that* $\mathbf{u}\in H^2(\Omega)$ *there exists a positive constant such that*

$$\tilde{R}_3^h \lesssim h_{\max,\Gamma}\Big(\|\sigma(\mathbf{u}-\mathbf{u}^h)\|_{0,s}^2 + \|\,\sigma_{\mathrm{n}} - {\sigma_{\mathrm{n}}}^h\,\|_{0,s}^2 + \|\phi-\phi^h\|_{-1,s} + \|\phi-\phi^h\|_{0,s}^2\Big).$$

Proof. Noting that $p^0\mathbf{n} = -2\sigma(\mathbf{u})\,\mathbf{n} - \sigma_{\mathrm{n}}\,\mathbf{n} + V\phi\mathbf{n} + \alpha\big(K+\frac{I}{2}\big)\phi\mathbf{n}$, we obtain on each $s\in\mathcal{S}_{\Gamma,\tilde{h}}$

$$\begin{aligned}
\tilde{R}_3^h &= \| -2\sigma(\mathbf{u}^h)\,\mathbf{n} - p^0\mathbf{n} + {\sigma_{\mathrm{n}}}^h\,\mathbf{n} - V\phi^h\,\mathbf{n} - \alpha(K+\frac{I}{2})\phi^h\,\mathbf{n}\|_{0,s}^2\\
&= \| -2\sigma(\mathbf{u}-\mathbf{u}^h)\,\mathbf{n} - (\sigma_{\mathrm{n}} - {\sigma_{\mathrm{n}}}^h)\mathbf{n} + V(\phi-\phi^h)\,\mathbf{n} + \alpha(K+\frac{I}{2})(\phi-\phi^h)\,\mathbf{n}\|_{0,s}^2\\
&\lesssim \| -2\sigma(\mathbf{u}-\mathbf{u}^h)\,\mathbf{n}\|_{0,s}^2 + \|(\sigma_{\mathrm{n}} - {\sigma_{\mathrm{n}}}^h)\mathbf{n}\|_{0,s}^2\\
&\qquad + \|V(\phi-\phi^h)\,\mathbf{n}\|_{0,s} + \alpha\|(K+\frac{I}{2})(\phi-\phi^h)\,\mathbf{n}\|_{0,s}^2.
\end{aligned} \tag{3.35}$$

Summing (3.35) over all elements $s\in\mathcal{S}_{\Gamma,\tilde{h}}$ we obtain by (3.3) that

$$\begin{aligned}\tilde{R}_3^h \lesssim & h_{\max,\Gamma}\Big(\| -2\sigma(\mathbf{u}-\mathbf{u}^h)\cdot\mathbf{n}\|_{0,s}^2 + \|(\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h)\mathbf{n}\|_{0,s}^2 \\ & + \|V(\phi-\phi^h)\,\mathbf{n}\|_{0,s} + \alpha\|(K+\frac{I}{2})(\phi-\phi^h)\,\mathbf{n}\|_{0,s}^2\Big) \\ \lesssim & h_{\max,\Gamma}\Big(\|\sigma(\mathbf{u}-\mathbf{u}^h)\|_{0,s}^2 + \|\,\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h\,\|_{0,s}^2 + \|\phi-\phi^h\|_{-1,s} + \|\phi-\phi^h\|_{0,s}^2\Big).\end{aligned}$$

□

Lemma 3.15. *Let $\mathcal{I}_h : \mathbf{C}(\Gamma) \to S^h$ denote the Lagrange interpolation operator, then there holds*

$$\begin{aligned}\tilde{R}_5^h \lesssim & h_{\max,\Gamma}\|\mathbf{u}-\mathbf{u}^h\|_{-1,\Gamma}^2 + h_{\max,\Gamma}\|\mathbf{u}-\mathbf{u}^h\|_{0,\Gamma}^2 + h_{\max,\Gamma}\|\,\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h\,\|_{0,\Gamma}^2 \\ & + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\mathcal{I}_h\,\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h\,\|_{1/2,\Gamma}^2 + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\,\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h\,\|_{1/2,\Gamma}^2.\end{aligned} \tag{3.36}$$

Proof. Noting that $\frac{1}{\rho_0\omega^2}(p^0+\alpha\frac{\partial p^0}{\partial n}) = \frac{1}{\rho_0\omega^2}\big((K-\frac{I}{2})\,\sigma_{\mathrm{n}}-\alpha W\,\sigma_{\mathrm{n}}\,\big)+V(\mathbf{u}\cdot\mathbf{n})+\alpha(K'+\frac{I}{2})(\mathbf{u}\cdot\mathbf{n})$ we obtain on each $s\in\mathcal{S}_{\Gamma,\tilde{h}}$

$$\begin{aligned}&\Big\|\frac{1}{\rho_0\omega^2}(p^0+\alpha\frac{\partial p^0}{\partial n}) - V(\mathbf{u}^h)\cdot\mathbf{n} - \alpha(K'+\frac{I}{2})(\mathbf{u}^h)\cdot\mathbf{n} \\ &\qquad - \frac{1}{\rho_0\omega^2}(K-\frac{I}{2})\,{\sigma_{\mathrm{n}}}^h + \frac{\alpha}{\rho_0\omega^2}W\,{\sigma_{\mathrm{n}}}^h\,\Big\|_{0,s}^2 \\ &= \Big\|V(\mathbf{u}-\mathbf{u}^h)\cdot\mathbf{n} + \alpha(K'+\frac{I}{2})(\mathbf{u}-\mathbf{u}^h)\cdot\mathbf{n} \\ &\quad + \frac{1}{\rho_0\omega^2}(K-\frac{I}{2})(\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h) - \frac{\alpha}{\rho_0\omega^2}W(\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h)\Big\|_{0,s}^2 \\ &\lesssim \big\|V(\mathbf{u}-\mathbf{u}^h)\cdot\mathbf{n}\big\|_{0,s} + \alpha\big\|(K'+\frac{I}{2})(\mathbf{u}-\mathbf{u}^h)\cdot\mathbf{n}\big\|_{0,s} \\ &\qquad + \frac{1}{\rho_0\omega^2}\big\|(K-\frac{I}{2})(\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h)\big\|_{0,s} + \frac{\alpha}{\rho_0\omega^2}\big\|W(\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h)\big\|_{0,s}^2.\end{aligned} \tag{3.37}$$

Summing (3.37) over all elements $s\in\mathcal{S}_{\Gamma,\tilde{h}}$ and applying (3.3), (3.28) and (3.29) we obtain (3.36). □

An upper bound for all indicators

From Lemmas 3.10, 3.14 and 3.15 we obtain the following upper bound for the error estimator (3.5).

Theorem 3.16. *Let $\mathcal{I}_h : \mathbf{C}(\Gamma) \to S^h$ denote the Lagrange interpolation operator, then there holds*

$$\begin{aligned}\left(\tilde{R}_1^h + \tilde{R}_2^h + \tilde{R}_3^h + \tilde{R}_4^h + \tilde{R}_5^h\right) \lesssim & \|\sigma(\mathbf{u}-\mathbf{u}^h)\|_{0,\Omega}^2 + h_{\max,\Omega}\|\sigma(\mathbf{u}-\mathbf{u}^h)\|_{0,\Gamma}^2 + h_{\max,\Omega}^2\|\mathbf{u}\|_{0,\Omega}^2 \\ & + h_{\max,\Omega}^2\|\mathbf{u}-\mathbf{u}^h\|_{0,\Omega}^2 + h_{\max,\Gamma}\|(\mathbf{u}-\mathbf{u}^h)\|_{0,\Gamma}^2 \\ & + h_{\max,\Gamma}\|\mathbf{u}-\mathbf{u}^h\|_{0,\Gamma}^2 + h_{\max,\Gamma}\|\mathbf{u}-\mathbf{u}^h\|_{-1,\Gamma}^2 \\ & + h_{\max,\Gamma}\|\phi-\phi^h\|_{0,\Gamma}^2 + h_{\max,\Gamma}\|\phi-\phi^h\|_{-1,\Gamma}^2 \\ & + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\mathcal{I}_h\phi-\phi\|_{1/2,\Gamma}^2 + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\phi-\phi^h\|_{1/2,\Gamma}^2 \\ & + h_{\max,\Gamma}\|\,\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h\,\|_{0,\Gamma}^2 \\ & + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\mathcal{I}_h\,\sigma_{\mathrm{n}}-\sigma_{\mathrm{n}}\,\|_{1/2,\Gamma}^2 + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\,\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h\,\|_{1/2,\Gamma}^2.\end{aligned}$$

Remark 3.17. *Using regularity assumptions on the solution, i.e.,* $(\mathbf{u}, \sigma_{\mathrm{n}}, \phi) \in [H^2(\Omega)]^d \times H^{3/2}(\Gamma) \times H^{3/2}(\Gamma)$ *and the approximation properties of the Lagrangian interpolation operator yields*

$$\begin{aligned}\|\,\sigma_{\mathrm{n}}-\mathcal{I}_h\,\sigma_{\mathrm{n}}\,\|_{1/2,\Gamma}^2 &\lesssim h_{\max,\Gamma}^2\|\,\sigma_{\mathrm{n}}\,\|_{3/2,\Gamma}^2, \\ h_{\max,\Gamma}\|\,\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h\,\|_{0,\Gamma}^2 &\lesssim h_{\max,\Gamma}^4\|\,\sigma_{\mathrm{n}}\,\|_{3/2,\Gamma}^2.\end{aligned}$$

Due to (3.33)*, there exists $c > 0$ such that*

$$c h_{\max,\Gamma}^2 \le \|\,\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h\,\|_{1/2,\Gamma}^2.$$

Thus, using the above inequality, Remark 3.12 , Assumption 3.32 and (3.33) *we obtain that*

$$\begin{aligned}& h_{\max,\Gamma}\|\phi-\phi^h\|_{0,\Gamma}^2 + h_{\max,\Gamma}\|\phi-\phi^h\|_{-1,\Gamma}^2 \\ & + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\mathcal{I}_h\phi-\phi\|_{1/2,\Gamma}^2 + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\phi-\phi^h\|_{1/2,\Gamma}^2 \\ & + h_{\max,\Gamma}\|\,\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h\,\|_{0,\Gamma}^2 \\ & + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\mathcal{I}_h\,\sigma_{\mathrm{n}}-\sigma_{\mathrm{n}}\,\|_{1/2,\Gamma}^2 + \frac{h_{\max,\Gamma}}{h_{\min,\Gamma}}\|\,\sigma_{\mathrm{n}}-{\sigma_{\mathrm{n}}}^h\,\|_{1/2,\Gamma}^2 \\ \lesssim\; & h_{\max,\Gamma}^2\|\phi\|_{3/2,\Gamma}^2 + h_{\max,\Gamma}^4\|\phi\|_{3/2,\Gamma}^2 + h_{\max,\Gamma}^6\|\phi\|_{3/2,\Gamma}^2\end{aligned}$$

$$
\begin{aligned}
&h^2_{\max,\Gamma}\| \sigma_{\rm n} \|^2_{3/2,\Gamma} + h^4_{\max,\Gamma}\| \sigma_{\rm n} \|^2_{3/2,\Gamma} + \| \sigma_{\rm n} - {\sigma_{\rm n}}^h \|^2_{1/2,\Gamma} \\
&\lesssim h^2_{\max,\Gamma}\|\phi\|^2_{3/2,\Gamma} + \|\phi - \phi^h\|^2_{0,\Gamma}. + h^2_{\max,\Gamma}\| \sigma_{\rm n} - {\sigma_{\rm n}}^h \|^2_{1/2,\Gamma} + \| \sigma_{\rm n} - {\sigma_{\rm n}}^h \|^2_{1/2,\Gamma} \\
&\lesssim \|\phi - \phi^h\|^2_{0,\Gamma} + \| \sigma_{\rm n} - {\sigma_{\rm n}}^h \|^2_{1/2,\Gamma},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\sigma(\mathbf{u} - \mathbf{u}^h)\|^2_{0,\Omega} + h_{\max,\Omega}\|\sigma(\mathbf{u} - \mathbf{u}^h)\|^2_{0,\Gamma} + h^2_{\max,\Omega}\|\mathbf{u} - \mathbf{u}^h\|^2_{0,\Omega} + h_{\max,\Gamma}\|\mathbf{u} - \mathbf{u}^h\|^2_{-1,\Gamma} \\
&\quad + h_{\max,\Gamma}\|(\mathbf{u} - \mathbf{u}^h)\|^2_{0,\Gamma} + h^2_{\max,\Omega}\|\mathbf{u}\|^2_{0,\Omega} \\
&\lesssim \|\sigma(\mathbf{u} - \mathbf{u}^h)\|^2_{0,\Omega} + \|\mathbf{u} - \mathbf{u}^h\|^2_{1,\Omega} \\
&\lesssim \|\mathbf{u} - \mathbf{u}^h\|^2_{1,\Omega}.
\end{aligned}
$$

Remark 3.18. *Together with Theorem 3.16 and the inequalities in Remarks 3.17 we obtain the following efficiency result for the formulation* (VP_2)

$$
\tilde{R}^h_1 + \tilde{R}^h_2 + \tilde{R}^h_3 + \tilde{R}^h_4 + \tilde{R}^h_5 \lesssim \|\mathbf{u} - \mathbf{u}^h\|^2_{1,\Omega} + \| \sigma_{\rm n} - {\sigma_{\rm n}}^h \|^2_{1/2,\Gamma} + \|\phi - \phi^h\|^2_{1/2,\Gamma}.
$$

Take into account that for this result we assumed regularity conditions of the solution and it is restricted to quasi-uniform meshes.

Chapter 4

Hierarchical Error Estimator

The purpose of this chapter is to establish an a posteriori error estimator using hierarchical basis techniques for the formulations (VP_1) and (VP_2). The method presented below is based on the method suggested by Maischak et al. [57] and Mund and Stephan [65].

The method proposed by Maischak et al. is applied to an *indefinite problem* derived using a BE method. The problem is called indefinite because the sesquilinear form $\mathcal{A}$ of the variational problem is of the form

$$\mathcal{A} = \mathcal{D} + \mathcal{K} \tag{4.1}$$

where $\mathcal{D}$ is a positive definite sesquilinear form and $\mathcal{K}$ is a compact and non positive definite sesquilinear form. The method suggested by Mund and Stephan is applied to a definite system derived for an FE/BE coupling method. Thus, the idea is to combine both methods and to adjust them to our formulations.

We also present the reliability and efficiency of the error estimates. This proof is based on two conditions imposed in [57]. One is the *saturation condition*, which guarantees the convergence of the Galerkin method in the norm induced by the form $\mathcal{D}(\cdot,\cdot)$. The second condition ensures the convergence in the hierarchical multi-level method of the compact part of the problem in the norm induced by $\mathcal{D}(\cdot,\cdot)$.

To apply the results of Maischak et al. [57], $\mathcal{D}$ and $\mathcal{A}$ should be continuous in a Hilbert space $\mathcal{H}$, i.e. there exist constants $\nu_{\mathcal{A}}, \nu_{\mathcal{D}} > 0$ such that

$$|\mathcal{D}(\eta,\xi)| \le \nu_{\mathcal{D}} \|\eta\|_{\mathcal{H}} \|\xi\|_{\mathcal{H}}, \qquad |\mathcal{A}(\eta,\xi)| \le \nu_{\mathcal{A}} \|\eta\|_{\mathcal{D}} \|\xi\|_{\mathcal{D}}, \tag{4.2}$$

for all $\eta, \xi \in \mathcal{H}$, where $\|\cdot\|_{\mathcal{D}} = \mathcal{D}(\cdot,\cdot)^{1/2}$ is the norm induced by the sesquilinear form $\mathcal{D}$.

4.1 Notation and Definitions

Remark 4.1. *For convenience, in this chapter we employ a different notation to distinguish the discretizations of our domain and the approximation spaces. Previously, this has been depended on the length h. In this case, h will depend on the level j corresponding to the level of discretization.*

Let $\{\mathcal{T}_j\}_j$ be a sequence of uniform partitions of Ω where $\mathcal{T}_{j+1}$ is obtained by subdividing all elements in $\mathcal{T}_j$. Let h_j be the diameter of the elements in $\mathcal{T}_j$. Let $\mathcal{W}_j := [\mathcal{W}_j]^d$ be the space of piecewise linear functions of dimension $d = 2, 3$ w.r.t. $\mathcal{T}_j$. Let n_j be the number of nodes which belong to the mesh $\mathcal{T}_{j+1}$ but not to $\mathcal{T}_j$ and let $\{\mathbf{b}_{j+1,i}\}_{i=1}^{n_j}$ be the piecewise linear functions with value 1 at one of these nodes and which vanish at all other nodes in $\mathcal{T}_{j+1}$.

Furthermore, we consider a sequence $\{\mathcal{S}_j\}_j$ of uniform partitions of Γ where $\mathcal{S}_{j+1}$ is obtained by subdividing all elements in $\mathcal{S}_j$. Let $h_{\Gamma,j}$ be the size of the elements in $\mathcal{S}_j$ and $\mathcal{B}_j$ the space of piecewise linear functions. Let m_j be the number of nodes which belong to the mesh $\mathcal{S}_{j+1}$ but not to $\mathcal{S}_j$ and let $\{\beta_{j+1,i}\}_{i=1}^{n_j}$ be the piecewise linear functions with value 1 at one of these nodes and which vanish at all other nodes in $\mathcal{S}_{j+1}$ (for the 2D case see Fig. 4.1).

We recall that $\mathcal{S}_j$ is the set of faces of $\mathcal{T}_j$ which are adjacent to the boundary Γ.

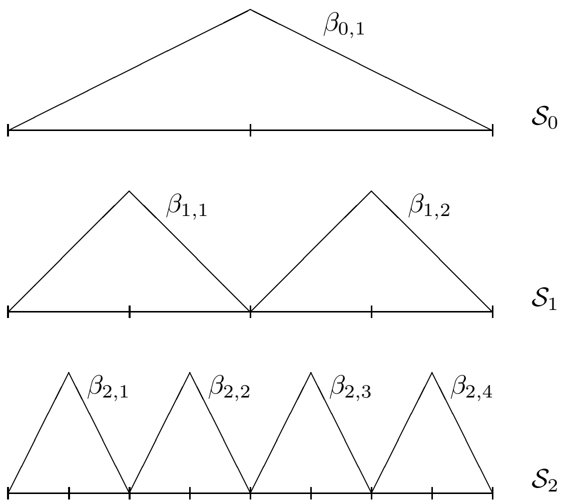


Fig. 4.1: Example the boundary meshes $\mathcal{S}_j$ and their corresponding basis function $\beta_{j,i}$.

For the three-dimensional case, an element $s_i^j \in \mathcal{S}_j$ is divided into four elements $s_{i_1}^{j+1}$, $s_{i_2}^{j+1}$,$s_{i_3}^{j+1}$ and $s_{i_4}^{j+1}$ (see Fig. 4.2) and we add five basis functions $\beta_{j+1,1}$, $\beta_{j+1,2}$, $\beta_{j+1,3}$, $\beta_{j+1,4}$ and $\beta_{j+1,5}$.

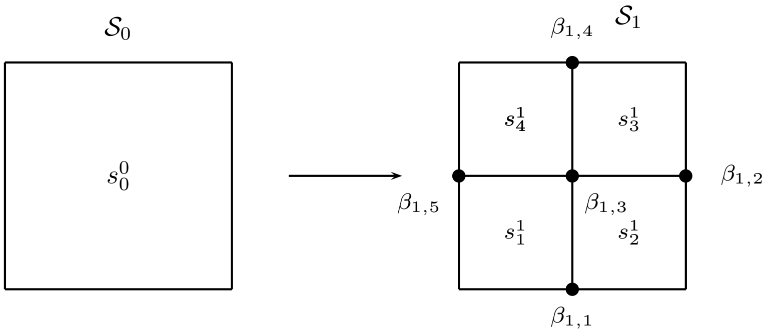


Fig. 4.2: Example of a two-level decomposition boundary mesh for the three-dimensional case.

We remember that $\mathcal{H}_1$ and $\mathcal{H}_2$ are the definition spaces of the variational formulations (VP_1) and (VP_2) (see Sec. 2.3), respectively. Let

$$\mathcal{H}_{1,0} \subset \cdots \subset \mathcal{H}_{1,j} \subset \mathcal{H}_{1,j+1} \subset \cdots \subset \mathcal{H}_1,$$
$$\mathcal{H}_{2,0} \subset \cdots \subset \mathcal{H}_{2,j} \subset \mathcal{H}_{2,j+1} \subset \cdots \subset \mathcal{H}_2,$$

be a sequence of finite dimensional subspaces, nested in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. For each of the subspaces

$$\mathcal{H}_{1,j} = \mathcal{W}_j \times \mathcal{B}_j, \qquad \mathcal{H}_{2,j} = \mathcal{W}_j \times \mathcal{B}_j \times \mathcal{B}_j,$$

we associate the following two-level subspace decomposition of $\mathcal{W}_j$ and $\mathcal{B}_j$

$$\mathcal{W}_j = \mathcal{W}_{j-1} + \sum_{i=1}^{n_j} X_{j,i} \qquad \mathcal{B}_j = \mathcal{B}_{j-1} + \sum_{i=1}^{m_j} Y_{j,i} \tag{4.3}$$

where $X_{j,i}$ and $Y_{j,i}$ are subspaces defined as $X_{j,i} = \text{span}\{\mathbf{b}_{j,i}\} \subset \mathcal{W}_j$ and $Y_{j,i} = \text{span}\{\beta_{j,i}\} \subset \mathcal{B}_j$.

Now we are ready to establish our estimator. We first consider the non-symmetric formulation (VP_1) and then we extend these results to the symmetric formulation (VP_2) in Section 4.3.

4.2 Non-symmetric Formulation (VP_1)

For the formulation (VP_1) in (4.1), we take as $\mathcal{D}$ the following sesquilinear form

$$\mathcal{D}(\mathbf{u},\phi;\mathbf{v},\psi) := a_0(\mathbf{u},\mathbf{v}) + |\alpha|\langle W_0\phi,\bar{\psi}\rangle$$

where W_0 is the hypersingular operator with kernel γ_0 (see (1.1)) and $a_0(\cdot,\cdot)$ is defined on p. 19. Note that $\mathcal{D}$ is continuous and positive definite in $\mathcal{H}_1$, due to the second Korn's inequality (2.17) and due to the coercivity of the operator W_0 (see Costabel and Stephan [23]). Therefore, $\mathcal{D}$ induces a norm in $\mathcal{H}_1$ defined as

$$\|(\mathbf{v},\psi)\|_{\mathcal{D}} := \big(a_0(\mathbf{v},\mathbf{v}) + |\alpha|\langle W_0\psi,\bar{\psi}\rangle\,\big)^{1/2},$$

the sesquilinear forms $a_0(\cdot,\cdot)$ and $\langle W_0\cdot,\cdot\rangle$ induce the norms

$$\|\mathbf{v}\|_{a_0} := a_0(\mathbf{v},\mathbf{v})^{1/2}, \qquad \|\psi\|_{W_0} := \langle W_0\psi,\bar{\psi}\rangle^{1/2}.$$

As $\mathcal{K}(\cdot\,;\cdot)$ we take

$$\begin{aligned}\mathcal{K}(\mathbf{u},\phi;\mathbf{v},\psi) :=& -\rho\omega^2 a_1(\mathbf{u},\mathbf{v}) + \langle V\phi\mathbf{n},\bar{\mathbf{v}}\rangle + \alpha\langle(K+\frac{I}{2})\phi\mathbf{n},\bar{\mathbf{v}}\rangle \\ &- \langle\mathbf{u}\cdot\mathbf{n},\bar{\psi}\rangle + \frac{1}{\rho\omega^2}\langle(K'-\frac{I}{2}) - \alpha(W+W_0)\,\phi,\bar{\psi}\rangle,\end{aligned} \tag{4.4}$$

where $a_1(\cdot,\cdot)$ is defined on p. 19. From Costabel and Stephan [23, Section 5] and Lemma 2.12 it follows that the sesquilinear form $\mathcal{K}(\cdot;\cdot)$ is compact.

Orthogonal projections

Let

$$P_{j-1} : \mathcal{W}_j \to \mathcal{W}_{j-1}, \qquad\qquad p_{j-1} : \mathcal{B}_j \to \mathcal{B}_{j-1},$$

$$P_{j,i} : \mathcal{W}_j \to X_{j,i}, \qquad\qquad p_{j,i} : \mathcal{B}_j \to Y_{j,i},$$

be the Galerkin projections with respect to the sesquilinear forms $a_0(\cdot,\cdot)$ and $\langle W_0\cdot,\cdot\rangle$, i.e.,

$$a_0(P_{j-1}\mathbf{u},\mathbf{v}) = a_0(\mathbf{u},\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{W}_j, \qquad a_0(P_{j,i}\mathbf{u},\mathbf{v}) = a_0(\mathbf{u},\mathbf{v}) \quad \forall \mathbf{v} \in X_{j,i},$$

$$\langle W_0\, p_{j-1}\,\phi, \bar{\psi}\rangle = \langle W_0\phi, \bar{\psi}\rangle \quad \forall \psi \in \mathcal{B}_j, \qquad \langle W_0\, p_{j,i}\,\phi, \bar{\psi}\rangle = \langle W_0\phi, \bar{\psi}\rangle \quad \forall \psi \in Y_{j,i}.$$

Now we define the *two-level additive Schwarz operators*: $\mathbf{P}^j : \mathcal{W}_j \to \mathcal{W}_j$ and $\mathbf{p}^j : \mathcal{B}_j \to \mathcal{B}_j$ as

$$\mathbf{P}^j := P_{j-1} + \sum_{i=1}^{n_j} P_{j,i}, \qquad\qquad \mathbf{p}^j := p_{j-1} + \sum_{i=1}^{m_j} p_{j,i}. \tag{4.5}$$

Note that $\mathbf{P}^j$ and $\mathbf{p}^j$ depend on the decomposition (4.3) and the sesquilinear form $a_0(\cdot,\cdot)$ and $\langle W_0\cdot,\bar{\cdot}\rangle$.

The following lemmas state that the operator $\mathbf{P}^j$ has a bounded condition number and $\mathbf{p}^j$ has an only moderately growing condition number.

Lemma 4.2. *There are constants $C_1, C_2 > 0$ which depend only on the smallest angle of the elements in $\mathcal{T}_{j-1}$ and on the diameter of Ω such that*

$$C_1\|\mathbf{v}\|_{a_0}^2 \le \|P_{j-1}\mathbf{v}\|_{a_0}^2 + \sum_{i=1}^{n_j} \|P_{j,i}\mathbf{v}\|_{a_0}^2 \le C_2\|\mathbf{v}\|_{a_0}^2 \tag{4.6}$$

for all $\mathbf{v} \in \mathcal{W}_j$.

Proof. For $d = 2, 3$ this result is proved in Zhang [84, Theorem 3.1], see also Mund and Stephan [65].

Lemma 4.3. *Let $h_{\Gamma,j-1}$ be the length of the smallest element in $\mathcal{S}_{j-1}$. For any $\epsilon > 0$ there are constants $c_1, c_2 > 0$ independent of $h_{\Gamma,j-1}$ such that*

$$c_1 h_{\Gamma,j-1}^{\epsilon}\|\psi\|_{W_0}^2 \le \|p_{j-1}\psi\|_{W_0}^2 + \sum_{i=1}^{m_j} \|p_{j,i}\psi\|_{W_0}^2 \le c_2 h_{\Gamma,j-1}^{-\epsilon}\|\psi\|_{W_0}^2, \quad \forall\, \psi \in \mathcal{B}_j. \tag{4.7}$$

Proof. For $d = 2$ this result is proved in Tran et al. [79], see also Mund [62]. For the case $d = 3$ the result is proved in Heuer and Stephan [40], Stephan et al. [76] and Ainsworth and Guo [1].

Remark 4.4. *We assume that the estimate* (4.7) *holds for* $\epsilon = 0$, *i.e., that the two level additive Schwarz operator* $\mathbf{p}^j$ *has a bounded condition number, i.e.,*

$$c_1\|\psi\|^2_{W_0} \leq \|p_{j-1}\psi\|^2_{W_0} + \sum_{i=1}^{m_j} \|p_{j,i}\psi\|^2_{W_0} \leq c_2\|\psi\|^2_{W_0} \tag{4.8}$$

for all $\psi \in \mathcal{B}_j$.

4.2.1 A posteriori error estimate

In this section we prove an a posteriori error estimate for the solution of (VP_1) in the norm generated by $\mathcal{D}$.

Let $(\mathbf{u}, \phi) \in \mathcal{H}_1$ be the solution of (VP_1) and let $(\mathbf{u}_j, \phi_j) \in \mathcal{H}_{1,j}$ be the Galerkin approximation of $(\mathbf{u}, \phi)$, i.e.,

$$\mathcal{A}_1(\mathbf{u}_j, \phi_j; \mathbf{v}, \psi) = \mathcal{L}(\mathbf{v}, \psi) \qquad \forall (\mathbf{v}, \psi) \in \mathcal{H}_{1,j}. \tag{4.9}$$

We assume the following saturation conditions (see Maischak et al. [57]):

Assumption 4.2.1 (Saturation condition). *There exist an index* $j_0 > 0$ *and a constant* $0 < \beta < 1$ *such that*

$$\|(\mathbf{u}, \phi) - (\mathbf{u}_{j+1}, \phi_{j+1})\|_{\mathcal{D}} \leq \beta \|(\mathbf{u}, \phi) - (\mathbf{u}_j, \phi_j)\|_{\mathcal{D}} \qquad \forall j \geq j_0.$$

Due to the quasi-optimal convergence and the convergence rate of the Galerkin solution for (VP_1) (see Section 2.4), we can assume the saturation condition to be satisfied in this case.

Assumption 4.2.2. *There exists a sequence* $\{\delta_j\} \in \mathbb{R}$ *with* $\lim_{j\to\infty} \delta_j = 0$ *such that*

$$|\mathcal{K}(\mathbf{u}_{j+1} - \mathbf{u}_j, \phi_{j+1} - \phi_j; \mathbf{u}_{j+1} - \mathbf{u}_j, \phi_{j+1} - \phi_j)| \leq \delta_j \|\mathbf{u}_{j+1} - \mathbf{u}_j, \phi_{j+1} - \phi_j\|^2_{\mathcal{D}}. \tag{4.10}$$

Next we prove the Assumption 4.2.2 for the sesquilinear form $\mathcal{K}(\cdot;\cdot)$ defined in (4.4). This assumption guarantees the quasi-optimal convergence of Galerkin solutions in the norm $\|\cdot\|_{\mathcal{D}}$.

Lemma 4.5. *Let* $(\mathbf{u}_j, \phi_j)$ *and* $(\mathbf{u}_{j+1}, \phi_{j+1})$ *be the solutions of problem* (4.9). *Let* h_j *be the diameter of* $\mathcal{T}_j$. *Then there holds*

$$|\mathcal{K}(\mathbf{u}_{j+1}-\mathbf{u}_j, \phi_{j+1}-\phi_j; \mathbf{u}_{j+1}-\mathbf{u}_j, \phi_{j+1}-\phi_j)| \le h_j^{1/2} \|\mathbf{u}_{j+1}-\mathbf{u}_j, \phi_{j+1}-\phi_j\|_{\mathcal{D}}^2. \quad (4.11)$$

Proof. We can decompose the sesquilinear form (4.4) into the following forms

$$\mathcal{K}(\mathbf{u},\phi;\mathbf{v},\psi) = \mathcal{K}_1(\mathbf{u},\mathbf{v}) + \mathcal{K}_2(\mathbf{u},\phi;\mathbf{v},\psi) + \mathcal{K}_3(\phi,\psi),$$

where

$$\begin{aligned}
\mathcal{K}_1(\mathbf{u},\mathbf{v}) :=& -\rho\omega^2 a_1(\mathbf{u},\mathbf{v}) = -\rho\omega^2 \int_\Omega \mathbf{u}\cdot\bar{\mathbf{v}}\,\mathrm{d}x,\\
\mathcal{K}_2(\mathbf{u},\phi;\mathbf{v},\psi) :=& \langle V\phi\,\mathbf{n}, \bar{\mathbf{v}}\rangle + \alpha\langle (K+\frac{I}{2})\phi\,\mathbf{n}, \bar{\mathbf{v}}\rangle - \langle \mathbf{u}\cdot\mathbf{n}, \bar{\psi}\rangle,\\
\mathcal{K}_3(\phi,\psi) :=& \frac{1}{\rho\omega^2}\langle\, (K'-\frac{I}{2})\phi, \bar{\psi}\rangle - \alpha\langle (W+W_0)\phi, \bar{\psi}\rangle.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and since $\mathbf{u}_j \in \mathcal{W}_j$ is the Galerkin projection of $\mathbf{u}_{j+1}$ with respect to $a_0(\cdot,\cdot)$ we can use the approximation properties (2.29) and obtain

$$\begin{aligned}
|\mathcal{K}_1(\mathbf{u}_{j+1}-\mathbf{u}_j, \mathbf{u}_{j+1}-\mathbf{u}_j)| &= |\rho\omega^2 a_1(\mathbf{u}_{j+1}-\mathbf{u}_j, \mathbf{u}_{j+1}-\mathbf{u}_j)|\\
&\lesssim \|\mathbf{u}_{j+1}-\mathbf{u}_j\|_0^2\\
&\lesssim h_j \|\mathbf{u}_{j+1}-\mathbf{u}_j\|_1^2.
\end{aligned} \quad (4.12)$$

Since ϕ_j is the Galerkin projection of ϕ_{j+1} with respect to the norm induced by $\langle W_0\cdot,\cdot\rangle$, we obtain using the Aubin-Nitsche Lemma (cf. Hsiao and Wendland [46]) that

$$\begin{aligned}
&|\mathcal{K}_2(\mathbf{u}_{j+1}-\mathbf{u}_j, \phi_{j+1}-\phi_j; \mathbf{u}_{j+1}-\mathbf{u}_j, \phi_{j+1}-\phi_j)|\\
&\quad= \Big|\Big\langle\, (V+\alpha(K+\frac{I}{2}))(\phi_{j+1}-\phi_j)\,\mathbf{n}, \bar{\mathbf{u}}_{j+1}-\bar{\mathbf{u}}_j\Big\rangle - \Big\langle (\mathbf{u}_{j+1}-\mathbf{u}_j)\cdot\mathbf{n}, \bar{\phi}_{j+1}-\bar{\phi}_j\Big\rangle\Big|
\end{aligned}$$

$$
\begin{aligned}
\leq & \left|\left\langle V(\phi_{j+1}-\phi_j)\,\mathbf{n}, \bar{\mathbf{u}}_{j+1}-\bar{\mathbf{u}}_j\right\rangle\right| + |\alpha|\,\left|\left\langle (K+\frac{I}{2})(\phi_{j+1}-\phi_j)\,\mathbf{n}, \bar{\mathbf{u}}_{j+1}-\bar{\mathbf{u}}_j\right\rangle\right| \\
& + \left|\left\langle (\mathbf{u}_{j+1}-\mathbf{u}_j)\cdot\mathbf{n}, \bar{\phi}_{j+1}-\bar{\phi}_j\right\rangle\right| \\
\lesssim & \|V(\phi_{j+1}-\phi_j)\|_{-1/2}\|\mathbf{u}_{j+1}-\mathbf{u}_j\|_{1/2,\Gamma} \\
& + |\alpha|\|(K+\frac{I}{2})(\phi_{j+1}-\phi_j)\|_{-1/2}\|\mathbf{u}_{j+1}-\mathbf{u}_j\|_{1/2,\Gamma} + \|\mathbf{u}_{j+1}-\mathbf{u}_j\|_{1/2,\Gamma}\|\phi_{j+1}-\phi_j\|_{-1/2} \\
\lesssim & \|\phi_{j+1}-\phi_j\|_{-3/2}\|\mathbf{u}_{j+1}-\mathbf{u}_j\|_1 + |\alpha|\|\phi_{j+1}-\phi_j\|_{-1/2}\|\mathbf{u}_{j+1}-\mathbf{u}_j\|_1 \\
& + \|\mathbf{u}_{j+1}-\mathbf{u}_j\|_1\|\phi_{j+1}-\phi_j\|_{-1/2} \\
\lesssim & h_{\Gamma,j}\|\phi_{j+1}-\phi_j\|_{1/2}\|\mathbf{u}_{j+1}-\mathbf{u}_j\|_1 \\
\lesssim & h_{\Gamma,j}\|\phi_{j+1}-\phi_j\|_{1/2}^2 + h_{\Gamma,j}\|\mathbf{u}_{j+1}-\mathbf{u}_j\|_1^2.
\end{aligned} \tag{4.13}
$$

Analogously, we have that

$$
\begin{aligned}
|\mathcal{K}_3(\phi_{j+1}-\phi_j,\phi_{j+1}-\phi_j)| &= |\frac{1}{\rho\omega^2}\langle((K'-\frac{I}{2})-\alpha(W+W_0))\,(\phi_{j+1}-\phi_j), \bar{\phi}_{j+1}-\bar{\phi}_j\rangle| \\
&\lesssim \|(K'-\frac{I}{2})(\phi_{j+1}-\phi_j)\|_{-1/2}\,\|\phi_{j+1}-\phi_j\|_{1/2} \\
&\quad + |\alpha|\|(W+W_0)\,(\phi_{j+1}-\phi_j)\|_{-1/2}\,\|\phi_{j+1}-\phi_j\|_{1/2} \\
&\lesssim \|\phi_{j+1}-\phi_j\|_{-1/2}\,\|\phi_{j+1}-\phi_j\|_{1/2} + \|\phi_{j+1}-\phi_j\|_0\,\|\phi_{j+1}-\phi_j\|_{1/2} \\
&\lesssim h_{\Gamma,j}^{1/2}\|\phi_{j+1}-\phi_j\|_{1/2}^2.
\end{aligned} \tag{4.14}
$$

Since the norms induced by $a_0(\cdot,\cdot)$ and $\langle W_0\cdot,\bar{\cdot}\rangle$ are equivalent to the norms $\|\cdot\|_{1,\Omega}$ and $\|\cdot\|_{1/2,\Gamma}$ respectively, (4.11) follows from (4.12)-(4.14). □

From the saturation condition (4.2.1) and the triangle inequality it follows that there exist $j_0 \in \mathbb{N}$, $0 \leq \beta \leq 1$ with

$$
\begin{aligned}
\frac{1}{1+\beta}\|(\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j)\|_{\mathcal{D}} \leq & \|(\mathbf{u},\phi)-(\mathbf{u}_j,\phi_j)\|_{\mathcal{D}} \\
\leq & \frac{1}{1-\beta}\|(\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j)\|_{\mathcal{D}}
\end{aligned} \tag{4.15}
$$

for all $j \geq j_0$.

Now we can formulate our a posteriori estimate.

Theorem 4.6. *Assume that Assumptions 4.2.1 and 4.2.2 hold. Then for any $\epsilon > 0$ there exist constants $\tilde{C}_1(j), \tilde{C}_2(j) > 0$ and a parameter $j_0 \in \mathbb{N}$ such that*

$$\tilde{C}_1(j)\Big(\sum_{i=1}^{n_j}\Theta_{j,i}^2+\sum_{i=1}^{m_j}\mathcal{V}_{j,i}^2\Big)^{1/2} \le \|(\mathbf{u},\phi)-(\mathbf{u}_j,\phi_j)\|_{\mathcal{D}} \le \tilde{C}_2(j)\Big(\sum_{i=1}^{n_j}\Theta_{j,i}^2+\sum_{i=1}^{m_j}\mathcal{V}_{j,i}^2\Big)^{1/2}, \quad (4.16)$$

for all $j \ge j_0$, where

$$\Theta_{j,i} := \frac{\big|\mathcal{R}(\mathbf{u}_j,\phi_j;\mathbf{b}_{j+1,i},0)\big|}{\|\mathbf{b}_{j+1,i}\|_{a_0}} \quad (4.17)$$

and

$$\mathcal{V}_{j,i} := \frac{\big|\mathcal{R}(\mathbf{u}_j,\phi_j;\mathbf{0},\beta_{j+1,i})\big|}{\|\beta_{j+1,i}\|_{W_0}},$$

where

$$\begin{aligned}\mathcal{R}(\mathbf{u}_j,\phi_j;\mathbf{b}_{j+1,i},\beta_{j+1,i}) :=& \mathcal{L}(\mathbf{b}_{j+1,i},\beta_{j+1,i}) - \mathcal{A}_1(\mathbf{u}_j,\phi_j;\mathbf{b}_{j+1,i},\beta_{j+1,i})\\ =&(\mathbf{f},\mathbf{b}_{j+1,i}) - \langle p^0\mathbf{n},\bar{\mathbf{b}}_{j+1,i}\rangle - a_0(\mathbf{u}_j,\mathbf{b}_{j+1,i}) + \rho\omega^2 a_1(\mathbf{u}_j,\mathbf{b}_{j+1,i})\\ &- \langle V\phi_j\mathbf{n},\bar{\mathbf{b}}_{j+1,i}\rangle - \alpha\langle (K+\frac{I}{2})\phi_j,\bar{b}_{j+1,i}\rangle\\ &+ \langle\frac{\partial p^0}{\partial n},\bar{\beta}_{j+1,i}\rangle - \rho_0\omega^2\langle\mathbf{u}_j\cdot\mathbf{n},\bar{\beta}_{j+1,i}\rangle\\ &+ \langle (K'-\frac{I}{2})\phi_j,\bar{\beta}_{j+1,i}\rangle - \alpha\langle W\phi_j,\bar{\beta}_{j+1,i}\rangle.\end{aligned}$$

Proof. To follow the proof given in Maischak et al. [57, Theorem 1]. We define $(\mathbf{e}_{j+1},\varepsilon_{j+1}) \in \mathcal{H}_{j+1} = \mathcal{W}_{j+1}\times\mathcal{B}_{j+1}$ by

$$\mathcal{D}(\mathbf{e}_{j+1},\varepsilon_{j+1};\mathbf{v},\psi) = \mathcal{L}(\mathbf{v},\psi) - \mathcal{A}_1(\mathbf{u}_j,\phi_j;\mathbf{v},\psi) \qquad \forall(\mathbf{v},\psi)\in\mathcal{H}_{j+1}. \quad (4.18)$$

We have to show the following inequalities

$$\frac{1}{\nu_{\mathcal{A}_1}}\|\mathbf{e}_{j+1},\varepsilon_{j+1}\|_{\mathcal{D}} \le \|(\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j)\|_{\mathcal{D}} \le \frac{1}{1-\delta_j}\|\mathbf{e}_{j+1},\varepsilon_{j+1}\|_{\mathcal{D}}, \quad (4.19)$$

where the left inequality follows from definition (4.18) and the continuity of $\mathcal{A}_1$ (see (4.2)). By (4.18), Assumption 4.2.2 and

$$\mathcal{L}((\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j);(\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j)) = 0,$$

the right inequality follows from

$$
\begin{aligned}
\|(\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j)\|_{\mathcal{D}}^2 &\le |\mathcal{D}((\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j);(\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j))| \\
&= |\mathcal{A}_1((\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j);(\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j)) \\
&\qquad -\mathcal{K}((\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j);(\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j))| \\
&= |\mathcal{D}(\mathbf{e}_{j+1},\varepsilon_{j+1};(\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j)) \\
&\qquad -\mathcal{K}((\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j);(\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j))| \\
&\le \|\mathbf{e}_{j+1},\varepsilon_{j+1}\|_{\mathcal{D}}\|(\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j)\|_{\mathcal{D}} \\
&\quad +\delta_j\|(\mathbf{u}_{j+1},\phi_{j+1})-(\mathbf{u}_j,\phi_j)\|_{\mathcal{D}}^2.
\end{aligned}
$$

Applying the definition of the additive Schwarz operators (4.5) on this results in

$$
\begin{aligned}
\mathcal{D}(\mathbf{P}^{j+1}\mathbf{e}_{j+1},\mathbf{p}^{j+1}\varepsilon_{j+1};\mathbf{e}_{j+1},\varepsilon_{j+1}) &= a_0(\mathbf{P}^{j+1}\mathbf{e}_{j+1},\mathbf{e}_{j+1})+\langle W_0\mathbf{p}^{j+1}\varepsilon_{j+1},\varepsilon_{j+1}\rangle \\
&= a_0(P_j\mathbf{e}_{j+1},\mathbf{e}_{j+1})+\langle W_0 p_j\varepsilon_{j+1},\varepsilon_{j+1}\rangle \\
&\quad +\sum_{i=1}^{n_j}\|P_{j+1,i}\,\mathbf{e}_{j+1},\mathbf{e}_{j+1}\|_{a_0}^2+\sum_{i=1}^{m_j}\|p_{j+1,i}\,\varepsilon_{j+1},\varepsilon_{j+1})\|_{W_0}^2.
\end{aligned}
\tag{4.20}
$$

Since $\mathcal{D}$ is hermitian and $(P_j\mathbf{e}_{j+1},p_j\varepsilon_{j+1})\in\mathcal{H}_j$, together with (4.18) and (4.9) we have that

$$
\begin{aligned}
\mathcal{D}(P_j\mathbf{e}_{j+1},p_j\varepsilon_{j+1};\mathbf{e}_{j+1},\varepsilon_{j+1}) &= \overline{\mathcal{D}(\mathbf{e}_{j+1},\varepsilon_{j+1};P_j\mathbf{e}_{j+1},p_j\varepsilon_{j+1})} \\
&= \overline{\mathcal{L}(P_j\mathbf{e}_{j+1};p_j\varepsilon_{j+1})-\mathcal{A}_1(\mathbf{u}_j,\phi_j;\,P_j\mathbf{e}_{j+1},p_j\varepsilon_{j+1})}=0.
\end{aligned}
$$

Therefore, due to Lemmas 4.2 and 4.3 there holds

$$
\begin{aligned}
\min\{C_1,c_1\}h_{j+1}^{\epsilon}\|\mathbf{e}_{j+1},\varepsilon_{j+1}\|_{\mathcal{D}}^2 &\le \sum_{i=1}^{n_j}\|P_{j+1,i}\,\mathbf{e}_{j+1}\|_{a_0}^2+\sum_{i=1}^{m_j}\|p_{j+1,i}\,\varepsilon_{j+1}\|_{W_0}^2 \\
&\le \max\{C_2,c_2\}h_{j+1}^{-\epsilon}\|\mathbf{e}_{j+1},\varepsilon_{j+1}\|_{\mathcal{D}}^2.
\end{aligned}
\tag{4.21}
$$

Finally, in (4.15) using (4.19) and (4.21) we get our error estimate

$$
\tilde{C}_1(j)\Big(\sum_{i=1}^{n_j}\|P_{j+1,i}\mathbf{e}_{j+1}\|_{a_0}+\sum_{i=1}^{m_j}\|p_{j+1,i}\varepsilon_{j+1}\|_{a_0}\Big)^{1/2}
$$

$$\leq \|(\mathbf{u},\phi)-(\mathbf{u}_j,\phi_j)\|_{\mathcal{D}} \leq \tilde{C}_2(j)\Big(\sum_{i=1}^{n_j}\|P_{j+1,i}\mathbf{e}_{j+1}\|_{a_0}+\sum_{i=1}^{m_j}\|p_{j+1,i}\varepsilon_{j+1}\|_{a_0}\Big)^{1/2},$$

where

$$\tilde{C}_1(j)=\frac{h_{j+1}^{\epsilon/2}}{\nu_{\mathcal{A}_1}(1+\beta)\sqrt{\max\{C_2,c_2\}}} \text{ and } \tilde{C}_2(j)=\frac{h_{j+1}^{-\epsilon/2}}{(1-\delta_j)(1-\beta)\sqrt{\min\{C_1,c_1\}}}.$$

We note that

$$\begin{aligned}P_{j+1,i}\,\mathbf{e}_{j+1}=&\frac{a_0(\mathbf{e}_{k+1},\mathbf{b}_{j+1,i})}{a_0(\mathbf{b}_{j+1,i},\mathbf{b}_{j+1,i})}\mathbf{b}_{j+1,i} \;=\frac{\mathcal{D}(\mathbf{e}_{j+1},\varepsilon_{j+1};\mathbf{b}_{j+1,i},0)}{a_0(\mathbf{b}_{j+1,i},\mathbf{b}_{j+1,i})}\mathbf{b}_{j+1,i}\\ =&\frac{\mathcal{L}(\mathbf{b}_{j+1,i},0)-\mathcal{A}_1(\mathbf{u}_j,\phi_j;\mathbf{b}_{j+1,i},0)}{\|\mathbf{b}_{j+1,i}\|_{a_0}^2}\mathbf{b}_{j+1,i}.\end{aligned}$$

Hence

$$\|P_{j+1,i}\,\mathbf{e}_{j+1}\|_{a_0}=\frac{\Big|\mathcal{L}(\mathbf{b}_{j+1,i},0)-\mathcal{A}_1(\mathbf{u}_j,\phi_j;\mathbf{b}_{j+1,1},0)\Big|}{\|\mathbf{b}_{j+1,i}\|_{a_0}}$$

where

$$\begin{aligned}\mathcal{L}(\mathbf{b}_{j+1,i},0)-\mathcal{A}_1(\mathbf{u}_j,\phi_j;\mathbf{b}_{j+1,i},0)=&(\mathbf{f},\mathbf{b}_{j+1,i})_0-\langle p^0\mathbf{n},\bar{\mathbf{b}}_{j+1,i}\rangle\\ &-a_0(\mathbf{u}_j,\mathbf{b}_{j+1,i})+\rho\omega^2 a_1(\mathbf{u}_j,\mathbf{b}_{j+1,i})\\ &-\langle V(\phi_j)\mathbf{n};\bar{\mathbf{b}}_{j+1,i}\rangle-\alpha\langle(K+\frac{I}{2}I)(\phi_j)\mathbf{n},\bar{\mathbf{b}}_{j+1,i}\rangle.\end{aligned}$$

and

$$\|p_{j+1,i}\,\varepsilon_{j+1}\|_{W_0}=\frac{|\mathcal{L}(\mathbf{0},\beta_{j+1,i})-\mathcal{A}_1(\mathbf{u}_j,\phi_j;\mathbf{0},\beta_{j+1,i})|}{\|\beta_{j+1,i}\|_{W_0}},$$

where

$$\begin{aligned}\mathcal{L}(\mathbf{0},\beta_{j+1,i})-\mathcal{A}_1(\mathbf{u}_j,\phi_j;\mathbf{0},\beta_{j+1,i})=&\langle\frac{\partial p^0}{\partial n},\bar{\beta}_{j+1,i}\rangle-\rho_0\omega^2\langle\mathbf{u}_j\cdot\mathbf{n},\bar{\beta}_{j+1,i}\rangle\\ &+\langle(K'-\frac{I}{2})\phi_j,\bar{\beta}_{j+1,i}\rangle-\alpha\langle W\phi_j,\bar{\beta}_{j+1,i}\rangle.\end{aligned}$$

$\square$

4.3 Symmetric Formulation (VP_2)

In this section we formulate a hierarchical error estimator for the formulation (VP_2) (see p.21) applying a similar procedure as in the above section. Considering that this case has one more unknown on the boundary than the case (VP_1), the hierarchical error estimator will contain one term more induced by the unknown $\sigma_{\rm n} \in H^{1/2}(\Gamma)$.

Let $(\mathbf{u}, \sigma_{\rm n}, \phi) \in \mathcal{H}_2$ be the solution of (VP_2) and let $(\mathbf{u}_j, \sigma_{{\rm n}j}, \phi_j) \in \mathcal{H}_{2,j}$ be the Galerkin approximation of $(\mathbf{u}, \sigma_{\rm n}, \phi) \in \mathcal{H}_2$, i.e.,

$$\mathcal{A}_2(\mathbf{u}_j, \sigma_{{\rm n}j}, \phi_j; \mathbf{v}, \chi, \psi) = \mathcal{L}(\mathbf{v}, \chi, \psi) \qquad \forall (\mathbf{v}, \chi, \psi) \in \mathcal{H}_{2,j}. \tag{4.22}$$

In this case we define by $\mathcal{D}(\cdot, \cdot)$ as

$$\mathcal{D}(\mathbf{u}, \sigma_{\rm n}, \phi; \mathbf{v}, \chi, \psi) := 2a_0(\mathbf{u}, \mathbf{v}) + |\alpha| \langle W_0\, \sigma_{\rm n}, \bar{\chi} \rangle + |\alpha| \langle W_0 \phi, \bar{\psi} \rangle$$

and

$$\begin{aligned} \mathcal{K}(\mathbf{u}, \sigma_{\rm n}, \phi; \mathbf{v}, \chi, \psi) := &- 2\rho\omega^2 a_1(\mathbf{u}, \mathbf{v}) - \langle \sigma_{\rm n}\, \mathbf{n}, \bar{\mathbf{v}} \rangle + \langle V\phi\mathbf{n}, \bar{\mathbf{v}} \rangle + \alpha \langle (K + \frac{I}{2})\phi\mathbf{n}, \bar{\mathbf{v}} \rangle \\ &- \langle \mathbf{u} \cdot \mathbf{n}, \bar{\chi} \rangle + \frac{1}{\rho\omega^2} \langle ((K' - \frac{I}{2}) - \alpha W)\, \phi, \bar{\chi} \rangle - \alpha \langle W_0\, \sigma_{\rm n}, \bar{\chi} \rangle \\ &+ \langle V\mathbf{u} \cdot \mathbf{n}, \bar{\psi} \rangle + \alpha \langle (K' + \frac{I}{2})\mathbf{u} \cdot \mathbf{n}, \bar{\psi} \rangle \\ &+ \frac{1}{\rho\omega^2} \langle ((K - \frac{I}{2}) - \alpha W\,)\ \sigma_{\rm n}, \bar{\psi} \rangle - \alpha \langle W_0 \phi, \bar{\psi} \rangle. \end{aligned} \tag{4.23}$$

Due to the quasi-optimal convergence and the convergence rates of the Galerkin solution for (VP_2) (see Section 2.4), we can also assume the saturation condition (4.2.1) to be satisfied in this case. Now we verify Assumption 4.2.2 for $\mathcal{K}(\cdot;\cdot)$ defined by (4.23).

Lemma 4.7. *Let* $(\mathbf{u}_j, \sigma_{{\rm n}j}, \phi_j)$ *and* $(\mathbf{u}_{j+1}, \sigma_{{\rm n}j+1}, \phi_{j+1})$ *be solutions of problem* (4.22). *Let* h_j *be the diameter of* $\mathcal{T}_j$, *then there holds*

$$\begin{aligned} &\big|\mathcal{K}(\mathbf{u}_{j+1} - \mathbf{u}_j, \sigma_{{\rm n}j+1} - \sigma_{{\rm n}j}, \phi_{j+1} - \phi_j; \mathbf{u}_{j+1} - \mathbf{u}_j, \sigma_{{\rm n}j+1} - \sigma_{{\rm n}j}, \phi_{j+1} - \phi_j)\big| \\ &\qquad \le h_j^{1/2} \|\mathbf{u}_{j+1} - \mathbf{u}_j, \sigma_{{\rm n}j+1} - \sigma_{{\rm n}j}, \phi_{j+1} - \phi_j\|_{\mathcal{D}}^2. \end{aligned} \tag{4.24}$$

where $\mathcal{K}$ is defined by (4.23).

Proof. We can decompose (4.23) in $\mathcal{K} = \sum_{i=1}^{7} \mathcal{K}_i$ where

$$\begin{aligned}
\mathcal{K}_1(\mathbf{u};\mathbf{v}) :=& -2\rho\omega^2 a_1(\mathbf{u},\mathbf{v}),\\
\mathcal{K}_2(\phi;\mathbf{v}) :=& \langle V\phi\mathbf{n},\bar{\mathbf{v}}\rangle + \alpha\langle (K+\frac{I}{2})\phi\mathbf{n},\bar{\mathbf{v}}\rangle,\\
\mathcal{K}_3(\sigma_{\mathrm{n}};\chi) :=& \frac{1}{\rho\omega^2}\langle((K-\frac{I}{2})-\alpha W)\ \sigma_{\mathrm{n}},\bar{\psi}\rangle,\\
\mathcal{K}_4(\mathbf{u},\sigma_{\mathrm{n}};\mathbf{v},\psi) :=& -\langle\sigma_{\mathrm{n}}\,\mathbf{n},\bar{\mathbf{v}}\rangle - \langle\mathbf{u}\cdot\mathbf{n},\bar{\chi}\rangle,\\
\mathcal{K}_5(\phi;\psi) :=& \frac{1}{\rho\omega^2}\langle((K'-\frac{I}{2})-\alpha W)\,\phi,\bar{\chi}\rangle,\\
\mathcal{K}_6(\mathbf{u};\chi) :=& \langle V\mathbf{u}\cdot\mathbf{n},\bar{\psi}\rangle + \alpha\langle(K'+\frac{I}{2})\mathbf{u}\cdot\mathbf{n},\bar{\psi}\rangle,\\
\mathcal{K}_7(\sigma_{\mathrm{n}},\phi;\psi,\chi) :=& -\alpha\langle W_0\,\sigma_{\mathrm{n}},\bar{\chi}\rangle - \alpha\langle W_0\phi,\bar{\psi}\rangle.
\end{aligned}$$

Recall that $(\mathbf{u}_j,\sigma_{\mathrm{n}j},\phi_j)\in\mathcal{W}_j\times\mathcal{B}_j\times\mathcal{B}_j$ is the Galerkin approximation of $(\mathbf{u}_{j+1},\sigma_{\mathrm{n}j+1},\phi_{j+1})$. For the following inequalities, we also employ the Cauchy-Schwarz inequality and the Aubin-Nitsche Lemma [46].

Then for $\mathcal{K}_3(\cdot,\cdot)$ it follows that

$$\begin{aligned}
|\mathcal{K}_3(\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j};\phi_{j+1}-\phi_j)| &= \frac{1}{\rho\omega^2}\big|\,\langle((K-\frac{I}{2})-\alpha W)\,(\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}),\bar{\phi}_{j+1}-\bar{\phi}_j\rangle\big|\\
&\lesssim \big\|(K-\frac{I}{2})(\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j})\big\|_{-1/2}\|\phi_{j+1}-\phi_j\|_{1/2}\\
&\quad+\big\|W(\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j})\big\|_{-1/2}\|\phi_{j+1}-\phi_j\|_{1/2}\\
&\lesssim \|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|_{-1/2}\,\|\phi_{j+1}-\phi_j\|_{1/2} + \|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|_0\,\|\phi_{j+1}-\phi_j\|_{1/2}\\
&\lesssim h_{\Gamma,j}\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|_{-1/2}^2 + h_{\Gamma,j}\|\phi_{j+1}-\phi_j\|_{1/2}^2\\
&\quad+h_{\Gamma,j}^{1/2}\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|_0^2 + h_{\Gamma,j}^{1/2}\|\phi_{j+1}-\phi_j\|_{1/2}^2\\
&\lesssim h_{\Gamma,j}^{1/2}(\,\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|_{1/2}^2 + \|\phi_{j+1}-\phi_j\|_{1/2}^2\,).
\end{aligned}$$

For $\mathcal{K}_4(\cdot;\cdot)$ we have

$$\begin{aligned}
&|\mathcal{K}_4(\mathbf{u}_{j+1}-\mathbf{u}_j,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j};\mathbf{u}_{j+1}-\mathbf{u}_j,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j})|\\
&\quad=|\langle(\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j})\mathbf{n},\bar{\mathbf{u}}_{j+1}-\bar{\mathbf{u}}_j\rangle - \langle(\mathbf{u}_{j+1}-\mathbf{u}_j)\cdot\mathbf{n},\bar{\sigma}_{\mathrm{n}j+1}-\bar{\sigma}_{\mathrm{n}j}\rangle,
\end{aligned}$$

$$
\begin{aligned}
&= |2\langle(\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j})\mathbf{n}, \bar{\mathbf{u}}_{j+1}-\bar{\mathbf{u}}_j\rangle| \\
&\lesssim \|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|_{-1/2}\|\mathbf{u}_{j+1}-\mathbf{u}_j\|_{1/2} \\
&\lesssim h_{\Gamma,j}\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|^2_{1/2}+h_{\Gamma,j}\|\mathbf{u}_{j+1}-\mathbf{u}_j\|^2_1 \\
&\lesssim h_{\Gamma,j}\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|^2_{1/2}+h_{\Gamma,j}\|\mathbf{u}_{j+1}-\mathbf{u}_j\|^2_1.
\end{aligned}
$$

$$
\begin{aligned}
&|\mathcal{K}_5(\phi_{j+1}-\phi_j;\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j})| = \frac{1}{\rho\omega^2}|\langle\,\big((K'-\frac{I}{2})-\alpha W\big)\,(\phi_{j+1}-\phi_j),\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\rangle| \\
&\lesssim \|(K'-\frac{I}{2})(\phi_{j+1}-\phi_j)\|_{-1/2}\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|_{1/2}+\|W(\phi_{j+1}-\phi_j)\|_{-1/2}\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|_{1/2} \\
&\lesssim \|\phi_{j+1}-\phi_j\|_{-1/2}\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|_{1/2}+\|\phi_{j+1}-\phi_j\|_0\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|_{1/2} \\
&\lesssim h_{\Gamma,j}\big(\|\phi_{j+1}-\phi_j\|^2_{1/2}+\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|^2_{1/2}\big)+h^{1/2}_{\Gamma,j}\big(\|\phi_{j+1}-\phi_j\|^2_0+\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|^2_{1/2}\big) \\
&\lesssim h^{1/2}_{\Gamma,j}(\,\|\phi_{j+1}-\phi_j\|^2_{1/2}+\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|^2_{1/2}\,),
\end{aligned}
$$

and using the same procedure as in (4.13) we get for $\mathcal{K}_6(\cdot,\cdot)$ that

$$
\begin{aligned}
|\mathcal{K}_6(\mathbf{u}_{j+1}-\mathbf{u}_j;\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j})| &= \big|\,\langle V+\alpha(K'+\frac{I}{2})\,(\mathbf{u}_{j+1}-\mathbf{u}_j)\cdot\mathbf{n},\bar{\sigma}_{\mathrm{n}j+1}-\bar{\sigma}_{\mathrm{n}j}\rangle\big| \\
&= \big|\langle V+\alpha(K'+\frac{I}{2})(\bar{\sigma}_{\mathrm{n}j+1}-\bar{\sigma}_{\mathrm{n}j}),(\mathbf{u}_{j+1}-\mathbf{u}_j)\cdot\mathbf{n}\rangle\big| \\
&\lesssim h_{\Gamma,j}\big(\,\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|^2_{1/2}+\|\mathbf{u}_{j+1}-\mathbf{u}_j\|^2_{1,\Omega}\,\big).
\end{aligned}
$$

Finally, for $\mathcal{K}_7(\cdot;\cdot)$ we have

$$
\begin{aligned}
&\mathcal{K}_7(\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j},\phi_{j+1};\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j},\phi_{j+1}-\phi_j) \\
&\quad = -\,\langle W_0(\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}),\bar{\sigma}_{\mathrm{n}j+1}-\bar{\sigma}_{\mathrm{n}j}\rangle-\langle W_0(\phi_{j+1}-\phi_j),\bar{\phi}_{j+1}-\bar{\phi}_j\rangle \\
&\quad \lesssim \|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|_0\,\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|_{1/2}+\|\phi_{j+1}-\phi_j\|_0\,\|\phi_{j+1}-\phi_j\|_{1/2} \\
&\quad \lesssim h^{1/2}_{\Gamma,j}\big(\|\,\sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}\,\|^2_{1/2}+\|\phi_{j+1}-\phi_j\|^2_{1/2}\,\big).
\end{aligned}
$$

For $\mathcal{K}_1(\cdot,\cdot)$ and $\mathcal{K}_2(\cdot,\cdot)$ we use the analogue procedures presented in (4.12) and (4.13), then we can collect the above inequalities and obtain that

$$\left|\mathcal{K}(\mathbf{u}_{j+1}-\mathbf{u}_j, \sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}, \phi_{j+1}-\phi_j; \mathbf{u}_{j+1}-\mathbf{u}_j, \sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}, \phi_{j+1}-\phi_j)\right| \\ \leq h_j^{1/2}\|\mathbf{u}_{j+1}-\mathbf{u}_j, \sigma_{\mathrm{n}j+1}-\sigma_{\mathrm{n}j}, \phi_{j+1}-\phi_j\|^2_{\mathcal{D}}.$$

□

Now we can formulate the hierarchical error estimate for the formulation (VP_2).

Theorem 4.8. *Assume that Assumptions 4.2.1 and 4.2.2 hold. Then for any* $\epsilon > 0$ *there exist constants* $\hat{C}_1(j), \hat{C}_2(j) > 0$ *and a parameter* $j_0 \in \mathbb{N}$ *such that*

$$\hat{C}_1(j)\Big(\sum_{i=1}^{n_j}\tilde{\Theta}^2_{j,i} + \sum_{i=1}^{m_j}\tilde{\Psi}^2_{j,i} + \sum_{i=1}^{m_j}\tilde{\mathcal{V}}^2_{j,i}\Big)^{1/2} \\ \leq \|(\mathbf{u}, \sigma_{\mathrm{n}}, \phi) - (\mathbf{u}_j, \sigma_{\mathrm{n}j}, \phi_j)\|_{\mathcal{H}} \leq \hat{C}_2(j)\Big(\sum_{i=1}^{n_j}\tilde{\Theta}^2_{j,i} + \sum_{i=1}^{m_j}\tilde{\Psi}^2_{j,i} + \sum_{i=1}^{m_j}\tilde{\mathcal{V}}^2_{j,i}\Big)^{1/2} \tag{4.25}$$

for all $j \geq j_0$, *where*

$$\tilde{\Theta}_{j,i} = \frac{\left|\mathcal{R}(\mathbf{u}_j, \sigma_{\mathrm{n}j}, \phi_j;\, \mathbf{b}_{j+1,i}, 0, 0)\right|}{\|\mathbf{b}_{j+1,i}\|_{a_0}},$$
$$\tilde{\Psi}_{j,i} = \frac{|\mathcal{R}(\mathbf{u}_j, \sigma_{\mathrm{n}j}, \phi_j;\, \mathbf{0}, \beta_{j+1,i}, 0)|}{\|\beta_{j+1,i}\|_{W_0}},$$
$$\tilde{\mathcal{V}}_{j,i} = \frac{|\mathcal{R}(\mathbf{u}_j, \sigma_{\mathrm{n}j}, \phi_j; \mathbf{0}, 0, \zeta_{j+1,i})|}{\|\zeta_{j+1,i}\|_{W_0}},$$

and

$$\begin{aligned}
&\mathcal{R}(\mathbf{u}_j, \sigma_{\mathrm{n}j}, \phi_j; \mathbf{b}_{j+1,i}, \beta_{j+1,i}, \zeta_{j+1,i}) \\
&=\mathcal{L}(\mathbf{b}_{j+1,i}, \beta_{j+1,i}, \zeta_{j+1,i}) - \mathcal{A}_2(\mathbf{u}_j, \sigma_{\mathrm{n}j}, \phi_j; \mathbf{b}_{j+1,i}, \beta_{j+1,i}, \zeta_{j+1,i}) \\
&=2(\mathbf{f}, \mathbf{b}_{j+1,i})_0 - \langle p^0\mathbf{n}, \bar{\mathbf{b}}_{j+1,i}\rangle \\
&\quad - 2a_0(\mathbf{u}_j, \mathbf{b}_{j+1,i}) + 2\rho\omega^2 a_1(\mathbf{u}_j, \mathbf{b}_{j+1,i}) + \langle \sigma_{\mathrm{n}j}\,\mathbf{n}, \mathbf{b}_{j+1,i}\rangle \\
&\quad - \langle V(\phi_j)\mathbf{n}, \bar{\mathbf{b}}_{j+1,i}\rangle - \alpha\langle (K + \frac{I}{2})(\phi_j)\mathbf{n}, \bar{\mathbf{b}}_{j+1,i}\rangle \\
&\quad - \frac{1}{\rho_0\omega^2}\langle \frac{\partial p^0}{\partial n}, \bar{\beta}_{j+1,i}\rangle + \langle \mathbf{u}_j\cdot\mathbf{n}, \bar{\beta}_{j+1,i}\rangle \\
&\quad - \frac{1}{\rho_0\omega^2}\langle (K' - \frac{I}{2})\phi_j, \bar{\beta}_{j+1,i}\rangle + \alpha\frac{1}{\rho_0\omega^2}\langle W\phi_j, \bar{\beta}_{j+1,i}\rangle \\
&\quad + \frac{1}{\rho_0\omega^2}\,\langle p^0 + \alpha\frac{\partial p^0}{\partial n}, \bar{\zeta}_{j+1,i}\rangle - \langle V(\mathbf{u}_j)\cdot\mathbf{n}, \bar{\zeta}_{j+1,i}\rangle - \alpha\langle (K' + \frac{I}{2})(\mathbf{u}_j)\cdot\mathbf{n}, \bar{\zeta}_{j+1,i}\rangle
\end{aligned}$$

$$- \frac{1}{\rho_0\omega^2}\langle (K-\frac{I}{2})\,\sigma_{\mathrm{n}j},\bar{\zeta}_{j+1,i}\rangle + \alpha\frac{1}{\rho_0\omega^2}\langle W\,\sigma_{\mathrm{n}j},\bar{\zeta}_{j+1,i}\rangle .$$

Proof. For this case we do not give the proof because it is analogous to presented in Theorem 4.6. We recall that in this case one adds a term $\tilde{\Psi}_{j,i}$ in the error estimator. This is derived by applying the two-level additive Schwarz operator $p^j : \mathcal{B}_j \to \mathcal{B}_j$ to ξ_{j+1}, where ξ_{j+1} is implicit in the following definition: Let

$$(\mathbf{e}_{j+1},\xi_{j+1},\varepsilon_{j+1}) \in \mathcal{H}_{j+1} = \mathcal{W}_{j+1}\times\mathcal{B}_{j+1}\times\mathcal{B}_{j+1}$$

be such that

$$\mathcal{D}(\mathbf{e}_{j+1},\xi_{j+1},\varepsilon_{j+1};\mathbf{v},\chi,\psi) = \mathcal{L}(\mathbf{v},\chi,\psi) - \mathcal{A}_2(\mathbf{u}_j,\sigma_{\mathrm{n}j},\phi_j;\,\mathbf{v},\chi,\psi) \qquad \forall(\mathbf{v},\chi,\psi)\in\mathcal{H}_{j+1}.$$

Also, the constants $\hat{C}_1$ and $\hat{C}_2(j)$ are given by

$$\hat{C}_1(j) = \frac{h_{j+1}^{\epsilon/2}}{\nu_{\mathcal{A}_2}(1+\beta)\sqrt{\max\{C_2,c_2\}}} \text{ and } \hat{C}_2(j) = \frac{h_{j+1}^{-\epsilon/2}}{(1-\delta_j)(1-\beta)\sqrt{\min\{C_1,c_1\}}}.$$

□

4.4 Adaptive Strategy

Based on the a posteriori error estimates (4.16) and (4.25), we formulate an h-adaptive refinement strategy, which consists of calculating the local error indicators where to perform a local refinement. The idea is to calculate a solution $(\mathbf{u}^h,\phi^h)$ and $(\mathbf{u}^h,{\sigma_{\mathrm{n}}}^h,\phi^h)$, in each problem (VP_1) and (VP_2), respectively, and to find those elements in which the local hierarchical error indicator is largest.

Given a mesh $\mathcal{T}_j$ one computes the refined mesh $\mathcal{T}_{j+1}$. Each element $\tau\in\mathcal{T}_j$ is subdivided into n_τ new nodes ($n_\tau = 3$ for the two-dimensional case or $n_\tau = 19$ nodes for the three-dimensional case). Also, some of the new nodes are on the boundary mesh of $\mathcal{T}_{j+1}$, so, let n_s be the number of these. Each new node $\mathbf{x}_i$ $(i = 1,\ldots,n_\tau)$ corresponds to a function $\mathbf{b}_{j+1,i}$

and each new node $\mathbf{x}_{b,i}$ $(i = 1, \ldots, n_s)$ on the boundary mesh $\mathcal{S}_{j+1}$ corresponds to a function $\beta_{j+1,i}, \xi_{j+1,i} \in \mathcal{B}_{j+1}$.

Then, we compute the nodal error indicator for the non-symmetric formulation (VP_1) as in (4.16)

$$\eta_{H_1}^{\tau} := \Big(\sum_{i=1}^{n_\tau} \Theta_{i,j}^2 + \sum_{i=1}^{n_s} \mathcal{V}_{i,j}^2 \Big)^{1/2}, \tag{4.26}$$

and for the symmetric formulation (VP_2)

$$\eta_{H_2}^{\tau} := \Big(\sum_{i=1}^{n_\tau} \tilde{\Theta}_{i,j}^2 + \sum_{i=1}^{n_s} \tilde{\Psi}_{i,j}^2 + \sum_{i=1}^{n_s} \tilde{\mathcal{V}}_{i,j}^2 \Big)^{1/2}.$$

The global error is estimated by

$$\eta_{H_i} = \Big(\sum_{j=1}^{n} (\eta_{H_i}^{\tau_j})^2 \Big)^{1/2}, \qquad i = 1, 2,$$

where n is the element number of $\mathcal{T}_j$.

Our adaptive refinement strategy is as follows:

Algorithm 2 Adaptive algorithm

Require: *TOL= error tolerance, δ = parameter of refinement*

for $j = 1, 2, \cdots$ **do**

1. Compute the Galerkin solution $(\mathbf{u}_j, \phi_j)$ or $(\mathbf{u}_j, \sigma_{\mathrm{n}j}, \phi_j)$ of the fully-discrete systems (VP_1^h) or (VP_2^h), respectively.
2. Compute for each $\tau \in \mathcal{T}_j$ the local error indicators for (VP_1)

$$\eta_{H_1}^{\tau} := \Big(\sum_{i=1}^{n_\tau} \Theta_{i,j}^2 + \sum_{i=1}^{n_s} \mathcal{V}_{i,j}^2 \Big)^{1/2}$$

or for the case (VP_2)

$$\eta_{H_2}^{\tau} := \Big(\sum_{i=1}^{n_\tau} \tilde{\Theta}_{i,j}^2 + \sum_{i=1}^{n_s} \tilde{\Psi}_{i,j}^2 + \sum_{i=1}^{n_s} \tilde{\mathcal{V}}_{i,j}^2 \Big)^{1/2}$$

and set

$$\eta_{\max_k}^{i} := \max_{\tau \in \mathcal{T}_j} \eta_H^{\tau} \qquad k = 1, 2$$

3. Refine any $\tau \in \mathcal{T}_j$ such that $\delta \cdot \eta_{\max_k}^{j} \leq \eta_{H_k}^{\tau}$.
4. Stop if $\sum_{\tau \in \mathcal{T}_j} \eta_{H_k}^{\tau} \leq$ TOL.

Note: In this algorithm j refers to the number of refinement of the mesh in the adaptive method. In this case, the mesh $\mathcal{T}_j$ is not necessarily uniform as presented in the theory given above.

4.5 Comparison of Hierarchical and Residual Estimators

In this section we investigate the relation of the hierarchical and residual error indicators. The following is a extension of analysis made by Carstensen et al. [14]. This present hierarchical and residual error estimates for a hypersingular equation.

First we need the following estimates on the $H^s(\Omega)$- and $H^{1/2}(\Gamma)$-norms of basis functions.

Lemma 4.9. *Ciarlet [20, Eq. 3.2.33]. Let $\mathcal{T}_j$ be a discretization of Ω and $\mathcal{W}_j$ the corresponding continuous and linear functions defined on $\mathcal{T}_j$. For each $\mathbf{b}_j \in \mathcal{W}_j$ we set $h_b := \operatorname{diam}(\operatorname{supp} \mathbf{b}_j)$. Then there are constants $c_1, c_2 > 0$ independent of h_b such that*

$$\|\mathbf{b}\|_{0,\Omega} \leq c h_b \|\mathbf{b}\|_{1,\Omega}.$$

For the three-dimensional case we have the following result, which can be extended to the two-dimensional case.

Lemma 4.10. *Carstensen et al. [14, Lema 5.3]. Let $\mathcal{S}_j$ be a discretization of Γ and $\mathcal{B}_j$ the corresponding continuous and linear functions defined on $\mathcal{S}_j$. For each $\beta_j \in \mathcal{B}_j$ we set $h_{\Gamma,\beta} := \operatorname{diam}(\operatorname{supp} \beta_j)$. Then there are constants $c_1, c_2 > 0$ independent of $h_{\Gamma,\beta}$ such that*

$$c_1 h_{\Gamma,\beta}^{1-s} \leq \|\beta\|_{H^s(\Gamma)} \leq c_2 h_{\Gamma,\beta}^{1-s} \text{ if } \Gamma \subset \mathbb{R}^3$$
$$c_1 h_{\Gamma,\beta}^{1/2-s} \leq \|\beta\|_{H^s(\Gamma)} \leq c_2 h_{\Gamma,\beta}^{1/2-s} \text{ if } \Gamma \subset \mathbb{R}^2$$

for each $0 \leq s \leq 1$.

Next, we present a result which compares the residual estimator with the hierarchical estimator.

Theorem 4.11. *Let $\tau \in \mathcal{T}_j$ and $\eta_{H_1}^{\tau}$ the local hierarchical error indicator defined in (4.26) associated only to the the basis function $(\mathbf{b}_{j+1,i}, \beta_{j+1,i}) \in \mathcal{W}_{j+1} \times \mathcal{B}_{j+1}$ defined on the new points of τ. Let $\eta_{R_1}(\tau)$ be the residual error indicator defined by (3.14) associated to an element $\tau \in \mathcal{T}_j$. Then assuming regularity conditions on the solution $(\mathbf{u}_j, \phi_j)$ there exists a constant $c > 0$ such that*

$$\eta_{H_1}^{\tau} \leq c \sum_{\tau_n \in \omega_\tau} \eta_{R_1}(\tau_n). \tag{4.27}$$

where ω_τ denoted the set of neighbour elements of τ. Note that the indicator $\eta_{H_1}^{\tau}$ depends on the basis function $(\mathbf{b}_{j+1,i}, \beta_{j+1,i})$, while the local indicator $\eta_R(\tau_s^r)$ depends on the elements $\tau_s^r \in \mathcal{T}_j$.
Consequently, there exists another constant $c_2 > 0$ such that the hierarchical and residual error estimator defined by (4.16) *and* (3.4)*, respectively, satisfy*

$$\eta_{H_1} \leq c_2 \eta_{R_1} \tag{4.28}$$

Remark 4.12. *Although the inequality* (4.28) *can be prove by Theorem 4.6, Theorem 3.4 and Remark 3.13 note that Theorem 4.11 together with Theorem 3.4 and Remark 3.13 proves the first inequality of Theorem 4.6. It means that the hierarchical error estimate can be efficient without a saturation condition 4.2.1 and witout the condition 4.2.2 for the case of a quasi-regular mesh and assuming regularity of the solutions. Conversely, Theorem 4.11 and Theorem 4.6 confirm the reliability of residual error estimator, i.e. , it confirms the inequality of Theorem 3.4.*

Proof of Theorem 4.11. Using (4.26) we have that

$$\eta_{H_1}^{\tau} = \Big(\sum_{i=1}^{n_\tau} \Theta_{j,i}^2 + \sum_{i=1}^{n_s} \mathcal{V}_{j,i}^2 \Big)^{1/2}$$

where

$$\Theta_{j,i}^2 := \frac{|\mathcal{L}(\mathbf{b}_{j+1,i}, 0) - \mathcal{A}_1(\mathbf{u}_j, \phi_j\,;\, \mathbf{b}_{j+1,1}, 0)|^2}{\|\mathbf{b}_{j+1,i}\|_{a_0}^2},$$

and

$$\mathcal{V}_{j,i}^2 = \frac{|\mathcal{L}(\mathbf{0}, \beta_{j+1,i}) - \mathcal{A}_1(\mathbf{u}_j, \phi_j; 0, \beta_{j+1,i})|^2}{\|\beta_{j+1,i}\|_{W_0}^2}.$$

By Green's Formula there holds

$$\begin{aligned} -(\mathbf{f}, \mathbf{b}_{j+1,i})_0 - a(\mathbf{u}_j, \mathbf{b}_{j+1,i}) &= (\operatorname{div} \sigma(\mathbf{u}_j) + \rho\omega^2 \mathbf{u}_j - \mathbf{f}, \mathbf{b}_{j+1,i})_{0,\,\operatorname{supp} \mathbf{b}_{j+1,i}} \\ &\quad + ([\![\sigma(\mathbf{u}_j)\,\mathbf{n}]\!], \mathbf{b}_{j+1,i})_{\operatorname{supp} \mathbf{b}_{j+1,i} \setminus \Gamma} + (\sigma(\mathbf{u}_j)\,\mathbf{n}, b_{j+1,i})_{\operatorname{supp} \mathbf{b}_{j+1,i} \cap \Gamma} \end{aligned}$$

Then, since $\|\mathbf{b}_{j+1,i}\|_{a_0}^2 \simeq \|\mathbf{b}_{j+1,i}\|_1^2$ and Lemma 4.9, it follows that

$$\begin{aligned}
&\frac{1}{\|\mathbf{b}_{j+1,i}\|_{a_0}} \big|\big(\operatorname{div}\sigma(\mathbf{u}_{j+1} + \rho\omega^2\mathbf{u}_j - \mathbf{f}, \mathbf{b}_{j+1,i}\big)_{0,\operatorname{supp}\mathbf{b}_{j+1,i}}\big| \\
&\quad \leq \frac{1}{\|\mathbf{b}_{j+1}\|_1} \sum_{\tau\in\mathcal{T}_j} \Big|(\operatorname{div}\sigma(\mathbf{u}_j) + \rho\omega^2\mathbf{u}_j - \mathbf{f}, \mathbf{b}_{j+1,i})_{\tau\cap\operatorname{supp}\mathbf{b}_{j+1,i}}\Big| \\
&\quad \leq \frac{1}{\|\mathbf{b}_{j+1,i}\|_1} \sum_{\tau\in\mathcal{T}_j} \|\operatorname{div}\sigma(\mathbf{u}_j) + \rho\omega^2\mathbf{u}_j - \mathbf{f}\|_{\tau\cap\operatorname{supp}\mathbf{b}_{j+1,i}} \|\mathbf{b}_{j+1,i}\|_{\tau\cap\operatorname{supp}\mathbf{b}_{j+1,i}} \\
&\quad \leq \sum_{\tau\in\mathcal{T}_j} h_\tau \|\operatorname{div}\sigma(\mathbf{u}_j) + \rho\omega^2\mathbf{u}_j - \mathbf{f}\|_{\tau\cap\operatorname{supp}\mathbf{b}_{j+1,i}}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\|\mathbf{b}_{j+1,i}\|_{a_0}} |([\![\sigma(\mathbf{u}_j)\,\mathbf{n}]\!],\mathbf{b}_{j+1,i})_{\operatorname{supp}\mathbf{b}_{j+1,i\setminus\Gamma}}| \\
&\qquad \leq \frac{1}{\|\mathbf{b}_{j+1,i}\|_{a_0}} \sum_{\tau\in\mathcal{T}_j} |\langle[\![\sigma(\mathbf{u}_j)\,\mathbf{n}]\!], \mathbf{b}_{j+1,i}\rangle_{0,\partial\tau_{\setminus\Gamma}\cap\operatorname{supp}\mathbf{b}_{j+1,i\setminus\Gamma}}| \\
&\qquad \leq \frac{1}{\|\mathbf{b}_{j+1,i}\|_{a_0}} \sum_{\tau\in\mathcal{T}_j} \|[\![\sigma(\mathbf{u}_j)\,\mathbf{n}]\!]\|_{0,\partial\tau_{\setminus\Gamma}} \|\mathbf{b}_{j+1,i}\|_{0,\partial\tau_{\setminus\Gamma}\cap\operatorname{supp}\mathbf{b}_{j+1,i}} \\
&\qquad \leq \sum_{\tau\in\mathcal{T}_j} h_{\partial\tau}^{1/2} \|[\![\sigma(\mathbf{u}_j)\,\mathbf{n}]\!]\|_{0,\partial\tau_{\setminus\Gamma}\cap\operatorname{supp}\mathbf{b}_{j+1,i}}.
\end{aligned}$$

Now, the boundary terms yield

$$\begin{aligned}
&\frac{1}{\|\mathbf{b}_{j+1,i}\|_{a_0}} \big|\langle\sigma(\mathbf{u}_j)\,\mathbf{n} - p^0\mathbf{n} - (V - \alpha(K + \frac{I}{2}))\phi_j\mathbf{n}, \mathbf{b}_{j+1,i}\rangle\,\big| \\
&\lesssim \frac{1}{\|\mathbf{b}_{j+1,i}\|_{a_0}} \sum_{\tau\in\mathcal{T}_j} \big|\,\langle\sigma(\mathbf{u}_j)\,\mathbf{n} - p^0\mathbf{n} - (V - \alpha(K + \frac{I}{2}))\phi_j\,\mathbf{n}, \mathbf{b}_{j+1,i}\rangle_{\partial\tau_s^r\cap\Gamma\cap\operatorname{supp}\mathbf{b}_{j+1,i}}\big| \\
&\lesssim \frac{1}{\|\mathbf{b}_{j+1,i}\|_{a_0}} \sum_{\tau\in\mathcal{T}_j} \|\sigma(\mathbf{u}_j)\,\mathbf{n} - p^0\mathbf{n} - (V - \alpha(K + \frac{I}{2}))\phi_j\,\mathbf{n}\|_{0,\partial\tau\cap\Gamma\cap\operatorname{supp}\mathbf{b}_{j+1,i}} \\
&\qquad\qquad \|\mathbf{b}_{j+1,i}\|_{0,\partial\tau\cap\Gamma\cap\operatorname{supp}\mathbf{b}_{j+1,i}} \\
&\lesssim \sum_{\tau\in\mathcal{T}_j} h_{\partial\tau}^{1/2} \|\sigma(\mathbf{u}_j)\,\mathbf{n} - p^0\mathbf{n} - (V - \alpha(K + \frac{I}{2}))\phi_j\,\mathbf{n}\|_{0,\partial\tau\cap\Gamma\cap\operatorname{supp}\mathbf{b}_{j+1,i}}.
\end{aligned}$$

With the above inequalities, applying the Hölder inequality and taking the square we have

$$\Theta_{j,i}^2 \lesssim \sum_{\tau\in\mathcal{T}_j} h_\tau^2 \|\operatorname{div}\sigma(\mathbf{u}_j) + \rho\omega^2\mathbf{u}_j - \mathbf{f}\|_{0,\tau\cap\operatorname{supp}\mathbf{b}_{j+1,i}}^2$$

$$+ \sum_{\tau \in \mathcal{T}_j} h_{\partial\tau} \| [\![\sigma(\mathbf{u})\, \mathbf{n}]\!] \|^2_{0,\partial\tau_{\setminus\Gamma} \cap \operatorname{supp} \mathbf{b}_{j+1,i}}$$

$$+ \sum_{\tau \in \mathcal{T}_j} h_{\partial\tau} \| \sigma(\mathbf{u})\, \mathbf{n} - p^0 \mathbf{n} - (V - \alpha(K + \frac{I}{2}))\phi_j\, \mathbf{n} \|^2_{0,\partial\tau \cap \Gamma \cap \operatorname{supp} \mathbf{b}_{j+1,i}}.$$

From $\|\beta_{j+1}\|_{W_0} \approx \|\beta_{j+1}\|_{1/2}$ and applying Lemma 4.10 it follows that

$$\begin{aligned}
\mathcal{V}_{j,i} &= \frac{|\mathcal{R}(\mathbf{u}_j, \phi_j; \mathbf{0}, \beta_{j+1,i})|}{\|\beta_{j+1,i}\|^2_{W_0}} \\
&\lesssim \frac{1}{\|\beta_{j+1,i}\|_{W_0}} |\langle \frac{\partial p^0}{\partial n}, \beta_{j+1,i} \rangle - \rho_0 \omega^2 \langle \mathbf{u}_j \cdot \mathbf{n}, \beta_{j+1,i} \rangle + \langle ((K' - \frac{I}{2}) - \alpha W)\phi_j, \beta_{j+1,i} \rangle | \\
&\lesssim \frac{1}{\|\beta_{j+1,i}\|_{1/2}} \| \frac{\partial p^0}{\partial n} - \rho_0 \omega^2 \mathbf{u}_j \cdot \mathbf{n} + ((K' - \frac{I}{2}) - \alpha W)\phi_j \|_{0,\operatorname{supp} \beta_{j+1,i}} \|\beta_{j+1,i}\|_0 \\
&\lesssim \frac{1}{\|\beta_{j+1,i}\|_{1/2}} \sum_{s \in \mathcal{S}_\Gamma} \| \frac{\partial p^0}{\partial n} - \rho_0 \omega^2 \mathbf{u}_j \cdot \mathbf{n} + ((K' - \frac{I}{2}) - \alpha W)\phi_j \|_{0,s \cap \beta_{j+1,i}} \|\beta_{j+1,i}\|_{0,s \cap \beta_{j+1,i}} \\
&\lesssim \sum_{s \in \mathcal{S}_\Gamma} h_{s_r}^{1/2} \| \frac{\partial p^0}{\partial n} - \rho_0 \omega^2 \mathbf{u}_j \cdot \mathbf{n} + ((K' - \frac{I}{2}) - \alpha W)\phi_j \|_{0,s \cap \beta_{j+1,i}}.
\end{aligned}$$

Then

$$\begin{aligned}
\left(\eta^\tau_{H_1}\right)^2 =& \sum_{i=1}^{n_\tau} \Theta^2_{j,i} + \sum_{i=1}^{n_s} \mathcal{V}^2_{j,i} \\
\lesssim& \sum_{i=1}^{n_\tau} \Bigg[\sum_{\tau \in \mathcal{T}_j} h^2_\tau \| \operatorname{div} \sigma(\mathbf{u}_{j,i}) + \rho\omega^2 \mathbf{u} - \mathbf{f} \|^2_{\tau \cap \operatorname{supp} \mathbf{b}_{j+1,i}} \\
&+ \sum_{\tau \in \mathcal{T}_j}^{n_s} h_{\partial\tau} \| [\![\sigma(\mathbf{u})\, \mathbf{n}]\!] \|^2_{\partial\tau \setminus \Gamma \cap \operatorname{supp} \mathbf{b}_{j+1,i}} \\
&+ \sum_{\tau \in \mathcal{T}_j}^{n_s} h_{\partial\tau} \| \sigma(\mathbf{u}) \cdot \mathbf{n} - p^0 \mathbf{n} - (V - \alpha(K + \frac{I}{2}))\phi_j \mathbf{n} \|^2_{\partial\tau^r_s \cap \Gamma \cap \operatorname{supp} \mathbf{b}_{j+1,i}} \Bigg] \\
&+ \sum_{i=1}^{n_s} \sum_{s \in \mathcal{S}_j}^{m_s} h_s \| \frac{\partial p^0}{\partial \mathbf{n}} - \rho_0 \omega^2 \mathbf{u}_j \cdot \mathbf{n} + ((K' - \frac{I}{2}) - \alpha W)\phi_j \|^2_{0,s \cap \beta_{j+1,i}} \\
\lesssim& \sum_{\tau_n \in \omega_\tau} \eta_{R_1}(\tau_n)^2.
\end{aligned}$$

where ω_τ denoted the set of neighbour elements of $\tau \in \mathcal{T}_j$. Hence, we have the desired estimate (4.27).

The second estimate can now be proven. Given an $\tau \in \mathcal{T}_j$ and because each subdivision midpoint of element τ is associated to a indicator $\eta^\tau_{H_1}$, there holds

$$(\eta_{H_1}^{\tau})^2 \lesssim n_\tau \, (\eta_{R_1}(\tau))^2 + \sum_{\tau_n \in \omega_\tau} \eta_{R_1}(\tau_n)^2$$

where n_τ is the number of midpoints obtained by subdividing the element τ. Using the above estimate together with the definitions (4.6) and (3.4), and denoting l_τ as the number of neighbour elements of τ, it follows

$$\begin{aligned}
\eta_{H_1} = \Big(\sum_{\tau \in \mathcal{T}_j} \eta_{H_1}^{\tau}(\tau)^2 \Big)^{1/2} &\lesssim \Big(\sum_{\tau \in \mathcal{T}_j} \big[n_\tau \, \eta_{R_1}(\tau)^2 + \sum_{\tau_n \in \omega_\tau} \eta_{R_1}(\tau_n)^2 \big] \Big)^{1/2} \\
&\lesssim \Big(\sum_{\tau \in \mathcal{T}_j} (n_\tau + l_\tau) \eta_{R_1}(\tau_i)^2 \Big)^{1/2} \\
&\lesssim \Big(\sum_{\tau \in \mathcal{T}_j} \eta_{R_1}(\tau_i)^2 \Big)^{1/2} = \eta_{R_1}.
\end{aligned}$$

□

Remark 4.13. *Given that the sesquilinear forms corresponding to the symmetric formulation* (VP_2) *are similar to the non-symmetric formulation* (VP_1)*, we can extend the above theorem for the symmetric formulation* (VP_2)*. The proof would be similar to that developed for the non-symmetric case.*

Chapter 5

Numerical Results in 2D

In this chapter we present the numerical results of an FE/BE coupling for a fluid structure interaction problem as introduced in Section 2.3. The non-symmetric (VP_1) and symmetric (VP_2) formulations derived in Section 2.3 are implemented and solved in the scientific program MaiProgs [58]. These formulations depend on the parameter α, and the first Section of this chapter presents the performance of the non-stable method (VP_1) for $\alpha = 0$ and stable methods using $\alpha = i/k$ with different wave numbers k . We recall that $\alpha = i/k$ is an optimal value in the sense that the condition number of the system is minimized, according to the considerations made by Kress [50] and Meyer et al. [61]. In Section 5.2, we show the convergence using h-uniform and adaptive refinements for the non-symmetric formulation (VP_1) and the symmetric formulation (VP_2) with $\alpha = i/k$. We implement the hierarchical error estimators presented in Theorems 4.6 and 4.8 and the residual error estimator presented in Theorems 3.4 and 3.5 for the non-symmetric and symmetric formulation, respectively . Furthermore, we analyze and compare the behavior of different formulations and the corresponding estimators.

Throughout each of the following sections, we solve the following model problem:

Numerical example in 2D (Ex. 2D). *Consider a square-shaped, homogeneous, isotropic, elastic scatterer made of steel with $\bar{\Omega} = [-1, 1]^2$. The scatterer possesses the following material parameters: Poisson's ratio $\nu = 0.28$, Young's module $E = 200GPa$ and $\rho = 7800kg/m^3$. The scatterer is submerged in sea water and is subject to a plane incident wave $p^0(x, y) = e^{ikx}$. Furthermore, we assume for sea water a density $\rho_0 = 1020Kg/m^3$ and a sound velocity $c_0 = 1500m/s$.*

Remark 5.1. *We remark that for Example (Ex. 2D) we do not know the exact solution of the system. The error and convergence analysis for the numerical solutions is performed using estimates of the exact norms of* $\mathbf{u} \in [H^1(\Omega)]^2$, $\sigma_{\rm n} \in H^{1/2}(\Gamma)$ *and* $\phi \in H^{1/2}(\Gamma)$. *These estimates are obtained by extrapolation using Aitken's* Δ^2 *process (see Press et al. [70, section 5.1]) with a sequence of norms, resulting from an h-uniformly refinement.*

5.1 Behavior of the Systems using $\alpha = 0$ and $\alpha = i/k$.

In this Section we compare the performance obtained with the stable and non-stable procedures using the symmetric and non-symmetric formulations (VP_1) and (VP_2) with the above fixed input data. In the following experiments we use a fixed triangulation with 800 inner elements (80 boundary elements) for different wave numbers $k \in \{0.01 \cdot n : n = 1, \ldots, 800\}$ with $\alpha = 0$ and $\alpha = i/k$.
We analyze the results on the surface Γ using the **symmetric stable formulation** (VP_2). This formulation has the advantage that we can analyze, in addition to the unknown density ϕ, the normal traction on the surface $\sigma_{\rm n}$. Fig. 5.1a and 5.1b show the L^2-norms of the variables $\sigma_{\rm n}$ and ϕ, respectively, for different wavenumbers k. In Fig. 5.1b we clearly observe the influence of the constant α on the system. For $\alpha = 0$ we observe peaks in the L^2-norm of ϕ for some values of k, which correspond to so the called *critical frequencies*. These frequencies are the eigenvalues of the interior Helmholtz problem of Dirichlet (1.12), (for more details see sections 2.1, 1.1.1 and 1.1.2).
Note that for $\alpha = 0$, ϕ is the density of the single layer potential, whereas for $\alpha = i/k$ it represents the density of a combined single and double layer (see Section 1.1.2). Thus, it is natural, without taking into account the critical frequencies, that the corresponding curves are different. Fig. 5.1b confirms that for $\alpha = i/k$ the L^2-norm keeps stable. In the internal clipping of Fig. 5.1b which is the same graphic but at a greater scale, we can confirm that the peaks that occur for $\alpha = 0$ are bounded.
However, contrary to the behavior of ϕ, we expect that the solution $\sigma_{\rm n}$ coincides for the two procedures, stable and non-stable, because the constant α was introduced to alleviate the non-uniqueness of the density ϕ and should not alter the results of displacement $\mathbf{u}$ or normal traction $\sigma_{\rm n}$. This is evident in Fig. 5.1a, which shows in a smaller scale the similarity of the

curves, except at some wavenumbers. In its corresponding internal clipping the complete curve is shown.

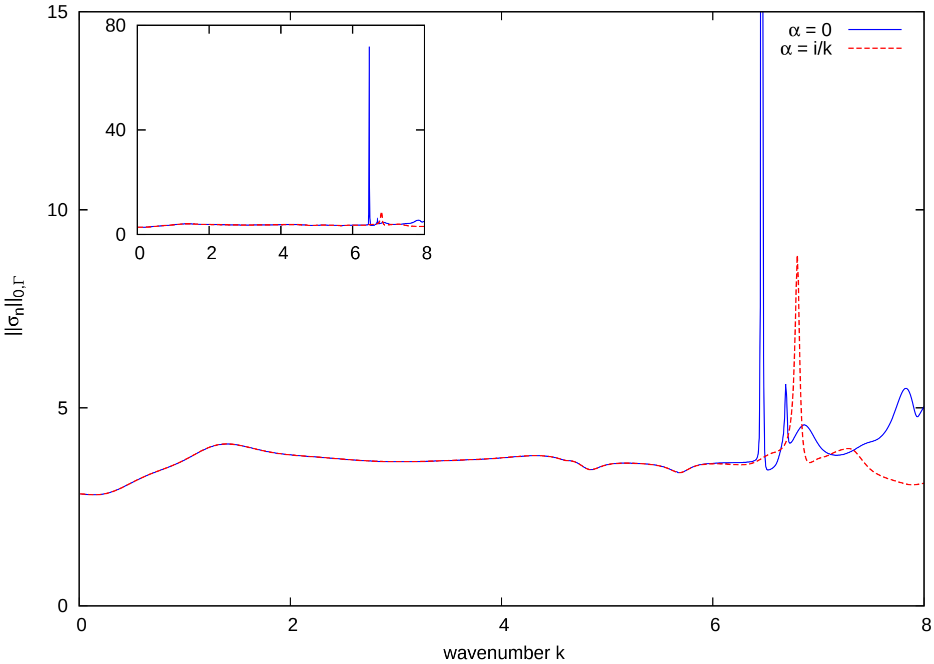


Fig. 5.1a: comparison of $\| {\sigma_{\rm n}}^h \|_{0,\Gamma}$ for the stable and non-stable case using (VP_2).

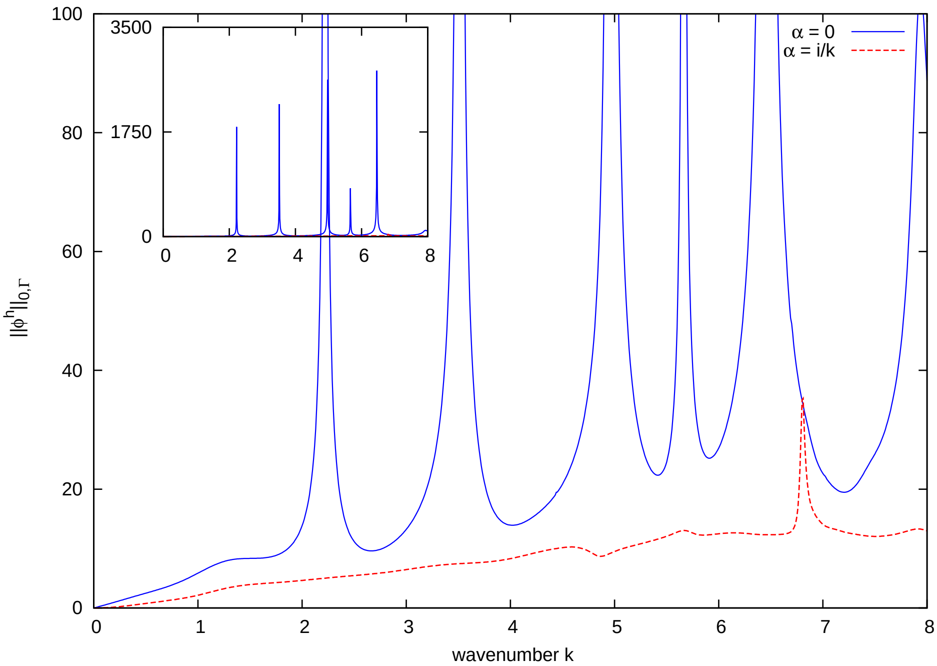


Fig. 5.1b: Comparison of $\|\phi^h\|_{0,\Gamma}$ vs. wavenumber for the stable and non-stable case using (VP_2).

In order to compare the non-stable and stable formulations Fig. 5.2 compares the density ϕ of the formulations (VP_1) and (VP_2) with $\alpha = i/k$ for different wavenumbers k. Note that ϕ coincides in both formulations, which is expected because the symmetric formulation (VP_2) is an extension of the non-symmetric formulation (VP_1). We recall that for the same value of α, both formulations seek the same variable ϕ (see Section 2.2, p. 16,17).

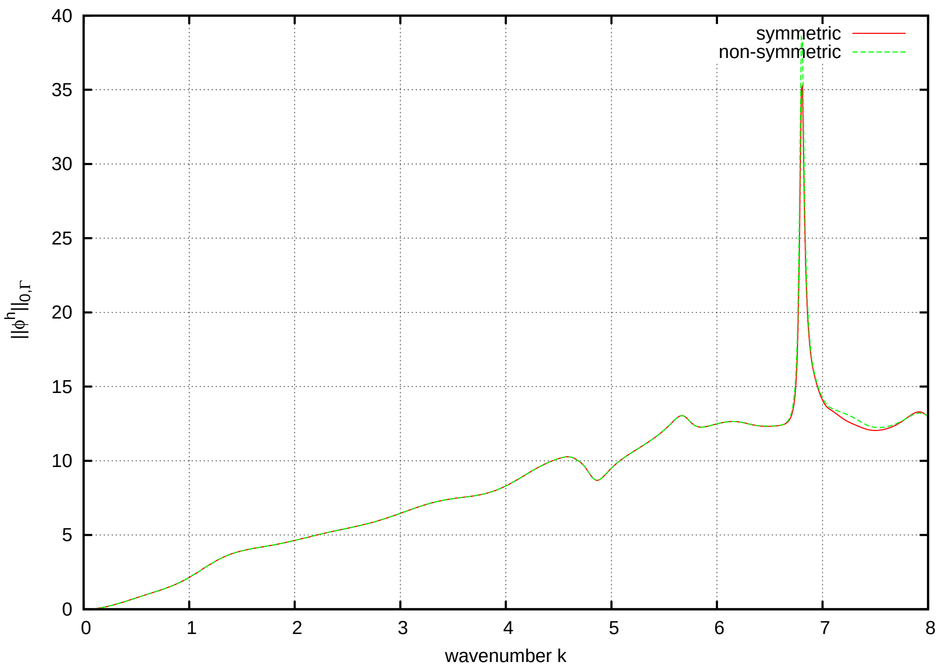


Fig. 5.2: Comparison of $\|\phi^h\|_{0,\Gamma}$ for the non-symmetric (VP_1) and symmetric (VP_2) formulations using $\alpha = i/k$.

In conclusion, these experiments verify the stability of our methods for $\alpha \neq 0$, in the sense that there is no abrupt behavior of the solutions at the critical frequencies.

5.2 Convergence, Error Indicators and Adaptive Methods for the Non-symmetric Formulation

In this Section we present results for the approximations of the non-symmetric stable formulation (VP_1).

Let $\|\cdot\|_{W_0}$ and $\|\cdot\|_{1,W_0}$ denote norms defined by

$$\begin{aligned} \|\psi\|^2_{W_0} &:= \langle W_0\psi, \bar{\psi}\rangle \quad \forall \psi \in H^{1/2}(\Gamma), \\ \|(\mathbf{v},\psi)\|^2_{1,W_0} &:= \|\mathbf{v}\|^2_1 + \|\psi\|^2_{W_0} \quad \forall (\mathbf{v},\psi) \in H^1(\Omega)\times H^{1/2}(\Gamma), \end{aligned}$$

where W_0 is the hypersingular operator with kernel γ_0 (see Section 1.1). In order, to calculate the error estimators of the formulation (VP_1) in the norm of $\mathcal{H}^1 := H^1(\Omega)\times H^{1/2}(\Gamma)$ we use the equivalent norm $\|\cdot\|_{1,W_0}$.

Remark 5.2. *Note that the bilinear form $\langle W_0\cdot,\cdot\rangle$ is Hermitian and positive definite in $H^{1/2}(\Gamma)$, and therefore $\|\cdot\|_{W_0}$ is equivalent to the norm in $H^{1/2}(\Gamma)$.*

We are interested in the following errors

$$\begin{aligned} e &:= \|(\mathbf{u},\phi) - (\mathbf{u}^h,\phi^h)\|_{1,W_0}, \\ e_{\mathbf{u}} &:= \|\mathbf{u} - \mathbf{u}^h\|_1, \\ e_{\phi} &:= \|\phi - \phi^h\|_{W_0}. \end{aligned}$$

The experimental convergence rate is given by

$$\theta_N = \frac{\log(e_j/e_{j+1})}{\log(N_{j+1}/N_j)},$$

where N_j is the degree of freedom of the discrete system. We recall that we do not know the exact solution of the system, and therefore the norms of $\mathbf{u}\in H^1(\Omega)$ and $\phi \in H^{1/2}(\Gamma)$ are extrapolated using Aitken's delta-squared process from the sequence of norms obtained from h-uniformly refined meshes.

In Fig. 5.3 the convergence of e in the norm $\|\cdot\|_{1,W_0}$ is shown for a h-uniform refinement using $\alpha = 0$ and $\alpha = i/k$ and for the wave numbers $k = 2, 3.5$ and 5. As expected for $\alpha = 0$ the method does not converge and for $\alpha = i/k$ the method converges, corresponding to the theoretical results given in Theorems 2.14 and 2.20. We chose the values k close to a critical frequency of the system (see Fig. 5.1b) to show that the non-stable method does not result in a convergent method there.

In Table 5.1 and 5.2 we present the convergence history of our example for a sequence of quasi-uniform triangulations using $k = 3.5$. We remark that the convergence rate $\mathcal{O}(h)$ predicted in Section 2.4.5 for this formulation is approximately obtained. Furthermore, we see that the dominant error is e_ϕ.

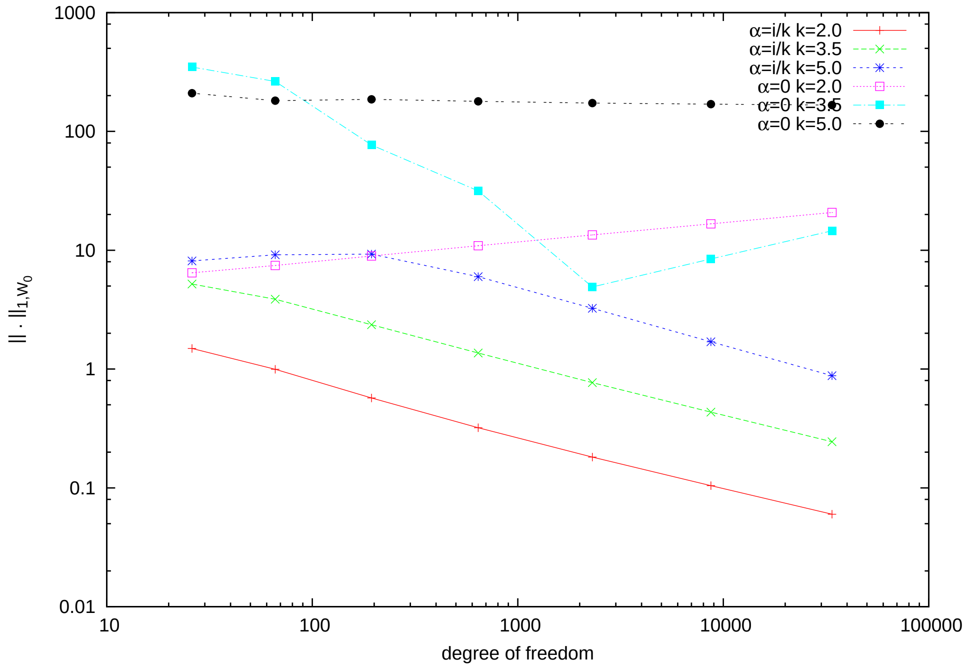


Fig. 5.3: Error e in the norm $\|\cdot\|_{1,W_0}$ vs. degree of freedom for (VP_1) using $\alpha = 0$ and $\alpha = i/k$.

5.2.1 Hierarchical error estimators

Now we approximate the solution of (VP_1) applying the hierarchical error estimator stated in Theorem 4.6 using an h-uniform and adaptive refinement. For the adaptive refinement we

h	N	e	θ_N
1	26	5.1908	-
1/2	66	3.8688	0.3155
1/4	194	2.3607	0.4582
1/8	642	1.3636	0.4586
1/16	2306	0.7685	0.4485
1/32	8706	0.4338	0.4305
1/64	33794	0.2447	0.4222
Theory			0.5

Table 5.1: Error and convergence rates using (VP_1) with $k = 3.5$ and $\alpha = i/k$.

h	N_u	$e_{\mathbf{u}}$	θ_{N_u}	N_ϕ	e_ϕ	θ_{N_ϕ}
1	18	.3888E-10	0	8	5.1908	0
1/2	50	.3464E-10	0.1130	16	3.8600	0.4241
1/4	162	.2613E-10	0.2397	32	2.3607	0.7127
1/8	578	.1609E-10	0.3814	64	1.3636	0.7918
1/16	2178	.0879E-10	0.4553	128	0.7685	0.8274
1/32	8450	.0464E-10	0.4718	256	0.4338	0.8251
1/64	33282	.0244E-10	0.4685	512	0.2447	0.8261
Theory			0.5			0.75

Table 5.2: Error and convergence rates of $\mathbf{u}$ and ϕ using (VP_1) with $k = 3.5$, $\alpha = i/k$.

apply the adaptive strategy as shown in Algorithm 2 using $\delta = 0.8$ (parameter of refinement). Fig. 5.4 appears the error e using an h-uniform and an adaptive refinement with their respective two-level hierarchical error estimators

$$\eta^j_{H_1} = \Big(\sum_{i=1}^{n_j} \Theta^2_{j,i} + \sum_{i=1}^{m_j} \mathcal{V}^2_{j,i} \Big)^{1/2}$$

derived in Theorem 4.6, where j is the level of discretization (see Section 4.1). The error estimator η_{H_1} behaves proportional to the error of the exact solution $(\mathbf{u}, \phi)$ for different choices of k. Table 5.3 shows the hierarchical error estimator η_{H_1} and the indicators

$$\Theta_j = \big(\sum_{i=1}^{n_j} \Theta^2_{j,i} \big)^{1/2}, \qquad \mathcal{V}_j = \big(\sum_{i=1}^{m_j} \mathcal{V}^2_{j,i} \big)^{1/2},$$

using h_j-uniform refinement and their effectivity indices q of each estimator. We see that the effectivity indices remain constant for the different triangulations, which confirms the reliability and efficiency stated in Theorem 4.6 of the error estimators η_{H_1}, Θ and $\mathcal{V}$ and the predicted order of convergence of our discrete problem. Moreover, implicitly the efficiency of the estimator $\mathcal{V}$ (Table 5.3[Column 6]) indicates that the operator p^j has bounded condition number (see Lemma 4.3) confirming the conjecture presented in Remark 4.4.

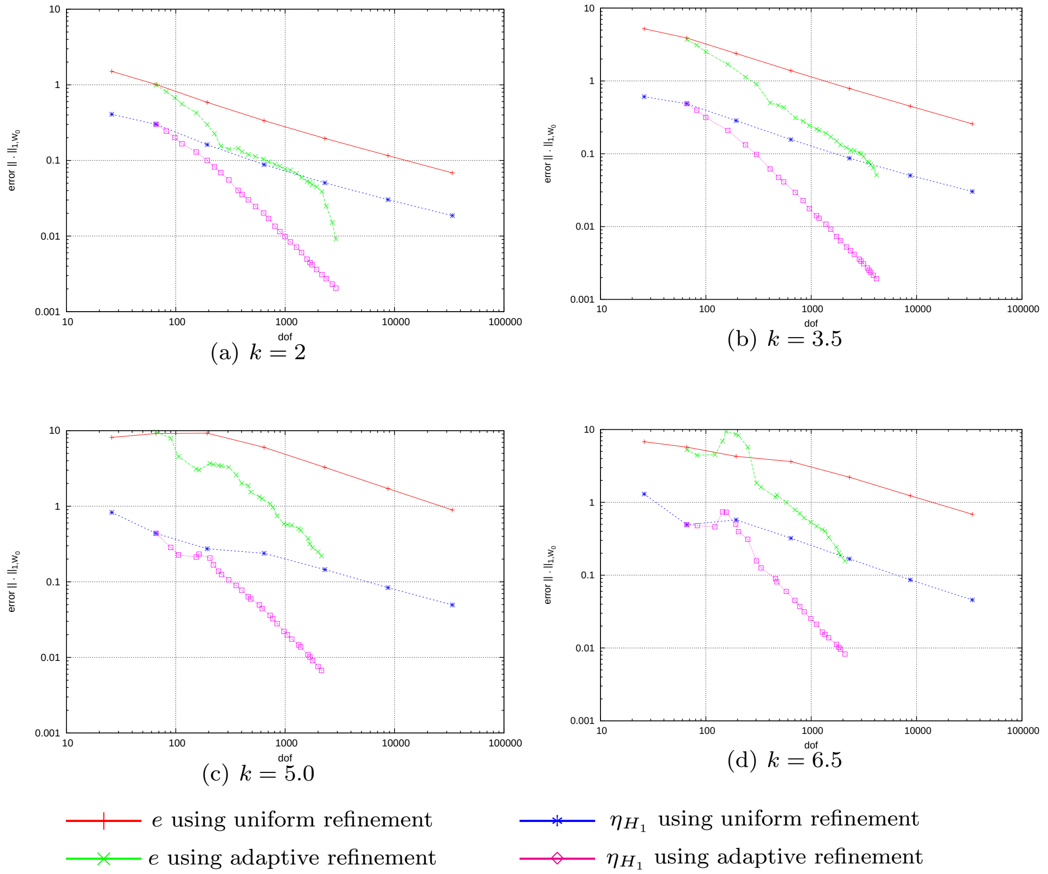


Fig. 5.4: Errors and hierarchical error estimators for the formulation (VP_1) with $\alpha = i/k$.

η_{H_1}.	$q = \eta_{H_1}/e$	Θ	$q = \Theta/e_{\mathbf{u}}$	$\mathcal{V}$	$q = \mathcal{V}/e_\phi$
0.6075	0.1170	0.1709E-5	4.3955E4	0.6075	0,1170
0.4873	0.1259	0.2014E-5	5.8141E4	0.4873	0.1259
0.2854	0.1209	0.1737E-5	6.6466E4	0.2854	0.1209
0.1567	0.1149	0.1184E-5	7.3595E4	0.1567	0.1149
0.0869	0.1130	0.0674E-5	7.6677E4	0.0869	0.1130
0.0503	0.1160	0.0357E-5	7.7051E4	0.0503	0.1160
0.0303	0.1239	0.0187E-5	7.6709E4	0.0303	0.1239

Table 5.3: Hierarchical error estimators and effectivity indices with $k = 3.5$ and $\alpha = i/k$ for an h-uniform refinement for (VP_1).

5.2.2 Residual error estimators

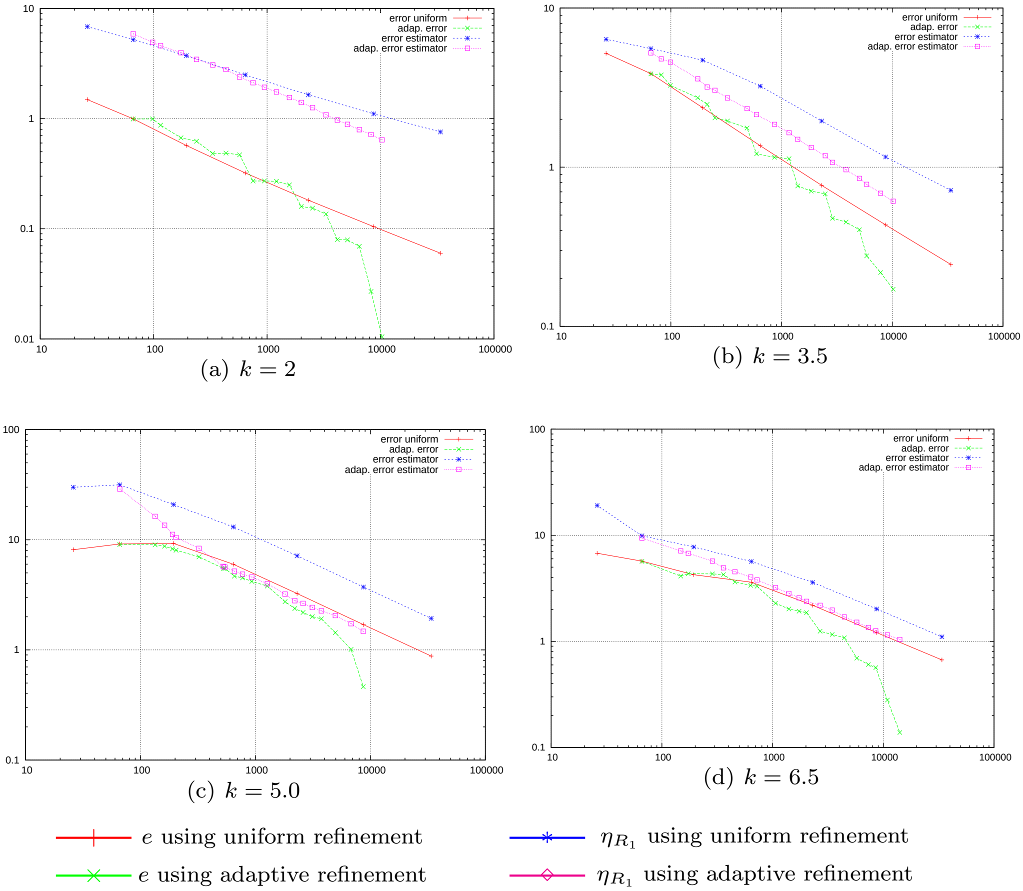
Now we approximate the residual error η_{R_1} as stated in Theorem 3.4 and apply the adaptive strategy shown in Algorithm 1 using the non-symmetric formulation (VP_1). In Fig. 5.5 the error e using an h-uniform and an adaptive refinement, with their respective residual error estimator

$$\eta_{R_1} = (R_1^h + R_2^h + R_3^h + R_4^h)^{1/2}$$

are displayed for different wave numbers k. Table 5.4 shows the residual error estimator η_{R_1} and its effectivity index $q = \eta_{R_1}/e$ calculated for $k = 3.5$ and $k = 5$. We see that for $k = 3.5$, the error e has a slightly better convergence than the error estimator η_{R_1}. This difference may be explained by the lack of regularity of the solution $(\mathbf{u}, \phi)$ for this wave number. However for $k = 5$ the equivalence between the error e and η_{R_1} is entirely confirmed. Note that Theorem 3.4 is confirmed, because the rate of convergence of e is greater than of the error estimator η_{R_1}. Remark 3.13 and its results concerning the upper bound of the estimator η_{R_1} by means of the error e is satisfied, since the effectivity indices are quasi-constant (see Table 5.4[Columns 4,6]).

5.2.3 Residual-hierarchical adaptive strategy

In the following we want to compare the residual and the hierarchical adaptive strategy. In Fig. 5.6 appears the error e using an h-uniform refinement and the errors obtained using an adaptive refinement based on residual and hierarchical error estimators, respectively. We



(a) $k = 2$ (b) $k = 3.5$ (c) $k = 5.0$ (d) $k = 6.5$

Fig. 5.5: Errors and residual error estimators for the formulation (VP_1) with $\alpha = i/k$.

		$k = 3.5$		$k = 5.0$	
h	N	η_{R_1}	$q = \eta_{R_1}/e$	η_{R_1}	$q = \eta_{R_1}/e$
1	26	6.3622	1.2257	29.9753	3.6887
1/2	66	5.5580	1.4366	31.5406	3.4438
1/4	194	4.7024	1.9920	20.8456	2.2508
1/8	642	3.2281	2.3674	13.0691	2.1793
1/16	2306	1.9494	2.5368	7.1497	2.2010
1/32	8706	1.1588	2.6715	3.7212	2.1949
1/64	33794	0.7152	2.9234	1.9314	2.1949

Table 5.4: Residual error estimator η_{R_1} and effectivity index q calculated for $k = 3.5$ and $k = 5$ with $\alpha = i/k$ using (VP_1), uniform refinement.

can see that the hierarchical method has better convergence than the residual method. Fig. 5.7 shows different meshes obtained in the process of using the hierarchical and residual refinement, respectively. Note that the hierarchical method refines exclusively the corners of the domain and the midpoints of the sides in the direction perpendicular to the wave front, which is where one would expect greater difficulty to approximate the solutions due to the interaction between the fluid and the material. Even when the residual method also attacks these areas, it refines also within the domain Ω, which would explain the advantage of the hierarchical method compared to the residual method to solve the Example (Ex. 2D) using the formulation (VP_1).

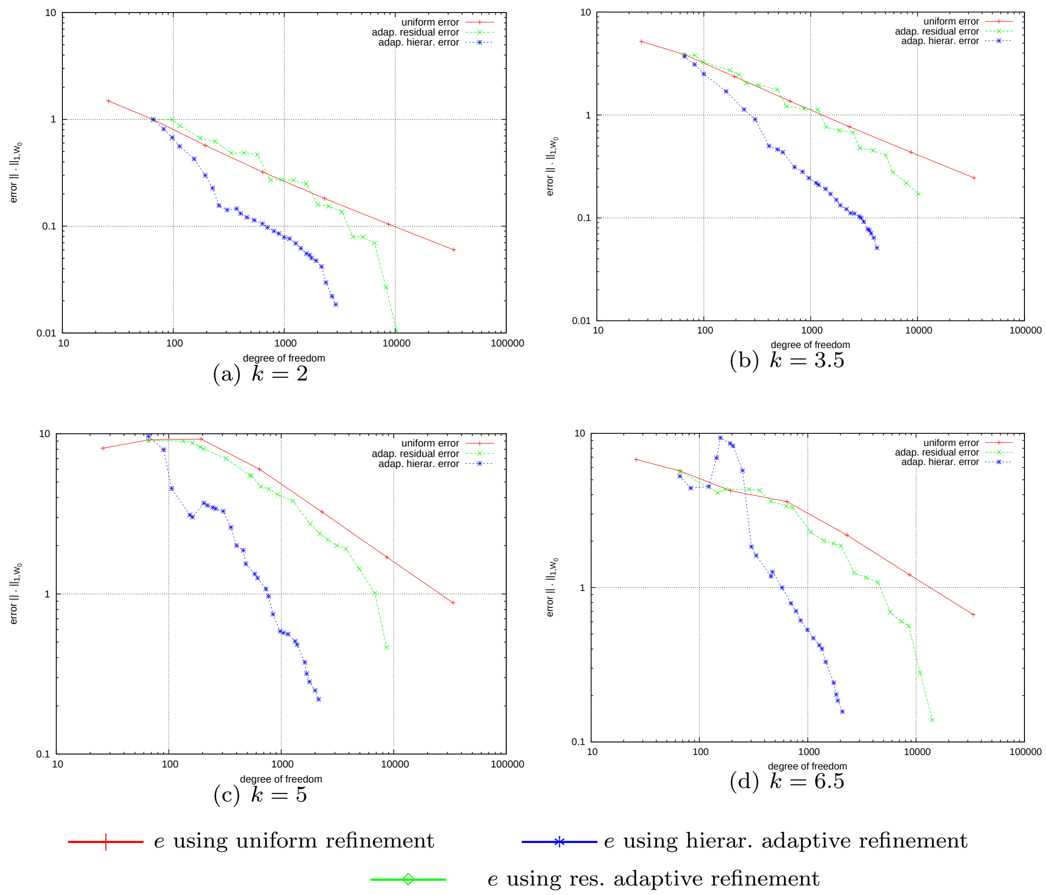
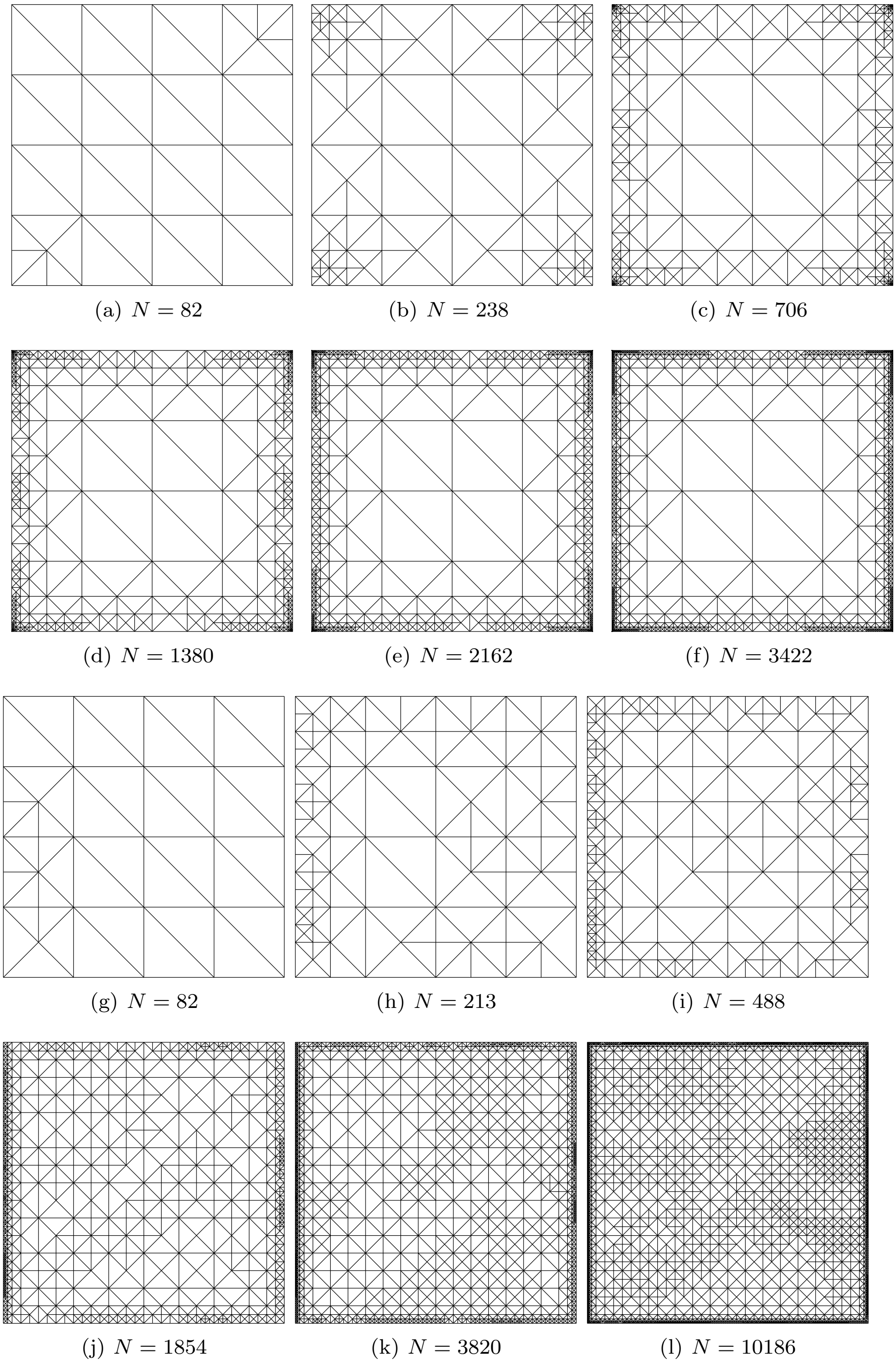


Fig. 5.6: Comparison of h-uniform, hierarchical and residual adaptive strategies using (VP_1), $\alpha = i/k$ and parameter of refinement $\delta = 0.8$.



(a) $N = 82$ (b) $N = 238$ (c) $N = 706$

(d) $N = 1380$ (e) $N = 2162$ (f) $N = 3422$

(g) $N = 82$ (h) $N = 213$ (i) $N = 488$

(j) $N = 1854$ (k) $N = 3820$ (l) $N = 10186$

Fig. 5.7: Adaptives meshes using (VP_1) with $\alpha = i/k$ and parameter of refinement $\delta = 0.8$: Fig. (a)- (f) hierarchical refinement, Fig. (g)-(l) residual refinement.

5.3 Convergence, Error indicators and Adaptive Methods for the Symmetric Formulation

In this Section we present the obtained results using the stable symmetric formulation (VP_2). We add the following notation to the ones used in the previous section: Let $\|\cdot\|_{1,W_0}$ denote a norm defined by

$$\|(\mathbf{v},\chi,\psi)\|_{1,W_0}^2 = \|\mathbf{v}\|_1^2 + \langle W_0\psi,\bar{\psi}\rangle + \langle W_0\chi,\bar{\chi}\rangle \quad \forall(\mathbf{v},\chi,\psi)\in\mathcal{H}^2$$

where W_0 is the hypersingular operator with γ_0. Let

$$\begin{aligned} e &:= \|(\mathbf{u},\sigma_{\mathrm{n}},\phi) - (\mathbf{u}^h,{\sigma_{\mathrm{n}}}^h,\phi^h)\|_{1,W_0,W_0}, \\ e_{\mathbf{u}} &:= \|\mathbf{u}-\mathbf{u}^h\|_{W_0}, \\ e_{\sigma_{\mathrm{n}}} &:= \|\,\sigma_{\mathrm{n}} - {\sigma_{\mathrm{n}}}^h\,\|_{W_0}, \\ e_\phi &:= \|\phi-\phi^h\|_{W_0}. \end{aligned}$$

To calculate the error in the norm of $\mathcal{H}^2 = H^1(\Omega)\times H^{1/2}(\Gamma)\times H^{1/2}(\Gamma)$ we use the equivalent norm $\|\cdot\|_{1,W_0}$ (see Remark 5.2) . Recall that the exact norms of $\mathbf{u}\in H^1(\Omega)$, $\sigma_{\mathrm{n}}\in H^{1/2}(\Gamma)$ and $\phi\in H^{1/2}(\Gamma)$ are extrapolated values only.

Fig. 5.8 shows the convergence of e for an h-uniform refinement using $\alpha=0$ and $\alpha=i/k$ and for the wave numbers $k=2,3.5$ and 5. As expected for $\alpha=0$, the method does not converge and for $\alpha=i/k$ the method converges, corresponding to the theoretical results given in Lemma 2.15 and Theorem 2.20. Tables 5.5 and 5.6 present the convergence history of our example for a sequence of quasi-uniform triangulations. We remark that the convergence rate $\mathcal{O}(h)$ predicted in Section 2.4.5 for this formulation is obtained for all unknowns. Furthermore, we see that the dominant errors in this formulation are $e_{\sigma_{\mathrm{n}}}$ and e_ϕ.

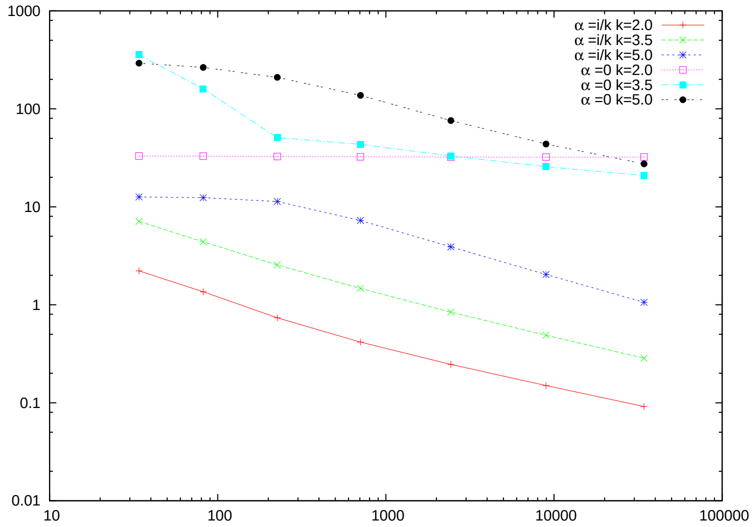


Fig. 5.8: Error e using the formulation (VP_2) with $\alpha = 0$ and $\alpha = i/k$.

h	N	e	θ_N
1	34	5.9250	-
1/2	82	4.1101	0.4154
1/4	226	2.4840	0.4967
1/8	706	1.4339	0.4824
1/16	2434	0.8073	0.4642
1/32	8962	0.4550	0.4399
1/64	34306	0.2563	0.4276
Theory			0.5

Table 5.5: Error and convergence rate of $(\mathbf{u}^h, {\sigma_{\mathrm{n}}}^h, \phi^h)$ using (VP_2) with $k = 3.5$ and $\alpha = i/k$.

$N_{\mathbf{u}}$	$e_{\mathbf{u}}$	$\theta_{N_{\mathbf{u}}}$	$N_{\sigma_{\mathrm{n}}}$	$e_{\sigma_{\mathrm{n}}}$	$\theta_{\sigma_{\mathrm{n}}}$	e_ϕ	θ_ϕ
18	0.3929E-10	-	8	2.8681	-	5.1845	0
50	0.3473E-10	0.1210	16	1.3734	1.0624	3.8738	0.4204
162	0.2617E-10	0.2407	32	0.7715	0.8321	2.3612	0.7142
578	0.1610E-10	0.3816	64	0.4437	0.7980	1.3635	0.7922
2178	0.0880E-10	0.4554	128	0.2474	0.8426	0.7684	0.8274
8450	0.0464E-10	0.4719	256	0.1374	0.8483	0.4338	0.8250
33282	0.0244E-10	0.4686	512	0.0763	0.8487	0.2447	0.8260
Theory		0.5			0.75		0.75

Table 5.6: Errors and convergence rates of each variable $\mathbf{u}^h, {\sigma_{\mathrm{n}}}^h, \phi^h$ using (VP_2) with $k = 3.5$ and $\alpha = i/k$.

5.3.1 Hierarchical error estimators

We compute the hierarchical error estimator stated in Theorem 4.8 using an h-uniform and an adaptive refinement. For the adaptive refinement we apply the adaptive strategy shown in Algorithm 2 using as parameter of refinement $\delta = 0.9$.
In Fig. 5.9 the error e using an h-uniform and adaptive refinement is compared, with their respective hierarchical error estimators η_{H_2} given by

$$\eta^j_{H_2} = \Big(\sum_{i=1}^{n_j} \tilde{\Theta}^2_{j,i} + \sum_{i=1}^{m_j} \tilde{\Psi}^2_{j,i} + \sum_{i=1}^{m_j} \tilde{\mathcal{V}}^2_{j,i}\Big)^{1/2}, \qquad j = 1, \dots, 7.$$

The error estimator η_{H_2} behaves equivalently to the error convergence of the solution $(\mathbf{u}, \sigma_{\mathrm{n}}, \phi)$ for different cases of k. Table 5.7 shows the hierarchical error estimator η_{H_2} and the estimators

$$\tilde{\Theta}_j = (\sum_{i=1}^{n_j} \tilde{\Theta}^2_{j,i})^{1/2}, \quad \tilde{\Psi}_j = (\sum_{i=1}^{m_j} \tilde{\Psi}^2_{j,i})^{1/2} \qquad \tilde{\mathcal{V}}_j = (\sum_{i=1}^{m_j} \tilde{\mathcal{V}}^2_{j,i})^{1/2}$$

using h-uniform refinement and the effectivity indices q of each estimator. We see that the effectivity indexes remain constant for the different triangulations, which confirms the reliability and efficiency of the error estimator η_{H_2} and the estimators Θ and $\mathcal{V}$ stated in Theorem 4.8. We recall that Θ is proportionally to the error of the unknown $\mathbf{u}$, Ψ to the error of σ_{n} and $\mathcal{V}$ to the error of ϕ.
Unlike the formulation (VP_1), in Table 5.8 the error indicators, Ψ and $\mathcal{V}$ are very small compared to the errors indicators $e_{\sigma_{\mathrm{n}}}$ and e_ϕ, respectively, and in turn their effectivity indices are large values. This causes that for the adaptive refinement it is difficult to find the most problematic places of these unknowns and the efficiency of the method is not as good, as those obtained in the formulation (VP_1).

5.3.2 Residual error estimators

Now we approximate the solution of (VP_2) applying the residual error estimator as stated in Theorem 3.5 and the adaptive strategy shown in Algorithm 1 with $\delta = 0.9$. Fig. 5.10

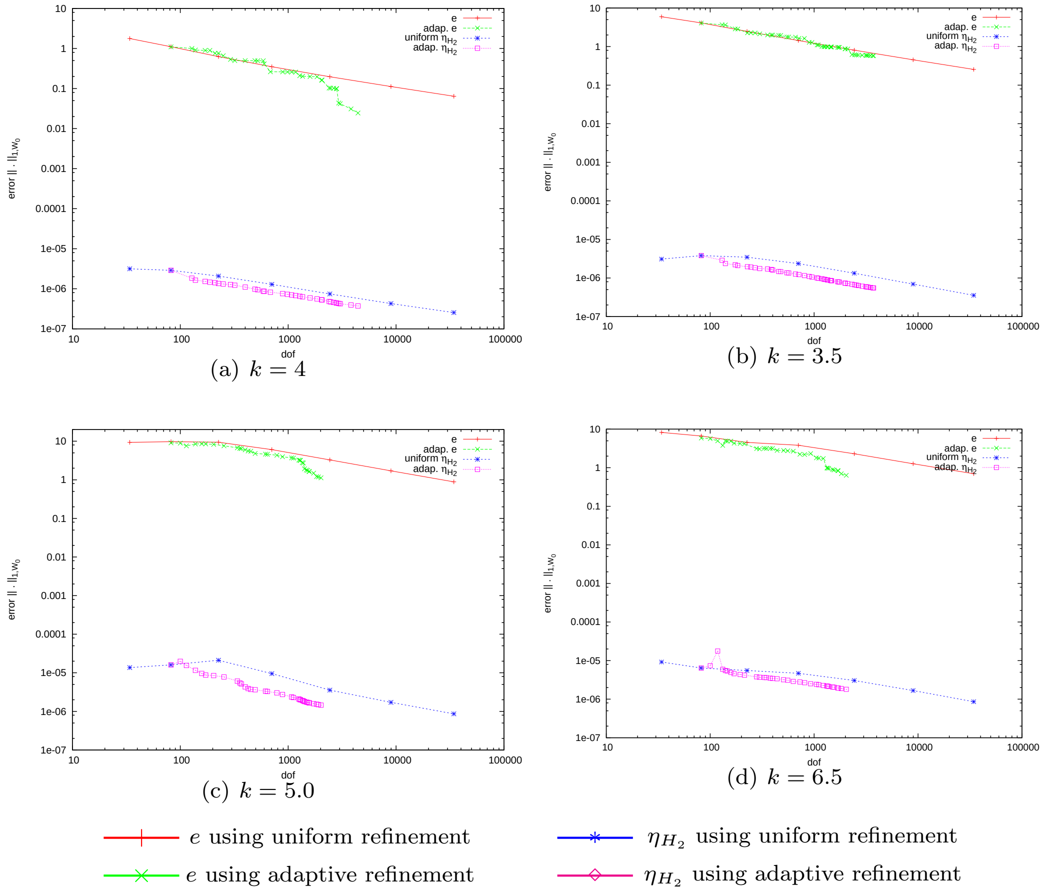


Fig. 5.9: Errors and hierarchical error estimators for the formulation (VP_2) with $\alpha = i/k$.

j	η_{H_2}	$q = \eta_{H_2}/e$
1	0.3107e-5	0.0524e-5
2	0.3812e-5	0.0927e-5
3	0.3471e-5	0.1397e-5
4	0.2365e-5	0.1649e-5
5	0.1334e-5	0.1652e-5
6	0.0696e-5	0.1530e-5
7	0.0356e-5	0.1389e-5

Table 5.7: Hierarchical error estimator η_{H-2} and effectivity index using the formulation (VP_2) with $k = 3.5$ and $\alpha = i/k$.

j	Θ_j	$q = \Theta_j/e_{\mathbf{u}}$	Ψ_j	$q = \Psi_j/e_{\sigma_n}$	$\mathcal{V}_j$	$q = \mathcal{V}_j/e_\phi$
1	0.3107e-5	0.7907e+5	0.2122e-10	0.0740e-10	0.1675e-10	0.3231e-11
2	0.3812e-5	1.0978e+5	0.1730e-10	0.1259e-10	0.0634e-10	0.1638e-11
3	0.3471e-5	1.3265e+5	0.1015e-10	0.1316e-10	0.0277e-10	0.1171e-11
4	0.2365e-5	1.4684e+5	0.0557e-10	0.1256e-10	0.0157e-10	0.1153e-11
5	0.1334e-5	1.5156e+5	0.0309e-10	0.1249e-10	0.0098e-10	0.1276e-11
6	0.0696e-5	1.5002e+5	0.0179e-10	0.1302e-10	0.0064e-10	0.1469e-11
7	0.0356e-5	1.4574e+5	0.0108e-10	0.1413e-10	0.0042e-10	0.1728e-11

Table 5.8: Hierarchical error indicators and their effectivity indexes using the formulation (VP_2) with $k = 3.5$ and $\alpha = i/k$.

shows the error e using an uniform and adaptive refinement, with their respective residual error estimator

$$\eta_{R_2} = (\tilde{R}_1^h + \tilde{R}_2^h + \tilde{R}_3^h + \tilde{R}_4^h + \tilde{R}_5^h)^{1/2}$$

for different wave numbers k. Table 5.9 shows the residual error estimator η_{R_2} and its effectivity index $q = \eta_{R_2}/e$ calculated for $k = 3.5$ and $k = 5$. We can see that the behavior is similar to the non-symmetric formulation (VP_1). As in the formulation (VP_1), for $k = 3.5$ the error e has a slightly better convergence than the error estimator η_{R_2}, and for $k = 5$, the equivalence between the error e and η_{R_2} is entirely confirmed. Thus, for this formulation Theorem 3.5 and Remark 3.18 are satisfied.

Note that the indices of efficiency of the residual estimator η_{R_2} in the formulation (VP_1) (Table 5.4) and (VP_2) (Table 5.9) are very similar. This means that we should expect similar convergence rates in the adaptive method, which is confirmed in Fig. 5.5 and 5.10 and in theirs respective meshes obtained in the adaptive refinement showed in Fig. 5.7 and 5.13. Note the similarity of the areas of refinement.

5.3.2.1 Residual-hierarchical adaptive strategy

In the following we compare the residual and the hierarchical adaptive strategies for the formulation (VP_2) with the date given in Example (Ex. 2D). Fig. 5.11 displays the error e using an h-uniform refinement and the errors obtained using an adaptive refinement based on residual and hierarchical error estimators, respectively. Contrary to what happened with formulation (VP_1), the hierarchical and the residual refinement strategies show almost the same behavior.

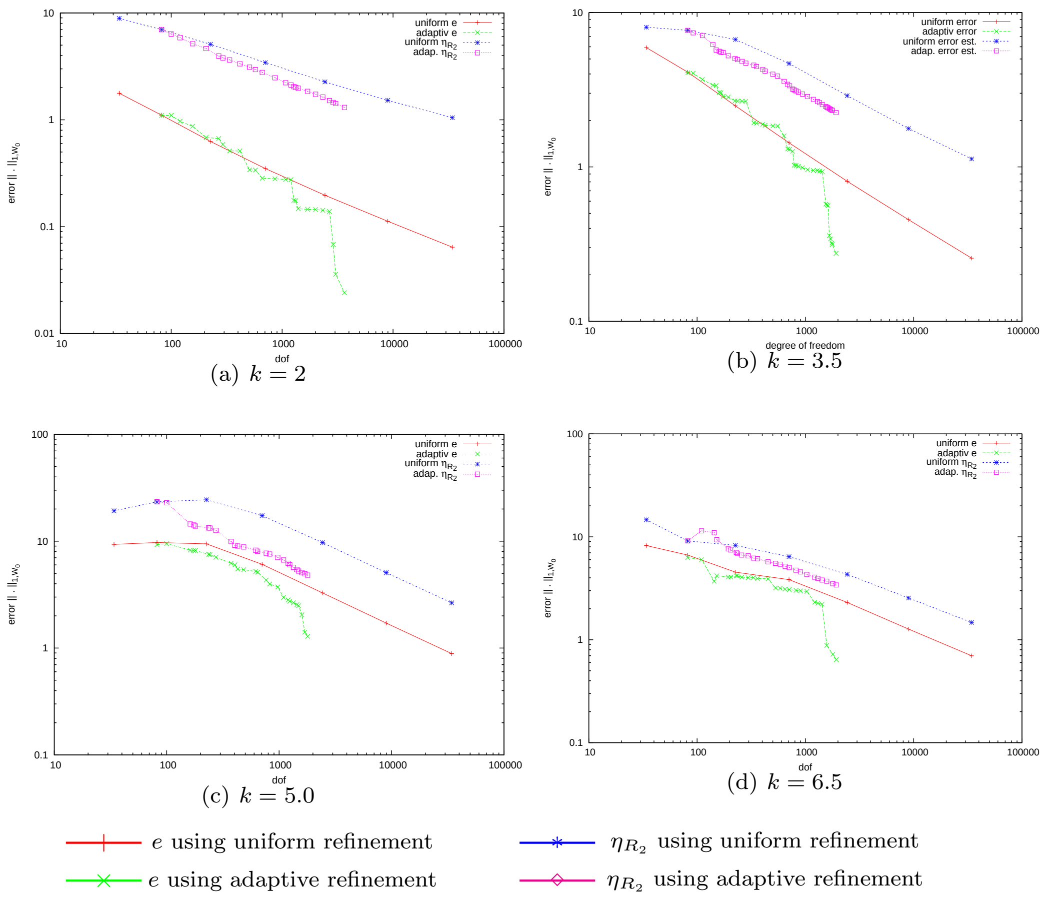


Fig. 5.10: Errors and residual error estimator using (VP_2), $\alpha = i/k$.

$k = 3.5$		$k = 5.0$	
η_{R_2}	η_{R_2}/e	η_{R_2}	η_{R_2}/e
1.3556	1.3556	19.2485	2.0610
1.8644	1.8644	23.3774	2.4047
2.6932	2.6932	24.4007	2.5826
3.2610	3.2610	17.3447	2.8555
2.8919	3.5822	9.6990	2.9545
1.7716	3.8935	5.0793	2.9670
1.1263	4.3944	2.6479	2.9846

Table 5.9: residual error estimators and effectivity index using (VP_2) with $\alpha = i/k$, $k = 3.5$ and $k = 5$.

Fig. 5.12 shows different meshes obtained in the adaptive refinement process using the hierarchical and residual estimators for $k = 3.5$, respectively. For this value, the residual strategy performs better than the hierarchical strategy. Note that for this value the hierarchical adaptive strategy has almost the same convergence order as the h-uniform method.

For the cases $k = 5$, for $k = 2, 5$ and 6.5 the hierarchical and residual strategies behave equally. Fig. 5.13 shows different meshes obtained in the adaptive process for $k = 2$. Both strategies focus their refinement on the borders of the domain. However, the residual strategy refines a litte more in the inside of Ω than the hierarchical strategy using (VP_1) (c.f. Fig. 5.7 and Fig. 5.13).

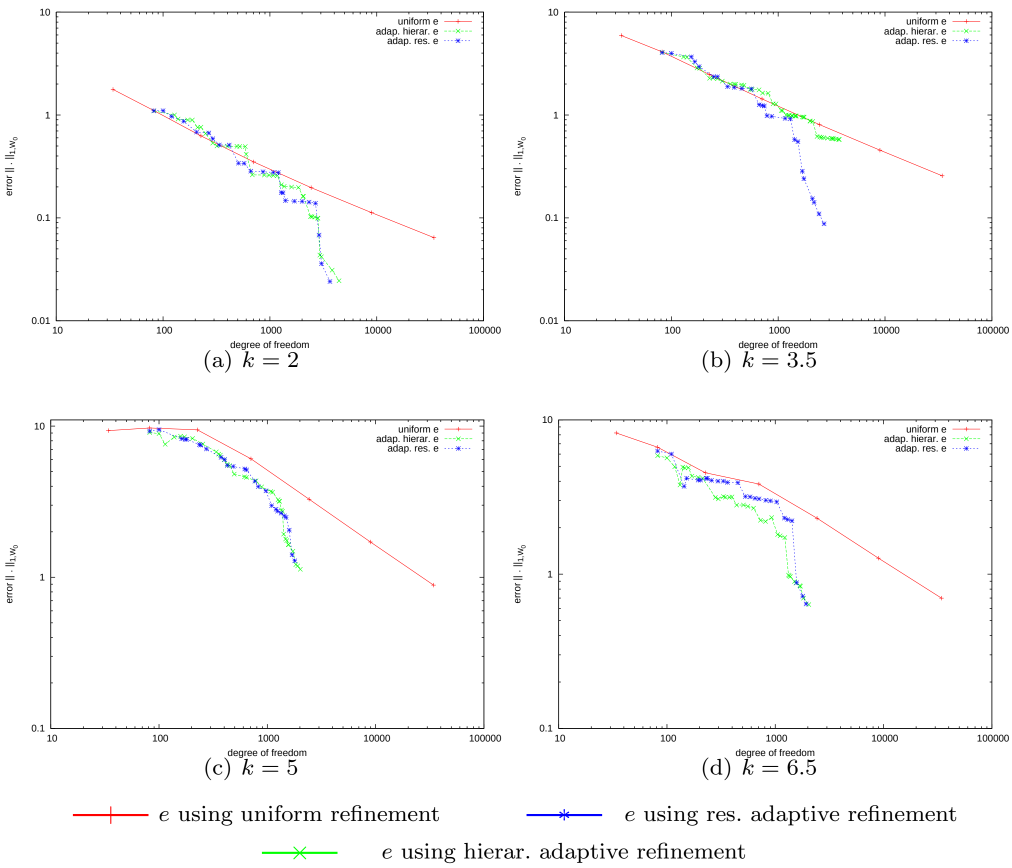
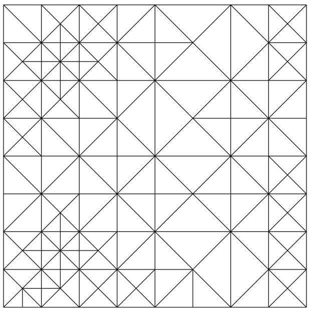
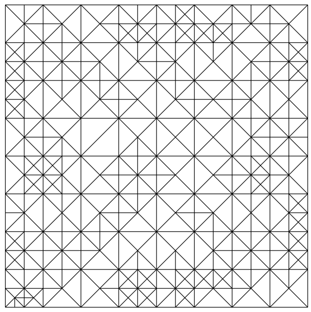
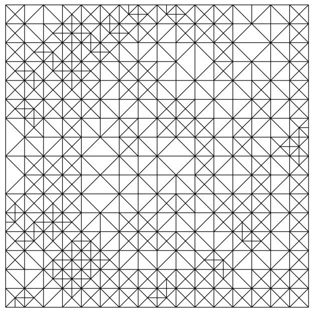
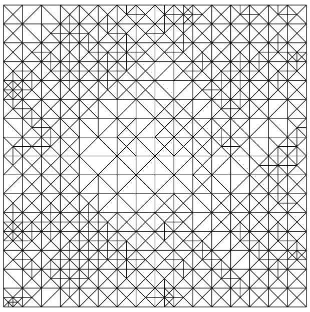


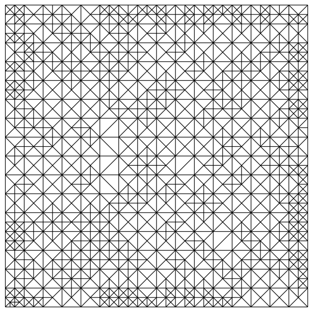
Fig. 5.11: Comparison of h-uniform, hierarchical and residual adaptive strategies using (VP_2), $\alpha = i/k$ and parameter of refinement $\delta = 0.95$.

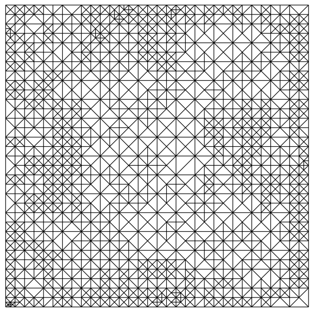

(a) $N = 276$

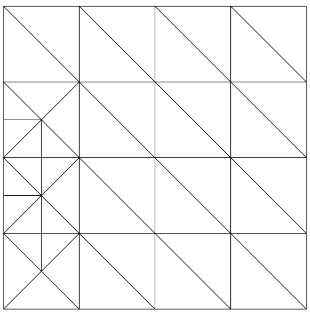

(b) $N = 628$

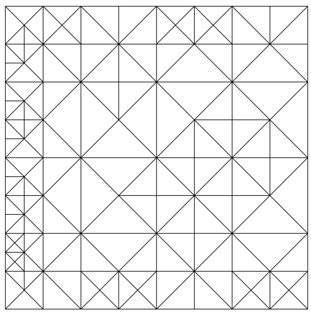

(c) $N = 1128$

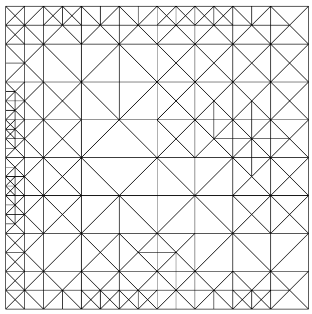

(d) $N = 1482$

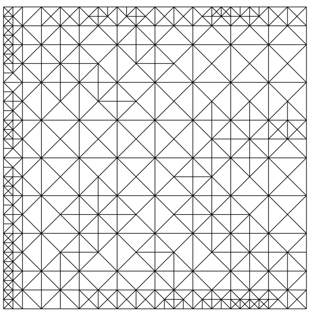

(e) $N = 2016$

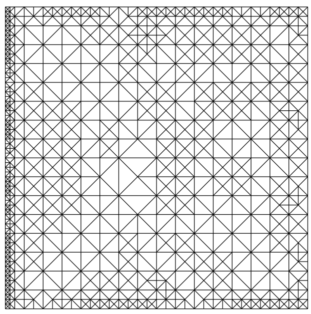

(f) $N = 2770$


(g) $N = 100$


(h) $N = 270$


(i) $N = 562$


(j) $N = 872$


(k) $N = 1740$

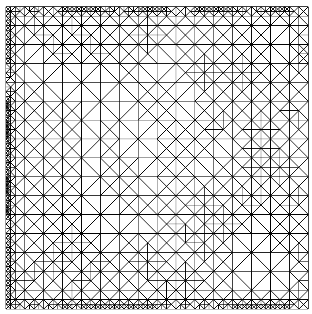
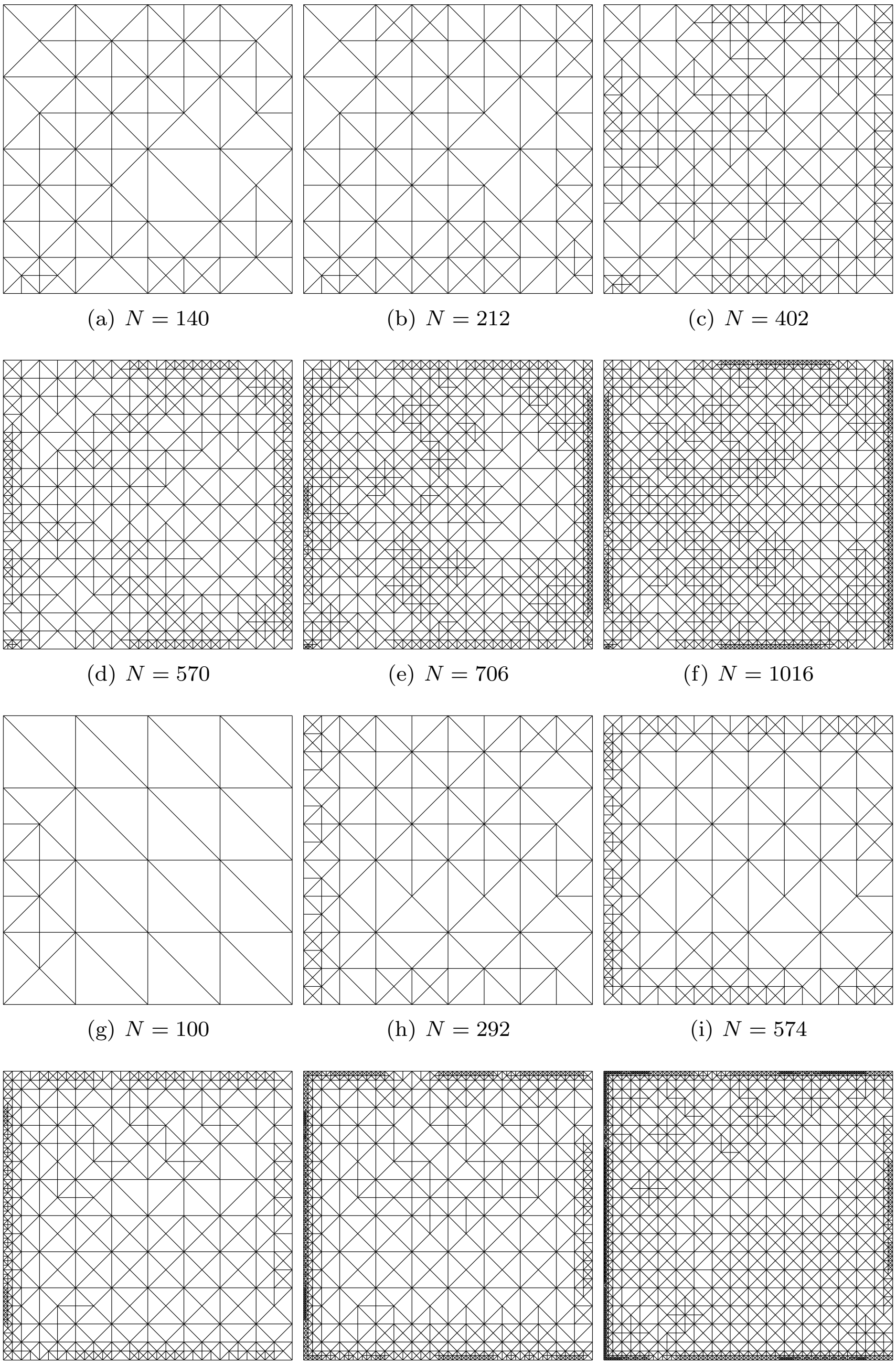

(l) $N = 2690$

Fig. 5.12: Adaptives meshes using (VP_2) with $k = 3.5$ and parameter of refinement $\delta = 0.95$. Fig. (a)- (f) hierarchical refinement, Fig. (g)-(l) residual refinement.



(a) $N = 140$ (b) $N = 212$ (c) $N = 402$

(d) $N = 570$ (e) $N = 706$ (f) $N = 1016$

(g) $N = 100$ (h) $N = 292$ (i) $N = 574$

(j) $N = 1200$ (k) $N = 2326$ (l) $N = 3648$

Fig. 5.13: Adaptives meshes using (VP_2) with $k = 2.0$ and parameter of refinement $\delta = 0.95$. Fig. (a)- (f) hierarchical refinement, Fig. (g)-(l) residual refinement.

Chapter 6

Numerical results in 3D

In this chapter we present the numerical results of an FE/BE coupling method for a Fluid Structure interaction problem in three-dimensional case. The method is an extension of the one used in the two-dimensional case in Chapter 5. We present similar results as for the two-dimesional case, however, for reasons of computing time and required memory by the computer, some results are shown only for one wave number k.

In the first section we present the performance of a non-stable ($\alpha = 0$) and a stable ($\alpha = i/k$) method with different wave numbers k. We recall that for $\alpha = i/k$ the system is stable, according to the considerations made by Kress [50] and the experiments of Bielak et al. [6] and Meyer et al. [61]. In the following sections, we show the convergence using h-uniform and adaptive refinements for each stable non-symmetric formulation (VP_1) and stable symmetric formulation (VP_2) with $\alpha = i/k$. Also, it shows the results of implementation of the hierarchical error estimator presented in Theorems 4.6 and 4.8 and residual error estimator presented in Theorems 3.4 and 3.5 for the non-symmetric and symmetric formulation, respectively.

The formulations are implemented and solved in the scientific program MaiProgs [58]. We use *hexahedral* elements for the discretization of the domain Ω and squares for the discretization of the boundary Γ. To approximate the solution we take piecewise linear test functions and trial functions.

Throughout each of the following sections, we solve the following problem, which is an extension of Example (Ex. 2D) presented in the previous chapter.

Numerical example in 3D (Ex. 3D). *Consider a cube-shaped, homogeneous, isotropic, elastic scatterer made of steel,* $\bar{\Omega} = [-1, 1]^3$*. The scatterer possesses the following material* *parameters, Poisson's ratio* $\nu = 0.28$*, Young's module* $E = 200GPa$ *and* $\rho = 7800kg/m^3$*. It is submerged by sea water and is subject to a plane incident wave* $p^0(x, y, z) = e^{ikx}$*. Furthermore, the density of sea water is* $\rho_0 = 1020kg/m^3$ *and the sound velocity is* $c_0 = 1500m/s$*.*

Remark 6.1. *We remark that for Example (Ex. 3D) we do not know the solution of the system. For the analysis of error and convergence of the approximated solutions, we take as the exact norm of* $\mathbf{u} \in H^1(\Omega)$ *or* $\sigma_{\rm n} \in H^{1/2}(\Gamma)$ *and* $\phi \in H^{1/2}(\Gamma)$ *the extrapolated norm using the Aitken's delta-squared process to a sequence of norms obtained by h-uniform refined meshes.*

6.1 Behavior of the Systems using $\alpha = 0$ and $\alpha = i/k$

In this section we compare the performance obtained using the stable and non-stable procedures using the symmetric and non-symmetric formulations (VP_1) and (VP_2) with the above fixed input data given in Example (Ex. 3D). In the following experiments we have computed the formulations (VP_1) and (VP_2) for a fixed mesh with 864 hexahedral elements (216 boundary elements) for different wave numbers $k \in \{n\, h_k : n = 1, \ldots, 400 \text{ and } h_k = 0.02\}$ and using $\alpha = 0$ and $\alpha = i/k$.
We analyze the results on the surface using the symmetric stable formulation (VP_2). This formulation has the advantage that we can analyze the normal traction on the surface $\sigma_{\rm n}$, in addition to the unknown density ϕ. Fig. 6.1a and 6.1b show the L^2-norms of the variables $\sigma_{\rm n}$ and ϕ, respectively, for different wave numbers k. In Fig. 6.1b we observe the influence of the constant α on the system. As in the two-dimensional case (see Fig. 5.1a) for $\alpha = 0$ we observe peaks in the L^2-norm of ϕ for some values of k, which correspond to critical frequencies. We recall that for $\alpha = 0$, ϕ is the density of the single layer potential, whereas for $\alpha = i/k$ ϕ represents the density of a combined single and double layer potential (see Section 1.1.2). Thus, it is natural, without taking into account the critical frequencies, that the corresponding curves are different.

From Fig. 6.1b we can confirm that for $\alpha = i/k$ the L^2-norm keeps stable. In the internal clipping of Fig. 6.1b, which is the same graphic but at a greater scale, we can confirm that the peaks occurring for $\alpha = 0$ are bounded.

However, contrary to ϕ, we expect that the normal traction σ_{n} coincides for the two procedures, since the constant α was introduced to ensure the uniqueness of the density ϕ and should not alter the results of the displacement $\mathbf{u}$ and the normal traction σ_{n}, as can be seen in Fig. 6.1a except at some wave numbers.

In order to compare the non-stable and stable formulations Fig. 6.2 shows the L^2-norm of the density ϕ approximated by of the formulations (VP_1) and (VP_2) with $\alpha = i/k$ for different wave numbers k. Note that the norm of ϕ coincides in both formulations. We recall that for the same value of α, both formulations return the same solution ϕ^h as expected. (see Problem (2.2), Section 2.2).

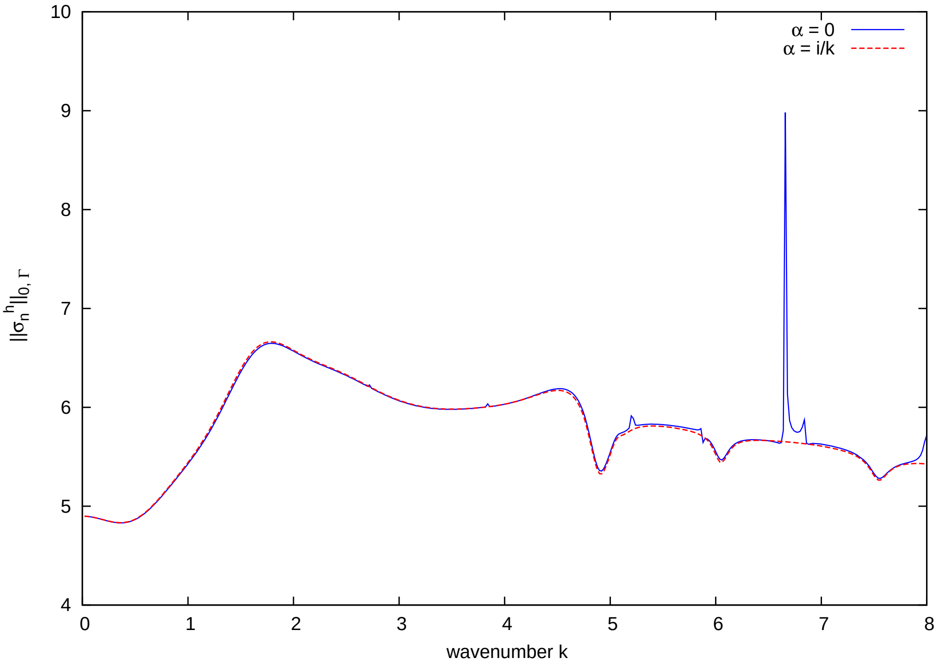


Fig. 6.1a: Comparison of $\| \sigma_{\mathrm{n}}{}^h \|_0$ for the non-stable and stable procedure using (VP_2).

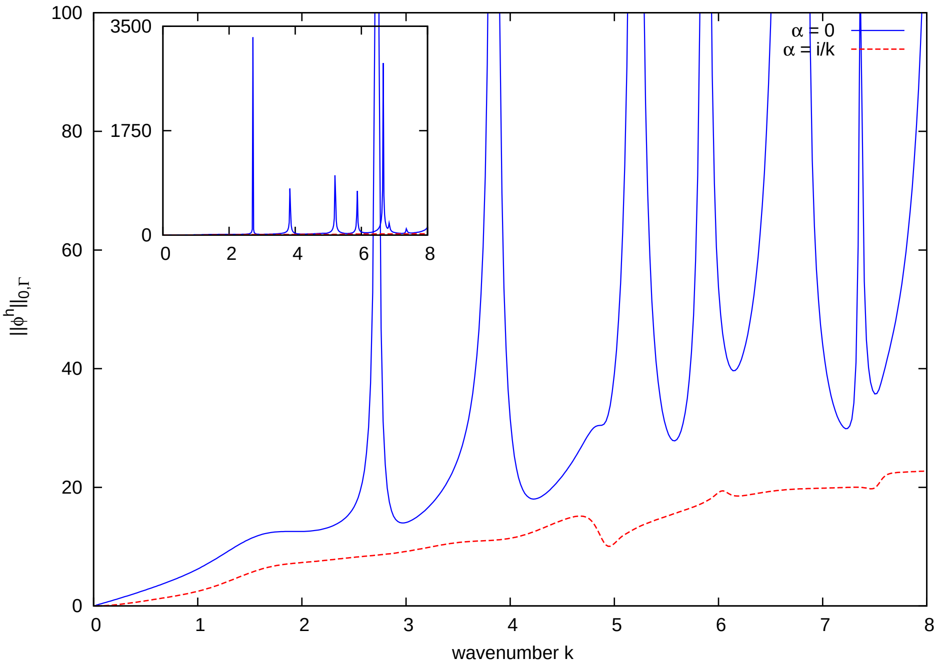


Fig. 6.1b: Comparison of $\|\phi^h\|_0$ for the non-stable and stable procedure using (VP_2).

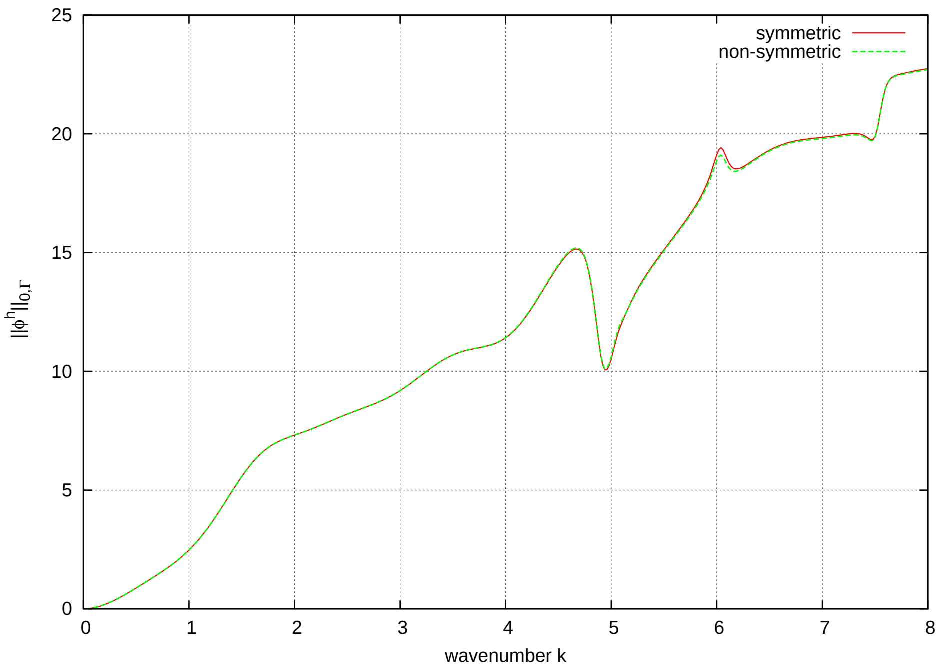


Fig. 6.2: Comparison of $\|\phi^h\|_{0,\Gamma}$ for the non-symmetric (VP_1) and symmetric (VP_2) formulations with $\alpha = i/k$.

6.2 Convergence, Error Indicators and Adaptive Methods for the Non-symmetric Formulation (VP_1)

In this section we present the numerical results obtained by solving the non-symmetric stable formulation (VP_1) for the three-dimensional case with the wave number $k = 5.2$, which is a critical value (cf. Fig. 6.1b). We use the same notation used in Section 5.2 for the numerical results in the two-dimensional case. For an h-uniform refinement we compute the error e, together with the residual and hierarchical error estimators.

In Table 6.1 and Fig. 6.3 we present the convergence history of our Example (Ex. 3D) for a sequence of quasi-uniform triangulations. It is confirmed that for $k = 5.2$, even though it is a critical frequency (see Fig. 6.1b), our stable formulations (VP_1) converges. We remark that the rate of convergence $\mathcal{O}(h)$ predicted in Section 2.4.5 for this formulation is approximately obtained for the unknowns. Additionally, Table 6.2 shows the errors $e_{\mathbf{u}}$ and e_ϕ and the experimental rates of convergence $\theta_{\mathbf{u}}$ and θ_ϕ for the unknowns $\mathbf{u}$ and ϕ, respectively. We can see that e_ϕ is the dominant error.

h	N	e	θ_N
1	107	17.8383	-
1/2	473	23.5836	-0.1878
1/3	1247	19.7620	0.1823
1/4	2573	15.9756	0.2936
1/5	4595	11.1997	0.3227
1/6	7457	13.2485	0.3469
1/7	11303	9.6006	0.3704
1/8	16277	8.3095	0.3960
1/9	22523	7.2355	0.4261
1/10	30185	6.3177	0.4632
1/11	39407	5.5132	0.5108

Table 6.1: Error e and convergence rates θ_N using (VP_1) with $k = 5.2$ and $\alpha = i/k$.

h	N_u	$e_{\mathbf{u}}$	θ_{N_u}	N_ϕ	e_ϕ	θ_{N_ϕ}
1	81	0.1318E-9	-	26	17.8383	-
1/2	375	0.0710E-9	0.4031	98	23.5837	-0.2104
1/3	1029	0.0789E-9	-0.1037	218	19.7620	0.2211
1/4	2187	0.0814E-9	-0.0408	386	15.9757	0.3723
1/5	3993	0.0721E-9	0.2004	602	13.2485	0.4212
1/6	6591	0.0630E-9	0.2690	866	11.1997	0.4620
1/7	10125	0.0551E-9	0.3111	1178	9.6006	0.5007
1/8	14739	0.0484E-9	0.3455	1538	8.3096	0.5416
1/9	20577	0.0427E-9	0.3791	1946	7.2356	0.5882
1/10	27783	0.0377E-9	0.4156	2402	6.3177	0.6444
1/11	36501	0.0332E-9	0.4584	2906	5.5132	0.7151

Table 6.2: Error and convergence rates of $\mathbf{u}$ and ϕ using (VP_1) with $k = 5.2$, $\alpha = i/k$.

Table 6.3 shows the residual error estimator

$$\eta_{R_1} = (R_1^h + R_2^h + R_3^h + R_4^h)^{1/2}$$

and the effectivity indices obtained in an h-uniform refinement. This confirms the efficiency of our residual estimator and the predicted order of convergence of our discrete problem. One can see that our estimator η_{R_1} is proportional to e, since the effectivity index $q = \eta_{R_1}/e \approx 0.6$ shown in Table 6.3 is quasi-constant, so we verify Theorem 3.4 and Remark 3.13 for the three-dimensional version and the non-symmetric formulation (VP_1).

In Table 6.4 we show the hierarchical error estimator η_{H_1} stated in Theorem 4.6

h	N	η_{R_1}	$q = \eta_{R_1}/e$
1	107	78.6972	4.4117
1/2	473	28.6879	1.2164
1/3	1247	17.0375	0.8621
1/4	2573	11.8714	0.7431
1/5	4595	9.2147	0.6955
1/6	7457	7.5367	0.6729
1/7	11303	6.3806	0.6646
1/8	16277	5.5392	0.6666
1/9	22523	4.9021	0.6775
1/10	30185	4.4047	0.6972
1/11	39407	4.0066	0.7267

Table 6.3: Residual error estimator η_R and effectivity index q calculated for $k = 5.2$ with $\alpha = i/k$ using (VP_1).

$$\eta^j_{H_1} = \Big(\sum_{i=1}^{n_j} \varTheta^2_{j,i} + \sum_{i=1}^{m_j} \mathcal{V}^2_{j,i}\Big)^{1/2},$$

using an h-uniform refinement. For this case the estimate is not exactly proportional to the error e, as seen in column 4 of Table 6.4, however, the rate of convergence of this estimator is approximately $\theta_{\eta_H} \approx 0.24$ which would be in an acceptable range of the a priori estimate of $\mathcal{O}(h)$ made for this case.

In the same table we can observe the indicators $\varTheta_j$ and $\mathcal{V}_j$ defined in Theorem 4.6

$$\varTheta_j = \Big(\sum_{i=1}^{n_j} \varTheta^2_{j,i}\Big)^{1/2}, \qquad \mathcal{V}_j = \Big(\sum_{i=1}^{m_j} \mathcal{V}^2_{h,i}\Big)^{1/2},$$

together with the effectivity indices.

j	h	N	η_{H_1}	$q = \eta_{H_1}/e$	$\varTheta_j$	$q = \varTheta_j/e$	$\mathcal{V}_j$	$q = \mathcal{V}_j/e$
1	1	107	1.0259	1.0259	0.3216E-4	2.4404E5	16.7457	0.9387
2	1/2	473	0.3213	0.3213	0.2217E-4	3.1207E5	6.9958	0.2966
3	1/3	1247	0.3415	0.3415	0.1386E-4	1.7567E5	6.3250	0.3201
4	1/4	2573	0.3877	0.3877	0.1043E-4	1.2818E5	5.7748	0.3615
5	1/5	4595	0.4349	0.4349	0.0858E-4	1.1894E5	5.3921	0.4070
6	1/6	7457	0.4819	0.4819	0.0728E-4	1.1559E5	5.0346	0.44495
7	1/7	11303	0.5321	0.5321	0.0645E-4	1.1695E5	4.7730	0.4972
8	1/8	16277	0.5820	0.5820	0.0580E-4	1.1972E5	4.5137	0.5432

Table 6.4: Hierarchical error estimator η_{H_1} and effectivity index q calculated for $k = 5.2$ with $\alpha = i/k$ using (VP_1).

Finally, Fig. 6.3 shows the error e and the a posteriori error estimators η_{R_1} and η_{H_1} for different degrees of freedom.

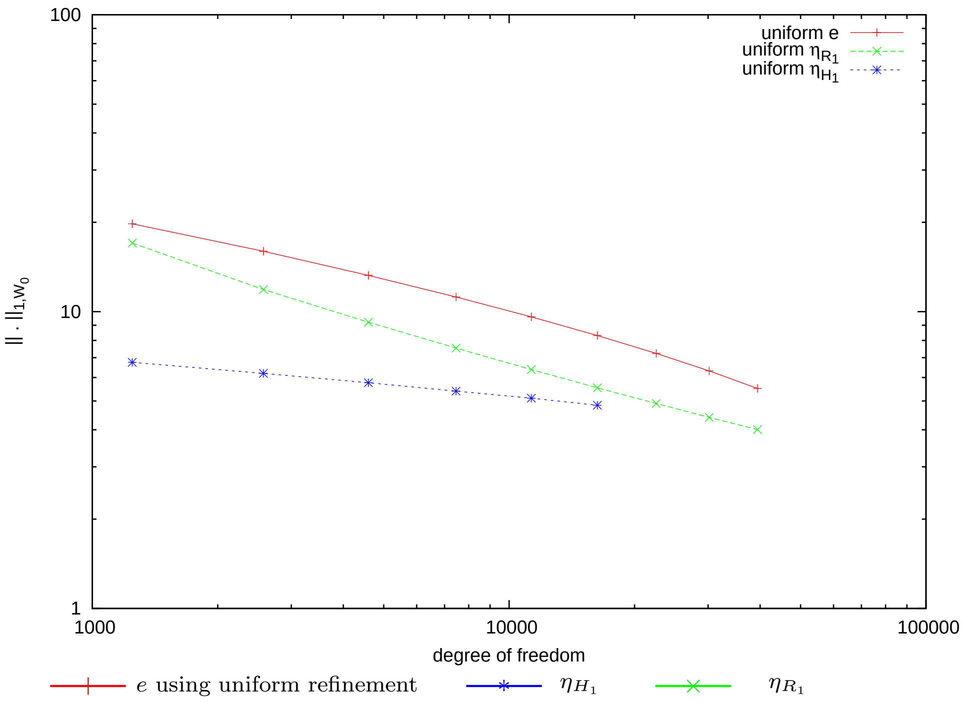


Fig. 6.3: Error and hierachical and residual error estimators using an h-uniform strategy and (VP_1) with $k = 5.2$ and $\alpha = i/k$.

6.3 Convergence, Error Indicators and Adaptive Methods for Symmetric Formulation $(\boldsymbol{VP_2})$

In this section we present results obtained approximately for the symmetric stable formulation (VP_2) for the three-dimensional case. As for the non-symmetric formulation, we show the results obtained using $k = 5.2$ for an h-uniform refinement with residual and hierarchical error estimators. We use the same notation as in Section 5.3 for the numerical results in the two-dimensional case and symmetric formulation (VP_2).

In Table 6.5 and Fig. 6.4 we present the results for our example for a sequence of quasi-uniform triangulations using $k = 5.2$. It is confirmed that for this value, even though it is a critical frequency (see Fig. 6.1b), our stable symmetric formulation converges. We remark that the convergence rate $\mathcal{O}(h)$ predicted in Section 2.4.5 for this formulation is approximately obtained for the unknowns. Additionally, Table 6.6 shows the error obtained for each of the unknowns $\mathbf{u}$, σ_{n} and ϕ. One can see that the dominant error is again given by e_ϕ.

h	N	e	θ_N
1	133	23.4089	-
1/2	571	24.6002	-0.0341
1/3	1465	19.7807	0.2314
1/4	2959	15.9455	0.3066
1/5	5197	13.1749	0.3389
1/6	8323	11.0840	0.3669
1/7	12481	9.4423	0.3956
1/8	17815	8.1065	0.4287
1/9	24469	6.9842	0.4696
1/10	32587	6.0125	0.5229

Table 6.5: Error e and convergence rates Θ_N using (VP_2) with $k = 5.2$ and $\alpha = i/k$.

In Table 6.7 we show the residual error estimator obtained for h-uniform refinement. Here, we confirm the efficiency of our residual estimator

$$\eta_{R_2} = \left(\tilde{R}_1^h + \tilde{R}_2^h + \tilde{R}_3^h + \tilde{R}_4^h + \tilde{R}_5^h\right)^{1/2}$$

stated in Theorem 3.5 and the predicted order of convergence of the discrete problem. We can see that our estimator η_{R_2} is proportional to e, since the effectivity index $q = \eta_R/e \approx 1.3$

h	N_u	$e_{\mathbf{u}}$	θ_{N_u}	$N_{\sigma_n,\phi}$	e_{σ_n}	$\theta_{N_{\sigma_n}}$	e_ϕ	θ_{N_ϕ}
1	81	0.6538E-10	-	26	10.6959	-	20.8224	-
1/2	375	0.9083E-10	-0.2145	98	6.2467	0.4053	23.7939	-0.1005
1/3	1029	0.8825E-10	0.0286	218	3.7152	0.6499	19.4286	0.2535
1/4	2187	0.8198E-10	0.0976	386	2.7096	0.5524	15.7136	0.3714
1/5	3993	0.7157E-10	0.2256	602	2.1542	0.5162	12.9976	0.4270
1/6	6591	0.6207E-10	0.2840	866	1.7797	0.5252	10.9402	0.4739
1/7	10125	0.5397E-10	0.3260	1178	1.5007	0.5541	9.3223	0.5201
1/8	14739	0.4707E-10	0.3641	1538	1.2798	0.5970	8.0049	0.5713
1/9	20577	0.4113E-10	0.4041	1946	1.0971	0.6547	6.8975	0.6328
1/10	27783	0.3593E-10	0.4506	2402	0.9404	0.7323	5.9385	0.7111

Table 6.6: Error and convergence rates of $\mathbf{u}$ and ϕ using (VP_2) with $k = 5.2$, $\alpha = i/k$.

shown in Table 6.7 is quasi-constant. This verifies Theorem 3.5 and Remark 3.18 for the three-dimensional version and the symmetry formulation.

h	N	η_{R_2}	$q = \eta_{R_2}/e$
1	133	126.5869	5.4076
1/2	571	52.9663	2.1531
1/3	1465	32.7106	1.6537
1/4	2959	23.3245	1.4628
1/5	5197	18.1849	1.3803
1/6	8323	14.8992	1.3442
1/7	12481	12.6254	1.3371
1/8	17815	10.9673	1.3529
1/9	24469	9.7105	1.3903
1/10	32587	8.7284	1.4517

Table 6.7: Residual error estimator η_{R_2} and effectivity index q calculated using (VP_2) for $k = 5.2$ with $\alpha = i/k$.

In Table 6.8 we show the hierarchical error estimator stated in Theorem 4.8 using an h-uniform refinement. For this case the estimate is not exactly proportional to the error e, as seen in column 4 of Table 6.4, however, the rate of convergence of this estimator is approximately $\theta_{\eta_H} \approx 0.26$ which would be in an acceptable range of the a priori estimate of $\mathcal{O}(h)$ made for this case.

In the same table we can observe the values of the indicators Θ_j, Ψ_j and $\mathcal{V}_j$ defined in Theorem 4.8

$$\Theta_j = \Big(\sum_{i=1}^{n_j} \Theta_{j,i}^2\Big)^{1/2}, \quad \Psi_j = \Big(\sum_{i=1}^{m_j} \Psi_{j,i}^2\Big)^{1/2}, \qquad \mathcal{V}_j = \Big(\sum_{i=1}^{m_j} \mathcal{V}_{j,i}^2\Big)^{1/2},$$

together with their respective effectivity indices.

j	η_{H_2}	$q = \eta_{H_2}/e$	Θ_j	$q = \Theta_j/e$	Ψ_j	$q = \Theta_j/e$	$\mathcal{V}_j$	$q = \mathcal{V}_j/e$
1	3.2454	0.1386	0.5085E-4	7.7771E5	0.8597	0.0804	2.1277	0.1022
2	2.1290	0.0865	0.4209E-4	4.6340E5	0.4606	0.0737	1.4332	0.0602
3	1.5513	0.0784	0.2681E-4	3.0382E5	0.2801	0.0754	1.0606	0.0546
4	1.4151	0.0887	0.2002E-4	2.4419E5	0.2170	0.0801	0.9769	0.0622
5	1.2545	0.0952	0.1636E-4	2.2859E5	0.1836	0.0853	0.8679	0.0668
6	1.1390	0.1028	0.1398E-4	2.2521E5	0.1641	0.0922	0.7885	0.0721

Table 6.8: Hierarchical error estimator η_{H_2} and effectivity index q calculated for $k = 5.2$ with $\alpha = i/k$ using (VP_2).

Finally, Fig. 6.3 shows the error e and the a posteriori error estimators η_{R_2} and η_{H_2} for different degrees of freedom.

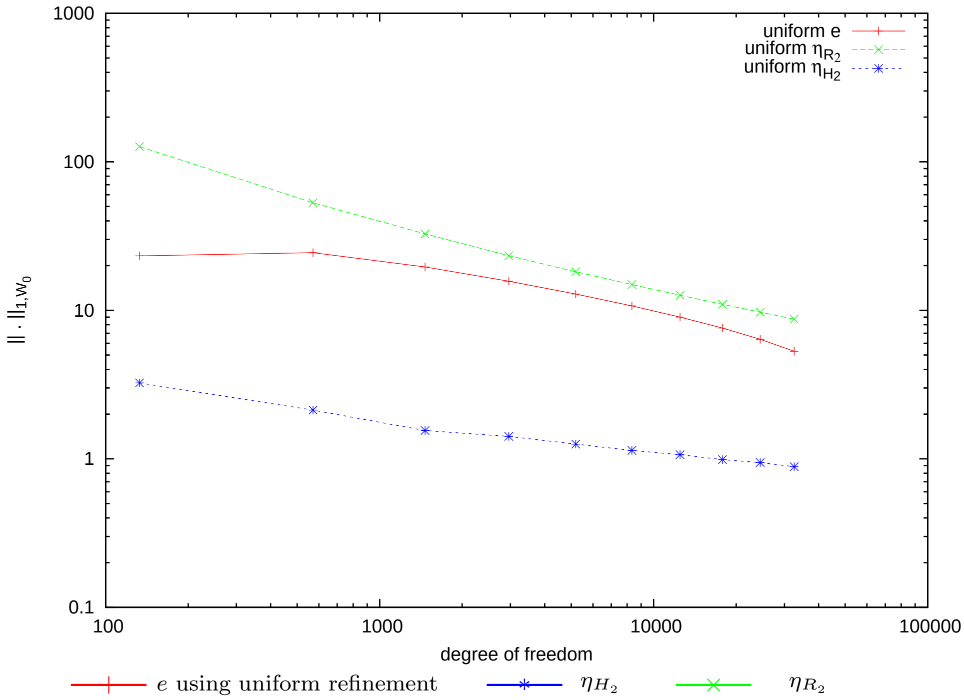


Fig. 6.4: Error and hierachical and residual error estimators using an h-uniform strategy and (VP_2) with $k = 5.2$ and $\alpha = i/k$.

6.4 Adaptive Method using Non-symmetric Formulation

In this sections we present results obtained using the residual adaptive strategy 1 (p. 39) using the non symmetric formulation (VP_1).

Fig. 6.5 shows the errors obtained using h-uniform and h-adaptive refinements. We can see that the rate of convergence for both methods is the same. we would expect that the adaptive method is more efficient than the uniform method for problems with singularities. However for this case the solution is smooth and the difference between one method and the other can not be observed. In the same figure also the residual error estimates for both procedures is shown.

Fig. 6.6 shows different meshes obtained during the adaptive process. One can see that the refinement is focused on one side of the cube and into the cube (in the center with respect to x-axis) in the proximity of the border. This is plausible as the incident wave is in x-direction.

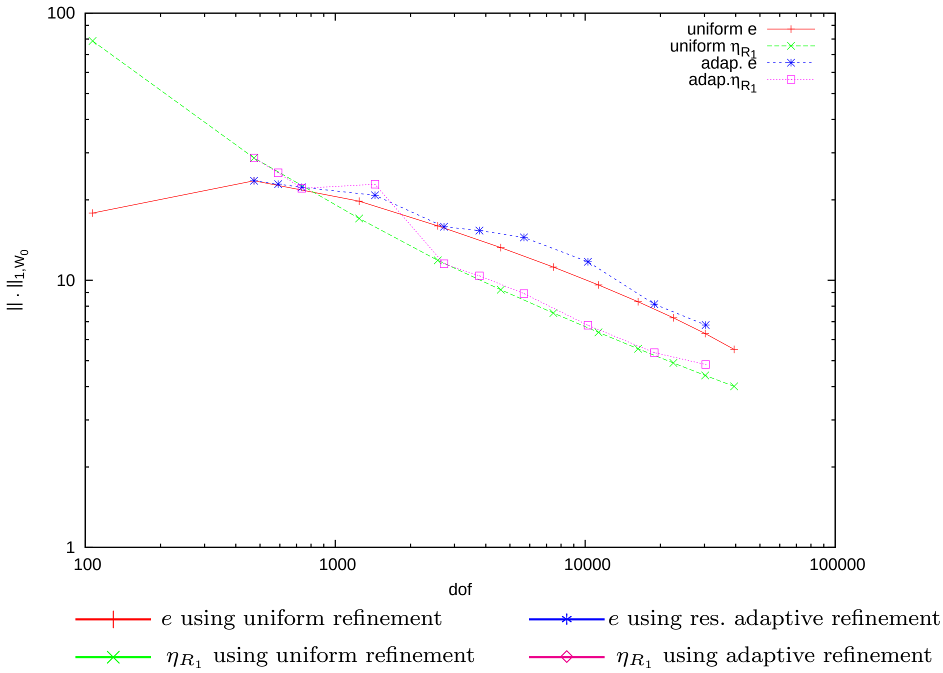


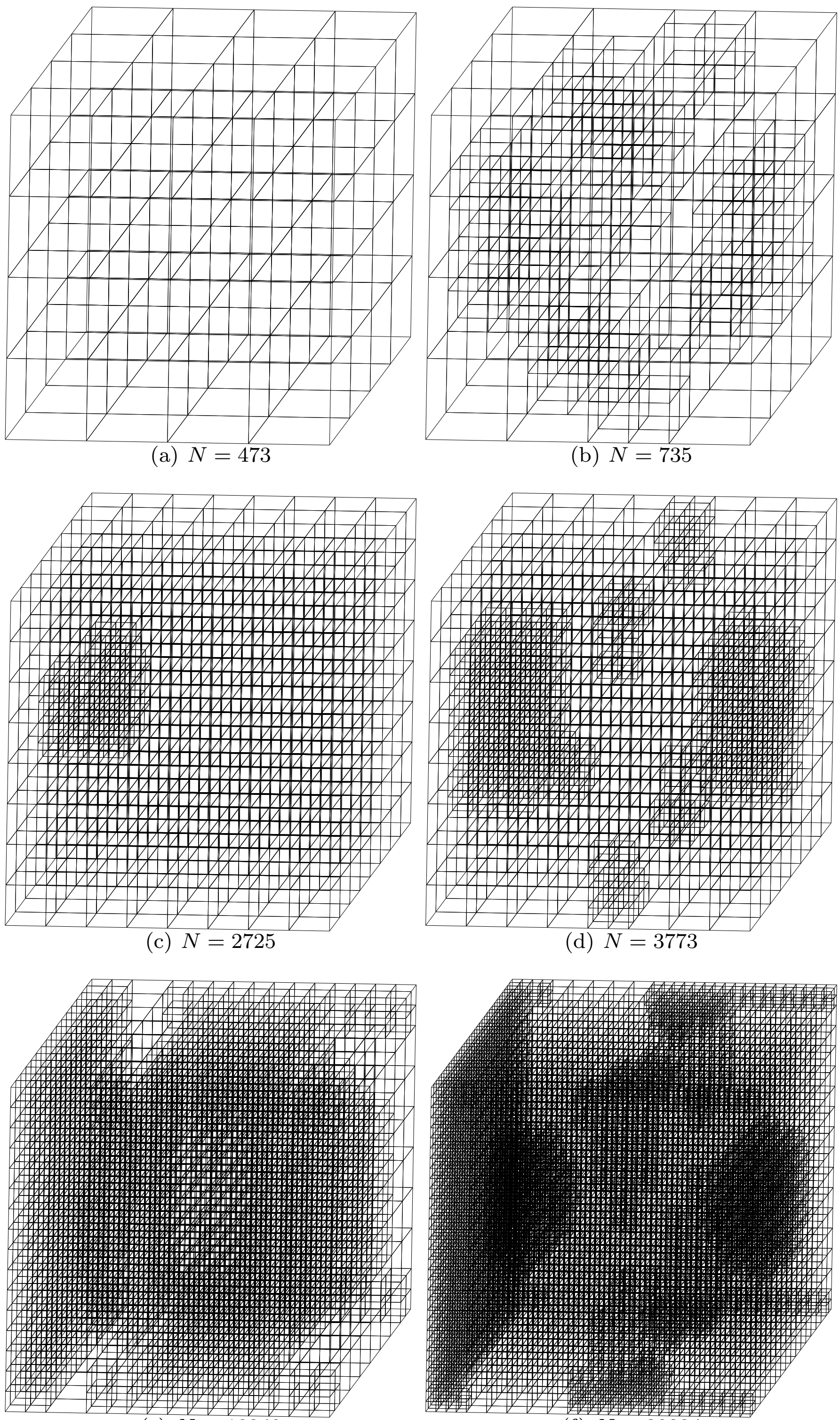
Fig. 6.5: Errors e and residual error estimators η_{R_1} of uniform and adaptive refinements using (VP_1) with $k = 5.2$, $\alpha = i/k$ and parameter of refinement $\delta = 0.95$.

6.5 Adaptive Method using Non-symmetric Formulation, L-Block

In this section we present the results using the same data presented in Example (Ex.3D), but taking as domain an L-Block, $\Omega := [-1,-1] \setminus ([0,1]^2 \times [-1,1])$

The refinement algorithm proceeds by first refining the 10% of the elements on which the local contributions of the residual error estimator are the greatest and then by further refining in order to eliminate hanging nodes that violate the one-constraint rule, i.e. only one edge has at most two smaller neighboring edges on the other element.

We extrapolate the error using a sequence of uniform meshes and compare the error of the adaptive and uniform sequences in Fig. 6.7. After several refinements the error of the adaptive algorithm is less than the error in the uniform refinement. Our adaptive algorithm produces a sequence of refined meshes, which is shown in Figure 6.8. The algorithm refines towards the side the side that coincides with the wavefront and on the corners of the domain Ω.



(a) $N = 473$ (b) $N = 735$

(c) $N = 2725$ (d) $N = 3773$

(e) $N = 10249$ (f) $N = 30324$

Fig. 6.6: Sequence of meshes for the adaptive strategy using (VP_1) with $k = 5.2$, $\alpha = i/k$ and $\delta = 0.9$.

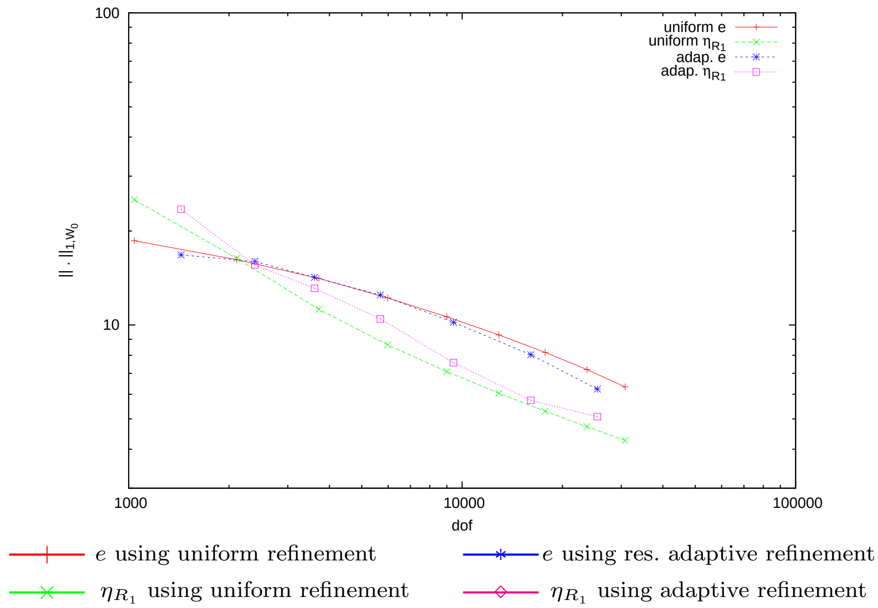
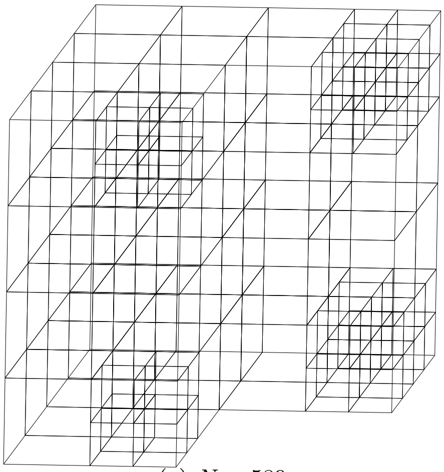
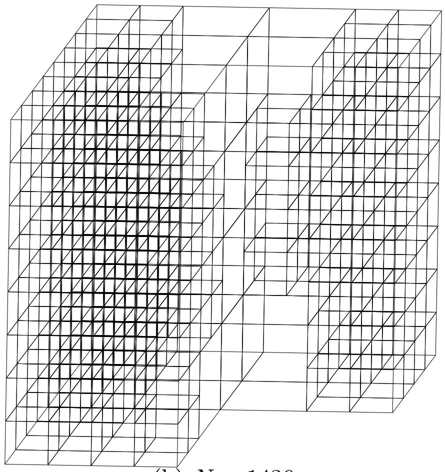


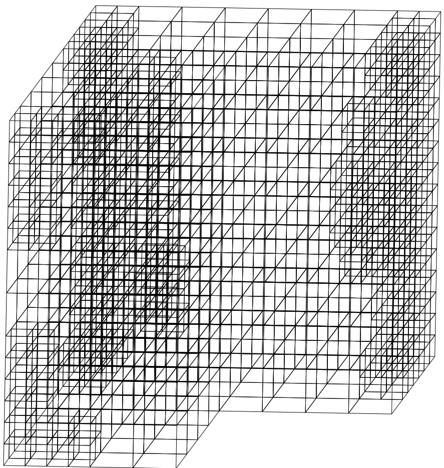
Fig. 6.7: Error e and error residual estimators η_{R_1} of uniform and adaptive refinements in a L-Block using (VP_1) with $k = 5.2$ $\alpha = i/k$ and parameter of refinement $\delta = 0.9$



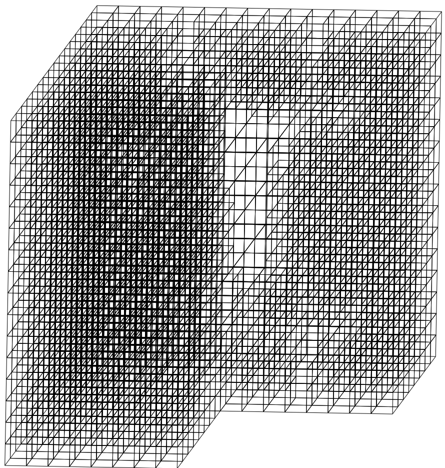
(a) $N = 589$



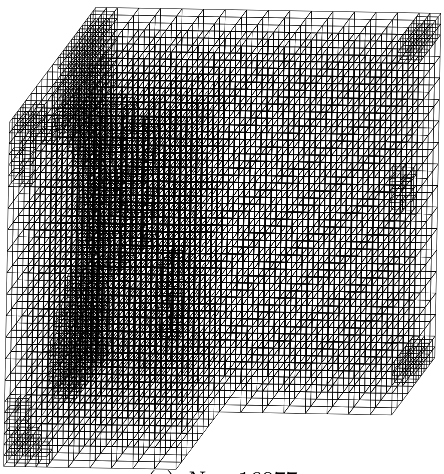
(b) $N = 1436$



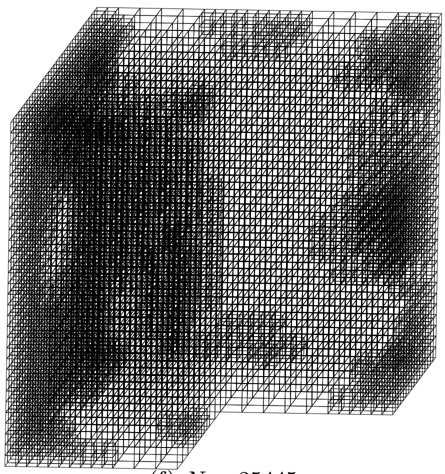
(c) $N = 3609$



(d) $N = 9420$



(e) $N = 16077$



(f) $N = 25445$

Fig. 6.8: Sequence of meshes for the adaptive strategy using (VP_1) with $k = 5.2$, $\alpha = i/k$ and parameter of refinement $\delta = 0.9$ on a L-Block.

Appendix A

Hanging nodes

The use of hanging nodes for the hp-finite element method is presented in [29], [68] and [71]. In [29] describes an implementation in Fortran 90 for the case two-dimensional case. In our numerical experiments, the adaptivity method and construction of mesh using hanging nodes has been implemented in MaiProgs so far, see Oestmann [69] for the h-version using linear polynomials and Leydecker [52] for the *hp*-version using Nedelec's and Raviart-Tomas elements, in this has used an extension of the "one-constraint-rule" by Demkowicz et al. [28] for the implementation in 3D.

Due to the hanging nodes need a special treatment in theory since the degrees of freedom are reduced and the problem arises whether this fact leads to a reduction in the order of approximation, in recent times the research has focused not only on its implementation, but also in the error analysis and convergence, for more detail see [13] [74] [41].

In the following we describe the implementation and some features of hanging nodes:

Definition A.1. Let $\mathcal{T}_h$ a triangulation of $\Omega \subset \mathbb{R}^3$. We denote with $\mathcal{N}$ the set of all nodes of the actual mesh. Let $n \in \mathcal{N}$ be a node such that it is an endpoint on one side of an element $T_i \in \mathcal{T}_h$. We say that n is a *regular node*, if for each of the elements to which belongs, n is an end point of each side to which it belongs. The set of regular nodes is denoted by $\mathcal{N}_r$

Definition A.2. Let $n \in \mathcal{N}$, n is a *hanging node*, if n belongs to one side of a element of $\mathcal{T}_h$ and is not an endpoint of the side. This would be the case when the node n is a midpoint of one side of an element $T_i \in \mathcal{T}_h$ (see Fig.A.1) or is a midpoint of one face of an neighbor element (see Fig. A.1). The set of hanging nodes is denoted by $\mathcal{N}_h$.

Each node $n \in \mathcal{N}$ is a regular node or a node hanging, i.e. $\mathcal{N} = \mathcal{N}_r \cup \mathcal{N}_h$

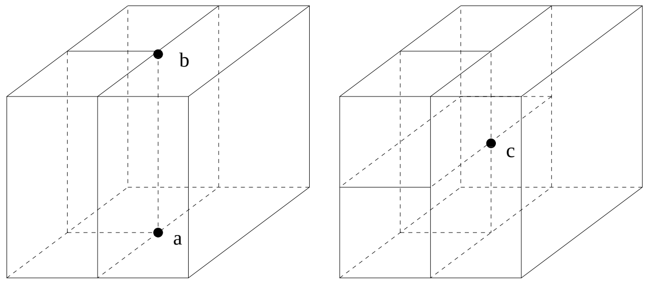


Fig. A.1: Hanging nodes: a,b midpoint of one side, c midpoint of a face of element.

A.1 Algorithms

For the refinement in 3D for hexahedrals, we have to ensure the *one-constraint rule*, see e.g. Demkowicz et. al [29] or Oestmann [69]. This means that only one hanging node on an edge is allowed. The **refinement algorithm** is the following

1. Initialize a regular mesh without hanging nodes.
2. Calculate the local error indicators on every element $T_i \in \mathcal{T}_h$.
3. Mark the elements to be refined.
4. Check if the "one-constraint rule" is fulfilled. If there is more than one hanging node on one edge mark the neighboring elements for refinement.
5. Go to 4. until no more extra refinements are necessary.
6. Initialize the new mesh.
7. Calculate the new approximation and go to 2.

Note that the refinement is performed on the elements of volume $T_i \in \mathcal{T}_i$, thus, of the mesh that results from items 2-4, it is estimated the mesh for boundary elements. For the refinement of a hexahedral there are seven different strategies see Fig. A.2 , however in this work only we apply the refinement strategy in direction xyz.

A.2 Approximation of degrees of freedom

The set of degrees of freedom of a current mesh $\mathcal{T}_h$ can be divided based on the definitions of regular nodes and hanging on two different subsets, the set of *regular* degrees of freedom $\mathcal{N}^r$ and the *dependent* degrees of freedom $\mathcal{N}^i$.

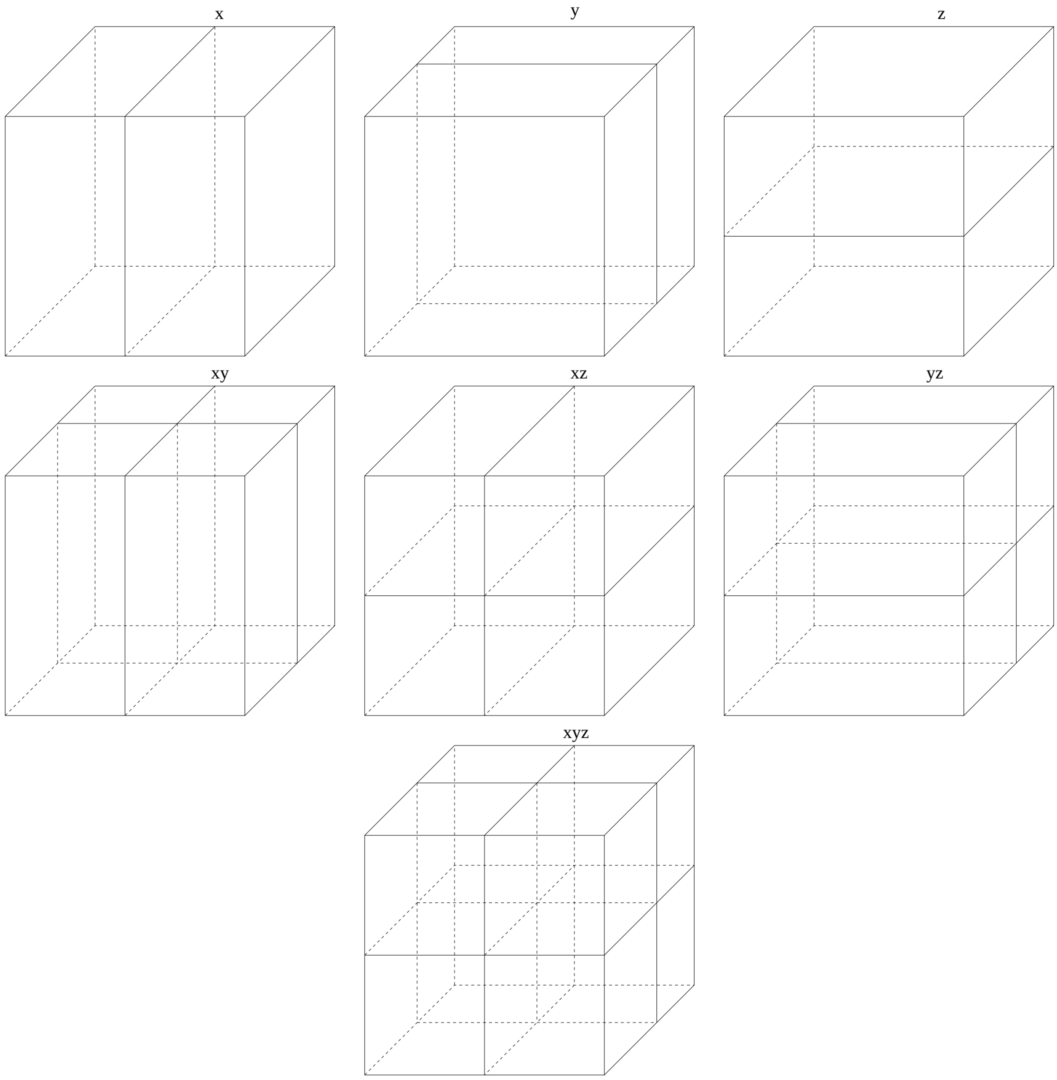


Fig. A.2: 3D: Different possibilities of refinement taking into account the directions.

The set of regular degrees of freedom are the dependent degrees of freedom of the mesh $\mathcal{T}_h$. Let $W_h^1(\Omega)$ the space of linear polynomials on Ω. A base of $W_h^1(\Omega)$ is defined thought the linear functions $\mathbf{v}_{x_i}$ for regular degree of freedom $x_i \in \mathcal{N}^r$ with

$$\mathbf{v}(x_i) = \delta_{i,j} \quad \forall x_j \in \mathcal{N}^d$$

We have the following representation for a function $u_h \in W_h^1(\mathcal{T}_h)$ with $N = \dim W_h^1(\mathcal{T}_h)$ for a regular mesh

$$\mathbf{u}_h(x) = \sum_{i01}^{N} u_k \mathbf{v}_{x_k}(x)$$

where u_k is the value of $\mathbf{u}_h$ in the point x_k.

In a mesh with hanging nodes are not defined in each a shape function. For each dependent degree of freedom $x_i \in \mathcal{N}^i$ there is a set I_j of regular degrees of freedom $x_j \in \mathcal{N}^r$ $(j = 1, \ldots, N^i)$ and a vector w_j with weight, such that

$$\mathbf{u}_{x_i} = \sum_{j=1}^{N^a} w_j \mathbf{u}_{x_j},$$

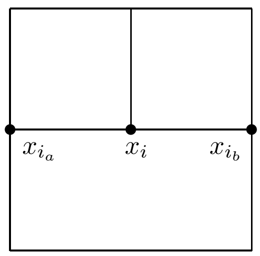


Fig. A.3: Hanging nodes on a boundary face

i.e., the dependent degrees of freedom are linear combination of adjacent owner regular degrees of freedom. For example, for the dependent degree of freedom x_1 in Fig. A.3 we have the representation

$$\mathbf{u}_{x_i} = \frac{1}{2}\mathbf{u}_{x_{i_a}} \mathbf{v}_{x_{i_a}} + \frac{1}{2}\mathbf{u}_{x_{i_b}} \mathbf{v}_{x_{i_b}}$$

with weight $w_{i_a} = w_{i_b} = \frac{1}{2}$ for the endpoint of edge, this situation applies for dependent nodes on the boundary. In the case of a interior face is considered the four points on the face

$$\tilde{\mathbf{u}}_{x_i} = \frac{1}{4} u_{x_{i_a}} \mathbf{v}_{x_{i_a}} + \frac{1}{4} u_{x_{i_b}} \mathbf{v}_{x_{i_b}} + \frac{1}{4}\mathbf{u}_{x_{i_c}} \mathbf{v}_{x_{i_c}} + \frac{1}{4}\mathbf{u}_{x_{i_d}} \mathbf{v}_{x_{i_d}}$$

If the corner of the edge or the face of dependent degree of freedom is also dependent (and this is still the "one-constraint-rule", is sufficient), these be added to the regular degrees of freedom for the determination of the dependent degree of freedom. Their corresponding weights are multiplied.

We have for a function $\mathbf{u} \in W_h^1(\mathcal{T}_h)$ of a triangulation with hanging nodes the following representation

$$\mathbf{u}_h(x) = \sum_{x_i \in \mathcal{N}^r} \mathbf{u}_i \mathbf{v}_i(x) + \sum_{x_j \in \mathcal{N}^i} \mathbf{u}_j \mathbf{v}_j(x) = \sum_{x_i \in \mathcal{N}^r} \mathbf{u}_i \mathbf{v}_i(x) + \sum_{x_j \in \mathcal{N}^r} \sum_{x_k \in I_j} w_k \mathbf{u}_k \mathbf{v}_j(x)$$

from the regular degrees of freedom and a linear combination of regular degrees of freedom of the dependent degrees of freedom.

References

1. M. Ainsworth and B. Guo. Analysis of iterative sub-structuring techniques for boundary element approximation of the hypersingular operator in three dimensions. *Applicable Analysis: An International Journal*, 81(2):241–280, 2002.
2. I. Babuška and A. K. Aziz. Survey lectures on the mathematical foundations of the finite element method. In *The mathematical foundations of the finite element method with applications to partial differential equations (Proc. Sympos., Univ. Maryland, Baltimore, Md., 1972)*, pages 1–359. Academic Press, New York, 1972. With the collaboration of G. Fix and R. B. Kellogg.
3. I. Babuška and M. Suri. The *h-p* version of the finite element method with quasi-uniform meshes. *RAIRO Modél. Math. Anal. Numér.*, 21(2):199–238, 1987.
4. R. E. Bank and R. K. Smith. A posteriori error estimates based on hierarchical bases. *SIAM Journal on Numerical Analysis*, 30(4):921, 1993.
5. J. Bielak and R. C. MacCamy. Symmetric finite element and boundary integral coupling methods for fluid-solid interaction. *Quart. Appl. Math.*, XLIX(1):107–119, 1991.
6. J. Bielak, R. C. MacCamy, and X. Zeng. Stable coupling method for the interface scattering problems by combined integral equations and finite elements. *Journal of computational physics*, 119:374–384, 1995.
7. D. Braess. *Finite elements*. Cambridge University Press, 2007.
8. H. Brakhage and P. Werner. über das dirichletsche außenraumproblem für die helmholtzsche schwingungsgleichung. *Archiv der Mathematik*, 16(1):325–329, Dec. 1965.
9. H. Brakhage and P. Werner. Über das dirichletsche außenraumproblem für die helmholtzsche schwingungsgleichung. *Archiv der Mathematik*, 16(1):325–329, Dec. 1965.
10. A. J. Burton and G. F. Miller. The application of integral equation methods to the numerical solution of some exterior Boundary-Value problems. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 323(1553):201–210, June 1971.
11. C. Carstensen. A posteriori error estimate for the symmetric coupling of finite elements and boundary elements. *Computing*, 57(4):301–322, Dec. 1996.
12. C. Carstensen, S. A. Funken, and E. P. Stephan. On the adaptive coupling of FEM and BEM in 2-d-elasticity. *Numer. Math.*, 77(2):187–221, 1997.

13. C. Carstensen and J. Hu. Hanging nodes in the unifying theory of a posteriori finite element error control. *J. Comput. Math.*, 27(2-3):215–236, 2009.
14. C. Carstensen, M. Maischak, D. Praetorius, and E. Stephan. Residual-based a posteriori error estimate for hypersingular equation on surfaces. *Numerische Mathematik*, 97(3):397–425, May 2004.
15. C. Carstensen and E. P. Stephan. Adaptive coupling of boundary elements and finite elements. *RAIRO Modél. Math. Anal. Numér.*, 29(7):779–817, 1995.
16. C. Carstensen and E. P. Stephan. A posteriori error estimates for boundary element methods. *Mathematics of Computation*, 64(210):483–500, 1995.
17. C. Carstensen and E. P. Stephan. Adaptive boundary element methods for some first kind integral equations. *SIAM Journal on Numerical Analysis*, 33(6):2166, 1996.
18. Y. C. Chang and L. Demkowicz. Solution of viscoelastic scattering problems in linear acoustics using *hp* boundary/finite element method. *International Journal for Numerical Methods in Engineering*, 44(12):1885–1907, 1999.
19. A. Chernov. *Nonconforming boundary elements and finite elements for interface and contact problems with friction.* PhD thesis, Technische Informationsbibliothek u. Universitätsbibliothek, Hannover, 2006.
20. P. G. Ciarlet. *The finite element method fot elliptic problems.* North-Holland, 1978.
21. D. Colton and R. Kress. *Inverse Acoustic and Electromagnetic Scattering Theory.* Springer, 2nd edition, 1998.
22. D. L. Colton and R. Kress. *Integral Equation Methods in Scattering Theory (Pure and Applied Mathematics.* John Wiley & Sons Inc, June 1983.
23. M. Costabel and E. Stephan. A direct boundary integral equation method for transmission problems. *Journal of Mathematical Analysis and Applications*, 106(2):367–413, Mar. 1985.
24. M. Costabel and E. Stephan. Coupling of finite elements and boundary elements for inhomogeneous transmission problems in $\mathbf{R}^3$. In *The mathematics of finite elements and applications, VI (Uxbridge, 1987)*, pages 289–296. Academic Press, London, 1988.
25. M. Costabel and E. P. Stephan. Duality estimates for the numerical solution of integral equations. *Numerische Mathematik*, 54(3):339–353, May 1989.
26. A. G. Dallas. *Analysis of a Limiting-Amplitude Problem in Acousto-Elastic Interactions, 1.* NTIS, Springfield, Va., 1989.
27. R. Dautray and J. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology: Functional and Variational Methods.* Springer, 1st. ed. 1988. 2nd printing edition, Nov. 1999.
28. L. Demkowicz, A. Bajer, W. Rachowicz, and K. Gerdes. 3d hp-adaptive finite element package (3Dhp90). Technical report, University of Texas, nov 2002.
29. L. Demkowicz, K. Gerdes, C. Schwab, A. Bajer, and T. Walsh. HP90: a general and flexible fortran 90 hp -FE code. *Computing and Visualization in Science*, 1(3):145–163, Nov. 1998.

30. L. Demkowicz, J. T. Oden, M. Ainsworth, and P. Geng. Solution of elastic scattering problems in linear acoustics using h-p boundary element methods. *Journal of Computational and Applied Mathematics*, 36(1):29–63, Aug. 1991.
31. J. D.S. Low-frequency scattering by a body in lubricated contact. *Q J Mechanics Appl Math*, 36(1):111–138, Feb. 1983.
32. A. C. Eringen and E. S. Şuhubi and. *Elastodynamics: v2. Linear theory.* Acad. Press, New York, NY [u.a.], 1975.
33. D. Feit and M. C. Junger. *Sound, structures and their interaction.* MIT Press, Cambridge, Mass., 2. ed edition, 1986.
34. G. N. Gatica, G. C. Hsiao, and S. Meddahi. A residual-based a posteriori error estimator for a two-dimensional fluid-solid interaction problem. *Numerische Mathematik*, 114:63–106, 2009.
35. G. C. Gaunaurd. Elastic and acoustic resonance wave scattering. *Applied Mechanics Reviews*, 42(6):143–192, June 1989.
36. W. Hackbusch. *Theorie und Numerik elliptischer Differentialgleichungen.* Teubner Studienbücher Mathematik. [Teubner Mathematical Textbooks]. B. G. Teubner, Stuttgart, second edition, 1996.
37. M. Hamdi and P. Jean. A mixed functional for the numerical resolution of fluid-structure interaction problems. In *Aero- and Hydro Acoustic, IUTAM Symposium, Comte-Bellot, G. and Willians, J.E.*, pages 269–276. Springer-Verlag, 1985.
38. M. A. Hamdi. Une formulation variationnelle par équations intégrales pour la résolution de l'équation de Helmholtz avec des conditions aux limites mixtes. *C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre*, 292(1):17–20, 1981.
39. T. Hargé. Valeurs propres d'un corps élastique. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(13):857–859, 1990.
40. N. Heuer and E. P. Stephan. Iterative substructuring for hypersingular integral equations in $\bbb r^3$. *SIAM Journal on Scientific Computing*, 20(2):739–749, 1998.
41. V. Heuveline and F. Schieweck. On the inf-sup condition for higher order mixed FEM on meshes with hanging nodes. *M2AN*, 41(1):1–20, 2007.
42. H. Holm, M. Maischak, and E. Stephan. The hp-version of the boundary element method for helmholtz screen problems. *Computing*, 57(2):105–134, June 1996.
43. G. Hsiao. On the boundary-field equations methods for fluid-structure interactions. In *Problems and Methods in Mathematical Physics*, pages 79–88. Teubner Veriagsgesellschaft, Stuttgart, Leipzig, 1994.
44. G. Hsiao, R. Kleinman, and G. Roach. Weak solutions of fluid-solid interaction problems. *Math. Nachr.*, 218(1):139–163, 2000.
45. G. Hsiao, R. Kleinman, and L. Schuetz. On variational formulations of boundary value problems for fluid-solid interactions. In *McCarthy, M.F., Hayes, M.A. (eds) Elastic Wave Propagation*, pages 321–326. Elsevier, North-Holland, 1989.
46. G. C. Hsiao and W. L. Wendland. The Aubin-Nitsche lemma for integral equations. *J. Integral Equations*, 3(4):299–315, 1981.

47. G. C. Hsiao and W. L. Wendland. *Boundary integral equations.* Springer, 2008.

48. F. Ihlenburg. *Finite Element Analysis of Acoustic Scattering.* Springer, Berlin, 1 edition, Aug. 1998.

49. C. Johnson and P. Hansbo. Adaptive finite element methods in computational mechanics. *Comput. Methods Appl. Mech. Engrg.*, 101(1-3):143–181, 1992. Reliability in computational mechanics (Kraków, 1991).

50. R. Kress. Minimizing the condition number of boundary integral operators in acoustic and electromagnetic scattering. *Q J Mechanics Appl Math*, 38(2):323–341, May 1985.

51. G. Krishnasamy, L. W. Schmerr, T. J. Rudolphi, and F. J. Rizzo. Hypersingular boundary integral equations: Some applications in acoustic and elastic wave scattering. *Journal of Applied Mechanics*, 57(2):404, 1990.

52. F. Leydecker. *hp-version of the boundary element method for electromagnetic problems.* PhD thesis, Technische Informationsbibliothek u. Universitätsbibliothek, 2006.

53. J. L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications [by] J. L. Lions [and] E. Magenes. Translated from the French by P. Kenneth*, volume I. Springer-Verlag, Berlin, New York,, 1972.

54. C. J. Luke and P. A. Martin. Fluid-solid interaction: acoustic scattering by a smooth elastic obstacle. *SIAM J. Appl. Math.*, 55(4):904–922, 1995.

55. M. Maischak. The analytical computation of the galerkin elements for the laplace, lamé and helmholtz equation in 2d-bem. DFG Schwerpunkt Randelementmethoden, 1995.

56. M. Maischak. The analytical computation of the galerkin elements for the laplace, lamé and helmholtz equation in 3d-bem. DFG Schwerpunkt Randelementmethoden, 1995.

57. M. Maischak, P. Mund, and E. P. Stephan. Adaptive multilevel BEM for acoustic scattering. *Computer Methods in Applied Mechanics and Engineering*, 150(1-4):351–367, Dec. 1997.

58. M. Maischask. *MaiProgs.* Institut fü Angewandte Mathematik, Leibniz Universität Hannover, http://www.ifam.uni-hannover.de/ maiprogs/.

59. R. McCamy and E. Stephan. A boundary element method for an exterior problem for three-dimensional Maxwell's equations. *Applicable Anal.*, 16(2):141–163, 1983.

60. W. Meyer, W. Bell, B. Zinn, and M. Styllybrass. Boundary integral solutions of three dimensional acoustic radiation problems. *Journal of sound and vibration*, 59(2):245–262, 1978.

61. W. Meyer, W. Bell, B. Zinn, and M. Styllybrass. Prediction of the sound field radiated from axisymmetric surfaces. *J. Acoust. Soc. Am*, 65(3):631–638, 1979.

62. P. Mund. *Zwei-Level-Verfahren für Randintegralgleichungen mit Anwendungen auf die nichtlineare FEM-BEM-Kopplung.* Dr. rer nat, Leibniz Hannover Universität, 1997.

63. P. Mund, E. Stephan, and J. W. e. Two-level methods for the single layer potential in r^3. *Computing*, 60(3):243–266, 1998.

64. P. Mund and E. P. Stephan. Adaptive coupling and fast solution of FEM-BEM equations for parabolic-elliptic interface problems. *Math. Methods Appl. Sci.*, 20(5):403–423, 1997.

65. P. Mund and E. P. Stephan. An adaptive Two-Level method for the coupling of nonlinear FEM-BEM equations. *SIAM Journal on Numerical Analysis*, 36(4):1001, 1999.
66. P. Mund and E. P. Stephan. An adaptive two-level method for hypersingular integral equations in $\mathbf{R}^3$. In *Proceedings of the 1999 International Conference on Computational Techniques and Applications (Canberra)*, volume 42, pages C1019–C1033, 2000.
67. J. C. Nédélec. Integral equations with non integrable kernels. *Integral Equations and Operator Theory*, 5:562–572, 1982.
68. J. Oden, L. Demkowicz, W. Rachowicz, and T. Westermann. A posteriori error analysis in finite elements: The element residual method for symmetrizable problems with applications to compressible euler and navier-stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 82(1-3):183–203, Sept. 1990.
69. S. Oestmann. *Fehlerkontrollierte adaptive FEM-BEM Kopplungsmethoden und Anwendungen*. PhD thesis, Technische Informationsbibliothek u. Universitätsbibliothek, 2006.
70. W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling. *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge University Press, 2 edition, Oct. 1992.
71. W. Rachowicz, J. T. Oden, and L. Demkowicz. Toward a universal h-p adaptive finite element strategy part 3. design of h-p meshes. *Computer Methods in Applied Mechanics and Engineering*, 77(1-2):181–212, Dec. 1989.
72. P. Raviart and V. Girault. *Finite element methods for Navier-Stokes equations*. Springer series in computational mathematics : SCM, ISSN 0179-3632. Springer, Berlin [u.a.], 1986.
73. S. Sauter and C. Schwab. *Randelementmethoden: Analyse, Numerik und Implementierung schneller Algorithmen*. Vieweg+Teubner, 1 edition, June 2004.
74. D. Schötzau, C. Schwab, and R. Stenberg. Mixed *hp*-fem on anisotropic meshes. ii: Hanging nodes and tensor products of boundary layer meshes. *Numer. Math.*, 83(4):667–697, 1999.
75. E. P. Stephan. A boundary integral equation procedure for the mixed boundary value problem of the vector helmholtz equation. *Applicable Analysis: An International Journal*, 35(1):59–74, 1990.
76. E. P. Stephan, M. Maischak, and T. Tran. Domain decomposition algorithms for an indefinite hypersingular integral equation in three dimensions. In *Domain decomposition methods in science and engineering XVII*, volume 60 of *Lect. Notes Comput. Sci. Eng.*, pages 647–655. Springer, Berlin, 2008.
77. E. P. Stephan and M. Suri. The *h-p* version of the boundary element method on polygonal domains with quasiuniform meshes. *RAIRO Modél. Math. Anal. Numér.*, 25(6):783–807, 1991.
78. J. R. Stewart and T. J. R. Hughes. An a posteriori error estimator and *hp*-adaptive strategy for finite element discretizations of the Helmholtz equation in exterior domains. *Finite Elem. Anal. Des.*, 25(1-2):1–26, 1997. Adaptive meshing, Part 1.
79. T. Tran, E. p. Stephan, and P. Mund. Hierarchical basis preconditioners for first kind integral equations. *Applicable Analysis*, 65(3):353–372, 1997.
80. R. Verfürth. *A Review of A Posteriori Error Estimation and Adaptive Mesh- Refinement Techniques*. Teubner Verlag, 1996.

81. W. L. Wendland. On asymptotic error estimates for combined BEM and FEM. In *Finite element and boundary element techniques from mathematical and engineering point of view*, volume 301 of *CISM Courses and Lectures*, pages 273–333. Springer, Vienna, 1988.

82. H. Yserentant. On the multi-level splitting of finite element spaces. *Numerische Mathematik*, 49(4):379–412, July 1986.

83. X. Zeng, J. Bielak, and R. C. MacCamy. Unified symmetric finite element and boundary integral variational coupling methods for linear fluid-structure interaction. *Numer. Methods Partial Differential Equations*, 8(5):451–467, 1992.

84. X. Zhang. Multilevel schwarz methods. *Numerische Mathematik*, 63(1):521–539, Dec. 1992.

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