Alternative approaches to maximally supersymmetric field theories

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Zusammenfassung

Diese Arbeit beschäftigt sich mit der Untersuchung und Anwendung alternativer Beschreibungsmöglichkeiten für maximal supersymmetrische Feldtheorien in vier Dimensionen: $\mathcal{N}$=4 Super-Yang-Mills-Theorie und $\mathcal{N}$=8 Supergravitation.

Während die Twistorstringtheorie Baumgraphen in der $\mathcal{N}$=4 Super-Yang-Mills-Theorie beschreiben kann, ist für die $\mathcal{N}$=8 Supergravitation keine derartige Formulierung bekannt. Zwar enthält die Twistorstringtheorie neben dem $\mathcal{N}$=4 Super-Yang-Mills(SYM)-Teil noch weitere Vertexoperatoren, doch beschreiben diese die Zustände in einer $\mathcal{N}$=4 konformen Supergravitation und müssen modifiziert werden, um für die Beschreibung einer Einsteinschen Supergravitation geeignet zu sein. Eine veränderte Version der Twistorstringtheorie, in der die konforme Symmetrie für die gravitativen Vertexoperatoren gebrochen wird, ist kürzlich vorgeschlagen worden. Der erste Teil der Dissertation diskutiert strukturelle Aspekte und die Konsistenz der modifizierten Theorie. Dabei zeigt sich, dass der Großteil der Amplituden nicht konsistent konstruiert werden kann. Dies wird darauf zurückgeführt, dass die Modifikation der Theorie die Dimension des Modulraumes der algebraischen Kurven im Twistorraum auf unzulässige Weise reduziert.

Die Frage einer möglichen Endlichkeit der $\mathcal{N}$=8 Supergravitation ist eng mit der Existenz gültiger Counterterme in der Störungsentwicklung der Theorie verknüpft. Der zum sogenannten R^4-Counterterm gehörende Vorfaktor ist kürzlich in einer expliziten Rechnung zu null bestimmt worden. Dieses Verhalten weist darauf hin, dass die verwendete Formulierung eine Symmetrie der Theorie nicht berücksichtigt. Eine der möglicherweise vernachlässigten Symmetrien ist die versteckte $E_{7(7)}$-Symmetrie. Für das Auftreten dieser Symmetrie in einer Theorie ist die Gültigkeit der doppelt-weichen skalaren Limes-Relation notwendig. Im zweiten Teil der Dissertation werden mit Hilfe der Stringtheorie die Amplituden für eine durch Hinzufügen eines R^4-Counterterms veränderte Supergravitationswirkung berechnet, um die Gültigkeit zu überprüfen. Es wird gezeigt, dass aus dem doppelt-weichen Limes keine $E_{7(7)}$-Einschränkungen an den R^4-Counterterm hergeleitet werden können. Entgegen der Erwartung für eine $E_{7(7)}$-symmetrische Theorie verschwindet der einfach-weiche skalare Limes der Amplituden jedoch nicht. Dies legt nahe, dass die $E_{7(7)}$-Symmetrie durch den R^4-Counterterm gebrochen wird.

Der dritte Teil der Dissertation beschäftigt sich mit der Grassmannschen Formulierung der $\mathcal{N}$=4 SYM-Theorie. Jede Amplitude in der $\mathcal{N}$=4 SYM-Theorie kann als Linearkombination bestimmter infrarot(IR)-divergenter Integrale ausgedrückt werden. Die Koeffizienten dieser Integrale, die führenden Singularitäten, bestimmen die Struktur der Amplituden vollständig. Aus Feldtheorierechnungen ist bekannt, dass die führenden Singularitäten nicht voneinander unabhängig, sondern durch die sogenannten IR-Gleichungen verknüpft sind. Weiterhin vermutet man, dass die führenden Singularitäten sich als Linearkombinationen von Residuen eines mehrdimensionalen komplexen Integrals in der alternativen Grassmannschen Formulierung darstellen lassen. Diese Residuen sind ebenfalls nicht unabhängig, sondern durch verallgemeinerte Formen des Cauchyschen Satzes, die verallgemeinerte Residuentheoreme, miteinander verknüpft. Beispiele weisen darauf hin, dass die IR-Gleichungen in der Sprache der Residuen aus den verallgemeinerten Residuentheoremen folgen. Es wird gezeigt, dass die verallgemeinerten Residuentheoreme in der Grassmannschen Formulierung nicht nur mit den IR-Gleichungen korrespondieren, sondern mit einem größeren Satz von Bedingungen, der aus Betrachtungen zur dualen konformen Anomalie von Ein-Schleifen-Amplituden hergeleitet werden kann. Eine explizite Form der Abbildung sowohl zwischen den dualen konformen Bedingungen als auch den IR-Gleichungen wird hergeleitet und diskutiert.

Abstract

The central objective of this work is the exploration and application of alternative possibilities to describe maximally supersymmetric field theories in four dimensions: $\mathcal{N}$=4 super Yang-Mills theory and $\mathcal{N}$=8 supergravity.

While twistor string theory has been proven very useful in the context of $\mathcal{N}$=4 SYM, no analogous formulation for $\mathcal{N}$=8 supergravity is available. In addition to the part describing $\mathcal{N}$=4 SYM theory, twistor string theory contains vertex operators corresponding to the states of $\mathcal{N}$=4 conformal supergravity. Those vertex operators have to be altered in order to describe (non-conformal) Einstein supergravity. A modified version of the known open twistor string theory, including a term which breaks the conformal symmetry for the gravitational vertex operators, has been proposed recently. In a first part of the thesis structural aspects and consistency of the modified theory are discussed. Unfortunately, the majority of amplitudes can not be constructed, which can be traced back to the fact that the dimension of the moduli space of algebraic curves in twistor space is reduced in an inconsistent manner.

The issue of a possible finiteness of $\mathcal{N}$=8 supergravity is closely related to the question of the existence of valid counterterms in the perturbation expansion of the theory. In particular, the coefficient in front of the so-called R^4 counterterm candidate has been shown to vanish by explicit calculation. This behavior points into the direction of a symmetry not taken into account, for which the hidden on-shell $E_{7(7)}$ symmetry is the prime candidate. The validity of the so-called double-soft scalar limit relation is a necessary condition for a theory exhibiting $E_{7(7)}$ symmetry. By calculating the double-soft scalar limit for amplitudes derived from an $\mathcal{N}$=8 supergravity action modified by an additional R^4 counterterm, one can test for possible constraints originating in the $E_{7(7)}$ symmetry. In a second part of the thesis, the appropriate amplitudes are calculated employing the low-energy limit of string theory, and the double-soft limit relation is indeed shown to hold. However, if the modified action has $E_{7(7)}$ symmetry, the single-soft scalar limit of any amplitude should vanish. This not being the case suggests that the $E_{7(7)}$ symmetry is broken by the R^4 counterterm.

Finally, the Grassmannian formulation of $\mathcal{N}$=4 SYM is investigated in a third part of the thesis. Any amplitude in $\mathcal{N}$=4 SYM theory can be expressed as a linear combination of certain infrared (IR) divergent integrals. Being known as leading singularities, the coefficients of these integrals completely determine the structure of an amplitude. From field-theory calculations it is known that the leading singularities are not independent, but are subject to a set of so-called IR equations. The alternative Grassmannian formulation is conjectured to describe the leading singularities as certain linear combinations of residues of a multidimensional complex integral. These residues are not independent but are related by generalized residue theorems (GRTs), which are multidimensional generalizations of Cauchy's theorem. Indeed, expressing the leading singularities known from field-theory calculations in terms of these residues supports the conjecture that the IR equations can be derived from GRTs. Here it is shown that GRTs in the Grassmannian formulation do not only give rise to IR equations, but to a larger set of constraints, which can be derived by considering the dual conformal anomaly of one-loop amplitudes. Explicit maps between GRTs and both, dual conformal constraints and IR equations, are deduced and discussed.

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BCFW	Britto, Cachazo, Feng and Witten
IR	infrared
GRT	generalized residue theorem
KLT	Kawai, Lewellen and Tye
(N)MHV	(next-to-) maximally helicity violating
SUSY	supersymmetry
SWI	supersymmetric Ward identity
SYM	super Yang-Mills

1 Introduction

Whenever today's particle physicists make predictions for scattering experiments, they use the *standard model* of particle physics, which in turn is based on *special relativity*. The standard model is a quantum theoretical framework, which incorporates three of the four known fundamental forces, the electromagnetic, weak and strong force. In contrast, *general relativity* describes the fourth force, gravitation, completely classically.

Both approaches, the standard model and general relativity, are very successful. All predictions drawn from those theories are in concordance with any experiment performed so far. While it would already be very natural to treat all four forces in a unified framework from an aesthetic point of view, there is also a physical necessity to search for a quantum theory of gravity. Classical general relativity fails to describe gravity at very high energy densities, which occur shortly after the big bang or at very small distances.

Since the standard model of particle physics is formulated in the language of quantum field theory, a quantum field theory of gravity would be desirable. Unfortunately, gravitation resists any naïve attempt to be incorporated into this framework. The quantum field theory analogue of Einstein's general relativity features an infinite number of divergences, which would have to be removed by an infinite number of renormalization parameters. A quantum field theory exhibiting this behavior does not lead to physically sensible predictions and is called *unrenormalizable*.

The probably most advanced concept incorporating gravity in a quantum-theoretical framework arose in the seventies: *string theory*. While already showing beautiful signatures of a unified theory, string theories come with a number of technical drawbacks, the most famous one being the requirement of living in more than four spacetime dimensions. Although this can be cured by a process called compactification, a distinguished string theory framework which reproduces the standard model and general relativity is still lacking.

However, there is a class of theories which are related to both, the unified string theory framework and usual field theories: *maximally supersymmetrically extended field theories*. In four dimensions they are called $\mathcal{N}$=4 super Yang-Mills theory for particles up to spin one, while the gravitational version for particles up to spin two is referred to as $\mathcal{N}$=8 supergravity. Arising as supersymmetrized versions of usual Yang-Mills theory and general relativity respectively, these theories can on the other hand be shown to agree with the low-energy limit of certain string theories.

While $\mathcal{N}$=4 super Yang-Mills theory is a consistent quantum field theory, the status of $\mathcal{N}$=8 supergravity is questionable: it is suspected to be incomplete as a quantum theory due to being non-renormalizable by power-counting. Although power-counting arguments deliver rather reliable hints for other quantum field theories, they do not seem to be sufficient in this context. Explicit calculations for certain amplitudes show seemingly unrelated divergences to cancel miraculously, which points into the direction of a renormalizable (or even finite) theory. Proving finiteness of $\mathcal{N}$=8 supergravity would render it the first consistent quantum field theory of gravity.

The nature of the two maximally supersymmetric theories is quite different:

- $\mathcal{N}$=4 super Yang-Mills (SYM) theory in four dimensions is one of the best explored interacting quantum field theories. Known since the late seventies, it has been proven to be a consistent quantum field theory free of problematic divergences. Witten's twistor string theory description triggered the discovery of a variety of new features during recent years. In particular, a connection of the usual and a new dual superconformal symmetries have been found to jointly represent the Yangian symmetry of $\mathcal{N}$=4 SYM. A novel description, the Grassmannian formulation, makes these symmetries manifest and was proposed in 2009. Very recently it has been shown that the Grassmannian formulation is implied by Yangian symmetry and closely related to other descriptions, *e.g.* the link representation and Hodges twistor diagrams. In addition to explaining the stunning simplicity of amplitudes, the Grassmannian mechanism supports the conjecture that all amplitudes in the whole theory are completely determined by their correct analytical behavior and symmetries.

- Although found only shortly after $\mathcal{N}$=4 SYM theory, $\mathcal{N}$=8 supergravity is less well explored due to its algebraic complexity. Calculations in the current spacetime formulation are cumbersome, but the results turn out to be very simple. This usually hints at a symmetry of the theory not being accounted for in the formalism employed. While the famous hidden E_7 symmetry of $\mathcal{N}$=8 supergravity is one of the candidate symmetries which could help to explain the astonishing simplicity of amplitudes, it has also been discussed recently, whether perhaps even this symmetry group needs to be extended to account for all symmetries.

  Both observations, the simplicity of the amplitudes and the miraculous cancellations mentioned above, are neither visible nor transparent in the current local spacetime formulation of $\mathcal{N}$=8 supergravity. Parallel to the situation in $\mathcal{N}$=4 SYM theory, it is suggestive to assume that amplitudes of the theory are fixed by their analytic behavior and symmetry to a large extent.

In this thesis, alternative approaches to both $\mathcal{N}$=4 SYM and $\mathcal{N}$=8 supergravity shall be investigated and employed. This will be done in three projects listed below.

- While twistor string theory has been proven very useful in the context of $\mathcal{N}$=4 SYM, no analogous formulation for $\mathcal{N}$=8 supergravity is available. However, twistor string theory is known to contain states corresponding to the particle content of linearized conformal supergravity. A modification for the twistor string theory which breaks the conformal invariance has been suggested and the resulting theory has been proposed to describe $\mathcal{N}$=8 supergravity. While this description seems to reproduce one particular amplitude in $\mathcal{N}$=8 supergravity, further calculations had not been performed initially. Motivated by the inconsistency of higher-point amplitude calculations, the structure of the modified theory is investigated. The problem can be traced back to overconstraining the moduli space of the algebraic curves in twistor space, which support the spacetime amplitudes. Furthermore, the constraints are shown to lead to trivial multiplets, which render the theory physically meaningless.

Nevertheless, one particular amplitude in conformal supergravity can be shown to consistently result in the known expression from $\mathcal{N}$=8 supergravity without the modification breaking conformal invariance. This raises the question for other amplitudes to be accessible in the framework of twistor string theory.

- The question of finiteness of $\mathcal{N}$=8 supergravity is closely related to the existence of suitable counterterms in the perturbative expansion of the theory. One example for the miraculous cancellations mentioned above is the vanishing of the coefficient for an $\mathcal{R}^4$ counterterm in $\mathcal{N}$=8 supergravity. Although shown by explicit calculations, the reason for the cancellations has not yet been understood. Since the $\mathcal{R}^4$ term respects $\mathcal{N}$=8 supersymmetry, this result poses the question of whether another symmetry could be responsible for the miraculous cancellations of the three-loop divergences. In the second project of this thesis, possible restrictions arising from the non-compact part of $E_{7(7)}$ symmetry are discussed. This symmetry can be accessed by investigating scattering amplitudes involving scalars. If the momentum of a scalar goes to zero, it is referred to as a *soft scalar*. It is this process of almost vanishing momenta, which gives access to the coset symmetry. In particular, a double-soft scalar limit relation derived recently has to be satisfied in order for the theory to be compatible with $E_{7(7)}$ symmetry. Calculations involving matrix elements derived from an action with an $\mathcal{R}^4$ counterterm are difficult to perform. In order to circumvent these problems, one can make use of the fact that the $\mathcal{R}^4$ term occurs as leading correction in the low-energy expansion of closed-string tree-level amplitudes. Although the considered matrix elements obey the double-soft scalar relation, they do not show the correct behavior in the single-soft limit. The expected vanishing by the action of $E_{7(7)}$ symmetry does not occur, thus questioning the $E_{7(7)}$ compatibility of the R^4 counterterm.

- Various intricate relations between amplitudes have been derived in $\mathcal{N}$=4 SYM theory. In particular, coefficients in the box expansion of one-loop amplitudes are known to satisfy certain infrared consistency conditions, which can be derived from demanding cancellations of infrared divergences in dimensionally regularized box integrals. Recently, infrared (IR) equations have been shown to be implied by the even larger set of dual conformal constraints, which originate in the necessity of cancellations of anomalies of the dual superconformal symmetry. The infrared structure of $\mathcal{N}$=4 SYM amplitudes has been conjectured to be geometrically represented by a generalization of Cauchy's theorem in the Grassmannian formulation of $\mathcal{N}$=4 SYM theory. Restricting the attention to next-to-maximally helicity-violating amplitudes, certain examples have been considered for one-loop IR equations and shown to indeed correspond to particular combinations of generalized Cauchy theorems in a Grassmannian geometry. In this third project, the precise mapping of not only one-loop IR equations but also dual superconformal constraints onto a certain class of generalized residue theorems is found and discussed. Although the investigated structure is speculated to contain information about the two- and higher loop infrared behavior of $\mathcal{N}$=4 super Yang-Mills theory in the NMHV sector, the lack of an integral basis at two and higher loops as well as the missing higher-loop field-theory calculations prevent the establishment of a map beyond one-loop level.

After discussing basic concepts of supersymmetric field theories in the introductory section 2, several alternatives to the usual spacetime description of these theories are introduced in section 3. Section 4 is devoted to the investigation of the possible twistor-string description of $\mathcal{N}$=8 supergravity, followed by the study of the constraints from the hidden E_7 symmetry on the appearance of a possible $\mathcal{R}^4$ counterterm in $\mathcal{N}$=8 supergravity in section 5. In section 6 the connection between one-loop infrared equations in $\mathcal{N}$=4 super-Yang-Mills theory and generalized residue theorems in its Grassmannian formulation are explored. Finally, the thesis is concluded in section 7.

2 Supersymmetric field theories

Starting with fields, symmetries and the S-matrix, this introductory section shall disscuss the necessary building blocks for maximal supersymmetric field theories and set the conventions used below. After the spinor helicity formalism is described in subsection 2.2, Lie algebras are explained and extended with supersymmetry in order to be applied to supersymmetric Ward identities lateron. Finally, basic properties of the two maximally supersymmetric field theories in four dimensions, $\mathcal{N}$=4 super Yang-Mills theory and $\mathcal{N}$=8 supergravity, are discussed in subsections 2.6 and 2.7 respectively.

2.1 Field theories

2.1.1 Fields and symmetries

Let x^μ with $\mu \in \{0, \ldots, d-1\}$ be local coordinates parameterizing a real manifold $\mathbb{R}^d$. A ***physical theory*** is defined to be a collection of local fields $\Phi = \{\Phi_1(x^\mu), \ldots, \Phi_n(x^\mu)\}$ whose dynamics, the explicit dependence on the coordinates, is governed by the equations of motion. For all theories considered in this thesis, the equations of motion can be derived by the variational principle from an action functional $S[\Phi]$.

In the language of differential geometry fields are defined to be sections of tensor bundles over the manifold $\mathbb{R}^d$. In particular, a scalar field is a map from the manifold to the real numbers

$$\varphi : \mathbb{R}^d \quad \rightarrow \quad \mathbb{R} \tag{2.1}$$

while a vector or tensor field are mappings to the tangent or cotangent spaces:

$$\begin{aligned} A_\mu : \mathbb{R}^d \quad &\rightarrow \quad T^*\mathbb{R}^d \\ g_{\mu\nu} : \mathbb{R}^d \quad &\rightarrow \quad T^*\mathbb{R}^d \otimes T^*\mathbb{R}^d. \end{aligned} \tag{2.2}$$

Symmetries of a theory map one set of classical field configurations Φ to another set Φ', which is again a solution to the equations of motion. There are two types of symmetries of an action functional S: ***external*** or ***spacetime symmetries*** act on the fields Φ and can be compensated by choosing a different set of local coordinates x^μ. On the contrary, ***internal symmetries*** act on the fibers of the tensor bundle and not on the manifold directly.

A symmetry is called ***global***, if the same transformation is applied for all points $x^\mu \in \mathcal{M}^d$ and ***local***, if the symmetry is parameterized by x^μ. The process of converting a global symmetry into a local one is referred to as ***gauging***.

Considering the space of all possible symmetry transformations

$$\kappa : \Phi \rightarrow \Phi' = \kappa[\Phi] \tag{2.3}$$

it should have an associative product structure with unit element (the trivial transformation) and inverse element (the backward transformation). One important example, which in addition requires the symmetry to be continuous, are the Lie groups discussed in subsection 2.3.

An object X, which a symmetry transformation can act on is called

- ***invariant*** under a symmetry action κ if

$$\kappa[X] = X \tag{2.4}$$

- ***covariant*** under a symmetry action κ, if the induced action on X is (multi)linear.

A physical theory is called invariant under a symmetry transformation, if its equation of motion are covariant. This is equivalent to invariance of the action $S[\Phi]$ up to total derivatives in the lagrangian and a constant rescaling.

Not all symmetries need to be manifest in the action S of a theory. For example it is not always possible to find representations for the fields which satisfy all symmetry relations explicitly. In particular for theories with a very rich symmetry structure there are several formulations, which make different symmetries manifest. If a symmetry transformations leaves the action invariant only after employing the equations of motion, this symmetry is called ***on-shell*** or ***hidden*** symmetry. Analogously, the symmetries which do not require the equations of motion to leave the action invariant are referred to as ***off-shell***.

2.1.2 Amplitudes and S-matrix

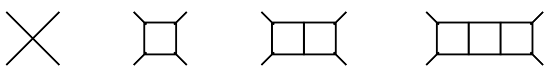
In order to relate a physical theory to experiments, one can calculate ***amplitudes*** from the collection of fields and the equations of motion originating in the action S. An amplitude is a functional depending on the external momenta and the type of the particles, which take part in the scattering process:

$$\mathcal{A}(1_{\text{in}}, 2_{\text{in}}, \ldots, n_{\text{in}}, 1_{\text{out}}, \ldots, m_{\text{out}}) \quad \rightarrow \quad \mathbb{C}\,. \tag{2.5}$$

The square of the absolute value of a particular amplitude is proportional to the probability for the corresponding scattering process to take place. Labelling the different scattering processes by their ingoing and outgoing particles, one can organize those probabilities in a ***S-matrix*** $\mathcal{S}$. Its elements, $\mathcal{S}_{\text{out}|\text{in}}$, contain just the momentum dependent part of eq. (2.5) accompanied by a momentum conserving δ-function [1].

In the arguments of amplitude expressions below, no distinction will be made between ingoing and outgoing particles. Instead, outgoing particles are treated as ingoing but with their momenta reversed.

Amplitudes can be determined from the action $S[\Phi]$ by the Feynman path integral approach [2]. The resulting perturbative expansion can be graphically represented in terms of Feynman diagrams, in which the order in perturbation theory corresponds to the number of closed loops. A general four-particle amplitude can be depicted in the following way:



$$\mathcal{A} \quad = \quad \mathcal{A}^{\text{tree}} + \quad \mathcal{A}^{1loop} \quad + \mathcal{A}^{2loop} + \text{ higher loops } \quad \cdots\,. \tag{2.6}$$

The leading contributions are called tree amplitudes while higher amplitudes are referred to by their number of loops.

Amplitudes can be considered for special kinematical configurations, which are called soft and collinear limit. The soft limit refers to the situation in which the four-momentum of one particle taking part in the scattering process goes to zero. In this case, one can show for the highly symmetric theories discussed below that the amplitude factorizes [3, 4]

$$\mathcal{A}(1,\dots,n) \stackrel{p_n \to 0}{=} f_{\text{soft}} \cdot \mathcal{A}(1,\dots,n-1)\,, \tag{2.7}$$

where f_{soft} is the soft factor. Similarly, if the momenta of two participating particles a and b become collinear

$$p_{\text{coll}} = p_a + p_b, \quad p_a \to z p_{\text{coll}},\; p_b \to (1-z)p_{\text{coll}},\; z \in [0,1] \tag{2.8}$$

one finds

$$\mathcal{A}_n(\dots,a,b,\dots) \stackrel{p_a || p_b}{=} f_{\text{coll}} \cdot \mathcal{A}_{n-1}(\dots,p_{\text{coll}},\dots)\,, \tag{2.9}$$

where f_{coll} is the splitting factor. The soft and collinear limit of amplitudes will be referred to as analytic behavior below.

Whereas symmetries of a theory leave the action $S[\Phi]$ invariant up to a total derivative and constant rescaling, the corresponding amplitudes are annihilated by acting with the (appropriate form of) symmetry generators. In particular, for a theory invariant under a symmetry with generator κ, the amplitudes satisfy

$$\kappa\, \mathcal{A}(1,\dots,n) = \mathcal{A}(\kappa[1],2,\dots,n) + \mathcal{A}(1,\kappa[2],\dots,n) + \dots + \mathcal{A}(1,\dots,\kappa[n]) = 0\,. \tag{2.10}$$

and thereby relate certain entries in the S-matrix, thus decreasing the number of independent elements. One particularly important example in the context of maximally supersymmetric field theories are the supersymmetric Ward identities discussed in subsection 2.5.

Symmetries of amplitudes have been the starting point for an approach called ***analytic S-matrix*** at the end of the sixties [5]. Since those symmetries reduce the number of independent elements in the S-matrix and furthermore certain entries are related by analytic relations eqs. (2.7) and (2.9), the question was raised, whether the whole S-matrix could be fixed by those constraints. Initially designed to yield an analytic S-matrix for the strong interaction, this idea did not prove a useful concept in this context: it is a very hard task to approach a realistic quantum field theory from this side. However, if it comes to theories which are strongly constrained by symmetries, an analytic S-matrix seems accomplishable. In particular for the $\mathcal{N}$=4 super Yang-Mills theory discussed in subsection 2.6 below, it is believed that symmetries and analytic properties are strong enough to finally determine the precise form of any amplitude in this theory completely.

2.2 Spinor helicity formalism

Most of the explicit expressions for amplitudes in four dimensional maximally supersymmetric field theories below will be presented in the spinor helicity formalism [6, 7, 8], which is responsible for the extremely compact expressions.

The universal cover of the Lorentz group $SO(p,q)$ is the spin group $Spin(p,q)$. For different signatures in four dimensions the universal covers are

$$\begin{aligned} SO(4) &\to Spin(4) \cong SU(2)\times SU(2) \\ SO(1,3) &\to Spin(1,3) \cong SL(2,\mathbb{C}) \\ SO(2,2) &\to Spin(2,2) \cong SL(2,\mathbb{R})\times SL(2,\mathbb{R})\,. \end{aligned} \tag{2.11}$$

Focussing on four dimensional spacetime with Minkowski signature below, a *spinor* of the Lorentz group $SO(1,3)$ transforms in a representation of the corresponding spin group $Spin(1,3)$. Starting from the Clifford algebra $\mathfrak{cl}(1,3)$ of γ-matrices

$$\gamma_0 \;=\; \begin{pmatrix} \mathbb{1} & \mathbf{0} \\ \mathbf{0} & -\mathbb{1} \end{pmatrix} \qquad \gamma_\nu \;=\; \begin{pmatrix} \mathbf{0} & \sigma_i \\ -\sigma_i & \mathbf{0} \end{pmatrix}\,, \tag{2.12}$$

a representation of the algebra $\mathfrak{spin}(1,3)$ can be obtained via

$$\Sigma_{\mu\nu} = -\frac{i}{4}[\gamma_\mu, \gamma_\nu]\,, \tag{2.13}$$

where the matrices σ_ν are the 2×2 Pauli matrices accompanied by the unit matrix

$$\sigma_0 = \mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.14}$$

The four-dimensional *Dirac* representation obtained in this way is reducible and can be split into two *Weyl representations*, which are distinguished by their positive or negative eigenvalue of the chirality operator

$$\gamma_5 \;= i\,\gamma_0\gamma_1\gamma_2\gamma_3 \;=\; \begin{pmatrix} \mathbf{0} & \mathbb{1} \\ \mathbb{1} & \mathbf{0} \end{pmatrix}. \tag{2.15}$$

Correspondingly, projection operators for the left-handed and right-handed Weyl representations read

$$R \;=\; \frac{\mathbb{1}+\gamma_5}{2}\,, \quad L = \frac{\mathbb{1}-\gamma_5}{2}\,. \tag{2.16}$$

Spinor indices for the two Weyl representations are $A = 1,2$ and $A' = 1', 2'$, which are raised and lowered with the two-dimensional antisymmetric tensors ε_{AB} and $\varepsilon_{A'B'}$. Conventionally $\varepsilon_{AB} = \varepsilon_{A'B'} = i\sigma_2$ with $\varepsilon^{AB}\varepsilon_{BC} = \delta^A_C$ and analogously for the primed tensor. In spinor language the Minkowski metric diag$(+,-,-,-)$ can be expressed as

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \varepsilon_{AB}\varepsilon_{A'B'}\mathrm{d}x^{AA'}\mathrm{d}x^{BB'}\,. \tag{2.17}$$

Depending on the number of dimensions [9], it is possible to either additionally or alternatively impose a Majorana condition on the Dirac spinor χ, which demands it to equal its charge conjugate

$$\chi^c = \chi\,. \tag{2.18}$$

In four dimensions,

$$\chi^c = C\gamma_0\chi^* \,, \tag{2.19}$$

where C denotes the charge conjugation operator

$$C = \begin{pmatrix} \varepsilon_{AB} & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \tag{2.20}$$

satisfying

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T \qquad \text{and} \qquad C\gamma_0(C\gamma_0)^* = \mathbb{1} \,. \tag{2.21}$$

Spinors can be either Majorana or Weyl but not both in four dimensions. While Majorana spinors are used in the derivation [10] of the action eq. (2.71) of $\mathcal{N}$=4 SYM theory below, the remainder of this subsection will be concerned with Weyl spinors exclusively.

According to the above discussion, any Lorentz vector index ν can be decomposed into spinor indices

$$\not{p}^{AA'} = p^\nu \sigma_\nu^{AA'} \,, \tag{2.22}$$

where for a general vector

$$\not{p}^{AA'} = \rho^A \rho^{A'} + \sigma^A \sigma^{A'} \tag{2.23}$$

with ρ and σ being commuting Weyl spinors. If the vector p^ν is real, ρ^A and $\sigma^{A'}$ are related to $\rho^{A'}$ and σ^A by complex conjugation. If p^ν is a null-vector, the determinant of $\not{p}$ vanishes

$$p^\nu\, p_\nu = \det(\not{p}^{AA'}) = 0 \quad \text{and therefore} \quad \operatorname{rank}(\not{p}^{AA'}) < 2 \,. \tag{2.24}$$

Thus the matrix $\not{p}$ can be decomposed into spinors π_A and $\bar{\pi}^{A'}$ transforming in representations of $SL(2,\mathbb{C})$ and a second copy $\overline{SL(2,\mathbb{C})}$ respectively

$$\not{p}_{AA'} = \pi_A \bar{\pi}_{A'} \,, \tag{2.25}$$

where the two spinors are related by complex conjugation

$$\pi_A = (\bar{\pi}^{A'})^* \,. \tag{2.26}$$

Correspondingly, for a momentum null-vector p^ν, the massless Dirac equation splits into two pieces

$$\not{p}^{AA'} \mu_A(p) = 0 \qquad \text{and} \qquad \not{p}_{AA'} \bar{\mu}^{A'}(p) = 0. \tag{2.27}$$

Noting that $\mu^A \mu_A = 0$ and $\bar{\mu}_{A'} \bar{\mu}^{A'} = 0$ due to the antisymmetry of the spinor product, solutions to eq. (2.27) have to be proportional to π_A and $\pi^{A'}$

$$\pi_A \sim \mu_A \qquad \text{and} \qquad \pi^{A'} \sim \mu^{A'} \,. \tag{2.28}$$

A convenient normalized choice for the spinors μ_A and $\bar{\mu}^{A'}$ satisfying eq. (2.27) is

$$\mu_A(p) = \begin{pmatrix} \sqrt{p^+} \\ \sqrt{p^-} e^{i\varphi} \end{pmatrix} \qquad \text{and} \qquad \bar{\mu}^{A'}(p) = \begin{pmatrix} \sqrt{p^+} \\ \sqrt{p^-} e^{-i\varphi} \end{pmatrix} , \tag{2.29}$$

where

$$e^{\pm i\varphi} = \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} = \frac{p^1 \pm ip^2}{\sqrt{p^+p^-}}, \quad p^\pm = p^0 \pm p^3 \,. \tag{2.30}$$

The spinors μ_A and $\bar{\mu}^{A'}$ are related to the positive and negative energy solutions $u(p)$ and $v(p)$ of the massless Dirac equation in four dimensions, where the conventions from [1] are used. Since the projection operators for positive and negative energy

$$P_{\text{pos}}(p) \sim u(p) \otimes \overline{u(p)} \qquad \text{and} \qquad P_{\text{neg}}(p) \sim v(p) \otimes \overline{v(p)} \tag{2.31}$$

are both proportional to $\not{p}$, the solutions of definite helicity

$$u_+(p) = Ru(p),\, u_-(p) = Lu(p),\, v_+(p) = Lv(p) \qquad \text{and} \qquad v_-(p) = Rv(p) \tag{2.32}$$

and their conjugates can be chosen equally

$$u_\pm(p) = v_\mp(p) \qquad \text{and} \qquad \overline{u_\pm(p)} = \overline{v_\mp(p)} \,. \tag{2.33}$$

Comparing eq. (2.32) with the definition of the Weyl representation above, one can identify the solutions to eq. (2.27) with the momentum spinors of the corresponding particles. Assuming the particles taking part in the scattering process to have null-momenta $p_i, i = 1, \ldots, n$ and using the notations

$$\begin{aligned} |i^\pm\rangle \;\equiv\; |p_i^\pm\rangle \;\equiv\; u_\pm(p_i) \;=\; v_\mp(p_i), \qquad \langle i^\pm| \equiv \langle p_i^\pm| \equiv \overline{u_\pm(p_i)} \;=\; \overline{v_\mp(p_i)}\,, \\ \langle i| = \langle i^-|, \quad [i| = \langle i^+|, \quad |i\rangle = |i^+\rangle \quad \text{and} \quad |j] = |j^-\rangle, \end{aligned} \tag{2.34}$$

the basic spinor products are defined as

$$\langle ij\rangle \;\equiv\; \langle i^-|j^+\rangle \;=\; \overline{u_-(p_i)}u_+(p_j) \;=\; \mu_i^A \mu_{jA}, \quad [ij] \;\equiv\; \langle i^+|j^-\rangle \;=\; \overline{u_+(p_i)}u_-(p_j) \;=\; \bar{\mu}_{iA'}\bar{\mu}_j^{A'}. \tag{2.35}$$

Furthermore, it will prove useful below to introduce spinor strings

$$\begin{aligned} \langle i|m|j] = \langle i|(p_m)_\nu|j] = \langle i|(\not{p}_m)_{AA'}|j] = \mu_i^A\, \mu_{mA}\, \bar{\mu}_{mA'}\, \bar{\mu}_j^{A'} = \langle im\rangle[mj] \\ [i|m|j\rangle = [i|(p_m)^\nu|j\rangle = [i|(\not{p}_m)^{AA'}|j\rangle = \bar{\mu}_{iA'}\, \bar{\mu}_m^{A'}\, \mu_m^A\, \mu_{jA} = [im]\langle mj\rangle \,. \end{aligned} \tag{2.36}$$

Spinor brackets are antisymmetric

$$\langle ij\rangle = -\langle ji\rangle, \quad [ij] = -[ji] \qquad \text{and} \qquad \langle ii\rangle = [ii] = 0\,, \tag{2.37}$$

as required by the determinant in eq. (2.24) and satisfy the Schouten identities

$$\langle ij\rangle\,\langle kl\rangle = \langle ik\rangle\,\langle jl\rangle + \langle il\rangle\,\langle kj\rangle \qquad \text{and} \qquad [ij]\,[kl] = [ik]\,[jl] + [il]\,[kj]\,. \tag{2.38}$$

Momentum conservation reads

$$\sum_{i=1}^{n} \langle ji\rangle[ik] \;=\; 0 \tag{2.39}$$

in spinor-helicity language. Spinor brackets can be related to the common Lorentz notation employing the γ-matrices eq. (2.12). In particular,

$$\langle ij\rangle[ji] = \langle i^-|j^+\rangle\langle j^+|i^-\rangle = \mathrm{tr}\left(\tfrac{1}{2}(1-\gamma_5)\not{p}_i\not{p}_j\right) = 2p_i\cdot p_j = s_{ij}, \tag{2.40}$$

where kinematical invariants are defined via

$$[\![i]\!]_n = (p_i + p_{i+1} + \cdots + p_{i+n-1})^2\,, \qquad s_j = s_{j\,j+1} = [\![j]\!]_2\,, \qquad t_j = [\![j]\!]_3\,. \tag{2.41}$$

For the massless ($p_i^2 = 0$) theories considered here, kinematical invariants reduce to

$$s_1 = [\![1]\!]_2 = s_{12} = 2p_1\cdot p_2 \qquad \text{and} \qquad t_2 = [\![2]\!]_3 = s_{23}+s_{24}+s_{34} = 2(p_2\cdot p_3+p_2\cdot p_4+p_3\cdot p_4). \tag{2.42}$$

Fierz identities are given by

$$\langle i^+|\gamma^\mu|j^+\rangle\langle k^+|\gamma_\mu|l^+\rangle \;=\; 2[ik]\langle lj\rangle \tag{2.43}$$

where the Gordon identity and the projection operator take the following form:

$$\langle i^\pm|\gamma^\mu|i^\pm\rangle \;=\; 2p_i^\mu, \qquad |i^\pm\rangle\langle i^\pm| \;=\; \tfrac{1}{2}(1\pm\gamma_5)\not{p}_i\,. \tag{2.44}$$

Employing the identities above, one can show that

$$\begin{aligned}\langle ij\rangle[jk]\langle kl\rangle[li] =&\,\mathrm{tr}\left(\tfrac{1}{2}(1-\gamma_5)\not{p}_i\not{p}_j\not{p}_k\not{p}_l\right)\\ =&\,\frac{1}{2}\left[s_{ij}s_{kl} - s_{ik}s_{jl} + s_{il}s_{jk} - \varepsilon(i,j,k,l)\right],\end{aligned} \tag{2.45}$$

where

$$\varepsilon(i,j,k,l) = 4i\varepsilon_{\mu\nu\rho\sigma}p_i^\mu p_j^\nu p_k^\rho p_l^\sigma = [ij]\langle jk\rangle[kl]\langle li\rangle - \langle ij\rangle[jk]\langle kl\rangle[li]\,. \tag{2.46}$$

For numerical calculations one would like to have expressions for the spinor brackets in terms of the momenta $p_1,\ldots,p_n$. With definitions eq. (2.29) above, solutions to the Dirac equation can be chosen as [7]

$$u_+(p) = v_-(p) = \frac{1}{\sqrt{2}}\begin{pmatrix}\mu_A\\ \mu_A\end{pmatrix} \qquad \text{and} \qquad u_-(p) = v_+(p) = \frac{1}{\sqrt{2}}\begin{pmatrix}\bar{\mu}_{A'}\\ -\bar{\mu}_{A'}\end{pmatrix}. \tag{2.47}$$

Plugging those choices into the definition of the spinor brackets eq. (2.35) leads for two positive energies p_i^0, $p_j^0 > 0$ to

$$\begin{aligned}\langle ij\rangle =&\sqrt{p_i^-p_j^+}e^{i\varphi_i} - \sqrt{p_i^+p_j^-}e^{i\varphi_j} \;=\; \sqrt{|s_{ij}|}e^{i\phi_{ij}},\\ [ij] =&-\sqrt{p_i^-p_j^+}e^{-i\varphi_i} + \sqrt{p_i^+p_j^-}e^{-i\varphi_j} \;=\; \sqrt{|s_{ij}|}e^{-i(\phi_{ij}+\pi)}\,,\end{aligned} \tag{2.48}$$

where

$$\cos\phi_{ij} \;=\; \frac{p_i^1p_j^+ - p_j^1p_i^+}{\sqrt{|s_{ij}|p_i^+p_j^+}}\,, \qquad \sin\phi_{ij} \;=\; \frac{p_i^2p_j^+ - p_j^2p_i^+}{\sqrt{|s_{ij}|p_i^+p_j^+}}\,. \tag{2.49}$$

So the spinor products are square roots of usual Lorentz scalar products up to a phase. The expressions above can be analytically continued to negative energies. In case of negative energy the p_i have to be replaced by $-p_i$ and for each negative-energy particle an extra factor of i has to be added to the definition of $\langle ij\rangle$ and $[ij]$.

Despite of many advantages of the spinor-helicity formalism, the resulting compact expressions come with a redundant set of spinor brackets, which are subject to nonlinear relations eqs. (2.38) and (2.39).

2.3 Lie algebras, Lie superalgebras and supersymmetry

As implied by the general symmetry considerations in subsection 2.1.1 above, Lie algebras [11, 12] generate the symmetry groups underlying almost all physical theories[1]. A ***Lie algebra*** is a vector space $\mathfrak{g}$ over a field $\mathbb{K}$ with a bilinear (eq. (2.51)), antisymmetric (eq. (2.52)) multiplication satisfying the Jacobi identity (eq. (2.53)):

$$\begin{aligned} [\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} &\longrightarrow \mathfrak{g} \\ (\mathfrak{X},\mathfrak{Y}) &\longmapsto [\mathfrak{X},\mathfrak{Y}]. \end{aligned} \tag{2.50}$$

The multiplication is referred to as Lie bracket, where $\forall\,\mathfrak{A},\mathfrak{B},\mathfrak{C}\in\mathfrak{g}$ and $\forall\alpha,\beta\in\mathbb{K}$:

$$[\alpha\mathfrak{A}+\beta\mathfrak{B},\mathfrak{C}] = \alpha[\mathfrak{A},\mathfrak{C}]+\beta[\mathfrak{B},\mathfrak{C}] \tag{2.51}$$

$$[\mathfrak{A},\mathfrak{B}] = -[\mathfrak{B},\mathfrak{A}] \tag{2.52}$$

$$[\mathfrak{A},[\mathfrak{B},\mathfrak{C}]]+[\mathfrak{C},[\mathfrak{A},\mathfrak{B}]]+[\mathfrak{B},[\mathfrak{C},\mathfrak{A}]] = 0\,, \tag{2.53}$$

which implies: $[\mathfrak{A},\mathfrak{A}]=0 \quad \forall\,\mathfrak{A}\in\mathfrak{g}$. A ***Lie group*** G is a real or complex manifold, endowed with a group structure, whose group multiplication

$$\begin{aligned} G\times G &\longrightarrow G \\ (g,h) &\longmapsto g\cdot h^{-1} \end{aligned} \tag{2.54}$$

is differentiable for all $g,h\in G$. The elements of the Lie algebra $\mathfrak{G}\in\mathfrak{g}$ form a basis of the Lie group G and are related to it by the exponential map

$$\begin{aligned} \mathfrak{g} &\to G \\ \mathfrak{G} &\mapsto \exp\mathfrak{G}. \end{aligned} \tag{2.55}$$

In order to classify Lie algebras, it is necessary to introduce some more notions: the action of a finite-dimensional, complex Lie algebra $\mathfrak{g}$ on itself, the vectorspace $\mathfrak{g}$, is called the ***adjoint action***:

$$\begin{aligned} \mathbf{ad}:\quad \mathfrak{g}\times\mathfrak{g} &\longrightarrow \mathfrak{g} \\ (\mathfrak{A},\mathfrak{B}) &\longmapsto \mathbf{ad}_{\mathfrak{A}}(\mathfrak{B}) = [\mathfrak{A},\mathfrak{B}]. \end{aligned} \tag{2.56}$$

[1] Lie algebras have been named after the Norwegian mathematician Sophus Lie, who studied continuous and discrete symmetries in the context of partial differential equations. In order to apply those transformation groups, he linearized the transformations and investigated the infinitesimal generators, which finally lead to the notion of a Lie algebra.

The ***Cartan subalgebra*** $\mathfrak{h}$ is the maximal abelian diagonalizable subalgebra of $\mathfrak{g}$, whose dimension is called the ***rank*** of the Lie algebra $\mathfrak{g}$. Restricting the adjoint action to the Cartan subalgebra

$$\begin{aligned} \mathbf{ad}: \quad \mathfrak{h}\times\mathfrak{g} &\longrightarrow \mathfrak{g} \\ (\mathfrak{H},\mathfrak{G}) &\longmapsto \mathbf{ad}_{\mathfrak{H}}(\mathfrak{G}) = [\mathfrak{H},\mathfrak{G}]\,, \end{aligned} \tag{2.57}$$

one can show that all adjoint actions induced by elements of $\mathfrak{h}$ commute. Hence all maps $\mathbf{ad}|_{\mathfrak{h}}$ have a common eigenvector, whose eigenvalue depends on the element $\mathfrak{H}\in\mathfrak{h}$. For any given eigenvector $\mathfrak{E}_\alpha \in \mathfrak{g}$ of the adjoint action $\mathbf{ad}|_{\mathfrak{H}}$ the eigenvalues are given by the functional α

$$\begin{aligned} \alpha\,:\,\mathfrak{h} &\to \mathbb{C} \\ \mathbf{ad}_{\mathfrak{H}}(\mathfrak{E}_\alpha) &= \alpha(\mathfrak{H})\mathfrak{E}_\alpha \end{aligned} \tag{2.58}$$

and are called ***roots*** or ***weights of the adjoint representation***. The number of roots can be determined by $\dim(\mathfrak{g}) - \mathrm{rank}\,(\mathfrak{g})$.

Since one can prove that there is only a finite number of roots, Lie algebras can be classified by those values. In particular, any simple root α_i with $i = 1,\ldots,\mathrm{rank}(\mathfrak{g})$ corresponds to a Chevalley triple, which is an $\mathfrak{su}(2)$ subalgebra $(\mathfrak{E}_{+\alpha_i},\,\mathfrak{E}_{-\alpha_i},\,\mathfrak{H}_{\alpha_i})$ whose elements satisfy

$$\left[\mathfrak{E}_{+\alpha_i},\,\mathfrak{E}_{-\alpha_j}\right] = \delta_{ij}\mathfrak{H}_{\alpha_i} \qquad \text{and} \qquad \left[\mathfrak{H}_{\alpha_i},\,\mathfrak{E}_{\pm\alpha_j}\right] = \pm\alpha_j(\mathfrak{H}_{\alpha_i})\mathfrak{E}_{\pm\alpha_j}\,. \tag{2.59}$$

A ***representation*** $\mathcal{D}$ of a Lie algebra $\mathfrak{g}$ is a homomorphism from $\mathfrak{g}$ to the group of automorphisms $\mathrm{Aut}(W)$ of a vector space W:

$$\mathfrak{G} \longmapsto \mathcal{D}(\mathfrak{G}) \text{ where } \mathcal{D}(\mathfrak{G}) \in \mathrm{Aut}(W),\;\; \mathfrak{G}\in\mathfrak{g} \tag{2.60}$$

The multiplication in $\mathfrak{g}$ corresponds to the successive application of automorphisms in W:

$$\mathcal{D}(\mathfrak{G}\mathfrak{K}) = \mathcal{D}(\mathfrak{G})\mathcal{D}(\mathfrak{K}) \;\;\forall\,\mathfrak{G},\mathfrak{K}\in\mathfrak{g}. \tag{2.61}$$

Representations are labeled by the dimension of the carrier space W, which is written as a bold number. The representation $\mathbf{6}$ of $SU(4)$ corresponds to 6×6 matrices acting on a two-index total antisymmetric tensor, whose indices can take four values.

Superalgebras and supersymmetry The idea of supersymmetry arose by systematically exploring possible extensions to the known symmetries of the S-matrix in a quantum field theory. While Coleman and Mandula showed [13] that the symmetries of the S-matrix have to be a direct product of Poincaré symmetry and an internal compact symmetry group, an extension is possible by accompanying the commuting generators of the Poincaré group with anticommuting generators. Those generators, initially introduced in [14] became known as supercharges lateron [15]. Initiated by those findings, all physically possible superalgebras have been classified in reference [16].

A ***Lie superalgebra*** $\mathfrak{g}$ is an associative $\mathbb{Z}_2$-graded algebra. It is a vector space which is a direct sum of two vector spaces $\mathfrak{g}_0$ and $\mathfrak{g}_1$. In $\mathfrak{g}$, a multiplication $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is defined for $\mathfrak{G}_i\in\mathfrak{g}_i$ with the following properties:

- $\mathbb{Z}_2$-gradation:
$$[\![\mathfrak{G}_i, \mathfrak{G}_j]\!] \in \mathfrak{g}_{i+j\ (mod\, 2)} \tag{2.62}$$

- graded antisymmetry:
$$[\![\mathfrak{G}_i, \mathfrak{G}_j]\!] = -(-1)^{i \cdot j} [\![\mathfrak{G}_j, \mathfrak{G}_i]\!]$$
If $i \cdot j = 0$, then $[\![\,,\,]\!]$ defines the usual commutator $[\,,\,]$, while for $i \cdot j = 1$ it is an anticommutator $\{\,,\,\}$.

- generalized Jacobi-identity:
$$(-1)^{i \cdot k} [\![\mathfrak{G}_i, [\![\mathfrak{G}_j, \mathfrak{G}_k]\!]]\!] + (-1)^{j \cdot i} [\![\mathfrak{G}_j, [\![\mathfrak{G}_k, \mathfrak{G}_i]\!]]\!] + (-1)^{k \cdot j} [\![\mathfrak{G}_k, [\![\mathfrak{G}_i, \mathfrak{G}_j]\!]]\!] = 0\,.$$

While the bosonic part $\mathfrak{g}_0$ is a Lie algebra, the fermionic part $\mathfrak{g}_1$ is not. All superalgebras considered in the following are Lie superalgebras.

One substantial part of any supersymmetry algebra are the fermionic supercharges $\mathfrak{Q}$ relating bosons to fermions. In particular, acting on a particle state with a certain helicity, $\mathfrak{Q}$ raises the helicity by $\frac{1}{2}$ while $\overline{\mathfrak{Q}}$ lowers it by one half (see eq. (2.64) below). Starting from a non-supersymmetric Lie algebra, there is a minimal number of supercharges which has to be added for the supersymmetry algebra to be consistent. A supersymmetric generalization of the Poincaré algebra is called extended if it exhibits more than this minimal number of supercharges. For example the minimal supersymmetrized version of Yang-Mills theory in four dimensions exhibits one supercharge for each possible spinor index $A = 1, 2$, $A' = 1, 2$. Thus the minimal set consists of four supercharges. This theory is referred to as $\mathcal{N}$=1 SYM theory, where $\mathcal{N}$ labels the number of minimal sets. Correspondingly, the $\mathcal{N}$=4 supersymmetrically extended Yang-Mills theory discussed below does contain 16 supercharges: four types of supercharges for each possible spinor index.

If there is more than one minimal set, it is possible to mix the supercharges among each other, which amounts to a rotation in an $\mathcal{N}$-dimensional complex space. The corresponding $SU(\mathcal{N})$ symmetry in the fermionic sector is called R-symmetry.

Searching for physically sensible theories with supersymmetry in four spacetime dimensions strongly constrains the number of candidate Lie superalgebras [15, 17]. Instead of giving a complete classification here, the $\mathcal{N}$=4 superconformal algebra $\mathfrak{su}(2,2|4)$ which generates the symmetry group of $\mathcal{N}$=4 SYM theory will be presented and discussed as an example in a representation acting on amplitudes in subsection 2.6 below.

2.4 On-shell superspace

The $SU(\mathcal{N})$ R-symmetry rotating the supercharges $\mathfrak{Q}_{Aa}, \overline{\mathfrak{Q}}^{A'a}$, $a \in \{1, \dots, \mathcal{N}\}$, can be made manifest in the on-shell superspace [18, 19]. States in this space are designed such that they diagonalize the usual momentum operator, but at the same time are also eigenstates of the supermomentum corresponding to the supertranslations. However, since $\mathfrak{Q}$ and $\overline{\mathfrak{Q}}$ anticommute, it is not possible to construct states diagonalizing both operators: here eigenstates of $\overline{\mathfrak{Q}}$ will be chosen. Using $\mathcal{N}$ complex-valued Grassmann parameters η to label the positions in the complex vector space spanned by the supercharges and acted on with rotations $\mathfrak{R}$, on-shell states are defined as

$$|\mu, \bar{\mu}, \eta\rangle = e^{\overline{\mathfrak{Q}}^{A'a} \bar{\omega}_{A'} \eta_a} |\mu, \bar{\mu}, +s\rangle \tag{2.63}$$

where $s = \frac{\mathcal{N}}{4}$ is the maximal possible helicity in the theory considered and the spinor $\bar{\omega}_{A'}$ is chosen[2] such that $[\bar{\omega}, \bar{\mu}] = 0$. Leaving out the momentum spinors $\mu, \bar{\mu}$ in the labeling of states below, conventions for the supercharges are

$$\begin{aligned} \mathfrak{Q}_{A\,a}|+s\rangle = 0, \quad & \mathfrak{Q}_{A\,a}|-s\rangle = \mu_A|-s+\tfrac{1}{2}\rangle_a \\ \overline{\mathfrak{Q}}^{A'\,a}|-s\rangle = 0, \quad & \overline{\mathfrak{Q}}^{A'\,a}|+s\rangle = \bar{\mu}^{A'}|+s-\tfrac{1}{2}\rangle^a . \end{aligned} \tag{2.64}$$

Supersymmetry with parameter $\bar{\zeta}_{A'}$acts on those states as

$$e^{\overline{\mathfrak{Q}}^{A'\,a} \bar{\zeta}_{A'\,a}} |\eta\rangle = |\eta + [\bar{\zeta}\bar{\mu}]\rangle \,, \tag{2.65}$$

in other words, the operator $\overline{\mathfrak{Q}}$ is really a supertranslation by shifting the state $|\eta\rangle$.

Given the Grassmannian variables η, one can define a superwavefunction, which is of highest helicity s of the theory. Expanding into powers of η, one obtains

$$\begin{aligned} \Phi(p, \eta) =& A(p) + \eta_{a_1} A^{a_1}(p) + \frac{1}{2}\eta_{a_1}\eta_{a_2} A^{a_1 a_2}(p) + \frac{1}{3!}\eta_{a_1}\eta_{a_2}\eta_{a_3} A^{a_1 a_2 a_3}(p) + \\ & \cdots \\ & + \frac{1}{(4s)!}\eta_{a_1}\eta_{a_2}\eta_{a_3}\cdots\eta_{a_{4s}} A^{a_1 a_2 a_3 \cdots a_{4s}}(p) \,. \end{aligned} \tag{2.66}$$

The superwavefunction introduced above provides the opportunity to write amplitudes in maximally supersymmetric theories in a completely supersymmetric way. This implies that it is not necessary to fix which states from the multiplet take part in a scattering process. One rather obtains an expression from which, by acting with an appropriate choice of derivatives with respect to η, the final result for a certain choice of particles can be obtained. As this mechanism is specific to the theory under consideration and closely related to the supersymmetric Ward identities discussed in the next subsection, it will be explored for $\mathcal{N}$=4 SYM theory and $\mathcal{N}$=8 supergravity separately in subsections 2.6.2 and 2.7.3 respectively.

2.5 Supersymmetric Ward identities and different MHV sectors

As for any manifest symmetry of an action, Ward identities relating different amplitudes can be derived. This is in particular useful for supersymmetry: considering the action (2.64) of generators $\mathfrak{Q}$ and $\overline{\mathfrak{Q}}$ on states, amplitudes with different types of particles are related [20, 21, 22, 23]. It is the supersymmetric Ward identities, which simplify the investigation of supersymmetric theories at the level of amplitudes.

[2]The condition $[\bar{\omega}, \bar{\mu}] = 0$ fixes $\bar{\omega}_{A'}$ only up to an additive shift: $\bar{\omega}_{A'} \sim \bar{\omega}_{A'} + c\bar{\mu}_{A'}$. However, this does not alter the state $|\eta\rangle$.

Supersymmetric Ward identities (SWI) can be derived in concordance with eq. (2.10) using the fact that supercharges annihilate the vacuum of the theory, $\mathfrak{Q}|0\rangle = 0$, such that

$$0 = \langle 0| \left[\mathfrak{Q}(\xi), \beta_1\beta_2\cdots\beta_n\right]|0\rangle = \sum_{i=1}^{n}\langle 0|\beta_1\beta_2\cdots[\mathfrak{Q}(\xi),\beta_i]\cdots\beta_n|0\rangle\,. \tag{2.67}$$

Here the β_i are arbitrary fields corresponding to states from the multiplet under consideration, $\mathfrak{Q}(\xi) = \langle\mathfrak{Q}\xi\rangle$ is a bosonized supersymmetry operator, which has been obtained by spinor contraction with the anticommuting supersymmetry parameter ξ, and $\langle 0|\beta_1\beta_2\cdots\beta_n|0\rangle$ will be called the *source term* for the SWI. Source terms need to have an odd number of fermions, because amplitudes derived by acting on terms with an even number of fermions will vanish trivially. A standard result implied by eq. (2.67) is the disappearance of all amplitudes with helicity structure $\langle +++\cdots+\rangle$ and $\langle -++\cdots+\rangle$ and their parity conjugates [7]. Here and in the following particles labeled by $+$ and $-$ are implicitely understood to be of (positive and negative) maximal helicity s. If there are other particles involved in an amplitude, the participating fields will be stated explicitly.

MHV and N^{p}MHV amplitudes Amplitudes in maximally supersymmetric field theories can be classified by the number of particles with negative helicity $-s$. Where k denotes the number of these particles, the label p in N^{p}MHV is related to it via

$$p = k - 2. \tag{2.68}$$

Maximally helicity violating (MHV) amplitudes are the simplest nontrivial amplitudes in maximally supersymmetric field theories. In case of particles with maximal helicity s, they have the structure[3]

$$\langle --++\cdots+\rangle. \tag{2.69}$$

With little effort one can show that any SWI for maximally helicity violating amplitudes (MHV) relates precisely two amplitudes, which in turn means that a known MHV amplitude implies expressions for all amplitudes related by SWI. This in turn means that the knowledge of one MHV amplitude determines the complete set of MHV amplitudes for a particular number of legs [7]. While it is easy to tell whether a certain amplitude resides in the MHV sector for pure gluon amplitudes in $\mathcal{N}$=4 SYM and pure graviton amplitudes in $\mathcal{N}$=8 supergravity, for amplitudes containing other particles a more general notion is necessary. Using the on-shell formalism introduced in the previous subsection, one can add up the number of Grassmann variables for each particle in the whole amplitude. The total number of η's in the MHV sector will be $8s$, where the maximal helicity s has been defined after eq. (2.63).

While in the four- and five-point case the only nonvanishing configurations are MHV (or anti-MHV), the advent of a sixth leg introduces a new class of helicity structures, the so-called next-to-MHV (NMHV) amplitudes. Here it is necessary to distinguish three different

[3]MHV amplitudes are simply N^{0}MHV amplitudes with $k = 0 + 2 = 2$ particles of helicity $-s$.

helicity orderings

$$X:\ (---+++)\qquad Y:\ (--+-++)\qquad Z:\ (-+-+-+)\,. \tag{2.70}$$

Expressions for the amplitudes are distinct for the different orderings X, Y and Z. However, since there is no procedural difference in deriving the expressions, amplitudes and supersymmetry relations will be generally illustrated for the helicity configuration X in section 5 below.

The simple relation of amplitudes via SWI as encountered in the MHV sector does not carry over to the NMHV sector: here each supersymmetric Ward identity relates three amplitudes, which requires two known amplitudes in order to determine a third one[4].

In terms of numbers of Grassmann variables, the change from the MHV to the NMHV sector is obvious. Adding one particle of maximal negative helicity increases the number of η's by $4s$. Continuing this counting one immediately finds that an N^{p}MHV amplitude exhibits $4s(p+2) = 4sk$ Grassmann variables η in the on-shell formalism.

Three-particle amplitudes vanish for real external momenta in maximally supersymmetric theories. However, considering complex external momenta instead renders those amplitudes nontrivial. In particular the amplitude $\langle -++\rangle$ is referred to as $\overline{\text{MHV}}$ and can be shown to be the lowest nontrivial amplitude accessible in twistor string theory (see subsection 3.2.2).

2.6 $\mathcal{N}$=4 super Yang-Mills theory

2.6.1 Fields and action

The maximally supersymmetric gauge theory in four dimensions containing particles up to spin one can be obtained as dimensional reduction of $\mathcal{N}$=1 supersymmetric SYM theory in ten dimensions. Its field content is the $\mathcal{N}$=4 irreducible supersymmetric multiplet consisting of a gauge potential A_μ, four chiral and anti-chiral spinors χ^a_A and $\tilde{\chi}^{A'}_a$ and six real scalars ϕ_{ab}. Those are combined in the action [10, 24][5]

$$\begin{aligned} S = \int \mathrm{d}^4x\ \mathrm{Tr}\Big\{ &-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}D_\mu\phi_{ab}D^\mu\phi^{ab} - \tfrac{1}{4}\,[\phi_{ab},\phi_{cd}]\,[\phi^{ab},\phi^{cd}] \\ &+ i\bar{\chi}\gamma^\mu D_\mu R\chi - \tfrac{i}{2}\left(\bar{\tilde{\chi}}^a[R\chi^b,\phi_{ab}] - \bar{\chi}_a[L\tilde{\chi}_b,\phi^{ab}]\right)\Big\}\ , \end{aligned} \tag{2.71}$$

where the spinors χ and $\tilde{\chi}$ are related by $\tilde{\chi}_a \;=\; C(\bar{\chi}^a)^T$ and combine into four gluini λ^a

$$\lambda \;=\; \begin{pmatrix} R\chi^a \\ L\tilde{\chi}_a\,. \end{pmatrix} \tag{2.72}$$

The four-dimensional charge conjugation operator C as well as the γ-matrices have been defined in eq. (2.20) and eq. (2.12) respectively. $\mathcal{N}$=4 SYM theory is a gauge theory with the Yang-Mills gauge group $SU(N_{\text{color}})$. The trace in eq. (2.71) is taken over matrices

[4] An example for $\mathcal{N}$=1 supersymmetry can be found in eq. (5.33).

[5] Note that the roles of R and L are interchanged here compared the first reference.

T^{α_i} which transform in the adjoint representation of this group and are normalized to $\mathrm{Tr}(T^\alpha T^\beta) = \delta^{\alpha\beta}$.

Here, Greek indices are four-dimensional Lorentz-indices, and spinor indices $A, B, \ldots$ and $A', B', \ldots$ and have been suppressed in the above action. Latin indices $a, b, \ldots$ label the internal R-symmetry group $SU(4)$ and are raised and lowered with the $SU(4)$-invariant tensor:

$$\phi^{ab} = \frac{1}{2}\varepsilon^{abcd}\phi_{cd} \qquad \text{and} \qquad \phi_{ab} = \frac{1}{2}\varepsilon_{abcd}\phi^{cd}\,. \tag{2.73}$$

The covariant derivative contains the Yang-Mills coupling g and is defined as

$$D_\mu = \partial_\mu - igA_\mu\,, \tag{2.74}$$

with the help of which the field strength F can be related to the gauge connection:

$$F_{\mu\nu} = ig^{-1}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\,. \tag{2.75}$$

Supersymmetry transformations are parameterized by four complex spinors α^a which satisfy the same condition as χ^a:

$$\delta A_\mu = i\left(\bar{\alpha}_c\gamma_\mu R\chi^c - \bar{\chi}_c\gamma_\mu R\alpha^c\right)\,, \tag{2.76}$$

$$\delta\phi_{ab} = i\left(\bar{\alpha}_b L\tilde{\chi}_a - \bar{\alpha}_a L\tilde{\chi}_b + \varepsilon_{abcd}\bar{\tilde{\alpha}}^c R\chi^d\right)\,, \tag{2.77}$$

$$\delta R\chi^c = \sigma_{\mu\nu}F^{\mu\nu}R\alpha^c - \gamma^\mu D_\mu\phi^{cd}L\tilde{\alpha}_d + \tfrac{1}{2}[\phi^{ck}, \phi_{kd}]R\alpha^d\,, \tag{2.78}$$

$$\delta L\tilde{\chi}_a = \sigma_{\mu\nu}F^{\mu\nu}L\tilde{\alpha}_a + \gamma^\mu D_\mu\phi_{ab}R\alpha^b + \tfrac{1}{2}[\phi_{ak}, \phi^{kb}]L\tilde{\alpha}_b\,. \tag{2.79}$$

The action eq. (2.71) and the corresponding equations of motion can be shown to be invariant under the $\mathcal{N}$=4 super Poincaré algebra. Besides of the usual Lorentz and translational symmetries $\mathfrak{L}, \bar{\mathfrak{L}}$ and $\mathfrak{P}$ there are in addition the internal $SU(4)$ R-symmetry[6] $\mathfrak{R}$ and the supertranslations $\mathfrak{Q}, \bar{\mathfrak{Q}}$. Since the theory is pure in the sense that it contains only the massless supermultiplet (see table 1 below), the generators of super Poincaré symmetry can be accompanied by superboosts $\mathfrak{S}, \bar{\mathfrak{S}}, \mathfrak{K}$, the dilatation $\mathfrak{D}$, a central charge $\mathfrak{C}$ and a hypercharge $\mathfrak{B}$ to collectively represent the *superconformal* symmetry. If all of the above operators are present, the algebra will be $\mathfrak{u}(2,2|4)$, while vanishing central charge $\mathfrak{C}$ leads to the projective group $\mathfrak{pu}(2,2|4)$. Leaving out the hypercharge operator $\mathfrak{B}$, one will obtain the algebra $\mathfrak{su}(2,2|4)$, which can be further reduced to $\mathfrak{psu}(2,2|4)$ by removing $\mathfrak{C}$.

If it comes to exploring the symmetries and the calculation and representation of amplitudes in $\mathcal{N}$=4 SYM theory, the manifestly Lorentz covariant formulation eq. (2.71) is not the most appropriate. Instead of insisting on Lorentz covariance, the on-shell superspace introduced in subsection 2.4 will prove a valuable tool to explore symmetries of amplitudes in $\mathcal{N}$=4 SYM theory. Considering the $SU(4)$ R-symmetry, the on-shell superspace will

[6] The automorphism group of $\mathcal{N}$=4 supersymmetry algebra is the group $U(4)$. However, due to its adjoint action on the fields, the sign of the determinant can not be recognized, which in turn implies that only the subgroup $SU(4)$ is realized.

exhibit four Grassmann variables, with the help of which the multiplet can be represented (cf. eq. (2.66)) as [18]

$$\Phi(p,\eta) = g^+(p) + \eta_a \lambda^{a+}(p) + \frac{1}{2}\eta_a\eta_b\phi^{ab}(p) + \frac{1}{3!}\eta_a\eta_b\eta_c\varepsilon^{abcd}\lambda_d^- + \frac{1}{4!}\eta_a\eta_b\eta_c\eta_d\varepsilon^{abcd}g^- \,, \quad (2.80)$$

where the gluons $g^\pm$ are the gauge bosons for the gauge potential A_μ. Corresponding to eq. (2.80), the states transform in representations of $SU(4)$ as shown in table 1. In the

Particle	g^+	λ^{a+}	ϕ_{ab}	λ_a^-	g^-
$SU(4)$-representation	**1**	**4**	**6**	**4**	**1**

Table 1: *Particles and $SU(4)$ representations of the $\mathcal{N}$=4 multiplet*

language of spinor-helicity superspace, the symmetry generators for the superconformal group can be expressed as[7]

$$\begin{aligned}
\mathfrak{L}_B^A &= \mu^A\partial_B - \tfrac{1}{2}\delta_B^A\mu^C\partial_C, & \bar{\mathfrak{L}}^{A'}{}_{B'} &= \bar{\mu}^{A'}\bar{\partial}_{B'} - \tfrac{1}{2}\delta_{B'}^{A'}\bar{\mu}^{C'}\bar{\partial}_{C'},\\
\mathfrak{D} &= \tfrac{1}{2}\partial_C\mu^C + \tfrac{1}{2}\bar{\mu}^{C'}\bar{\partial}_{C'}, & \mathfrak{R}^a{}_b &= \eta^a\partial_b - \tfrac{1}{4}\delta_b^a\eta^c\partial_c,\\
\mathfrak{Q}^{Ab} &= \mu^A\eta^b, & \mathfrak{S}_{Ab} &= \partial_A\partial_b,\\
\bar{\mathfrak{Q}}_b^{A'} &= \bar{\mu}^{A'}\partial_b, & \bar{\mathfrak{S}}_{A'}^b &= \eta^b\bar{\partial}_{A'},\\
\mathfrak{P}^{AB'} &= \mu^A\bar{\mu}^{B'}, & \mathfrak{K}_{AB'} &= \partial_A\bar{\partial}_{B'},
\end{aligned} \quad (2.81)$$

where

$$\partial_A = \partial/\partial\mu^A,\ \ \bar{\partial}_{A'} = \partial/\partial\bar{\mu}_{A'} \qquad \text{and} \qquad \partial_a = \partial/\partial\eta^a. \quad (2.82)$$

The non-trivial commutation relations between those generators are

$$\{\mathfrak{Q}_{Aa}, \overline{\mathfrak{Q}}_{A'}^b\} = \delta_a^b\mathfrak{P}_{AA'}, \qquad \{\mathfrak{S}_{Aa}, \overline{\mathfrak{S}}_{A'}^b\} = \delta_a^b\mathfrak{K}_{AA'}, \quad (2.83)$$

$$\begin{aligned}
&[\mathfrak{P}_{AA'}, \mathfrak{S}_B^a] = \varepsilon_{A'B'}\overline{\mathfrak{Q}}_{A'}^a, \qquad [\mathfrak{K}_{AA'}, \mathfrak{Q}_{Ba}] = \varepsilon_{AB}\overline{\mathfrak{S}}_{A'}^a,\\
&[\mathfrak{P}_{AA'}, \overline{\mathfrak{S}}_{B'a}] = \varepsilon_{A'B'}\mathfrak{Q}_{Aa}, \qquad [\mathfrak{K}_{AA'}, \overline{\mathfrak{Q}}_{B'a}] = \varepsilon_{A'B'}\mathfrak{S}_{Aa},\\
&[\mathfrak{K}_{AA'}, \mathfrak{P}^{BB'}] = \delta_A^B\delta_{A'}^{B'}\mathfrak{D} + \mathfrak{L}_A{}^B\delta_{A'}^{B'} + \overline{\mathfrak{L}}_{A'}{}^{B'}\delta_A^B,\\
&\{\mathfrak{Q}_{Aa}, \mathfrak{S}_B^b\} = \varepsilon_{AB}\mathfrak{R}^b{}_a + \mathfrak{L}_{AB}\delta_a^b + \varepsilon_{AB}\delta_a^b(\mathfrak{D}+\mathfrak{C}),\\
&\{\overline{\mathfrak{Q}}_{A'}^a, \overline{\mathfrak{S}}_{B'b}\} = \varepsilon_{A'B'}\mathfrak{R}^a{}_b + \overline{\mathfrak{L}}_{A'B'}\delta_b^a + \varepsilon_{A'B'}\delta_b^a(\mathfrak{D}-\mathfrak{C})
\end{aligned} \quad (2.84)$$

where indices $a, b, \ldots$ are $SU(4)$-indices and capital indices $A, A', \ldots$ refer to spinor indices as defined in eq. (2.22). The bar over an generator distinguishes between the two spinor representations.

[7] A nice introduction to superconformal symmetry acting on amplitudes in the language of the spinor-helicity superspace can be found in [25].

2.6.2 Tree-level amplitudes in $\mathcal{N}$=4 SYM theory

An amplitude in $\mathcal{N}$=4 SYM can be color-decomposed as[8]

$$\mathcal{A}_n(1,2,\ldots,n) \;= \delta(\sum p_i)\; g_{YM}^{n-2} \sum_{\sigma\in S_n/\mathbb{Z}_n} \mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\, A_n(\sigma(1),\sigma(2),\ldots,\sigma(n)), \tag{2.85}$$

where the summation is over all $(n-1)!$ non-cyclic permutations of $i = 1,2,\ldots,n$. As before, the number i is understood as a collective label for the momentum p_i and helicity h_i of particle i, *e.g.* $1 \equiv (p_1, h_1)$ and matrices T^{α_i} have been defined after eq. (2.72).

The subamplitudes A_n are independent of the color structure and can be shown to exhibit the following properties for any type of particles [26]:

- invariance under gauge transformations
- reflection identity: $A_n(1,2,\ldots,n) \;=\; (-1)^n A_n(n,n-1,\ldots,2,1)$
- invariance under cyclic permutations: $A_n(1,2,\ldots,n) \;=\; A_n(2,3,\ldots,n,1)$
- photon decoupling (or dual Ward) identity:

$$\begin{aligned} A_n(1,2,3,\ldots,n) + A_n(2,1,3,\ldots,n) + A_n(2,3,1,\ldots,n) \\ + \cdots + A_n(2,3,\ldots,1,n) \;=\; 0. \end{aligned} \tag{2.86}$$

For the MHV configurations $(--+\cdots+)$ discussed in subsection 2.5, amplitudes with all external legs being gluons $g^{\pm}$, are given by [27]:

$$A_n(g_1^-, g_2^-, g_3^+, \ldots, g_n^+) \;=\; i\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\,. \tag{2.87}$$

However, in terms of the superfield $\Phi(i) = \Phi(p_i, \eta_i)$ introduced in eq. (2.80), it is possible to write down a superamplitude

$$\mathcal{A}_n(p,\eta) = \mathcal{A}(\Phi(1),\Phi(2),\ldots,\Phi(n))\,, \tag{2.88}$$

which reads for tree-level MHV amplitudes

$$\mathcal{A}_n^{\mathrm{MHV}} = i\,\frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\,. \tag{2.89}$$

Here the second δ-function ensures the conservation of the supermomentum $q = \sum_{i=1}^n \mu_i^A \eta_i^a$, where momentum spinors μ_i^A and Grassmann variables η have been introduced in eq. (2.22) and subsection 2.4 respectively. Evaluating the Grassmannian δ-function will result in four spinor brackets corresponding to the contraction of $SU(4)$-indices. For two negative helicity

[8] Here the full amplitude including the color information is labeled by $\mathcal{A}_n$, while the subamplitude is assigned the non-calligraphic A_n. However, as the full amplitude will not occur again, the letter $\mathcal{A}_n$ will label the superamplitude starting from eq. (2.89). Furthermore, as only color-stripped subamplitudes are considered below, they will simply be referred to as amplitudes for convenience.

gluons at positions 1 and 2, the numerator factor yields $\langle 12\rangle^4$, thus confirming eq. (2.87). For any other combination of particles resulting in an amplitude which is an $SU(4)$ singlet, the numerator factor $\langle 12\rangle^4$ will be replaced by the appropriate expression in concordance with the supersymmetric Ward identities. Equation (2.89) furthermore reflects the fact that in the MHV sector the knowledge of exactly one amplitude is sufficient to determine all others by means of supersymmetric Ward identities elaborated on in subsection 2.5.

For the discussions in section 5 below it is useful to introduce the generating functional for MHV tree amplitudes [28]

$$\Omega_n^{\rm SYM} = \frac{1}{16}\frac{A_n(g_1^-, g_2^-, g_3^+, g_4^+, \dots, g_n^+)}{\langle 12\rangle^4}\prod_{a=1}^{4}\sum_{i,j=1}^{n}\langle ij\rangle\eta_{ia}\eta_{ja}\,. \tag{2.90}$$

This functional is a sum of all valid MHV amplitudes with n legs. The desired expression for a particular MHV amplitude can be extracted by acting with derivatives

$$\begin{aligned} g^+(i) &\leftrightarrow 1, & \lambda^{a+}(i) &\leftrightarrow \frac{\partial}{\partial\eta_{ia}}, & \phi^{ab}(i) &\leftrightarrow \frac{\partial^2}{\partial\eta_{ia}\partial\eta_{ib}}, \\ \lambda_a^-(i) &\leftrightarrow -\frac{1}{6}\varepsilon_{abcd}\frac{\partial^3}{\partial\eta_{ib}\partial\eta_{ic}\partial\eta_{id}}, & g^-(i) &\leftrightarrow \frac{1}{24}\varepsilon_{abcd}\frac{\partial^4}{\partial\eta_{ia}\partial\eta_{ib}\partial\eta_{ic}\partial\eta_{id}} \end{aligned} \tag{2.91}$$

on $\Omega_n^{\rm SYM}$. Corresponding to the Grassmann variables η in the generating functional, the total number of derivatives needs to add up to eight.

Including now all further N^{p}MHV sectors ($p \geq 1$), the superamplitude can be written as

$$\mathcal{A}_n(p,\eta) = i\,\delta^{(4)}(p)\delta^{(8)}(q)\left[\mathcal{P}_n^{(0)} + \mathcal{P}_n^{(4)} + \cdots + \mathcal{P}_n^{(4n-16)}\right], \tag{2.92}$$

where $\mathcal{P}_n^{(m)}$ denotes a homogeneous polynomial of degree m in the Grassmann variables η. The first term $\mathcal{P}_n^{(0)}$ corresponds to the denominator eq. (2.89) above: in this case $\mathcal{A}_n(p,\eta)$ is homogeneous of degree eight in η because of $\delta^{(8)}$ and thus represents the MHV amplitude in $\mathcal{N}$=4 SYM theory as explained in subsection 2.5. Other contributions $\mathcal{P}_n^{(m)}$ describe to N^{p}MHV sectors with $p = \frac{m}{4}$.

Limiting the consideration to the tree-level, expressions for low-point NMHV pure gluon amplitudes are presented in ref. [26]. Recently, Drummond and Henn [29] introduced a recursive procedure delivering expressions for all tree-level N^{p}MHV superamplitudes, which will be described below. For tree amplitudes, eq. (2.92) can be conveniently rewritten as

$$\mathcal{A}_n(p,\eta) = \mathcal{A}_n^{\rm MHV}(p,\eta)\left[1 + \mathcal{P}_n^{\rm NMHV} + \mathcal{P}_n^{\rm N^2MHV} + \cdots + \mathcal{P}_n^{\rm N^{n-4}MHV}\right]. \tag{2.93}$$

Since $\mathcal{A}_n^{\rm MHV}$ is known, the first interesting part is the NMHV superamplitude $\mathcal{A}_n^{\rm NMHV}$, which can be determined by

$$\mathcal{A}_n^{\rm NMHV} = \mathcal{A}_n^{\rm MHV}\mathcal{P}_n^{\rm NMHV} = \mathcal{A}_n^{\rm MHV}\sum_{1<s<t<n} R_{n;st} = i\,\frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\sum_{1<s<t<n} R_{n;st}\,, \tag{2.94}$$

where $R_{n;st}$ are invariants of the dual superconformal symmetry described in the subsection below. Since dual (super-)conformal invariants are of degree four in Grassmann variables

η, the correct overall degree of 12 for an NMHV amplitude is obtained (see eq. (2.92)). The index n on $R_{n;st}$ refers to the number of legs in the amplitude and the labels $s, t = 2, \ldots, n-1$ are assumed to be separated by at least two[9]: $t - s \geq 2$.

It turns out that this procedure can be repeated. In the next step, the N^{2}MHV amplitude

$$\begin{aligned}\mathcal{A}_n^{\mathrm{N^2MHV}} &= \mathcal{A}_n^{\mathrm{MHV}} \mathcal{P}_n^{\mathrm{N^2MHV}} \\ &= \mathcal{A}_n^{\mathrm{MHV}} \mathcal{P}_n^{\mathrm{NMHV}} \left[\sum R^* + \sum R\right]\end{aligned} \tag{2.95}$$

can be obtained from the NMHV amplitude by mulitplying with a factor containing a sum of dual superconformal invariants $R_{n;st}$ and a summation over generalized conformal[10] invariants R^*. From the schematic form of eq. (2.95) one can guess the general pattern: in order to get from N^{p}MHV to N^{p+1}MHV, one has to multiply the expression for the N^{p}MHV amplitude with a factor consisting of two sums of (generalized) conformal invariants. As the determination of summation limits and the precise form of the conformal invariants R^* is involved, the technical details of the prescription are left to reference [29]. Nevertheless, expressions for all tree amplitudes in any MHV sector can be obtained explicitly from combinations of dual (super-)conformal invariants.

2.6.3 Dual (super)conformal symmetry

In addition to the superconformal symmetry described above, a dual version thereof has been revealed recently [25] as a symmetry of $\mathcal{N}$=4 SYM theory. In contrast to the ordinary superconformal symmetry, its dual counterpart is realized on planar amplitudes as in eq. (2.10), but is not a symmetry of the action eq. (2.71) . While the usual superconformal symmetry is believed to be an exact symmetry of the quantum theory, the dual superconformal symmetry becomes anomalous at loop level.

The natural description of dual superconformal symmetry takes place in a space dual to the on-shell superspace. The idea of the dual superspace emerged while searching for a space, in which the momentum and supermomentum conservation conditions implied by the two δ-functions in eq. (2.92) are naturally taken care of. Coordinates for the dual superspace are x and θ, which are related to the momentum spinors μ_i^A, $\bar{\mu}_i^{A'}$ and the Grassmann variables η via

$$\begin{aligned}\sum_{i=1}^{n} \mu_i^A \bar{\mu}_i^{A'} = 0 \quad &\Rightarrow \quad x_i^{AA'} - x_{i+1}^{AA'} = \mu_i^A \bar{\mu}_i^{A'}\,, \\ \sum_{i=1}^{n} \mu_i^A \eta_i^a = 0 \quad &\Rightarrow \quad \theta_i^{a\,A} - \theta_{i+1}^{a\,A} = \mu_i^A \eta_i^a\,,\end{aligned} \tag{2.96}$$

[9]This can be understood graphically by interpreting $R_{n;st}$ as the box coefficient for a 3-mass IR-divergent integral as explained in subsection 2.6.4 below.

[10]The quantity R^* is not invariant under the full dual superconformal group, but only under the dual conformal subgroup, as has been shown in [29].

and take care for the conservation of the momenta after imposing the cyclicity conditions

$$x_{n+1} \equiv x_1\,, \qquad \theta_{n+1} \equiv \theta_1 \tag{2.97}$$

where n is the number of particles involved in the scattering process for which an amplitude shall be considered[11].

Employing the variables defined above, it was proven in [25] that tree-level amplitudes are annihilated by generators

$$\begin{aligned}
\hat{\mathfrak{P}}_{AA'} &= \sum_i \partial_{iAA'}, \qquad \hat{\mathfrak{Q}}_{Ab} = \sum_i \partial_{iAb}, \qquad \hat{\bar{\mathfrak{Q}}}^{\,b}_{A'} = \sum_i [\theta_i^{Ab}\partial_{iAA'} + \eta_i^b \partial_{iA'}],\\
\hat{\mathfrak{L}}_{AB} &= \sum_i [x_{i(A}{}^{A'}\partial_{iB)A'} + \theta^A_{i(A}\partial_{iB)A} + \mu_{i(A}\partial_{iB)}], \quad \hat{\bar{\mathfrak{L}}}_{A'B'} = \sum_i [x_{i(A'}{}^{A}\partial_{iB')A} + \bar{\mu}_{i(A'}\partial_{iB')}],\\
\hat{\mathfrak{R}}^a{}_b &= \sum_i [\theta_i^{Aa}\partial_{iAb} + \eta_i^a\partial_{ib} - \tfrac{1}{4}\delta^a_b\theta_i^{Ac}\partial_{iAc} - \tfrac{1}{4}\eta_i^c\partial_{ic}],\\
\hat{\mathfrak{S}}^a_A &= \sum_i [\theta^b_{iA}\theta_i^{Ba}\partial_{iBb} - x_{iA}{}^{B'}\theta_i^{Ba}\partial_{BB'} - \mu_{iA}\theta_i^{Ga}\partial_{iG} - x_{i+1\,A}{}^{B'}\eta_i^a\partial_{iB'} + \theta^B_{i+1\,A}\eta_i^a\partial_{iB}],\\
\hat{\bar{\mathfrak{S}}}_{A'a} &= \sum_i [x_{iA'}{}^{B}\partial_{iBa} + \bar{\mu}_{iA'}\partial_{ia}], \quad \hat{\mathfrak{D}} = \sum_i [x_i^{AA'}\partial_{iAA'} + \tfrac{1}{2}\theta_i^{Aa}\partial_{iAa} + \tfrac{1}{2}\mu_i^A\partial_{iA} + \tfrac{1}{2}\bar{\mu}_i^{A'}\partial_{iA'}],\\
\hat{\mathfrak{K}}_{AA'} &= \sum_i [x_{iA}{}^{B'}x_{iA'}{}^{B}\partial_{iBB'} + x_{iA'}{}^{B}\theta^b_{iA}\partial_{iBb} + x_{iA'}{}^{B}\mu_{iA}\partial_{iB} + x_{i+1\,A}{}^{B'}\bar{\mu}_{iA'}\partial_{iB'} + \bar{\mu}_{iA'}\theta^b_{i+1\,A}\partial_{ib}].
\end{aligned} \tag{2.98}$$

where

$$\partial_{iAA'} = \frac{\partial}{\partial x_i^{AA'}}, \qquad \partial_{iAa} = \frac{\partial}{\partial \theta_i^{Aa}}, \tag{2.99}$$

and derivatives ∂_{iA}, $\partial_{iA'}$ and $\partial\eta_i^a$ have been defined without the particle label i in eq. (2.82). Algebras eq. (2.81) and eq. (2.98) overlap on the operators $\bar{\mathfrak{Q}} = \hat{\bar{\mathfrak{S}}}$ and $\bar{\mathfrak{S}} = \hat{\bar{\mathfrak{Q}}}$ [25]. Furthermore, generators $\hat{\bar{\mathfrak{K}}}$ and $\hat{\bar{\mathfrak{S}}}$ are exact symmetries of tree amplitudes, but become anomalous starting at the one-loop level because of the presence of infrared divergences. It is those anomalies, which the conformal equations discussed in subsection 2.6.5 below are implied by.

Tree-level amplitudes in $\mathcal{N}$=4 SYM theory have been expressed in the last subsection in terms of the quantities $R_{n;st}$, which are invariant under dual superconformal transformations [25]. In terms of variables eq. (2.96), they are given by

$$R_{r;st} = \frac{\langle s\ s-1\rangle\langle t\ t-1\rangle\delta^{(4)}(\langle r|x_{rs}x_{st}|\theta_{tr}\rangle + \langle r|x_{rt}x_{ts}|\theta_{sr}\rangle)}{x^2_{st}\langle r|x_{rs}x_{st}|t\rangle\langle r|x_{rs}x_{st}|t-1\rangle\langle r|x_{rt}x_{ts}|s\rangle\langle r|x_{rt}x_{ts}|s-1\rangle}\,. \tag{2.100}$$

where $x_i - x_j = x_{ij}$, $\theta_i - \theta_j = \theta_{ij}$ and $r,\, s,\, t = 1, \ldots, n$. Correspondingly, $\langle r|$ and $\langle s|$ are momentum spinors for particles with momentum p_r, p_s and $\langle ..|..|..|..\rangle$ is the obvious generalization of the spinor strings, which have been introduced in eq. (2.34) above. The

[11] Variables $x^{AA'}$ shall not be confused with local coordinates as used in subsection 2.1.1. They denote the difference of consecutive momenta here and below.

explicit form of the generalized dual conformal invariants R^* necessary for amplitudes beyond NMHV in eq. (2.95) is similar to eq. (2.100). However, as those invariants will not be used below and the summation conventiens and boundary conditions need elaborate explanantions, the discussion is left to reference [29].

Commutators of the generators of ordinary (2.81) and dual superconformal (2.98) symmetries have been shown to result in an infinite-dimensional Yangian symmetry of tree-level amplitudes in [30].

2.6.4 One-loop amplitudes and IR equations in $\mathcal{N}$=4 SYM

Any one-loop amplitude in $\mathcal{N}$=4 SYM theory can be expanded [31, 32] into a basis of scalar box integrals[12] I_i. However, in the context of leading singularities discussed below (subsection 3.3.1) and dual conformal symmetry it is more convenient to express one-loop amplitudes in terms of box functions F_i

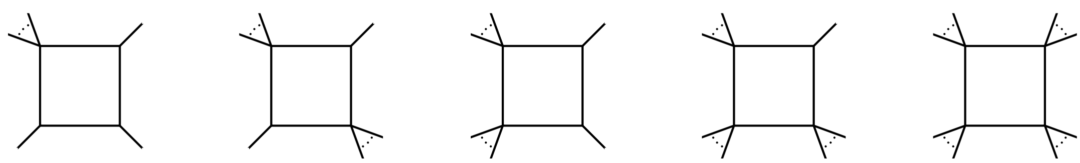
$$A^{\text{1-loop}} = \sum_{\{r,s,t,u\}} C_{r,s,t,u} F_{r,s,t,u} \tag{2.101}$$

which differ from the scalar box integrals I_i by a kinematical factor R_i [32]

$$F_i = -\frac{I_i}{2\sqrt{R_i}} \,. \tag{2.102}$$

The coefficients C_i in the expansion eq. (2.101) are called box coefficients.

Considering whether there are one or more legs attached to the corners of a box naturally leads to the following categories:



1-mass (1m) 2-mass easy (2me) 2-mass hard (2mh) 3-mass (3m) 4-mass (4m)

(2.103)

The corresponding box functions and their coefficients are conveniently labeled by the distribution of legs onto the corners of the boxes. Given the known number of legs n, it is sufficient to note the first legs attached to the four corners of the box. If not stated differently, the first entry of the four-element list $r, s, t, u \in \{1, \ldots, n\}$ is assumed to label the massless leg in the upper right corner. Despite being redundant, the type of box will be noted as a superscript on the coefficient (see eq. (2.108) below for explicit examples). Employing those conventions, eq. (2.101) reads

$$A_n^{\text{1-loop}}(\mu, \bar{\mu}, \eta, \epsilon) = \sum_{\{r,s,t,u\}} C_{r,s,t,u}(\mu, \bar{\mu}, \eta) F_{r,s,t,u}(\mu, \bar{\mu}, \epsilon) \,. \tag{2.104}$$

[12] For scalar particles, those correspond to box-shaped one-loop Feynman diagrams.

While box coefficients $C_{r,s,t,u}$ as well as their higher-loop generalizations are IR-finite and rational functions, box functions are IR-divergent objects, which have to be dimensionally regularized [33]

$$\begin{aligned} F^{1\mathrm{m}}(p,q,r,P) &= -\frac{r_\Gamma}{\epsilon^2}\left((-s)^{-\epsilon}+(-t)^{-\epsilon}-(-P^2)^{-\epsilon}\right) \\ F^{2\mathrm{me}}(p,P,q,Q) &= -\frac{r_\Gamma}{\epsilon^2}\left((-s)^{-\epsilon}+(-t)^{-\epsilon}-(-P^2)^{-\epsilon}-(-Q^2)^{-\epsilon}\right) \\ F^{2\mathrm{mh}}(p,q,P,Q) &= -\frac{r_\Gamma}{\epsilon^2}\left(\frac{1}{2}(-s)^{-\epsilon}+(-t)^{-\epsilon}-\frac{1}{2}(-P^2)^{-\epsilon}-\frac{1}{2}(-Q^2)^{-\epsilon}\right) \\ F^{3\mathrm{m}}(p,P,R,Q) &= -\frac{r_\Gamma}{\epsilon^2}\left(\frac{1}{2}(-s)^{-\epsilon}+\frac{1}{2}(-t)^{-\epsilon}-\frac{1}{2}(-P^2)^{-\epsilon}-\frac{1}{2}(-Q^2)^{-\epsilon}\right), \end{aligned} \quad (2.105)$$

where $I^{4\mathrm{m}}$ has not been listed because of its IR finiteness. Here $2\epsilon = 4 - D$ and

$$r_\Gamma = \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)/\Gamma(1-2\epsilon)\,. \quad (2.106)$$

In the notation $F(K_1, K_2, K_3, K_4)$ of eq. (2.105), capital letters define sums of consecutive massless momenta, and lower case letters correspond to single (null) momenta. In addition, the two main kinematical invariants are defined as $s = (K_1 + K_2)^2$ and $t = (K_2 + K_3)^2$.

Box coefficients $C_{r,s,t,u}$ are not independent objects, but are connected by infrared (IR) equations [34, 35]. Those equations originate in the fact that all infrared divergences have to cancel in the scheme of dimensional regularization. More explicitly, the relation of the infrared divergences of the one-loop amplitude to the tree amplitude can be derived from considering the factorization properties of one-loop amplitudes for an internal soft gluon [36, 37].

All IR equations are encoded in the IR consistency condition

$$A^{\text{1-loop}}\Big|_{IR} = -\frac{r_\Gamma}{\epsilon^2}\sum_{i=1}^{n}(-[\![i]\!]_2)^{-\epsilon}A^{\text{tree}}, \quad (2.107)$$

collecting the various IR-divergent contributions to the dimensionally regularized one-loop amplitudes of $\mathcal{N}$=4 SYM where the subscript IR denotes the IR-divergent part of the amplitude.

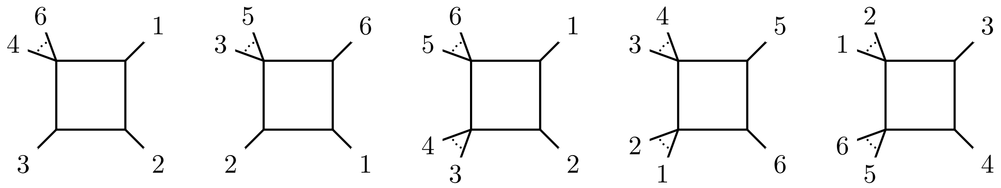
In order to obtain a particular IR equation, one picks a kinematical invariant, whose IR behavior shall be considered, and expands the left hand side of eq. (2.107) using eqs. (2.104) and (2.105). There are two different situations:

- If the considered kinematical invariant is of the form $[\![i]\!]_2$, the sum of box coefficients from the left hand side has to be proportional to the tree amplitude A^{tree}.
- For kinematical invariants of the form $[\![i]\!]_m$ with $m > 2$ there is no contribution from the right-hand side of eq. (2.107). Thus the total sum of box coefficients will vanish, which leads to a relation purely between box coefficients themselves.

Conveniently, kinematical invariants $[\![i]\!]_m$ with $(2 \le m \le \lfloor n/2 \rfloor$ defined in eq. (2.41) are used to label the corresponding IR equation. Considerations can be limited to $m = 2, ..., \lfloor n/2 \rfloor$, because all other situations are related to those by $[\![i]\!]_m = [\![i+m]\!]_{(n-m)}$.

The resulting IR equations are relations between IR-finite quantities, tree amplitudes and one-loop box coefficients. It is not difficult to count the number of IR equations for a particular number of legs. Taking again the momentum conservation into account, there are $\frac{n(n-3)}{2}$ independent IR equations in total, which split up into n equations involving the tree amplitude and $\frac{n(n-5)}{2}$ pure one-loop equations.

The derivation of an IR equation is probably best explained by using an example: considering the kinematical invariant $s_{12} = [\![1]\!]_2$ in a six-particle one-loop calculation, one will have to scan all possible box configurations for the occurrence of this term. In particular, this would happen at



$I^{1\text{m}}(1,2,3,4)$ $\quad I^{1\text{m}}(6,1,2,3)$ $\quad I^{2\text{mh}}(1,2,3,5)$ $\quad I^{2\text{mh}}(5,6,1,3)$ $\quad I^{2\text{mh}}(3,4,5,1)$

(2.108)

Taking the prefactors in eq. (2.107) into account, the IR equation corresponding to $[\![i]\!]_2$ reads:

$$C^{1m}_{1234} + C^{1m}_{6123} + \tfrac{1}{2}C^{2mh}_{1235} - \tfrac{1}{2}C^{2mh}_{5613} - \tfrac{1}{2}C^{2mh}_{3451} = A^{\text{tree}}\,. \tag{2.109}$$

2.6.5 Dual conformal constraints

The IR equations discussed in the previous subsection represent only a subset of all relations between box coefficients. As already mentioned above, operators $\hat{\hat{\mathfrak{K}}}$ and $\hat{\hat{\mathfrak{S}}}$ of the dual superconformal symmetry are anomalous starting at one-loop level due to the occurence of infrared divergences in the box functions $F_{r,s,t,u}$. Below only the dual conformal group subgroup of dual superconformal symmetry shall be considered, thus the considerations will be limited to the anomaly corresponding to $\hat{\hat{\mathfrak{K}}}$. Constraints on amplitudes implied by the anomalous symmetry have been investigated in [38, 39], whose main results shall be reviewed here as a basis for the investigation in section 6.

Two important results have been obtained for a slightly modified version[13] $\hat{\hat{\mathfrak{K}}}'$ of the original operator $\hat{\hat{\mathfrak{K}}}$. The conformal anomaly of the one-loop amplitude is proportional to the tree amplitude [25]

$$\hat{\hat{\mathfrak{K}}}'\, A^{\text{1-loop}} = -\frac{4r_\Gamma}{\epsilon} A^{\text{tree}} \sum_{i=1}^{n} x^\mu_{i+1}(-x^2_{i\,i+2})^{-\epsilon} \tag{2.110}$$

where the factor r_Γ and momentum differences $x^{AA'} = x^\mu$ have been defined in eqs. (2.106) and (2.96) respectively. The invariance of box coefficients under dual superconformal sym-

[13] Certain objects can be shown to be covariant only rather than invariant under the action of the symmetry generator $\hat{\hat{\mathfrak{K}}}$ in the form given in eq. (2.98). However, in order to simplify the calculations by using objects invariant under the symmetry, one defines $\hat{\hat{\mathfrak{K}}}' \to \hat{\hat{\mathfrak{K}}} + \sum_{i=1}^{n} x^\mu_i$ as suggested in [30].

metry

$$\hat{\bar{\mathfrak{K}}}' C_{r,s,t,u} = 0 \tag{2.111}$$

has been proven in [40].

Applying the symmetry generator $\hat{\bar{\mathfrak{K}}}'$ to both sides of eq. (2.104), yields eq. (2.110) on the left hand side, while on the right hand side the operator passes through the coefficients $C_{r,s,t,u}$ because of eq. (2.111) and thus directly acts on the box functions $F_{r,s,t,u}$, whose anomaly structure resembles those of their infrared divergences eq. (2.105). By carefully sorting out coefficients, $n(n-4)$ independent conformal constraints have been derived in [38]. In parallel to the IR equations, conformal equations fall into two categories. In terms of

$$\mathcal{E}_{i,k} := \sum_{j=k+1}^{i+n-2} C_{i,k,j,i-1} - \sum_{j=i+1}^{k-1} C_{i,j,k,i-1}\,, \tag{2.112}$$

one finds

- $n(n-3)$ combinations of box coefficients to vanish

  $$\mathcal{E}_{i,k} = 0\,, \tag{2.113}$$

  where $i = 1, ..., n$, $k = i+2, ..., i+n-3$. Since there are $2n$ algebraic identities among them, there are $n(n-5)$ independent constraints. These constraints imply the $\frac{n(n-5)}{2}$ IR equations in which the tree amplitude is not involved, but also have $\frac{n(n-5)}{2}$ new constraints.

- n combinations of (2-mass hard and 1-mass) box coefficients to equal the tree amplitude

  $$\mathcal{E}_{i,i-2} = -\mathcal{E}_{i-1,i} = -2A_n^{\text{tree}}\,, \tag{2.114}$$

  where $i = 1, ..., n$ and eq. (2.112) has been generalized for the boundary cases

  $$\mathcal{E}_{i,i-2} = -\mathcal{E}_{i-1,i} := -\sum_{j=i+1}^{i+n-3} C_{i-2,i-1,i,j}\,. \tag{2.115}$$

  These conformal equations are in one to one correspondence with the n IR equations with two-particle channels.

How are dual conformal constraints explicitly related to the IR equations discussed in the previous subsection? As shown in [38], any IR equation labeled by a kinematic invariant $[\![i]\!]_m$ with $m = 2, ..., \lfloor n/2 \rfloor$ can be written down as a particular combination of $\mathcal{E}(i,k)$

$$[\![i]\!]_2 : \mathcal{E}_{i,i+2} + \mathcal{E}_{i+2,i} - \mathcal{E}_{i+3,i} = -2A_n^{\text{tree}}\,, \tag{2.116}$$

and

$$[\![i]\!]_m : \mathcal{E}_{i,i+m} + \mathcal{E}_{i+m,i} - \mathcal{E}_{i+1,i+m} - \mathcal{E}_{i+m+1,i} = 0 \text{ for } m \geq 3\,. \tag{2.117}$$

2.7 $\mathcal{N}$=8 supergravity

After presenting the $\mathcal{N}$=8 supersymmetric multiplet and discussing the action of $\mathcal{N}$=8 supergravity in the next subsection, the necessary steps to obtain this $SU(8)$ invariant theory and the origin of the hidden $E_{7(7)}$ symmetry will be described in the next subsection 2.7.2. Amplitudes in $\mathcal{N}$=8 supergravity are explored in subsection 2.7.3. A detailed discussion of recent results, finiteness and counterterms in $\mathcal{N}$=8 supergravity can be found as an introductory subsection 5.1 to the investigation of the $\mathcal{R}^4$ counterterm.

2.7.1 Fields and action

The maximally supersymmetric field theory in four dimensions containing particles up to spin two can be obtained from compactifying the $\mathcal{N}$=1, $d = 11$ supergravity on a torus T^7 [41, 42, 43]. The physical field content consists of a vierbein (or graviton), 8 gravitini, 28 abelian gauge fields, 56 Majorana gauginos, and 70 real (or 35 complex) scalars, which can be collected into a single massless $\mathcal{N}$=8 (on-shell) supermultiplet:

Particle	B^+	F^{a+}	B^{ab+}	F^{abc+}	X^{abcd}	F^-_{abc}	B^-_{ab}	F^-_a	B^-
$SU(8)$-representation	**1**	**8**	**28**	**56**	**70**	**56**	**28**	**8**	**1**

Table 2: *Particles and $SU(8)$ representations of the $\mathcal{N}$=8 supergravity multiplet*

Interactions of those particles are governed by the action of $\mathcal{N}$=8 supergravity given in [44]. As this action is very long and complicated and needs several additional definitions in order to be written in an accessible form, it will not be stated here. However, it is the $\mathcal{N}$=8 supersymmetric completion of the Einstein-Hilbert action eq. (5.1) and has the following form:

$$S^{\text{SUGRA}} = \int \mathrm{d}^4x \Big(eR + (\mathcal{N}\text{=8 supersymmetric completion})\Big) \tag{2.118}$$

where $e = \sqrt{-g}$ is the determinant of the vierbein e^m_μ. The Ricci scalar and Ricci tensor

$$R \;=\; g^{\mu\nu} R_{\mu\nu} \quad \text{and} \quad R_{\mu\nu} \;=\; R^\sigma{}_{\mu\sigma\nu}\,, \tag{2.119}$$

are derived from the Riemann tensor

$$R^\rho{}_{\sigma\mu\nu} \;=\; \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{2.120}$$

where

$$\Gamma^\rho_{\mu\nu} = \tfrac{1}{2} g^{\rho\kappa}(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu})\,. \tag{2.121}$$

2.7.2 Coset structure and hidden symmetry

Contrary to the situation in $\mathcal{N}$=4 SYM theory, the $SU(8)$ R-symmetry in table 2 is not visible directly in the four-dimensional compactified theory. After compactifying $\mathcal{N}$=1, $d = 11$ supergravity on the torus T^7 all but one type of particles transform in representations of

$SU(8)$. Vector bosons behave exceptional: in the ungauged compactified theory they form an antisymmetric tensor representation of $SO(8)$. However, employing their Bianchi identities and equations of motion, a much larger symmetry leading to the notion of generalized electric-magnetic duality transformations can be realized [44]. Upon closer investigation of these duality transformations, it is possible to maximally enlarge the corresponding duality group by adding further scalars. Not all of those additional scalars are physical. After gauging a resulting local $SU(8)$ symmetry in order to reduce the degrees of freedom of the generalized duality group, 70 physical scalars remain. These scalars parameterize the coset $\frac{E_{7(7)}}{SU(8)}$ [43, 44], where $E_{7(7)}$ denotes a non-compact real form of E_7, which has $SU(8)$ as its maximal compact subgroup. In other words, the scalars X can be identified with the non-compact generators $\mathfrak{X}$ of $E_{7(7)}$. The resulting gauge is called *unitary.*

In unitary gauge the 63 compact generators $\mathfrak{R}_a^b$ of $SU(8)$ can be joined with 70 generators $\mathfrak{X}_{a_1\ldots a_4}$ to form the adjoint representation of $E_{7(7)}$. Here $\mathfrak{X}_{a_1\ldots a_4}$ transforms under $SU(8)$ in the four-index antisymmetric tensor representation $(a, b = 1, \ldots, 8)$. The commutation relations between those generators are given schematically by

$$[\mathfrak{R},\mathfrak{R}] \sim \mathfrak{R}\,, \quad [\mathfrak{X},\mathfrak{R}] \sim \mathfrak{X}\,, \qquad \text{and} \qquad [\mathfrak{X},\mathfrak{X}] \sim \mathfrak{R}\,. \tag{2.122}$$

While the first commutator is just the usual $SU(8)$ Lie algebra, the second one follows straightforwardly from the identification of $\mathfrak{X}$ with the **70** of $SU(8)$. The nontrivial statement about $E_{7(7)}$ invariance resides in the third commutator in eq. (2.122). Assuming the two generators to be represented as $\mathfrak{X}_1^{a_1\ldots a_4}$ and $\mathfrak{X}_{2\,a_5\ldots a_8}$, where the upper-index version can be obtained by employing the $SU(8)$-invariant tensor,

$$\mathfrak{X}^{a_1\ldots a_4} \;=\; \frac{1}{24}\varepsilon^{a_1a_2a_3a_4a_5a_6a_7a_8}\mathfrak{X}_{a_5\ldots a_8}\,, \tag{2.123}$$

the third relation reads explicitly (see *e.g.* ref. [19]),

$$-i\,[\mathfrak{X}_1^{a_1\ldots a_4}, \mathfrak{X}_{2\,a_5\ldots a_8}] \;=\; \varepsilon^{b\,a_2a_3a_4}_{a_5a_6a_7a_8}\mathfrak{R}_b^{a_1} + \varepsilon^{a_1b\,a_3a_4}_{a_5a_6a_7a_8}\mathfrak{R}_b^{a_2} + \ldots + \varepsilon^{a_1a_2a_3a_4}_{a_5a_6a_7b}\mathfrak{R}^b_{a_8}\,. \tag{2.124}$$

Here $\varepsilon^{a_1a_2a_3a_4}_{a_5a_6a_7a_8} = 1, -1, 0$ if the upper index set is an even, odd or no permutation of the lower set, respectively. For a more general discussion of the properties of $E_{7(7)}$, see appendix B of ref. [43].

2.7.3 Tree-level amplitudes in $\mathcal{N}$=8 supergravity

For amplitudes in $\mathcal{N}$=8 supergravity the color trace, which forces particles in gauge-theory subamplitudes to remain in a certain cyclic order, does not exist. Instead, supergravity amplitudes are symmetric under exchange of particles with the same helicity. The full amplitude $\mathcal{M}_n^{\text{SUGRA}}(1, 2, \ldots, n)$ can be written as

$$\mathcal{M}_n^{\text{SUGRA}}(1, 2, \ldots, n) \;=\; \left(\frac{\kappa}{2}\right)^{(n-2)} M_n^{\text{SUGRA}}(1, 2, \ldots, n), \tag{2.125}$$

where the subamplitude[14] M_n^{SUGRA} and $\mathcal{M}_n^{\text{SUGRA}}$ differ by powers of the gravitational coupling constant $\kappa = \sqrt{32\pi G_N}$ only. The four- and five-point MHV amplitudes for gravitons

[14] As for $\mathcal{N}$=4 SYM theory, subamplitudes M will be simply referred to as amplitudes below.

$B^\pm$ are given by [45]

$$M_4^{\rm SUGRA}(B_1^-,B_2^-,B_3^+,B_4^+) = i\,\langle 12\rangle^8 \frac{[12]}{\langle 34\rangle\,N(4)}\,,$$
$$M_5^{\rm SUGRA}(B_1^-,B_2^-,B_3^+,B_4^+,B_5^+) = i\,\langle 12\rangle^8 \frac{\varepsilon(1,2,3,4)}{N(5)}\,, \tag{2.126}$$

where $\varepsilon(i,j,m,n)$ has been defined in eq. (2.46) and

$$N(n) \equiv \prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\langle ij\rangle\,. \tag{2.127}$$

The higher-point MHV graviton amplitudes were first written down in reference [45]. Explicit expressions for other helicity configurations are rare. However, in [46] a prescription is given how to calculate any $\mathcal{N}$=8 supergravity tree-level amplitude by employing "gravity subamplitudes", Britto-Cachazo-Feng-Witten (BCFW) recursion relations, and superconformal invariants [25, 29]. This formalism makes use of the field theory subsector of the KLT relations described in subsection 2.8.

In the same manner as for $\mathcal{N}$=4 SYM (see eq. (2.90)) it is possible to write down a generating functional for MHV amplitudes in $\mathcal{N}$=8 supergravity [28]

$$\Omega_n^{\rm SUGRA} = \frac{1}{256}\frac{M_n(B_1^-,B_2^-,B_3^+,B_4^+,\ldots,B_n^+)}{\langle 12\rangle^8}\prod_{A=1}^{8}\sum_{i,j=1}^{n}\langle ij\rangle\eta_{iA}\eta_{jA}\,, \tag{2.128}$$

from which the particular expressions can be obtained by acting with the appropriate derivatives with respect to the anticommuting variables η:

$$B_i^+\leftrightarrow 1,\qquad F_i^{a+}\leftrightarrow\frac{\partial}{\partial\eta_{i\,a}},\qquad\cdots,\qquad X^{abcd}\leftrightarrow\frac{\partial^4}{\partial\eta_{i\,a}\partial\eta_{i\,b}\partial\eta_{i\,c}\partial\eta_{i\,d}},\qquad\cdots,$$
$$\cdots,\quad F_{i\,a}^-\leftrightarrow -\frac{1}{7!}\varepsilon_{abcdefgh}\frac{\partial^7}{\partial\eta_{i\,b}\partial\eta_{i\,c}\ldots\partial\eta_{i\,h}},\quad\cdots,\quad B_i^-\leftrightarrow\frac{1}{8!}\varepsilon_{abcdefgh}\frac{\partial^8}{\partial\eta_{i\,a}\partial\eta_{i\,b}\ldots\partial\eta_{i\,h}}\,. \tag{2.129}$$

Again the number of derivatives is connected to the helicity of the state. However, for an MHV amplitude in $\mathcal{N}$=8 supergravity there have to be 16 derivatives. As in the $\mathcal{N}$=4 situation, the resulting amplitudes automatically obey the MHV supersymmetric Ward identities (see subsection 2.5). For example, a two-gravitino two-graviton amplitude will read:

$$\begin{aligned}\langle F^{5+}\,F_5^-\,B^+\,B^-\rangle \equiv\;& M_4(F_1^{5+},F_{2,5}^-,B_3^+,B_4^-)\\ =\;& -\left(\frac{\partial}{\partial\eta_{15}}\right)\left(\frac{1}{7!}\varepsilon_{12345678}\frac{\partial^7}{\partial\eta_{21}\ldots\partial\eta_{24}\partial\eta_{26}\ldots\partial\eta_{28}}\right)\\ &\times\left(\frac{1}{8!}\varepsilon_{12345678}\frac{\partial^8}{\partial\eta_{41}\ldots\partial\eta_{48}}\right)\Omega_4\,.\end{aligned} \tag{2.130}$$

2.8 KLT relations in the field-theory limit

Although KLT relations originate in the close connection of open and closed string amplitudes as will be discussed in subsection 3.1.2, they have proven to be an indispensable tool for calculations in maximally supersymmetric field theories. This traces back to the fact that $\mathcal{N}$=4 SYM and $\mathcal{N}$=8 supergravity actions arise as the zeroth-order terms in the low-energy effective actions of open and closed string theory respectively, as will be described in subsection 3.1.3. Here the results for pure field theory will be stated, and the identification of states between the two maximally supersymmetric field theories will be discussed in detail.

The field-theory KLT relations connect amplitudes in $\mathcal{N}$=8 supergravity with permuted sums of products of amplitudes in $\mathcal{N}$=4 SYM theory. While the relations are remarkably simple for lower-point amplitudes, their complexity grows with the number of legs:

$$\begin{aligned}
M_4^{\text{tree}}(1,2,3,4) &= -is_{12}A_4^{\text{tree}}(1,2,3,4)\,A_4^{\text{tree}}(1,2,4,3)\,,\\
M_5^{\text{tree}}(1,2,3,4,5) &= is_{12}s_{34}A_5^{\text{tree}}(1,2,3,4,5)\,A_5^{\text{tree}}(2,1,4,3,5)\\
&\qquad + is_{13}s_{24}A_5^{\text{tree}}(1,3,2,4,5)\,A_5^{\text{tree}}(3,1,4,2,5)\,,\\
M_6^{\text{tree}}(1,2,3,4,5,6) &= -is_{12}s_{45}A_6^{\text{tree}}(1,2,3,4,5,6)\,\big[s_{35}A_6^{\text{tree}}(2,1,5,3,4,6)\\
&\qquad\qquad +(s_{34}+s_{35})A_6^{\text{tree}}(2,1,5,4,3,6)\big]\\
&\qquad + \mathcal{P}(2,3,4)\,,\\
M_n^{\text{tree}}(1^-,2^-,3^+,\ldots,n^+) &= -i\,\langle 12\rangle^8\\
\times\Bigg[\frac{[12][n{-}2\;n{-}1]}{\langle 1\;n{-}1\rangle\,N(n)}&\Bigg(\prod_{i=1}^{n-3}\prod_{j=i+2}^{n-1}\langle ij\rangle\Bigg)\prod_{l=3}^{n-3}\Big(-\langle n|\sum_{s=l+1}^{n-1}(k_s)_\mu|l]\Big)\;+\;\mathcal{P}(2,3,\ldots,n{-}2)\Bigg]\,.
\end{aligned} \tag{2.131}$$

Besides the form containing permutations above, a closed expression for all n has been found in reference [47].

Although the KLT relations are often applied to pure-graviton and pure-gluon amplitudes, their use is not limited to these scenarios. Any pair of consistent $\mathcal{N}$=4 amplitudes is related to an amplitude in $\mathcal{N}$=8 supergravity and vice versa.

Given two states of the $\mathcal{N}$=4 SYM multiplet, one can immediately determine which type of particle has to appear at a certain position on the supergravity side by adding the helicities and combining the indices, according to the tensor-product decomposition of the Fock space,

$$[\mathcal{N}{=}8] \;\leftrightarrow\; [\mathcal{N}{=}4]_L \otimes [\mathcal{N}{=}4]_R\,. \tag{2.132}$$

Remarkably, the opposite statement is true as well: given a certain operator, corresponding to a particular state in $\mathcal{N}$=8 supergravity, the helicity, global symmetry properties, and the consistent action of supercharges in either of the theories are sufficient to unambiguously determine the decomposition into $\mathcal{N}$=4 SYM states [28] as listed in table 3 below. Therein the particle labeling is the same as tables 1 and 2. Quantities with indices $a, b, \ldots$ correspond to the first $SU(4)$, while tilded quantities with indices $r, s, \ldots$ are in the second $SU(4)$.

B^{+}	$=$	$g^{+}\,\tilde{g}^{+}$	B^{-}	$=$	$g^{-}\,\tilde{g}^{-}$
F^{a+}	$=$	$\lambda^{a+}\,\tilde{g}^{+}$	F^{-}_{a}	$=$	$\lambda^{-}_{a}\,\tilde{g}^{-}$
F^{r+}	$=$	$g^{+}\,\tilde{\lambda}^{r+}$	F^{-}_{r}	$=$	$g^{-}\,\tilde{\lambda}^{-}_{r}$
B^{ab+}	$=$	$\phi^{ab}\,\tilde{g}^{+}$	B^{-}_{ab}	$=$	$\phi_{ab}\,\tilde{g}^{-}$
B^{ar+}	$=$	$\lambda^{a+}\,\tilde{\lambda}^{r+}$	B^{-}_{ar}	$=$	$-\lambda^{-}_{a}\,\tilde{\lambda}^{-}_{r}$
B^{rs+}	$=$	$g^{+}\,\tilde{\phi}^{rs}$	B^{-}_{rs}	$=$	$g^{-}\,\tilde{\phi}_{rs}$
F^{abc+}	$=$	$\varepsilon^{abcd}\,\lambda^{-}_{d}\,\tilde{g}^{+}$	F^{-}_{abc}	$=$	$-\varepsilon_{abcd}\,\lambda^{d+}\,\tilde{g}^{-}$
F^{abr+}	$=$	$\phi^{ab}\,\tilde{\lambda}^{r+}$	F^{-}_{abr}	$=$	$\phi_{ab}\,\tilde{\lambda}^{-}_{r}$
F^{ars+}	$=$	$\lambda^{a+}\,\tilde{\phi}^{rs}$	F^{-}_{ars}	$=$	$\lambda^{-}_{a}\,\tilde{\phi}_{rs}$
F^{rst+}	$=$	$\varepsilon^{rstu}\,g^{+}\,\tilde{\lambda}^{-}_{u}$	F^{-}_{rst}	$=$	$-\varepsilon_{rstu}\,g^{-}\,\tilde{\lambda}^{u+}$
X^{abcd}	$=$	$\varepsilon^{abcd}\,g^{-}\,\tilde{g}^{+}$	X_{abcd}	$=$	$\varepsilon_{abcd}\,g^{+}\,\tilde{g}^{-}$
X^{abcr}	$=$	$\varepsilon^{abcd}\,\lambda^{-}_{d}\,\tilde{\lambda}^{r+}$	X_{abcr}	$=$	$\varepsilon_{abcd}\,\lambda^{d+}\,\tilde{\lambda}^{-}_{r}$
X^{abrs}	$=$	$\phi^{ab}\,\tilde{\phi}^{rs}$	X_{abrs}	$=$	$\phi_{ab}\,\tilde{\phi}_{rs}$
X^{arst}	$=$	$\varepsilon^{rstu}\,\lambda^{a+}\,\tilde{\lambda}^{-}_{u}$	X_{arst}	$=$	$\varepsilon_{rstu}\,\lambda^{-}_{a}\,\tilde{\lambda}^{u+}$
X^{rstu}	$=$	$\varepsilon^{rstu}\,g^{+}\,\tilde{g}^{-}$	X_{rstu}	$=$	$\varepsilon_{rstu}\,g^{-}\,\tilde{g}^{+}$

Table 3: *KLT decomposition of particles in $\mathcal{N}$=8 supergravity*

2.9 Conformal supergravity

A conformal theory for gravity [48, 49] can be constructed starting from the Weyl tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - (g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) + \tfrac{1}{3}Rg_{\mu[\rho}g_{\sigma]\nu} \tag{2.133}$$

which is the part of the Riemann tensor[15] invariant under conformal transformations of the metric

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{2.134}$$

with Ω a smooth and strictly positive function. The Weyl tensor is completely traceless.

A conformally invariant action can be obtained by squaring the Weyl-tensor

$$\begin{aligned} S_{\text{CSG}} &= \int \mathrm{d}^4x\sqrt{-g}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \\ &= \int \mathrm{d}^4x\sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{2}R^2\right), \end{aligned} \tag{2.135}$$

where the last line can be obtained by noting that the Gauss-Bonnet term

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4\,R_{\mu\nu}R^{\mu\nu} + R^2 \tag{2.136}$$

[15] Ricci scalar, Ricci tensor and the Riemman tensor have been defined around eq. (2.120).

can be written as a total covariant derivative. The action eq. (2.135) does not exhibit a dimensionful coupling constant, as is dictated by conformal invariance. Considering the two-derivative nature of the Riemann tensor, it is obvious that the equations of motion derived from the action eq. (2.135) contain four derivatives. This is a rather special feature, whose consequences challenge the physical sensibility of the theory as will be shown below.

Expanding the action eq. (2.135) around the flat Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and fixing the diffeomorphism invariance, the equations of motion at linearized level read [50]

$$\Box^2 \bar{h}_{\mu\nu} = 0 \quad \text{where} \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{4} h^{\gamma}{}_{\gamma} \tag{2.137}$$

and determine the traceless part of $h_{\mu\nu}$ exclusively.

The general solution to an equation of the form $\Box^2 \phi = 0$ is

$$\phi(x) = \int \mathrm{d}^4 k \ \left(a(k)\exp(ik \cdot x) + b(k) A \cdot x \exp(ik \cdot x)\right) + c.c., \tag{2.138}$$

where $A \cdot k \neq 0$ and *c.c.* denotes the complex conjugated expression. The part containing $\exp(ikx)$ will be called a plane wave ϕ_p, while a solution proportional to $A \cdot x \exp(ikx)$ will be denoted as conformal wave ϕ_c. Spacetime translations on ϕ_p and ϕ_c act as [51]

$$\begin{pmatrix} P & 0 \\ * & P \end{pmatrix} \begin{pmatrix} \phi_p \\ \phi_c \end{pmatrix}, \tag{2.139}$$

where P is the usual translation operator and $*$ an additional nonzero contribution. Therefore ϕ_p and ϕ_c are neither eigenstates of the translation operator nor do they transform independently

$$\begin{aligned} \phi_p &\to \phi'_p, \\ \phi_c &\to \phi'_c + \phi''_p, \end{aligned} \tag{2.140}$$

and thus constitute a doublet. This implies that ϕ_p and ϕ_c can not be treated separately without breaking translational invariance.

For the metric perturbation $\bar{h}_{\mu\nu}$ eq. (2.137) can be solved starting from the ansatz

$$\bar{h}_{\mu\nu} = (a_{\mu\nu} + b_{\mu\nu} A_\rho x^\rho) \exp(ik \cdot x) + \text{c.c.} \tag{2.141}$$

with $a^{\mu}{}_{\mu} = b^{\mu}{}_{\mu} = 0$. Taking again the gauge freedom into account, one can show that $\bar{h}_{\mu\nu}$ describes six degrees of freedom: a masless spin-1 state containing two degrees of freedom and a massless spin-2 doublet with four degrees of freedom [50].

Although quantization is a concept which is usually applied to a two-derivative theory with a two-dimensional phase-space, one can still try to derive facts about the Hilbert-space structure of a higher derivative theory by introducing an auxiliary field and reformulating the theory as a pair of related two-derivative theories. The Lagrangian

$$\mathcal{L} = \frac{1}{2} B \Box^2 B \tag{2.142}$$

for a scalar field B results in an equation of motion of the form eq. (2.138) and therefore describes a doublet. Introducing an auxiliary field A it can be rewritten as

$$\mathcal{L} = -\partial A \cdot \partial B - \frac{1}{2}\lambda^2 A^2 \,, \tag{2.143}$$

where λ with $[\lambda] = 1$ ensures the canonical dimension for the field A. Upon use of the equations of motion

$$\Box B = \lambda^2 A \qquad \text{and} \qquad \Box A = 0 \tag{2.144}$$

one can eliminate A in order to restore eq. (2.142) up to a factor $\frac{1}{\lambda^2}$.

Performing now the usual quantization procedure with canonical momenta

$$\Pi_\phi = \partial\mathcal{L}/\partial(\partial_t \phi) \,, \tag{2.145}$$

leads to [48]

$$\begin{aligned} [\Pi_A, A(x')]|_{t=t'} &= [-\partial_t A, A(x')]|_{t=t'} = -i\delta^3(x-x') \\ [\Pi_B, B(x')]|_{t=t'} &= [-\partial_t B, B(x')]|_{t=t'} = -i\delta^3(x-x') \,. \end{aligned} \tag{2.146}$$

Employing the Green function for the Laplace operator $\Box\Delta(x-y) = \delta^4(x-y)$ and eq. (2.143), one derives from the above commutators

$$[A(x), A(y)] = 0, \quad [A(x), B(y)] = -i\Delta(x-y), \quad [B(x), B(y)] = -i\lambda^2\tilde{\Delta}(x-y) \tag{2.147}$$

where $\Box\tilde{\Delta} = \Delta$.

Fixing the vector $A_\mu = (1,0,0,0)$ (see 2.138) without loss of generality, the fields A and B are at oscillator level given by

$$\begin{aligned} A(x) &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3\sqrt{2\omega_k}}(a_k \exp(-ikx) + a_k^\dagger \exp(ikx)) \\ B(x) &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3\sqrt{2\omega_k}}\left\{\left(b_k - \lambda^2\left(\frac{1}{4\omega_k^2} - \frac{it}{2\omega_k}\right)a_k\right)\exp(-ikx)\right. \\ &\qquad\qquad \left. + \left(b_k^\dagger - \lambda^2\left(\frac{1}{4\omega_k^2} + \frac{it}{2\omega_k}\right)a_k^\dagger\right)\exp(-ikx)\right\} \,, \end{aligned} \tag{2.148}$$

where $\omega_k = \sqrt{(\vec{k})^2}$. Here the dependence of the fields A and B is expressed in the fact that the coefficients in the expansion of B contain the oscillators from A. Commutation relations are given by

$$[a_k, a_{k'}^\dagger] = [b_k, b_{k'}^\dagger] = 0 \qquad \text{and} \qquad [a_k, b_{k'}^\dagger] = [b_k, a_{k'}^\dagger] = \delta^3(k-k') \,. \tag{2.149}$$

Constructing now the Fock-space, one defines the vacuum and the one-particle states as

$$a_k|0\rangle = b_k|0\rangle = 0, \quad |k\rangle_a = a_k^\dagger|0\rangle \qquad \text{and} \qquad |k\rangle_b = b_k^\dagger|0\rangle \,. \tag{2.150}$$

Employing the commutation relations eq. (2.149) one can show the norm of the states to vanish ${}_a\langle k|k'\rangle_a = {}_b\langle k|k'\rangle_b = 0$ but the off-diagonal elements to yield

$${}_a\langle k|k'\rangle_b = \delta^3(k-k') \,. \tag{2.151}$$

Considering the momentum operator

$$P^\mu = \int \frac{\mathrm{d}^3k}{(2\pi)^3} k^\mu \left(a_k^\dagger b_k + b_k^\dagger a_k + \delta^{\mu 0} \frac{\lambda^2}{2\omega_k^2} a_k^\dagger a_k \right) \tag{2.152}$$

one can see the doublet structure already encountered in the classical solution eq. (2.138) to carry over to the quantum theory: the states eq. (2.150) transform in the same manner as the plane- and conformal wave part in eq. (2.140) above

$$\begin{aligned} P^\mu |k\rangle_a &= k^\mu |k\rangle_a \\ P^\mu |k\rangle_b &= k^\mu |k\rangle_b + \delta^{\mu 0} \frac{\lambda^2}{2\omega_k^2} |k\rangle_a \,. \end{aligned} \tag{2.153}$$

Besides of their unusual translational properties, states eq. (2.150) are not eigenstates of the energy-momentum operator. With a linear transformation on $|k\rangle_a$ and $|k\rangle_b$ one could cure this problem, but one of the new states would have negative norm-square then. States showing the behavior described above are called *dipole ghosts*.

Ghost states with negative squared norm and mixing of physical states is a serious problem for a theory: the unitarity principle is violated and therefore the physical interpretation of any result calculated in the theory remains questionable. In references [52, 53] those theories are discussed at great length without a definite conclusion.

When it comes to the explicit calculation of amplitudes, it is no problem to develop Feynman rules. In order to do so, one has to fix the gauge, which is done preferably with a higher-derivative gauge fixing term [54]. It is possible to find a gauge corresponding to the well-known deDonder gauge in ordinary gravity. The resulting propagator resembles the structure of the propagator in usual two-derivative gravity:

$$\begin{aligned} &\text{gravity} && \sim \frac{\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\sigma}}{k^2} \\ &\text{conformal gravity} && \sim \frac{\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - 5\,\eta_{\mu\rho}\eta_{\nu\sigma}}{k^4} \,. \end{aligned} \tag{2.154}$$

Despite of the drawbacks of the non-supersymmetric theory, several candidates for a maximal supersymmetric generalization of conformal gravity have been identified [49, 55, 56]. Linearized conformal supergravity is the theory constructed from the $\mathcal{N}$=4 supersymmetric generalization of the Weyl tensor eq. (2.133). However, this theory is not invariant under the full $\mathcal{N}$=4 superconformal group. In an attempt to cure this problem two additional theories have been identified by applying superconformal transformations to the linearized theory and adding correction terms iteratively:

- a "minimal" version of conformal supergravity interacting with $\mathcal{N}$=4 super Yang Mills with gauge group $SU(2) \times U(1)$
- a "non-minimal" $\mathcal{N}$=4 conformal supergravity.

While this iterative process has been completed for the minimal version, in non-minimal conformal supergravity not all terms necessary for closure of the superconformal algebra have been identified.

In order to compare with states represented by vertex operators in twistor string theory lateron, it is sufficient to limit the consideration to the linearized $\mathcal{N}$=4 conformal supergravity. The supersymmetric generalization of the Weyl tensor is a scalar superfield $W(x^{AA'}, \bar{\theta}^A_a, \theta^{aA'})$ satisfying the condition

$$\varepsilon^{abcd} D_{cA'} D^{A'}_e D^{B'}_d D_{fB'} W = \varepsilon_{efgh} \bar{D}^a_A \bar{D}^{gA} \bar{D}^{bB} \bar{D}^h_B \overline{W} \tag{2.155}$$

where $D_{aA'}$ and $\bar{D}^a_A$ are the usual superspace derivatives given by

$$D_{aA'} = \frac{\partial}{\partial \theta^{aA'}} + \bar{\theta}^A_a \partial_{AA'} \qquad \text{and} \qquad \bar{D}^a_A = \frac{\partial}{\partial \bar{\theta}^A_a} \tag{2.156}$$

in anti-chiral coordinates. The condition for W to be chiral can then be expressed as $\bar{D}^A_a W = 0$, which shows that W is independent of $\bar{\theta}$. Expanding the superfield W into its fermionic coordinates yields

$$\begin{aligned} W(x,\theta) \;=\; & \varphi + \theta^{aA'} \Lambda_{aA'} + (\theta^2)^{(ab)} E_{(ab)} + (\theta^2)^{(A'B')}_{[ab]} T^{[ab]}_{(A'B')} + (\theta^3)^{(A'B'C')}_d (\partial\eta)^d_{(A'B'C')} \\ & + (\theta^3)^{A'c}_{[ab]} \xi^{[ab]}_{A'c} + (\theta^4)^{a(A'B')}_b (\partial V)^b_{(A'B')a} + (\theta^4)^{(A'B'C'D')} \Psi_{A'B'C'D'} + (\theta^4)^{[ab]}_{[cd]} d^{[cd]}_{[ab]} \\ & + (\theta^5)^{A'[ab]}_c \partial_{AA'} \bar{\xi}^{Ac}_{[ab]} + (\theta^5)^{a(A'B'C')} (\partial\rho)_{a(A'B'C')} + (\theta^6)_{(ab)} \partial_\mu \partial^\mu \bar{E}^{(ab)} \\ & + (\theta^6)_{[ab](A'B')} \partial^{AA'} \partial^{BB'} \bar{T}^{[ab]}_{(AB)} + (\theta^7)_{aA'} (\partial^\mu \partial_\mu) \partial^{AA'} \bar{\Lambda}^a_A + (\theta^8)(\partial_\mu \partial^\mu)^2 \bar{\varphi}, \end{aligned} \tag{2.157}$$

where Ψ is the self-dual (SD) part of the Weyl-tensor

$$C_{\alpha\beta\gamma\delta} = \Psi^{ASD}_{(ABCD)} \varepsilon_{A'B'} \varepsilon_{C'D'} + \Psi^{SD}_{(A'B'C'D')} \varepsilon_{AB} \varepsilon_{CD} \,. \tag{2.158}$$

Moreover, the condition eq. (2.155) implies several Bianchi-identities, which allow writing components of the expansion in terms of the potentials η, V and ρ conveniently. In terms of W and the anti-chiral superfield $\overline{W}$ the linearized action for $\mathcal{N}$=4 conformal supergravity reads

$$S = \int \mathrm{d}^4x \, \mathrm{d}^8\theta W^2 + \text{c.c.} \tag{2.159}$$

which, upon performing the θ-integrations results in the component action

$$\begin{aligned} S = \int \mathrm{d}^4x \, \Big(& \varphi(\partial^\mu \partial_\mu)^2 \bar{\varphi} + \Lambda_{A'a} (\partial^\mu \partial_\mu) \partial^{A'A} \bar{\Lambda}^a_A + E_{(ab)} \partial_\mu \partial^\mu \bar{E}^{(ab)} \\ & + T^{[ab]}_{(A'B')} \partial^{A'A} \partial^{B'B} \bar{T}_{(AB)[ab]} + \xi^{[ab]}_{A'c} \partial^{A'A} \bar{\xi}^c_{A[ab]} + d^{[ab]}_{[cd]} d^{[cd]}_{[ab]} \\ & + (\partial V)^b_{(A'B')a} (\partial V)^{(A'B')a}_b + \Psi_{(A'B'C'D')} \Psi^{(A'B'C'D')} + (\partial\eta)^a_{(A'B'C')} (\partial\rho)^{(A'B'C')}_a \Big) \\ & + \text{c.c.} \,. \end{aligned} \tag{2.160}$$

Repeating the analysis sketched for the metric perturbation $\bar{h}_{\mu\nu}$ above, one can determine the degrees of freedom for each of the fields or potentials in eq. (2.160). The field content of linearized conformal supergravity is collected in table 4[16]. Whenever the same helicity occurs in a row twice, the corresponding particles constitute a dipole ghost.

[16]the field ρ can be expressed in terms of $\bar{\eta}$ by $\rho_{\mu aA'} = \sigma^\mu_{AA'} (\partial_\mu \bar{\eta}^A_{\nu a} - \partial_\nu \bar{\eta}^A_{\mu a} + \frac{1}{2} \varepsilon_{\mu\nu\tau\kappa} \partial^\tau \bar{\eta}^{\kappa A}_a)$

Fields	Helicity	$SU(4)$ representation
φ	$0, 0$	$\mathbf{1}$
$\Lambda_a^{A'}$	$-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}$	$\overline{\mathbf{4}}$
$E_{(ab)}$	0	$\overline{\mathbf{10}}$
$T^{[ab]}_{(A'B')}$	$-1, -1, 0$	$\mathbf{6}$
$\xi^{[ab]}_{A'c}$	$-\frac{1}{2}$	$\mathbf{20}$
$\eta_\mu^{aA'}$	$-\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}$	$\mathbf{4}$
$V_{\mu a}^b$	$1, -1$	$\mathbf{15}$
$d^{[cd]}_{[ab]}$	aux. field	$\mathbf{20}$
$e_\mu^{A'A}$	$2, 2, 1, -1, -2, -2$	$\mathbf{1}$
$\bar{\eta}_{\mu a}^A$	$\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{3}{2}$	$\overline{\mathbf{4}}$
$\bar{\xi}^{Ac}_{[ab]}$	$\frac{1}{2}$	$\overline{\mathbf{20}}$
$\bar{T}^{[ab]}_{(AB)}$	$1, 1, 0$	$\mathbf{6}$
$\bar{E}^{(ab)}$	0	$\mathbf{10}$
$\bar{\Lambda}_A^a$	$\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$	$\mathbf{4}$
$\bar{\varphi}$	$0, 0$	$\mathbf{1}$

Table 4: *Helicities and $SU(4)$ representations of states in $\mathcal{N}$=4 conformal supergravity.*

3 Maximally supersymmetric field theories in different formulations

3.1 String theory

String theory will be used in this work twofold: on the one hand side its twistor version will be investigated in the context of supergravities in section 4, while the low-energy limit of usual string theories will serve as a calculational tool for accessing otherwise unaccessible amplitudes in maximally supersymmetric field theories in section 5. However, the basic idea of string theory shall be discussed here in order to set the grounds for the introduction of twistor string theory in subsection 3.2.2. In addition, the low-energy limit of type I and type II string theory will be shown to be connected to $\mathcal{N}$=4 SYM and $\mathcal{N}$=8 supergravity in subsection 3.1.3.

3.1.1 Actions and amplitudes

In order to discuss the concept of string theory [57, 58, 59, 60, 9] below, it is useful to start with a pointlike particle. The action for a free point particle

$$S'_{pp} = -m \int_{\mathcal{C}} ds = -m \int \mathrm{d}\tau \sqrt{-g_{\mu\nu}(X) \frac{dX^\mu}{d\tau} \frac{dX^\nu}{\mathrm{d}\tau}} \tag{3.1}$$

is proportional to the length of its worldline, which in turn is parameterized by τ. In eq. (3.1), $g_{\mu\nu}$ denotes the metric in the target space, the space in which the motion of the particle takes place. By introducing a one-dimensional metric $\gamma_{\tau\tau}(\tau)$ on the worldline, the point particle action can be rewritten as

$$S_{pp} = \frac{1}{2} \int \mathrm{d}\tau \left(\xi^{-1} g_{\mu\nu}(X) \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} - \xi m^2 \right), \tag{3.2}$$

where $\xi = \sqrt{-\gamma_{\tau\tau}(\tau)}$, and eq. (3.1) can be restored by use of the equation of motion. For a point particle, X is a map from a one-dimensional parameter space into a d-dimensional target space.

Promoting the zero-dimensional point particle to an one-dimensional object, a string, the worldline is replaced by a two-dimensional parameter space Σ with coordinates (τ, σ), which is called worldsheet:
In analogy to the point particle case, equations of motion for a string moving in the target space can be obtained from minimizing the action, which is now proportional to the area of the worldsheet. Introducing again a metric γ on the worldsheet, one can conveniently write the action in conformal gauge [60] where $\gamma^{\alpha\beta} = e^{\phi(\sigma,\tau)} \eta^{\alpha\beta}$ and $\gamma = \det \gamma_{\alpha\beta}$

$$S_P = -\frac{1}{4\pi\alpha'} \int_\Sigma d\tau d\sigma \sqrt{-\gamma} \gamma^{\alpha\beta} g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu , \tag{3.3}$$

which reduces to

$$S_P = -\frac{1}{4\pi\alpha'} \int_\Sigma d\tau d\sigma \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu = -\frac{1}{2\pi\alpha'} \int_\Sigma d^2 z \partial X^\mu \bar{\partial} X_\mu \tag{3.4}$$

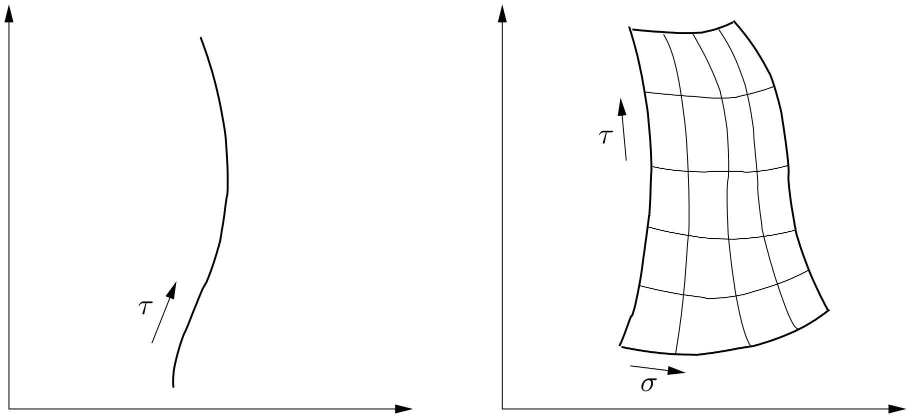


Figure 1: *Worldline of a particle and worldsheet of an open string*

for flat Minkowski target space. The second form is written in complex coordinates, which will be used below: $d^2z = dzd\bar{z} = 2d\tau d\sigma$, where $z = \tau + i\sigma$ and $\bar{z} = \tau - i\sigma$ [60]. Correspondingly, derivatives are defined as $\partial = \partial_z = \frac{1}{2}(\partial_\tau - i\partial_\sigma)$ and $\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_\tau + i\partial_\sigma)$. Being invariant under local diffeomorphisms, local Weyl-transformations of the metric γ and global Poincaré spacetime transformations, the second invariance renders the theory a *conformal field theory* [61, 62] on the worldsheet.

The parameter α' in eq. (3.3) is the inverse tension of the string, has dimension of $(\text{length})^2$ and is of the order of magnitude of squared Planck length: $\sqrt{\alpha'} \sim l_{\text{Planck}} = 1.6 \times 10^{-35} m$.

Strings come in either open or closed form. The closed form is obtained by identifying the endpoints of the string, thus the worldsheet swept out will be a cylinder. Accordingly, if the endpoints of the string are distinct, the string is called open. While it is possible to consider string theories with closed strings exclusively, open strings can always join their endpoints to form a closed string. Speaking of an open string theory therefore refers to the open subsector of a theory containing open and closed strings.

Amplitudes in string theories are calculated by inserting vertex operators into the worldsheet, evaluating their operator product expansions in the corresponding conformal field theory and integrating over all possible insertion points. For open-string tree amplitudes, any worldsheet can be conformally transformed into a disk with the vertex operators being inserted at the boundary. Correspondingly, closed string tree amplitudes can be calculated by inserting vertex operators at the surface of a sphere:

While for a point-particle quantum field theory the vanishing of the vacuum expectation value of a field $\langle \psi^\mu \rangle = 0$ is equivalent to the statement that the vacuum state ψ_0^μ is a solution to the classical equations of motion, the corresponding condition for a classical solution in string theory is the vanishing of the vacuum expectation value for any vertex operator of a physical state:

$$\langle V^\mu \rangle = 0. \tag{3.5}$$

Worldsheet conformal invariance implies equation (3.5) at string tree level. Employing a

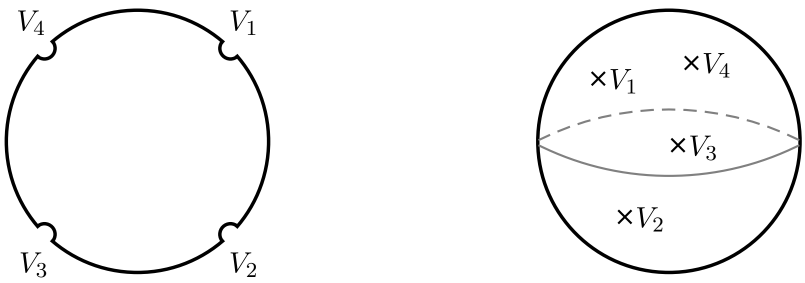


Figure 2: *Tree-level amplitudes in open and closed string theories*

similar argument, it is furthermore possible to see that a solution to string theory at the classical level corresponds to a conformally invariant σ-model[17].

Demanding local conformal worldsheet invariance corresponds to the *local* vanishing of the trace of the energy momentum tensor. This condition translates into remarkable equations in the target space: Einstein's equation in the presence of source terms and an antisymmetric tensor generalization of Maxwell's equations arise as conditions for the conformal invariance of the worldsheet theory [60].

While eq. (3.3) describes bosonic degrees of freedom on the target space exclusively, one would like to incorporate fermions into the string framework. Besides of fixing the dimension of the target space to $d = 10$, the introduction of fermionic fields avoids the appearance of particles with a negative mass: tachyons. One possible way to generalize eq. (3.3) is the introduction of ten-dimensional[18] Majorana-Weyl spinors θ_a^A with supersymmetry index $a \in \{1, \dots, \mathcal{N}\}$ and spinor index A

$$S = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-\gamma}\gamma^{\alpha\beta} g_{\mu\nu} \partial_\alpha \Pi^\mu \partial_\beta \Pi^\nu + \text{extra terms} \tag{3.6}$$

where

$$\Pi_a^\mu = \partial_\alpha \Pi^\mu - i\bar{\theta}^a \Gamma^\mu \partial_\alpha \theta^a \quad \text{with} \quad \bar{\theta} = \theta^\dagger \Gamma^0 \tag{3.7}$$

and $\Gamma^0 \cdots \Gamma^9$ are the ten-dimensional analogue of the γ-matrix algebra eq. (2.12). The extra terms in eq. (3.6) are combinations of the Majorana-Weyl spinors θ and the field X and are required ensure the so-called κ-symmetry of the action. This hidden symmetry, whose algebra closes only after employing the equations of motion, is a necessary consistency requirement. An elaborate discussion of the derivation and the precise form of the extra terms can be found in [58].

The requirement of κ-symmetry of eq. (3.6) has an additional consequence: only $\mathcal{N} = 0, 1, 2$ spacetime supersymmetries are allowed. Accordingly, variables θ^1 and θ^2 appropriate

[17] A proof and further treatment can be found in chapter 3.4.3 of [58].

[18] In ten dimensions it is possible to impose the Majorana and Weyl condition on a spinor simultaneously.

for $\mathcal{N}$=2 supersymmetry are considered, where the theories with $\mathcal{N}$=0, 1 can be obtained by setting one or both of them to zero.

Majorana-Weyl spinors in ten dimensions must be assigned a definite handedness. The relative choice of handedness for θ^1 and θ^2 distinguishes between different types of string theories.

- **Type I string theory:** For open strings, the spinors θ^1 and θ^2 have to exhibit the same handedness. In addition, open string boundary conditions reduce the expected $\mathcal{N}$=2 worldsheet supersymmetry to $\mathcal{N}$=1, which explains the name type I string theory. The theory is allowed to have a Yang-Mills gauge group, which in ten dimensions is fixed to be $SO(32)$ by consistency conditions at the quantum level. Type I string theory is a theory of open (and closed) strings allowing orientable as well as unorientable worldsheets.

- **Type II closed string theories:** Considering closed strings, either choice of relative handedness is possible. For opposite handedness of θ^1 and θ^2 the resulting theory involves oriented strings and exhibits $\mathcal{N}$=2 spacetime supersymmetry. This nonchiral theory is called type IIA theory. Requiring θ^1 and θ^2 to have equal handedness leads to the oriented chiral type IIB string theory, which is $\mathcal{N}$=2 target-space supersymmetric as well.

3.1.2 KLT relations

Tree-level amplitudes in closed and open string theories are linked by the Kawai-Lewellen-Tye (KLT) relations [63], which arise from the fact that any closed-string vertex operator can be represented as a product of two open-string vertex operators,

$$V^{\text{closed}}(z_i, \bar{z}_i) = V^{\text{open}}_{\text{left}}(z_i)\, V^{\text{open}}_{\text{right}}(\bar{z}_i)\,. \tag{3.8}$$

As mentioned above, insertion points z_i, $\bar{z}_i$ of vertex operators are integrated over a two-sphere in the closed-string amplitude, while in the open-string case the real z_i are integrated over the boundary of a disk. Any open string amplitude can be written as a multiple integral, whose integrand factorizes into two terms corresponding to the left and right sector and depending on z and $\bar{z}$ respectively. Performing the integrals will convolute the two sectors, however the action of the convolution can be made explicit as has been shown by KLT in ref. [63]. In particular, any closed string amplitude can be written as

$$M^n_{\text{closed}} = \sum_{P,P'} A^n_{\text{open}}(P) A^n_{\text{open}}(P') e^{i\pi F(P,P')}\,, \tag{3.9}$$

where P and P' are all possible cyclic permutations of the set $\{1 \dots n\}$. On closer inspection it turns out that the phase factor $e^{i\pi F(P,P')}$ is completely independent of the choice of string theories for the amplitudes: it will serve for combining two type I open string amplitudes into a closed type II amplitude as well as for determining heterotic string amplitudes by combining a bosonic theory with a supersymmetric one.

In addition, it can be shown that the number of permutations to sum over can be largely reduced by carefully examining branch cuts and contour deformations of the phase factor in complex plane and taking the $SL(2)$ symmetry of the worldsheets into account. Doing so, the final result for four-, five- and six-point amplitudes reads

$$M_4(1,2,3,4) = \frac{-i}{\alpha'\pi}\sin(\alpha'\pi s_{12})\, A_4(1,2,3,4)\, A_4(1,2,4,3)\,, \tag{3.10}$$

$$M_5(1,2,3,4,5) = \frac{i}{\alpha'^2\pi^2}\sin(\alpha'\pi s_{12})\sin(\alpha'\pi s_{34})\, A_5(1,2,3,4,5)\, A_5(2,1,4,3,5) + \mathcal{P}(2,3)\,, \tag{3.11}$$

$$\begin{aligned} M_6(1,2,3,4,5,6) = \frac{-i}{\alpha'^3\pi^3}\sin(\alpha'\pi s_{12})\sin(\alpha'\pi s_{45})\, A_6(1,2,3,4,5,6) \\ \times \big[\sin(\alpha'\pi s_{35})\, A_6(2,1,5,3,4,6) \\ + \sin(\alpha'\pi(s_{34}+s_{35}))\, A_6(2,1,5,4,3,6)\big] \\ + \mathcal{P}(2,3,4)\,. \end{aligned} \tag{3.12}$$

Formulae for higher-point amplitudes can be derived straightforwardly [63].

3.1.3 Low-energy effective actions of type I and type II string theories

The worldsheet actions eq. (3.3) and eq. (3.6) combined with appropriate boundary conditions contain the complete information about a free string theory. However, starting from these actions the mass spectrum of the theory can be shown to exhibit besides of massless modes an infinite number of massive modes with $m^2 = \frac{l}{\alpha'}$, $l = 1, 2, \ldots$. The mass difference between massless and massive particles in string theory is related to the to the Planck length via

$$\triangle m^2 \sim \frac{1}{\alpha'} \sim \frac{1}{l^2_{\text{Planck}}}. \tag{3.13}$$

Therefore, the first massive strings are expected to have a mass of order of the Planck mass $m_{\text{Planck}} = 1.22086 \times 10^{19} \frac{\text{GeV}}{c^2}$. This mass being several orders of magnitude larger than the masses encountered in the standard model, it is reasonable to concentrate on the massless sector of string theory.

For this sector it is possible to find a target space action, which is constructed such as to reproduce the amplitudes calculated in the worldsheet formalism. This low-energy effective action is a perturbative expansion in two parameters. While the string coupling constant g_s does play a role for string loop amplitudes not considered here, the expansion in the inverse tension α' is the key parameter in the relation between string theories and maximally supersymmetric field theories. While low-energy effective actions for string theories are well known to $\mathcal{O}(\alpha')$, the knowledge is limited to special terms beyond leading order.

In the limit $\alpha' \to 0$ the tension of the string becomes infinite, which prohibits any finite energy vibrations of the string. Without excited modes however, one will recover the solution to the equations of motion of a pointlike particle. Thus the limit $\alpha' \to 0$ corresponds to a point particle description.

While low-energy effective actions in ten dimensions are constructed by finding terms reproducing the known amplitude results, the process is more subtle for the desired low-energy effective action in four dimensions. Since in a trivial torus compactification from ten to four dimensions the collective[19] radius parameter r has to tend to zero in order to decouple motions in the internal six dimensions from the desired spacetime description, there are now two competing expansions around zero values: the expansion in small inverse string tension α' and the expansion in the compactification radius r.

In reference [64] this simultaneous limit has been studied. While it had already been shown that type I and type II string theories approach supersymmetrical Yang-Mills theory and extended supergravity in ten dimensions in the limit $\alpha' \to 0$ respectively, a careful treatment of the additional $r \to 0$ limit reveals that the leading terms in the four-dimensional low-energy effective actions for compactified type I and type II string theories coincide with $\mathcal{N}$=4 SYM and $\mathcal{N}$=8 supergravity theory in four dimensions. While in $\mathcal{N}$=4 SYM theory the leading string correction at $\mathcal{O}(\alpha'^2)$ is the well-known F^4-term [65, 66, 67], the leading correction to the $\mathcal{N}$=8 supergravity action occurs at $\mathcal{O}(\alpha'^3)$ and is the famous R^4 term [68] introduced in eq. (5.6) in subsection 5.1 below.

Of course, one of the most obvious question is: which of the properties and symmetries of supersymmetric field theories and string theories are combined in the low-energy effective actions? Although there are many results in this field of ongoing research, there is one fact which will be made use of below. Stieberger and Taylor have explicitly proven for open string theory on the disk that the form of the supersymmetric Ward identities (see subsection 2.5) to all orders in α' is identical to that in the corresponding four-dimensional field-theoretical limit [69]. Thus KLT relations are valid order by order in α' for string-corrected amplitudes.

3.2 Twistor space and twistor string theory

In this subsection, an introduction to twistor space and to twistor string theory is provided. After starting with the geometrical foundations of the (super-)twistor construction in subsection 3.2.1, twistor string theory will be discussed in subsection 3.2.2. Limiting the consideration to the foundations in the current subsection, the status of twistor string theory and recent developments will be elaborated on in subsection 4.1 below.

3.2.1 Twistor space, geometrical twistor construction

Twistor space is a concept to replace spacetime with a more fundamental structure. While spacetime locality is obscured, the twistor description of a theory is in particular adequate for theories exhibiting conformal symmetry, as for example $\mathcal{N}$=4 SYM theory. In the same manner as spinors transform in representations of the Lorentz group, twistors transform in representations of the conformal group.

The special nature of transformations of objects from spacetime into twistor space leads to a remarkable feature: spacetime fields describing particles, which in spacetime

[19] Without loss of generality one can assume all compact dimensions to be curled up with the same radius.

are constrained by their respective equations of motion, can be treated as objects free of constraints in twistor space, as will be explained in detail below[20].

In order to rigorously define the twistor correspondence, it is necessary to work with a complexified spacetime and furthermore introduce the notion of flag manifolds.

Complexified spacetime The complexified four-dimensional Minkowski spacetime $\mathbb{CM}$ is defined to be $\mathbb{C}^4$ with the metric $\eta = \mathrm{diag}(+,-,-,-)$. In analogy to the real situation considered in subsection 2.2, the complex Lorentz group $SO(1,3,\mathbb{C}) = SO(4,\mathbb{C})$ is isomorphic to $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, which again allows the decomposition eq. (2.22).

One additional structure will prove useful below: two-dimensional null-planes in complexified Minkowski space. A 2-plane in spacetime is null, if $\eta_{\mu\nu} v^\mu w^\nu = 0$ for all tangent vectors v, w. In order to further characterize the null-planes, one can associate to each null-plane a spacetime bivector $\pi = v \wedge w \, : \, \pi^{\mu\nu} = v^{[\mu} w^{\nu]}$, which determines the orthogonal complement to the tangent space of the null-plane. The bivector in turn can be decomposed into a self-dual (α-plane) and an anti-self-dual part (β-plane) by means of eq. (2.22)

$$\pi^{ABA'B'} = \varepsilon^{AB} \rho^{A'} \rho^{B'} + \varepsilon^{A'B'} \sigma^A \sigma^B . \tag{3.14}$$

Alternatively, an α-plane can be described in spacetime as the solution to

$$\sigma^A = x^{AA'} \rho_{A'} \tag{3.15}$$

for two fixed homogeneous coordinates ρ, σ. Solving (3.15) yields

$$x^{AA'} = x_0^{AA'} + \sigma^A \rho^{A'} , \tag{3.16}$$

which is the definition of an α-plane, as can be seen by orthogonality with the self-dual part of the bivector π:

$$\pi^{(SD)}_{ABA'B'} \sigma^A \rho^{A'} = \varepsilon_{AB} \rho_{A'} \rho_{B'} \sigma^A \rho^{A'} = 0 . \tag{3.17}$$

Flag manifolds A flag manifold of the n-dimensional vector space V is defined as

$$F_{d_1 \cdots d_m}(V) := \{(S_1, \ldots, S_m) | S_i \subset V, \dim S_i = d_i, S_1 \subset S_2 \subset \cdots \subset S_m\}, \tag{3.18}$$

where $\{d_1, \ldots, d_m\}$ is a sequence of positive integers satisfying $1 \leq d_1 < \cdots < d_m \leq n$ and the S_i are subspaces of V. Common examples are $F_1(\mathbb{C}^n) = \mathbb{C}P^{n-1}$ and the Grassmannian $F_k(\mathbb{C}^n) = G_{k,n}(\mathbb{C})$. In terms of homogeneous spaces, flag manifolds of $\mathbb{C}^n$ can equally well be defined by

$$F_{d_1 \cdots d_m}(\mathbb{C}^n) := \frac{U(n)}{U(d_1) \times U(d_2 - d_1) \times \cdots \times U(d_m - d_{m-1}) \times U(n - d_m)} , \tag{3.19}$$

which rather straightforwardly leads to the formula for their dimension

$$\dim_{\mathbb{C}} F_{d_1 \cdots d_m}(\mathbb{C}^n) = d_1(n - d_1) + (d_2 - d_1)(n - d_2) + \cdots + (d_m - d_{m-1})(n - d_m) . \tag{3.20}$$

The relation between twistor space and spacetime can be defined by a double fibration involving certain flag manifolds of the twistor space.

[20]This introduction to twistor space and the geometrical twistor construction partially follows refs. [70, 71]. The foundations of twistor theory have been laid in ref. [72].

Twistor space and double fibration Let $\mathbb{T} = \mathbb{C}^4$ be a four-dimensional complex vector space called twistor space. The twistor correspondence can then be expressed in the following double fibration

$$\begin{array}{ccc} & F_{12}(\mathbb{T}) & \\ {}^{\mu_1}\swarrow & & \searrow^{\mu_2} \\ F_1(\mathbb{T}) = \widetilde{\mathbb{PT}} & & F_2(\mathbb{T}) = \widetilde{\mathbb{CM}} \end{array} \tag{3.21}$$

where $F_{12}(\mathbb{T})$ is simultaneously fibered over $F_1(\mathbb{T})$ and $F_2(\mathbb{T})$ with corresponding projections $\mu_i(S_1, S_2) = S_i$ for $i = 1, 2$.

The three-dimensional complex space $F_1(\mathbb{T})$ is called *projective twistor space* and is the set of one-dimensional subspaces of $\mathbb{T} = \mathbb{C}^4$, which is $\mathbb{C}P^3 =: \widetilde{\mathbb{PT}}$. The space $F_2(\mathbb{T}) = G_{2,4}(\mathbb{C}) =: \widetilde{\mathbb{CM}}$ is referred to as *compactified complexified four-dimensional spacetime.* One can show the space of two-dimensional subspaces of $\mathbb{T}$, $G_{2,4}(\mathbb{C})$, to be isomorphic to $\widetilde{\mathbb{CM}}$ by identifying both spaces with the Klein quadric in $\mathbb{C}P^5$:

$$\widetilde{\mathbb{CM}} \cong G_{2,4}(\mathbb{C}) \cong \{X^{\alpha\beta} | X^{[\alpha\beta}X^{\gamma\delta]} = 0\} \subset \mathbb{C}P^5, \tag{3.22}$$

where $\{X^{\alpha\beta} = X^{[\alpha\beta]} | \alpha, \beta = 0,\ 1,\ 2,\ 3\}$ are homogeneous coordinates on $\mathbb{C}P^5$. Finally, the space $F_{12}(\mathbb{T})$ is referred to as *correspondence space*. Its local structure can be revealed by fixing a two-dimensional subspace S_2^0 of $\mathbb{T}$ and considering the corresponding subspace $f_{12} \in F_{12}$ over S_2^0

$$f_{12} = \{S_1 \in \mathbb{T} : S_1 \in S_2^0\}\,. \tag{3.23}$$

Since $S_2^0 \cong \mathbb{C}^2$, f_{12} is isomorphic to the set of one-dimensional subspaces of $\mathbb{C}^2$, which in turn is $\mathbb{C}P^1$. Remembering the set of all subspaces S_2^0 to be given by $G_{2,4}(\mathbb{C})$, one finds locally:

$$F_{12} = G_{2,4} \times \mathbb{C}P^1\,. \tag{3.24}$$

Local coordinates and twistor correspondence Given the local structure (3.24) of F_{12}, one can choose $x = x^{AA'}$, $A, A' = 1, 2$ as coordinates for $G_{2,4}$ and $\pi = [\pi_{A'}]$ for $\mathbb{C}P^1$, which results in the coordinate mapping

$$(x, [\pi]) \mapsto \left(\begin{bmatrix} x^{AA'}\pi_{A'} \\ \pi_{A'} \end{bmatrix}, \begin{bmatrix} x^{AA'} \\ \mathbb{1}_2 \end{bmatrix} \right) = \left(S_1^{(x,[\pi])}, S_2^{(x,[\pi])} \right) \in F_{12}\,. \tag{3.25}$$

Using the local coordinate representation, one can rewrite (3.21) as

$$\begin{array}{ccc} & (x, [\pi]) \in F_{12}(\mathbb{T}) & \\ {}^{\mu_1}\swarrow & & \searrow^{\mu_2} \\ (x \cdot \pi, \pi) \in \widetilde{\mathbb{PT}} & & x \in \widetilde{\mathbb{CM}}\,. \end{array} \tag{3.26}$$

The relation between $\widetilde{\mathbb{PT}}$ and $\widetilde{\mathbb{CM}}$ is called *twistor correspondence*. Employing the local coordinates defined above, the correspondence can be expressed as the *incidence relation*

$$\omega^A = x^{AA'}\pi_{A'} \tag{3.27}$$

where $Z^\alpha = (\omega^A, \pi_{A'})$ with $\alpha = 0, 1, 2, 3$ are homogeneous coordinates for $\widetilde{\mathbb{PT}}$. Holding a spacetime point x fixed in (3.27) and allowing the twistor coordinates to vary, will result in $\mu_1(\mu_2^{-1}(x)) \cong \mathbb{C}P^1$. If one considers the coordinates Z^α as non homogeneous, a point x in $\widetilde{\mathbb{CM}}$ can be represented as two-dimensional subspace of the (non-projective) twistor space. This subspace can be represented as a twistor-space bivector[21] $X^{\alpha\beta} = Z^{[\alpha}Y^{\beta]}$ satisfying the simplicity condition

$$X^{[\alpha\beta}X^{\gamma\delta]} = 0\,. \tag{3.28}$$

The other way around, interpreting the incidence relation for fixed Z^α, defines an α-plane (3.14) in $\widetilde{\mathbb{CM}}$: $\mu_2(\mu_1^{-1}(Z)) \cong \mathbb{C}P^2$. So one has the following correspondence between points and subsets of spacetime and twistor space:

$$\begin{aligned} \mathbb{C}P^1 \subset \widetilde{\mathbb{PT}} \quad &\leftrightarrow \quad \textit{point in } \widetilde{\mathbb{CM}} \\ \textit{point in } \widetilde{\mathbb{PT}} \quad &\leftrightarrow \quad \mathbb{C}P^2 \in \widetilde{\mathbb{CM}}\,. \end{aligned} \tag{3.29}$$

Compact spacetime and infinity twistor In the paragraphs above, the twistor correspondence has been established for compactified four-dimensional complexified spacetime $\widetilde{\mathbb{CM}}$, which is invariant under the full conformal group and can be obtained from the usual Poincaré-invariant spacetime $\mathbb{CM}$ by adding a point at infinity

$$\widetilde{\mathbb{CM}} = \mathbb{CM} \cup \infty\,. \tag{3.30}$$

What is the analogue of the point at infinity in projective twistor space $\widetilde{\mathbb{PT}}$? The incidence relation is degenerated for $\pi_{A'} = 0$, because for any $x \in \mathbb{CM}$ eq. (3.27) would lead to $\omega^A = 0$. So the subspace corresponding to $\infty \in \widetilde{\mathbb{CM}}$ is the $\mathbb{C}P^1 \subset \widetilde{\mathbb{PT}}$ defined by $\pi_{A'} = 0$. Accordingly, one can identify the projective twistor space of $\mathbb{CM}$ with a subset of $\mathbb{C}P^3$:

$$\mathbb{CM} \quad \leftrightarrow \quad \mathbb{PT} \cong \mathbb{C}P^3 - \mathbb{C}P^1\,. \tag{3.31}$$

In order to remain in a compact twistor setup and simultaneously work with a non-compact spacetime, one can mark any point $i \in \widetilde{\mathbb{CM}}$ to be the point at infinity. The subgroup of the conformal group $SL(4,\mathbb{C})$ which leaves the point i invariant is the Poincaré group. By marking this point and requiring its invariance one implicitly sets a length scale for spacetime, which is chosen to be unity. Again, this point i can be represented as a bivector $I^{\alpha\beta} \in \widetilde{\mathbb{PT}}$ satisfying the simplicity condition (3.28). This bivector is called the *infinity twistor*.

[21] Those bivectors are exactly the homogeneous coordinates of $\mathbb{C}P^5$ which have been introduced after eq. (3.22).

Identifying the complex projective line $\pi_{A'} = 0$ with the point $\infty \in \widetilde{\mathbb{CM}}$ fixes the explicit form of the infinity twistor:

$$I^{\alpha\beta} = \varepsilon^{AB} \quad \leftrightarrow \quad I_{\alpha\beta} = \varepsilon^{A'B'} . \tag{3.32}$$

Alternatively, there is also a representation of the infinity twistor as a 2-form τ on $\mathbb{T}$:

$$\tau = \frac{1}{2} I_{\alpha\beta} \mathrm{d}Z^\alpha \wedge \mathrm{d}Z^\beta , \tag{3.33}$$

where the dual of the infinity twistor is defined to be $I_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} I^{\gamma\delta}$. It is furthermore useful to define a one-form $k = I_{\alpha\beta} Z^\alpha \mathrm{d}Z^\beta = \varepsilon^{A'B'} \pi_{A'} \wedge \mathrm{d}\pi_{B'}$ from which τ can be obtained via $\tau = \frac{1}{2}\mathrm{d}k$. The one-form k will be one major ingredient for the modified version of a twistor string theory discussed in section 4 below.

Real spacetimes Although the discussion of the twistor construction in the last paragraphs has been performed for complexified Minkowski spacetime, the final goal is to calculate physical objects in real spacetimes. In particular, real slices are defined as the set of fixed points of an antiholomorphic involution $\tau : \widetilde{\mathbb{CM}} \to \widetilde{\mathbb{CM}}$, where the specific choice of τ determines the resulting spacetime signature [70, 71]. Choosing appropriate coordinates, this involution can be expressed as complex conjugation. On each of those real slices the group $SL(4, \mathbb{C})$ reduces to one of its real forms. The involution τ on $\mathbb{CM}$ induces a map on twistor space. For Lorentzian $(+, -, -, -)$ and split $(+, +, -, -)$ signature real spacetimes, the involutions and maps will be characterized below.

- **Lorentzian signature** The involution resulting in a real slice of $\mathbb{CM}$ with Lorentzian signature is given by hermitian conjugation:

$$\tau_L : \widetilde{\mathbb{CM}} \to \widetilde{\mathbb{CM}}, \quad \sigma(x) = x^\dagger , \tag{3.34}$$

where suitable local coordinates are defined as[22]

$$x^{AA'} = \sigma_\mu^{AA'} x^\mu \quad \text{with} \quad \sigma_\mu^{AA'} = (\sigma_0, \sigma_3, \sigma_1, \sigma_2). \tag{3.35}$$

The real subgroup of $SL(4, \mathbb{C})$ preserving the Lorentzian slice is the real conformal group $SU(2, 2) = Spin(2, 4)$. In order to define the corresponding induced map τ_L on twistor space, one has to take care of an additional subtlety: the involution τ_L switches the role of α-planes and β-planes in twistor space, because the indices A and A' are interchanged, which in turn carries over to a change in indices of the bivector defining the null planes. The space of β-planes in $\widetilde{\mathbb{CM}}$ is the dual twistor space $\mathbb{T}^*$.

Furthermore, $SU(2, 2)$ preserves a Hermitian metric $\Sigma_{\alpha\bar{\beta}}$ on $\mathbb{T}$ (and $\widetilde{\mathbb{PT}}$), which in turn defines a map from the complex conjugated twistor space $\bar{\mathbb{T}}$ to the dual twistor space $\mathbb{T}^*$:

$$\sigma_\Sigma : \bar{\mathbb{T}} \to \mathbb{T}^* \quad \text{with} \quad \bar{Z}^{\bar{\alpha}} = (Z^\alpha)^* \to \Sigma_{\alpha\bar{\beta}} \bar{Z}^{\bar{\beta}} . \tag{3.36}$$

[22]see eq. (2.14) for the definition for Pauli matrices

Employing the map σ_Σ one can identify any complex conjugated twistor with its dual twistor. Thus one finds the action of the map τ on twistor space to be

$$\tau_L : \widetilde{\mathbb{PT}} \to \widetilde{\mathbb{PT}}^*, \tau_L : Z^\alpha \to \bar{Z}_\alpha = \Sigma_{\alpha\bar{\beta}}\bar{Z}^{\bar{\beta}}. \tag{3.37}$$

- **Split signature** The real slice with split signature is just the subspace of $\mathbb{CM}$ where $x^{AA'}$ is real. So the involution τ_S is given by complex conjugation

$$\tau_S : \widetilde{\mathbb{CM}} \to \widetilde{\mathbb{CM}}, \quad \sigma(x) = x^*, \tag{3.38}$$

where again $x^{AA'} = \sigma_\mu^{AA'} x^\mu$ but now appropriate local coordinates are obtained from $\sigma_\mu^{AA'} = (\sigma_0, i\sigma_3, \sigma_1, \sigma_2)$. The subgroup preserving the Kleinian metric is $SL(4,\mathbb{R}) = Spin(3,3)$.

In split signature the ordinary complex conjugation on $\mathbb{CM}$ is carries over to ordinary component-by-component complex conjugation on twistor space

$$\tau_S : \widetilde{\mathbb{PT}} \to \widetilde{\mathbb{PT}} \qquad \tau_S : Z^\alpha \to (Z^\alpha)^*. \tag{3.39}$$

Hence the real slice of $\widetilde{\mathbb{CM}}$ corresponds to $\mathbb{T}_\mathbb{R} = \mathbb{R}^4 \in \mathbb{T}$ and therefore $\widetilde{\mathbb{PT}}_\mathbb{R} = \widetilde{\mathbb{R}P}^3 \in \widetilde{\mathbb{PT}}$.

Penrose-Ward transformation The correspondence eq. (3.29) above between points and subsets can be extended to more advanced geometrical analytical objects, such as holomorphic vector bundles and homology classes.

Starting from an object in twistor space, the pull-back of this object onto the correspondence space must be constant along the fiber $F_{12} \to \widetilde{\mathbb{PT}}$. Projected onto spacetime, this constancy condition can be expressed in form of a partial differential equation. In other words: objects constrained by (particular) partial differential equations in spacetime can be studied as free objects in twistor space.

Identifying the constrained spacetime objects with fields describing particles and obeying certain field equations makes clear where the power of the twistor construction comes from: the corresponding free objects in twistor space in turn are elements of certain cohomology groups, for which correlators can be defined. Hereby the field equations are taken into account already by the formalism itself.

Explicitly, the Penrose-Ward transformation [70] relates spacetime fields ϕ of helicity h defined on a suitable region $U \subset \widetilde{\mathbb{CM}}$ and satisfying the massless field equation with elements of the cohomology group $H^1(\hat{U}, O(2h-2))$, where $\hat{U} = \mu_2^{-1}(\mu_1(U)) \subset \widetilde{\mathbb{PT}}$. An element of $H^1(\hat{U}, O(2h-2))$ takes values in the bundle $O(2h-2)$ and is therefore homogeneous of degree $2h-2$ in $Z^\alpha \in \widetilde{\mathbb{PT}}$.

$$\begin{array}{ccc} & U' & \\ {}^{\mu_1}\swarrow & & \searrow^{\mu_2} \\ \hat{U} \subset \widetilde{\mathbb{PT}} & & U \subset \widetilde{\mathbb{CM}} \end{array} \tag{3.40}$$

Any region $\hat{U} \subset \widetilde{\mathbb{PT}}$ can be covered by two open subsets V_1 and V_2 (for example $V_1 = \{\pi_{1'} = 0\}$ and $V_2 = \{\pi_{2'} = 0\}$). Employing this covering, any element of the sheaf cohomology group $H^1(\hat{U}, O(2h-2))$ can be represented as a holomorphic function $f(Z^\alpha)$ of degree $2h-2$ defined on the region $V_1 \cap V_2$. Being a Čech cocycle, $f(Z^\alpha)$ is subject to the coboundary transformation

$$f \to f' = f + f_1 - f_2 \,, \tag{3.41}$$

where f_1 and f_2 are holomorphic functions of degree $2h-2$ on V_1 and V_2 respectively.

The first step in performing the transformation is to start from a function $f(x^{AA'}\pi_{A'}, \pi_{A'})$ on $\hat{U} \subset \widetilde{\mathbb{PT}}$ and to pull it back to a holomorphic function $g(x^a, \pi_{A'})$ of degree $2h-2$ on $\mu_1^{-1}(V_1) \cap \mu_1^{-1}(V_2) \subset U'$. The second step consists in the integration of $\pi_{A'}$ over a suitable circle Γ in $V_1 \cap V_2 \cap \{\omega^A = x^{AA'}\pi_{A'}\}$, where the incidence relation is taken care of by the choice of the contour [73]. The integration measure on the $\mathbb{C}P^1$ to integrate over is $\triangle\pi = \pi_{A'}\mathrm{d}\pi^{A'}$. However, since one has to chose either V_1 or V_2 as a coordinate chart, the integration measure can be taken to be $\triangle\pi = \pi_{1'}\mathrm{d}\pi_{2'}$ on V_2.

One additional condition has to be taken care of: in order for the result to be independent of continuous deformations of the contour Γ, the exterior derivative of the integrand has to vanish, which is equivalent to requiring the integrand to be of homogeneity degree 0. Therefore a suitable number of weighting factors $\pi_{A'}$ or $\frac{\partial}{\partial\omega^A}$ have to be adjoined in order to raise or lower the degree respectively. This particular choice of factors for raising and lowering the degree of homogeneity is implied by using $\pi_{A'}$ as local coordinates for the fiber $\mathbb{C}P^1$, which in turn dictates the form of the incidence relation eq. (3.27). Choosing ω^A instead of $\pi_{A'}$ as a factor to raise the degree of homogeneity would spoil Poincaré invariance due to the position dependence of ω. Accordingly, the incidence relation singles out $\frac{\partial}{\partial\omega^A}$ as translational invariant object.

For $h \leq 0$ the Penrose-Ward transformation is

$$\phi_{(A'_1 \ldots A'_n)}(x^b) = \int_\Gamma \pi_{A'_1} \cdots \pi_{A'_n} g(x^b, \pi_{B'}) \triangle\pi \,, \tag{3.42}$$

while for positive helicity

$$\phi_{(A_1 \ldots A_n)}(x^b) = \int_\Gamma \frac{\partial}{\partial\omega^{A_1}} \cdots \frac{\partial}{\partial\omega^{A_n}} g(x^b, \pi_{B'}) \triangle\pi \,. \tag{3.43}$$

As already mentioned above, the spacetime fields ϕ satisfy massless field equations,

$$\nabla^{AA'}\phi_{(A'_1 \ldots A'_n)} = 0, \quad \nabla^{AA'}\phi_{(A_1 \ldots A_n)} = 0 \quad \text{and} \quad \Box\phi = 0 \tag{3.44}$$

as can be seen from Eqs. (3.42) and (3.43) by performing the differentiation under the integral sign and employing the consistency condition of the pull-back $f \to g$:

$$\nabla_{AA'} g = \pi_{A'} \frac{\partial}{\partial\omega^A} f \,. \tag{3.45}$$

The twistor-analogue of a wavefunction for a scalar particle has to be a holomorphic function of degree -2 representing a Čech-cocycle on twistor space, whose Penrose-Ward transformation is the usual plane wave solution for the massless field equation in spacetime

$$\phi(x) = \exp(ix^a p_a) \,. \tag{3.46}$$

A consistent choice for the twistor-space wave function reads

$$\phi(Z) = \int \frac{\mathrm{d}k}{k} \prod_{A'=1}^{2} \delta(k\pi^{A'} - p^{A'}) \exp(ik\omega^A p_A)\,, \tag{3.47}$$

where p_A and $p^{A'}$ are momentum spinors[23] defined in eq. (2.22). The variable k to be integrated over takes the role of the $GL(1)$-weight ensuring the projective nature of $\mathbb{PT}$. If an appropriate δ-function ensuring the proportionality of $p^{A'}$ and $\pi^{A'}$ is present as in eq. (3.47), a possible choice of k is

$$k = \left(\frac{p}{\pi}\right) = \left(\frac{p\xi}{\pi\xi}\right) = \left(\frac{p_{A'}\xi^{A'}}{\pi_{B'}\xi^{B'}}\right)\,, \tag{3.48}$$

where ξ is a reference spinor satisfying $\xi \cdot \pi \neq 0$. One can perform the k-integration and thereby remove one of the δ-functions to obtain another convenient form of the twistor-space wavefunction:

$$\phi(Z) = \left(\frac{p}{\pi}\right) \delta(\pi p) \exp\left(i\omega p \left(\frac{p}{\pi}\right)\right)\,. \tag{3.49}$$

A couple of identities involving the weighting factor $\left(\frac{p}{\pi}\right)$ will prove useful,

$$\begin{aligned} \left(\frac{p}{\pi}\right) \delta(\pi p) = \left(\frac{p_{1'}}{\pi_{1'}}\right) \delta(\pi p) = \left(\frac{p_{2'}}{\pi_{2'}}\right) \delta(\pi p)\,, \\ \pi_{B'} \left(\frac{p}{\pi}\right) \delta(\pi p) = p_{B'} \delta(\pi p) \quad p_{B'} \left(\frac{\pi}{p}\right) \delta(\pi p) = \pi_{B'} \delta(\pi p)\,, \end{aligned} \tag{3.50}$$

in showing that eq. (3.49) indeed transforms into eq. (3.45),

$$\begin{aligned} \phi(Z) &= \int_{\omega^A = x^{AA'}\pi_{A'}} \mathrm{d}\pi_{2'}\pi_{1'} \left(\frac{p}{\pi}\right) \exp\left(i\,\omega p \left(\frac{p}{\pi}\right)\right) \delta(\pi p) \\ &= \int_{\omega^A = x^{AA'}\pi_{A'}} \mathrm{d}\pi_{2'} \exp\left(i\,\omega p \left(\frac{p}{\pi}\right)\right) \delta\left(\pi_{2'} - \pi_{1'}\frac{p_{2'}}{p_{1'}}\right) \\ &= \int_{\omega^A = x^{AA'}\pi_{A'}} \mathrm{d}\pi_{2'} \exp\left(i\,x^{CC'} p_C \pi_{C'} \left(\frac{p}{\pi}\right)\right) \delta\left(\pi_{2'} - \pi_{1'}\frac{p_{2'}}{p_{1'}}\right) \\ &= \exp(i\,x^{CC'} p_C p_{C'}) = \exp(i\,x^a p_a). \end{aligned} \tag{3.51}$$

Supertwistor space In order to obtain a supersymmetric extension of twistor space, one can accompany the spacetime coordinates $x^{AA'}$ by anticommuting spacetime spinors $\theta^A_a, \tilde\theta^{aA'}$, $a = \{1, \ldots \mathcal{N}\}$, which transform in an $\mathcal{N}$-dimensional representation of an R-symmetry group. While this group is $SU(\mathcal{N})$ for Lorentzian signature, it will be $GL(\mathcal{N}, \mathbb{C})$ or $SL(\mathcal{N}, \mathbb{C})$ for split signature.

The complexified superconformal group $SL(4|\mathcal{N}, \mathbb{C})$, which is an extended version of $SL(4, \mathbb{C})$, is naturally realized on $\mathbb{C}^{4|\mathcal{N}}$ with coordinates $Z^I = (Z^\alpha, \psi^a) = (\omega^A, \pi_{A'}, \psi^a)$.

[23] Although momentum spinors are usually denotet by greek letters, here and in the twistor-related literature p_A and $p^{A'}$ are used in order to avoid confusion with the common choice of twistor variables π, ω.

The supertwistor space $\mathbb{T}_{[\mathcal{N}]}$ is the subset of the supersymmetrically extended complexified spacetime, on which $Z^\alpha \neq 0$:

$$\mathbb{T}_{[\mathcal{N}]} = \mathbb{C}^{4|\mathcal{N}} - \mathbb{C}^{0|\mathcal{N}} . \tag{3.52}$$

Accordingly, one can define the projective supertwistor space as equivalence class under complex scalings

$$\widetilde{\mathbb{PT}}_{[\mathcal{N}]} = \mathbb{C}P^{3|\mathcal{N}} = Z^I \sim \lambda Z^I | Z^I \in \mathbb{T}_{[\mathcal{N}]}, \lambda \in \mathbb{C}^\times . \tag{3.53}$$

In order to consider double fibrations involving flag manifolds of the supertwistor space, the definition of flag manifolds has to be extended.
A **flag supermanifold** over the vector space $\mathbb{C}^{m|n}$ is defined as

$$F_{d_1 \cdots d_k}(\mathbb{C}^{m|n}) := \{(S_1, \dots, S_k) | S_i \subset \mathbb{C}^{m|n}, \operatorname{rank} S_i = d_i = p_i|q_i, S_1 \subset S_2 \subset \cdots \subset S_k\} . \tag{3.54}$$

In analogy to the purely even situation eq. (3.19), an equivalent definition in terms of a quotient of groups $U(m|n)$ exists.

Contrary to the situation in the non supersymmetric case there are now several possibilities to choose fibrations over $\mathbb{T}_{[\mathcal{N}]}$ corresponding to anti-chiral, chiral and full Minkowski-superspace respectively. Leading to different incidence relations, the resulting twistor correspondences will be distinct as well. Only the compactified anti-chiral Minkowski superspace $\widetilde{\mathbb{CM}}^-_{[\mathcal{N}]}$ will be used below, so the following considerations will be limited to this case. The correspondence between projective supertwistor space and chiral Minkowski superspace can be depicted as

$$\begin{array}{ccc} & F_{1|0,2|0}(\mathbb{T}_{[\mathcal{N}]}) & \\ \mu_1 \swarrow & & \searrow \mu_2 \\ F_{1|0}(\mathbb{T}_{[\mathcal{N}]}) = \widetilde{\mathbb{PT}}_{[\mathcal{N}]} & & F_{2|0}(\mathbb{T}_{[\mathcal{N}]}) = \widetilde{\mathbb{CM}}^-_{[\mathcal{N}]} . \end{array} \tag{3.55}$$

Anti-chiral Minkowski superspace with coordinates $(x^{AA'}, \theta^{aA'})$ can be defined as the fibration of $\mathbb{C}P^{1|0}$ over $\widetilde{\mathbb{PT}}_{[\mathcal{N}]}$ employing the incidence relation

$$(\omega^A, \pi_{A'}, \psi^a) = (x^{AA'}\pi_{A'}, \pi_{A'}, \theta^{aA'}\pi_{A'}) . \tag{3.56}$$

What is the analogue of the infinity twistor eq. (3.32) in supertwistorspace? In order to find the supertwistor space corresponding to the uncompactified anti-chiral Minkowski space $\mathbb{CM}^-_{[\mathcal{N}]}$, one has to again fix the infinity twistor in $\mathbb{PT}_{[\mathcal{N}]}$. The infinity twistor corresponds to a superline $\mathbb{C}P^{1|\mathcal{N}}$ and one can choose coordinates such that

$$I_{[\mathcal{N}]} = \{\pi_{A'} = 0\}. \tag{3.57}$$

Considering now the incidence relation eq. (3.56), one can indeed see that this choice is correct: by choosing $\pi_{A'} = 0$, the coordinates $x^{AA'}$ and $\theta^{aA'}$ in anti-chiral Minkowski superspace have to take infinite values for fixed finite nonzero values of ω^A and ψ^a. As eq. (3.57) is a condition on $\pi_{A'}$ only, one can choose the same form for the infinity twistor as in the non supersymmetric case: $I^{\alpha\beta} = \varepsilon^{AB}, \quad I_{\alpha\beta} = \varepsilon^{A'B'}$ (cf. eq. (3.32)).

In parallel to the superwavefunction eq. (2.80) one can equally well define a superwavefunction on supertwistor space

$$\phi(Z) = \int \frac{\mathrm{d}k}{k} \prod_{A'=1}^{2} \delta(k\pi^{A'} - p^{A'}) \exp(ik\omega^A p_A) u(k\psi), \tag{3.58}$$

where the expansion in fermionic coordinates is given by

$$u_\phi(k\psi) = \phi_0 + k\phi_{1a}\psi^a + \tfrac{k^2}{2}\phi_{2ab}\psi^a\psi^b + \tfrac{k^3}{3!}\phi_{3abc}\psi^a\psi^b\psi^c + \tfrac{k^4}{4!}\phi_{4abcd}\psi^a\psi^b\psi^c\psi^d. \tag{3.59}$$

Here k is the factor of $GL(1,\mathbb{R})$-weight -1 introduced in (3.48) which keeps $\phi_{1a} \dots \phi_{4abcd}$ weightless.

3.2.2 Twistor string theory

The original proposal for a twistor string theory by Witten [51] used a topological B-model with target space $\mathbb{C}P^{3|4}$, which is related to four-dimensional spacetime by Penrose's twistor construction [72] described in subsection 3.2.1. In this context, open string states can be identified with gauge fields in spacetime and closed strings correspond to conformal supergravity. The theory describes a duality between the perturbative expansion of $\mathcal{N}$=4 super-Yang-Mills and the D-instanton expansion of string theory. As a consequence, amplitudes localize on holomorphically embedded algebraic curves in twistor space.

An alternative string theory in twistor space was formulated by Berkovits [74] shortly after, in which both gauge theory and conformal supergravity states come from open string vertex operators. It is the Berkovits formulation of twistor string theory which is introduced below.

Open twistor string theory is a theory of maps from a worldsheet Σ onto a supertwistor space with coordinates $Z^I = (\omega^A, \pi_{A'}, \psi^a)$, $\bar{Z}^I$ defined in the previous paragraph. Two different but equivalent formulations are available for the open string theory. In the first formulation, which will be used later on, the worldsheet Σ is endowed with Euclidean signature. Employing a twistor correspondence which results in anti-chiral superspace as introduced in subsection 3.2.1, this implies a complex target space $\mathbb{C}P^{3|4}$, where Z^I is the complex conjugate of $\bar{Z}^I$. This is the setup used in references [51, 74, 75, 73]. Alternatively one can consider a worldsheet with Lorentzian signature. In this scenario Z^I and $\bar{Z}^I$ are real independent coordinates, such that the target space will be $\mathbb{R}P^{3|4} \times \mathbb{R}P^{3|4}$.

The worldsheet action for open twistor string theory is given by

$$S = \int \mathrm{d}^2 z \left(Y_I \bar{\partial} Z^I + \bar{Y}_I \partial \bar{Z}^I + \bar{A} J + A \bar{J} \right) + S_C, \tag{3.60}$$

where Z and $\bar{Z}$ have conformal dimensions $(0,0)$, and Y, $\bar{Y}$ are their conjugate variables with conformal dimensions $(1,0)$ and $(0,1)$ respectively. Coupling to currents

$$J = Y_I Z^I, \qquad \bar{J} = \bar{Y}_I \bar{Z}^I, \tag{3.61}$$

the worldsheet gauge fields A and $\bar{A}$ ensure that the theory is defined on projective super-twistor space $\mathbb{C}P^{3|4}$. Consequently, the action (3.60) exhibits a local $GL(1,\mathbb{C})$-symmetry:

$$Z^I \to gZ^I, \qquad Y_I \to g^{-1}Y_I, \qquad \bar{Z}^I \to \bar{g}\bar{Z}^I, \qquad \bar{Y}_I \to \bar{g}^{-1}\bar{Y}_I, \tag{3.62}$$

$$\bar{A} \to \bar{A} - g^{-1}\bar{\partial}g, \qquad A \to A - \bar{g}^{-1}\partial\bar{g}. \tag{3.63}$$

The last part of the action, S_C, denotes a conformal field theory with central charge $c_c = 28$, which is assumed to include a current algebra of some gauge group G. This additional system is required to cancel the conformal anomaly of the worldsheet theory as will be shown below.

The equations of motion originating in (3.60) for the fields Z^I and Y_I

$$\begin{aligned} (\bar{\partial} + \bar{A})Z^I &= (\partial + A)Z^I = 0, \\ (\bar{\partial} - \bar{A})Y_I &= (\partial + A)Y_I = 0 \end{aligned} \tag{3.64}$$

are accompanied by constraints from the Lagrange multipliers A and $\bar{A}$

$$\bar{J} = 0 \qquad J = 0 \tag{3.65}$$

and the condition for the endpoints of the open string

$$nY_I\delta Z^I + \bar{n}\bar{Y}_I\delta Z^I = 0\,. \tag{3.66}$$

Equation (3.66) can be solved by

$$\bar{Z} = UZ, \qquad Yn = -U^{-1}\bar{Y}\bar{n}\,, \tag{3.67}$$

where $U = e^{2i\alpha}$ for some function α varying over the boundary and being furthermore continuous up to multiples of π. Furthermore $|U| = 1$ on the boundary since Z^I and $\bar{Z}^I$ are complex conjugated. Using the gauge freedom to set the gauge fields A and $\bar{A}$ to zero, forces the information about the gauge structure of the theory to reside in the boundary condition (3.66) or in the transition functions for different coordinate patches in the case of a nontrivial worldsheet topology. Some topologies including the annulus are studied in [75], while the implications of the boundary for tree amplitudes will be discussed in subsection 3.2.5 below.

The boundary condition (3.66), restricts the open string to live on a subspace $\mathbb{R}P^{3|4}$, with isometry group $SL(4|4,\mathbb{R})$ instead of $SL(4|4,\mathbb{C})$. Therefore, open string operators can be expressed in terms of Z, Y and a set of variables originating from S_C exclusively. Additionally, the scaling symmetry group is broken into $GL(1,\mathbb{R})$ by the boundary condition. One further consequence of the target space being $\mathbb{R}P^{3|4}$, is the fact that the open string theory corresponds to a spacetime theory defined in Kleinian (split) signature $(++--)$. The following amplitude calculations assume that the analytical continuation to Lorentzian signature exists and is well-defined.

Generators for the Virasoro and $GL(1,\mathbb{R})$-symmetry on the boundary are

$$T = Y_I\partial Z^I + T_C, \qquad J = Y_I Z^I, \tag{3.68}$$

where T_C is the stress tensor for the current algebra. Quantization produces the well-known (b, c) ghost system of conformal weight $(2, -1)$ in combination with ghosts (u, v) of weight $(1, 0)$ for the $GL(1, \mathbb{R})$-gauge symmetry. The BRST-charge on the boundary is then given as

$$Q = \int \mathrm{d}z(cT + vJ + cu\partial v + cb\partial c) \tag{3.69}$$

where the requirement of $c_c = 28$ now becomes clear: by cancelling the contributions to the conformal anomaly from the bc-system with $c_{bc} = -26$ and the uv-system, $c_{uv} = -2$, the additional system S_c ensures the nilpotence of the operator Q.

Interestingly, there are no contributions to the conformal anomaly from Z^I and Y_I, because fermionic and bosonic components are of equal count. This arguments carries over to the current J, which explains the absence of an $GL(1, \mathbb{R})$-anomaly.

3.2.3 Twistor-string vertex operators for $\mathcal{N}$=4 SYM

Vertex operators representing physical states need to be primary fields with respect to the generators (3.68). The simplest form will represent SYM-states and can be constructed combining the currents j_r from the gauge group G in S_C with any field $\phi(Z)$ having zero conformal dimension and being invariant under $GL(1, \mathbb{R})$:

$$V_\phi = j_r \phi^r(Z), \quad r = 1, \ldots, \dim G\,. \tag{3.70}$$

The particle spectrum represented by a twistor string vertex operator can be determined applying the Penrose transformation described in subsection 3.2.1 to the twistor wavefunction $\phi^r(Z)$. Being neutral under $GL(1, \mathbb{R})$-scalings, it takes values in the bundle $O(0) = O(2h - 2)$ over $\widetilde{\mathbb{PT}}$. Correspondingly, it describes a particle of helicity 1.

As already discussed in subsection 3.2.1, the superwavefunction $\phi^r(Z)$ in supertwistor space (3.58) contains an expansion $u(k\psi)$ into fermionic variables (3.59). Correspondingly, the vertex operator V_ϕ represents the $\mathcal{N}$=4 multiplet

$$\phi^r \,:\, ((1,\,\mathbf{1}), (\tfrac{1}{2},\,\bar{\mathbf{4}}), (0,\,\mathbf{6}), (-\tfrac{1}{2},\,\mathbf{4}), (-1,\,\mathbf{1}))\,, \tag{3.71}$$

where the bold number states the $SU(4)_R$ representation, which in turn is determined by the requirement of ϕ^r transforming as a singlet under the R-symmetry group $SU(4)_R$.

3.2.4 Twistor-string vertex operators for $\mathcal{N}$=4 conformal supergravity

In addition to the vertex operators introduced in the previous section, one can construct further vertex operators corresponding to states in linearized conformal supergravity (see subsection 2.9) as initially explored by Berkovits and Witten in [76]. By combining the fields Y and ∂Z of conformal dimension one with vertex functions $f^I(Z)$ and $g_I(Z)$ of zero conformal dimension one obtains

$$V_f = Y_I f^I(Z) \quad \text{and} \quad V_g = g_I(Z)\partial Z^I\,. \tag{3.72}$$

In order to ensure neutrality under $GL(1,\mathbb{R})$-scalings for V_f and V_g, f and g have to carry charge 1 and -1 to compensate for the contributions from Y and ∂Z respectively. Furthermore, in order to be primary with respect to the Virasoro and $GL(1,\mathbb{R})$-generators, the following physicality conditions have to be satisfied:

$$\partial_I f^I = \partial_\alpha f^\alpha(Z) - \partial_a f^a(Z) = 0, \tag{3.73}$$

$$Z^I g_I = Z^\alpha g_\alpha + Z^a g_a = 0. \tag{3.74}$$

In addition to the above constraints, f^I and g_I exhibit the gauge invariances [74]

$$\delta f^I = Z^I \Lambda \quad \text{and} \quad \delta_1 g_I = \partial_I \chi\,, \tag{3.75}$$

which correspond to changing the vertex operators by an BRST-exact term.

Without taking the index I into account, f^I has $GL(1,\mathbb{R})$-weight 1 and thus corresponds to a particle of helicity $\frac{3}{2}$. The $SL(2,\mathbb{C})$-indices alter the helicity by either adding or subtracting $\frac{1}{2}$. Therefore, one is left with a helicity 2 and a helicity 1 state from each of the bosonic parts A and A'. The helicity 1 functions are removed by virtue of (3.73) and (3.75), leaving f^A and $f_{A'}$ to serve as highest helicity states for two positive helicity $\mathcal{N}$=4 graviton multiplets. From the fermionic part f^a one obtains in addition four $\mathcal{N}$=4 multiplets with leading helicity $\frac{3}{2}$. A similar analysis shows the vertex operators of type V_g to describe the helicity conjugated part to the spectrum obtained from V_f.

A complete set of vertex operators satisfying all constraints was found by Dolan and Ihry in [77]. Considering the sector originating from vertex operators of type V_f first, there are three consistent choices:

- $V_{f_p}(z) = f^A Y_A$ yields one graviton multiplet, $((2,\,\mathbf{1}), (\frac{3}{2},\,\bar{\mathbf{4}}), (1,\,\mathbf{6}), (\frac{1}{2},\,\mathbf{4}), (0,\,\mathbf{1}))$ of positive helicity, where the vertex function f^A is

$$f^A = \int \frac{\mathrm{d}k}{k^2}\, p^A \prod_{B'=1}^{2} \delta(k\pi^{B'} - p^{B'})\, \exp(ik\omega^D p_D)\, u(k\psi). \tag{3.76}$$

- $V_{f_c}(z) = f_{A'} Y^{A'} + \tilde{f}^A Y_A$ delivers a second graviton multiplet of positive helicity as above, where

$$\tilde{f}^A = -is^A \bar{s}^{A'} \int \frac{\mathrm{d}k}{k^3} \frac{\partial}{\partial \pi^{A'}} \prod_{B'=1}^{2} \delta(k\pi^{B'} - p^{B'})\, \exp(ik\omega^D p_D)\, u(k\psi) \tag{3.77}$$

  and

$$f_{A'} = \bar{s}_{A'} \int \frac{\mathrm{d}k}{k^2} \prod_{B'=1}^{2} \delta(k\pi^{B'} - p^{B'})\, \exp(ik\omega^D p_D)\, u(k\psi). \tag{3.78}$$

  Here spinors s^A and $\bar{s}_{A'}$ are chosen to satisfy $p^A s_A = 1$ and $p_{A'}\bar{s}^{A'} = 1$.

- $V_{f_f}(z) = f^m Y_m + \hat{f}^A Y_A$ finally describes a gravitino multiplet of positive helicity transforming in the $\mathbf{4}$ of $SU(4)_R$: $((\frac{3}{2},\,\mathbf{4}), (1,\,\mathbf{15}\oplus\mathbf{1}), (\frac{1}{2},\,\overline{\mathbf{20}}\oplus\bar{\mathbf{4}}), (0,\,\mathbf{10}\oplus\mathbf{6}), (-\frac{1}{2},\,\mathbf{4}))$. Explicit expressions of f^m and $\hat{f}^A$ will not be used below, but can be found in reference [77].

Switching to vertex operators of type V_g, one finds the following three functions to satisfy all constraints:

- $V_{g_p}(z) = g^{A'}\partial\pi_{A'}$ corresponds to a negative helicity graviton multiplet, $((0,\,\mathbf{1}), (-\frac{1}{2},\,\bar{\mathbf{4}}), (-1,\,\mathbf{6}), (-\frac{3}{2},\,\mathbf{4}), (-2,\,\mathbf{1}))$, where the vertex function $g^{A'}$ reads
$$g^{A'} = \int \mathrm{d}k\, k\, \pi^{A'} \prod_{B'=1}^{2} \delta(k\pi^{B'} - p^{B'})\, \exp(ik\omega^D p_D)\, u(k\psi). \tag{3.79}$$

- $V_{g_c}(z) = g_A\partial\omega^A + \tilde{g}^{A'}\partial\pi_{A'}$ delivers a second graviton multiplet of the above type, where
$$g_A = is_A \int \mathrm{d}k \prod_{B'=1}^{2} \delta(k\pi^{B'} - p^{B'})\, \exp(ik\omega^D p_D)\, u(k\psi) \tag{3.80}$$
and
$$\tilde{g}^{A'} = -i\bar{s}^{A'} s_A \omega^A \int \mathrm{d}k\, k \prod_{B'=1}^{2} \delta(k\pi^{B'} - p^{B'})\, \exp(ik\omega^D p_D)\, u(k\psi). \tag{3.81}$$

- The multiplet containing gravitini of negative helicity $((\frac{1}{2},\,\bar{\mathbf{4}}), (0,\,\overline{\mathbf{10}} \oplus \mathbf{6}), (-\frac{1}{2},\,\mathbf{20} \oplus \mathbf{4}), (-1,\,\mathbf{15} \oplus \mathbf{1}), (-\frac{3}{2},\,\bar{\mathbf{4}}))$ is given by $V_{g_f}(z) = g_m\partial\psi^m + \hat{g}^{A'}\partial\pi_{A'}$. Again, explicit expressions for g_m and $\hat{g}^{A'}$ will not be needed and can be found in reference [77].

However, spacetime states represented by V_{f_p} and V_{f_c} (and similarly V_{g_p} and V_{g_c}) are not independent particles, but comprise a so-called dipole ghost as discussed in subsection 2.9.

Applying the twistor version of a spacetime translation to V_{f_p} results in a shifted plane wave solution, while the action on V_{f_c} is twofold (see subsection 2.9). Besides of the expected shifted particle one additionally obtains a contribution proportional to a plane wave:
$$\begin{aligned} V_{f_p} &\to V'_{f_p}, \\ V_{f_c} &\to V'_{f_c} + V''_{f_p}. \end{aligned} \tag{3.82}$$
Performing the above consideration for vertex operators of type V_g, one can identify V_{f_p} and V_{g_p} with a plane wave and V_{f_c} and V_{g_c} with the conformal wave part of (2.138) by comparison of their properties under translations.

The preceding analysis suggests that the degrees of freedom corresponding to the plane wave part reside in f^A and $g_{A'}$, while the functions $f_{A'}$ and g^A contain the degrees of freedom of the conformal wave part of the particles. In the discussion of the modified twistor string theory suggestion for $\mathcal{N}=8$ supergravity below, it will be shown in subsection 4.3 that the degrees of freedom cannot be separated into two parts of the bosonic twistor. While the decomposition of the bosonic index into $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ is natural in twistor space, the components A and A' mix in spacetime as shown in [72]. In particular, the two bosonic parts of the wavefunction are coupled by the equation of motion in spacetime (see eq. (4.18)). Nevertheless, the terms $f_{A'}Y^{A'}$ and $g^A\partial Z_A$ are closely related to the conformal wave part of the particle, because of their behavior under translations as shown in [76].

3.2.5 Amplitudes in twistor string theory

The correlation function for a set of integrated vertex operators derived from the worldsheet action (3.60) is given by

$$\mathcal{M} = \int [\mathrm{d}\bar{A}][\mathrm{d}Z][\mathrm{d}Y]e^{-S}V_1V_2\dots V_n\,. \tag{3.83}$$

Again, as in discussion of the boundary conditions for the open string, it is possible to employ the gauge freedom to set A and $\bar{A}$ to zero, which results in purely holomorphic and antiholomorphic fields Z and $\bar{Z}$ respectively. Considering tree amplitudes in the following, the worldsheet is a disk. This in turn makes it possible to employ the so-called doubling trick: one can identify the fields Z and $\bar{Z}$ with a theory of holomorphic fields on the sphere $S^2 = \mathbb{C}P^1$. The analogue of the function U introduced in eq. (3.66) is the patching function relating the two open covers of $\mathbb{C}P^1$. Therefore the gauge structure is classified by the topologically nontrivial $GL(1)$-configurations on S^2.

Nontrivial $GL(1)$-configurations on the sphere S^2 correspond to instantons, which are labeled by their instanton number $d \in \mathbb{N}$. While in each instanton sector d any gauge configuration can be written as a fluctuation $\delta\bar{A}$ of zero instanton number around a fixed configuration $\bar{A}_d$, it is not possible to relate fixed configurations from different sectors in this way. So the functional integration over $\bar{A}$ in the correlation function eq. (3.83) splits into a summation over the different instanton sectors. On the two-sphere, one can express any fluctuation as $\delta\bar{A} = \bar{\partial}\Omega$, where Ω is some complex function, which leads after summing over all instanton sectors to the measure

$$[\mathrm{d}\bar{A}] = \sum_{d=0}^{\infty}[\mathrm{d}\Omega]\det(\bar{\partial}) \tag{3.84}$$

Now one can rewrite the determinant as the (u,v)-ghost system and absorb it into the definition of the action S. In addition, the volume $[\mathrm{d}\Omega]$ of the gauge group $GL(1)$ will be absorbed into a prefactor, which will be cancelled in the calculation below:

$$\mathcal{M} = \sum_{d=0}^{\infty}\int [\mathrm{d}Z][\mathrm{d}Y]e^{-S}V_1V_2\dots V_n\,. \tag{3.85}$$

The remaining functional integration over the fields Z and Y

$$[\mathrm{d}Z][\mathrm{d}Y] = \prod_{I,J=1}^{8}[\mathrm{d}Z^I][\mathrm{d}Y_J] \tag{3.86}$$

can be performed by expanding them into modes of $(\partial + A)(\bar{\partial} + \bar{A})$ and integration over the mode coefficients. The expansion into homogeneous coordinates u^j, $\bar{u}^j$, $j = 1,2$ of $\mathbb{C}P^1$ yields

$$\begin{aligned}
Z^\alpha &= \sum_{\{b_1\dots b_d\}}\beta^\alpha_{b_1\cdots b_d}\,u^{b_1}\cdots u^{b_d} + \sum_{\{b_1\dots b_d\}} b^{\alpha,i_1\dots i_k}_{b_1\dots b_d j_1\dots j_k}\bar{u}_{i_1}\cdots\bar{u}_{i_k}u^{b_1}\cdots u^{b_d}u^{j_1}\cdots u^{j_k}\\
\psi^\alpha &= \sum_{\{b_1\dots b_d\}}\gamma^\alpha_{b_1\cdots b_d}\,u^{b_1}\cdots u^{b_d} + \sum_{\{b_1\dots b_d\}} c^{\alpha,i_1\dots i_k}_{b_1\dots b_d j_1\dots j_k}\bar{u}_{i_1}\cdots\bar{u}_{i_k}u^{b_1}\cdots u^{b_d}u^{j_1}\cdots u^{j_k}
\end{aligned} \tag{3.87}$$

The first terms in the expansion eq. (3.87) are solutions to the equation of motion for Z^I:

$$(\bar{\partial} + \bar{A})Z^I = 0 \tag{3.88}$$

and are referred to as zero modes. While they are constant in the case of $d = 0$, they are holomorphic in u for $d > 0$. Analogously, the expansion of Y_I reads

$$Y_I = (Y_\alpha, Y_a) = \sum_{\{b_1 \dots b_d\}} (\tilde{b}^{i_1 \dots i_k}_{\alpha, b_1 \dots b_d j_1 \dots j_k} \tilde{c}^{i_1 \dots i_k}_{a, b_1 \dots b_d j_1 \dots j_k}) \bar{u}_{i_1} \cdots \bar{u}_{i_k} u^{b_1} \cdots u^{b_d} u^{j_1} \cdots u^{j_k}. \tag{3.89}$$

Since the coefficients $\tilde{b}$ and $\tilde{c}$ are conjugate to the coefficients of the non-zero modes of Z^I, the integration over $(b, \tilde{b})$ and $(c, \tilde{c})$ corresponds to a Wick contraction of Z and Y. Once all Wick contractions are performed, the remaining nonzero mode parts of Z will give no contribution to the correlator and one can replace the remaining Z's with their zero mode part. The remaining coefficients β and γ imply a holomorphic map from $\mathbb{C}P^1$ to complex projective twistor space $\mathbb{PT}$. The such defined homogeneous polynomial of degree d can be interpreted as a holomorphic curve of the same degree, parameterized by β and γ.

In order to connect to the usual string theory description in terms of complex worldsheet variables, one has to express the variables u^j, $\bar{u}^j$ of $\mathbb{C}P^1$ in terms of $(z, \bar{z})$. While this seems topologically impossible at the first glance, one has to remember how the $\mathbb{C}P^1$-geometry arose. Because the open string vertex operators depend on the real axis of the worldsheet only, it is sufficient to restrict the attention to just one coordinate patch of S^2. The zero-mode expansion can then be written as

$$Z^I(z) = \sum_{d=0}^{\infty} Z^I_{-d} z^d \tag{3.90}$$

which, after having taken care for the $SL(2, \mathbb{R})$-symmetry on the complex worldsheet and remembering the absorbed factor of the $GL(1, \mathbb{R})$-volume above leads to the final expression:

$$\mathcal{M} = \sum_{d=0}^{\infty} \int \sum_{m=0}^{d} \frac{\mathrm{d}^8 Z_{-m}}{SL(2, \mathbb{R}) GL(1, \mathbb{R})} \, e^{-S} V_1 \dots V_n \,, \tag{3.91}$$

where Wick contractions transform into an operator product expansion on the worldsheet

$$\langle Z^I(z) Y_J(w) \rangle = \frac{\delta^I{}_J}{z - w} \tag{3.92}$$

and are understood to be performed on the vertex operators V_i before evaluating the integral.

How does this calculational procedure connect to different helicity sectors for tree amplitudes? In Wittens original twistor string article [51], he conjectured N^{p}MHV tree amplitudes to be supported on algebraic curves of degree

$$d = p + 1 \,, \tag{3.93}$$

where p is allowed to take the values $-1, 0, 1, \dots$. Limiting the consideration to particles of maximal helicity s, the number of negative helicity particles is given by $p + 2$. Here the examples of $d = 0$ and $d = 1$ will be discussed.

d = 0 Fermionic integration in this case does provide four integrations, which means that contributions to the $d = 0$ part of the amplitude occur, if there is exactly one particle with negative helicity (see equation (3.59)). This is the $\overline{\text{MHV}}$ sector described in subsection 2.5. Assuming the fermionic integration to be performed already, one will have to integrate the coefficients Z_0^I over all twistor space, since a zero dimensional algebraic curve is just a point

$$\int \prod_{A,A'=1}^{2} \frac{\mathrm{d}\omega_0^A \, \mathrm{d}\pi_0^{A'}}{SL(2,\mathbb{C})GL(1,\mathbb{R})} \prod_{i=1}^{n} \mathrm{d}z_i \tag{3.94}$$

In the case of three particles, the integration over the insertion points z_i on the worldsheet will be cancelled by $SL(2,\mathbb{C})$-invariance.

d = 1 Here eight fermionic moduli have to be integrated over, which corresponds to amplitudes containing two particles of negative helicity, which are the MHV amplitudes. Assuming the two negative helicity particles being labeled by 1 and 2, the fermionic integration will result in a factor $(z_1 - z_2)^4$ (see for example [75]). For the $d = 1$ case, zero modes have to be expanded to first order: $Z^I(z) = Z_0^I + Z_{-1}^I z_i$, resulting in the measure

$$\int \prod_{A,A'=1}^{2} \frac{\mathrm{d}\omega_0^A \, \mathrm{d}\omega_{-1}^A \, \mathrm{d}\pi_0^{A'} \, \mathrm{d}\pi_{-1}^{A'}}{SL(2,\mathbb{C})GL(1,\mathbb{R})} \prod_{i=1}^{n} \mathrm{d}z_i. \tag{3.95}$$

3.3 Dual S-matrix description for $\mathcal{N}$=4 SYM theory

3.3.1 Leading singularities

Leading singularities [78, 79, 80, 81] arise as invariants of $\mathcal{N}$=4 SYM scattering amplitudes and can be evaluated employing the generalized unitarity method [32, 82, 83, 84, 85, 86].

This method relies on the calculation of scattering amplitudes across a given branch cut in the space of kinematical invariants. For one-loop amplitudes this has initially been done by cutting[24] two propagators and integrating over the Lorentz-invariant phase space of the product of the remaining tree-level amplitudes. Applying this technique once for the usual Feynman diagrams and a second time for box integrals in the one-loop expansion eq. (2.101), it is possible to calculate the box coefficients using this method.

For more general scenarios, it is possible to not only cut one single but all available propagators. The singularity thus obtained leads to the notion of the *leading singularity*, which by a slight inconsistency of terminology refers to the discontinuity across a leading singularity. In other words, leading singularities are the highest-codimension singularities at l loop-level, which can be obtained by cutting $4l$ propagators in the generalized unitarity method.

Leading singularities are well-defined and IR-finite objects. It turns out that the box coefficients $C_{r,s,t,u}$ from expanding a one-loop amplitude in $\mathcal{N}$=4 SYM into scalar box functions are exactly the one-loop leading singularities. Generalizing this fact, it was conjectured

[24] Cutting a propagator refers to choosing a kinematical configuration for complexified momenta, which puts the propagator on its pole $p^2 = 0$.

that leading singularities together with their corresponding integral basis can determine amplitudes of $\mathcal{N}$=4 SYM theory at any loop-order [19] completely.

A general leading singularity consists of subamplitudes from all MHV sectors including the special $\overline{\text{MHV}}$ $=\text{N}^{-1}$MHV three-point amplitude, which has been discussed[25] in subsection 2.5. For the MHV sector one can show that all leading singularities are equal to the tree amplitude, which explains why MHV amplitudes can be expanded as

$$A^{\text{MHV}} = A^{\text{MHV}}_{\text{tree}}(1 + f_{1-\text{loop}} + f_{2-\text{loop}} + \cdots)\,. \tag{3.96}$$

While box coefficients and thus one-loop leading singularities are related by infrared (IR) equations and subject to dual conformal constraints, this behavior is expected to extend to higher-loop leading singularities [34, 35]. However, higher-loop generalizations of the one-loop IR equations are not well understood due to the lack of a complete integral basis. For the same reason, higher-loop leading singularities have not been studied in detail so far.

3.3.2 Functional and residues

Many of the properties and symmetries of $\mathcal{N}$=4 SYM amplitudes discussed in subsection 2.6 above are neither manifest nor visible in the usual, Feynman-diagram based approach. As already discussed in subsection 2.1.1, this drawback can be avoided in an S-matrix theory. One would like to find a formalism which directly produces amplitudes or building blocks of those, without making reference to Feynman diagrams during the calculation. The price to pay is that spacetime locality emerges in a complicated way from those formulations.

In a suggestion of Arkani-Hamed *et al.* the building blocks to be delivered by an alternative formulation are the leading singularities described in the previous subsection [87]. More precisely, the Grassmannian integral

$$\mathcal{L}_{n;k}(Z_a) = \frac{1}{\text{vol}(GL(k))}\int \frac{d^{k\times n}C_{\alpha a}}{(12\cdots k)\,(23\cdots(k+1)\,)\,\cdots(n1\cdots(k-1)\,)}\prod_{\alpha=1}^{k}\delta^{4|4}(C_{\alpha a}Z_a) \tag{3.97}$$

is conjectured to yield the leading singularities for an n-particle amplitude being homogeneous of degree $4k$ in Grassmann coordinates η and thus residing in the N^{k-2}MHV sector. Here $\alpha = 1\ldots k$, $a = 1\ldots n$ and Z_a are the supertwistor variables defined in subsection 3.2.1. The objects in the denominator are called minors and are defined as the determinant of a $k\times k$ submatrix of $C_{\alpha a}$ consisting of the columns $m_1,\cdots,m_k$:

$$(m_1\cdots m_k) \equiv \varepsilon^{\alpha_1\cdots\alpha_k}C_{\alpha_1 m_1}\cdots C_{\alpha_k m_k}\,. \tag{3.98}$$

Starting from the functional eq. (3.97), a couple of steps are necessary to understand its interpretation and the consequences for leading singularities. The lines below roughly follow [87], however, a lot of calculational details are omitted here and can be found in that reference.

[25]The $\overline{\text{MHV}}$-amplitude corresponds to an algebraic curve of degree $d = 0$ in twistor space, which is just a point. This scenario is mentioned shortly at the end of subsection 3.2.5 above.

Considering the product of $\delta^{4|4}$-functions, it is invariant under the $GL(k)$-transformation

$$C_{\alpha a} \to L_\alpha^\beta C_{\beta a} \tag{3.99}$$

for any invertible $k \times k$-matrix L_α^β. Carrying the character of a gauge symmetry, it is necessary to fix this symmetry before evaluating the integral in order to avoid divergences. The gauge-fixing can most easily be done by interpreting the matrix $C_{\alpha a}$ as a collection of k-vectors and use the $GL(k)$-symmetry to bring k of them to the k-dimensional orthogonal Cartesian basis. The remaining elements of the matrix will be labeled by c_{Ii}, where now $I = 1 \dots k$ and $i = 1 \dots (n-k)$. This version of gauge fixing is not unique, but it will simplify calculations below:

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1\,(n-k)} \\ 0 & 1 & \cdots & 0 & c_{21} & c_{22} & \cdots & c_{2\,(n-k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & c_{k1} & c_{k2} & \cdots & c_{k\,(n-k)} \end{pmatrix}. \tag{3.100}$$

In terms of the entries of the gauge-fixed matrix one can now write down the gauge-fixed version of eq. (3.97):

$$\mathcal{L}_{n;k}(Z_a) = \int \frac{d^{k\times(n-k)}c_{Ii}}{(12\cdots k)\,(23\cdots(k+1)\,)\,\cdots(n1\cdots(k-1)\,)} \prod_I \delta^{4|4}(Z_I + c_{Ii}Z_i)\,, \tag{3.101}$$

which can be brought back into spinor-helicity momentum space with momentum spinors μ and $\bar{\mu}$ via the usual half-Fourier transform

$$\mathcal{L}_{n;k}(\mu,\bar{\mu},\eta) = \int \frac{d^{k\times(n-k)}c_{Ii}\,\delta^2(\mu_i - c_{Ii}\mu_I)\delta^2(\bar{\mu}_I + c_{Ii}\bar{\mu}_i)\delta^4(\eta_I + c_{Ii}\eta_i)}{(12\cdots k)\,(23\cdots(k+1)\,)\,\cdots(n1\cdots(k-1)\,)}\,. \tag{3.102}$$

Taking the available bosonic δ-functions and the number of variables c_{Ii} into account and subtracting four for the remaining momentum conserving δ-function, one ends up with a total number of $d := (k-2)\times(n-k-2)$ integration variables. In a last step one can unify the integration variables into a d-vector τ by replacing

$$\delta^2(\mu_i - c_{Ii}\mu_I)\delta^2(\bar{\mu}_I + c_{Ii}\bar{\mu}_i) = \delta^4(\sum_a p_a)J(\mu,\bar{\mu}) \int d^{(k-2)(n-k-2)}\tau_\gamma \delta(c_{Ii} - c_{Ii}(\tau_\gamma)) \tag{3.103}$$

where $J(\mu,\bar{\mu})$ is a Jacobian factor[26] and the index γ runs from 1 to d. The final expression then reads:

$$\mathcal{L}_{n;k} = L_{n;k} \times \delta^4(\sum_a p_a) \tag{3.104}$$

where $L_{n;k}$ is defined as

$$L_{n;k} = J(\mu,\bar{\mu}) \int \frac{d^{(k-2)\times(n-k-2)}\tau}{[(12\cdots k)\,(23\cdots(k+1)\,)\,\cdots(n1\cdots(k-1)\,)]\,(\tau)} \prod_I \delta^4(\eta_I + c_{Ii}(\tau)\eta_i)\,. \tag{3.105}$$

[26]In the original version of the paper [87] the sign of the Jacobian factor has been ignored. However, once it comes to calculating box diagrams in terms of residues below, the relative sign will play a role and even result in modified expressions compared to those given in the original work.

The degree of the minors eq. (3.98) is given by $\min[k-2, n-k-2]$. This renders the NMHV situation as the easiest nontrivial one: for $k = 3$ the minors are linear expressions in τ. Note that the integrand is a holomorphic function of the (complexified) variables τ_γ. In this light, eq. (3.104) is a contour integral in $\mathbb{C}^d$.

3.3.3 Multiresidues

As stated initially, eq. (3.104) is conjectured to calculate the leading singularities of an amplitude. Explicitly, its value is determined by the residues enclosed in the contour chosen for the evaluation. However, dealing with an expression in multiple complex variables, the common notion of a residue has to be extended. Following ref. [87], the generalization is presented for a complex function of the form

$$f(\tau_1, \ldots, \tau_d) = \frac{g(\tau_1, \ldots, \tau_d)}{P_1(\tau_1, \ldots, \tau_d) \cdots P_n(\tau_1, \ldots, \tau_d)}, \tag{3.106}$$

where p are polynomials linear in $\tau_1, \ldots, \tau_d$ and $n > d$. As mentioned above, integrands of this form occur in eq. (3.105) in the NMHV sector, where all minors are linear in the integration variables τ_γ. A residue occurs if d of the polynomial factors vanish:

$$P_{i_1} = \cdots = P_{i_d} = 0. \tag{3.107}$$

Solving the above system of equations determines the point $\tau^* = \tau_1^*, \ldots, \tau_d^* \in \mathbb{C}^d$. The residue is then defined by

$$\mathrm{Res}[f](\tau_1, \ldots, \tau_d) = \frac{g(\tau_1^*, \ldots, \tau_d^*)}{\prod_{i \neq (i_1, \ldots, i_d)} p_i(\tau_1^*, \ldots, \tau_d^*) \det \left(\frac{\partial(p_{i_1}, \cdots p_{i_d})}{\partial(\tau_1, \ldots, \tau_d)} \right) \Big|_{\tau_1^*, \ldots, \tau_d^*}}. \tag{3.108}$$

In general, choosing d out of n minors to vanish will result in $\binom{n}{d}$ different residues.

In the NMHV situation a residuum is obtained by choosing $d = n - 5$ minors to vanish simultaneously. Due to the linear degree of the minors there is exactly one solution and thus one residue for each of these choices. Correspondingly, any residue in the NMHV sector can be identified by noting the vanishing minors, where each minor (cf. eq. (3.98)) is referred to by its first entry m_1. The resulting list with $n - 5$ elements is enclosed in curly brackets and is referred to as Plücker label

$$\{i_1, \ldots, i_d\}. \tag{3.109}$$

In the 8-point NMHV situation a residue can be calculated by choosing $3 = n - 5$ minors to vanish. For example the residue obtained by setting the denominator terms (123), (234) and (781) to zero, will be referred to as $\{1, 2, 7\}$.

One property of the multiresidues follows from inspecting eq. (3.108): besides of being defined at the point $(\tau_1^*, \ldots, \tau_d^*)$, the sign of the residue depends on the order in which the polynomials p_i are taken to zero. This in turn implies total antisymmetry for the Plücker label

$$\{i_1, \ldots, i_d\} = \mathrm{sgn}(\sigma)\{\sigma(i_1), \sigma(i_2), \ldots \sigma(i_d)\}. \tag{3.110}$$

By choosing certain minors to vanish, one implicitly defines a contour for the integration. In particular, the residue is determined by enclosing each complex component of the vector τ^* by an infinitesimal circle. While in the complex plane the residue is encircled by S^1, this geometrical picture can not be extended to arbitrary complex dimensions d. While the contour is now $(S^1)^d \cong T^d$ of real dimension d, the subspace necessary to surround the point $(\tau_1^*, \ldots, \tau_d^*)$ has real dimension $2d-1$. It is only in the complex plane where those two dimensions agree. In order to emphasize this difference for complex dimensions larger than one, the contour is referred to as the *distinguished boundary.*

While the discussion above is valid for the NMHV sector, new effects show up for $k > 3$. The fact that minors are now polynomials of higher degree causes two changes: the Plücker label has to be accompanied by an additional index denoting which of the solutions of the polynomial set of equations is meant. In addition, dealing with polynomials of higher degree will also include special situations like doubled zeros. This leads to the notion of composite residues, which are discussed in [87] and not used below.

3.3.4 Tree-level amplitude

Finally, the NMHV tree amplitude can be expressed in terms of residues. With the sums

$$E = \sum_{k \text{ even}} \{k\}$$
$$O = \sum_{k \text{ odd}} \{k\} \tag{3.111}$$

and the product[27]

$$\{i_1\} \star \{i_2\} = \left\{ \begin{array}{l} \{i_1, i_2\} \text{ if } \; i_1 < i_2 \\ 0 \quad \text{otherwise} \end{array} \right\} \tag{3.112}$$

the BCFW form of the tree amplitude is given by

$$A^{\text{tree}}_{\text{BCFW}} = E \star O \star E \star \cdots , \tag{3.113}$$

and the parity-conjugated (P(BCFW)) form is obtained from

$$A^{\text{tree}}_{\text{P(BCFW)}} = (-1)^{n-5}\, O \star E \star O \star \cdots . \tag{3.114}$$

The names BCFW and P(BCFW) form refer to different solutions of BCFW shifts, which have been used to obtain the particular forms above. Both versions are identical, which can be easily shown numerically. However, it is highly nontrivial to show the equivalence using analytic forms of the spacetime leading singularities corresponding to the residues. The identities necessary to show

$$A_{\text{BCFW}} = A_{\text{P(BCFW)}} \tag{3.115}$$

are called *remarkable identities*, ensure the absence of spurious poles, parity invariance and cyclicity of tree amplitudes and are discussed in subsection 6.2.

[27] Here a slightly different but equivalent notation for residues is used as compared to [87]: $\{i\}\{j\} := \{i, j\}$ and $\{i + j\}\{k\} := \{i, j\} + \{i, k\}$

3.3.5 Box coefficients and residues in the NMHV sector

Using the definitions from the last subsection, it is now possible to compare known results for leading singularities from field-theoretical calculations with residues determined with the help of eq. (3.108). Comparing tree and one-loop results for amplitudes with six, seven or eight legs reveals that the particular leading singularities are indeed given by a certain linear combination of residues.

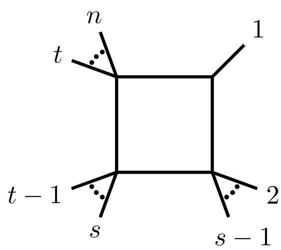
Although the labeling of NMHV residues with $n-5$ coordinates is favorable for lower-point amplitudes, general considerations are more accessible employing the *complementary labeling.* The usual labeling can be obtained from the 5-number complementary labeling by the *bar operation*

$$\{j_1,\ldots,j_{(n-5)}\} = \overline{\{i_1,\ldots,i_5\}} = \{\Xi\}\cdot \mathrm{sgn}(i_1,\ldots,i_5)\cdot \mathrm{sgn}(i_1,\ldots,i_5,\Xi) \qquad (3.116)$$

where Ξ is the ordered complement

$$\{1,\ldots,n\}\backslash\{i_1,\ldots,i_5\}\,. \qquad (3.117)$$

In the NMHV sector, there is a clear map between box coefficients and residues [87]. The simplest situation occurs for the 3-mass box, where the corresponding box coefficient is given by



$$C^{3m}_{12st} \mathrel{\widehat{=}} \overline{\{s-2,s-1,t-2,t-1,n\}}, \qquad (3.118)$$

and any other 3-mass boxes can be obtained by cyclic shifts.

The expressions for other box coefficients can be easily obtained as sums of (degenerate) 3-mass boxes by employing the results from [88]:

$$\begin{aligned}
C^{1m}_{r,r+1,r+2,r+3} &= C^{2me}_{r+2,r+3,r,r+1} + C^{3m}_{r+1,r+2,r+3,r} \\
C^{2mh}_{r,r+1,r+2,s} &= C^{3m}_{r+1,r+2,s,r} + C^{3m}_{r,r+1,r+2,s} && (s>r+3,\ r>s+1) \\
C^{2me}_{r,r+1,s,s+1} &= \sum_{\substack{u,v\\ u\geq r+2\\ u+2\leq v\leq s}} C^{3m}_{r,r+1,u,v} + \sum_{\substack{u,v\\ u\geq s+2\\ u+2\leq v\leq r}} C^{3m}_{s,s+1,u,v} && (s>r+2, r>s+2),
\end{aligned} \qquad (3.119)$$

where all indices have to be understood modulo n. Writing '>' means '$>\bmod n$' and the summations have also to be adapted accordingly. If not stated otherwise, the modulo-n notation will be understood implicitly below.

3.3.6 Generalized residue theorems

As discussed in reference [87], residues of eq. (3.105) are not independent objects, but are subject to generalized (or global) residue theorems (GRTs). In particular, all NMHV GRTs can be generated from basic GRTs, which have the form,

$$\sum_{j=1}^{n} \{j, i_1, ..., i_{n-6}\} = 0\,. \tag{3.120}$$

where $(i_1, ..., i_{n-6})$ is referred to as *source term* and will be used to uniquely label basic GRTs below. It is not difficult to see that any GRT constrains the sum of 6 residues to vanish. In order to avoid confusion, source terms are enclosed by usual brackets (), while residues will be enclosed by curly brackets {}.

4 A twistor-string description for $\mathcal{N}$=8 supergravity?

4.1 Twistor string theory: state of the art

The twistor string theory [51, 74] introduced in subsection 3.2.2 reproduces the known results for tree-level amplitudes in $\mathcal{N}$=4 SYM theory. However, at the loop level, $\mathcal{N}$=4 SYM is intrinsically coupled to conformal supergravity: the complete collection of states described by the twistor string theory can propagate around internal loops. Although it is possible to calculate loop amplitudes in twistor string theory [75], field-theory results for conformal supergravity coupled to $\mathcal{N}$=4 SYM are not available due to the nonunitary nature of the former (see subsection 2.9).

Tree amplitudes with external conformal supergravity states have been calculated employing twistor string theory [77, 76]. In parallel to the situation described in the last paragraph, comparison with field-theory results is difficult: the Feynman formalism for conformal supergravity has not been explored due to the higher derivative and nonunitary nature of the theory. So the only calculations in such theories have been performed in the twistor string formalism, although the interpretation of S-matrix elements in a nonunitary theory remains unclear.

With the objective of curing those drawbacks and realizing a description of Einstein gravity (or supersymmetric extensions thereof), Abou-Zeid, Hull and Mason suggested a new family of twistor strings in [73]. In order to do so, it is necessary to reduce the conformal symmetry of the gravity part of twistor string theory to Poincaré symmetry. One way to achieve this is to fix the twistor analogue of the light cone at infinity (eq. (3.30)), which has to be added in order to compactify Minkowski spacetime. As discussed in subsection 3.2.1, in twistor space it is represented by the infinity twistor.

In the proposal of Abou-Zeid, Hull and Mason (AHM), the Berkovits open twistor string theory is modified by introducing additional worldsheet gauge fields, which are coupled to currents preserving the infinity twistor eq. (3.32). While this mechanism can be applied in a wide range of situations describing several different supersymmetric extension of Einstein gravity, one particular suggestion was expected to describe $\mathcal{N}$=8 supergravity. An initial computation of a three-point MHV graviton correlator supported the speculation. However, in [89] Nair showed the conjugated ($\overline{\text{MHV}}$) amplitude to vanish, thus questioning the original interpretation.

In the subsections below the properties of the modified theory introduced above will be investigated further. After describing the modification and its implications in detail, the equations of motion and the gauge invariances of the negative-helicity graviton multiplet are examined by translating them into Minkowski spacetime. Doing so, one finds that the graviton multiplet contains no on-shell degrees of freedom. Furthermore, the equation of motion for the additional gauge field changes the localization properties of the original theory: for amplitudes localizing in higher instanton sectors the moduli space of algebraic curves is now reduced, which suggests the vanishing of these correlators. From the restricted set of algebraic curves it follows that the amplitudes can only depend nontrivially on one type of spinor momenta, rendering the theory to be chiral.

Finally, the equality of a certain three-point conformal supergravity amplitude with the known Einstein supergravity result will be presented in subsection 4.5.

4.2 Additional worldsheet symmetries

In the Berkovits open string theory the $GL(1)$-structure of projective twistor space has been incorporated into the action (3.60) by employing gauge fields A and $\bar{A}$ without kinetic terms, thus taking the role of Lagrange multipliers. In the same manner new gauge fields corresponding to conserved currents have been suggested by AHM in order to preserve the infinity twistor.

The derivation of additional constraints from the modification will be restricted to holomorphic quantities because the antiholomorphic part works analogously and is fixed by boundary conditions in the open string case (see subsection 3.2.2).

Assuming the target space of the open string theory (3.60) to be equipped with a one-form k_I, the corresponding bosonic current $K = k_I(Z)\partial Z^I$ can be coupled to a gauge field $\bar{B}$ to result in the modified action[28]:

$$S = \int \mathrm{d}^2 z \left(Y_I \bar{\partial} Z^I + \bar{A} J + \bar{B} K \right) + S_c + \text{barred part.} \tag{4.1}$$

In order for K to be well defined on the target twistor space $\mathbb{R}P^{3|4}$, its interior product with the Euler operator $\Upsilon = Z^I \frac{\partial}{\partial Z^I}$ has to vanish, which implies

$$Z^I k_I = 0. \tag{4.2}$$

The above condition fixes k_I to have $GL(1,\mathbb{R})$-charge -1, and therefore K has homogeneity degree 0. In order to guarantee vanishing of the $GL(1,\mathbb{R})$-anomaly, one has to require K to have conformal weight 1, which determines k_I to be a worldsheet scalar. As a consequence, all commutators of currents J (eq. (3.61)) and K vanish so that J and K generate an Abelian Kac-Moody algebra with central charge zero. Together with the cancellation between bosons and fermions in the YZ-system this is sufficient to guarantee the absence of a $GL(1,\mathbb{R})$-anomaly. Moreover, in order to have vanishing conformal anomaly (cf. 3.69), the central charge of the current system S_c is now determined to be $c_c = 30$ by taking into account the additional ghosts from the additional gauge symmetry $c_{\text{additional}} = -2$.

Parallel to the situation in the original open twistor string theory, vertex operators for physical fields have to be chosen to be primary with respect to the symmetry generators T, J and K. Therefore, conditions (3.73) and gauge invariances (3.75) have to be accompanied by additional constraints

$$f^I k_I = 0, \qquad f^I k_{[I,J]} = 0, \tag{4.3}$$

while g_I obtains a further gauge symmetry $\delta_2 g_I = \eta k_I$. Vertex operators V_ϕ are not affected by the additional symmetry. Thus one additional goal of the modification is implemented:

[28]The meaning of the bar is related to the choice of the worldsheet metric and the specific form of the twistor correspondence employed as has been discussed in 3.2.2. Here, the bar denotes comlex conjugation.

the conformal symmetry of the gravity part of twistor string theory is broken, while it remains intact for the SYM part.

The appropriate expression for the nonzero components of the infinity twistor is (cf. 3.32)

$$I^{A'B'} = \varepsilon^{A'B'}, \tag{4.4}$$

and consequently $I^{AB} = \varepsilon^{AB}$. In order to keep I_{IJ} invariant, the one-form k_I is chosen to be

$$k_I = -\Theta(Z) I_{IJ} Z^I, \tag{4.5}$$

where Θ denotes a function with homogeneity degree -2 compensating the contributions from other components in the one-form above, which can be chosen as $\Theta = \Theta(\pi)$, see ref. [73]. Condition (4.2) and the second part of (4.3) are then satisfied trivially, such that one is left with

$$\partial_I f^I = 0, \tag{4.6}$$
$$\delta f^I = Z^I \Lambda, \tag{4.7}$$
$$f^I I_{IJ} Z^J \Theta(Z) = f_{A'} \pi^{A'} = 0 \tag{4.8}$$

for the positive helicity graviton. The other graviton g is constrained in the following way:

$$Z^I g_I = 0, \tag{4.9}$$
$$\delta_1 g_I = \partial_I \chi, \tag{4.10}$$
$$\delta_2 g_I = \eta k_I = \Theta I_{IJ} Z^J \eta. \tag{4.11}$$

Taking eq. (4.4) into account, the nonzero components of (4.11) are

$$\delta_2 g^{A'} = \Theta \cdot \eta\, \varepsilon^{A'B'} \pi_{B'} \equiv \eta_\Theta \pi^{A'}. \tag{4.12}$$

The fermionic multiplets are not affected by the additional gauge symmetry.

Abou-Zeid, Hull and Mason use the above constraints and gauge invariances to set $f_{A'}$ and $g^{A'}$ to zero. Their interpretation is that one of the degrees of freedom contained in the bosonic part of f^I and g_I, respectively, is removed. Summing up the remaining states, there are six $\mathcal{N}$=4 vector multiplets missing in order to reproduce the $\mathcal{N}$=8 supergravity spectrum. Assuming the gauge group G of S_c to be six-dimensional, one obtains the correct number of states:

Helicity	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
g_A	1	4	6	4	1				
g_a		4	16	24	16	4			
ϕ^r			**6**	**24**	**36**	**24**	**6**		
f^a				4	16	24	16	4	
f^A					1	4	6	4	1
$\mathcal{N}$=8	1	8	28	56	70	56	28	8	1

Note that in the above table the negative helicity $\mathcal{N}$=4 graviton multiplet is closely related to a conformal wave, which should not be the case in an Einstein gravity theory. However, the two bosonic parts of a twistor are coupled by their equation of motion and therefore do not exhibit independent degrees of freedom (see eq. (2.140)). The implications resulting from this structure will be discussed in the next subsection.

4.3 Degrees of freedom

While the leading helicity degrees of freedom resulting from f^I subject to gauge invariances and constraints (4.6-4.8) have been shown to describe an Einstein graviton in [90], the corresponding investigation for the negative helicity graviton will be carried out in this subsection.

Following the ideas of [91], g_I and the appropriate gauge invariances and constraints (4.9-4.11) will be Penrose transformed into Minkowski space. The consideration can be limited to the bosonic part $\alpha = (A, A')$, because the fermionic degrees of freedom g_a are independent of the bosonic ones.

Penrose transforming the graviton vertex function $g_\alpha = (g_A,\, g_{A'})$ of $GL(1,\mathbb{R})$-weight -5 results in

$$g_\alpha \leftrightarrow \Gamma_{\alpha(B'C'D')} = \begin{pmatrix} \psi_{A(B'C'D')} \\ \phi_{A'(B'C'D')} \end{pmatrix}, \tag{4.13}$$

where the last part is the decomposition of α into $(A,\, A')$. The spacetime analogue of (4.9) reads

$$Z^\alpha g_\alpha = 0 \quad \leftrightarrow \quad \begin{pmatrix} 0 \\ \phi^{A'}_{(A'C'D')} \end{pmatrix} = 0, \tag{4.14}$$

which can be rewritten as

$$\phi^{A'}_{(A'C'D')} = \varepsilon^{B'A'}\phi_{A'(B'C'D')} = 0. \tag{4.15}$$

In equation (4.12), η_Θ has $GL(1,\mathbb{R})$-weight -6 in order to match the homogeneity degree of g_α. Therefore, the Penrose transform of (4.12) yields

$$I_{\alpha\beta}Z^\beta\eta_\Theta \quad \leftrightarrow \quad \begin{pmatrix} 0 \\ \eta_{(A'B'C'D')} \end{pmatrix}. \tag{4.16}$$

Furthermore, equations of motion for a massless particle have to be obeyed:

$$\nabla^{BB'}\Gamma_{\alpha(B'C'D')} = 0, \tag{4.17}$$

which reads in components[29]:

$$\nabla^{BB'}\psi_{A(B'C'D')} = 0 \quad \text{and} \quad \nabla^{BB'}\phi_{A'(B'C'D')} + \varepsilon^{AB}\psi_{A(B'C'D')} = 0. \tag{4.18}$$

[29]The derivative $\nabla^{BB'}$ acts on twistor indices via the local twistor connection.

The constraint (4.15) is solved by a totally symmetric function $\phi_{(A'B'C'D')}$, which can be set to zero via (4.16). Plugging this result into the equation of motion (4.18), one obtains

$$\nabla^{BB'}\psi_{A(B'C'D')} = 0 \quad \text{and} \quad \varepsilon^{AB}\psi_{A(B'C'D')} = 0. \tag{4.19}$$

Interpreting the above equation leads to an obvious conclusion: while $\phi_{A'(B'C'D')}$ can be gauged to zero, the field $\psi_{A(B'C'D')}$ vanishes on-shell and thus the corresponding twistor function g_α does not describe any physical degrees of freedom. Similar computations can be performed for the other components in the negative helicity graviton multiplet, showing the corresponding g_α to either be pure gauge or to vanish on-shell.

4.4 An overconstrained system?

In order to further discuss the modified theory, implications of the constraints arising from the equation of motion for the gauge field $\bar{B}$ will be investigated. Varying (4.1) with respect to $\bar{B}$ yields

$$K = k_I \partial Z^I = \Theta(Z) I_{IJ} Z^J \partial Z^J \sim \pi_{A'}\partial\pi^{A'} \;=\; 0. \tag{4.20}$$

Due to its purely classical nature, this constraint does only affect zero modes of π. As long as the amplitude resides in the $d = 0$ sector, $\pi^{A'}$ does not depend on the worldsheet coordinate z, such that equation (4.20) is satisfied by $\partial\pi^{A'} = 0$ trivially. However, in the $(d = 1)$-instanton sector, the equation of motion

$$\pi_{A'}\partial\pi^{A'} = (\pi_{0A'} + \pi_{-1A'}z)(\pi^{A'}_{-1}) = \pi_{0A'}\pi^{A'}_{-1} = 0 \tag{4.21}$$

enforces proportionality of $\pi^{A'}_0$ and $\pi^{A'}_{-1}$. Considering the $(d = 2)$-instanton sector, one now obtains

$$\begin{aligned}\pi_{A'}\partial\pi^{A'} &= (\pi_{0A'} + \pi_{-1A'}z + \pi_{-2A'}z^2)(\pi^{A'}_{-1} + 2\pi^{A'}_{-2}z)\\ &= \pi_{0A'}\pi^{A'}_{-1} + 2\pi_{0A'}\pi^{A'}_{-2}z + \pi_{-1A'}\pi^{A'}_{-2}z^2 = 0.\end{aligned} \tag{4.22}$$

In order to satisfy the above equation, each part of the sum must vanish separately, leading to

$$\pi^{A'}_0 = m_{-1}\pi^{A'}_{-1} \quad \text{and} \quad \pi^{A'}_0 = m_{-2}\pi^{A'}_{-2}, \tag{4.23}$$

where m_{-i} denote factors of proportionality. Generalizing to the d-instanton sector, one can show that all coefficients in the expansion have to be proportional to $\pi^{A'}_0$:

$$\pi^{A'}_0 = m_{-i}\pi^{A'}_{-i}, \quad \forall\, i = 1, \ldots d. \tag{4.24}$$

What does the proportionality imply? While the ω-part of twistor space is not modified, the expansion for π looks different compared to (3.90):

$$\omega^A = \omega^A_0 + \omega^A_{-1}z + \omega^A_{-2}z^2 + \cdots + \omega^A_{-d}z^d, \tag{4.25}$$

$$\begin{aligned}\pi_{A'} &= \pi_{0A'} + \pi_{-1A'}z + \pi_{-2A'}z^2 + \cdots + \pi_{-dA'}z^d\\ &= \pi_{0A'}(1 + m_{-1}z + m_{-2}z^2 + \cdots + m_{-d}z^d).\end{aligned} \tag{4.26}$$

While the degree of the algebraic curve is not altered, the dimension of its moduli space in twistor space is reduced. Featuring $4(d+1)$ dimensions in the unconstrained case, there are now d additional conditions from (4.24) leaving $3d+4$ integrations: $2(d+1)$ integrations over the ω-zero modes, two integrations over $\pi_0^{A'}$ and d integrations over the factors of proportionality, m_{-i}. So not all algebraic curves of degree d in twistor space are considered, but a subset thereof.

While one could imagine the constraints from additional symmetries to be incorporated in this way, problems arise from a different direction: the calculation of twistor string amplitudes relies on a well balanced number of integrations and δ-functions, which after performing all integrals results in the momentum conserving δ-function. Implementing the additional constraints originating from the equation of motion for the field $\bar{B}$ by inserting additional δ-functions into the correlator, one would obtain an overconstrained system for $d \neq 0$. The last condition explains why the three-point correlator calculated in [73] does not exhibit the inconsistencies occurring in higher amplitude calculations: the $d=0$ sector is unaffected by the modification of the original twistor theory.

In order to further investigate the consequences of the additional conditions described above, a constrained correlator shall be investigated more closely. Leaving aside integrations over the moduli space of algebraic curves and the insertion points z for a moment, an n-particle amplitude localizing in instanton sector d is proportional to

$$\mathcal{M} \sim \prod_{i=1}^{n} \int \mathrm{d}k_i \; \delta^2(k_i \pi_i^{A'} - p_i^{A'}) \prod_{j=1}^{d} \; \delta(\pi_0 \pi_{-j}), \tag{4.27}$$

where n integrations and $2n$ δ-functions originate from the vertex functions, while d δ-functions additionally ensure proportionality according to (4.24). Before performing the integrals, all π_i can be replaced employing (4.26), yielding

$$\mathcal{M} \sim \prod_{i=1}^{n} \int \mathrm{d}k_i \; \delta^2(k_i \pi_0^{A'} A_i - p_i^{A'}) \prod_{j=1}^{d} \; \delta(\pi_0 \pi_{-j}), \quad \text{where} \quad A_i = \sum_{l=1}^{d} m_{-l} z_i^l. \tag{4.28}$$

In order to proceed further, one has to assume $A_i \neq 0 \;\forall\, i$. This is a reasonable assumption, because otherwise, by virtue of the first δ-function, the corresponding momentum would vanish, resulting in a trivial dependence of the amplitude on $p_i^{A'}$. Thus, the above equation can be rewritten as

$$\begin{aligned}
\mathcal{M} &\sim \prod_{i=1}^{n} \int \mathrm{d}k_i \, \frac{1}{\pi_0^{1'} A_i} \, \delta\left(k_i - \frac{p_i^{1'}}{\pi_0^{1'} A_i} \right) \delta(k_i \pi_0^{2'} A_i - p_i^{2'}) \prod_{j=1}^{d} \; \delta(\pi_0 \pi_{-j}) \\
&\sim \prod_{i=1}^{n} \frac{1}{\pi_0^{1'} A_i} \, \delta\left(\frac{\pi_0^{2'} p_i^{1'}}{\pi_0^{1'}} - p_i^{2'} \right) \prod_{j=1}^{d} \; \delta(\pi_0 \pi_{-j}) \\
&\sim \prod_{i=1}^{n} \frac{1}{A_i} \, \delta(\pi_0 \, p_i) \prod_{j=1}^{d} \; \delta(\pi_0 \pi_{-j}).
\end{aligned} \tag{4.29}$$

This result is in concordance with the conclusion of the previous paragraph: the first set of δ-functions in the above equation implies the proportionality of all primed momenta $p_i^{A'}$ to $\pi_0^{A'}$ and consequently to each other. Hence all spinor brackets $[ij]$ disappear, which implies chirality of all $(d > 0)$-interactions in the gauged string theory.

4.5 A surprising result in conformal supergravity

The three-point correlator describing the scattering of one conformal wave graviton with negative helicity and two positive helicity plane wave gravitons

$$\langle V_{f_{p1}}\, V_{f_{p2}}\, V_{g_{c3}}\rangle = \langle f_1^A Y_A f_2^B Y_B (g_{3C}\partial\omega_3^C + \tilde{g}_3^{C'}\partial\pi_{3C'})\rangle \tag{4.30}$$

localizes in the zero instanton sector. Following the procedure described in subsection 3.2.2 above, Wick contractions have to be performed in the next step employing the operator product expansion (3.92). Since contractions give nonzero results only if taken between quantities carrying the same type of indices, there is no quantity which can be combined with $\partial\pi_{3C'}$ to give a nonzero result. But any $d = 0$ correlator containing an uncontractable expression of the form ∂Z will vanish, because the zero-modes of Z is not a function of the worldsheet coordinate z in zeroth order of the instanton expansion. Starting therefore from

$$\langle f_1^A Y_A\, f_2^B Y_B\, g_{3C}\partial\omega_3^C\rangle \tag{4.31}$$

and applying all possible Wick contractions results in

$$\frac{1}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)} \left\langle f_1^A f_2^B \partial_{[A} g_{3B]}\right\rangle = \frac{1}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)} \left\langle f_1^A f_{2A}\, \partial^B g_{3B}\right\rangle \tag{4.32}$$

after partial integration. Evaluating the above expression by plugging in the appropriate vertex functions (3.76) and (3.80), integrating over the moduli-space $\mathbb{R}P^{3|4}$ and taking the worldsheet $SL(2, \mathbb{C})$- and target space $GL(1, \mathbb{R})$-invariances into account yields

$$\langle V_{f_{p1}}\, V_{f_{p2}}\, V_{g_{c3}}\rangle = \delta^4\left(\sum_i P_i\right) \frac{\langle 12\rangle^8}{\langle 12\rangle^2\langle 13\rangle^2\langle 23\rangle^2}, \tag{4.33}$$

which surprisingly agrees with the result from Einstein gravity [45].

Equation (4.32) can be found as an intermediate result in the calculation of AHM. For their as well as Dolan and Ihry's choice of vertex operators the expression $\partial^B g_B$ can be shown to represent a plane wave in spacetime. Therefore, the calculation in either scenario results in (4.33). So the correlator evaluated in AHM's article reproduces the result of a three-graviton amplitude in conformal supergravity, with two positive-helicity plane wave and one negative-helicity conformal wave graviton. The reason for this result is obvious: since the calculated correlator localizes in the $d = 0$ sector, the constraints from the additional gauge symmetry are satisfied trivially, as pointed out in the previous paragraph. Therefore, the integration measure remains untouched compared to the unconstrained case.

4.6 Discussion

The consequences of gauging an additional current in Berkovits' open string theory as proposed by Abou-Zeid, Mason and Hull are shown to require some adjustments in the interpretation of the resulting theories compared to the original article [73].

The negative-helicity $\mathcal{N}$=4 supergraviton multiplet is shown to vanish on-shell, if the additional current (4.5) is gauged. Since the equation of motion for the new gauge field $\bar{B}$ restricts the possible interactions of the theory significantly above the $d = 0$ instanton level, the only remaining interactions are chiral. Moreover, the equations of motion of $\bar{B}$ render the correlators for $d > 0$ overconstrained, which questions the existence of interactions above three-point tree-level.

In the light of the chiral interactions discussed in subsection 4.4 above, the interpretation of the spectrum in the theory proposed to describe $\mathcal{N}$=8 supergravity is unclear. Physical states described by the theory are an $\mathcal{N}$=4 gravity multiplet, four $\mathcal{N}$=4 gravitini multiplets of each chirality and six $\mathcal{N}$=4 SYM multiplets. All interactions of the theory have to be chiral, and probably there are no interactions above three-point tree-level. Nevertheless, the vanishing of only one of the $\mathcal{N}$=4 supermultiplets necessary to build up the complete spectrum seems to rule out the interpretation as $\mathcal{N}$=8 supergravity or a self-dual version thereof.

Scattering of two plane wave gravitons with a conformal wave graviton part of opposite helicity in the Berkovits open twistor string is shown to agree with the corresponding gravity three-point interaction in Einstein gravity. This poses the question whether other tree-level amplitudes in supergravity might be constructed in a similar manner.

5 $E_{7(7)}$ and $\mathcal{R}^4$ counterterm in $\mathcal{N}$=8 supergravity

5.1 Counterterms in gravity and $\mathcal{N}$=8 supergravity

Divergences in the perturbative expansion of a quantum field theory can be removed by suitable counterterms respecting the (quantum) symmetries of the theory. However, the process of renormalization of a quantum theory of gravity is made special by the fact that coupling constant is dimensionful. Employing the usual power-counting method [1] to determine the superficial degree of divergence for terms in the perturbative expansion of ordinary Einstein gravity, it turns out that there is an infinite number of terms to renormalize. Thus an infinite number of constants in front of appropriate counterterms has to be determined. A theory with this behavior needs an ultraviolet completion and is called power-counting non-renormalizable.

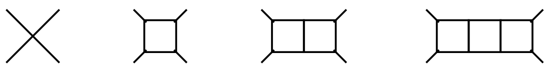
On-shell counterterms for gravitational theories need to be composed from contractions of the Riemann tensor $R_{\mu\nu\rho\sigma}$, because the Ricci tensor and the Ricci scalar vanish on-shell[30]. By dimensional analysis one can immediately determine the loop-level, at which a certain counterterm candidate appears, as will be shown on the example of the Einstein-Hilbert action

$$S_{\text{EH}} = \int \mathrm{d}^4x \kappa \sqrt{-g}\, R\,, \tag{5.1}$$

where the coupling constant $\kappa = \frac{1}{16\pi G_{\text{Newton}}}$ has mass dimension $[\kappa] = 2$ because of $[G_{\text{Newton}}] = -2$. The Ricci scalar has mass dimension

$$[R^{\mu}_{\nu\rho\sigma}] = [\partial_\rho \Gamma^{\mu}_{\nu\sigma}] = [g^{\mu\kappa}\partial_\rho\partial_\nu g_{\kappa\sigma}] = 2 \tag{5.2}$$

which in combination with κ compensates the contribution from the integration measure $[\mathrm{d}^4x] = -4$ thus rendering the action a scalar. However, any additional loop in the perturbative expansion of the theory will come with an extra factor of $1/\kappa$, which has to be compensated by an extra power of $R_{\mu\nu\rho\sigma}$. Accordingly, the number of loops can be obtained from the power of the Riemann tensor by subtracting one:



$\sim R \qquad \sim R^2 \qquad \sim R^3 \qquad \sim R^4$ (5.3)

The pure gravity theory eq. (5.1) can be shown to be free of ultraviolet divergences at one loop, which can be understood from the fact that the Gauss-Bonnet-term

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4\, R_{\mu\nu}R^{\mu\nu} + R^2 \tag{5.4}$$

is a total derivative in four dimensions. However, the addition of scalars or other particles renders the theory non-renormalizable [92]. Proceeding to the two-loop level, the counterterm

$$R^3 \;\equiv\; R_{\mu\nu}^{\lambda\rho}\, R_{\lambda\rho}^{\sigma\tau}\, R_{\sigma\tau}^{\mu\nu} \tag{5.5}$$

[30] Conventions for the Riemann tensor and related objects are defined in eq. (2.120)

has been shown to respect all symmetries and to exist on-shell [93, 94]. The nonzero coefficient has been determined by [95] and later confirmed by [96].

Adding supersymmetry improves the ultraviolet behavior of many quantum field theories. In particular, the counterterm eq. (5.5) is forbidden in any supersymmetric version of four-dimensional gravity, provided that all particles are in the same multiplet as the graviton. That is because the operator R^3 generates a scattering amplitude of helicity structure $(- + \ldots +)$ [97, 98] which can be shown to vanish by supersymmetric Ward identities (see subsection 2.5).

The next higher possible counterterm [48, 99, 100, 101, 102]

$$R^4 \equiv t_8^{\mu_1\nu_1\ldots\mu_4\nu_4} t_8^{\rho_1\sigma_1\ldots\rho_4\sigma_4} R_{\mu_1\nu_1\rho_1\sigma_1} R_{\mu_2\nu_2\rho_2\sigma_2} R_{\mu_3\nu_3\rho_3\sigma_3} R_{\mu_4\nu_4\rho_4\sigma_4}\,, \tag{5.6}$$

where t_8 is antisymmetric in each index pair $\mu_i\nu_i$ and symmetric under the interchange of two pairs $\mu_i\nu_i \leftrightarrow \mu_j\nu_j$ [66], is known as the square of the Bel-Robinson tensor [103] or simply the R^4 term. This counterterm can appear at three loops (cf. eq. (5.3)) in $\mathcal{N}$=8 supergravity.

The existence of an $\mathcal{N}$=8 supersymmetric extension to eq. (5.6) follows from the appearance of the R^4 term in the low-energy effective action (see subsection 3.1.3) of the $\mathcal{N}$=8 supersymmetric closed superstring [68]. Showing up at order α'^3, it represents the leading correction to pure $\mathcal{N}$=8 supergravity [64]. The $\mathcal{N}$=8 supersymmetric multiplet of operators containing R^4 will be denoted by $\mathcal{R}^4$ below.

In the notation of refs. [104, 105], the R^4 term appearing in the tree-level closed superstring effective action in ten dimensions is

$$e^{-2\phi}(t_8 t_8 - \frac{1}{8}\varepsilon_{10}\varepsilon_{10})R^4\,, \tag{5.7}$$

where ϕ is the (ten-dimensional) dilaton. The second term in 5.7 is absent in the four-dimensional compactifications to be considered below. The dilaton is also the string loop-counting parameter, which implies that terms in the effective action at L loops are proportional to $\exp(-2(1-L)\phi)$ in the string frame. The corresponding term in the one-loop effective action in the IIA string theory differs from the IIB case in the sign of the $\varepsilon_{10}\varepsilon_{10}$ term, and is proportional to $(t_8 t_8 + \frac{1}{8}\epsilon_{10}\epsilon_{10})R^4$. Although the $\varepsilon_{10}\varepsilon_{10}$-terms are absent in four dimensions, the different possible dependences of R^4 terms on the dilaton persist. Compared to the situation in 10 dimensions, they are even more complicated, because the dilaton can be expressed as a linear combination of the 70 scalars of $\mathcal{N}$=8 supergravity. Because the $\mathcal{R}^4$ terms itself are $SU(8)$-invariant so should be their prefactors. Green and Sethi [106] found powerful constraints on the possible dependences of prefactors in ten dimensions using supersymmetry alone. Indeed, only tree-level ($e^{-2\phi}$) and one-loop (constant) terms are allowed. It would be very interesting to examine the analogous supersymmetry constraints in four dimensions.

The issue of possible counterterms in maximal $\mathcal{N}$=8 supergravity [42, 43] is under perpetual investigation. Many of the current arguments rely on (linearized) superspace formulations and nonrenormalization theorems [107, 108], which in turn depend on the existence of an off-shell superspace formulation. While it was a common belief for some time that a superspace formulation of maximally-extended supersymmetric theories could be achieved employing off-shell formulations with at most half of the supersymmetry realized, an off-shell harmonic superspace with $\mathcal{N}$=3 supersymmetry for $\mathcal{N}$=4 super-Yang-Mills (SYM) theory was constructed in [109]. Assuming the existence of a similar description realizing six of the eight supersymmetries of $\mathcal{N}$=8 supergravity would postpone the onset of possible counterterms at least to the five-loop level, while realizing seven of eight would postpone it to the six-loop level [108]. However, an explicit construction of such superspace formalisms has not yet been achieved in the gravitational case.

Accompanying the above superspace considerations, there are also string- and M-theoretic arguments for the excellent ultraviolet behavior observed in $\mathcal{N}$=8 supergravity. While a nonrenormalization theorem developed in the pure spinor formalism for the closed superstring [110] has been used by Green, Russo and Vanhove [111] to argue that the first divergence in $\mathcal{N}$=8 supergravity might be delayed until nine loops, an analysis of dualities and volume-dependence in compactified string theory by the same authors [112] indicates a divergence already at seven loops. Arguments based on M-theory dualities suggest the possibility of finiteness to all loop orders [113, 114]. However, the applicability of arguments based on string and M-theory to $\mathcal{N}$=8 supergravity is subject to resolving the issues related to the decoupling of massive string states discussed in [115].

5.2 Is supersymmetry sufficient to show the finiteness of $\mathcal{N}$=8 supergravity?

Besides of the formal considerations described above it is possible to explore the divergence structure of $\mathcal{N}$=8 supergravity through direct computation of on-shell multi-loop graviton scattering amplitudes. While the two-loop four-graviton scattering amplitude [116] provided first hints that the $\mathcal{R}^4$ counterterm might have a vanishing coefficient at three loops, the full three-loop computation demonstrated this vanishing explicitly [117, 118].

A similar cancellation at four loops [119] is not so surprising for the four-point amplitude, because operators of the form $\partial^2 R^4$ can be eliminated in favor of R^5 using equations of motion [120]. The R^5-term in turn has no $\mathcal{N}$=8 supersymmetric completion [121, 122], which is in concordance with the absence of R^5 terms in the closed-superstring effective action [123].

Collecting the results from explicit multi-loop calculations, $\mathcal{N}$=8 supergravity amplitudes show an ultraviolet behavior which is even better than finite: it seems to be as good as for $\mathcal{N}$=4 super-Yang-Mills theory. Besides of explicit calculations, there have been a variety of attempts to understand the ultraviolet structure of $\mathcal{N}$=8 supergravity more directly at the amplitude level. The most famous result is the "no triangle" hypothesis [47, 124], which has been proven in [19, 125]. This theorem implies many, though not all, of the cancellations seen at higher loops [126]. Whereas some of the observed one-loop cancellations are not just due to supersymmetry, but to other properties of gravitational theories [127], others can be related to their non-color-ordered nature [128].

It is in particular those one-loop considerations and the work of ref. [108], which suggest that the conjectured finiteness of $\mathcal{N}$=8 supergravity might not be dictated by conventional $\mathcal{N}$=8 supersymmetry alone. However, since the construction of $\mathcal{N}$=8 supergravity it has been realized that another symmetry plays a key role — the exceptional, non-compact symmetry $E_{7(7)}$.

The general role of the $E_{7(7)}$ symmetry, regarding the finiteness of maximal supergravity, has been a topic of constant discussion. While a manifestly $E_{7(7)}$-invariant counterterm was presented long ago at eight loops [101, 129], newer results using the light-cone formalism cast a different light on the question [130]. In particular, the action of $E_{7(7)}$ on the Lagrangian in light-cone gauge [131] and covariantly [132, 133] have been studied recently.

Since no supersymmetry argument seems to restrict the appearance of the $\mathcal{R}^4$ counterterm, it is suggestive to investigate whether constraints could originate directly from the exceptional symmetry. As will be explained in detail in the next subsection, the non-compact part of $E_{7(7)}$ symmetry controls the soft emission of scalars in $\mathcal{N}$=8 supergravity. In particular there are two distinguished features: the single-soft scalar limit of an amplitude vanishes [28], while the double-soft scalar limit results in a weighted and $SU(8)$-rotated sum of amplitudes differing in the number of legs by two [19]. If one could show vanishing of the single-soft limit and validity of the double-soft limit relation for all amplitudes derived from a modified $\mathcal{N}$=8 supergravity action, in this case perturbing it by the $\mathcal{R}^4$ term, then this action would be compatible with $E_{7(7)}$.

However, in order for this reasoning to hold, $E_{7(7)}$ should remain a good symmetry at the quantum level. Although there is evidence in favor of this, no all-order proof is known. At one loop, the cancellation of anomalies for currents from the $SU(8)$ subgroup of $E_{7(7)}$ was demonstrated quite a while ago [134]. The analysis was subtle because a Lagrangian for the vector particles cannot be written in a manifestly $SU(8)$-covariant fashion. Thus the vectors contribute to anomalies, cancelling the more-standard contributions from the fermions. More recently, the question of whether the full $E_{7(7)}$ is a good quantum symmetry has been re-examined: He and Zhu showed that the infrared-finite part of single-soft scalar emission vanishes at one loop for an arbitrary number of external legs [135] as it does at tree level. Earlier, Kallosh, Lee and Rube [136] showed the vanishing of the four-point one-loop amplitude in the single-soft limit for complex momenta. A similar argument by Kaplan [137] shows that the double-soft scalar limit relation in $\mathcal{N}$=8 supergravity can also be extended to one loop. These results support the speculation that the full $E_{7(7)}$ is a good quantum symmetry of the theory, at least at the one-loop level.

5.3 $E_{7(7)}$ and soft scalar limits

Amplitudes in $\mathcal{N}$=8 supergravity are invariant under $SU(8)$ rotations by construction. On the other hand, the action of the coset symmetry $\frac{E_{7(7)}}{SU(8)}$ on amplitudes is not obvious. One can understand the connection by recalling that the vacuum state of the theory is specified by the expectation values of the physical scalars. Because the scalars are Goldstone bosons, their soft emission in an amplitude changes this expectation value and moves the theory to another point in the vacuum manifold.

Arkani-Hamed, Cachazo and Kaplan (ACK) [19] provided a very useful tool to investigate how the non-compact part of $E_{7(7)}$ symmetry controls the soft emission of scalars in $\mathcal{N}$=8 supergravity. Using the BCFW recursion relations [138, 139] they showed how generic amplitudes with one soft scalar particle vanish as the soft momentum approaches zero,

$$M_{n+1}(1,2,\ldots,n+1) \xrightarrow[p_1\to 0]{} 0\,. \tag{5.8}$$

This vanishing was first observed by Bianchi, Elvang and Freedman [28] and associated with the fact that the scalars parameterizing the coset manifold $E_{7(7)}/SU(8)$ obey relations similar to soft pion theorems [3, 140]. On the other hand, in the case of soft pion emission, the amplitude can remain nonvanishing as the (massless) pion momentum vanishes, due to graphs in which the pion is emitted off an external line; a divergence in the adjacent propagator cancels a power of pion momentum in the numerator from the derivative interaction. In the supergravity case, it was found that the external scalar emission graphs actually vanish on-shell in the soft limit [28].

Moving on to double-soft emission, several different situations have to be distinguished, which are labeled by the number of common indices between the sets $\{a_1,a_2,a_3,a_4\}$ and $\{a_5,a_6,a_7,a_8\}$ of scalar indices in the commutation relation (consult subsection 2.7.2 for a detailed discussion of the coset structure of $\mathcal{N}$=8 supergravity)[31]

$$-i\,[\mathfrak{X}_1^{a_1\ldots a_4},\mathfrak{X}_{2\,a_5\ldots a_8}] \;=\; \varepsilon^{b\,a_2a_3a_4}_{a_5a_6a_7a_8}\mathfrak{R}^{a_1}_b \;+\; \varepsilon^{a_1b\,a_3a_4}_{a_5a_6a_7a_8}\mathfrak{R}^{a_1}_b \;+\;\ldots\;+\; \varepsilon^{a_1a_2a_3a_4}_{a_5a_6a_7b}\mathfrak{R}^{b}_{a_8}\,. \tag{5.9}$$

Four common indices allow the creation of an $SU(8)$ singlet, corresponding to the emission of a single soft graviton. This case is not interesting because $[\mathfrak{X},\mathfrak{X}]$ vanishes. Similarly, if the scalars share one or two indices, the situation corresponds to a single soft limit in one of the subamplitudes generated by the BCFW recursion relations; thus this limit vanishes and does not probe the commutator in eq. (5.9). Another way to see the vanishing is to reconsider eq. (5.9) explicitly: there are simply not enough indices to saturate the right-hand side. The only interesting configuration occurs if the two generators $\mathfrak{X}_1$ and $\mathfrak{X}_2$ (and thus the scalars X_1 and X_2) agree on exactly three of their indices[32]. This result is in accordance with the commutation relation eq. (5.9), where three equal indices are necessary for the commutator of two non-compact generators to yield a result proportional to an $SU(8)$ generator.

Performing an explicit calculation of an $(n+2)$-point supergravity tree amplitude M_{n+2} containing two scalars sharing three indices and considering the double-soft limit on X_1 and X_2 results in the double-soft limit relation [19]

$$M_{n+2}(1,2,\ldots) \xrightarrow[p_1,p_2\to 0]{} \frac{1}{2}\sum_{i=3}^{n+2}\frac{p_i\cdot(p_2-p_1)}{p_i\cdot(p_1+p_2)}\mathfrak{R}(\eta_i)M_n(3,4,\ldots)\,, \tag{5.10}$$

[31] As defined after eq. (2.124), $\varepsilon^{a_1a_2a_3a_4c}_{a_5a_6a_7a_8b}=1,-1,0$ if the upper index set is an even, odd or no permutation of the lower set respectively.

[32] Working in unitary gauge, expressions $\mathfrak{X}$ for generators and X for scalars can be used equivalently. However, in amplitudes the X will be used to denote scalar particles although the resulting $SU(8)$ rotation is of course implied by the commutator of the corresponding generators.

where

$$\mathfrak{R}(\eta_i)^b_c \;=\; \mathfrak{R}\left([\mathfrak{X}^{a_1\ldots a_4}, \mathfrak{X}_{a_5\ldots a_8}]\right)^b_c \;=\; \varepsilon^{a_1a_2a_3a_4c}_{a_5a_6a_7a_8b} \times \eta_{ic}\partial_{\eta_{ib}} \tag{5.11}$$

acts on $(M_n)^c_b$. The right-most form of the generator is given in parallel to eq. (2.81) in the on-shell formalism. The n-point amplitude M_n has open $SU(8)$ indices due to the particular choice of indices of the scalars.

In the double-soft limit (5.10), the amplitude with two soft scalars sharing three indices becomes a sum of amplitudes with only hard momenta; in each summand one leg gets $SU(8)$ rotated by an amount depending on its momentum.

The relation (5.10) has been proven by ACK at tree level for pure $\mathcal{N}$=8 supergravity. In the following subsections suitable α'-corrected amplitudes, derived from an action containing the $\mathcal{R}^4$ term, will be constructed. For those amplitudes, the single- and double-soft limit will be taken numerically in order to test the $E_{7(7)}$ invariance of the $\mathcal{R}^4$ counterterm.

5.4 String-theory corrections to field-theory amplitudes

While the relation between open string theory and $\mathcal{N}$=4 SYM theory was discussed on the level of the action in subsection 3.1.3, the low-energy expansion for amplitudes is necessary to understand how string theory can help to calculate otherwise unaccessible field theory amplitudes.

Open-string tree amplitudes $\mathcal{A}_n$ have the same color decomposition (2.85), with $A_n^{\rm SYM}$ replaced by the color-ordered string subamplitude A_n. At the four-point level, the two subamplitudes are related by the Veneziano formula [141],

$$\begin{aligned} A_4(1^-,2^-,3^+,4^+) =&\; V^{(4)}_{\rm open}(s_1,s_2)\, A_4^{\rm SYM}(1^-,2^-,3^+,4^+) \\ =&\; \frac{\Gamma(1+\alpha' s_1)\Gamma(1+\alpha' s_2)}{\Gamma(1+\alpha' s_1+\alpha' s_2)}\, A_4^{\rm SYM}(1^-,2^-,3^+,4^+)\,. \end{aligned} \tag{5.12}$$

Expanding the form-factor $V^{(4)}$ in powers of α' in order to single out the low-energy contributions one finds

$$V^{(4)}_{\rm open}(s_1,s_2) \;=\; 1-\alpha'^2\zeta(2)s_1s_2+\alpha'^3\zeta(3)s_1s_2(s_1+s_2)+\mathcal{O}(\alpha'^4), \tag{5.13}$$

where the leading correction to the pure Yang-Mills amplitude arises from the F^4 interaction term of four gauge field-strength tensors (see subsection 3.1.3).

The full open string amplitude is quite simple in the four-point case (5.12). On the other hand, its generalizations to more external legs turn out to involve generalized hypergeometric functions [142]. Any n-point open string amplitude can be expressed in terms of $(n-3)!$ hypergeometric basis integrals. Expanding those functions in powers of α' yields expressions for the string-corrected five- and six-point MHV amplitudes

$$\begin{aligned} A_5 &= \left[V^{(5)}_{\rm open}(s_j) - \frac{i\,\alpha'^2}{2}\,\varepsilon(1,2,3,4)P^{(5)}_{\rm open}(s_j)\right] A_5^{\rm SYM} \\ A_6 &= \left[V^{(6)}_{\rm open}(s_j,t_j) - \frac{i\,\alpha'^2}{2}\sum_{k=1}^{5}\varepsilon_k P^{(6)}_{k\;\rm open}(s_j,t_j)\right] A_6^{\rm SYM}\,, \end{aligned} \tag{5.14}$$

where

$$\varepsilon_1 = \varepsilon(2,3,4,5), \quad \varepsilon_2 = \varepsilon(1,3,4,5), \quad \varepsilon_3 = \varepsilon(1,2,4,5), \quad \varepsilon_4 = \varepsilon(1,2,3,5), \quad \varepsilon_5 = \varepsilon(1,2,3,4)\,. \tag{5.15}$$

Expansions in α' are given by [143]

$$\begin{aligned} V^{(5)}_{\text{open}}(s_i) &= 1 - \frac{\alpha'^2\zeta(2)}{2}(s_1s_2 + s_2s_3 + s_3s_4 + s_4s_5 + s_5s_1) \\ &\quad + \frac{\alpha'^3\zeta(3)}{2}\big(s_1^2s_2 + s_2^2s_3 + s_3^2s_4 + s_4^2s_5 + s_5^2s_1 + s_1s_2^2 + s_2s_3^2 + s_3s_4^2 + s_4s_5^2 + s_5s_1^2 \\ &\qquad + s_1s_3s_5 + s_2s_4s_1 + s_3s_5s_2 + s_4s_1s_3 + s_5s_2s_4\big) + \mathcal{O}(\alpha'^4)\,, \end{aligned} \tag{5.16}$$

$$P^{(5)}_{\text{open}}(s_i) = \zeta(2) - \alpha'\zeta(3)(s_1 + s_2 + s_3 + s_4 + s_5) + \mathcal{O}(\alpha'^2)\,, \tag{5.17}$$

and explicit expressions for $V^{(6)}$ and $P_k^{(6)}$ can be found in the same reference.

Stieberger and Taylor have pushed the calculations even further [144]. In the process of determining all pure-gluon NMHV six-point amplitudes, they computed the following additional auxiliary amplitudes for helicity configuration X defined in eq. (2.70):

$$\langle \phi^-\phi^-\phi^-\phi^+\phi^+\phi^+\rangle\,, \quad \langle \phi^-\phi^-\lambda^-\lambda^+\phi^+\phi^+\rangle\,, \quad \text{and} \quad \langle \phi^-\phi^-g^-g^+\phi^+\phi^+\rangle\,, \tag{5.18}$$

as well the analogous quantities for Y and Z. As defined in table 1 in subsection 2.6, λ denotes a gluino and ϕ a scalar. In order to get an impression of the complexity of the result, here the pure-gluon NMHV six-point amplitude in helicity configuration X [144] will be provided, which will be expressed employing the following kinematic variables:

$$\alpha_X = -[12]\langle 34\rangle[6|X|5\rangle\,, \qquad \beta_X = [12]\langle 45\rangle[6|X|3\rangle\,, \qquad \gamma_X = [61]\langle 34\rangle[2|X|5\rangle\,, \tag{5.19}$$

where $X \equiv p_6 + p_1 + p_2$. The subamplitude reads[33]

$$\begin{aligned} &A_6(g_1^+, g_2^+, g_3^-, g_4^-, g_5^-, g_6^+) = \\ &\quad \frac{1}{s_5}\left(N_1^X\frac{\alpha_X^2}{s_1^2s_3^2} + N_2^X\frac{\beta_X^2}{s_1^2} + N_3^X\frac{\gamma_X^2}{s_3^2} + N_4^X\frac{\alpha_X\beta_X}{s_1^2s_3} + N_5^X\frac{\alpha_X\gamma_X}{s_1s_3^2} + N_6^X\frac{\beta_X\gamma_X}{s_1s_3}\right)\,, \end{aligned} \tag{5.20}$$

where the expansion of the functions N^X to $\mathcal{O}(\alpha'^2)$ is:

$$\begin{aligned} N_1^X &= -\alpha'^2\zeta(2)\ s_1s_3 + \dots\,, \\ N_2^X &= \frac{s_1}{s_2s_4t_1} - \alpha'^2\zeta(2)\ \left(\frac{s_1s_6}{s_2s_4} + \frac{s_1^2}{s_4t_1} + \frac{s_1s_5}{s_2t_1}\right) + \dots\,, \\ N_3^X &= \frac{s_3}{s_2s_6t_2} - \alpha'^2\zeta(2)\ \left(\frac{s_3s_4}{s_2s_6} + \frac{s_3s_5}{s_2t_2} + \frac{s_3^2}{s_6t_2}\right) + \dots\,, \\ N_4^X &= \alpha'^2\zeta(2)\ \left(\frac{s_1t_2}{s_2} + \frac{s_1t_3}{s_4}\right) + \dots\,, \\ N_5^X &= \alpha'^2\zeta(2)\ \left(\frac{s_3t_1}{s_2} + \frac{s_3t_3}{s_6}\right) + \dots\,, \\ N_6^X &= \frac{t_3}{s_2s_4s_6} + \alpha'^2\zeta(2)\ \left(\frac{s_1+s_3-s_5}{s_2} - \frac{t_1t_3}{s_2s_4} - \frac{t_2t_3}{s_2s_6} - \frac{t_3^2}{s_4s_6}\right) + \dots\,. \end{aligned} \tag{5.21}$$

[33] Note the shifted ordering of helicities compared to eq. (2.70). A cyclic shift $(1,2,3,4,5,6) \to (3,4,5,6,1,2)$ has to be performed in order to match the results analytically with ref. [144].

As discussed in subsection 3.1.3, the low-energy limit of closed type II string theory in four dimensions is $\mathcal{N}$=8 supergravity. The first correction to the low-energy effective action can be determined from the expression for the closed string four-point amplitude, or Virasoro-Shapiro amplitude [145, 146],

$$\begin{aligned} M_4(1^-,2^-,3^+,4^+) &= V^{(4)}_{\text{closed}}(s_1,s_2)\, M_4^{\text{SUGRA}}(1^-,2^-,3^+,4^+) \\ &= \frac{\Gamma(1+\alpha' s_1)\Gamma(1+\alpha' s_2)\Gamma(1-\alpha' s_1-\alpha' s_2)}{\Gamma(1-\alpha' s_1)\Gamma(1-\alpha' s_2)\Gamma(1+\alpha' s_1+\alpha' s_2)}\, M_4^{\text{SUGRA}}(1^-,2^-,3^+,4^+)\,. \end{aligned} \tag{5.22}$$

The expansion of $V^{(4)}_{\text{closed}}$ has the first nonvanishing correction at $\mathcal{O}(\alpha'^3)$,

$$V^{(4)}_{\text{closed}}(s_1,s_2) \;=\; 1 + 2\alpha'^3\zeta(3)\, s_1 s_2(s_1+s_2) + \mathcal{O}(\alpha'^4)\,, \tag{5.23}$$

which corresponds to a supersymmetrized version of eq. (5.6) in the low energy effective action (cf. subsection 3.1.3). In other words, keeping terms up to order $\mathcal{O}(\alpha'^3)$ in the closed-string amplitudes is equivalent to working with a theory whose effective action is given by

$$S_{\text{corr}} = \int \mathrm{d}^4x\sqrt{-g}(\mathcal{R} + \alpha'^3\mathcal{R}^4) + \mathcal{O}(\alpha'^4)\,. \tag{5.24}$$

Below it will turn out that the investigation of the double-soft scalar limit requires at least six-point NMHV amplitudes in $\mathcal{R}^4$-modified $\mathcal{N}$=8 supergravity. While α'-corrected six-point amplitudes in open string theory ($\mathcal{N}$=4 SYM) are already very cumbersome to calculate, the situation is even worse for closed string theory ($\mathcal{N}$=8 supergravity). For higher-point tree amplitudes it is therefore more convenient to rely on the KLT relations (subsection 3.1.2), which express closed string amplitudes as simple quadratic combinations of open string amplitudes.

As discussed in subsection 2.8, different cyclic orderings are required as input to the KLT relations. All possible different cyclic orderings are available for the open-string six-point amplitudes[34] computed in (5.18). However, the limited number of explicit open-string amplitude calculations constrains the available supergravity amplitudes. Thus the α'-corrected $\mathcal{N}$=8 supergravity amplitude has to be chosen carefully, which will be discussed in the next subsection.

5.5 Setting up the calculation

Arkani-Hamed, Cachazo and Kaplan have proven eq. (5.10) analytically, by employing BCFW recursion relations for $\mathcal{N}$=8 supergravity with $E_{7(7)}$ realized on-shell. Because invariance under $E_{7(7)}$ is a necessary condition for the relation to be valid, eq. (5.10) provides a useful tool for testing other theories, or operators, for their symmetry properties under $E_{7(7)}$. In particular, if the double-soft limit of all $(n+2)$-point amplitudes derived from

[34] I am grateful to Stephan Stieberger and Tomasz Taylor for providing me with expressions for the amplitudes from ref. [144] through order α'^3.

eq. (5.24) coincides with the $SU(8)$-rotated sum of the corresponding n-point amplitudes, that would be strong evidence that $E_{7(7)}$ symmetry does not restrict the appearance of $\mathcal{R}^4$ as a counterterm in $\mathcal{N}$=8 supergravity.

The analytical approach that ACK used to prove eq. (5.10) does not hold for the α'-corrected $\mathcal{N}$=8 amplitudes. Higher-dimension operators lead to poorer large-momentum behavior, so that amplitudes shifted by large complex momenta will not fall off fast enough for the BCFW recursion relations to be valid. Instead one has to find explicit lengthy expressions for suitable string theory amplitudes, from which the α'-corrected amplitudes corresponding to eq. (5.24) can be deduced, and their double-soft limits inspected numerically.

After discussing the low-energy-expansion of KLT relations, the constraints on the α'-corrected $\mathcal{N}$=8 supergravity amplitude originating from the single- and the double-soft limit relation eq. (5.10) in subsection 5.5.1 will be explored. Appropriate $\mathcal{N}$=8 amplitudes will be identified and decomposed into $\mathcal{N}$=4 SYM matrix elements using the KLT relations. The required α'-corrected $\mathcal{N}$=4 SYM matrix elements can be related to the available open string amplitudes by carefully examining the NMHV supersymmetric Ward identities. In subsections 5.5.2 and 5.5.3, the $\mathcal{N}$=1 supersymmetric Ward identities will be reviewed in detail and used to obtain expressions for the $\mathcal{N}$=4 amplitudes, which finally serve as input to the KLT relations, in subsection 5.6.

As discussed in subsection 3.1.2, equalities (3.10-3.12) are exact relations between string theory amplitudes, and so they are valid order by order in α'. In order to calculate the string correction to an $\mathcal{N}$=8 supergravity amplitude at a certain order in α' from known α'-corrected expressions in $\mathcal{N}$=4 SYM, one has to determine all combinations of terms from the expansions of the amplitudes and the sine functions, whose multiplication results in the correct power of α'. For instance the second-order correction to the five-point amplitude in supergravity corresponds to terms of $\mathcal{O}(\alpha'^4)$, due to the prefactor of $\frac{1}{\alpha'^2}$. Taking the absence of first-order corrections to $\mathcal{N}$=4 SYM amplitudes into account, four combinations have to be considered in eq. (3.11), according to the following table:

$\sin(\alpha'\pi s_{12})$	$\sin(\alpha'\pi s_{34})$	$A_5(1,2,3,4,5)$	$A_5(2,1,4,3,5)$
$\mathcal{O}(\alpha'^1)$	$\mathcal{O}(\alpha'^1)$	$\mathcal{O}(\alpha'^0)$	$\mathcal{O}(\alpha'^2)$
$\mathcal{O}(\alpha'^1)$	$\mathcal{O}(\alpha'^1)$	$\mathcal{O}(\alpha'^2)$	$\mathcal{O}(\alpha'^0)$
$\mathcal{O}(\alpha'^3)$	$\mathcal{O}(\alpha'^1)$	$\mathcal{O}(\alpha'^0)$	$\mathcal{O}(\alpha'^0)$
$\mathcal{O}(\alpha'^1)$	$\mathcal{O}(\alpha'^3)$	$\mathcal{O}(\alpha'^0)$	$\mathcal{O}(\alpha'^0)$

yielding

$$\begin{aligned} M_5^{\mathcal{O}(\alpha'^2)} = i s_{12} s_{34} \Big[& A_5^{\rm SYM}(1,2,3,4,5)\, A_5^{\mathcal{O}(\alpha'^2)}(2,1,4,3,5) \\ & + A_5^{\mathcal{O}(\alpha'^2)}(1,2,3,4,5)\, A_5^{\rm SYM}(2,1,4,3,5) \\ & - \tfrac{\pi^2}{6}(s_{12}^2 + s_{34}^2) A_5^{\rm SYM}(1,2,3,4,5)\, A_5^{\rm SYM}(2,1,4,3,5) \Big] + \mathcal{P}(2,3)\,. \end{aligned} \quad (5.25)$$

The above expression can be shown to vanish analytically, in accordance with the higher-point generalization of eq. (5.23), or alternatively eq. (5.24), the statement that the first correction to the closed-string effective action is at $\mathcal{O}(\alpha'^3)$.

5.5.1 Choosing a suitable amplitude

The simplest scenario one might think of, in order to test the double-soft scalar limit relation (5.10), is to start with a five-point amplitude, which in turn leads to a sum of three-point amplitudes on the right-hand side of the relation. Three-point amplitudes are special as they require a setup with complex momenta in order to be non-trivial. However, here another constraint has to be taken into account: amplitudes shall be tested that receive nonvanishing corrections from the $\mathcal{R}^4$ term. Because the interactions originating in this counterterm candidate start at the four-point level, it is not sufficient to consider three-point amplitudes on the right-hand side.

Therefore the investigation has to be performed for at least six-point amplitudes, which should reduce to a sum of four-point amplitudes in the double-soft limit. Requiring again that the four-point amplitudes on the right-hand side of eq. (5.10) are nonvanishing implies that they are MHV (or equivalently anti-MHV). Fortunately, corrections to all MHV amplitudes with four legs are known up to $\mathcal{O}(\alpha'^3)$, indeed to arbitrary order in α', using eq. (5.22) and the MHV supersymmetry Ward identities.

On the left-hand side of eq. (5.10) the situation is more intricate. The four particles that appear already on the right-hand side are now accompanied by two additional scalars. According to eq. (2.129), the number of η derivatives acting on the generating functional is increased by eight, four for each scalar, so that the resulting amplitude resides in the NMHV sector. In addition, the two scalars have to share three $SU(8)$ indices, as elaborated on in subsection 2.7.2. Sorting out the distribution of the scalars' indices into two $SU(4)$ subgroups, there are finally five possible distinct choices[35] satisfying the constraints. They are listed here, together with their respective KLT decompositions according to table 3 in subsection 2.8:

$$\langle X^{abrs} X_{abrt} \cdot\cdot\cdot\cdot \rangle \;\to\; \langle \phi^{ab}\, \phi_{ab} \cdot\cdot\cdot\cdot \rangle_L \times \langle \phi^{rs}\, \phi_{rt} \cdot\cdot\cdot\cdot \rangle_R, \tag{5.26}$$
$$\langle X^{abrc} X_{abrs} \cdot\cdot\cdot\cdot \rangle \;\to\; \langle \varepsilon^{abcd} \lambda^-_d\, \phi_{ab} \cdot\cdot\cdot\cdot \rangle_L \times \langle \lambda^{r+}\, \phi_{rs} \cdot\cdot\cdot\cdot \rangle_R, \tag{5.27}$$
$$\langle X^{abrc} X_{abrd} \cdot\cdot\cdot\cdot \rangle \;\to\; \langle \lambda^-_d\, \lambda^{c+} \cdot\cdot\cdot\cdot \rangle_L \times \langle \lambda^{r+}\, \lambda^-_r \cdot\cdot\cdot\cdot \rangle_R, \tag{5.28}$$
$$\langle X^{abcr} X_{abcs} \cdot\cdot\cdot\cdot \rangle \;\to\; \langle \lambda^-_d\, \lambda^{d+} \cdot\cdot\cdot\cdot \rangle_L \times \langle \lambda^{r+}\, \lambda^-_s \cdot\cdot\cdot\cdot \rangle_R, \tag{5.29}$$
$$\langle X^{abcd} X_{abcr} \cdot\cdot\cdot\cdot \rangle \;\to\; \langle g^-\, \lambda^{d+} \cdot\cdot\cdot\cdot \rangle_L \times \langle g^+\, \lambda^-_r \cdot\cdot\cdot\cdot \rangle_R\,. \tag{5.30}$$

Here the ellipses are understood to be filled with four particles such that the L- and R-amplitudes on the right-hand side of the KLT relation each transform as an $SU(4)$ singlet. In each of equations (5.28) to (5.30) a factor of $\varepsilon^{abcd}\varepsilon_{abcd}$ was left out. Because these indices are not summed over, this factor is equal to unity. Note that $\langle X^{abcd} X_{abce} \cdot\cdot\cdot\cdot \rangle$ is absent because the five $SU(4)$ indices a, b, c, d, e cannot be made all distinct.

In order to proceed, supersymmetric Ward identities have to be used to relate one of the five decompositions (5.26)–(5.30) to the available open-string six-point results (see

[35] Another five combinations can be obtained by switching the left and right $SU(4)$.

eq. (5.18) in section 5.4):

$$\begin{aligned}&\langle g^- \, g^- \, g^- \, g^+ \, g^+ \, g^+ \rangle\,, \quad \langle \phi^- \, \phi^- \, \phi^- \, \phi^+ \, \phi^+ \, \phi^+ \rangle, \\ &\langle \phi^- \, \phi^- \, \lambda^- \, \lambda^+ \, \phi^+ \, \phi^+ \rangle \quad \text{and} \quad \langle \phi^- \, \phi^- \, g^- \, g^+ \, \phi^+ \, \phi^+ \rangle\,. \end{aligned} \tag{5.31}$$

Supersymmetric Ward identities can be classified by the amount of supersymmetry employed (*e.g.*, $\mathcal{N}$=1, 2, 4), as well as the number of legs and the sector (MHV, NMHV, *etc.*) characterizing the amplitudes. Dealing with six-point NMHV amplitudes exclusively below, the notation $\mathcal{N}$=4 SWI will refer to the set of supersymmetric Ward identities relating six-point NMHV amplitudes built from the full $\mathcal{N}$=4 multiplet $(g^\pm, \lambda^\pm_m, \phi^\pm_n)$, where $m = 1, 2, 3, 4$ and $n = 1, 2, 3$. (Note that a superscript $\pm$ on ϕ implies a complex field with a index labeling different from the real ϕ_{ab} used above.) In the original article [144], $\mathcal{N}$=2 supersymmetric Ward identities have been served to relate the latter three amplitudes in eq. (5.31) to the pure-gluon one. So the obvious idea is to search among the decompositions (5.26)–(5.30) for one in which the amplitudes contain particles from a single $\mathcal{N}$=2 multiplet and its CPT conjugate, $(g^\pm, \lambda^\pm_m, \phi^\pm)$ with $m = 1, 2$.

However, the third amplitude in eq. (5.31) contains only one type of fermion, which points into the direction of a $\mathcal{N}$=1 multiplet. Setting up the calculation employing $\mathcal{N}$=1 SWI exclusively is simpler than using $\mathcal{N}$=2 SWI: for six-point NMHV amplitudes an explicit solution to the $\mathcal{N}$=1 SWI is known [21, 28].[36]

The decompositions (5.26) to (5.30) are not all equally suited to the use of an $\mathcal{N}$=1 SWI. For example, the left $SU(4)$ amplitude of eq. (5.27) contains three distinct $SU(4)$ indices, a, b, d, thus requiring a full $\mathcal{N}$=4 multiplet. The other four decompositions contain amplitudes which can be constructed from SWI with less supersymmetry. Indeed, the decomposition (5.30) contains only one index for the left $SU(4)$ amplitude, and one for the right one; this decomposition is the one which will be used in the following subsections. As will be explained below, it is possible to obtain everything necessary for testing the single- and double-soft limit through eq. (5.30), by using a two-step procedure employing two different sets of $\mathcal{N}$=1 SWI based on the multiplets $(g^\pm, \lambda^\pm)$ and $(\phi^\pm, \lambda^\pm)$.

The next three paragraphs elaborate on the $\mathcal{N}$=1 NMHV SWI for $(g^\pm, \lambda^\pm)$ in particular, and then describe the analogous set of $\mathcal{N}$=1 SWI for the multiplet $(\phi^\pm, \lambda^\pm)$. Then, in subsection 5.6, these ingredients will be assembled in order to test the $E_{7(7)}$ symmetry.

5.5.2 $\mathcal{N}$=1 supersymmetric Ward identities in the NMHV sector

As an example, the set of amplitudes involving four gluons (g^+, g^-) and a single pair of gluinos (λ^+, λ^-) shall be investigated. For simplicity, the $SU(4)$ index will be dropped.

[36] Very recently the supersymmetric Ward identities in maximally supersymmetric $\mathcal{N}$=4 super-Yang-Mills theory and $\mathcal{N}$=8 supergravity were solved, for arbitrary n-point N^{p}MHV amplitudes [147] in terms of basis amplitudes, in a manifestly supersymmetric form. These results may prove very useful in extending the considerations of this paper to a larger number of legs.

The states are related by $\mathcal{N}$=1 supersymmetry via

$$\begin{aligned}\big[\mathfrak{Q}(\xi), g^+(p)\big] &= [p\xi]\,\lambda^+(p),\\ \big[\mathfrak{Q}(\xi), \lambda^+(p)\big] &= -\langle p\xi\rangle g^+(p),\\ \big[\mathfrak{Q}(\xi), g^-(p)\big] &= \langle p\xi\rangle \lambda^-(p),\\ \big[\mathfrak{Q}(\xi), \lambda^-(p)\big] &= -\,[p\xi]\,g^-(p),\end{aligned} \tag{5.32}$$

where $\mathfrak{Q}(\xi) = \langle \mathfrak{Q}\xi\rangle$ as defined in subsection 2.5.

For each NMHV helicity sector, there are 20 distinct amplitudes related by $\mathcal{N}$=1 SWI: a pure-gluon amplitude, a pure-gluino amplitude, nine two-gluino four-gluon amplitudes, and nine four-gluino two-gluon amplitudes, as shown in figure 3. In the following, amplitudes drawn from helicity configuration X in eq. (2.70) are discussed. For the two other configurations Y and Z, the relations are completely analogous.

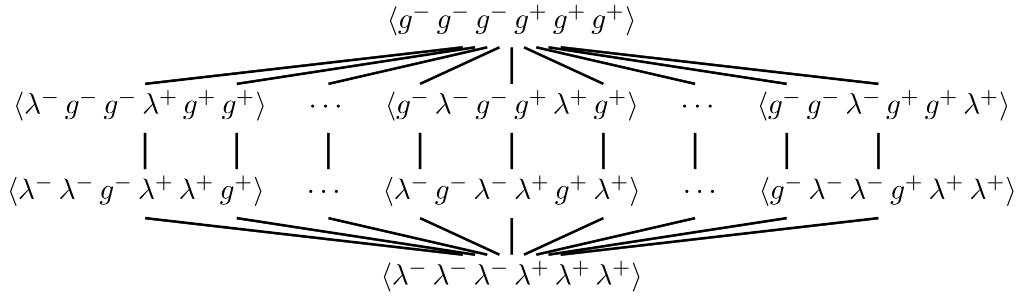


Figure 3: *Amplitudes related by $\mathcal{N}$=1 supersymmetric Ward identities.*

Amplitudes in adjacent rows of figure 3 are related by the $\mathcal{N}$=1 SWI. Acting for example with the supersymmetry operator $\mathfrak{Q}(\xi)$ on the source term $\langle g^-\, g^-\, g^-\, \lambda^+\, g^+\, g^+\rangle$ yields

$$\begin{aligned}&\langle 4\eta\rangle\langle g^-\, g^-\, g^-\, g^+\, g^+\, g^+\rangle - \langle 1\eta\rangle\langle \lambda^-\, g^-\, g^-\, \lambda^+\, g^+\, g^+\rangle\\ &- \langle 2\eta\rangle\langle g^-\, \lambda^-\, g^-\, \lambda^+\, g^+\, g^+\rangle - \langle 3\eta\rangle\langle g^-\, g^-\, \lambda^-\, \lambda^+\, g^+\, g^+\rangle = 0\,,\end{aligned} \tag{5.33}$$

which relates the pure-gluon amplitude to the two-gluino four-gluon ones from the second row in figure 3. Due to the freedom in choosing the two-component supersymmetry parameter η, the result is a system of equations which has rank 2. In order to find all relations between the pure gluon amplitude (first row) and the amplitudes in the second row, the action of $\mathfrak{Q}(\xi)$ on all possible source terms featuring one gluino and five gluons,

$$\begin{aligned}&\langle \lambda^-\, g^-\, g^-\, g^+\, g^+\, g^+\rangle, \quad \langle g^-\, \lambda^-\, g^-\, g^+\, g^+\, g^+\rangle, \quad \langle g^-\, g^-\, \lambda^-\, g^+\, g^+\, g^+\rangle,\\ &\langle g^-\, g^-\, g^-\, \lambda^+\, g^+\, g^+\rangle, \quad \langle g^-\, g^-\, g^-\, g^+\, \lambda^+\, g^+\rangle, \quad \langle g^-\, g^-\, g^-\, g^+\, g^+\, \lambda^+\rangle,\end{aligned} \tag{5.34}$$

has to be considered. The resulting system, linking ten amplitudes from the first and second rows, turns out to have rank eight, thus requiring two known amplitudes in order to derive all the others.

Repeating the analysis for the second and third rows, there are notably more identities to consider. They are generated by acting with $\mathfrak{Q}(\xi)$ on any of the 18 different source terms built from three gluinos and the same number of gluons, *e.g.* $\langle \lambda^- \lambda^- g^- g^+ \lambda^+ g^+ \rangle$. Interestingly this system connecting 18 unknown amplitudes is of rank 16, meaning that again two amplitudes have to be known in order to fix all the others.

Finally, the relations between the third row and the pure-gluino amplitude (fourth row) mirror the situation found for the top of the diagram and are also of rank eight.

Combining all of the above into one large system of equations, the total rank of the supersymmetric Ward identities pictured in figure 3 turns out to be 18. So, given any two of the 20 distinct amplitudes, one can calculate any other from this set employing the complete collection of $\mathcal{N}$=1 SWI. The corresponding result has already been found by Grisaru and Pendleton in the context of $\mathcal{N}$=1 supergravity [21], and recast recently in modern spinor-helicity form [28].

More explicitly, any two-gluino four-gluon amplitude $F_{i,I}$, with the gluinos situated at positions i and I, can be expressed in terms of the pure-gluon and pure-gluino amplitude as

$$F_{i,I} = \frac{4\langle Ij\rangle[ij]\langle g^- g^- g^- g^+ g^+ g^+\rangle - \varepsilon_{ijk}\langle jk\rangle\varepsilon_{IJK}[JK]\langle \lambda^-\lambda^-\lambda^-\lambda^+\lambda^+\lambda^+\rangle}{-2\sum_{m,n\in\{i,j,k\}}\langle mn\rangle[nm]}\,, \tag{5.35}$$

where i, j, k and I, J, K mark the set of negative and positive helicity particles respectively, and the numerator contains implicit sums over j, k, J, K. For example,

$$F_{3,4} = \langle g^- g^- \lambda^- \lambda^+ g^+ g^+\rangle = \frac{\langle 4|(1+2)|3]\langle g^- g^- g^- g^+ g^+ g^+\rangle + \langle 12\rangle[56]\langle \lambda^-\lambda^-\lambda^-\lambda^+\lambda^+\lambda^+\rangle}{t_1}\,. \tag{5.36}$$

A similar formula for all four-gluino two-gluon amplitudes can be found in the appendix of ref. [28].

5.5.3 The second $\mathcal{N}$=1 SUSY diamond

Recall [144] that the pure-gluon amplitude can be calculated from the latter three amplitudes in eq. (5.31), namely

$$\langle \phi^-\phi^-\phi^-\phi^+\phi^+\phi^+\rangle, \quad \langle \phi^-\phi^-\lambda^-\lambda^+\phi^+\phi^+\rangle \quad \text{and} \quad \langle \phi^-\phi^- g^- g^+\phi^+\phi^+\rangle\,. \tag{5.37}$$

The question that immediately arises is whether this set forms a basis for the complete set of all six-point NMHV $\mathcal{N}$=2 amplitudes[37] in helicity configuration X? Having not been aware of a direct answer to that question, the following approach was: as mentioned already in subsection 5.5.1, a second set of six-point NMHV $\mathcal{N}$=1 supersymmetric Ward identities was employed in addition to the $\mathcal{N}$=1 SWI for $(g^\pm, \lambda^\pm)$ described in the previous subsection.

In figure 4 the collection of six-point NMHV $\mathcal{N}$=2 amplitudes is depicted in helicity configuration X. Every black dot denotes a particular amplitude. The top point represents the pure-gluon amplitude $\langle g^- g^- g^- g^+ g^+ g^+\rangle$, the lowest point refers to the pure-scalar

[37] The term $\mathcal{N}$=2 amplitudes refers to all possible amplitudes that can be constructed exclusively from particles from a single $\mathcal{N}$=2 multiplet and its CPT conjugate, $(g^\pm, \lambda^\pm_m, \phi^\pm)$ with $m = 1, 2$ [148].

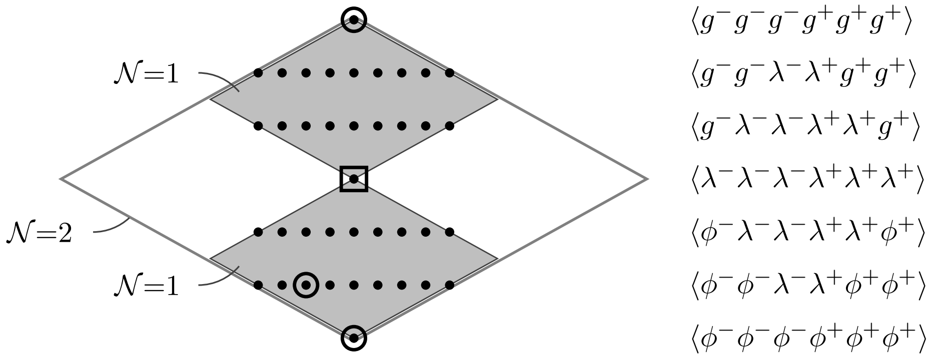


Figure 4: *Amplitudes involving particles from a single $\mathcal{N}$=2 multiplet containing two $\mathcal{N}$=1 subsets.*

amplitude $\langle \phi^- \phi^- \phi^- \phi^+ \phi^+ \phi^+ \rangle$, and the central point denotes the pure-gluino amplitude $\langle \lambda^- \lambda^- \lambda^- \lambda^+ \lambda^+ \lambda^+ \rangle$. Supersymmetric Ward identities relate certain amplitudes from adjacent rows and the elements of eq. (5.31) are encircled. The upper diamond-shaped region corresponds precisely to figure 3: it is the subset of six-point NMHV $\mathcal{N}$=1 amplitudes built from the multiplet ($g^\pm$, $\lambda^\pm$) within the $\mathcal{N}$=2 amplitudes. (There are additional states in the full $\mathcal{N}$=2 diamond in figure 4, of course, even in the second row.)

However, the upper diamond-shaped region is not the only subset of six-point NMHV $\mathcal{N}$=2 amplitudes which can be related by $\mathcal{N}$=1 supersymmetric Ward identities. Stretching between the pure-gluino and the pure-scalar amplitude there is a second region (referred to as the lower diamond in the following), which satisfies relations similar to those in the upper $\mathcal{N}$=1 diamond. The modified supersymmetry operator $\tilde{\mathfrak{Q}}$ will now act on a multiplet consisting of scalars (ϕ^+, ϕ^-) and gluinos (λ^+, λ^-) via

$$\begin{aligned}
\Big[\tilde{\mathfrak{Q}}(\xi), \phi^+(p)\Big] &= \langle p\xi\rangle \lambda^+(p), \\
\Big[\tilde{\mathfrak{Q}}(\xi), \lambda^+(p)\Big] &= -\,[p\xi]\,\phi^+(p), \\
\Big[\tilde{\mathfrak{Q}}(\xi), \phi^-(p)\Big] &= [p\xi]\,\lambda^-(p), \\
\Big[\tilde{\mathfrak{Q}}(\xi), \lambda^-(p)\Big] &= -\langle p\xi\rangle \phi^-(p)\,,
\end{aligned} \tag{5.38}$$

which can be easily derived by identifying the supercharges of $\mathcal{N}$=2 supersymmetry, $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$, with $\mathfrak{Q}$ and $\tilde{\mathfrak{Q}}$ respectively.

Writing down the set of supersymmetric Ward identities generated by acting with a supersymmetry generator $\tilde{\mathfrak{Q}}$ on the source term $\langle \phi^- \phi^- \phi^- \lambda^+ \phi^+ \phi^+ \rangle$, one encounters the same structure derived in eq. (5.33):

$$\begin{aligned}
&[4\eta]\langle \phi^- \phi^- \phi^- \phi^+ \phi^+ \phi^+ \rangle - [1\eta]\langle \lambda^- \phi^- \phi^- \lambda^+ \phi^+ \phi^+ \rangle \\
&- [2\eta]\langle \phi^- \lambda^- \phi^- \lambda^+ \phi^+ \phi^+ \rangle - [3\eta]\langle \phi^- \phi^- \lambda^- \lambda^+ \phi^+ \phi^+ \rangle = 0.
\end{aligned} \tag{5.39}$$

In fact, one can show that the complete system of supersymmetric Ward identities and amplitudes for the lower diamond, ranging from the pure-gluino to the pure-scalar amplitude, can be obtained from the original $\mathcal{N}$=1 system considered in figure 3 by exchanging

$$\begin{array}{ccc} \mathfrak{Q} & \leftrightarrow & \tilde{\mathfrak{Q}} \\ [\ \ ] & \leftrightarrow & \langle\ \ \rangle \\ g^+ & \leftrightarrow & \phi^+ \\ g^- & \leftrightarrow & \phi^- . \end{array} \tag{5.40}$$

This symmetry corresponds geometrically to reflecting figure 4 about a horizontal line passing through the central point $\langle\lambda^-\lambda^-\lambda^-\lambda^+\lambda^+\lambda^+\rangle$.

The second system of supersymmetric Ward identities in the lower diamond is obviously of the same rank as the original system. However, in contrast to the upper diamond it contains now *two* of the known amplitudes from ref. [144],

$$\langle\phi^-\phi^-\phi^-\phi^+\phi^+\phi^+\rangle \qquad \text{and} \qquad \langle\phi^-\phi^-\lambda^-\lambda^+\phi^+\phi^+\rangle\,, \tag{5.41}$$

which allows the calculation of any other amplitude in the lower $\mathcal{N}$=1 set. In particular, the pure-gluino amplitude $\langle\lambda^-\lambda^-\lambda^-\lambda^+\lambda^+\lambda^+\rangle$ ($\square$ in figure 4), which is the element connecting the upper and lower set of equations, can be determined. Having done so, there are now two known amplitudes from the upper $\mathcal{N}$=1 diamond, the pure-gluino and the pure-gluon amplitude [144], which in turn is the precondition for determining any amplitude from the upper $\mathcal{N}$=1 region. In other words: any six-point NMHV amplitude in the two shaded regions in figure 4 can be calculated from eq. (5.31).

In the next section, the ellipses on the left-hand side of the decomposition (5.30) will be completed by two gravitini and two gravitons. The resulting amplitude will be KLT factorized in such a way that the desired six-point closed-string ($\mathcal{N}$=8 supergravity) amplitude can be related to a set of two-gluino four-gluon $\mathcal{N}$=4 SYM amplitudes. The SYM amplitudes are available in turn by the two-step procedure described above.

5.6 $E_{7(7)}$ symmetry for α'-corrected amplitudes?

As explained in the last section, the most accessible way of testing the double-soft scalar limit relation is to calculate the $\mathcal{N}$=8 supergravity amplitude,

$$\langle X^{1234}\, X_{1235}\, F^{5+} F_4^-\, B^+\, B^-\rangle = \text{KLT}\Big[\langle g^-\,\lambda^{4+}\,g^+\,\lambda_4^-\,g^+\,g^-\rangle_L \times \langle g^+\,\lambda_5^-\,\lambda^{5+}\,g^-\,g^+\,g^-\rangle_R\Big]\,, \tag{5.42}$$

a particular version of eq. (5.30). The determination of the right-hand side of eq. (5.42) will be done by employing the two-step procedure described in the last subsection.

How is it possible to obtain the pure-gluino amplitude $\langle\lambda^-\lambda^-\lambda^-\lambda^+\lambda^+\lambda^+\rangle$ from the amplitudes in eq. (5.41) in the first step? An expression relating any six-point NMHV two-fermion four-boson amplitude to the pure-fermion and pure-boson one has been given in eq. (5.35). Starting from eq. (5.36), employing the correspondence eq. (5.40) which transforms the pure-gluon amplitude into the pure-scalar one, and solving the resulting equation for the pure-gluino amplitude yields

$$\langle\lambda^-\lambda^-\lambda^-\lambda^+\lambda^+\lambda^+\rangle = \frac{(k_1+k_2+k_3)^2\langle\phi^-\phi^-\lambda^-\lambda^+\phi^+\phi^+\rangle - \langle 3|(1+2)|4]\langle\phi^-\phi^-\phi^-\phi^+\phi^+\phi^+\rangle}{\langle 56\rangle[12]}\,. \tag{5.43}$$

In the second step, eq. (5.35) will be used to obtain analytical expressions for all two-gluino four-gluon amplitudes, allowing the assembly of the $\mathcal{N}$=8 amplitude finally.

In the same manner as explained in subsection 5.5 for the expansion to $\mathcal{O}(\alpha'^2)$ of a five-point gravity amplitude, appropriate combinations of orders in α' have to be added and permuted on the right-hand side of eq. (5.42) in order to obtain the result including the $\mathcal{R}^4$ perturbation. Explicitly, the third order in α' can be obtained by evaluating

$$\begin{aligned} M_6^{\mathcal{O}(\alpha'^3)} = -is_{12}s_{45}\Big(& A_6^{\rm SYM}(1,2,3,4,5,6) \\ &\times\Big[s_{35}A_6^{\mathcal{O}(\alpha'^3)}(2,1,5,3,4,6)+(s_{34}+s_{35})A_6^{\mathcal{O}(\alpha'^3)}(2,1,5,4,3,6)\Big] \\ &+A_6^{\mathcal{O}(\alpha'^3)}(1,2,3,4,5,6) \\ &\times\Big[s_{35}A_6^{\rm SYM}(2,1,5,3,4,6)+(s_{34}+s_{35})A_6^{\rm SYM}(2,1,5,4,3,6)\Big]\Big) \\ &+\mathcal{P}(2,3,4)\,. \end{aligned} \tag{5.44}$$

All amplitudes needed on the right-hand side of eq. (5.44) are two-gluino four-gluon amplitudes for the helicity configurations X, Y or Z, which have been related by supersymmetry to the amplitudes considered in ref. [144].

Before discussing the double-soft limit relation, the single-soft limit will be examined in order to see whether the vanishing (5.8) observed in $\mathcal{N}$=8 supergravity still holds for the $\mathcal{R}^4$ matrix elements. For the four-point amplitude, the factor of $s_1s_2(s_1+s_2)$ in the $\mathcal{O}(\alpha'^3)$ term in $V^{(4)}_{\rm closed}$ in eq. (5.23) shows that the $\mathcal{R}^4$ matrix element vanishes at least as fast as the supergravity amplitude. Similarly, using the forms (5.14) for the open string five- and six-point MHV amplitudes, together with the appropriate KLT relations, it can be shown numerically that the single-soft limit of the five- and six-point MHV matrix elements of $\mathcal{R}^4$ vanish. That is, a sequence of kinematical configurations with the momentum of the scalar tending to zero is constructed, for which the $\mathcal{R}^4$ matrix elements vanish linearly. In the MHV case, it is sufficient to test the single-soft vanishing for one particular amplitude containing scalars, because all other MHV amplitudes are related by SWI involving ratios of spinor products that are constant in the soft limit.

On the other hand, examining non-MHV six-point $\mathcal{R}^4$ matrix elements (5.44) numerically, one finds that the single-soft limit does *not* vanish.[38] The question is whether this implies the breaking of $E_{7(7)}$ symmetry by the $\mathcal{R}^4$ term. In principle there could be modifications to the external scalar emission graphs that still allowed the symmetry to be intact (as happens in the pion case). However, the $\mathcal{R}^4$ term does not produce any nonvanishing on-shell three-point amplitudes. So it seems that the $E_{7(7)}$ symmetry is indeed broken, beginning at the level of the non-MHV six-point amplitude.

[38] Thanks to Juan Maldacena for suggesting to examine this limit, and for related discussions.

One might wonder why the breaking shows up only at this level. Considering the ten-dimensional term $e^{-2\phi} t_8 t_8 R^4$ discussed in the introduction, which becomes $e^{-6\phi} t_8 t_8 R^4$ after transforming to Einstein frame, one might suspect a violation of the single-soft limit from the non-derivative ϕ coupling already at the five-point level, expanding $e^{-6\phi} = 1 - 6\phi + \ldots$, and with R^4 producing two negative and two positive helicity gravitons. However, in four dimensions, the dilaton belongs to the **70** of $SU(8)$, while the gravitons are singlets, so an amplitude of the form $\langle \phi B^- B^- B^+ B^+ \rangle$ is forbidden by $SU(8)$. Adding another scalar corresponds to providing a quadratic $SU(8)$-invariant scalar prefactor for R^4, and first affects NMHV six-point amplitudes.

Despite the apparent breaking of the $E_{7(7)}$ symmetry exhibited by the single-soft limit of the NMHV six-point amplitude $\langle X^{1234}\, X_{1235}\, F^{5+} F_4^- \, B^+ \, B^- \rangle$ at $\mathcal{O}(\alpha'^3)$, the double-soft limit of this amplitude will now be examined. Given the particular choice of amplitude (5.42), it is straightforward to find an expression for the right-hand side of (5.10). The operator

$$\mathfrak{R}^4{}_5 = \varepsilon^{12345}_{12354}\ \eta_{i5} \partial_{\eta_{i4}} = -\, \eta_{i5} \partial_{\eta_{i4}} \tag{5.45}$$

will act on the remnant of the six-point amplitude as

$$\begin{aligned}
&- \sum_{i=3}^{6} \eta_{i5} \partial_{\eta_{i4}} \langle F^{5+} F_4^- \, B^+ \, B^- \rangle \\
&= \sum_{i=3}^{6} \eta_{i5} \partial_{\eta_{i4}} \left(\frac{\partial}{\partial \eta_{35}} \right) \left(\frac{1}{7!} \varepsilon_{12345678} \frac{\partial^7}{\partial \eta_{41} \ldots \partial \eta_{43} \partial \eta_{45} \ldots \partial \eta_{48}} \right) \\
&\qquad \times \left(\frac{1}{8!} \varepsilon_{12345678} \frac{\partial^8}{\partial \eta_{61} \ldots \partial \eta_{68}} \right) \Omega_4^{\mathrm{SUGRA}} \\
&= \langle F^{4+} F_4^- \, B^+ \, B^- \rangle - \langle F^{5+} F_5^- \, B^+ \, B^- \rangle \, .
\end{aligned} \tag{5.46}$$

Acting on particle 3, the operator changes the derivative with respect to η_{35} into a derivative with respect to η_{34}, thus effectively transforming the positive helicity gravitino F^{5+} into F^{4+}. Correspondingly, by acting on particle 4, again a derivative with respect to η_{45} will be changed into one with respect to η_{44}, this time transforming F_4^- into F_5^-.

Restoring the kinematical weight factors in eq. (5.10), the final comparison of the double-soft scalar limit will be made according to the following formula:

$$\begin{aligned}
&\langle X^{1234}\, X_{1235}\, F^{5+} F_4^- \, B^+ \, B^- \rangle \Big|_{\mathcal{O}(\alpha'^3)} \to \\
&\frac{1}{2} \left[\frac{p_3 \cdot (p_2 - p_1)}{p_3 \cdot (p_1 + p_2)} \langle F^{4+} F_4^- \, B^+ \, B^- \rangle \Big|_{\mathcal{O}(\alpha'^3)} - \frac{p_4 \cdot (p_2 - p_1)}{p_4 \cdot (p_1 + p_2)} \langle F^{5+} F_5^- \, B^+ \, B^- \rangle \Big|_{\mathcal{O}(\alpha'^3)} \right] .
\end{aligned} \tag{5.47}$$

Given the complexity of the higher-order α' corrections in the available amplitudes (see *e.g.* eq. (5.20) at only $\mathcal{O}(\alpha'^2)$), the analytical computation of the left-hand side of eq. (5.47) is very cumbersome. Instead, the computation and comparison have been performed numerically for a sufficient number of kinematical points.

For reference, numerical values at one sample double-soft kinematical point, with all outgoing momenta fulfilling $p_i^2 = 0$ and $\sum_{i=1}^{6} p_i^\mu = 0$ are given below:

$$\begin{aligned}
p_1 &= (-0.853702542142,\ +0.696134406758,\ -0.306157335124,\ +0.387907984368) \times 10^{-4},\\
p_2 &= (+0.711159367201,\ -0.099704627834,\ -0.295472686856,\ +0.639142021830) \times 10^{-4},\\
p_3 &= (+0.818866370407,\ +0.408234512914,\ -0.661447772542,\ -0.257630664418),\\
p_4 &= (-1.098195656456,\ -0.551965696904,\ -0.598319787466,\ +0.737143813124),\\
p_5 &= (-0.618073260483,\ +0.143671541012,\ +0.362410922160,\ -0.479615853707),\\
p_6 &= (+0.897416800850,\ +0.000000000000,\ +0.897416800850,\ +0.000000000000).
\end{aligned} \tag{5.48}$$

At this kinematical point, with a particular external-state phase convention, the left- and right-hand sides of the supergravity ($\mathcal{O}(\alpha'^0)$) version of the formula (5.47) are given respectively by

$$-0.30572232 \,-\, i\,0.89270274 \;\approx\; -0.30615989 \,-\, i\,0.89271337\,, \tag{5.49}$$

while the desired $\mathcal{O}(\alpha'^3)$ terms in eq. (5.47) are,

$$3.08397954 \,+\, i\,9.00278816 \;\approx\; 3.08775134 \,+\, i\,9.00339016\,. \tag{5.50}$$

The difference between the left- and right-hand sides is due merely to the finite separation of the point (5.48) from the double-soft limit. It can be made as small as desired by working closer to the limit, using higher precision kinematics to avoid roundoff error.

The result is surprising: for any double-soft kinematical configuration considered, the left- and the right-hand side of eq. (5.47) show complete agreement within numerical errors.

Given the available amplitudes from the two shaded regions in figure 4, one can perform further tests for other $\mathcal{N}$=8 amplitudes. In addition to eq. (5.47), the double-soft scalar limit has been tested for the following amplitudes

$$\begin{aligned}
&\langle X^{1234}\, X_{1235}\, F^{5+}\, F_4^-\, F^{4+}\, F_4^-\rangle\Big|_{\mathcal{O}(\alpha'^3)} \to\\
&\frac{1}{2}\Bigg[+\frac{p_3\cdot(p_2-p_1)}{p_3\cdot(p_1+p_2)}\langle F^{4+}F_4^-\,F^{4+}F_4^-\rangle\Big|_{\mathcal{O}(\alpha'^3)}\\
&\quad-\frac{p_4\cdot(p_2-p_1)}{p_4\cdot(p_1+p_2)}\langle F^{5+}F_5^-\,F^{4+}F_4^-\rangle\Big|_{\mathcal{O}(\alpha'^3)}\\
&\quad-\frac{p_6\cdot(p_2-p_1)}{p_6\cdot(p_1+p_2)}\langle F^{5+}F_4^-\,F^{4+}F_5^-\rangle\Big|_{\mathcal{O}(\alpha'^3)}\Bigg]
\end{aligned} \tag{5.51}$$

and

$$\begin{aligned}
&\langle X^{1234}\, X_{1235}\, X^{1235}\, X_{1235}\, X^{1235}\, X_{1234}\rangle\Big|_{\mathcal{O}(\alpha'^3)} \to\\
&\frac{1}{2}\Bigg[+\frac{p_3\cdot(p_2-p_1)}{p_3\cdot(p_1+p_2)}\langle X^{1234}\, X_{1235}\, X^{1235}\, X_{1234}\rangle\Big|_{\mathcal{O}(\alpha'^3)}\\
&\quad+\frac{p_5\cdot(p_2-p_1)}{p_5\cdot(p_1+p_2)}\langle X^{1235}\, X_{1235}\, X^{1234}\, X_{1234}\rangle\Big|_{\mathcal{O}(\alpha'^3)}\\
&\quad-\frac{p_6\cdot(p_2-p_1)}{p_6\cdot(p_1+p_2)}\langle X^{1235}\, X_{1235}\, X^{1235}\, X_{1235}\rangle\Big|_{\mathcal{O}(\alpha'^3)}\Bigg]\,.
\end{aligned} \tag{5.52}$$

Each limit shows complete agreement for any double-soft kinematical point tested.

5.7 Discussion

The computation shows that the double-soft limit of three distinct six-point $\mathcal{O}(\alpha'^3)$-corrected $\mathcal{N}=8$ matrix elements yields the corresponding weighted sum of four-point amplitudes, precisely as dictated by $E_{7(7)}$ invariance [19]. However, this is quite puzzling, given the nonvanishing single-soft limits of the same six-point amplitudes. The most likely possibility seems to be that the double-soft limits will begin to fail, but only beginning with the NMHV seven-point amplitudes. It would be very interesting to test this limit, but that is beyond the scope of this thesis.

Whether the three-loop cancellations [117, 118] can be explained by a simple symmetry argument that originates in the $\frac{E_{7(7)}}{SU(8)}$ coset symmetry of $\mathcal{N}=8$ supergravity remains still open. The results from the preceding subsections suggests that the $\mathcal{R}^4$ term produced by tree-level string theory can be ruled out in this way, but other dependences on scalars should be considered. The work of Green and Sethi [106] in ten dimensions indicates that supersymmetry may forbid any $\mathcal{R}^4$ term, but an argument using supersymmetry directly in four dimensions would be very welcome.

Of course, there are higher-dimension potential counterterms than $\mathcal{R}^4$, which are relevant beginning at five loops. It is possible that $E_{7(7)}$ and/or supersymmetry can be used to exclude these counterterms as well, up to a certain dimension or loop order. However, at eight loops a counterterm exists that is invariant under both supersymmetry and $E_{7(7)}$ [101, 129]. It is still possible that $E_{7(7)}$ plays a more subtle role in the excellent ultraviolet behavior of the theory, perhaps by relating somehow the coefficients of certain loop integrals making up the full multi-loop amplitude.

Completely understanding the role of $E_{7(7)}$ will very likely be part of a fundamental explanation of the conjectured finiteness of $\mathcal{N}=8$ supergravity. However, whether supersymmetry and the coset symmetry alone are sufficient ingredients remains to be shown.

6 Mapping IR equations and dual conformal constraints to generalized residue theorems

Leading singularities (see subsection 3.3.1) in $\mathcal{N}$=4 SYM theory can be related to each other by infrared (IR) equations [36, 34, 35] for all loop orders. As discussed in subsection 2.6.4, at one-loop level these equations are simple linear relations among various box coefficients (or one-loop leading singularities) and the tree amplitude[39]. One-loop IR equations in turn are part of the constraints implied by the anomalous dual conformal symmetry [38, 39] as elucidated in subsection 2.6.5.

In [87], evidence has been put forward that the one-loop IR equations of $\mathcal{N}$=4 SYM can be traced back to general residue theorems (GRT) (cf. subsection 3.3.6) in the Grassmannian description of $\mathcal{N}$=4 SYM. As explained in subsection 3.3, in this formalism leading singularities are expressed as residues of the multi-dimensional complex contour integral

$$\mathcal{L}_{n;k}(Z_a) = \frac{1}{\mathrm{vol}(GL(k))} \int \frac{d^{k\times n} C_{\alpha a}}{(12\cdots k)\,(23\cdots(k+1)\,)\;\cdots(n1\cdots(k-1)\,)} \prod_{\alpha=1}^{k} \delta^{4|4}(C_{\alpha a} Z_a)\,. \tag{6.1}$$

Starting from known IR equations and expressing the leading singularities in terms of residues of eq. (6.1), it was shown for a couple of examples that those equations indeed can be traced back to GRTs. Despite these promising results, a general map between one-loop IR equations and GRTs has been missing so far. This map and its extension to the full set of one-loop dual conformal constraints is proposed below.

Although it is very likely that the map between one-loop dual conformal constraints and GRTs holds beyond the NMHV sector, the considerations in this section will be limited to NMHV amplitudes. In this situation, any integration contour for the evaluation of the Grassmannian integral is in one-to-one correspondence with a certain choice of denominator factors in eq. (6.1) to be set to zero. Starting from the N^{2}MHV level, a choice of vanishing minors does not determine a residue uniquely. Thus this identification can not be made straightforwardly any more. In addition, in the NMHV situation a definite map between one-loop leading singularities and residues is known, while beyond NMHV a complete identification has not yet been achieved.

One should note that only a subset of all available GRTs is used to derive all one-loop dual conformal constraints. Since it is now clear [149, 150] that residues in Grassmannian formulation correspond to all-loop leading singularities, some GRTs should have interpretations as relations involving higher-loop leading singularities. In order to map all one-loop constraints onto leading singularities, one needs to decide which residue occurs as a contribution to a leading singularity at a certain loop level *first*. Here the *invariant label* introduced below will serve as a criterion, while a similar classification has been performed by considering the twistor support of leading singularities in [150]. In the same reference it was shown that NMHV leading singularities can maximally occur at three-loop level.

[39] The tree amplitude can be considered as a special version of a leading singularity.

By systematically translating all box coefficients appearing in one-loop dual conformal constraints into residues in the Grassmannian formulation, it is indeed possible to find a general map between all one-loop dual conformal constraints and combinations of GRTs in the NMHV sector. In the same manner one can work out a general formula delivering the GRTs corresponding to the one-loop IR equations. Finally, the mapping is rounded off by identifying a mechanism which provides the highly nontrivial identities relating the BCFW and the P(BCFW) representations of the NMHV tree amplitude.

It is difficult to generalize the result to higher-loop level. The absence of a general notion for an integral basis, already seen at two loops, results in the lack of a clear identification of dual conformal constraints and IR equations beyond one-loop. While there are definitely additional relations originating in GRTs which correspond to these higher-loop constraints, the identification and exploration of those structures is left for future considerations.

6.1 Classification of residues

As discussed in subsection 3.3.3 above, any residue of an n-point NMHV amplitude is labeled by $n-5$ numbers determining the minors to be set to zero in eq. (3.105).

Employing the identifications eqs. (3.118) and (3.119), one can single out all residues which appear in the one-loop leading singularities. This will also include the constituents of the tree amplitude, as those are related to the one-loop leading singularities by eq. (2.107). For $n \leq 7$ this covers all possible residues.

Starting from $n = 8$, there are residues which only contribute to at least two-loop leading singularities, but do not participate in any one-loop leading singularities [87]. However, residues occurring already in one-loop singularities can contribute to two-loop (and higher) leading singularities.

As argued in [150], certain residues appear at three-loop level only for amplitudes with $n \geq 10$. This is also the maximal loop-level for NMHV leading singularities: as shown in the same reference, there are no leading singularities at four-loops and higher.

Assuming the Grassmannian conjecture to be true, it is clear that residues for any NMHV amplitude should be classified by whether they appear at the one-loop, two-loop or three-loop level *first*. Although they can contribute to higher-loop leading singularities, we will refer to those residues in a slightly inaccurate manner as one-loop, two-loop and three-loop residues respectively.

As will be proven below, residues can be classified by using invariant labels. Given any sequence of cyclically ordered numbers $(i_1, \ldots, i_p)$ where $i_l \in \{1, \ldots, n\}$ and $p \leq n$, the invariant label is defined as the set

$$\{(i_2 - i_1) \bmod n,\, (i_2 - i_1) \bmod n,\, \ldots, (i_1 - i_p) \bmod n\}\,. \tag{6.2}$$

As is obvious from the above definition, the name invariant label refers to its invariance under cyclic shifts of $(i_1, \ldots, i_p)$. Since any residue is a sequence of $n-5$ numbers from $1, \ldots, n$, one can determine an invariant label for each of them.

Starting from the fact that any invariant label for a residue is a decomposition of n into $n-5$ numbers, it is easy to prove that, given a sufficiently large n, there are exactly seven distinct invariant labels, which are listed in table 5.

type	invariant label
1	$\{1, ..., 1, 6\}$
2	$\{1, ..., 1, 2, 5\}$
3	$\{1, ..., 1, 3, 4\}$
4	$\{1, ..., 1, 2, 3, 3\}$
5	$\{1, ..., 1, 2, 2, 4\}$
6	$\{1, ..., 1, 2, 2, 2, 3\}$
7	$\{1, ..., 1, 2, 2, 2, 2, 2\}$.

Table 5: *Types of invariant labels for the NMHV sector*

In addition to the classification of the loop level, the invariant label contains further information: certain types of box coefficients correspond to particular types of one-loop residues, as will be discussed below.

The mapping of box coefficients to residues is given in terms of the complementary labeling defined in eq. (3.116) in subsection 3.3.5. Therefore in a first step classes of complementary labels corresponding to types of residues in table 5 above are identified. As expected from the definition of the invariant label, the criterion for classification is the number of successive subsequences. Again there are exactly seven types, corresponding to seven possible invariant labels in table 5:

Comparing the above table with the results for one-loop leading singularities eqs. (3.118) and (3.119), it is straightforward to see that types 1 to 4 correspond to one-loop residues. The invariant labels of types 1 to 4 contain exactly one even number.

Alternatively, one could have considered the BCFW and P(BCFW) forms of NMHV tree amplitudes given by eqs. (3.113) and (3.114). Using the fact that their residues are odd/even alternating sequences, it follows immediately that their invariant labels can contain one even number only. Since those representations of the tree amplitudes are built from one-loop residues, one again arrives at the conclusion that they belong to types 1 to 4.

Having identified all one-loop residues, the remaining types 5, 6 and 7 must correspond to higher-loop residues. In order to find the classification, the results from references [87, 150] stated earlier in this subsection are helpful: the fact that all leading singularities for $n \leq 7$ are combinations of one-loop residues nicely agrees with the lack of decompositions of types 5, 6 or 7 at $n \leq 7$ as shown in table 7.

In addition, since types 5 and 6 can appear for $n = 8, 9$ but the last type only appears for $n \geq 10$, one can fit this fact to the results of [150]: types 5 and 6 seem to correspond to two-loop residues while three-loop residues can be assigned to type 7.

It is remarkable to see that no further types appear as the number of particles increases, which agrees with the claim that all NMHV leading singularities are combinations of these

type	complementary sequence	length of succ. subseq.
1	$\overline{\{i,i+1,i+2,i+3,i+4\}}$	5
2	$\overline{\{i,i+1,i+2,i+3,j_{>i+4}\}}$ $\overline{\{j,i_{>j+1},i+1,i+2,i+3\}}$	4
3	$\overline{\{i,i+1,i+2,j_{>i+3},j+1\}}$ $\overline{\{j,j+1,i_{>j+2},i+1,i+2\}}$	3 and 2
4	$\overline{\{i,i+1,j_{>i+2},j+1,k_{>j+2}\}}$ $\overline{\{k,i_{>k+1},i+1,j_{>i+2},j+1\}}$ $\overline{\{i,i+1,k_{>i+2},j_{>k+1},j+1\}}$	2 and 2
5	$\overline{\{i,i+1,i+2,j_{>i+3},k_{>j+1}\}}$ $\overline{\{j,k_{>j+1},i_{>k+1},i+1,i+2\}}$ $\overline{\{j,i_{>j+1},i+1,i+2,k_{>i+3}\}}$	3
6	$\overline{\{i,i+1,j_{>i+2},k_{>j+1},l_{>k+1}\}}$ $\overline{\{j,k_{>j+1},l_{>k+1},i_{>l+1},i+1\}}$	2
7	$\overline{\{i,j_{>i+1},k_{>j+1},l_{>k+1},m_{>l+1}\}}$	0

Table 6: *Classes of complementary labels*

three types of residues [150].

Examining the first four types more carefully, one can find a subclassification of one-loop residues. By comparing them with results in subsection 3.3.5, one finds that 3-mass leading singularities can receive contributions from type 2 and 4 residues while residues for 2-mass hard leading singularities are of type 1 and 3. For 2-mass easy coefficients, the corresponding residues are of type 2, 3 and 4, and 1-mass leading singularities can have all four possible cases for one-loop residues. In summary, a complete classification of residues based on invariant labels is presented in table 8.

n	invariant label
6	$\{6\}$
7	$\{1,6\}$,$\{2,5\}$,$\{3,4\}$
8	$\{1,1,6\}$,$\{1,2,5\}$,$\{1,3,4\}$,$\{2,3,3\}$,$\{2,2,4\}$
9	$\{1,1,1,6\}$,$\{1,1,2,5\}$,$\{1,1,3,4\}$,$\{1,2,2,3\}$,$\{1,2,2,4\}$,$\{2,2,2,3\}$
10	$\{1,1,1,1,6\}$,$\{1,1,1,2,5\}$,$\{1,1,1,3,4\}$,$\{1,1,2,2,3\}$,$\{1,1,2,2,4\}$,$\{1,2,2,2,3\}$,$\{2,2,2,2,2\}$
$\vdots$	$\vdots$
n	$\{1,...,1,6\}$,$\{1,...,1,2,5\}$,$\{1,...,1,3,4\}$,$\{1,...,1,2,3,3\}$, $\{1,...,1,2,2,4\}$,$\{1,...,1,2,2,2,3\}$,$\{1,...,1,2,2,2,2,2\}$.

Table 7: *Invariant label of partitions of* $\{1,\ldots,n\}$ *into* $n-5$ *parts*

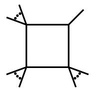
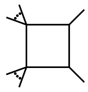
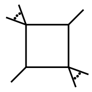
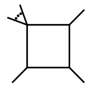
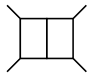
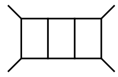
	3m	2mh	2me	1m	2-loop	3-loop
						 .
types	2 or 4	(1 + 3) or (3 + 3)	(1 + 2 +3) or (1 + 1)	1 to 4	5 and 6	7

Table 8: *Classification of residues*

6.2 Mapping generalized residue theorems to dual conformal constraints and IR equations

After having classified NMHV residues in the previous subsection, here all one-loop dual conformal constraints defined in subsection 2.6.5 are expressed in terms of residues. In the following, the source terms of the GRTs necessary to show (cf. eq. (2.113))

$$\mathcal{E}_{i,k} = 0 \tag{6.3}$$

for $i = 1, ..., n$ and $k = i+2, ..., i+n-3$ shall be investigated. Starting with $\mathcal{E}(1,4) = 0$ for an amplitude with $n = 9$ legs as an example, the corresponding expression in terms of residues reads

$$\begin{aligned}(&\{1,2,4,9\} + \{1,2,6,9\} + \{1,2,8,9\} + \{1,3,4,9\} \\ &+ \{1,3,6,9\} + \{1,3,8,9\} - \{1,4,5,9\} - \{1,4,7,9\} \\ &+\{1,5,6,9\} + \{1,5,8,9\} - \{1,6,7,9\} + \{1,7,8,9\}) = 0\,.\end{aligned} \tag{6.4}$$

The above equality can be obtained by adding GRTs with the following source terms:

$$-(1,4,9) - (1,6,8) - (1,8,9) = 0\,. \tag{6.5}$$

Here and below a global minus sign in the equation will not be noted for vanishing results.

For finding a general rule describing which GRTs have to be added, it is sufficient to restrict the attention to the case $i = 1$, because all other conformal constraints can be obtained by cyclical shifts. In table 9, a couple of lower-point examples are listed. Based on these examples and further tests, the general rule for obtaining the source terms in the fourth column for a certain m and n legs can be conjectured as

$$0 = \sum_{\mathcal{V}} (1, 2, .., (m-2), \underbrace{v_{(1)}, \ldots, v_{(n-m-5)}}_{\mathcal{V}}, n)\,, \tag{6.6}$$

where $\mathcal{V}$ is a strictly increasing succession of $n - m - 5$ numbers

$$v \in \{m+1, \ldots, n-1\}\,, \tag{6.7}$$

which has to be chosen such that the whole source term is strictly alternating between odd and even. The summation is over all possible $\mathcal{V}$s.

part.	$\mathcal{E}(i,k)$	$m{=}k{-}i$	source terms
7	$\mathcal{E}(1,3)$	2	$0=(7)$
8	$\mathcal{E}(1,3)$	2	$0=(3,8)+(5,8)+(7,8)$
	$\mathcal{E}(1,4)$	3	$0=(1,8)$
	$\mathcal{E}(1,5)$	4	trivial vanishing
	$\mathcal{E}(1,6)$	5	trivial vanishing
9	$\mathcal{E}(1,3)$	2	$0=(3,4,9)+(3,6,9)+(3,8,9)+(5,6,9)+(5,8,9)+(7,8,9)$
	$\mathcal{E}(1,4)$	3	$0=(1,4,9)+(1,6,9)+(1,8,9)$
	$\mathcal{E}(1,5)$	4	$0=(1,2,9)$
	$\mathcal{E}(1,6)$	5	trivial vanishing
	$\mathcal{E}(1,7)$	6	trivial vanishing
10	$\mathcal{E}(1,3)$	2	$0=(3,4,5,10)+(3,4,7,10)+(3,4,9,10)+(3,6,7,10)+(3,6,9,10)$ $+(3,8,9,10)+(5,6,7,10)+(5,6,9,10)+(5,8,9,10)+(7,8,9,10)$
	$\mathcal{E}(1,4)$	3	$0=(1,4,5,10)+(1,4,7,10)+(1,4,9,10)$ $+(1,6,7,10)+(1,6,9,10)+(1,7,9,10)$
	$\mathcal{E}(1,5)$	4	$0=(1,2,5,10)+(1,2,7,10)+(1,2,9,10)$
	$\mathcal{E}(1,6)$	5	$0=(1,2,3,10)$
	$\mathcal{E}(1,7)$	6	trivial vanishing
	$\mathcal{E}(1,8)$	7	trivial vanishing
⋮	⋮	⋮	⋮

Table 9: *Source terms for GRTs related to the vanishing of dual conformal constraints*

For example, in order to obtain the GRTs for the vanishing of $\mathcal{E}(1,5)$ in a scenario with $n=11$, one would start with $(1,2,\mathcal{V},11)$. According to eq. (6.7) the numbers $v\in\mathcal{V}$ have to be in the range $\{5,\ldots,10\}$. Thus all valid choices for $\mathcal{V}$ in this scenario are

$$(5,6),\ (5,8),\ (5,10),\ (7,8),\ (7,10),\ (9,10)\,, \tag{6.8}$$

which finally leads to

$$\begin{aligned}\mathcal{E}(1,5)&=(1,2,5,6,11)+(1,2,5,8,11)+(1,2,5,10,11)\\&\quad+(1,2,7,8,11)+(1,2,7,10,11)+(1,2,9,10,11)=0\,.\end{aligned} \tag{6.9}$$

Considering the other type of dual conformal constraint (cf. eq. (2.114)),

$$\mathcal{E}_{i,i-2}=-\mathcal{E}_{i-1,i}=-2A_n^{\text{tree}}\,, \tag{6.10}$$

one first needs to pick a form of the tree amplitude. For the investigation here it will be useful to choose

$$2A^{\text{tree}}=A^{\text{tree}}_{\text{BCFW}}+A^{\text{tree}}_{\text{P(BCFW)}} \tag{6.11}$$

where the two representations of the tree-level amplitudes have been defined in eqs. (3.113) and (3.114). The BCFW and the P(BCFW) form of the tree amplitude are cyclically invariant, but in order to show this, it is necessary to employ GRTs. For example, the equality of the seven-point BCFW form of the tree amplitude to its shifted version,

$$\begin{aligned}&\{2,3\}+\{2,5\}+\{2,7\}+\{4,5\}+\{4,7\}+\{6,7\}\\&=\{3,4\}+\{3,6\}+\{3,1\}+\{5,6\}+\{5,1\}+\{7,1\}\,,\end{aligned}\tag{6.12}$$

is not obvious. So one expects that only for one particular choice of i in eq. (6.10) is the expression eq. (6.11) obtained. Translating again box coefficients into residues confirms this expectation. Only for $i = 2$ is the chosen form of the tree amplitude reproduced,

$$\mathcal{E}(2,n) = -\mathcal{E}(1,2) = -(A^{\text{tree}}_{\text{BCFW}} + A^{\text{tree}}_{\text{P(BCFW)}})\,.\tag{6.13}$$

Having mapped all dual conformal constraints to sums of GRTs, it is interesting to make contact to the IR equations. As discussed in subsection 2.6.5, all IR equations are related to conformal equations via eqs. (2.116) and (2.117). It would be straightforward to just add and subtract the appropriate source terms corresponding to the different terms $\mathcal{E}(i,k)$. However, because of cancellations between different GRTs this does not result in the simplest possible expression. Therefore, the above analysis performed for the dual conformal constraints will be repeated for the IR equations below[40].

As an illustration for IR equations (see subsection 2.6.4 for an introduction)

$$A_n^{\text{1-loop}}\Big|_{IR} = -\frac{1}{\epsilon^2}\sum_{i=1}^{n}(-[\![i]\!]_2)^{-\epsilon}A_n^{\text{tree}},\tag{6.14}$$

the kinematic invariant $[\![1]\!]_2$ for the 7-point NMHV scenario is considered. By either scanning for the corresponding IR divergences or employing eq. (2.116) one obtains

$$C^{1\text{m}}_{1234} + C^{1\text{m}}_{7123} + \tfrac{1}{2}C^{2\text{mh}}_{1235} - \tfrac{1}{2}C^{2\text{mh}}_{6713} + \tfrac{1}{2}C^{2\text{mh}}_{1236} - \tfrac{1}{2}C^{2\text{mh}}_{3451} - C^{2\text{me}}_{7134} - \tfrac{1}{2}C^{3\text{m}}_{3461} - \tfrac{1}{2}C^{3\text{m}}_{7135}\tag{6.15}$$

for the left hand side of eq. (6.14). Translating into residues yields

$$\begin{aligned}&(\{7,1\}+\{5,1\}+\{3,1\}+\{4,5\})+(\{6,7\}+\{4,7\}+\{2,7\}+\{3,4\})\\&+\tfrac{1}{2}(\{2,5\}+\{5,6\})-\tfrac{1}{2}(\{7,3\}+\{3,4\})+\tfrac{1}{2}(\{2,3\}+\{3,6\})-\tfrac{1}{2}(\{4,5\}+\{5,1\})\\&-(\{7,1\})-\tfrac{1}{2}(\{3,1\})-\tfrac{1}{2}(\{7,5\})\end{aligned}\tag{6.16}$$

which, by use of GRTs, equals

$$\{2,3\}+\{2,5\}+\{2,7\}+\{4,5\}+\{4,7\}+\{6,7\}\,.\tag{6.17}$$

This is the expected BCFW form of the 7-point NMHV tree amplitude.

As a second example, consider the invariant $[\![1]\!]_4$ for 9 particles. The corresponding IR equation reads

$$\begin{aligned}&-\tfrac{1}{2}C^{2\text{mh}}_{1235} - \tfrac{1}{2}C^{2\text{mh}}_{5671} + C^{2\text{mh}}_{9125} + C^{2\text{mh}}_{4561} - \tfrac{1}{2}C^{2\text{mh}}_{8915} - \tfrac{1}{2}C^{2\text{mh}}_{3451} - C^{2\text{me}}_{1245} - \tfrac{1}{2}C^{3\text{m}}_{5681} + \tfrac{1}{2}C^{3\text{m}}_{9135}\\&+\tfrac{1}{2}C^{3\text{m}}_{4571} + C^{3\text{m}}_{1256} - C^{3\text{m}}_{5691} + C^{3\text{m}}_{9145} + \tfrac{1}{2}C^{3\text{m}}_{1257} + \tfrac{1}{2}C^{3\text{m}}_{4581} + \tfrac{1}{2}C^{3\text{m}}_{1258} + \tfrac{1}{2}C^{3\text{m}}_{5613} - \tfrac{1}{2}C^{3\text{m}}_{9157} = 0\,.\end{aligned}\tag{6.18}$$

[40]Note that the considerations are again limited to the starting point $i = 1$, because one can always obtain results for other i's by cyclic shifts.

Again translating into residues and sorting out the prefactors (but not using any GRT) leads to the following result:

$$\begin{aligned}\tfrac{1}{2}\big(&-\{1,2,3,5\}-\{1,2,3,7\}-\{1,2,4,5\}-\{1,2,4,7\}\\&+\{1,2,5,6\}+\{1,2,5,8\}+\{1,2,5,9\}-\{1,2,6,7\}\\&+\{1,2,7,8\}+\{1,2,7,9\}-\{1,5,6,7\}-\{2,5,6,7\}\\&-\{3,5,6,7\}-\{4,5,6,7\}+\{5,6,7,8\}+\{5,6,7,9\}\big)=0\,.\end{aligned}\tag{6.19}$$

It is not difficult to see that this equation arises by adding three GRTs with the following source terms,

$$-(1,2,5)-(1,2,7)-(5,6,7)=0\,,\tag{6.20}$$

where again the global minus sign will be neglected in the examples following below.

After having shown the correspondence in the above examples, the general analysis of GRTs for IR equations shall be performed. For $m=2$, the linear combination of residues should coincide with the expression for the tree amplitude. From eqns. (6.6), (6.13) and (2.116), it is straightforward to see which GRTs lead to the parity-invariant form of the tree amplitude eq. (6.11). With other choices of GRTs, one can also generate BCFW, P(BCFW) (eqs. (3.113) and (3.114)) or numerous other forms of the tree amplitude in terms of residues.

In case of $m>2$, there is no infrared divergence on the right hand side of eq. (6.14). Therefore, the sum is expected to vanish by use of certain combinations of GRTs. Several examples are listed in table 10. Inspecting the table 10, one initial observation can be made: any one-loop IR equation for an n-point amplitude can be represented as the sum of $n-6$ basic GRTs, each of which is a sum of 6 residues. However, some residues will cancel, thus the final number of terms is smaller than $6\cdot(n-6)$.

Furthermore, based on the examples in table 10 and further tests up to $n=20$, the general rule for $m=3$ IR equations is found to be

$$[\![1]\!]_3\ :\sum_{\mathcal{V}}(1,\underbrace{v_1,\ldots,v_{n-7}}_{\mathcal{V}})=0\,,\tag{6.21}$$

where $\mathcal{V}$ is a strictly increasing succession of $n-7$ numbers,

$$v_i\in\{4,\ldots,n-2\}\tag{6.22}$$

and again only strictly odd/even alternating source terms are allowed.

For $m>3$, another kind of source terms appears. While the type already encountered for $m=3$ starts from the left with $(1,\ldots)$, a second type has the form $(\ldots,n-2)$ starting from the right. Again by testing examples up to $n=20$, for $4\le m\le\lfloor n/2\rfloor$, one finds the following general rule,

$$\begin{aligned}[\![1]\!]_m\ :&\sum_{\mathcal{V}}(1,2,\ldots,(m-2)\,,\underbrace{v_1,\ldots,v_{n-m-4}}_{\mathcal{V}})\\&+\sum_{\mathcal{W}}(\underbrace{w_1,\ldots,w_{m-4}}_{\mathcal{W}}\,,(m+1),\ldots,(n-2))=0\,,\end{aligned}\tag{6.23}$$

particles	kin. inv	source terms
7	$[\![1]\!]_3$	$0 = (1)$
8	$[\![1]\!]_3$	$0 = (1,4) + (1,6)$
	$[\![1]\!]_4$	$0 = (1,2) + (5,6)$
9	$[\![1]\!]_3$	$0 = (1,4,5) + (1,4,7) + (1,6,7)$
	$[\![1]\!]_4$	$0 = (1,2,5) + (1,2,7) + (5,6,7)$
10	$[\![1]\!]_3$	$0 = (1,4,5,6) + (1,4,5,8) + (1,4,7,8) + (1,6,7,8)$
	$[\![1]\!]_4$	$0 = (1,2,5,6) + (1,2,5,8) + (1,2,7,8) + (5,6,7,8)$
	$[\![1]\!]_5$	$0 = (1,2,3,6) + (1,2,3,8) + (1,6,7,8) + (3,6,7,8)$
$\vdots$	$\vdots$	$\vdots$
12	$[\![1]\!]_3$	$0 = (1,4,5,6,7,8) + (1,4,5,6,7,10) + (1,4,5,6,9,10)$ $+(1,4,5,8,9,10) + (1,4,7,8,9,10) + (1,6,7,8,9,10)$
	$[\![1]\!]_4$	$0 = (1,2,5,6,7,8) + (1,2,5,6,7,10) + (1,2,5,6,9,10)$ $+(1,2,5,8,9,10) + (1,2,7,8,9,10) + (5,6,7,8,9,10)$
	$[\![1]\!]_5$	$0 = (1,2,3,6,7,8) + (1,2,3,6,7,10) + (1,2,3,6,9,10)$ $+(1,2,3,8,9,10) + (1,6,7,8,9,10) + (3,6,7,8,9,10)$
	$[\![1]\!]_6$	$0 = (1,2,3,4,7,8) + (1,2,3,4,7,10) + (1,2,3,4,9,10)$ $+(1,2,7,8,9,10) + (1,4,7,8,9,10) + (3,4,7,8,9,10)$

Table 10: *Source terms for GRTs related to the vanishing of the IR equation obtained from a particular kinematic invariant*

where $\mathcal{V}$ and $\mathcal{W}$ are again strictly increasing successions of numbers satisfying

$$m + 1 \leq v_i \leq n - 2 \qquad \text{and} \qquad 1 \leq w_i \leq m - 2 \tag{6.24}$$

and chosen to respect the odd/even alternating structure of source terms. The first term in eq. (6.23) is a generalization of eq. (6.21).

The labels of source terms in the general formula eq. (6.23) are strictly odd/even alternating, which nicely connects to the classification of residues: starting from a strictly alternating source term, the residues in the resulting GRT will be either strictly alternating themselves or have an "all-but-one" odd/even alternating structure such as *eoeoo*. Translating this into the language of invariant labels and comparing with table 5 singles out exactly types 1 to 4, as expected.

One of the most important implications of one-loop IR equations, which is completely obscured in the BCFW formalism, are the so-called "remarkable identities", which relate the BCFW representation of the tree amplitude eq. (3.113) with the P(BCFW) form eq. (3.114)

$$A_{\text{BCFW}} = A_{\text{P(BCFW)}}\,. \tag{6.25}$$

Being highly nontrivial identities in the BCFW approach, they have an astonishingly simple form in the language of residues.

While it was already shown in [87] that identities (cf. eq. (3.115))

$$E \star O \star E \star \cdots = (-1)^{(n-5)} O \star E \star O \star \cdots \tag{6.26}$$

are implied by GRTs, one can as well derive them directly from combinations of GRTs.

The statement is simple: adding all GRTs corresponding to all source terms of the form *oeoe*... for a particular number of legs produces the remarkable identity

$$A_{\text{BCFW}} = A_{\text{P(BCFW)}} \;:\; \underbrace{O \star E \star O \cdots}_{n-6 \text{ factors}} = 0, \tag{6.27}$$

where O, E and the star product have been defined in eq. (3.111) and eq. (3.112) respectively.

For example, adding GRTs with source terms

$$(1,2),\ (1,4),\ (1,6),\ (1,8),\ (3,4),\ (3,6),\ (3,8),\ (5,6),\ (5,8) \text{ and } (7,8)$$

produces the identity for $n = 8$:

$$\begin{aligned} &\{2,3,4\} + \{2,3,6\} + \{2,3,8\} + \{2,5,6\} + \{2,5,8\} \\ &\quad + \{2,7,8\} + \{4,5,6\} + \{4,5,8\} + \{4,7,8\} + \{6,7,8\} \\ = &-(\{1,2,3\} + \{1,2,5\} + \{1,2,7\} + \{1,4,5\} + \{1,4,7\} \\ &\quad + \{1,6,7\} + \{3,4,5\} + \{3,4,7\} + \{3,6,7\} + \{5,6,7\}) \end{aligned} \tag{6.28}$$

As expected by parity invariance, one obtains the same result by adding GRTs corresponding to all source terms of the form *eoeo*... .

6.3 Discussion

In this section it was investigated which global residue theorems in the Grassmannian formulation of $\mathcal{N}$=4 SYM theory imply the recently derived one-loop dual conformal constraints and the well-known one-loop IR equations in the NMHV sector. For both sets of equations the source terms for the corresponding GRTs can be obtained from the general rules eqs. (6.6) and (6.23). In addition, the remarkable identities relating the BCFW and the P(BCFW) form of the tree amplitude emerge from adding all GRTs with an odd/even alternating pattern of source terms eq. (6.27).

According to the classification of NMHV residues performed in subsection 6.1, all one-loop residues are of odd/even alternating or "all-but-one" odd/even alternating structure. This nicely fits to the general rules: all GRTs involved have source terms of strictly odd/even alternating structure, which relate exactly these types of residues.

Since there exist further GRTs beyond the ones employed in the mappings, these are presumably related to higher-loop dual conformal constraints or IR equations in the NMHV sector. From the classification it is obvious, which residues contribute to higher-loop leading singularities. However, without a general formalism to single out an integral basis for two loops and beyond, the identification of higher-loop dual conformal constraints or IR equations can not be performed.

Furthermore, although one-loop dual conformal constraints (and thus IR equations) should be related to GRTs beyond the NMHV sector, a general map has not been found for these amplitudes so far. In [151], general contours for N^2MHV tree amplitudes have been derived using ideas from localization in Grassmannian manifolds. It would be very interesting to generalize this analysis to the leading singularities of loop amplitudes beyond the NMHV sector. Once this is achieved, it should be possible to identify the GRT origin of dual conformal constraints for N^2MHV and beyond.

7 Conclusions and outlook

Although maximally supersymmetric field theories have been known for more than 30 years, their symmetries and properties are still subject of numerous investigations, discussions and speculations. As discussed in this thesis, many new results have been found for $\mathcal{N}$=4 SYM theory, but comparable achievements have not been obtained to the same extent for $\mathcal{N}$=8 supergravity. In this thesis, different approaches to overcoming this deficiency of $\mathcal{N}$=8 supergravity have been investigated.

In a first part, the suggestion for a twistor description for $\mathcal{N}$=8 supergravity was shown to be inconsistent in section 4. Even more, the results of subsection 4.4 question the existence of a twistor description for $\mathcal{N}$=8 supergravity in general: since twistor space is intimately tied to the concept of chirality, it does not seem to be the correct framework to describe $\mathcal{N}$=8 supergravity in its present form. Although alternative suggestions for a twistor description are known, it seems more likely that gravity is incorporated into twistor space along the lines of the nonlinear graviton approach proposed by Penrose. The concept of a metric deformation in usual spacetime translating into a marginal deformation of the complex structure on twistor space seems to be a natural and beautiful concept. In addition, it is necessary to note that twistor string theory in the formulation of Berkovits and Witten is an incomplete theory. It is in general not possible to limit the attention exclusively to the open sector of a string theory. However, the closed sector of twistor string theory has not been explored so far. If existent at all, a possible twistor description of $\mathcal{N}$=8 supergravity could hide in there, but more likely a completely new geometrical concept will have to replace twistor space.

In a second part, usual (non-twistor) string theory has been employed to investigate possible symmetry reasons for the vanishing of the prefactor of a possible $\mathcal{R}^4$ counterterm in $\mathcal{N}$=8 supergravity in section 5. While the results indeed suggest that the hidden $E_{7(7)}$ symmetry constrains the appearance of the $\mathcal{R}^4$ counterterm, the conditions investigated need further study in order to establish their sufficiency. In particular, since the calculations utilize scalars, which on the one hand side parameterize the coset of $\mathcal{N}$=8 supergravity and on the other hand are related to the dilaton from string theory, it is not clear in which manner this influences the result. While for some compactifications certain constraints on the prefactors in front of possible counterterms are known, this is not the case for the compactification of supergravity to four dimensions. With such a result at hand, it would be possible to draw a reliable conclusion.

Although the third part is concerned with the investigation of $\mathcal{N}$=4 SYM theory exclusively, it will be of use in further work on $\mathcal{N}$=8 supergravity: the IR structure is one of the first testing grounds for a possible prospective dual formulation of $\mathcal{N}$=8 supergravity. In section 6, one-loop dual conformal constraints and IR equations in $\mathcal{N}$=4 SYM theory have been explicitly mapped to generalized residue theorems in the novel Grassmannian formulation of $\mathcal{N}$=4 SYM. While this result was expected, the residue theorems seem to contain a lot more information: not all available generalized residue theorems are used in the mapping. Furthermore, the classification singles out certain residues as contributing to two-loop and three-loop leading singularities. Although this notion has to be sharply defined in the future, the infrared information about those quantities is readily available.

What are the next steps to further investigate the structure of $\mathcal{N}$=8 supergravity? The main reason for an unsatisfactory understanding of the simplicity of the amplitudes, the miraculous cancellations and the finiteness issue in $\mathcal{N}$=8 supergravity is the current local spacetime formulation. Since this description does not account for all symmetries of the theory, an alternative, probably nonlocal, description for $\mathcal{N}$=8 supergravity incorporating all known and possibly new symmetries needs to be found.
Considering the current state of knowledge, there seem to be two possibilities to achieve this:

- A number of steps towards another formulation have already been made: the description in *light-cone superspace* seems to be particularly suitable for the hidden E_7 symmetry [131, 130, 132]. Lightcone coordinates have also proven useful to reveal the structural similarity between the Lagrangian densities of $\mathcal{N}$=8 and $\mathcal{N}$=4 theories [152]. In another recent paper [133], an E_7 manifest formulation of the theory has been found. If the hidden E_7 is the missing symmetry to be incorporated, it will be promising to combine these approaches. However, while those formulations are very powerful for discussing particular aspects such as the finiteness of $\mathcal{N}$=8 supergravity, they tend to obscure the analytic behavior of amplitudes and symmetries which are capable to determine the theory completely.

- Due to the close relationship between $\mathcal{N}$=4 and $\mathcal{N}$=8 one hopes to find a description for $\mathcal{N}$=8 supergravity similar to the Grassmannian formulation for $\mathcal{N}$=4 SYM. With such a description, it would be an amazing result to completely determine the theory by its symmetries and the analytical behavior of its amplitudes, as is suspected for $\mathcal{N}$=4 SYM theory.

  However, in order to find such a formulation, one has to make explicit the symmetries to start with. Given the recent findings in $\mathcal{N}$=4 SYM theory, it is not clear whether the known $E_{7(7)}/SU(8)$ symmetries of $\mathcal{N}$=8 supergravity are exhaustive. Therefore, one will have to investigate the symmetries of $\mathcal{N}$=8 supergravity more thoroughly in a first step. Starting from symmetries in $\mathcal{N}$=4 SYM theory, this amounts to employing the KLT relations in order to investigate the echos of the newly discovered Yangian symmetries in the amplitudes of $\mathcal{N}$=8 supergravity. In a second step, a novel formulation for $\mathcal{N}$=8 supergravity needs to be found. The sought-after description is required to:

  - manifestly incorporate all symmetries of $\mathcal{N}$=8 , with the exception of Lorentz symmetry which, however, is implicit and can be restored,
  - be consistent with the KLT relations,

- incorporate the analytic behavior of the amplitudes, i.e. reproduce the infrared equations, like the Grassmannian formulation does in $\mathcal{N}$=4 SYM.

It would be even more beautiful, if the formulation did not only satisfy the above constraints, but if these requirements *imply* the novel formulation.

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Curriculum vitae

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