Singularities of unitary modular varieties

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Zusammenfassung

In dieser Arbeit werden Singularitäten von Ballquotienten untersucht. Ballquotienten erhält man als Quotienten des n-dimensionalen komplex hyperbolischen Raums $\mathbb{C}H^n$ nach einer speziellen arithmetischen Untergruppe Γ der Gruppe der Automorphismen. Diese sind quasi-projektive Varietäten nach Ergebnissen von W.L. Baily, Jr. und A. Borel. Es wird bewiesen, dass der Ballquotient kanonische Singularitäten besitzt, solange $n \geq 12$. Wir benutzen hierbei Techniken, die auf einer Arbeit von V.A. Gritsenko, K. Hulek und G.K. Sankaran beruhen, in der ähnliche Resultate für orthogonale modulare Varietäten bewiesen werden. Weiterhin konstruieren wir eine toroidale Kompaktifizierung $(\Gamma\backslash\mathbb{C}H^n)^*$ des Ballquotienten, wobei wir benutzen, dass der Ballquotient eine Darstellung als beschränktes symmetrisches Gebiet besitzt. Auch geben wir hierfür ein Resultat über kanonische Singularitäten an.

Für unsere Untersuchungen ist es von entscheidender Bedeutung, dass sich das Studium von Singularitäten lokal auf Quotienten V/G reduziert, wobei G eine endliche Gruppe ist die auf dem Tangentialraum V operiert. Auch muss das der Konstruktion zu Grunde liegende $\mathcal{O}$-Gitter betrachtet werden, wobei $\mathcal{O}$ der Ring der ganzen Zahlen des imaginär-quadratischen Zahlkörpers $\mathbb{Q}(\sqrt{D})$ ist. Eine genaue Untersuchung der auftretenden Darstellungen in Verbindung mit dem Reid-Tai Kriterium liefert eine Schranke für die Dimension n. Die Konstruktion einer toroidalen Kompaktifizierung benutzt Techniken der torischen Geometrie, wobei die Kompaktifizierung durch Arbeiten von A. Ash, D. Mumford, M. Rapoport und Y.-S. Tai gegeben ist. Wir werden eine solche Kompaktifizierung angeben, wobei die auftretenden torischen Varietäten durch Gitter vom Rang 1 gegeben sind. Zudem wird eine Schranke angegeben, so dass diese Kompaktifizierung kanonische Singularitäten besitzt. Hierfür werden Argumente benutzt, die sich vorheriger Resultate im nicht-kompakten Fall bedienen.

Schlagworte: Ballquotient, kanonische Singularitäten, toroidale Kompaktifizierung

Abstract

In this thesis we study the singularities of ball quotients obtained as quotients of the n-dimensional complex hyperbolic space $\mathbb{C}H^n$ by a special arithmetic subgroup Γ of the automorphism group. This is actually a quasi-projective variety by results of W.L. Baily, Jr. and A. Borel. We will prove that the ball quotient has canonical singularities if $n \geq 12$. We rely on techniques based on work of V.A. Gritsenko, K. Hulek and G.K. Sankaran where similar results for orthogonal modular varieties are obtained. Furthermore, we construct a toroidal compactification $(\Gamma \backslash \mathbb{C}H^n)^*$ from the ball quotient, using a representation of the latter as a bounded symmetric domain. We also state a result about canonical singularities for the compactified ball quotient.

For this study, it is of crucial importance that the investigation of singularities can locally be reduced to quotients V/G, with finite group G acting on the tangent space V. We also have to take into account the underlaying $\mathcal{O}$-lattice, where $\mathcal{O}$ is the ring of integers of an imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$. Examing the relevant representations, coupled with the Reid-Tai criterion yields a bound on the dimension n. Constructing a toroidal compactification of the ball quotient $\Gamma \backslash \mathbb{C}H^n$ requires framework provided by toric geometry and results of A. Ash, D. Mumford, M. Rapoport and Y.-S. Tai. We will state this compactification using that the construction for ball quotients only needs toric varieties induced by rank 1 lattices. Furthermore, we prove a bound, such that this compactification has canonical singularities. For this purpose we use arguments that are based on former results.

Keywords: Ball quotient, canonical singularities, toroidal compactification

Contents

Introduction

Modular varieties are much studied objects in algebraic geometry. V.A. Gritsenko, K. Hulek and G.K. Sankaran studied modular varities of orthogonal type (cf. [GHS07]). These modular varieties of orthogonal type appear when one wants to study certain moduli spaces, for example the moduli space of K3 surfaces, or the moduli space of polarised symplectic manifolds. For dimension $n \geq 9$ they prove that the corresponding compactified modular variety has only canonical singularities. This result was used to give a result on the Kodaira dimension of the moduli space of polarised K3 surfaces of degree $2d$. This moduli space is of general type if $d > 61$.

The definition of unitary modular varieties is similar, i.e. the quasi-projective varieties given by unitary groups instead of orthogonal groups. This thesis will concentrate on the special case of ball quotient and will not regard unitary groups in full generality.

Ball quotients also appear as moduli spaces, for example as the moduli space of cubic threefolds which was studied by D. Allcock [All03] or as the moduli space of cubic surfaces also investigated by D. Allcock together with J. Carlson and D. Toledo [ACT02]. In the case of threefolds there are results on some stability problems in terms of geometric invariant theory. For the surfaces they proved that the moduli space of semistable cubic surfaces is in fact biholomorphic to the Satake compactification of a special 4-dimensional ball quotient and investigated some relations between their orbifold structures.

Ball quotients in dimension 2, the ball quotient surfaces $B^2_{\mathbb{C}}/\Gamma$, were studied intensively by R.-P. Holzapfel (see [Hol98, Hol81] for example). He calculated some formulas for the Euler number $e(\overline{B^2_{\mathbb{C}}/\Gamma})$ and the index $\tau(\overline{B^2_{\mathbb{C}}/\Gamma})$ for a smooth model of the Baily-Borel compactification of the surface. Furthermore he studied arithmetic aspects of ball quotient surfaces and their singularities.

Since ball quotients have a representation as a bounded symmetric domain, one can apply toroidal compactification as studied by A. Ash, D. Mumford, M. Rapoport and Y.-S. Tai [AMRT75]. They deliver the tools using constructions from toric geometry. Y.-S. Tai proved that one can construct a projective compactification under some assumptions.

These results will be used in this thesis to construct a toroidal compactification of ball quotients and study the singularities of these projective varieties. M. Reid [Rei87] and Y.-S. Tai [Tai82] studied canonical singularities and proved a criterion to decide when only these singularities occur.

Chapter 1 provides the techniques and definitions that will also be used in the later chapters. After an introduction into quotient singularities, canonical singularities will be introduced. This is based on work of Y.-S. Tai [Tai82] and M. Reid [Rei87]. In particular the Reid-Tai criterion will be stated which can be used to determine canonical singularities in the situation of quotient singularities. The last section leads to the construction of toroidal compactifications where this thesis restricts to the case that the lattice on which the construction of the toric variety is based has rank 1.

In Chapter 2, there will be an investigation of representation theory of a cyclic group μ_r over a quadratic number field $\mathbb{Q}(\sqrt{D})$. Classical results about the decompositions of cyclotomic polynomials over the number field $\mathbb{Q}(\sqrt{D})$ will be used to state all irreducible representations of μ_r. There will be a criterion to determine the eigenvalues of such representations.

The ball quotient will be defined in Chapter 3. Using results of V.A. Gritsenko, K. Hulek and G.K. Sankaran (cf. [GHS07]), we provide assumptions which ensure that $\Gamma\backslash\mathbb{C}H^n$ has canonical singularities. This will be done by first reducing to the action of the stabilizer subgroup of a point $[\omega]$ on the tangent space at $[\omega]$ and making heavy use of the Reid-Tai criterion. For $D = -3$ we list all elements $g \in \Gamma$ that could possibly lead to non-canonical singularities.

We construct a toroidal compactification $(\Gamma\backslash\mathbb{C}H^n)^*$ of the quasi-projective variety $\Gamma\backslash\mathbb{C}H^n$ in Chapter 4. This will be done for the 0-dimensional boundary components as described in the book [AMRT75]. Finally, we prove a theorem about canonical singularities of the compactification $(\Gamma\backslash\mathbb{C}H^n)^*$.

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Notations

$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\left(\frac{\cdot}{\cdot}\right)$	Kronecker symbol
(a, b)	greatest common divisor for integers a, b
φ	Euler's phi function
ϕ_r	rth cyclotomic polynomial
$\{q\}$	fractional part of a rational number q
${}^{H}A$	complex conjugate transpose of a matrix A
$\mathbb{C}H^n$	n-dimensional complex hyperbolic space
$\mathbb{P}^n_{\mathbb{C}}$	n-dimensional complex projective space
$B^n_{\mathbb{C}}$	n-dimensional complex ball
$U(\Lambda)$	automorphism group of lattice Λ
$U(n, 1)$	unitary group of signture $(n, 1)$

Chapter 1

Preliminaries

This chapter will give some tools for the future study of ball quotients. We will start recalling some facts about singularities, where we first introduce quotient singularities and reduce them to a quotient of $\mathbb{C}^n$ by a finite subgroup of the group of invertible $n \times n$-matrices. Then we study canonical singularities and state a criterion for these singularities in case when they arise as quotient singularities. The last section will be devoted to the construction of a toroidal compactification of a bounded symmetric domain restricted to rank 1. For this compactification we state some properties.

1.1 Singularities

Singularities are a well-studied object in algebraic geometry. The goal of this section is to state some important results by M. Reid and Y.-S. Tai.

1.1.1 Quotient Singularities

First we will follow D. Prill [Pri67] and H. Cartan [Car57] who investigated quotient singularities.
Let M be a complex manifold and H a subgroup of the group of holomorphic homeomorphisms of M that acts properly discontinously on M. Now denote by M/H the space of orbits. This space turns out to be complex analytic.
As we want to study singularities we have to introduce an equivalence relation for complex analytic spaces X_1 and X_2. Let $p_1 \in X_1$ and $p_2 \in X_2$, then we say that two pairs (X_1, p_1) and (X_2, p_2) are *locally isomorphic* if there exist neighborhoods $U_1 \subset X_1$, $U_2 \subset X_2$ of p_1, p_2 and a map $f : U_1 \longrightarrow U_2$, such that

(i) f is biholomorphic, and

(ii) $f(p_1) = p_2$.

Now let G be a finite subgroup of the general linear group $\mathrm{GL}(n, \mathbb{C})$. We define

$$\mathbb{C}^n/G$$

to be the *normal analytic space*. We want to say what we mean by a quotient singularity:

Definition 1.1. A *quotient singularity* is a singularity which is locally isomorphic to a singularity M/H, where M and H a finite group as above.

Now we can classify quotient singularities in the following way.

Lemma 1.2. *Every quotient singularity is locally isomorphic to a pair* $(\mathbb{C}^n/G, 0)$*, where* $G \subset \mathrm{GL}(n, \mathbb{C})$ *is a finite subgroup and* $0 \in \mathbb{C}^n$*.*

Proof. [Car57, p.97] □

To give a more exact correspondance we need to define a special class of matrices.

Definition 1.3. An element $g \in \mathrm{GL}(n, \mathbb{C})$ is called a *quasi-reflection*, if all but one eigenvalues of g are equal to 1.

Now we can state a more precise result.

Lemma 1.4. *Let* G_1*,* $G_2 \subset \mathrm{GL}(n, \mathbb{C})$ *be two finite subgroups without quasi-reflections. Then the singularities* $(\mathbb{C}^n/G_1, 0)$ *and* $(\mathbb{C}^n/G_2, 0)$ *are locally isomorphic if and only if* G_1 *and* G_2 *are conjugated.*

Proof. [Pri67, Theorem 2] □

As before let G denote a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$. Now take N to be the subgroup of G generated by its quasi-reflections. This is a normal subgroup, and the quotient G/N is isomorphic to a group without quasi-reflections, which we will call K. Then $(\mathbb{C}^n/G, 0)$ and $(\mathbb{C}^n/K, 0)$ are locally isomorphic by the following proposition.

Proposition 1.5. *Every quotient singularity* $(\mathbb{C}^n/G, 0)$ *is locally isomorphic to a pair* $(\mathbb{C}^n/K, 0)$*, where* $K \subset \mathrm{GL}(n, \mathbb{C})$ *is a finite subgroup without quasi-reflections and* $0 \in \mathbb{C}^n$*.*

Proof. [Pri67, Proposition 6] □

Thus it is enough when one works with groups without quasi-reflections.
As we described the relations between groups and the corresponding singularities we now want to know which groups give rise to smooth quotients.

Corollary 1.6. *Let* $G \subset \mathrm{GL}(n, \mathbb{C})$ *be a finite group. Then the quotient* $(\mathbb{C}^n/G, 0)$ *is non-singular if and only if* G *is generated by quasi-reflections.*

Proof. [Pri67, Corollary of Theorem 2] □

1.1.2 Canonical Singularities

In the last section we introduced quotient singularities. Now we want to give a criterion when they are 'nice', namely canonical. In this section we first follow the article of M. Reid [Rei87].
Let X be a normal, quasi-projective variety over $\mathbb{C}$. We denote by K_X a canonical divisor of X. As we will mostly assume that X is singular, we will first tell what we mean by the canonical divisor K_X.
For this let $X_{\mathrm{sm}} := X - \mathrm{Sing}(X)$ be the smooth part of X, where $\mathrm{Sing}(X)$ denotes the singular locus of X. We have the natural inclusion map

$$\iota : X_{\mathrm{sm}} \hookrightarrow X. \tag{1.1}$$

Now we can define a canonical divisor on X.

Definition 1.7. A *canonoical divisor* K_X on X is a Weil divisor, such that K_X coincides with a canonical divisor $K_{X_{\mathrm{sm}}}$ of the smooth locus X_{sm}, when restricted to X_{sm}.

Now we have everything together that we need to define canonical singularities.

Definition 1.8. A variety X has *canonical singularities* if the following holds:

(i) rK_X is Cartier for some $r \geq 1$, and

(ii) if $f : \widetilde{X} \longrightarrow X$ is any resolution and $\{E_i\}$ is the family of exceptional prime divisors of the resolution f, then

$$rK_{\widetilde{X}} = f^*(rK_X) + \sum a_i E_i, \quad \text{where all } a_i \geq 0. \tag{1.2}$$

If one strengthens the conditions on the a_i, one can define stricter types of singularities.

Remark 1.9. If we assume in Definition 1.8 that all $a_i > 0$ then we say that X has *terminal singularities*.

In a local situation we can define what is meant if we say that a point is a canonical singularity.

Definition 1.10. A point $x \in X$ is a *canonical singularity* if there exists a neighbourhood of x that has canonical singularities.

There are some other definitions that should be made when one introduces canonical singularities.

Definition 1.11. (i) Let $p \in X$ be a singularity. Then we call the smallest r, such that rK_X is Cartier in a neighboorhood $U \ni p$ the *index* of the singularity p.

(ii) We define the *discrepancy* of a resolution $f : \widetilde{X} \longrightarrow X$ as

$$\Delta := K_{\widetilde{X}} - f^* K_X.$$

If one thinks of Δ as a $\mathbb{Q}$-divisor, one can write $\Delta = \frac{1}{r} \sum a_i E_i$.

From now on we will restrict to the case that $X = M/G$, where $M = \mathbb{C}^n$ and G is a finite subgroup of $\mathrm{GL}(M)$. In the following we will describe criteria when the quotient singularity $0 \in X$ is a canonical singularity. We make use of statements of Y.-S. Tai [Tai82] and M. Reid [Rei87].
We will use the abbreviation X_g for the quotient $M/\langle g \rangle$, where $\langle g \rangle$ denotes the cyclic subgroup of G generated by $g \in G$.
Let s be a G-invariant pluricanonical form on M, which we denote by $s \in H^0(M, \mathcal{O}_M(kK_M))^G$ for some integer k.
Then the quasi-projective variety X has canonical singularities if and only if s lifts holomorphically to every resolution $\widetilde{X}$. That means that if we regard the form s as a meromorphic form on $\widetilde{X}$ it does not have poles on any exceptional divisor E_i. This statement corresponds to $a_i \geq 0$ in the definition of canonical singularities. A resolution of X_g is denoted by $\widetilde{X}_g$.

Lemma 1.12. *A form $s \in H^0(M, \mathcal{O}_M(kK_M))^G$ extends to $\widetilde{X}$ if and only if it extends to $\widetilde{X}_g$ for every $g \in G$.*

Proof. [Tai82, Proposition 3.1] □

But when s is a G-invariant form, then it is automatically $\langle g \rangle$-invariant. This leads to

Proposition 1.13. *X has canonical singularities, if X_g has canonical singularities for every $g \in G$.*

Proof. This follows directly from the discussion above and Lemma 1.12. □

Note that the other direction is in general not true. But if we restrict to groups without quasi-reflections, we get a better result.

Proposition 1.14. *Let G be a finite group as above without quasi-reflections. Then X has canonical singularities if and only if X_g has canonical singularities for every $g \in G$.*

Proof. [Rei80, Remark 3.2] □

Finally we want to state a criterion for canonical singularities. Therefore let $g \in \mathrm{GL}(M)$ be an element of order m and $\zeta = \zeta_m$ be a primitive mth root of unity. Assume that

$$g \sim \begin{pmatrix} \zeta^{a_1} & & \\ & \ddots & \\ & & \zeta^{a_n} \end{pmatrix}, \tag{1.3}$$

where $0 \leq a_i < m$. For this element g and the eigenvalues ζ^{a_i} we make a definition.

Definition 1.15. We call

$$\Sigma(g) := \frac{1}{m} \sum_{i=1}^{n} a_i \tag{1.4}$$

the *Reid-Tai sum* of g.

The Reid-Tai sum is the right object to study, in order to decide if X has canonical singularities.

Theorem 1.16 (Reid-Tai criterion)**.** *Let G be a finite subgroup of* $\mathrm{GL}(M)$ *without quasi-reflections. Then $X = M/G$ has canonical singularities if and only if*

$$\Sigma(g) \geq 1$$

for every $g \in G$, $g \neq \mathrm{id}$.

Proof. [Rei87, (4.11)] and [Tai82, Theorem 3.3]. □

Remark 1.17. (i) The statement of the theorem is independent of the choice of ζ as with $g \in G$ every power of g lies in G.

(ii) We can assume that every element of G can be written in the form (1.3), because g has finite order as G is finite.

1.2 Toroidal Compactification

We will deal with quasi-projective non-compact varieties. Often one wants to be in a compact setting. Therefore one needs to compactify the quasi-projective variety. There are some different compactifications, e.g. the Baily-Borel compactification, or a toroidal compactification. As the Baily-Borel compactification is minimal but usually highly singular and we later want to describe the singularities of the compactification, we will deal with toroidal compactification.
As we will not need toroidal compactification in full generality we will restrict ourself to a special case and skip some of the details. A good reference for the following is the book of A. Ash, D. Mumford, M. Rapoport and Y.-S. Tai [AMRT75]. We will sometimes cite K. Hulek, C. Kahn and S.H. Weintraub [HKW93, I.3], as they stated the toroidal compactification of the moduli space of abelian surfaces and these results generalize to arbitrary quotients of bounded symmetric domains. Therefore we will give this book as a reference whenever the generalization holds.

1.2.1 Basics

Let $D \subset \mathbb{C}^n$ be a realization as a bounded symmetric domain and $\mathrm{Aut}(D)$ the automorphism group of D. Let $G \subset \mathrm{Aut}(D)$ be an arithmetic subgroup.
In the following we want to sketch the steps that lead to a special compactification of $X(G) := D/G$ which we will denote by $(D/G)^*$ and call a *toroidal compactification.*
As we will see we first have to compactify the quotient locally around a boundary component. We will speak of this as 'in direction of the boundary components of $X(G)$'. We denote by $\overline{D}$ the closure of D in the ambient space $\mathbb{C}^n$.
First give some definitions.

Definition 1.18. (i) We say that two points $p_1, p_2 \in \overline{D}$ are *equivalent*, denoted $p_1 \sim p_2$, if they could be connected by finitely many holomorphic curves

(ii) We define a *boundary component* of D to be an equivalence class of a point $p \in D$.

(iii) We will denote the set of all boundary components by $\mathcal{F}$.

(iv) A boundary component is called *proper* if it lies only on the border, i.e. on $\overline{D} - D$.

We have group G acting on the space D. This action extends naturally to $\overline{D}$ and therefore we give a definition what this actions means for boundary components.

Definition 1.19. Two boundary components $F_1, F_2 \in \mathcal{F}$ are called *congruent* if there exist an $g \in G$ with $g(F_1) = F_2$.

Usually one should also define a partial ordering on $\mathcal{F}$, called *adjacency.* This is not required here, as our boundary components will all be of the same dimension 0.

1.2.2 Local compactification

For the compactification we need to define stabilizing groups of the boundary components F. Therefore let $G_{\mathbb{R}} := G \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 1.20. (i) For the boundary component $F \in \mathcal{F}$ we can define

$$N(F) \quad := \quad \{g \in G_{\mathbb{R}};\ g(F) = F\}\,, \tag{1.5}$$

which is called the *stabiliser group of F.*

(ii) We call $F \in \mathcal{F}$ *rational* if $N(F)$ is defined over $\mathbb{Q}$.

In the following we will mention a correspondance of boundary components and subspaces. First we have to introduce the notion of isotropic subspaces.

Definition 1.21. A subspace $U \subset \mathbb{C}^n$ is called *isotropic (with respect to the form $\langle \cdot, \cdot \rangle$)*, if

$$\langle u_1, u_2 \rangle = 0$$

holds for all $u_1, u_2 \in U$.

To each rational boundary component $F \in \mathcal{F}$ one can associate a rational isotropic subspace U_F. Equivalently to the definition above we could call F *rational* if the corresponding subspace U_F is rational.
From the theory we have to restrict this group to the arithmetic subgroup G. Hence we will define

$$N(F)_{\mathbb{Z}} \;:=\; N(F) \cap G. \tag{1.6}$$

Now we will introduce some more groups, deduced from $N(F)$, which we will need to go on in the compactification process.

Definition 1.22. (i) First define some subgroups.

(a) We denote by $W(F)$ the unipotent radical of $N(F)$.

(b) Let $U(F)$ be the centre of $W(F)$, i.e.

$$U(F) = \{g \in W(F);\; gh = hg \text{ for all } h \in W(F)\}.$$

(ii) Now we will restrict to G as in the case of $N(F)$:

$$U(F)_{\mathbb{Z}} \;:=\; U(F) \cap G. \tag{1.7}$$

(iii) Define

$$G(F) \;:=\; N(F)_{\mathbb{Z}}/U(F)_{\mathbb{Z}}. \tag{1.8}$$

In the following we will discuss some quotients and give a concrete construction. As all groups mentioned above are subgroups of the automorphism group of D we can define the quotient of D by these groups.

Definition 1.23. The *partial quotient $D(F)$ of D(with respect to the the boundary component F)* is given by

$$D(F) \;:=\; D/U(F)_{\mathbb{Z}}. \tag{1.9}$$

Furthermore the *partial quotient map* will be denoted by

$$q(F) : D \longrightarrow D(F).$$

Now we can give a first theorem which shows how toroidal compactification will work. But before this we should mention that the group $U(F)_{\mathbb{Z}}$ is a $\mathbb{Z}$-lattice of rank, say r in the $\mathbb{C}$-vector space

$$U(F)_{\mathbb{C}} := U(F)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C},$$

which is therefore of dimension r.

Theorem 1.24. *Let F be a rational boundary component of D. Then there exists a trivial torus bundle $\mathcal{D}(F)$ with fibre*

$$T := U(F)_{\mathbb{C}}/U(F)_{\mathbb{Z}} \cong (\mathbb{C}^*)^r$$

over $F \times (W(F)/U(F))$, such that

(i) $D(F)$ is isomorphic to an open subset of $\mathcal{D}(F)$, and

(ii) the action of $G(F)$ on $D(F)$ extends to $\mathcal{D}(F)$.

Proof. [AMRT75, III.4] □

From now on we will restrict our discussion to the case that the rank of the lattice $U(F)_{\mathbb{Z}}$ is 1 and therefore $T \cong \mathbb{C}^*$ in the formulation of Theorem 1.24.
The following usually requires more knowledge of toric geometry. As we are restricting to the rank 1 case, however, we will not give precise definitions of the objects and the techniques we will use, but say what can occur in our case. For more details about toric geometry we refer the reader to the book of T. Oda [Oda88].
First we will sketch the general contruction. We have to choose a 'fan' $\Sigma := \Sigma(F)$ in $U(F)$ over $\mathbb{R}$ for every F. This has to be '*admissible*' in the sense of Namikawa [Nam80, Definition 7.3] and this restricts the choice of the fan, e.g. there has to be a compatibility with the group action.
A fan consists of '*strongly convex rational polyhedral cones*' σ which has to fulfill some additional properties. To a cone σ one can associate a so called '*dual cone*' $\sigma^\vee$ (cf. [Oda88, 1.1]).
From this admissible fan we can construct a '*toric variety*' T_Σ. In this construction the dual cone appears. After this we want to 'replace' the torus T mentioned before by the toric variety T_Σ.
Now we will discuss what can happen in the rank 1 case. The strongly convex rational polyhedral cones that can occur are

$$\{0\},\ \sigma_+ := \mathbb{R}_{\geq 0} \text{ and } \sigma_- := -\sigma_+.$$

Now we want to choose an admissible fan Σ which has to be a suitable subset of $\{\{0\}, \sigma_+, \sigma_-\}$. As we have rank 1 there are only two choices of the admissible fan, in fact

$$\Sigma = \{\{0\}, \sigma_\pm\}. \tag{1.10}$$

We already mentioned dual cones. Here

$$\sigma_\pm^\vee = \sigma_\pm, \{0\}^\vee = \mathbb{R} = \sigma_+ \cup \sigma_-.$$

Thus $\sigma_\pm$ is self-dual. It turns out directly that for each of the two choices for the fan Σ

$$T_\Sigma = \mathbb{C} = \operatorname{Spec} \mathbb{C}[x].$$

In our construction this toric variety is an essential part of the compactification of the space $X(G)$.

Definition 1.25. For a rational boundary component and the admissible fan $\Sigma(F)$ as above we define

(i) $\mathcal{D}_\Sigma(F) := \mathcal{D}(F) \times_T T_\Sigma$, and

(ii) $D_\Sigma(F) := (\overline{D(F)})^0$ as the interior of $\overline{D(F)}$, where $\overline{D(F)}$ denotes the closure of $D(F) \subset \mathcal{D}_\Sigma(F)$.

Now we are in the right set-up to give a result for the *partial compactification* of $D/N(F)_\mathbb{Z}$.

Proposition 1.26. *The induced action of $G(F)$ on $D(F)$ extends uniquely to a properly discontinous action on $D_\Sigma(F)$.*
The space

$$X_\Sigma(F) := D_\Sigma(F)/G(F)$$

is an analytic variety. Assume that $D_\Sigma(F)$ is smooth, then $X_\Sigma(F)$ has at worst finite quotient singularities.

Proof. [AMRT75, III.6 Proposition 2] □

Now we can define the boundary of this quotient space.

Definition 1.27. The *boundary* of $X_\Sigma(F)$ is defined as

$$\partial X_\Sigma(F) := X_\Sigma(F) - (D/N(F)_\mathbb{Z}).$$

It turns out from the theory that $\partial X_\Sigma(F)$ is a divisor on $X_\Sigma(F)$.
Now we can see $X_\Sigma(F)$ as a partial compactification of $D/N(F)_\mathbb{Z}$. But we want to have a partial compactification of $X(G)$. This could be achieved by the following.

Proposition 1.28. *Let $F \in \mathcal{F}$ be a rational boundary component. Then there exist a $N(F)_\mathbb{Z}$-invariant interior neighborhood U of F in D, such that there is a local isomorphism*

$$p(F) : D/N(F)_\mathbb{Z} \longrightarrow X(G),$$

when restricted to U, induced by the inclusion $U \subset D$.

Proof. [HKW93, Proposition 3.47] □

With the map from the previous Proposition we can attach the divisor $\partial X_\Sigma(F)$ to D/G with the help of the isomorphism $p(F)$ around F and achieve a *partial compactification of D/G in the direction of the boundary component F.*

1.2.3 Global construction

In the last section we have constructed a partial compactification in the direction of F. Now we will make precise what to do to glue all these partial compactifications together. Remember that we restricted ourself to the case $r = 1$.
First we collect all fans with respect to the rational boundary components:

$$\widetilde{\Sigma} \ := \ \{\Sigma(F);\ F \text{ is a rational boundary component}\}. \tag{1.11}$$

Similarly to the fans (where only certain choices of admissible fans were allowed), one has to define what is meant by an admissible collection of fans.

Definition 1.29. We call a collection of fans $\widetilde{\Sigma}$ (in the rank 1 case) *admissible*, if

(i) every $\Sigma(F) \in \widetilde{\Sigma}$ is admissible, i.e. it is of the form (1.10), and

(ii) for two rational boundary components F_1, F_2 with $F_1 = g(F_2)$ for an $g \in G$, it follows that $\Sigma(F_1) = g\Sigma(F_2)g^{-1}$.

Now we have to say how one gets from one partial compactification to another.

Proposition 1.30. *Let F_1 and F_2 be two rational boundary components, such that $F_1 = g(F_2)$ for an $g \in G$ as above. Additionally let $\Sigma(F_1)$, $\Sigma(F_2)$ be two fans in $U(F_1), U(F_2)$, such that $\Sigma(F_1) = g\Sigma(F_2)g^{-1}$. Then*

(i) the diagram

$$\begin{tikzcd} D_{\Sigma(F_2)}(F_2) \arrow[r, "\tilde{g}"] & D_{\Sigma(F_1)}(F_1) \\ D/U(F_2)_{\mathbb{Z}} \arrow[u, hook] \arrow[r, "g"] & D/U(F_1)_{\mathbb{Z}} \arrow[u, hook] \end{tikzcd}$$

commutes, and $\tilde{g}$ is an isomorphism, and

(ii) the diagram

$$\begin{tikzcd} X_{\Sigma(F_2)}(F_2) \arrow[r, "\overline{g}"] & X_{\Sigma(F_1)}(F_1) \\ D/N(F_2)_{\mathbb{Z}} \arrow[u, hook] \arrow[r, "g"] & D/N(F_1)_{\mathbb{Z}} \arrow[u, hook] \end{tikzcd}$$

commutes, and $\overline{g}$ is an isomorphism

Proof. [HKW93, Proposition 3.69] □

We can now define the toroidal compactification of D/G. Remember that $\widetilde{\Sigma}$ denotes an admissible collection of fans.

First we have to define

$$D(\widetilde{\Sigma}) := \bigsqcup_{F\in\mathcal{F} \text{ rational}} D_{\Sigma(F)}(F), \tag{1.12}$$

as a disjoint union, where $\Sigma(F) \in \widetilde{\Sigma}$. Next we want to identify points of $D(\widetilde{\Sigma})$ with the help of an equivalence relation.

Definition 1.31. Let $D(\widetilde{\Sigma})$ and $\widetilde{\Sigma}$ be as in (1.12). Let F_1, F_2 be rational boundary components. Then for $x_i \in D_{\Sigma(F_i)}(F_i)$, $i = 1, 2$, the points x_1 and x_2 are *equivalent*, denoted $x_1 \simeq x_2$, if there exist an $g \in G$ with the property $F_1 = g(F_2)$, such that

$$x_1 = \tilde{g}x_2,$$

for $\tilde{g}$ as in Proposition 1.30.

Finally we can define the toroidal compactification constructed by such a collection $\widetilde{\Sigma}$.

Definition 1.32. Let $\widetilde{\Sigma} = \{\Sigma(F)\}$ be an admissible collection of fans. Then we will call

$$X(G)^* = (D/G)^* := D(\widetilde{\Sigma})/\simeq$$

the *toroidal compactification* of D/G given by $\widetilde{\Sigma}$.

Remark 1.33. The map $p(F)$ from Proposition 1.28 can be extended to a map

$$\overline{p}(F) : X_{\Sigma(F)}(F) \longrightarrow X(G)^*.$$

Now we have constructed a toroidal compactification of D/G. This compactification has some nice properties.

Theorem 1.34. *Let $(D/G)^*$ be a toroidal compactification of D/G constructed by an admissible collection of fans $\widetilde{\Sigma} = \{\Sigma(F)\}$. Then the following statements hold*

(i) $(D/G)^*$ is compact,

(ii) D/G is an open and dense subset of $(D/G)^*$,

(iii) $\partial(D/G)^* := (D/G)^* - D/G$ is a Weil divisor,

(iv) the map $\overline{p}(F)$ is an isomorphism when restricted to a sufficiently small neighborhood of the boundary of $X_{\Sigma(F)}(F)$,

(v) $X(G)^*$ is the union of the images of $\overline{p}(F)$ for all F.

Proof. [HKW93, Theorem 3.82, Remark 3.77(i)] □

A property one usually wants to have for a compactification is to be a projective variety. The projectivity follows directly if the admissible collection of fans is '*projective*', i.e. that there exisits a '*polarization function*' in the sense of G. Faltings and C.-L. Chai [FC90, IV. Definition 2.4]. As we are in the rank 1 case every admissible collection of fans is automatically projective.

Theorem 1.35. *Let $(D/G)^*$ be a toroidal compactification defined by $\widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is a projective admissible collection of fans. Then $(D/G)^*$ is a projective variety.*

Proof. [AMRT75, IV.2 Theorem] $\square$

Chapter 2

Representation theory and number fields

The representation theory of cyclic groups over $\mathbb{Q}$ is well understood. In the case of quadratic number fields there are some results we will use to classify all irreducible representations of the cyclic group. In this chapter we will first recall some classical results on irreducibilty of cyclotomic polynomials which give all numberfields over which a given cyclotomic polynomial is reducible. Then we mention some correspondences for irreducible factors of cyclotomic polynomials corresponding to the quadratic number field. Finally we state all irreducible representations of the cyclic group over this field.

2.1 Cyclotomic polynomials

In this section we consider reducibility of cyclotomic polynomials over several quadratic number fields. It is well-known that the rth cyclotomic polynomial is irreducible over the rationals.
In this section we provide the full list of quadratic number fields over which the rth cyclotomic polynomial is reducible. Mostly we follow arguments given by L. Weisner [Wei28].
Let $r \in \mathbb{N}$ and $r \geq 2$.

Definition 2.1. The *rth cyclotomic polynomial* is defined as

$$\phi_r(x) \quad = \quad \prod_{\substack{i=1 \\ (i,r)=1}}^{r} (x - \zeta^i), \tag{2.1}$$

where $\zeta := \zeta_r$ is a primitive rth root of unity.

Since the exponents i of the roots of (2.1) are chosen coprime to r, the power ζ^i is also a primitive rth root of unity. Via this construction one gets all primitive

rth roots. Therefore the cyclotomic polynomial is independent of the choice of ζ_r. It is a classical result that $\phi_r(x)$ has integer coefficients for all r.
Another fact is

$$\phi_{2r}(x) = \phi_r(-x),$$

for odd r. This is true because the only primitive 2nd root of unity is $-1 \in \mathbb{Q}$.

Remark 2.2. A common choice for a primitive rth root of unity is

$$\zeta_r := e^{\frac{2\pi i}{r}}.$$

Now we can give some results on irreducibility. We start with a result for the rationals.

Lemma 2.3. *The polynomial ϕ_r is irreducible over $\mathbb{Q}$.*

Proof. [Was82, Chapter 2] □

Thus, over the rationals, there is nothing more to say about irreducibility. However, when one uses quadratic number fields instead of the rationals the situation becomes more complicated. We write $\mathbb{Q}(\sqrt{D})$ for the quadratic number field, where $D \in \mathbb{Z}\backslash\{0, 1\}$ is squarefree.
When we have two such number fields where a given cyclotomic polynomial is reducible we want to construct a third one with this property. The following lemma will make this more precise.

Lemma 2.4. *Assume that ϕ_r is reducible over $\mathbb{Q}(\sqrt{D_1})$ and $\mathbb{Q}(\sqrt{D_2})$, where $\mathbb{Q}(\sqrt{D_1})$ and $\mathbb{Q}(\sqrt{D_2})$ are distinct fields. Then ϕ_r is reducible over $\mathbb{Q}(\sqrt{D_1 \cdot D_2})$.*

Proof. [Wei28, Lemma 1] □

Another property to which we will refer later, is that a 'divisibility property' of two cyclotomic polynomials preserves reducibility over the same quadratic number field.

Lemma 2.5. *Assume that ϕ_r is reducible over $\mathbb{Q}(\sqrt{D})$ and choose r' with r divides r'. Then the cyclotomic polynomial $\phi_{r'}$ is reducible over $\mathbb{Q}(\sqrt{D})$.*

Proof. [Wei28, Lemma 2] □

We introduced some technical results which will be used to find 'enough' quadratic number fields.

2.1.1 Reducibility

We now state a list of quadratic number fields that leads to reducible cyclotomic polynomials. Therefore we get a lower bound for the number of all such quadratic number fields.
We now fix $r \geq 2$ and investigate the behaviour of ϕ_r for some number fields as before using results of [Wei28].
We have to use the prime factorization of r. In the following we assume that r is of the form

$$r = 2^{a_0} p_1^{a_1} \cdots p_s^{a_s}, \tag{2.2}$$

where the p_i are distinct odd prime numbers with $a_i \geq 1$ for $i \geq 1$ and $a_0 \geq 0$.
Already Gauss investigated the behaviour of cyclotomic polynomials over such fields. The rest of this section relies on one of Gauss' theorems.

Theorem 2.6 (Gauss)**.** *Let p be an odd prime number.Then the pth cyclotomic polynomial ϕ_p is reducible over the field $\mathbb{Q}\left(\sqrt{(-1)^{\frac{p-1}{2}} p}\right)$.*

Proof. [Gau66, 357.] □

Using this theorem one can state a first estimate for the number of those number fields. We use the lemma above and the fact that r has s distinct odd prime factors.

Lemma 2.7. *There are at least $2^s - 1$ quadratic number fields over which the cyclotomic polynomial ϕ_r is reducible.*

Proof. We know that ϕ_r is reducible over $\mathbb{Q}\left(\sqrt{(-1)^{\frac{p_i-1}{2}} p_i}\right)$ for all i by Gauss and Lemma 2.5 . By Lemma 2.4, the polynomial ϕ_r is reducible over $\mathbb{Q}(\sqrt{\Delta})$, where $\Delta = \Delta(r)$ is of the form

$$\Delta = \prod_{i \in A} (-1)^{\frac{p_i-1}{2}} p_i,$$

for all non-empty subsets $A \subset \{1, \ldots, s\}$ and odd prime numbers p_i as in (2.2).
Collectively there exist

$$\begin{aligned} \sum_{k=1}^{s} \binom{s}{k} &= \sum_{k=0}^{s} \binom{s}{k} - 1 \\ &= 2^s - 1 \end{aligned}$$

such quadratic number fields. □

Up to this point we have only used the odd prime numbers p_i in the factorization of r and not the factor 2^{a_0}. When $a_0 \geq 2$ there appear more quadratic number fields.
For this we state the explicit form of the 2^{a_0}th cyclotomic polynomial. First we have to state a well-known decomposition.

Remark 2.8. Let n be a positive integer. Then

$$x^n - 1 = \prod_{d|n} \phi_d.$$

It is now easy to calculate that

$$\phi_{2^{a_0}}(x) = x^{2^{a_0-1}} + 1. \tag{2.3}$$

This can be shown easily if one writes

$$\begin{aligned} x^{2^{a_0}} - 1 &= \prod_{d|2^{a_0}} \phi_d \\ \text{and } x^{2^{a_0-1}} - 1 &= \prod_{d|2^{a_0-1}} \phi_d. \end{aligned}$$

When one substitutes $y := x^{2^{a_0-1}}$, it follows that:

$$\begin{aligned} \phi_{2^{a_0}}(x) &= \frac{x^{2^{a_0}} - 1}{x^{2^{a_0-1}} - 1} \\ &= \frac{y^2 - 1}{y - 1} \\ &= y + 1 \\ &= x^{2^{a_0-1}} + 1. \end{aligned}$$

Now we first assume $a_0 \geq 2$ and construct a specific decomposition of the polynomial (2.3).

Lemma 2.9. *Let $a_0 \geq 2$. Then the cyclotomic polynomial $\phi_{2^{a_0}}$ is reducible over $\mathbb{Q}(i) = \mathbb{Q}(\sqrt{-1})$.*

Proof. An easy calculation shows that

$$\left(x^{2^{a_0-2}} + i\right) \cdot \left(x^{2^{a_0-2}} - i\right) = x^{2^{a_0-1}} + 1 = \phi_{2^{a_0}}(x).$$

$\square$

This lemma enables us to expand the list of quadratic number fields given in Lemma 2.7.

Lemma 2.10. *If $a_0 \geq 2$, then there exist at least $2^{s+1}-1$ quadratic number fields over that ϕ_r is reducible.*

Proof. Since 2^{a_0} divides r, the polynomial ϕ_r is reducible over $\mathbb{Q}(i)$ by Lemma 2.5. In addition, Lemma 2.7 implies reducibility over $\mathbb{Q}(\sqrt{\Delta})$. Hence the cyclotomic polynomial is reducible over

$$\mathbb{Q}(\sqrt{\pm\Delta}) \text{ and } \mathbb{Q}(i). \tag{2.4}$$

Altogether we have

$$2 \cdot (2^s - 1) + 1 = 2^{s+1} - 1 \tag{2.5}$$

distinct such quadratic number fields. In the formula (2.5) the $2^s - 1$ comes from Lemma 2.7, the factor 2 from the $\pm$ in (2.4) and the summand 1 appears because of the number field $\mathbb{Q}(i)$. □

When one even assumes $a_0 \geq 3$ one gets a further decomposition.

Lemma 2.11. *If $a_0 \geq 3$, then there exist at least $2^{s+2}-1$ quadratic number fields over that ϕ_r is reducible.*

Proof. For $a_0 \geq 3$ we want to factor the polynomial $\phi_{2^{a_0}}(x) = x^{2^{a_0-1}} + 1$ over $\mathbb{Q}(\sqrt{2})$ in a different way than we did before. We have

$$x^{2^{a_0-1}} + 1 \;=\; \left(x^{2^{a_0-2}} + \sqrt{2}x^{2^{a_0-3}} + 1\right) \cdot \left(x^{2^{a_0-2}} - \sqrt{2}x^{2^{a_0-3}} + 1\right).$$

With an analogous statement as in Lemma 2.10 we can construct the number fields

$$\mathbb{Q}(i),\ \mathbb{Q}(\sqrt{\pm 2}),\ \mathbb{Q}(\sqrt{\pm\Delta}) \text{ and } \mathbb{Q}(\sqrt{\pm 2\Delta}) \tag{2.6}$$

with the required property. Thus we get

$$\begin{aligned} 1 + 2 \cdot \left(\sum_{k=0}^{s+1} \binom{s+1}{k} - 1 \right) &= 1 + 2 \cdot 2^{s+1} - 2 \\ &= 2^{s+2} - 1 \end{aligned} \tag{2.7}$$

quadratic number fields. In (2.7) we get the summand 1 again from the field $\mathbb{Q}(i)$ and the factor 2 from the $\pm$. The expression in the brackets is the number of all non-empty subsets of a set with $s+1$ elements which is the number of all possible choices of a product with factors $2, p_1, \ldots, p_s$. □

Now we can state a bound on the number of number fields which lead to reducible cyclotomic polynomials ϕ_r, only depending on the prime factorization of r.

Corollary 2.12. *Let ϕ_r be the rth cyclotomic polynomial and $r = 2^{a_0} p_1^{a_1} \cdots p_s^{a_s}$, with $a_0 \geq 0$ and $a_i \geq 1$, $i = 1, \ldots, s$. Then the number of distinct quadratic number fields over which ϕ_r is reducible is at least*

(i) $2^s - 1$, *if* $4 \not| r$,

(ii) $2^{s+1} - 1$, *if* $4|r$,

(iii) $2^{s+2} - 1$, *if* $8|r$.

Proof. This follows at once from the Lemmas 2.7, 2.10 and 2.11. □

2.1.2 Galois theory

So far we know the minimal number of quadratic number fields which imply reducibility of ϕ_r, since we constructed them explicitly. Now we mention tools which will lead to the maximum number of such fields. To do this, we have to use Galois theory.

In the previous section we only considered quadratic number fields. Now we must use number fields of higher degree, namely the *cyclotomic fields* $\mathbb{Q}(\zeta_r)$, where ζ_r is a primtive rth root of unity. Note that the cyclotomic field is the splitting field of the polynomial ϕ_r. This follows easily because if one primitive rth root of unity lies in the field then every power of the root is also an element. Thus all zeroes of ϕ_r lie in the cylotomic field and it is therefore the splitting field of ϕ_r.

First we define G to be the Galois group

$$G = \mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q}).$$

There is a classical description of the group G given by the isomorphism

$$G \cong (\mathbb{Z}/r\mathbb{Z})^* . \tag{2.8}$$

Here $(\mathbb{Z}/r\mathbb{Z})^*$ denotes the group of units of the cyclic group $\mathbb{Z}/r\mathbb{Z}$.

In the following we want to count all intermediate fields of degree 2 over $\mathbb{Q}$ of the field extension $\mathbb{Q}(\zeta_r)/\mathbb{Q}$. This relies on the fact that the degree of the field extension is $[\mathbb{Q}(\zeta_r) : \mathbb{Q}] = \varphi(r)$, where φ denotes Euler's phi function.

Therefore assume the cyclotomic polynomial ϕ_r to be reducible of over a quadratic extension of $\mathbb{Q}$ which is not a subfield of $\mathbb{Q}(\zeta_r)$. Then it is not possible that $\mathbb{Q}(\zeta_r)$ is the splitting field of degree $\varphi(r)$ corresponding to the polynomial ϕ_r. This gives a contradiction. So finding all intermediate quadratic fields will be an upper bound on the number of fields over which ϕ_r is reducible.

Now we can rephrase this problem using Galois theory. The problem of finding these subfields is equivalent to finding all subgroups of G of index 2.

The following is a classical result that solves this problem.

Lemma 2.13. *Let r be as in (2.2). The number of subgroups of index 2 of the group $(\mathbb{Z}/r\mathbb{Z})^*$ is*

(i) $2^s - 1$, *if* $4 \nmid r$,

(ii) $2^{s+1} - 1$, *if* $4|r$ *but* $8 \nmid r$ *and*

(iii) $2^{s+2} - 1$, *if* $8|r$.

Proof. [Wei28, Section 6] □

2.1.3 A Result

In 2.1.1 and 2.1.2 we first constructed quadratic number fields such that the cyclotomic polynomial ϕ_r decomposes. Then we gave an upper bound on the number of number fields of degree 2 with this property. Now we will combine these two results and provide a complete list of such fields.

Theorem 2.14. *Let r be as in (2.2). Then the only quadratic number fields over which the cyclotomic polynomial ϕ_r is reducible are*

(i) $\mathbb{Q}(\sqrt{D})$, *where* $D = \Delta = \prod_{i\in A}(-1)^{\frac{p_i-1}{2}}p_i$, *for all non-empty subsets* $A \subset \{1,\ldots,s\}$, *if* $4 \nmid r$,

(ii) $\mathbb{Q}(i)$ *and* $\mathbb{Q}(\sqrt{D})$, *where* $D = \pm\Delta$, *for* Δ *as in (i), if* $4|r$ *and* $8 \nmid r$,

(iii) $\mathbb{Q}(i)$ *and* $\mathbb{Q}(\sqrt{D})$, *where* $D = \pm\Delta$ *or* $D = \pm 2\Delta$, *for* Δ *as in (i), if* $8|r$.

Proof. As the lower bound equals the upper bound by Corollary 2.12 and Lemma 2.13, this follows directly from the discussion in 2.1.1. □

For this result we give two easy examples, which will show how one can find number fields explicitly.

Examples 2.15. (a) $r = 35 = 5 \cdot 7$. Then the $2^2 - 1 = 3$ quadratic number fields mentioned in Theorem 2.14 are:

$$\mathbb{Q}(\sqrt{5}),\ \mathbb{Q}(\sqrt{-7}) \text{ and } \mathbb{Q}(\sqrt{-35}).$$

(b) $r = 36 = 2^2 \cdot 3^2$. Then we also have $2^{1+1} - 1 = 3$ such number fields, namely

$$\mathbb{Q}(i) \text{ and } \mathbb{Q}(\sqrt{\pm 3}).$$

2.2 Eigenvalues of cyclotomic polynomials

In the last section we discussed the quadratic number fields where ϕ_r splits. Now we will focus on the case where ϕ_r factorizes for such a number field. We are interested in the zeroes of the irreducible factors and how they behave for different number fields as in Theorem 2.14.
First recall the definition of the rth cyclotomic polynomial.

$$\phi_r(x) \quad = \quad \prod_{\substack{i=1 \\ (i,r)=1}}^{r} (x - \zeta^i),$$

where ζ is a primitive rth root of unity (cf. (2.1)). As already mentioned before the zeroes of ϕ_r are the primitive rth roots of unity.
Assume that $\phi_r(x)$ is reducible over a quadratic number field $\mathbb{Q}(\sqrt{D})$. Then it has to decompose into two irreducible polynomials of degree $\frac{\varphi(r)}{2}$ with coefficients in $\mathbb{Q}(\sqrt{D})$.
This can be best seen studying the diagram

$$\begin{tikzcd} \mathbb{Q}(\zeta) \arrow[d, dash, "\frac{\varphi(r)}{2}"] \\ \mathbb{Q}(\sqrt{D}) \arrow[d, dash, "2"] \\ \mathbb{Q} \end{tikzcd}$$

The numbers along the lines denote the degree of the field extensions. Since we assumed that ϕ_r is reducible, we will write

$$\phi_r(x) = \phi_r'(x) \cdot \phi_r''(x). \tag{2.9}$$

Assume that the degree of ϕ_r' is less than $\frac{\phi(r)}{2}$. Then the degree of the field extension $\mathbb{Q}(\sqrt{D}) \subset \mathbb{Q}(\zeta_r)$ would be also less than $\frac{\phi(r)}{2}$. This contradicts the degree of the splitting field over $\mathbb{Q}$. Therefore the polynomials ϕ_r' and ϕ_r'' are defined over a quadratic number field $\mathbb{Q}(\sqrt{D})$ as in Theorem 2.14 with $\deg \phi_r' = \deg \phi_r'' = \frac{\varphi(r)}{2}$.
Choose $\zeta_r := e^{\frac{2\pi i}{r}}$ to be a generator of the group of r-th roots of unity. Clearly ζ_r is primitive by definition. Without loss of generality we choose ϕ_r' such that

$$\phi_r'(\zeta_r) = 0. \tag{2.10}$$

This is equivalent to $\phi_r''(\zeta_r) \neq 0$.
We want to calculate all roots of ϕ_r' (and therefore also the roots of ϕ_r''). Now we need a criterion for the roots of ϕ_r that distinguishes zeroes of ϕ_r' from zeroes of ϕ_r''.

There is a natural map between Galois groups of the fields introduced before which induces a map that distinguishes the roots. The following diagram makes this more precise:

$$\begin{tikzcd}
\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q}) \arrow[r, "f"] \arrow[d, "\cong"] & \mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}) \arrow[r] \arrow[d, equal] & \mathrm{Gal}(\mathbb{Q}/\mathbb{Q}) \arrow[d, equal] \\
(\mathbb{Z}/r\mathbb{Z})^* \arrow[r, "\tilde{f}"] & \{\pm 1\} \arrow[r] & \{\mathrm{id}_{\mathbb{Q}}\}
\end{tikzcd} \tag{2.11}$$

In this diagram $+1$ denotes the action on $\mathbb{Q}(\sqrt{D})$ as the identity and -1 the action as conjugation, i.e.

$$-1: \; a + b\sqrt{D} \;\longmapsto\; a - b\sqrt{D}. \tag{2.12}$$

The Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q}) = \{\pm 1\}$ also acts on ϕ_r' and ϕ_r'' by the action on the coefficients. Therefore we obtain

$$+1: \; \phi_r' \;\mapsto\; \phi_r', \quad \phi_r'' \;\mapsto\; \phi_r'', \tag{2.13}$$
$$-1: \; \phi_r' \;\mapsto\; \phi_r'', \quad \phi_r'' \;\mapsto\; \phi_r'. \tag{2.14}$$

In the following, we refer to a map f which has the properties requested by the diagram. To be more precise, it must map $a \in \mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$ to $+1$ resp. -1 in $\mathrm{Gal}(\mathbb{Q}(\sqrt{D})/\mathbb{Q})$, i.e. $f(a) = +1$ resp. $f(a) = -1$.
If $f(a) = +1$ we want to get $\phi_r'(\zeta_r^{\tilde{a}}) = 0$, were $\tilde{a}$ is the image of a by the vertical isomorphism between the Galois group of the cyclotomic field and the group of units. From now on we identify a and $\tilde{a}$.
We can rephrase this question as finding all $a \in (\mathbb{Z}/r\mathbb{Z})^*$ with the property $\tilde{f}(a) = +1$.
For this we will state some well-known maps coming from number theory.
First we introduce a map defined only for odd primes.

Definition 2.16. Let $b \in \mathbb{Z}$ and p an odd prime number. Then the *Legendre symbol* is defined as

$$\left(\frac{b}{p}\right) := \begin{cases} 1, & \text{if } b \text{ is a quadratic residue modulo } p, \\ -1, & \text{if } b \text{ is a quadratic non-residue modulo } p, \\ 0, & \text{if } p \text{ divides } b. \end{cases}$$

We want to use not only primes but also integers. There is a well-known generalization of this symbol.

Definition 2.17. For all $b \in \mathbb{Z}$, $n = u \cdot \prod_{i=1}^{k} p_i^{e_i}$, with $e_i > 0$, u a unit and p_i a prime, we define the *Kronecker symbol*

$$\left(\frac{b}{\cdot}\right): \; \mathbb{Z} \longrightarrow \{0, \pm 1\}$$

by

$$\left(\frac{b}{n}\right) := \left(\frac{b}{u}\right) \cdot \prod_{i=1}^{k} \left(\frac{b}{p_i}\right)^{e_i}, \text{ where} \tag{2.15}$$

$$\left(\frac{b}{p_i}\right) \text{ denotes the Legendre symbol for } p_i > 2$$

$$\left(\frac{b}{2}\right) := \begin{cases} 1, & \text{if } b \equiv 1 \text{ or } b \equiv 7 \mod 8, \\ -1, & \text{if } b \equiv 3 \text{ or } b \equiv 5 \mod 8, \\ 0, & \text{if } 2|b, \end{cases} \tag{2.16}$$

$$\text{and } \left(\frac{b}{1}\right) := 1, \qquad \left(\frac{b}{-1}\right) := \begin{cases} 1, & \text{if } b > 0, \\ -1, & \text{if } b < 0. \end{cases} \tag{2.17}$$

The Kronecker symbol should be regarded as the map $\tilde{f}$ introduced in the diagram (2.11). But by definition it can take the value 0, which is not allowed by the diagram.
The following lemma shows that this can not occur in our situation.

Lemma 2.18. *Let $r \geq 2$ as in (2.2) and ϕ_r be the rth cyclotomic polynomial with corresponding Galois group* $\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q}) \cong (\mathbb{Z}/r\mathbb{Z})^*$. *For D as in Theorem 2.14 we have*

$$\left(\frac{D}{\cdot}\right) : \ (\mathbb{Z}/r\mathbb{Z})^* \longrightarrow \{\pm 1\}. \tag{2.18}$$

Proof. From Theorem 2.14 we know that $D|r$. Let a be an element of $(\mathbb{Z}/r\mathbb{Z})^*$, i.e. $(a, r) = 1$. Let $a = 2^{s_0} q_1^{s_1} \cdots q_l^{s_l}$ be the prime factorization of a with $s_0 \geq 0$ and $s_i \geq 1$, $i = 1, \dots, l$. We have to distinguish two cases which would lead to value 0 in the decomposition (2.15) of the Kronecker symbol:

(1) Assume that $s_0 > 0$ and $\left(\frac{D}{2}\right) = 0$. This implies that D is even. Then by the theorem $a_0 \geq 3$ and $8|r$. However, $2 \notin (\mathbb{Z}/r\mathbb{Z})^*$ for $8|r$, because $(2, r) = 2 \neq 1$.

(2) Now let $\left(\frac{D}{q_i}\right) = 0$ for one i. This is only possible when q_i devides D by the definition of the Legendre symbol. So one has $q_i|D$ for one i and $D|r$. Overall we get $q_i|r$ in contradiction to the definition of $(\mathbb{Z}/r\mathbb{Z})^*$.

□

This shows that the Kronecker symbol has the properties required. With these properties the map can also be called a *primitive Dirichlet character* mod r.
We will now state the easy fact that complex conjugation acts on the group of primitive roots of unity resp. on $(\mathbb{Z}/r\mathbb{Z})^*$.

Lemma 2.19. *Let r be a positive integer and $a \in (\mathbb{Z}/r\mathbb{Z})^*$. Then*

$$r - a \in (\mathbb{Z}/r\mathbb{Z})^* .$$

Proof. This follows by an easy calculation. □

With the Kronecker symbol we can decide which root belongs to which irreducible factor of the cyclotomic polynomial over some quadratic number field.
We will now state a criterion to distinguish the zeroes of the factors for some special cases with respect to a given root of ϕ_r.

Proposition 2.20. *Assume that ϕ_r decomposes over $\mathbb{Q}(\sqrt{D})$ into two polynomials ϕ_r' and ϕ_r'' of degree $\frac{\varphi(r)}{2}$ as in (2.9) with the property (2.10). Then*

(i) the complex conjugate ζ_r^{r-a} of ζ_r^a with $\phi_r'(\zeta_r^a) = 0$ is a zero of ϕ_r', in particular $\phi_r'(\zeta_r^{r-1}) = 0$, if $D > 0$.

(ii) the complex conjugate ζ_r^{r-a} of ζ_r^a with $\phi_r'(\zeta_r^a) = 0$ is a zero of ϕ_r'', in particular $\phi_r''(\zeta_r^{r-1}) = 0$, if $D < 0$.

Proof. We need to investigate the values of the Kronecker symbol for the different exponents.

(i) It is to show that $\left(\frac{D}{r-a}\right) = 1$ if $\left(\frac{D}{a}\right) = 1$. We know $D|r$ and thus

$$\begin{aligned}\left(\frac{D}{r-a}\right) &= \left(\frac{D}{-a}\right)\\ &= \left(\frac{D}{-1}\right)\cdot\left(\frac{D}{a}\right)\\ &= \left(\frac{D}{-1}\right).\end{aligned}$$

By definition of the Kronecker symbol $\left(\frac{D}{-1}\right) = 1$ if and only if $D > 0$.

(ii) Analogous to the previous argument.

□

We want to apply the results and techniques from above to the Examples 2.15. Therefore we will state how a cyclotomic polynomial factors in different number fields.

Example 2.21. (a) For $r = 35$ and $D = -7$ one can estimate the zeroes of ϕ_{35}' in this situation:

$$\zeta_{35}^a, \text{ for } a = 1, 2, 4, 8, 9, 11, 16, 18, 22, 23, 29, 32.$$

The remaining primitive 35th roots of unity are the zeroes of ϕ_{35}''.

(b) One achieves, for $r = 36$ and

(1) $D = -1$, the decomposition

$$\begin{aligned} \phi_{36}(x) &= \phi'_{36}(x)\phi''_{36}(x) \\ &= \left(x^6 - ix^3 - 1\right)\left(x^6 + ix^3 - 1\right) \ /\mathbb{Q}(i), \end{aligned}$$

with $\phi'_{36}(\zeta_{36}^a) = 0$ for $a = 1, 5, 13, 17, 25, 29$.

(2) $D = 3$, the decomposition

$$\begin{aligned} \phi_{36}(x) &= \tilde{\phi}'_{36}(x)\tilde{\phi}''_{36}(x) \\ &= \left(x^6 - x^3\sqrt{3} + 1\right)\left(x^6 + x^3\sqrt{3} + 1\right) \ /\mathbb{Q}(\sqrt{3}), \end{aligned}$$

with $\tilde{\phi}'_{36}(\zeta_{36}^a) = 0$ for $a = 1, 11, 13, 23, 25, 35$.

2.3 Representations over number fields

So far we introduced cyclotomic polynomials and studied their decompostion for some specific quadratic number fields. These results will be used in this section to state the irreducible representations of cyclic groups for number fields $\mathbb{Q}(\sqrt{D})$. It is well-known that over the rational numbers there is a unique irreducible faithful subrepresentation of the group of rth roots of unity and the eigenvalues of this subrepresentation are the primitive rth roots of unity.
We will see that this is not the case any more when the cyclotomic polynomial factors.
Denote by μ_r the cyclic group of rth roots of unity. We are interested in the action of μ_r on an r-dimensional vector space over a quadratic number field. The group μ_r can be identified with $\mathbb{Z}/r\mathbb{Z}$.
Let V be an r-dimensional $\mathbb{Q}(\sqrt{D})$-vector space and

$$\rho: \ \mu_r \longrightarrow \mathrm{Aut}(V) \tag{2.19}$$

be a representation of μ_r on the vector space V. Let $\zeta \in \mu_r$ be the generator of this cyclic group and therefore a primitive rth root of unity.
Now choose a basis $e_1, \ldots, e_r$ of V such that the action of $\rho(\zeta)$ is given by

$$\begin{aligned} \rho(\zeta)e_i &= e_{i+1}, \quad 1 \leq i \leq r-1, \\ \rho(\zeta)e_r &= e_1. \end{aligned}$$

Then $\rho(\zeta)$ is given by the matrix

$$M = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in \mathrm{Mat}(r, \mathbb{Q}(\sqrt{D})), \tag{2.20}$$

with respect to the basis $e_1, \ldots, e_r$.
In the following we will construct a subspace, such that we can decompose the vector space V.
For this define the vector $v := e_1 + \cdots + e_r$. It holds that $\rho(\zeta)v = v$, i.e. the space $\langle v \rangle_{\mathbb{Q}(\sqrt{D})}$ is an μ_r-invariant 1-dimensional subspace of V where $\rho(\zeta)$ has eigenvalue 1 and eigenvector v.
We choose a basis for which we describe the representation explicitly. Let

$$b_1 := e_2 - e_1, \ldots, b_{r-1} := e_r - e_{r-1}. \tag{2.21}$$

Using the action of $\rho(\zeta)$ on the e_i we can state the action on the b_i:

$$\begin{aligned} \rho(\zeta)b_i &= b_{i+1}, \quad i = 1, \ldots, r-2, \\ \rho(\zeta)b_{r-1} &= -(b_1 + \cdots + b_{r-1}). \end{aligned}$$

The b_i define a $(r-1)$-dimensional representation U with the decomposition of V given by

$$V = \langle v \rangle_{\mathbb{Q}(\sqrt{D})} \oplus U, \tag{2.22}$$

as a $\mathbb{Z}/r\mathbb{Z}$-module. The matrix that represents $\rho(\zeta)|_U$ is denoted by M_U and is given with respect to the basis $\{b_i\}$ by

$$M_U = \begin{pmatrix} 0 & & & & -1 \\ 1 & \ddots & & & \vdots \\ & \ddots & \ddots & & \vdots \\ & & 1 & 0 & \vdots \\ & & & 1 & -1 \end{pmatrix} \in \mathrm{Mat}(r-1, \mathbb{Q}(\sqrt{D})). \tag{2.23}$$

Thus we can assume

$$M \sim_{\mathbb{Q}(\sqrt{D})} \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & M_U \end{array} \right). \tag{2.24}$$

The characteristic polynomial $\chi_M(x) = \pm(x^r - 1)$ of M has a zero at 1 and hence we can divide the linear factor corresponding to the eigenvalue 1 out and get the characteristic polynomial of M_U. Note that the sign of χ_M depends on whether r is even or odd.

$$\begin{aligned} \chi_{M_U}(x) &= \frac{\chi_M(x)}{\chi_{I_1}(x)} = \frac{\pm(x^r - 1)}{x - 1} \\ &= \pm(x^{r-1} + x^{r-2} + \cdots + x + 1). \end{aligned}$$

With this result it is obvious that

$$\chi_{M_U}(x) = \pm \prod_{i=1}^{r-1} (x - \zeta^i).$$

For the rationals one could already say that the representation U is irreducible if r is a prime by Lemma 2.3. In the case of quadratic number fields we get a weaker result.

Proposition 2.22. *If r is a prime and the quadratic number field $\mathbb{Q}(\sqrt{D})$ is not one of the list in Theorem 2.14, then the representation U is irreducible over $\mathbb{Q}(\sqrt{D})$.*

Proof. This follows from the irreducibility of $\chi_{M_U}(x) = \phi_r(x)$, which has been showed in Theorem 2.14. □

Since we want to give a description of the irreducible representations over number fields in a general setting, we have to use more theory.
Assume that $r \in \mathbb{N}$, $r \geq 2$ is arbitrary. For a fixed r there can be at most $2 \cdot \#\{d \in \mathbb{N};\ d \text{ divides } r,\ d \leq r\}$ irreducible factors of χ_M. This relies on the fact that χ_M can be written as

$$\begin{aligned} \chi_M(x) &= \pm(x^r - 1) \\ &= \pm \prod_{d|r} \phi_d(x), \end{aligned}$$

and each of the ϕ_d is irreducible or decomposes into two irreducible polynomials of degree $\frac{\varphi(d)}{2}$ depending on the field (cf. section 2.2).
We need not a bound on the factors but rather an exact number which we will define as

$$\mathrm{irr}_D(r) := \text{number of irreducible factors of } x^r - 1 \text{ over } \mathbb{Q}(\sqrt{D}).$$

As indicated this number depends on the choosen number field and r. Trivially

$$\#\{d \in \mathbb{N};\ d|r,\ d \leq r\} \leq \mathrm{irr}_D(r) \leq 2 \cdot \#\{d \in \mathbb{N};\ d|r,\ d \leq r\}.$$

This enables us to state a result how V decomposes as a $\mathbb{Z}/r\mathbb{Z}$-module for a given number field.

Proposition 2.23. *There exist $\mathrm{irr}_D(r)$ irreducible subrepresentations $V_1, \ldots, V_{\mathrm{irr}_D(r)}$ of $\mathbb{Z}/r\mathbb{Z}$ over $\mathbb{Q}(\sqrt{D})$ with the property*

$$V = \bigoplus_{i=1}^{\mathrm{irr}_D(r)} V_i. \tag{2.25}$$

Every V_i corresponds to exactly one irreducible factor of $x^r - 1$, i.e. the characteristic polynomial of $\rho(\zeta)|_{V_i}$ is one of these factors.

Proof. We know that M is the matrix that represents ρ on V with $\chi_M(x) = \pm \prod_{d|r} \phi_d(x)$, where ϕ_d is the dth cyclotomic polynomial. If ϕ_d is reducible over $\mathbb{Q}(\sqrt{D})$ replace it by $\phi'_d \cdot \phi''_d$, where ϕ'_d and ϕ''_d are irreducible of degree $\frac{\varphi(d)}{2}$. Hence we can write χ_M as a product of irreducible polynomials.
Now consider χ_M over the splitting field $\mathbb{Q}(\zeta_r)$, where ζ_r is a primitive rth root of unity. This can be done via

$$V_{\mathbb{Q}(\zeta_r)} := V \otimes_{\mathbb{Q}(\sqrt{D})} \mathbb{Q}(\zeta_r). \tag{2.26}$$

With respect to the cyclotomic field $\mathbb{Q}(\zeta_r)$ the polynomial χ_M decomposes into linear factors, namely

$$\chi_M(x) \quad = \quad (x-1) \cdot (x - \zeta_r) \cdots (x - \zeta_r^{r-1}). \tag{2.27}$$

The matrix M can be diagonalised over $\mathbb{Q}(\zeta_r)$, so there corresponds to each zero ζ_r^i, $0 \leq i \leq r-1$, of the characteristic polynomial χ_M a 1-dimensional subspace $\widetilde{U}_i$, with the property $\mathbb{Z}/r\mathbb{Z} \cdot \widetilde{U}_i \subset \widetilde{U}_i$. Each subspace $\widetilde{U}_i$ is a $\mathbb{Q}(\zeta_r)$-vector space by construction.
Now we collect the $\widetilde{U}_i$ for each irreducible factor of χ_M that are given by d with the property that d divides r. We have

(1)

$$\widetilde{V}_d \quad = \quad \bigoplus_{\substack{l|\phi_d \\ l=(x-\zeta_r^i)}} \widetilde{U}_i,$$

if ϕ_d is irreducible over $\mathbb{Q}(\sqrt{D})$, resp.

(2)

$$\begin{aligned} \widetilde{V}'_d \quad &= \quad \bigoplus_{\substack{l|\phi'_d \\ l=(x-\zeta_r^i)}} \widetilde{U}_i \\ \text{and } \widetilde{V}''_d \quad &= \quad \bigoplus_{\substack{l|\phi''_d \\ l=(x-\zeta_r^i)}} \widetilde{U}_i, \end{aligned}$$

if ϕ_d is reducible over $\mathbb{Q}(\sqrt{D})$.

The spaces $\widetilde{V}_d$, resp. $\widetilde{V}'_d$ and $\widetilde{V}''_d$, are $\varphi(d) = \deg \phi_d$-, resp. $\frac{\phi(d)}{2} = \deg \phi'_d = \deg \phi''_d$-dimensional $\mathbb{Q}(\zeta_r)$-vector spaces with the property

$$V_{\mathbb{Q}(\zeta_r)} \cong \bigoplus_{d|r} \widetilde{V}_d. \tag{2.28}$$

In this description we have to replace $\widetilde{V}_d$ in (2.28) by $\widetilde{V}_d' \oplus \widetilde{V}_d''$ for the reducible cyclotomic polynomials.
To get a $\mathbb{Z}/r\mathbb{Z}$-invariant $\mathbb{Q}(\sqrt{D})$-subspace of V, as requested, we have to define

$$V_d := \left(\widetilde{V}_d\right)^{\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q}(\sqrt{D}))} \tag{2.29}$$

$$:= \left\{v \in \widetilde{V}_d;\ \sigma(v) = v \text{ for all } \sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q}(\sqrt{D}))\right\}. \tag{2.30}$$

The Galois group acts on $V_{\mathbb{Q}(\zeta_r)}$ by operating on the coefficients and permuting the basis elements. Analogously we obtain V_d' and V_d'' when ϕ_d is reducible. Standard calculations show that these objects are $\mathbb{Q}(\sqrt{D})$-vector spaces.
By a result of Silverman [Sil86, Lemma II.5.8.1] it holds that

$$V_d \otimes_{\mathbb{Q}(\sqrt{D})} \mathbb{Q}(\zeta_r) \cong \widetilde{V}_d.$$

So the $\mathbb{Q}(\sqrt{D})$- and the $\mathbb{Q}(\zeta_r)$-dimension of the associated spaces coincide. The same is true for V_d' and V_d''.
Hence we get exactly $\mathrm{irr}_D(r)$ irreducible subrepresentations of $\mathbb{Z}/r\mathbb{Z}$ with the property (2.25). □

As mentioned in the proof, there is a correspondence of representations and irreducible polynomials. For later purposes we will make a definition.

Definition 2.24. Let ϕ_d be the dth cyclotomic polynomial and $\mathbb{Q}(\sqrt{D})$ a given number field.

(i) Let ϕ_d be irreducible over $\mathbb{Q}(\sqrt{D})$. Then we denote the corresponding representation by V_d.

(ii) Let $\phi_d = \phi_d' \cdot \phi_d''$ the decomposition into irreducible factors for $\mathbb{Q}(\sqrt{D})$. Then the representations corresponding to ϕ_d' resp. ϕ_d'' will be denoted by V_d' resp. V_d''.

In the proof of Proposition 2.23 we have constructed an irreducible representation to each irreducible factor of the characteristic polynomial χ_M. So it remains to show that this construction leads to all possible irreducible representations of the cyclic group μ_r, i.e. we must show that there are no other non-isomorphic irreducible represenations over the field $\mathbb{Q}(\sqrt{D})$.
For this we first need to define what is meant by isomorphic representations.

Definition 2.25. Let G be a group, V_1, V_2 be vector spaces and

$$\rho_1 :\ G \longrightarrow \mathrm{Aut}(V_1),\ \rho_2 :\ G \longrightarrow \mathrm{Aut}(V_2)$$

representations of G in V_1 resp. V_2. Then ρ_1 and ρ_2 are called *isomorphic representations*, if

(i) there exists a vector space isomorphism $\alpha : V_2 \longrightarrow V_1$, and

(ii) $\alpha^{-1} \circ \rho_1(g) \circ \alpha = \rho_2(g)$ holds for all $g \in G$, i.e. the diagram

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] & V_1 \arrow[d, "\alpha^{-1}"] \\ V_2 \arrow[u, "\alpha"] \arrow[r, "\rho_2(g)"] & V_2 \end{tikzcd} \tag{2.31}$$

commutes for all $g \in G$.

The following proposition shows that we already found all irreducible representations of μ_r. For this we introduce the corresponding group algebra and some other facts and tools from representation theory which we will not define formally. For details we refer the reader to some books which contain representation theory, e.g. [Lan02] or [CR62].

Proposition 2.26. *There are no more non-isomorphic irreducible representations over $\mathbb{Q}(\sqrt{D})$ than the $\mathrm{irr}_D(r)$ ones already stated in Proposition 2.23.*

Proof. We first will recall the notion of the group algebra. In this case the group algebra is defined as all formal sums

$$R := \mathbb{Q}(\sqrt{D})\mathbb{Z}/r\mathbb{Z} := \left\{ \sum_{\zeta \in \mathbb{Z}/r\mathbb{Z}} \alpha_\zeta \cdot \zeta;\ \alpha_\zeta \in \mathbb{Q}(\sqrt{D}) \right\}.$$

Since R is a group algebra it is semi-simple by Maschkes Theorem [Lan02, XVIII Theorem 1.2]. The dimension of R is

$$\dim_{\mathbb{Q}(\sqrt{D})} \mathbb{Q}(\sqrt{D})\mathbb{Z}/r\mathbb{Z} = |\mathbb{Z}/r\mathbb{Z}| = r.$$

Among the representations of the group algebra and the group representations there is a one-to-one correspondance, and the representations of R are the R-modules (cf. [CR62, §10]). From [Lan02, XVII Theorem 4.3 and 4.4] one knows that R decomposes into

$$R \cong \prod_{i=1}^{s} R_i,$$

for some s, where R_i represents the simple modules. The R-modules are isomorphic if and only if the corresponding representations are (cf. again [CR62, §10]). Hence the group algebra decomposes into all irreducible representations.
Indeed, the dimension of the group algebra is r and the dimensions of the representations from Proposition 2.23 add up to $|\mathbb{Z}/r\mathbb{Z}| = r$. As long as all representaions are non-isomorphic there is nothing more to prove.

It remains to show that the irreducible subrepresentations from the proposition are all non-isomorphic. If there are isomorphic subrepresentations, they must have the same characteristic polynomial by the definition of isomorphic representations. This can not happen.
So there are no more representations left and the proposition is proved. □

Thus we could state all irreducible subrepresentations. In the following chapters we do not have to mention all representations, but rather only the faithful ones, i.e. the injective group homomorphisms.

Lemma 2.27. *Let $\rho: \mathbb{Z}/r\mathbb{Z} \longrightarrow \mathrm{Aut}(V)$ be a representation of the cyclic group $\mathbb{Z}/r\mathbb{Z}$. Then the following holds:*

(i) Assume that ϕ_r is irreducible over $\mathbb{Q}(\sqrt{D})$. Then there exists a unique faithful irreducible representation V_r.

(ii) Assume $\phi_r = \phi'_r \cdot \phi''_r$. Then there are exactly two faithful irreducible representations, namely V'_r and V''_r.

Proof. Let $d \in \{d < r;\ d \text{ divides } r\}$ and ζ be a generator of $\mathbb{Z}/r\mathbb{Z}$. Now define $A_d := \rho|_{V_d}(\zeta)$, where the eigenvalues of A_d are the zeroes of ϕ_d.

(i) Using the fact that ρ is a homomorphism we get

$$\rho|_{V_d}(\zeta^d) = (\rho|_{V_d}(\zeta))^d = A_d^d, \tag{2.32}$$

and the eigenvalues of A_d^d on V_d are 1. Thus the represenatation $\rho|_{V_d}$ is not injective, i.e. not faithful. An analogous argument for V_r gives the result.

(ii) If the cyclotomic polynomial ϕ_d is reducible, we have to replace V_d by V'_d resp. V''_d. A similar approach proves (ii).

□

These results give rise to a complete charaterization of the irreducible representations of $\mathbb{Z}/r\mathbb{Z}$.

Theorem 2.28. *Let $\rho: \mathbb{Z}/r\mathbb{Z} \longrightarrow \mathrm{Aut}(V)$ be a representation of $\mathbb{Z}/r\mathbb{Z}$ on the vector space V over $\mathbb{Q}(\sqrt{D})$. Then*

(i) there is a unique irreducible faithful representation V_r if ϕ_r is irreducible. The eigenvalues of $\rho|_{V_d}(\zeta_r)$ are the primitive r-th roots of unity.

(ii) there are two irreducible faithful representations V'_r, V''_r if ϕ_r is reducible. The eigenvalues of $\rho|_{V'_d}(\zeta_r)$ are the primitive rth roots of unity ζ_r^a with $\left(\frac{D}{a}\right) = 1$ for $a \in (\mathbb{Z}/r\mathbb{Z})^*$. The eigenvalues of $\rho|_{V''_d}(\zeta_r)$ are ζ_r^a for the remaining $a \in (\mathbb{Z}/r\mathbb{Z})^*$, i.e. the a with $\left(\frac{D}{a}\right) = -1$.
Restricting to specific D's make this more precise:

(a) If $D > 0$, then for each eigenvalue ζ_r^a of $\rho|_{V_d'}(\zeta_r)$ the complex conjugate ζ_r^{r-a} is an eigenvalue as well.

(b) If $D < 0$, then for each eigenvalue ζ_r^a of $\rho|_{V_d'}(\zeta_r)$ the complex conjugate ζ_r^{r-a} is an eigenvalue of $\rho|_{V_d''}(\zeta_r)$.

Proof. This follows directly from Theorem 2.14, the Propositions 2.20, 2.23 and Lemma 2.27. □

The cases (a) resp. (b) in the theorem lead to real resp. imaginary quadratic number fields.

Chapter 3

Ball quotients and singularities

In this chapter we will first introduce ball quotients and their automorphism group. Then we will study the representations that can occur and their contribution to the Reid-Tai sum. This will lead to results on canonical singularities for ball quotients. We will state a general result and restrict to a special case as this could not be covered by the general argument.
In this chapter we will use techniques introduced by S. Kondo [Kon93] and enhanced by V.A. Gritsenko, K. Hulek and G.K. Sankaran [GHS07].

3.1 Ball quotients

First we choose $\mathbb{Q}(\sqrt{D})$ to be an imaginary quadratic number field, where $D \in \mathbb{Z}$ is a squarefree integer with $D < 0$. For this quadratic number field we define

$$\mathcal{O} := \mathcal{O}_{\mathbb{Q}(\sqrt{D})}$$

to be its ring of integers.
A classical result from algebraic number theory gives

$$\mathcal{O} = \mathcal{O}_{\mathbb{Q}(\sqrt{D})} \quad = \quad \begin{cases} \mathbb{Z}[\sqrt{D}], & \text{if } D \equiv 2, 3 \mod 4, \text{ or} \\ \mathbb{Z}\left[\frac{1}{2}\left(1+\sqrt{D}\right)\right], & \text{if } D \equiv 1 \mod 4. \end{cases}$$

For later arguments we have to start with an lattice.

Definition 3.1. We will denote by Λ an $\mathcal{O}$-lattice of signature $(n, 1)$. The hermitian form given by this lattice will be denoted by $h(\cdot, \cdot)$.

Instead of a lattice we can speak of Λ as a free $\mathcal{O}$-module of rank $n+1$ with a hermitian form of signature $(n, 1)$, i.e. $\Lambda \cong \mathcal{O}^{n,1}$ where the exponent indicates the signature of the form.
From now on we fix Λ.

Over the complex numbers we can give an identification in terms of matrices. Therefore first define

$$I_{p,q} := \left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array} \right), \tag{3.1}$$

where I_i denotes the $i \times i$-identity matrix. Also note that we will denote ${}^H A :={}^T \overline{A}$ for an arbitrary matrix A.

Remark 3.2. (i) When we fix a basis we get an isomorphism

$$\psi : \Lambda \otimes_{\mathcal{O}} \mathbb{C} \cong \mathbb{C}^{n,1},$$

where $\mathbb{C}^{n,1}$ denotes the pair $(\mathbb{C}^{n+1}, \text{form of signature } (n,1))$. Therefore $\Lambda \otimes_{\mathcal{O}} \mathbb{C}$ could be regarded as a $(n+1)$-dimensional $\mathbb{C}$-vector space.

(ii) We will also denote the induced form on $\Lambda \otimes_{\mathcal{O}} \mathbb{C}$ by $h(\cdot, \cdot)$.

(iii) We can choose a basis of $\mathbb{C}^{n+1}$, such that the form $h(\cdot, \cdot)$ is given by $I_{n,1}$, i.e.

$$\begin{aligned} h(x,y) &= x_1\overline{y}_1 + \cdots + x_n\overline{y}_n - x_{n+1}\overline{y}_{n+1} \\ &= {}^H y I_{n,1} x. \end{aligned}$$

With this notations we can introduce the main object to study.

Definition 3.3. We call

$$\mathbb{C}H^n := \{[\omega] \in \mathbb{P}(\Lambda \otimes_{\mathcal{O}} \mathbb{C});\ h(\omega,\omega) < 0\}. \tag{3.2}$$

the *complex hyperbolic space of dimension* n.

Therefore $\mathbb{C}H^n$ can be regarded as an open subset of the complex projective n-space $\mathbb{P}^n_{\mathbb{C}}$. We can also see by definition, that $\mathbb{C}H^n$ has an natural underlying lattice structure given by Λ.

Remark 3.4. Note that

$$U(n,1) := \left\{A \in \mathrm{GL}(n+1,\mathbb{C});\ {}^H A I_{n,1} A = I_{n,1}\right\}$$

is the *unitary group of signature* $(n,1)$.

There are some well-known identifications of the space $\mathbb{C}H^n$.

Proposition 3.5.

$$\begin{aligned} \mathbb{C}H^n &\cong B^n_{\mathbb{C}} := \{z \in \mathbb{C}^n;\ |z| = z_1\overline{z_1} + \cdots + z_n\overline{z_n} < 1\} && (3.3) \\ &= \mathcal{H}_{n,1} := \left\{Z \in \mathrm{Mat}(n,1;\mathbb{C});\ {}^H Z Z - I_1 < 0\right\} \\ &\cong U(n,1)/(U(n) \times U(1)). && (3.4) \end{aligned}$$

Proof. The identification (3.3) follows directly from the definitions of these spaces, while the isomorphism (3.4) was shown by Shimura (cf. [Shi63]). We sketch Shimuras proof:
An element $\left(\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right) := U \in U(n,1)$ acts holomorphically on $z \in B^n_{\mathbb{C}}$ via $U(z) := \dfrac{Az+b}{cz+d}$. Define the map

$$\psi: \; U(n,1) \longrightarrow B^n_{\mathbb{C}}$$

by $\psi(U) := U(0) = \dfrac{b}{d}$.
It is easy to show that ψ is surjective. Since ψ is not injective we compute the kernel of this map. Easy calculations yield that

$$U(n,1)_0 := \ker(\psi) = U(n) \times U(1).$$

Hence the map $U(n,1)/(U(n)\times U(1)) \longrightarrow B^n_{\mathbb{C}}$ is bijective and the result follows. $\square$

Remark 3.6. One can generalize this identification to

$$\begin{aligned} \mathcal{H}_{p,q} &:= \left\{ Z \in \mathrm{Mat}(p,q;\mathbb{C});\; {}^{H}ZZ - I_q < 0 \right\} \\ &\cong U(p,q)/(U(p)\times U(q)). \end{aligned}$$

For details see G. Shimura [Shi63].

For the following we will consider the automorphism group of the lattice Λ.

Definition 3.7.

$$U(\Lambda) := \text{group of automorphisms of } \Lambda.$$

As we did before we can consider the corresponding group for the induced complex vector space.

Remark 3.8. It holds for a suitable choice of a basis

$$U(\Lambda)_{\mathbb{C}} := U(\Lambda) \otimes_{\mathcal{O}} \mathbb{C} \cong U(n,1). \tag{3.5}$$

So far we introduced the complex hyperbolic space and the automorphism group of the lattice. Now we can define quotients for suitable subgroups.

Definition 3.9. Let $\Gamma < U(\Lambda)$ be a subgroup of finite index. We define the n-dimensional *ball quotient*

$$\Gamma \backslash \mathbb{C}H^n \tag{3.6}$$

as the space of orbits.

As $\mathbb{C}H^n$ can be represented as a ball (cf. (3.3)) it makes sense to speak of (3.6) as a ball quotient. This turns out to be a quasi-projective variety by [BB66]. One can give a description of the ramification divisors. Let

$$f_\Gamma : \mathbb{C}H^n \longrightarrow \Gamma\backslash\mathbb{C}H^n$$

be the quotient map for Γ. The elements fixing a divisor in $\mathbb{C}H^n$ are the quasi-reflections. Therefore the ramification divisors of f_Γ are the fixed loci of elements of Γ acting as quasi-reflections.

3.2 A local description

In section 3.1 we considered some results for the complex ball resp. complex hyperbolic space. From now on we restrict to the local situation as we want to give results about canonical singularities. Therefore we will first define the fixgroup of a point in $\mathbb{C}H^n$ and study its action on the tangent space. Some of the proofs we give are similar to those of [GHS07, 2.1].
As before Γ is of finite index in $U(\Lambda)$. Now choose a point $[\omega] \in \mathbb{C}H^n$.

Definition 3.10. Let

$$G := \Gamma_{[\omega]} \quad := \quad \{g \in \Gamma;\ g[\omega] = [\omega]\} \tag{3.7}$$

be the fixgroup of $[\omega]$.

The group G is finite by [Hol98, 4.1.2] or [Shi71, pp. 1].
In the following we will define some sublattices. To construct these sublattices we need a specific complex line corresponding to $[\omega]$.
For this let $\omega \in \Lambda \otimes_{\mathcal{O}} \mathbb{C}$ be a representative and define the line though the point ω to be

$$\mathbb{W} := \mathbb{C} \cdot \omega. \tag{3.8}$$

Now we can define the following lattices.

Definition 3.11.

$$S := \mathbb{W}^\perp \cap \Lambda \text{ and } T := S^\perp \cap \Lambda, \tag{3.9}$$

with respect to the form $h(\cdot,\cdot)$.

These are sublattices of the lattice Λ.
To be in the complex setting we can make the following definitions for S and for T, similar to the complexification of Λ in the previous section.

$$S_{\mathbb{C}} := S \otimes_{\mathcal{O}} \mathbb{C} \text{ and } T_{\mathbb{C}} := T \otimes_{\mathcal{O}} \mathbb{C}. \tag{3.10}$$

First we want to prove that the intersection of the subspaces defined in (3.10) is trivial. This will be used to give a proof of one of the lemmas below.

Lemma 3.12. *The only point the two lattices have in common is the origin, i.e.*

$$S_{\mathbb{C}} \cap T_{\mathbb{C}} = \{0\}.$$

Proof. Let $x \in S_{\mathbb{C}} \cap T_{\mathbb{C}}$. Then one has $h(x,x) = 0$, since $x \in T_{\mathbb{C}} = S_{\mathbb{C}}^{\perp}$. Therefore it remains to show that the hermitian form is positive definite on $S_{\mathbb{C}}$. Now consider the complex line $\mathbb{W} \subset \Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathcal{O}} \mathbb{C}$. The space $\mathbb{W}$ is 1-dimensional and has $\{\omega\}$ as a $\mathbb{C}$-basis with $h(\omega,\omega) < 0$, since $[\omega]$ lies in $\mathbb{C}H^n$. Hence the hermitian form has signature $(0,1)$ on $\mathbb{W}$ and this implies that its signature on $\mathbb{W}^{\perp}$ is $(n,0)$. By definition one has $S_{\mathbb{C}} \subset \mathbb{W}^{\perp}$ and because of that $h(\cdot,\cdot)$ is positive definite on $S_{\mathbb{C}}$. □

In the following we will study the action of the fixgroup G. Therefore we have to prove that there is an action on the sublattices defined above.

Lemma 3.13. *The fixgroup G of $[\omega]$ acts on S and T.*

Proof. G acts on $\mathbb{W}$ and on Λ, hence on $S = \mathbb{W}^{\perp} \cap \Lambda$ and on $T = S^{\perp} \cap \Lambda$. □

As we proved that G acts on the sublattices and, by definition, on the whole space we will formalize this action. We defined G as the stabilzer of the projective point $[\omega]$ and therefore we have for a representative ω of $[\omega]$ the equation

$$g(\omega) = \alpha(g)\omega, \tag{3.11}$$

where the map

$$\alpha : G \longrightarrow \mathbb{C}^*$$

is a group homomorphism. As we have to use it for the arguments following we will denote the kernel of this map as

$$G_0 := \ker \alpha.$$

We now prove that the spaces $S_{\mathbb{C}}$ and $T_{\mathbb{C}}$ are closed under the action of the group G.

Lemma 3.14. *The spaces $S_{\mathbb{C}}$ and $T_{\mathbb{C}}$ are G-invariant subspaces of the vector space $\Lambda_{\mathbb{C}}$.*

Proof. Let $x \in T_{\mathbb{C}}$, $y \in S_{\mathbb{C}}$, $\omega \in \mathbb{W}$ and $g \in G$. We will give seperated proofs for these statements.

(1) First prove the statement for $S_{\mathbb{C}}$.

$$0 = h(y,\omega) = h(g(y), g(\omega)) = \overline{\alpha(g)} \cdot h(g(y),\omega). \tag{3.12}$$

As $\alpha(g) \neq 0$ we get $h(g(y),\omega) = 0$, i.e. $g(y) \in S_{\mathbb{C}}$.

(2) A similar statement holds for $T_{\mathbb{C}}$:

$$0 = h(x, y) = h(g(x), g(y)), \tag{3.13}$$

with $g(y) \in S_{\mathbb{C}}$ as shown in (1). Hence we have $g(x) \in S_{\mathbb{C}}^{\perp} = T_{\mathbb{C}}$.

$\square$

As G_0 is a subgroup of G it acts on S and T by Lemma 3.13. When we investigate this it turns out that the action of G_0 is special for T. Analogous to the complex case we define $S_{\mathbb{Q}(\sqrt{D})} := S \otimes_{\mathcal{O}} \mathbb{Q}(\sqrt{D})$ and $T_{\mathbb{Q}(\sqrt{D})} := T \otimes_{\mathcal{O}} \mathbb{Q}(\sqrt{D})$.

Lemma 3.15. *The group G_0 acts trivially on $T_{\mathbb{Q}(\sqrt{D})}$.*

Proof. Let $x \in T_{\mathbb{Q}(\sqrt{D})}$ and $g \in G_0$. Then

$$h(\omega, x) = h(g(\omega), g(x)) = h(\omega, g(x)).$$

Thus we have $T_{\mathbb{Q}(\sqrt{D})} \ni x - g(x) \in \mathbb{W}^{\perp} \cap \Lambda_{\mathbb{Q}(\sqrt{D})} = S_{\mathbb{Q}(\sqrt{D})}$. So by Lemma 3.12 we get $g(x) - x = 0$. $\square$

By (3.11) the quotient G/G_0 is a subgroup of $\operatorname{Aut} \mathbb{W}$ which is isomorphic to $\mathbb{C}^*$. Thus G/G_0 is cyclic.

Definition 3.16. The order of G/G_0 is defined as

$$r_\omega := \operatorname{ord}(G/G_0).$$

Thus we can identify G/G_0 with $\mathbb{Z}/r_\omega\mathbb{Z}$.
The subspace $T_{\mathbb{Q}(\sqrt{D})}$ splits as a module into a direct sum of $\mathbb{Q}(\sqrt{D})$-irreducible representations as stated in Theorem 2.28. For the notation of the representations we refer to chapter 2.

Lemma 3.17. *The space $T_{\mathbb{Q}(\sqrt{D})}$ decomposes as a G/G_0-module*

(i) into a direct sum of V_{r_ω}'s, i.e. $\varphi(r_\omega)$ divides $\dim T_{\mathbb{Q}(\sqrt{D})}$, if V_{r_ω} is irreducible over $\mathbb{Q}(\sqrt{D})$,

(ii) into a direct sum of V'_{r_ω}'s and V''_{r_ω}'s, in particular $\frac{\varphi(r)}{2}$ divides $\dim T_{\mathbb{Q}(\sqrt{D})}$, if there exist a decomposition $V_{r_\omega} = V'_{r_\omega} \oplus V''_{r_\omega}$ over $\mathbb{Q}(\sqrt{D})$.

Proof. As $G/G_0 \cong \mu_{r_\omega}$ and by the chinese remainder theorem $(\mathbb{Z}/r_\omega\mathbb{Z})^* \cong ((\mathbb{Z}/p_1\mathbb{Z})^*)^{a_1} \times \cdots \times ((\mathbb{Z}/p_t\mathbb{Z})^*)^{a_t}$ for suitable p_i and a_i. It remains to show, that the only element having 1 as an eigenvalue on $T_{\mathbb{C}}$ could be the identity element in G/G_0. Assume that $g \in G - G_0$ with $g(x) = x$ for a $x \in T_{\mathbb{C}}$. Then

$$h(\omega, x) = h(g(\omega), g(x)) = \alpha(g) \cdot h(\omega, x).$$

As we have $a(g) \neq 1$ by the choice of g we get $h(\omega, x) = 0$ and therefore $x \in S_{\mathbb{C}} \cap T_{\mathbb{C}} = \{0\}$. $\square$

If we consider the action of an element $g \in G$ we can state a similar result on the decomposition. Henceforth we will denote the order of $\alpha(g)$ by r.

Corollary 3.18. *For $g \in G$ the space $T_{\mathbb{Q}(\sqrt{D})}$ decomposes as a g-module into a direct sum of V_r's resp. V_r''s or V_r'''s of dimension $\varphi(r)$ resp. $\frac{\varphi(r)}{2}$.*

Proof. The same as in Lemma 3.17. □

3.2.1 Tangent space

Instead of studying this quotient globally we will restrict ourself to the action of the stabiliser subgroup on the tangent space $T_{[\omega]}\mathbb{C}H^n$.
As we will study the action of G on the tangent space of $\mathbb{C}H^n$ we need a description of $T_{[\omega]}\mathbb{C}H^n$ that enables us to calculate things.

Lemma 3.19. *The tangent space of $\mathbb{C}H^n$ at a point $[\omega]$ is given by*

$$\mathrm{Hom}(\mathbb{W}, \mathbb{C}^{n+1}/\mathbb{W}).$$

Proof. The space $\mathbb{C}H^n$ is an open subset of the Grassmannian variety $G(1, n+1)$ of 1-dimensional subspaces in $(n+1)$-dimensional complex vector space $\mathbb{C}^{n+1}$. Thus the tangent spaces of the the complex hyperbolic space and the Grassmannian coincide in $[\omega]$. With a result of [ACGH85, Chapter II §2] we get

$$T_{[\omega]}\mathbb{C}H^n = T_{[\omega]}G(1, n+1) \cong \mathrm{Hom}(\mathbb{W}, \mathbb{C}^{n+1}/\mathbb{W}). \tag{3.14}$$

□

As we will refer to this description of the tangent space we will denote it by

$$V := \mathrm{Hom}(\mathbb{W}, \mathbb{C}^{n+1}/\mathbb{W}). \tag{3.15}$$

Hence in the following we will investigate the quotient

$$G \backslash V$$

instead of $\Gamma \backslash \mathbb{C}H^n$. Here V as above and $G = \Gamma_{[\omega]}$ the stabiliser subgroup as already defined before.

Remark 3.20. Note that we can write $V = \mathbb{W}^\vee \otimes (\mathbb{C}^{n+1}/\mathbb{W})$.

Now let $g \in G$ be of order m. Then we can consider the eigenvalues

$$\zeta^{a_1}, \ldots, \zeta^{a_n}$$

of g on the tangent space V, where ζ denotes a primitive mth root of unity and $0 \leq a_i < m$. Now we are in the set-up of section 1.1.2 and can therefore apply the Reid-Tai criterion which essentially says that the Reid-Tai sum has to fulfill

$$\Sigma(g) := \sum_{i=1}^{n} \frac{a_i}{m} \geq 1$$

for every $g \in G$ for V/G to have canonical singularities, as long as G does not contain any quasi-reflections. For a detailed argument see Theorem 1.16.

3.3 A first result

The discussion in 3.2.1 enables us to state some results on canonical singularities of $G\backslash V$. For this we will calculate the contribution of certain irreducible representations to the Reid-Tai sum.
As we will see we can not give a result for general $D < 0$ but for $D < -3$. Therefore we define $D_0 := -3$.
First we will give a bound on r as $\alpha(g)$ is the eigenvalue from g on $\mathbb{W}$. This bound will be independent of n. For the action of g on the tangent space V there occurs $\alpha(g)^{-1}$ because of the dual of $\mathbb{W}$ by Remark 3.20. As already defined the order of $\alpha(g)$ is denoted by r.
For the following arguments we will denote the fractional part of a rational number q by $\{q\}$.

Lemma 3.21. *Suppose $g \in G$ does not act as a quasi-reflection on V. Then the Reid-Tai sum fulfills $\Sigma(g) \geq 1$, if*

(i) $\varphi(r) \geq 10$ for all $D < 0$,

(ii) $\varphi(r) = 4$ for $D < D_0$.

Proof. Let $\mathbb{V}_r^\omega$ be the copy of $V_r \otimes \mathbb{C}$ resp. $V_r' \otimes \mathbb{C}$ or $V_r'' \otimes \mathbb{C}$ containing ω. For a fixed primitive mth root of unity ζ we get a primitive rth root of unity when we consider $\zeta^{\frac{m}{r}}$.
Let $0 \leq k_i < r$ be the distinct numbers coprime to r. We will denote the set of all k_i with this property by $A_r := \{k_i;\ i = 1, \ldots, \varphi(r)\}$, where $\#A_r = \varphi(r)$. Let $\zeta^{\frac{mk_1}{r}}$ be the eigenvalue of g on $\mathbb{W}$, i.e. $\alpha(g) = \zeta^{\frac{mk_1}{r}}$. On the dual space $\mathbb{W}^\vee$ we will get the eigenvalue $\overline{\alpha(g)} = \overline{\zeta^{\frac{mk_1}{r}}} =: \zeta^{\frac{mk_2}{r}}$.
Now on the space $\mathbb{V}_r^\omega$ the element g will have the eigenvalues $\zeta^{\frac{mk_i}{r}}$ for $k_i \in A_r$ resp. in the case where ϕ_r is reducible over $\mathbb{Q}(\sqrt{D})$, as in Chapter 2, we have $k_i \in A \subset A_r$ for a suitable A with $k_1 \in A$ and $\#A = \frac{\varphi(r)}{2}$ (cf. Theorem 2.28). This A depends on D. Thus we will have eigenvalues $\zeta^{\frac{mk_i}{r}}$ for $k_i \in A_r - \{k_1\}$ resp. $k_i \in A - \{k_1\}$ on $V_r^\omega \cap \mathbb{C}^{n+1}/\mathbb{W}$. However, as $k_1 \in A$ we know $k_2 \notin A$, since

$\mathbb{V}_r^\omega$ is the subspace containing ω.
On $\mathrm{Hom}(\mathbb{W}, \mathbb{V}_r^\omega \cap \mathbb{C}^{n+1}/\mathbb{W}) \subset V$ the element g has eigenvalues $\zeta^{\frac{mk_2}{r}}\zeta^{\frac{mk_i}{r}}$ for $k_i \in A_r - \{k_1\}$ resp. $k_i \in A - \{k_1\}$. As we are interested in $\Sigma(g)$ we will get with the estimation

$$\Sigma(g) \;\geq\; \begin{cases} \sum_{k_i \in A_r - \{k_1\}} \left\{\dfrac{k_2+k_i}{r}\right\}, & \text{resp.} \\ \sum_{k_i \in A - \{k_1\}} \left\{\dfrac{k_2+k_i}{r}\right\}. & \end{cases} \tag{3.16}$$

Each summand will give rise to non-zero contribution to $\Sigma(g)$, as the only such summand would be $\left\{\dfrac{k_1+k_2}{r}\right\}$. We now give a finiteness result on r.

(i) We can give a very coarse approximation of the sums (3.16) by

$$\sum_{j=1}^{\frac{\varphi(r)}{2}-1} \frac{j}{r} = \frac{\left(\frac{\varphi(r)}{2}-1\right)\cdot\left(\frac{\varphi(r)}{2}\right)}{2r} \tag{3.17}$$

$$= \frac{\left(\frac{\varphi(r)^2}{4}-\frac{\varphi(r)}{2}\right)}{2r}$$

$$= \frac{r(p_1-1)^2\cdots(p_s-1)^2}{8p_1\cdots p_s} - \underbrace{\frac{(p_1-1)\cdots(p_s-1)}{4p_1\cdots p_s}}_{\leq\frac{1}{4}}, \tag{3.18}$$

with $r = p_1^{a_1}\cdots p_s^{a_s}$. This expression has a 'monotonicity' property and we can calculate all possibilities for p_i and a_i that do not contribute 1. Therefore (3.18) is greater or equal to 1, if

(a) $s \geq 4$,

(b) $s \geq 3$ unless $r = 2^a \cdot p^b \cdot q$ (assume $p < q$) and

a	b	p	q
<3	1	3	<11
1	1	3	11
1	1	3	13
1	2	3	5
1	1	5	7

(c) $r = p^a q^b$ unless

a	b	p	q
1	1	2	≤ 19
1	1	3	5,7
2	1	2	<11
2,3	2	2	3
1	2	2	5
3	1	2	<7
4	1	2	3
2	1	3	2
3	1	3	2

(d) $r = p^a$ unless

a	p
1	<11
2	3
≤ 5	2

So there are only finitely many cases left which we will study in more detail. For these cases we will define an expression to calculate the minimal contribution $\mathrm{mc}(r)$ of g on $\mathrm{Hom}(\mathbb{W}, \mathbb{V}_r^\omega \cap \mathbb{C}^{n+1}/\mathbb{W})$. This is done by

$$\mathrm{mc}(r) \quad := \quad \min_{\substack{D<0 \\ \text{suitable}}} \; \min_{k_2 \in A_r} \sum_{\substack{k_i \in A_r - \{k_1\} \\ \left(\frac{D}{k_i}\right) = \left(\frac{D}{r-k_2}\right)}} \left\{ \frac{k_2 + k_i}{r} \right\}. \tag{3.19}$$

By 'suitable' we mean that we only consider number fields that lead to reducibility of the representation. This expression calculates the contribution from all possible irreducible representations V_r' and V_r'' that can occur.
If V_r is irreducible for all $D < 0$ one has to omit the first 'min' and the Kronecker symbols in (3.19).
We only have to calculate $\mathrm{mc}(r)$ for the remaining r with $\varphi(r) \geq 10$. Computer calculation yields that for these r we get $\mathrm{mc}(r) \geq 1$. This can be done by a computer as there are, for each r, only finitely many well-known possibilities for a 'suitable' D. This number fields can be found by Theorem 2.14. Thus we proved (i).

(ii) Now we have to consider $r = 5, 8, 10, 12$. The corresponding representations are irreducible by the assumption on D. So we are in the first case of (3.16) and may write

$$\sum_{i=2}^{4} \left\{ \frac{k_2 + k_i}{r} \right\} \quad = \quad \left\{ \frac{2k_2}{r} \right\} + \left\{ \frac{k_2 + k_3}{r} \right\} + \left\{ \frac{k_2 + k_4}{r} \right\}. \tag{3.20}$$

Calculating this for all values of r and all possibilities of k_2, k_3 and k_4 produces a contribution of at least 1 to the Reid-Tai sum.

□

Calculating explicit values of some minimal contributions implies even more.

Remark 3.22. The same calculations of $\mathrm{mc}(r)$ shows that $\Sigma(g) \geq 1$ for $r = 9, 16, 18$ and no restriction on $D < 0$.

As we will always deal with irreducible representations and therefore have to switch beween V_d, V_d' and V_d'' we will make a definition that makes arguments shorter.

Definition 3.23. Let d be a positive integer.

(i) We will always denote by $\mathcal{V}_d$ the 'right' irreducible representation over $\mathbb{Q}(\sqrt{D})$, i.e. a copy of V_d resp. V_d' or V_d''.

(ii) Similarly to (i) we will define

$$\mathbb{V}_d := \mathcal{V}_d \otimes_{\mathbb{Q}(\sqrt{D})} \mathbb{C}.$$

By the 'right' representation we mean that there is always the choice to take V_d' or V_d'' in the reducible case. We will not specify this choice anymore and only refer to $\mathcal{V}_d$.
We will prove a result for $\varphi(r) = 1$. Here we have to restrict to the case $D \leq D_0$.

Lemma 3.24. *Assume that $g \in G$ does not act as a quasi-reflection on the tangent space V. Additionally let $r = 1, 2$ and $D \neq -1, -2$. Then $\Sigma(g) \geq 1$.*

Proof. As $r = 1, 2$ we have $\alpha(g) = \pm 1$. With an analogous statement as in [GHS07, Proposition 2.9] we get that g is not of order 2 and g^2 acts trivially on $T_{\mathbb{C}}$ but not on $S_{\mathbb{C}}$. Therefore let g act on the subspace $\mathrm{Hom}(\mathbb{W}, \mathbb{V}_d) \subset V$ as $\pm\mathcal{V}_d$ with $d > 2$, for a representation $\mathcal{V}_d$ from the decomposition of $S_{\mathbb{C}}$ as a g-module over $\mathbb{Q}(\sqrt{D})$. The contribution from this subspace to $\Sigma(g)$ is at least

$$\min_{\substack{D<0\\ \text{suitable}}} \min_{r\in\{1,2\}} \min_{\alpha=\pm 1} \sum_{\substack{(k_i,d)=1\\ \left(\frac{D}{k_i}\right)=\alpha}} \left\{ \frac{1}{r} + \frac{k_i}{d} \right\} \geq \sum_{j=1}^{\frac{\varphi(d)}{2}} \frac{j}{d}. \tag{3.21}$$

Again one has to modify this expression if $\mathcal{V}_d = V_d$.
One sees that the right hand side of this inequality is similar to the estimation in the proof of Lemma 3.21, so we achieve for $d = p_1^{a_1} \cdots p_s^{a_s}$ and

(a) $s \geq 4$,

(b) $s \geq 3$ unless $d = 2^a \cdot p^b \cdot q$ (assume $p < q$) and

a	b	p	q
<3	1	3	<11
1	1	3	11
1	2	3	5

(c) $d = p^a q^b$ unless

a	b	p	q
1	1	2	<17
1	1	3	5
2	1	2	<11
2	2	2	3
3	1	2	<7
4	1	2	3
2	1	3	2
3	1	3	2

(d) $d = p^a$ unless

a	p
1	<11
2	3
<5	2

a contribution of at least 1 to $\Sigma(g)$. For the remaining d we have to make a better estimation by calculating the left side of (3.21).
The only value of d for that the expression is less than 1 is $8 = 2^3$. But as it holds $\mathcal{V}_8 = V_8$ for $D \neq -1, -2$ we can choose conjugate eigenvalues ζ_8 and $\overline{\zeta}_8$ which add up to 1. $\square$

So far we only gave bounds according to r, the order of $\alpha(g)$. There we proved that we only have to deal with finitely many r's in the following.
Now we can state a theorem that leads to canonical singularities for general r.

Theorem 3.25. *Let $g \in G$ do not act as a quasi-reflection on V. If*

(i) $n \geq 11$ and $D < D_0$, then $\Sigma(g) \geq 1$.

(ii) $n \geq 7$ and $D < -15$, then $\Sigma(g) \geq 1$.

Proof. Let m be the order of g, choose ζ to be a primitive mth root of unity and $\mathbb{V}_d := V_d \otimes \mathbb{C}$ or $V_d' \otimes \mathbb{C}$ resp. $V_d'' \otimes \mathbb{C}$ as in Definition 3.23. On the space $\mathrm{Hom}(\mathbb{W}, \mathbb{V}_d) \subset V$ the element g has eigenvalues $\zeta^{\frac{mc}{r}} \zeta^{\frac{mk_i}{d}}$ for fixed $0 < c < r$ with $(c, r) = 1$ and the number of $0 < k_i < d$ coprime to d that occur as an exponent

is $\dim_{\mathbb{C}} \mathbb{V}_d$.
So the contribution of g on this subspace to the Reid-Tai sum is given by

$$\begin{aligned}
&\sum_{i=1}^{\varphi(d)} \left\{ \frac{c}{r} + \frac{k_i}{d} \right\}, \qquad \text{if } \dim_{\mathbb{C}} \mathbb{V}_d = \varphi(d), \text{ resp.} \\
&\sum_{k_i \in A} \left\{ \frac{c}{r} + \frac{k_i}{d} \right\}, \qquad \text{if } \dim_{\mathbb{C}} \mathbb{V}_d = \frac{\varphi(d)}{2},
\end{aligned}$$

for a suitable A as in the proof of Lemma 3.21 depending on the number filed $\mathbb{Q}(\sqrt{D})$, with $\#A = \frac{\varphi(d)}{2}$.
Each of these sums is greater or equal to $\sum_{j=1}^{\frac{\varphi(d)}{2}} \dfrac{j}{d}$ if we exclude $d \in \varphi^{-1}(\{2,4,6,8\})$.
Thus we know, that all d which contribute at least 1 to $\Sigma(g)$ are the d not mentioned in the list of the proof of Lemma 3.24.
Hence there are only finitely many d left. The remaining d, which has to be investigated by a more exact argument, are

$$\begin{aligned}
d \quad = \quad & 1, 2, \ldots, 10, 12, 14, 15, 16, 18, 20, 22, 24, 26, 28, 30, 36, \\
& 40, 42, 48, 54, 60, 66, 84, 90. \qquad (3.22)
\end{aligned}$$

Now we could calculate the contributions for each choice of d and each choice of r. But as this not feasible we define

$$c_{\min}(d) \quad := \quad \min_{0 \le a < d} \sum_{\substack{0<b<d \\ (b,d)=1}} \left\{ \frac{b+a}{d} \right\}, \text{ resp.} \qquad (3.23)$$

$$c^{\text{red}}_{\min}(d) \quad := \quad \min_{\substack{D<0 \\ \text{suitable}}} \min_{\alpha=\pm 1} \min_{0 \le a < d} \sum_{\substack{0<b<d \\ (b,d)=1 \\ \left(\frac{D}{b}\right)=\alpha}} \left\{ \frac{b+a}{d} \right\}. \qquad (3.24)$$

By this choice we mean that if there exist at least one imaginary quadratic number field for which the representaition V_d decomposes we have to calculate $c^{\text{red}}_{\min}(d)$ for those D. If there exists no such D we will use $c_{\min}(d)$.
Both expressions only depend on d by definition and are a lower bound for the contribution to $\Sigma(g)$. This was shown in [GHS07, Proof of Theorem 2.10]. By obvious reasons $c_{\min}(d) \ge c^{\text{red}}_{\min}(d)$ for all d.
Nevertheless we have to take the minimum over all such D. To avoid this first we will define a less exact argument $\tilde{c}^{\text{red}}_{\min}(d)$ with the property

$$c^{\text{red}}_{\min}(d) \quad \ge \quad \tilde{c}^{\text{red}}_{\min}(d) := \min_{0 \le a < d} \sum_{\substack{0<b\le \lfloor \frac{d}{2} \rfloor \\ (b,d)=1}} \left\{ \frac{b+a}{d} \right\}. \qquad (3.25)$$

This enables us to reduce the list of d's given by (3.22) using $\tilde{c}^{\mathrm{red}}_{\min}(d)$. So the only values for d not cotributing 1 are

$$d = 1, 2, \ldots 10, 12, 14, 15, 16, 18, 20, 22, 24, 30.$$

For the remaining d it is worth calculating $c_{\min}(d)$ resp. $c^{\mathrm{red}}_{\min}(d)$ if V_d is reducible for one $D < 0$. The values that are at least 1 are

$$c^{\mathrm{red}}_{\min}(22) = 14/11,\ c^{\mathrm{red}}_{\min}(18) = 1,\ c^{\mathrm{red}}_{\min}(16) = 5/4,$$
$$c_{\min}(10) = 6/5,\ c^{\mathrm{red}}_{\min}(9) = 1,\ c_{\min}(5) = 6/5,$$

while

$$\begin{gathered} c^{\mathrm{red}}_{\min}(30) = 11/15,\ c^{\mathrm{red}}_{\min}(24) = 5/6,\ c^{\mathrm{red}}_{\min}(20) = 4/5,\ c^{\mathrm{red}}_{\min}(15) = 11/15, \\ c^{\mathrm{red}}_{\min}(14) = 4/7,\ c^{\mathrm{red}}_{\min}(12) = 1/3,\ c^{\mathrm{red}}_{\min}(8) = 1/4, \\ c^{\mathrm{red}}_{\min}(7) = 4/7,\ c^{\mathrm{red}}_{\min}(6) = 0,\ c^{\mathrm{red}}_{\min}(4) = 0, c^{\mathrm{red}}_{\min}(3) = 0. \end{gathered} \tag{3.26}$$

do not contribute 1. As we know $T_{\mathbb{C}}$ decomposes into a direct sum of $\mathbb{V}_r$ while we can assume the space $S_{\mathbb{C}}$ decomposes into a direct sum of $\mathbb{V}_d$ where

$$d \in \{1, 2, 3, 4, 6, 7, 8, 12, 14, 15, 20, 24, 30\}$$

by the estimation above.
When we write down all possibilities for the compostions of these representations this leads to the equation

$$\begin{aligned} \dim \mathbb{V}_r \cdot \lambda + \nu_1 + \nu_2 + 2\nu_3 + 2\nu_4 + 2\nu_6 + \frac{6}{2}\nu_7 + \frac{4}{2}\nu_8 & \\ + \frac{4}{2}\nu_{12} + \frac{6}{2}\nu_{14} + \frac{8}{2}\nu_{15} + \frac{8}{2}\nu_{20} + \frac{8}{2}\nu_{24} + \frac{8}{2}\nu_{30} &= n + 1, \end{aligned} \tag{3.27}$$

where λ denotes the multiplicity of $\mathbb{V}_r$ in $T_{\mathbb{C}}$ and ν_d denotes the multiplicity of $\mathbb{V}_d$ in $S_{\mathbb{C}}$. Note that we can assume that V_d for $d \in \{7, 8, 12, 14, 15, 20, 24, 30\}$ is reducible. If it is not it would contribute at least 1 to $\Sigma(g)$, as shown in [GHS07, Theorem 2.10].
For $D < 0$ and $d = 3, 4, 6$ it is possible for V_d to be reducible. In that case we have to divide the dimension(i.e. 2) by 2 in equation (3.27). But as we restrict to the case $D < D_0$ this can not happen.
For the quotient $\Lambda_{\mathbb{C}}/\mathbb{V}^{\omega}_r$ we denote by ν_r the multiplicity of $\mathbb{V}_r$ in $\Lambda_{\mathbb{C}}/\mathbb{V}^{\omega}_r$ as a g-module, i.e. the number of copies of $\mathbb{V}_r$ without $\mathbb{V}^{\omega}_r$.
As we will only observe $\Lambda_{\mathbb{C}}/\mathbb{V}^{\omega}_r$ in the following we have to add $\mathrm{mc}(r)$ from (3.19), as the subspace $\mathrm{Hom}(\mathbb{W}, \mathbb{V}^{\omega}_r \cap \mathbb{C}^{n+1}/\mathbb{W}) \subset V$ will not appear in the further calculations. Now we can calculate the (minimal) contribution of $\mathrm{Hom}(\mathbb{W}, \mathbb{V}_d)$ to $\Sigma(g)$ by

$$\sum_{(a,d)=1} \left\{ \frac{a}{d} + \frac{k_1}{r} \right\} \text{ resp. } \min_{D<D_0} \min_{\alpha=\pm 1} \sum_{\substack{(a,d)=1 \\ \left(\frac{D}{a}\right)=\alpha}} \left\{ \frac{a}{d} + \frac{k_1}{r} \right\}. \tag{3.28}$$

According to Lemma 3.21 and Remark 3.22 we have to investigate the cases $r \in \{3,4,6\} = \varphi^{-1}(2)$, $r \in \{7,14\} \subset \varphi^{-1}(6)$ and $r \in \{15,20,24,30\} \subset \varphi^{-1}(8)$.

(1) Let $\varphi(r) = 2$. The contributions of the $\mathbb{V}_d$ with $\varphi(d) \geq 4$ are greater or equal to 1 and (3.27) becomes

$$\nu_1 + \nu_2 + 2\nu_3 + 2\nu_4 + 2\nu_6 \quad = \quad n + 1 - 2 = n - 1.$$

For the 6 possible cases of the choice of (r, k_1), namely $r \in \{3,4,6\}$ and $k_1 \in \{1, r-1\}$. The other contributions are at least

d	contribution
1	1/6
2	1/6
3	1/3
4	1/2
6	1/3

After all $\Sigma(g) \geq 1$ if $n - 1 \geq 6$.

(2) Let $r = 7, 14$. We can assume $D = -7$ as if this is not the case explicit calculations show that$\mathbb{V}_r^\omega$ will contribute at least 1 to $\Sigma(g)$. Equation (3.27) becomes

$$\nu_1 + \nu_2 + 2\nu_3 + 2\nu_4 + 2\nu_6 + 3\nu_7 + 3\nu_{14} \quad = \quad n + 1 - 3 = n - 2$$

and the contributions are

d	contribution
1	1/14
2	1/14
3	3/7
4	4/7
6	3/7
7	4/7
14	4/7

and 4/7 from $\mathbb{V}_r^\omega$. So we may assume that $\nu_3 = \nu_4 = \nu_6 = \nu_7 = \nu_{14} = 0$, because otherwise the contribution will be ≥ 1. So $\Sigma(g) \geq 1$, if $\nu_1 + \nu_2 \geq 6$ and $n \geq 8$.

(3) Let $r = 15, 20, 24, 30$. Analogously to the last case we can assume that $D = -5, -6, -15$.

(a) Let $D = -5$. Hence we get the equation

$$\nu_1 + \nu_2 + 2\nu_3 + 2\nu_4 + 2\nu_6 + 4\nu_{20} \quad = \quad n - 3.$$

The contributions are

d	contribution
1	$1/30$
2	$1/30$
3	$5/12$
4	$8/15$
6	$5/12$
20	$4/5$

and $4/5$ from $\mathbb{V}_r^\omega$. So $\Sigma(g) \geq 1$ unless $\nu_1 + \nu_2 \leq 5$ resp. $n \leq 8$.

(b) Let $D = -6$ and therefore we get the equation

$$\nu_1 + \nu_2 + 2\nu_3 + 2\nu_4 + 2\nu_6 + 4\nu_{24} \quad = \quad n - 3.$$

The contributions of $\mathbb{V}_{24}$ and $\mathbb{V}_r^\omega$ are $5/6$. So $\Sigma(g) \geq 1$ unless $\nu_1 + \nu_2 \leq 4$ resp. $n \leq 7$.

(c) The last case is $D = -15$. So we get the equation

$$\nu_1 + \nu_2 + 2\nu_3 + 2\nu_4 + 2\nu_6 + 4\nu_{15} + 4\nu_{30} \quad = \quad n - 3. \tag{3.29}$$

The contributions of $\mathbb{V}_{15}$, $\mathbb{V}_{30}$ and $\mathbb{V}_r^\omega$ are $11/15$. So $\Sigma(g) \geq 1$, if $\nu_1 + \nu_2 \geq 8$ resp. $n \geq 11$.

Thus (i) is proved.
Statement (ii) follows directly from the above discussion, because g contributes at least 1 on the subspace $\mathrm{Hom}(\mathbb{W}, \mathbb{V}_r^\omega \cap \mathbb{C}^{n+1}/\mathbb{W})$ for $\varphi(r) = 6, 8$, as the corresponding representations are irreducible in this case. □

So far we have only studied elements g that are not quasi-reflections. Nevertheless this enables us to state a first result on canonical singularities of the quasi-projective variety.

Corollary 3.26. *Let $D \neq -1, -2, -3$ and $n \geq 11$. Then $\Gamma \backslash \mathbb{C}H^n$ has canonical singularities away from the branch divisors.*

Proof. This directly follows from Theorem 3.25 and the Reid-Tai criterion. □

This is true because the quasi-reflection induce the branch divisors.

3.4 Quasi-reflections

In the previous section we only considered elements that were not quasi-reflections. Now we will turn our attention to quasi-reflections and elements whose power is a quasi-reflection.
We will start with a description how $\Lambda_{\mathbb{Q}(\sqrt{D})}$ decomposes as a g-module, for a quasi-reflection $h = g^k$. We can mention the possible decompositions with respect to a given imaginary quadratic number field.

Proposition 3.27. *Let $h = g^k$ be a quasi-reflection on V for $g \in G$ and $n \geq 2$. As a g-module we have*

$$\Lambda_{\mathbb{Q}(\sqrt{D})} \cong \mathcal{V}_{m_0} \oplus \bigoplus_j \mathcal{V}_{m_j}$$

for some $m_i \in \mathbb{N}$. Then

(i) $(m_0, k) = m_0$ and $2(m_j, k) = m_j$, or $2(m_0, k) = m_0$ and $(m_j, k) = m_j$ for $j \geq 1$ in the cases $D < D_0$ and $D = -2$,

(ii) $(m_0, k) = m_0$ and $l(m_j, k) = m_j$, or $l(m_0, k) = m_0$ and $(m_j, k) = m_j$, $l \in \{2, 4\}$, for $j \geq 1$ in the case $D = -1$,

(iii) $(m_0, k) = m_0$ and $l(m_j, k) = m_j$, or $l(m_0, k) = m_0$ and $(m_j, k) = m_j$, $l \in \{2, 3, 6\}$, for $j \geq 1$ in the case $D = -3$.

Proof. As a g-module $\Lambda_{\mathbb{Q}(\sqrt{D})}$ decomposes into $\mathcal{V}_r^\omega \oplus \bigoplus_i \mathcal{V}_{d_i}$ for some $d_i \in \mathbb{N}$. As h is a quasi-reflection on V, all but one eigenvalues on V must be 1.
First fix an i. Now define $\mathcal{V}_d := \mathcal{V}_{d_i}$ and $d' := \dfrac{d}{(k, d)}$, then the eigenvalues of h on $\mathcal{V}_d$ are primitive d'th roots of unity of multiplicity $\dfrac{\dim \mathcal{V}_d}{\dim \mathcal{V}_{d'}}$. We want to give restrictions on the d_i:

(1) $\dim \mathcal{V}_{d'} \leq 2$: Assume that the dimension is at least 3. One can choose three distinct eigenvalues ζ, ζ', ζ'' on $\mathcal{V}_{d'}$, such that h would have eigenvalues $\alpha(h)^{-1}\zeta, \alpha(h)^{-1}\zeta'$ and $\alpha(h)^{-1}\zeta''$ on V and at most one of these eigenvalues could be 1.

(2) $\frac{\dim \mathcal{V}_d}{\dim \mathcal{V}_{d'}} = 2 \Rightarrow \dim \mathcal{V}_{d'} = 1$: Assume $\dim \mathcal{V}_{d'} \geq 2$ under the given condition. Denote two of the $\dim \mathcal{V}_{d'}$ eigenvalues of multiplicity 2 of h on $\mathcal{V}_d$ by ζ, ζ'. So one would have the eigenvalues $\alpha(h)^{-1}\zeta$ and $\alpha(h)^{-1}\zeta'$ of multiplicity 2 on V.

(3) $\dim \mathcal{V}_d \geq 2, \dim \mathcal{V}_{d'} = 1 \Rightarrow$ the eigenvalue of h on $\mathcal{V}_d$ is $\alpha(h)$: If ζ is the eigenvalue of h on $\mathcal{V}_d$ with $\zeta \neq \alpha(h)$, then $\alpha(h)^{-1}\zeta \neq 1$ would be an eigenvalue on V of multiplicity $\dim \mathcal{V}_d \geq 2$.

(4) $\dim \mathcal{V}_{d'} = 2 \Rightarrow \dim \mathcal{V}_d = 2$: Let $\dim \mathcal{V}_d > 2$. There are two eigenvalues $\zeta \neq \zeta'$ of h on $\mathcal{V}_d$ of multiplicity greater or equal to 2. Hence we have on V the eigenvalues $\alpha(h)^{-1}\zeta$ and $\alpha(g)^{-1}\zeta'$ of the same multiplicity.

(5) The case $\dim \mathcal{V}_{d'} = \dim \mathcal{V}_d = 2$ could not occur: Let $\dim \mathcal{V}_{d'} = \dim \mathcal{V}_d = 2$ with eigenvalues ζ, ζ' of h on $\mathcal{V}_d$. Without loss of generality we can assume that $\zeta = \alpha(h)$. If not we would have eigenvalues $\alpha(h)^{-1}\zeta \neq 1$ and $\alpha(h)^{-1}\zeta' \neq 1$ on V. There could be no other summand $\mathcal{V}_{d_1}$ in the decomposition of $\Lambda_{\mathbb{Q}(\sqrt{D})}$, as this summand would give an eigenvalue $\neq 1$ (the dimension of $\mathcal{V}_{d'}$ has to be 1, but as $\zeta = \alpha(h)$ and ζ is a primitive d'th root of unity, this can not happen).
There are two eigenvalues of h on $\mathbb{V}_r^\omega$ (because of $\dim \mathcal{V}_{d'} = 2$) which we will call $\alpha(h)$ and ζ'' with multiplicity $\frac{\dim \mathcal{V}_r}{2}$ (the denominator is $\dim \mathcal{V}_{d'}$). Therefore the multiplicity of the eigenvalues have to be 1, because $\alpha(h)^{-1}\zeta'' \neq 1$ is an eigenvalue on V. But then we will have two eigenvalues $\neq 1$ on V (namely $\alpha(h)^{-1}\zeta'$ and $\alpha(h)^{-1}\zeta''$).

Hence there follows $\dim \mathcal{V}_{d'} = 1$.
Now we want to study $\mathcal{V}_r$. Let $r' := \dfrac{r}{(k,r)}$. We claim that $\dim \mathcal{V}_{r'} = 1$. Suppose $\dim \mathcal{V}_{r'} \geq 2$.

(6) $\dim \mathcal{V}_{r'} \leq 2$: Assume that $\dim \mathcal{V}_{r'} > 2$, i.e. h has on $\mathcal{V}_r^\omega$ at least three distinct eigenvalues $\alpha(h), \zeta, \zeta'$, which will give rise to eigenvalues $\alpha(h)^{-1}\zeta \neq 1$ and $\alpha(h)^{-1}\zeta' \neq 1$ on V.

(7) $\dim \mathcal{V}_{r'} = 2 \Rightarrow n = 1$: We know $\dim \mathcal{V}_{d'} = 1$ from above. Let ζ be the eigenvalue of h on $\mathcal{V}_d$ of multiplicity $\dim \mathcal{V}_d$. Clearly $\zeta \neq \alpha(h)$, because of dimension reasons. So we get the eigenvalue $\alpha(h)^{-1}\zeta$ on V, and hence $\Lambda_{\mathbb{Q}(\sqrt{D})} = \mathcal{V}_r^\omega$ and $\operatorname{rk}\Lambda = 2$.

By the assumption $n \geq 2$ we get $\dim \mathcal{V}_{r'} = 1$.
Putting this all together we get as a h-module

$$\Lambda_{\mathbb{Q}(\sqrt{D})} \;\cong\; \mathcal{V}_r^\omega \oplus \bigoplus_i \mathcal{V}_{d_i},$$

where the eigenvalues of $h = g^k$ on

(a) $\mathcal{V}_r^\omega$ are primitive r'th roots of unity ($\dim \mathcal{V}_{r'} = 1$) of multiplicity $\dim \mathcal{V}_r$.

(b) $\mathcal{V}_{d_i}$ are primitive d_i'th roots of unity ($\dim \mathcal{V}_{d_i'} = 1$) of multiplicity $\dim \mathcal{V}_{d_i}$.

□

The proof enables us to give an explicit decomposition of $\Lambda_{\mathbb{Q}(\sqrt{D})}$.

Remark 3.28. (i) As a h-module we have

(1) in the cases $D < D_0, D = -2$:

$$\Lambda_{\mathbb{Q}(\sqrt{D})} \cong \mathcal{V}_r^{\omega} \oplus \mathcal{V}_1^{a_1} \oplus \mathcal{V}_2^{a_2},\ a_i \geq 0, r \in \{1,2\},$$

(2) in case $D = -1$:

$$\Lambda_{\mathbb{Q}(\sqrt{D})} \cong \mathcal{V}_r^{\omega} \oplus \mathcal{V}_1^{a_1} \oplus \mathcal{V}_2^{a_2} \oplus \mathcal{V}_4^{a_4},\ a_i \geq 0, r \in \{1,2,4\},$$

(3) in case $D = -3$:

$$\Lambda_{\mathbb{Q}(\sqrt{D})} \cong \mathcal{V}_r^{\omega} \oplus \mathcal{V}_1^{a_1} \oplus \mathcal{V}_2^{a_2} \oplus \mathcal{V}_3^{a_3} \oplus \mathcal{V}_6^{a_6},\ a_i \geq 0, r \in \{1,2,3,6\}.$$

In particular h has order

(1) 2, if $D < D_0, D = -2$,

(2) $\mathrm{lcm}(r, (i)_{a_i>0}) = 2$ or 4, if $D = -1$,

(3) $\mathrm{lcm}(r, (i)_{a_i>0}) = 2, 3$ or 6, if $D = -3$.

(ii) The l in the proposition above is the lowest common multiple resp. 2 mentioned in (i).

By now we know the possible actions on the tangent space for a given D. This leads to the elements in $U(\Lambda)$ inducing quasi-reflections.

Corollary 3.29. *The quasi-reflections on V are induced by elements $h \in U(\Lambda)$, such that*

(i) $\pm h$ acts as a reflection on $\Lambda_{\mathbb{C}}$, if $D < D_0$ or $D = -2$,

(ii) $h^4 \sim I$, if $D = -1$,

(iii) $h^6 \sim I$, if $D = -3$.

Proof. One has to check all possibilities for $\alpha(h)$ and the order of the quasi-reflection on V. □

We want to the investigate the elements in the corollary in more detail depending on the number field $\mathbb{Q}(\sqrt{D})$.

Remark 3.30. Let $h \in U(\Lambda)$ such that the induced action on the tangent space, denoted by h', is a quasi-reflection. Therefore

$$h' \sim_{\mathbb{C}} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \zeta \end{pmatrix},\ \text{with } \zeta \neq 1.$$

Hence

$$h \sim_{\mathbb{C}} \left(\begin{array}{ccc|c} \alpha(h) & & & \\ & \ddots & & \\ & & \alpha(h) & \\ \hline & & & \lambda \end{array}\right).$$

(1) Let $D < D_0$ or $D = -2$. So $\alpha(h) = \pm 1$. If $\alpha(h) = +1$ we will have the eigenvalue $+1$ on $\Lambda_{\mathbb{C}}$ of multiplicity n and $\lambda = -1$ of multiplicity 1. For $\alpha(h) = -1$ it is the other way round. Hence $\pm h$ is a reflection on $\Lambda_{\mathbb{C}}$.

(2) Let $D = -1$, so $\alpha(h) = \pm 1, \zeta_4$.

order of h'	$\alpha(h)$	λ	order of h
2	± 1	cf. 1.	2
2	ζ_4	ζ_4^{-1}	4
4	± 1	$\pm\zeta_4$	4
4	ζ_4	± 1	4

(3) Let $D = -3$. Hence $\alpha(h) = \pm 1, \zeta_3, \zeta_6$.

order of h'	$\alpha(h)$	λ	order of h
2	± 1	cf. 1.	2
2	ζ_3	$\zeta_6 = -\zeta_3$	6
2	ζ_6	$-\zeta_6$	6
3	± 1	$\begin{cases} \zeta_3, & \alpha(h) = +1 \\ \zeta_6, & \alpha(h) = -1 \end{cases}$	$\begin{cases} 3 \\ 6 \end{cases}$
3	ζ_3	$+1, \zeta_3^{-1}$	3
3	ζ_6	$-1, \zeta_6^{-1}$	6
6	± 1	$\begin{cases} \zeta_6, & \alpha(h) = +1 \\ \zeta_3, & \alpha(h) = -1 \end{cases}$	6
6	ζ_3	ζ_6	6
6	ζ_6	$+1, \zeta_3$	6

In the following we will show that for some restrictions on n and D we always have canonical singularities. As we will do this by reducing to suitable quotient groups without quasi-reflections we first have to note

Remark 3.31. *V/G has canonical singularities if $V/\langle g\rangle$ has canonical singularities for all $g \in G$. This was shown by* [GHS07, Proof of Lemma 2.14].

We have stated all the results we need to produce a result for a general element g and the corresponding Reid-Tai sum. First we give some definitions and then prove the result.

Let $h = g^k$ be a quasi-reflection on the tangent space V and $g \in G$. We assume that $k > 1$ is minimal with this property. If one considers the quotient $V/\langle h\rangle$ by the subgroup generated by h, this quotient is smooth by Corollary 1.6.
Let h be of order l, where l is given by Remark 3.28, i.e. g has order $l \cdot k$. We want to have a look on the eigenvalues $\zeta^{a_1}, \ldots, \zeta^{a_n}$ of g on V, where ζ is a primitive $(l \cdot k)$th root of unity, and $0 \leq a_i < lk$.
We want to consider the action of the group $\langle g\rangle/\langle h\rangle$ on $V' := V/\langle h\rangle$. Clearly $V'/(\langle g\rangle/\langle h\rangle) \cong V/\langle g\rangle$. Now we want to use analogous arguments as before to describe the action of elements of $\langle g\rangle/\langle h\rangle$, namely $g^f\langle h\rangle$, on V'. Note that the differential of $g^f\langle h\rangle$ on V' has eigenvalues $\zeta^{fa_1}, \ldots, \zeta^{fa_{n-1}}, \zeta^{lfa_n}$. The nth eigenvalue correspond to the eigenvalue of h not equal to 1.
As we have an other action then before we modify the Reid-Tai sum.

Definition 3.32.

$$\Sigma'(g^f) := \left\{\frac{fa_n}{k}\right\} + \sum_{i=1}^{n-1}\left\{\frac{fa_i}{lk}\right\}. \tag{3.30}$$

One can show that this is the right definition to study quasi-reflections.

Lemma 3.33. *The quasi-projective variety $\Gamma\backslash\mathbb{C}H^n$ has canonical singularities, if*

(i) $\Sigma(g) \geq 1$ for all $g \in \Gamma$ no power of which is a quasi-reflection, and

(ii) $\Sigma'(g^f) \geq 1$ for $1 \leq f < k$, where $h = g^k$ is a quasi-reflection.

Proof. [GHS07, Lemma 2.14] □

We already proved some results for (i), so we now have to give a result for g inducing quasi-reflections.

Proposition 3.34. *Let $D < D_0$, $h = g^k$ be a quasi-reflection and $n \geq 12$. Then $\Sigma'(g^f) \geq 1$ for every $1 \leq f < k$.*

Proof. We know from the former results that all eigenvalues on $\mathcal{V}_r^\omega$ are $\alpha(h)$, where

$$\alpha(h) \quad = \quad \begin{cases} \pm 1, & D < D_0 \text{ and } D = -2, \\ \pm 1, \zeta_4, & D = -1, \\ \pm 1, \zeta_3, \zeta_6, & D = -3. \end{cases} \tag{3.31}$$

We also already decomposed $\Lambda_{\mathbb{C}}$ into $\mathbb{Q}(\sqrt{D})$ irreducible pieces and by Remark 3.30 there is exactly one eigenvalue on $\Lambda_{\mathbb{C}}$ that is $\lambda \neq \alpha(h)$, since only one eigenvalue on V is not 1. This eigenvalue λ will appear on one $\mathcal{V}_d$. As all eigenvalues of g on $\mathcal{V}_d$ are primitive dth roots of unity they all have the same order. We know that λ must have multiplicity 1 on $\Lambda_{\mathbb{C}}$ so $\dim \mathcal{V}_d = 1$. This implies

$$d = \begin{cases} 1, 2, \\ 1, 2, 4, \\ 1, 2, 3, 6. \end{cases} \tag{3.32}$$

Denote by v the eigenvector of g corresponding to the eigenvalue ζ^{a_n}. Then v clearly comes from $\mathcal{V}_d$ and therefore $\langle v \rangle = \mathrm{Hom}(\mathbb{W}, \mathbb{V}_d)$.
If δ is the primitive generator of $\mathcal{V}_d \cap \Lambda$ then $h(\delta, \delta) > 0$, since $\mathcal{V}_d \subset W^{\perp}_{\mathbb{Q}(\sqrt{D})}$, where $W_{\mathbb{Q}(\sqrt{D})} \otimes_{\mathbb{Q}(\sqrt{D})} \mathbb{C} \cong \mathbb{W}$ and $W_{\mathbb{Q}(\sqrt{D})}$ is a $\mathbb{Q}(\sqrt{D})$-vector space. The form $h(\cdot, \cdot)$ is negative definite on $\mathbb{W}$ as shown in the proof of Lemma 3.12. If we define the sublattice $\Lambda' \subset \Lambda$ as $\Lambda' := \delta^{\perp}$, this lattice has signature $(n-1, 1)$.
Now $\langle g \rangle / \langle h \rangle$ acts on Λ' as a subgroup of $U(\Lambda')$.
Therefore

$$\Sigma'(g^f) = \left\{ \frac{f a_n}{k} \right\} + \Sigma(g^f \langle h \rangle)$$

and $g^f \langle h \rangle \in U(\Lambda')$.
Analogously to the proof of [GHS07, Proposition 2.15] we can give the following argument: We claim that $g^f \langle h \rangle$ is not a quasi-reflection on Λ'. If it were, the eigenvalues of g^f on Λ' are as in Corollary 3.29. Thus the order of the eigenvalue on $\mathcal{V}_d$ is

$$d = \begin{cases} 1, 2, \\ 1, 2, 4, \\ 1, 2, 3, 6. \end{cases}$$

So $\operatorname{ord} g^f$ divides l, and therefore $g^f \in \langle h \rangle$. Hence the group $\langle g \rangle / \langle h \rangle$ has no quasi-reflections and we apply Theorem 3.25 for $n - 1 \geq 11$. □

Theorem 3.35. *Let $n \geq 12$ and $D < D_0$. Then $\Gamma \backslash \mathbb{C}H^n$ has canonical singularities.*

Proof. This follows directly from Lemma 3.33, Theorem 3.25 and Proposition 3.34. □

Remark 3.36. All the techniques provided in this section also work for arbitrary $D < 0$. The only reason for the restriction is Theorem 3.25, as there can occur contribution of 0 for $D = -1, -2, -3$ and some representations.

3.5 Non-canonical singularities

In the last section we proved a result that there exist a bound on n for $\mathbb{C}H^n / \Gamma$ to have canonical singularities when one restricts to $D < -3$. This restriction relies on the contribution of some representations to $\Sigma(g)$.

Now we want to give a bound for $D = -3$ by explicitly examining the cases that do not lead to canonical singularities. But even there will occur some elements in Γ that could give rise to non-canonical singularities. We will give a list of all these elements.
The whole section we will assume that $D = -3$.
Primarily we assume that g do not act as a quasi-reflection on V. We will keep the notation from the previous sections.
First we summarize some of the results we obtained in section 3.3.

Lemma 3.37. *The only values for r that do not fulfill the Reid-Tai inequality are $r = 3, 4, 6, 7, 12, 14, 15, 20, 24, 30$.*

Proof. We only need to cite previous results.

(1) $r = 1, 2$ follows from Lemma 3.24

(2) Using the arguments from the proof of Lemma 3.21(ii) for $r = 5, 8, 10$ imply the result.

(3) By Remark 3.22 we get $r = 9, 16, 18$.

(4) All r with $\varphi(r) \geq 10$ follow directly from Lemma 3.21

□

Now we have to check the remaining values for r by hand. This can be done using techniques introduced in the previous sections.

Lemma 3.38. *If $r = 7, 14, 15, 20, 24, 30$ then $\Sigma(g) \geq 1$.*

Proof. All we have to do is to calculate the contribution of g on $\mathrm{Hom}(\mathbb{W}, \mathbb{V}_r^w \cap \mathbb{C}^{n+1}/\mathbb{W})$ explicitly. This can be done using $\mathrm{mc}(r)$ from(3.19). We have to watch out, as we only do this for $D = -3$ as long as V_r is reducible over $\mathbb{Q}(\sqrt{-3})$. If it is not reducible a slight modification of $\mathrm{mc}(r)$ gives the right formula. Calculating these contributions one gets:

r	contribution to $\Sigma(g)$
3	0
4	1/2
6	0
7	15/7
12	1/2
14	15/7
15	6/5
20	16/5
24	3/2
30	6/5

The r we are looking for contribute at least 1. $\square$

Now there are only four different choices for r left, which would not contribute 1 to the Reid-Tai sum $\Sigma(g)$. There are two pairs that behave differently, namely $4, 12$ and $3, 6$. This is based on the fact that V_r is irreducible in the first case, but reducible in the second.
First we will give a bound on the dimension for $r = 4, 12$.

Proposition 3.39. *Let $r = 4, 12$ and $n \geq 7$, then $\Sigma(g) \geq 1$.*

Proof. Analogously to the proof of Theorem 3.25 we obtain (3.27) for the $D = -3$ case. But now we can be more exact what the dimensions are, as we know exactly which representations are irreducible resp. reducible over this specific number field:

$$\begin{aligned} \dim \mathbb{V}_r \cdot \lambda + \nu_1 + \nu_2 + \nu_3 + 2\nu_4 + \nu_6 + 6\nu_7 + 4\nu_8 & \\ +2\nu_{12} + 6\nu_{14} + 4\nu_{15} + 8\nu_{20} + 4\nu_{24} + 4\nu_{30} &= n + 1. \end{aligned} \tag{3.33}$$

Here it automatically follows that:

(1) $\nu_7 = \nu_8 = \nu_{14} = \nu_{20} = 0$, because they contribute at least 1 to $\Sigma(g)$ (cf. the discussion after (3.27)).

(2) $\nu_4 = \nu_{24} = \nu_{30} = 0$, as their contribution together with the contribution $\mathrm{mc}(r)$ add up to at least 1. This has been already calculated in (3.26).

(3) $\lambda = 1$, for the same reason as in (b)(remember that $\dim \mathbb{V}_r = 2$).

Now we can rewrite (3.33) as

$$\nu_1 + \nu_2 + \nu_3 + \nu_6 + 2\nu_{12} + 4\nu_{15} \quad = \quad n + 1 - 2 = n - 1. \tag{3.34}$$

To give a bound on n we will first state a list of the minimal possible values for the contribution of $\mathcal{V}_d$ to the Reid-Tai sum. To get better results we will distinguish the cases $r = 4, 12$, and therefore choose $\alpha(g) = \zeta_r,\ r = 4, 12$.

d	min. contr. for $r = 4$	min. contr. for $r = 12$
1	1/4	1/12
2	1/4	1/12
3	1/12	1/12
6	1/12	1/12
12	5/6	1/2
15	5/3	5/3

As $\mathrm{mc}(r) = 1/2$ we can give bounds using similar arguments as for Theorem 3.25:

(1) In case $r = 4$, the contribution to $\Sigma(g)$ is less than one, as long as

$$\nu_1, \nu_2 \leq 1,\ \nu_{12} = \nu_{15} = 0,\ \nu_3 + \nu_6 \leq 5.$$

Therefore $\Sigma(g) \geq 1$, if $n - 1 \geq 6 \Leftrightarrow n \geq 7$.

(2) Analogously to (1) we get in case $r = 12$:

$$\nu_{12} = \nu_{15} = 0,\ \nu_1 + \nu_2 + \nu_3 + \nu_6 \leq 5.$$

Thus n has to be at least 7.

□

Now there are only two possible values for r left, namely $r = 3, 6$.
But we want to reduce further. Up to now there are infinitely many possible representations occuring in the decomposition as a g-module.

Lemma 3.40. *Let $r = 3, 6$ and $d \neq 1, 2, 3, 4, 6, 12$. Then the contribution to $\Sigma(g)$ coming from* $\operatorname{Hom}(\mathbb{W}, \mathbb{V}_d \cap \mathbb{C}^{n+1}/\mathbb{W})$ *is at least* 1.

Proof. We will calculate the contribution of g from the subspace $\operatorname{Hom}(\mathbb{W}, \mathbb{V}_d \cap \mathbb{C}^{n+1}/\mathbb{W}) \subset V$ to the Reid-Tai sum in the case of $D = -3$ and $r = 3, 6$. First note that we only have to consider $d \leq 90$, because this was already done for (3.22).
Now there are only finitely many cases left. Using computer calculation we will get

(1) the value

$$\min_{(a,r)=1} \sum_{(b,d)=1} \left\{ \frac{a}{r} + \frac{b}{d} \right\}$$

for V_d irreducible over $\mathbb{Q}(\sqrt{-3})$, and

(2) the value

$$\min_{\alpha=\pm 1} \min_{(a,r)=1} \sum_{\substack{(b,d)=1 \\ \left(\frac{-3}{b}\right)=\alpha}} \left\{ \frac{a}{r} + \frac{b}{d} \right\}$$

for V_d reducible over $\mathbb{Q}(\sqrt{-3})$.

As this leads to the values of d we stated above we are done. □

So there are no irreducible representations of dimension greater than 2 leading to non-canonical singularities that we have to care about.
As we will need some estimations of these contributions for the non-canonical singularities to have something like a finiteness result we state some contributions calculated for the proof of Lemma 3.40.

Remark 3.41. The smallest possible contributions coming from 2-dimensional representations calculated in Lemma 3.40 are

d	min. contr. for $r = 3$	min. contr. for $r = 6$
4	2/3	2/3
12	2/3	2/3

We can now state all elements that do not give rise to canonical singularities by the Reid-Tai criterion.

For this we first fix $\alpha(g) =: \zeta_r$, $r = 3, 6$, for a primitive rth root of unity ζ_r. By Lemma 3.40 we can assume

$$g \sim_{\mathbb{Q}(\sqrt{-3})} \begin{pmatrix} I_{a_1} & & & & & & & \\ & -I_{a_2} & & & & & & \\ & & A^3_{a_3} & & & & & \\ & & & \overline{A}^3_{b_3} & & & & \\ & & & & A^6_{a_6} & & & \\ & & & & & \overline{A}^6_{b_6} & & \\ & & & & & & A^4_{a_4} & \\ & & & & & & & A^{12}_{a_{12}} \end{pmatrix}, \tag{3.35}$$

where $A^i_{a_i}$, $i = 3, 6$ denotes the $a_i \times a_i$-diagonal matrix with entries ζ_i and $\overline{A}^i_{b_i}$ the complex conjugate of this matrix. The two remaining $2a_i \times 2a_i$-matrices are defined as

$$A^i_{a_i} = \begin{pmatrix} \mathcal{V}_i & & \\ & \ddots & \\ & & \mathcal{V}_i \end{pmatrix}, \ i = 4, 12.$$

Remark 3.42. In the definition of $A^i_{a_i}$, for $i = 4, 12$ we only write $\mathcal{V}_i$. In case $i = 12$ there are two choices for $\mathcal{V}_{12}$, as there are two 2-dimensional irreducible representations. So by this notation we mean that there can occur both in $A^{12}_{a_{12}}$.

Proposition 3.43. *Let $r = 3, 6$ and g be as in (3.35). Then $\Sigma(g) \geq 1$ except for the following values of a_i and b_j:*

(i) All $a_3 \geq 1$ and

b_3	a_1	a_2	a_6	b_6	a_4	a_{12}
0	0	0,1 0	0	0 0,1	1	0
0	0	0,1 0	0	0 0,1	0	1
1	0	1,2,3 0 0	0 1 0	0 0 1,2,3	0	0
2	0	1 0 0	0	0 1 0	0	0
0	1	1,2,3 0 0	0 1 0	0 0 1,2,3	0	0
0	2	1 0 0	0	0 1 0	0	0
0	0	3,4,5 2 1	0 1,0 1	0	0	0
0	0	0	0 1,0 1	3,4,5 2 1	0	0

in case $r = 3$.

(ii) All $a_6 \geq 1$ and

a_2	b_6	b_3	a_3	a_1	a_4	a_{12}
0	*0*	*0,1* *0*	*0*	*0* *0,1*	*1*	*0*
0	*0*	*0,1* *0*	*0*	*0* *0,1*	*0*	*1*
1	*0*	*1,2,3* *0* *0*	*0* *1* *0*	*0* *0* *1,2,3*	*0*	*0*
2	*0*	*1* *0* *0*	*0*	*0* *1* *0*	*0*	*0*
0	*1*	*1,2,3* *0* *0*	*0* *1* *0*	*0* *0* *1,2,3*	*0*	*0*
0	*2*	*1* *0* *0*	*0*	*0* *1* *0*	*0*	*0*
0	*0*	*3,4,5* *2* *1*	*0* *1,0* *1*	*0*	*0*	*0*
0	*0*	*0*	*0* *1,0* *1*	*3,4,5* *2* *1*	*0*	*0*

in case $r = 6$.

Proof. First we state the induced action of g on the tangent space V. In case $r = 3$ this is

$$\begin{pmatrix} \overline{A}^3_{a_1} & & & & & & & \\ & \overline{A}^6_{a_2} & & & & & & \\ & & I_{a_3-1} & & & & & \\ & & & A^3_{b_3} & & & & \\ & & & & -I_{a_6} & & & \\ & & & & & A^6_{b_6} & & \\ & & & & & & \overline{\zeta_3}A^4_{a_4} & \\ & & & & & & & \overline{\zeta_3}A^{12}_{a_{12}} \end{pmatrix},$$

where without loss of generality we fix $\zeta_6 := -\zeta_3$.

The induced action in case $r = 6$ is given by

$$\begin{pmatrix} \overline{A}^6_{a_1} &&&&&&& \\ & \overline{A}^3_{a_2} &&&&&& \\ && -I_{a_3} &&&&& \\ &&& A^6_{b_3} &&&& \\ &&&& I_{a_6-1} &&& \\ &&&&& A^3_{b_6} && \\ &&&&&& \overline{\zeta_6} A^4_{a_4} & \\ &&&&&&& \overline{\zeta_6} A^{12}_{a_{12}} \end{pmatrix},$$

with $-\zeta_6 =: \zeta_3$.
We will only prove the case $r = 3$ as $r = 6$ is identical.
As we assume that g do not act as a quasi-reflection on V, the values that fulfill

$$a_1 + a_2 + 2a_4 + a_6 + 2a_{12} + b_3 + b_6 = 1$$

are not allowed.

(1) If the 2-dimensional representations occur, i.e. $a_{12} \neq 0$ or $a_4 \neq 0$, then they contribute at least $2/3$ by Remark 3.41. So only when $a_2 \leq 1$ or $b_6 \leq 1$ and not simultaneously equal 1 (each of the 1-dimensional contributions is at least $1/6$) they do not sum up to 1 for the Reid-Tai sum.

For the other cases we can assume that $a_4 = a_{12} = 0$.

(2) Let $b_3 > 0$ (and less than 3 as each ζ_3 contributes at least $1/3$), then at once $a_1 = 0$, as otherwise one could choose complex conjugate eigenvalues. If $b_3 = 1$ and $0 < a_2 \leq 3$ then $b_6 = 0$ (again choose complex conjugate eigenvalues) and $a_6 = 0$ as its contribution is $1/2$. Therefore if $a_6 = 1$, then automatically $a_2 = b_6 = 0$. Interchanging the roles of a_2 and b_6 gives the same result.
When one assumes $a_1 > 0$ the conclusion is the same.

(3) So we may assume $a_1 = b_3 = 0$. Let $a_2 > 0$(and less than 6 as the contribution is at least $1/6$), then $b_6 = 0$ and vice versa (cf. (2)). If $a_6 > 0$ then a_2 is at most 2.

These are all possible cases for the a_i and the b_i. □

As the techniques introduced to prove Proposition 3.34 were proven for general $D < 0$, we can give a result in case $D = -3$.

Corollary 3.44. *Let $D = -3$, $h = g^k$ be a quasi-reflection and $n \geq 8$. Then $\Sigma'(g^f) \geq 1$ for every $1 \leq f < k$ and h not corresponding to one of the list of Proposition 3.43.*

Proof. The proof is analogous to the proof of Proposition 3.34 □

Thus we can give a result on singularities of $\Gamma\backslash\mathbb{C}H^n$ in case $D = -3$. As shown before there are elements in arbitrary dimension that could lead to non-canonical singularities. But by Proposition 3.43 we can study the singularities if necessary.

Corollary 3.45. *Let $n \geq 8$. Then the quasi-projective variety $\Gamma\backslash\mathbb{C}H^n$ has canonical singularities up to singularities that can arise induced by elements as from Proposition 3.43.*

Proof. This follows with the results from this section used with Lemma 3.33 and the techniques from Proposition 3.34. □

Remark 3.46. If one wants to study the elements not leading to quasi-reflections one has to watch out. In Proposition 3.34 we do not study the action on V but on V'. Therefore there happens a base change, and they are not induced directly from the elements we already mentioned.

In principle we could give a statement also for the other two remaining cases which were not covered by Theorem 3.25, namely $D = -1$ and $D = -2$. One has to consider the 2-dimensional representations that decompose over the number field. But as seen in this section it is very messy to do. Furthermore, the case $r = 1, 2$ is not covered anymore since Lemma 3.24 is only true for $D \neq -1, -2$.

Chapter 4

Compactification

In the last chapter we studied the singularities of the quasi-projective variety $\Gamma\backslash\mathbb{C}H^n$. There we proved a bound on n (under some restrictions on D), such that the ball quotient has canonical singularities. However, $\Gamma\backslash\mathbb{C}H^n$ is a non-compact variety. So there arises the natural question of a compactification and the singularities occuring at the boundary. Based on [GHS07] we will construct locally a toroidal compactification $(\Gamma\backslash\mathbb{C}H^n)^*$ and study its singularities.

4.1 Definitions

As already introduced in section 1.1.2 we will construct a toroidal compactification of $\Gamma\backslash\mathbb{C}H^n$. Therefore we have to follow this construction.
We first recall that the hermitian form $h(\cdot,\cdot)$ given by the lattice Λ has signature $(n,1)$. Thus the isotropic subspaces corresponding to the form are 1-dimensional and the boundary components or *cusps* are for this reason 0-dimensional, i.e. points.
As we will give local arguments we fix a rational boundary component F of $\mathbb{C}H^n$.

Definition 4.1. We will denote the corresponding rational isotropic subspace by $E_{\mathbb{Q}(\sqrt{D})}$.

Of course $E_{\mathbb{Q}(\sqrt{D})} \subset \Lambda_{\mathbb{Q}(\sqrt{D})}$ is an 1-dimensional subspace of the $\mathbb{Q}(\sqrt{D})$-vector space $\Lambda_{\mathbb{Q}(\sqrt{D})}$, where $D < 0$.
As we want sometimes to restrict to the lattice we define

$$\begin{aligned} E &:= E_{\mathbb{Q}(\sqrt{D})} \cap \Lambda, \text{ and} \\ E^{\perp} &:= E^{\perp}_{\mathbb{Q}(\sqrt{D})} \cap \Lambda. \end{aligned}$$

These two are primitive sublattices of Λ.
We will calculate the groups $N(F)$, $W(F)$ and $U(F)$ defined in 1.1.2 to follow the compactification process. Therefore we have to fix a basis. We will assume that

the hermitian form $h(\cdot,\cdot)$ of signature $(n,1)$ over $\mathbb{Q}(\sqrt{D})$ is given by the matrix Q', i.e.

$$h(x,y) = {}^H y Q' x. \tag{4.1}$$

By the signature of the form we know that $Q' \sim_{\mathbb{C}} I_{n,1}$.
We have to give the construction over the number field $\mathbb{Q}(\sqrt{D})$. In the first step we will choose a suitable basis for the form.

Lemma 4.2. *There exists a basis $e_1,\dots,e_{n+1}$ of $\Lambda_{\mathbb{Q}(\sqrt{D})}$, such that*

(i) e_1 is a basis of $E_{\mathbb{Q}(\sqrt{D})}$ and $e_1,\dots,e_n$ is a basis of $E^{\perp}_{\mathbb{Q}(\sqrt{D})}$,

(ii) the hermitian form with respect to this basis is given by

$$Q' := (h(e_i,e_j))_{1\le i,j\le n+1} = \left(\begin{array}{c|c|c} 0 & 0 & a \\ \hline 0 & B & c \\ \hline \overline{a} & {}^H c & d \end{array}\right), \tag{4.2}$$

where $a \in \mathbb{Q}(\sqrt{D})$, $d \in \mathbb{Q}$, $c \in \mathbb{Q}(\sqrt{D})^{n-1}$, $B \in \mathrm{GL}(n-1,\mathbb{Q}(\sqrt{D}))$ and $B = {}^H B$.

Proof. The proof directly follows from the choice of the basis and the properties of the form:

(1) We get the zeroes by $h(E_{\mathbb{Q}(\sqrt{D})}, e) = 0$ for all $e \in E^{\perp}_{\mathbb{Q}(\sqrt{D})}$.

(2) The rest follows as $h(x,y) = \overline{h(y,x)}$.

□

But the matrix Q' we get is not good enough in our situation. Thus we have to make a suitable base change.
We briefly want to recall the well-known fact how a hermitian form behaves under base change. For this let $h : \mathbb{C}^k \times \mathbb{C}^k \longrightarrow \mathbb{C}$ be a hermitian form given by S with respect to the basis $\mathcal{B}$. Then S' represents the same hermitian form with respect to a basis $\mathcal{B}'$, if there exist a matrix $A \in \mathrm{GL}(k,\mathbb{C})$, such that

$$S' = {}^H ASA.$$

Now we will choose a basis for Q', such that such that the matrix corresponding to the hermitian form has an 'simple' structure.

Lemma 4.3. *There exists a basis $b_1,\dots,b_{n+1}$ of $\Lambda_{\mathbb{Q}(\sqrt{D})}$, such that*

(i) b_1 is a basis of $E_{\mathbb{Q}(\sqrt{D})}$ and $b_1,\dots,b_n$ is a basis of $E^{\perp}_{\mathbb{Q}(\sqrt{D})}$,

(ii) the hermitian form is written with respect to this basis as

$$Q := (h(b_i, b_j))_{1\leq i,j\leq n+1} = \left(\begin{array}{c|c|c} 0 & 0 & a \\ \hline 0 & B & 0 \\ \hline \overline{a} & 0 & 0 \end{array}\right), \tag{4.3}$$

where a and B are as in Lemma 4.2.

Proof. This proof is similar to the proof of [GHS07, Lemma 2.24]. The matrix B represents the hermitian form h on $E^{\perp}_{\mathbb{Q}(\sqrt{D})}/E_{\mathbb{Q}(\sqrt{D})}$ and is therefore invertible. Thus one can define

$$N := \left(\begin{array}{c|c|c} 1 & 0 & r' \\ \hline 0 & I_{n-1} & r \\ \hline 0 & 0 & 1 \end{array}\right), \tag{4.4}$$

where $r := -B^{-1}c \in \mathbb{Q}(\sqrt{D})^{n-1}$. Choose r' such that it satisfies the equation

$$d - {}^{H}cB^{-1}c + \overline{r'}a + \overline{a}r' = 0. \tag{4.5}$$

This is possible as the first two summands are real by definition and the other two are the complex conjugate of each other and therefore their sum is real. Now we apply base change which gives

$$ {}^{H}NQ'N \;=\; \left(\begin{array}{c|c|c} 0 & 0 & a \\ \hline 0 & B & Br+c \\ \hline \overline{a} & {}^{H}rB + {}^{H}c & \delta \end{array}\right), \tag{4.6}$$

with $\delta := \overline{a}r' + ({}^{H}rB + {}^{H}c)r + \overline{r'}a + {}^{H}rc + d$. But

$$Br + c = B(-B^{-1}c) + c = 0.$$

Because of the definition of r and r' we achieve

$$\begin{aligned} \delta &= \overline{a}r' + {}^{H}(-B^{-1}c)B(-B^{-1}c) + {}^{H}c(-B^{-1}c) + \overline{r'}a + {}^{H}(-B^{-1}c)c + d \\ &= \underbrace{\overline{a}r' + \overline{r'}a - {}^{H}c^{H}(B^{-1})c + d}_{=0,\text{ because of (4.5)}} + \underbrace{{}^{H}c^{H}(B^{-1})BB^{-1}c - {}^{H}cB^{-1}c}_{=0} \\ &= 0. \end{aligned}$$

Note that ${}^{H}(B^{-1}) = B^{-1}$. Altogether this gives

$$ {}^{H}NQ'N \;=\; \left(\begin{array}{c|c|c} 0 & 0 & a \\ \hline 0 & B & 0 \\ \hline \overline{a} & 0 & 0 \end{array}\right).$$

□

Remark 4.4. The columns $N_i,\ i = 1, \ldots, n+1$ of the matrix N in the proof of Lemma 4.3 are the desired basis b_i.

4.2 Groups

So far we have chosen a suitable basis for the form $h(\cdot,\cdot)$. To proceed with the compactification to get $(\Gamma\backslash\mathbb{C}H^n)^*$, we have to calculate

(1) the stabiliser subgroup $N(F) \subset \Gamma_{\mathbb{R}}$ of the isotropic subspace corresponding to the cusp F,

(2) the unipotent radical $W(F)$ of $N(F)$,

(3) the center $U(F)$ of $N(F)$.

as mentioned in 1.2.2. We will start by finding the stabiliser subgroup. Note that we have chosen a basis such that the rational isotropic subspace is generated by the basis element b_1.

Lemma 4.5. *Let $N(F) \subset \Gamma_{\mathbb{R}}$ be the stabiliser subgroup corresponding to the cusp F. Then*

$$N(F) \quad = \quad \left\{ g = \left(\begin{array}{c|c|c} u & v & w \\ \hline 0 & X & y \\ \hline 0 & 0 & z \end{array}\right) ; \begin{array}{c} z\overline{u} = 1, \ {}^H\!XBX = B, \\ {}^H\!XBy +^H vaz = 0, \\ {}^H\!yBy + \overline{za}w + za\overline{w} = 0 \end{array} \right\}. \qquad (4.7)$$

Proof. This can be easily shown by doing the following two calculations, that come from the definition of $N(F)$.

(1) Collect all $g \in \Gamma_{\mathbb{R}}$ which satisfy the equation

$$gb_1 = b_1.$$

This means that the isotropic subspace is g-invariant.

(2) Drop all g that do not respect the form defined by Q.

□

As we want to calculate the unipotent radical of $N(F)$ we first have to state a lemma which comes from a more general algebraic setting, namely for a hermitian matrix under some restrictions.

Lemma 4.6. *Let $A =^H A \in \mathrm{Mat}(n,\mathbb{C})$ be a definite, hermitian matrix, and B element $\mathrm{Mat}(n,\mathbb{C})$ a unipotent matrix, i.e. $B = I_n + N$, where $N \in \mathrm{Mat}(n,\mathbb{C})$ is a strict upper triangular matrix.*
Then $N = 0$, if B satisfies the equation ${}^H\!BAB = A$.

Proof. We will prove this lemma by induction on n.
First we assume $n = 2$ and write $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $A = (a_{i,j})_{1\leq i,j\leq 2}$. So we get for these choices

$$\begin{aligned}
{}^H BAB &= \begin{pmatrix} 1 & 0 \\ \bar{b} & 1 \end{pmatrix}\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1} & a_{1,2} \\ \bar{b}a_{1,1} + a_{2,1} & \bar{b}a_{1,2} + a_{2,2} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1} & a_{1,1}b + a_{1,2} \\ \bar{b}a_{1,1} + a_{2,1} & (\bar{b}a_{1,1} + a_{2,1})b + \bar{b}a_{1,2} + a_{2,2} \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.
\end{aligned}$$

So from the $(1, 2)$-entry we get $a_{1,1}b = 0$, i.e. $a_{1,1} = 0$ or $b = 0$. As A is definite we have $a_{1,1} \neq 0$ and therefore $b = 0$.
Now assume that the statement holds for $n - 1$. First note that the restriction A_{res} of A on the subspace $\mathbb{C}^{n-1} = \{z \in \mathbb{C}^n;\ z_n = 0\} \subset \mathbb{C}^n$ is definite, as A is definite. With this notation we write

$$A = \left(\begin{array}{c|c} A_{\text{res}} & a \\ \hline {}^H a & \alpha \end{array}\right),\ B = \left(\begin{array}{c|c} I + N' & \nu \\ \hline 0 & 1 \end{array}\right), \tag{4.8}$$

where $N =: \left(\begin{array}{c|c} N' & \nu \\ \hline 0 & 0 \end{array}\right)$. Now we will calculate the product of these matrices and make a similar argument as before.

$$\begin{aligned}
{}^H BAB &= \left(\begin{array}{c|c} I + {}^H N' & 0 \\ \hline {}^H \nu & 1 \end{array}\right)\left(\begin{array}{c|c} A_{\text{res}} & a \\ \hline {}^H a & \alpha \end{array}\right)\left(\begin{array}{c|c} I + N' & \nu \\ \hline 0 & 1 \end{array}\right) \\
&= \left(\begin{array}{c|c} (I + {}^H N')A_{\text{res}}(I + N') & (I + {}^H N')A_{\text{res}}\nu + (I + {}^H N')a \\ \hline ({}^H \nu A_{\text{res}} + {}^H a)(I + N') & ({}^H \nu A_{\text{res}} + {}^H a)\nu + {}^H \nu a + \alpha \end{array}\right) \qquad (4.9) \\
&= \left(\begin{array}{c|c} A_{\text{res}} & a \\ \hline {}^H a & \alpha \end{array}\right).
\end{aligned}$$

Here we denote by I the identity matrix I_{n-1}. When one compares the matrices one obtains from the first $(n - 1) \times (n - 1)$-entries

$$\underbrace{(I + {}^H N')}_{= {}^H (I+N')} A_{\text{res}}(I + N') = A_{\text{res}}.$$

Now we can apply the induction hypothesis and thus get $N' = 0$. Hence we can rewrite the matrix (4.9) and it becomes much simpler. The remainig condition from the equation is

$$\begin{aligned}
 & A_{\text{res}}\nu + a &= a \\
\Leftrightarrow\ & A_{\text{res}}\nu &= 0 \\
\Rightarrow\ & {}^H \nu A_{\text{res}}\nu &= 0
\end{aligned}$$

But, as already mentioned, the matrix A_{res} is definite because it is defined by A, therefore we get $\nu = 0$. Thus $N = 0$ and the statement is proved. $\square$

With the help of this lemma we can prove a proposition about the unipotent radical. Here we have to use that the elements of $W(F)$ are unipotent and then apply the lemma.

Proposition 4.7. *The unipotent radical is*

$$W(F) \;=\; \left\{ g = \left(\begin{array}{c|c|c} 1 & v & w \\ \hline 0 & I_{n-1} & y \\ \hline 0 & 0 & 1 \end{array}\right) ;\; \begin{array}{c} By +^H va = 0, \\ {}^H yBy + \overline{a}w + a\overline{w} = 0 \end{array} \right\} \tag{4.10}$$

Proof. The group $W(F)$ is by definition the subgroup of $N(F)$ consisiting of all unipotent elements of $N(F)$. Therefore an element $g \in W(F)$ has to be of the form

$$g = \left(\begin{array}{c|c|c} 1 & v & w \\ \hline 0 & X & y \\ \hline 0 & 0 & 1 \end{array}\right),$$

where $X = I_{n-1} + T$ with T strict upper triangular. So it remains to show that $T = 0$. As B is definite by definition and X is unipotent the statement follows from Lemma 4.6. $\square$

The fact that the matrix B is definite relies on its construction belonging to the matrix Q and Q defining a hermitian form.
Now we can find the remaining group we need to construct a toroidal compactification.

Lemma 4.8. *The centre of $W(F)$ is then given by the group*

$$U(F) \;=\; \left\{ g = \left(\begin{array}{c|c|c} 1 & 0 & iax \\ \hline 0 & I_{n-1} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) ;\; x \in \mathbb{R} \right\} \tag{4.11}$$

Proof. First note that the first condition of $W(F)$ gives:

$$v =^H \left(-\frac{1}{a}By\right). \tag{4.12}$$

Now we will use that $U(F)$ is the centre of $W(F)$, i.e.

$$\text{centre}(W(F)) = \{g \in W(F);\; gg' = g'g \text{ for all } g' \in W(F)\}.$$

When we calculate these products for $g, g' \in W(F)$ we get a condition and have to use property (4.12):

$$\begin{array}{rrcl} & vy' & = & v'y \\ \Longrightarrow & {}^H\!\left(-\frac{1}{a}By\right) y' & = & {}^H\!\left(-\frac{1}{a}By'\right) y \\ \Longleftrightarrow & {}^H y {}^H B y' & = & {}^H y' {}^H B y \\ \Longleftrightarrow & {}^H yBy' -^H \left({}^H yBy'\right) & = & 0, \end{array}$$

as $B =^H B$ by definition. Clearly the last equivalence implies that ${}^H yBy' \in \mathbb{R}$ for every y'.
The matrix B has full rank as it is invertible and thus

$$B \cdot \mathbb{C}^{n-1} = \mathbb{C}^{n-1}.$$

Therefore set $z' := By' \in \mathbb{C}^{n-1}$. Now we rephrase the property from above as

$$ {}^H yz' \text{ is real for all } z' \in \mathbb{C}^{n-1}. \tag{4.13}$$

As this is true for all vectors we can choose z' to be

$$z' =^T (0, \dots, 0, 1, 0, \dots, 0),$$

where the only coordinate not equal to 0 is the jth. For this choice in (4.13) only the jth coordinate of y remains and therefore $\overline{y}_j \in \mathbb{R}$.
Now let

$$z' =^T (0, \dots, 0, \sqrt{D}, 0, \dots, 0).$$

Then (4.13) becomes $\overline{y}_j \cdot \sqrt{D} \in \mathbb{R}$, and as $D < 0$ this means $y_j \in i\mathbb{R}$. Hence y_j has to lie in the intersection of these spaces.

$$y_j \in \mathbb{R} \cap i\mathbb{R} = \{0\}, \text{ because } z' \text{ varies in } \mathbb{C}^{n-1}.$$

As j is chosen arbitrary we can deduce that this is true for every entry, i.e. $y = 0$. But as $y = 0$ by (4.12) also $v = 0$.
So we have to study the remaining condition given in (4.10), which is $\overline{a}w + \overline{w}a = 0$. We want to describe w more specifically, i.e. in terms of a. For this we write $w = c + id$ and $a = e + if$. So we get

$$\overline{a}w + \overline{w}a = 2(ec + df) = 0.$$

Assuming $e \neq 0$ this implies $c = -d\frac{f}{e}$ and for this reason $w = -d\frac{f}{e} + id$, $d \in \mathbb{R}$. Therefore

$$w \in \mathbb{R}\left(-\frac{f}{e} + i\right) = \mathbb{R}\left(-f + ie\right) = i\mathbb{R}(e + if) = ia\mathbb{R}.$$

The case $f \neq 0$ is similar. □

Now we determined all groups we need to start the process of toroidal compactification. But first we have to restrict the center $U(F)$ from above to the arithmetic group Γ.

Lemma 4.9. *Th restriction of $U(F)$ to Γ induces*

$$U(F)_{\mathbb{Z}} = U(F) \cap \Gamma \cong \mathbb{Z}.$$

Proof. As $\Gamma \subset \mathrm{GL}(n+1, \mathcal{O})$ it is clear that $iax \in \mathcal{O}$. Also note that $x \in \mathbb{R}$ by Lemma 4.8 we have to distinguish two cases:

(i) $D \equiv 2, 3 \mod 4$. Therefore

$$iax = c + d\sqrt{D} \text{ for some } c, d \in \mathbb{Z}. \tag{4.14}$$

Additionally we know that $a \in \mathbb{Q}(\sqrt{D})$ and hence $a = e + f\sqrt{D}$ for some $e, f \in \mathbb{Q}$. Thus we can write equation (4.14) as

$$\begin{aligned} & i(e + f\sqrt{D})x = c + d\sqrt{D} \\ \Leftrightarrow \quad & iex + if\sqrt{D}x = c + d\sqrt{D} \\ \Leftrightarrow \quad & f\sqrt{-D}x + iex = c + d\sqrt{D}. \end{aligned}$$

Therefore $fx\sqrt{-D} \in \mathbb{R}$ and $iex \notin \mathbb{R}$, or to be more precise

$$fx\sqrt{-D} \in \mathbb{Z} \text{ and } iex \in \mathbb{Z}\sqrt{D},$$

so we get $x \in \frac{1}{f\sqrt{-D}}\mathbb{Z} \cap \frac{1}{e}\mathbb{Z}\sqrt{-D} = \frac{1}{f(-D)}\mathbb{Z}\sqrt{-D} \cap \frac{1}{e}\mathbb{Z}\sqrt{-D}$. As $e, f \in \mathbb{Q}$ choose $e = \frac{p}{q}$, $f = \frac{r}{s}$, both with coprime numerator and denominator, and $\tilde{x}\sqrt{-D} = x$, hence

$$\tilde{x} \in \frac{s}{r(-D)}\mathbb{Z} \cap \frac{q}{p}\mathbb{Z}.$$

We claim

$$\frac{s}{rD'}\mathbb{Z} \cap \frac{q}{p}\mathbb{Z} = \frac{\mathrm{lcm}(sp, rD'q)}{rD'p}\mathbb{Z}, \tag{4.15}$$

where for brevity we define

$$D' := -D, \;\; c_1 sp := \mathrm{lcm}(sp, rD'q), \;\; c_2 rD'q := \mathrm{lcm}(sp, rD'q).$$

Note that as c_1, c_2 are defined by the lowest common multiple, they are therefore coprime.

(1) We will first prove '$\supset$'. Let $\eta \in \frac{\mathrm{lcm}(sp, rD'q)}{rD'p}\mathbb{Z}$. Thus we can write with c_1, c_2 defined as above and $c \in \mathbb{Z}$:

$$\begin{aligned} \eta = \frac{\mathrm{lcm}(sp, rD'q)}{rD'p}c \quad &= \quad \frac{c_1 sp}{rD'p}c = \frac{c_2 rD'q}{rD'p}c \\ &= \quad \frac{c_1 s}{rD'}c = \frac{c_2 q}{p}c. \end{aligned}$$

We have to find $a = a(c), b = b(c) \in \mathbb{Z}$, such that we can represent η as $\frac{s}{rD'}a, \frac{q}{p}b$. Now let $a := c_1 c,\ b := c_2 c$ which are integers because they are products of integers. With this choice η lies in $\frac{s}{rD'}\mathbb{Z}$ and in $\frac{q}{p}\mathbb{Z}$ and therefore in

$$\frac{s}{rD'}\mathbb{Z} \cap \frac{q}{p}\mathbb{Z}.$$

(2) Now we deal with '$\subset$'. Now choose $\eta \in \frac{s}{rD'}\mathbb{Z} \cap \frac{q}{p}\mathbb{Z}$, i.e. there exist $a, b \in \mathbb{Z}$ with

$$\eta = \frac{s}{rD'}a = \frac{q}{p}b. \tag{4.16}$$

We have to show that there exist an $c(a, b) = c \in \mathbb{Z}$ with $\eta = \frac{\operatorname{lcm}(sp, rD'q)}{rD'p}c$. Now let $c := \frac{b}{c_2} = \frac{a}{c_1}$. Writing the first part of (4.16) with this choice of c leads to

$$\eta = \frac{s}{rD'}cc_1 = \frac{spc_1}{rD'p}c = \frac{\operatorname{lcm}(sp, rD'q)}{rD'p}c.$$

The other case is analogous. So it remains to show that this choices of c lead to integers. This can be seen in the following way:
By (4.16) we get $sap = qbrD'$, and multiplying this by c_1c_2 gives

$$\begin{aligned} sapc_1c_2 &= qbrD'c_1c_2 \\ \Longleftrightarrow \quad ac_2 \operatorname{lcm}(sp, rD'q) &= bc_1 \operatorname{lcm}(sp, rD'q) \\ \Longleftrightarrow \quad ac_2 &= bc_1. \end{aligned}$$

We know that c_1 and c_2 are coprime because they are defined by the lowest common multiple. From this and the equation above it follows that c_1 divides a and c_2 divides b. Thus $c \in \mathbb{Z}$ as required.

(ii) $D \equiv 1 \mod 4$. We have to use the same argument, only that

$$iax = c + d\frac{1+\sqrt{D}}{2} = \frac{2c+d}{2} + \frac{d\sqrt{D}}{2} \text{ for some } c, d \in \mathbb{Z}. \tag{4.17}$$

As in (i) we know that

$$\begin{aligned}
& fx\sqrt{-D} &&= \frac{2c+d}{2} \text{ and } iex = \frac{d}{2}\sqrt{D} \\
\Longleftrightarrow\quad & x &&= \frac{2c+d}{2fD'}\sqrt{-D} \text{ and } x = \frac{-d}{2e}\sqrt{-D} \\
\Longleftrightarrow\quad & && x \in \begin{cases} \frac{1}{2fD'}\sqrt{-D}2\mathbb{Z} \cap \frac{1}{2e}\sqrt{-D}2\mathbb{Z} \text{ (d even)} \\ \frac{1}{2fD'}\sqrt{-D}(\mathbb{Z}-2\mathbb{Z}) \cap \frac{1}{2e}\sqrt{-D}(\mathbb{Z}-2\mathbb{Z}) \text{ (d odd)} \end{cases} \\
\Longleftrightarrow\quad & && \tilde{x} \in \begin{cases} \frac{1}{fD'}\mathbb{Z} \cap \frac{1}{e}\mathbb{Z} \\ \frac{1}{2fD'}(\mathbb{Z}-2\mathbb{Z}) \cap \frac{1}{2e}(\mathbb{Z}-2\mathbb{Z}) \end{cases} \\
\Longleftrightarrow\quad & && \tilde{x} \in \begin{cases} \frac{s}{rD'}\mathbb{Z} \cap \frac{q}{p}\mathbb{Z} \\ \frac{s}{2rD'}(\mathbb{Z}-2\mathbb{Z}) \cap \frac{q}{2p}(\mathbb{Z}-2\mathbb{Z}) \end{cases} \\
\Longleftrightarrow\quad & && \tilde{x} \in \frac{s}{2rD'}\mathbb{Z} \cap \frac{q}{2p}\mathbb{Z}
\end{aligned}$$

with $e, f, D', \tilde{x}$ as in (i). By the same argument as above it follows that this is

$$\frac{\operatorname{lcm}(sp, rD'q)}{2rD'p}\mathbb{Z}.$$

$\square$

This leads to the construction of a toroidal compactification done in 1.2.2. We have a $\mathbb{Z}$-lattice $U(F)_{\mathbb{Z}}$ of rank 1 in the vector space $U(F)_{\mathbb{C}} = U(F) \otimes_{\mathbb{Z}} \mathbb{C}$. This is exactly the case in which we stated the toroidal compactification. So we can use the results mentioned in 1.2.

4.3 Constructing the compactification

In the last section we have given the foundations for toroidal compactification. In the following we will, as in section 1.2.2, construct the algebraic torus T and add a divisor to compactify $\Gamma\backslash\mathbb{C}H^n$ locally.
First we will choose coordinates on $\mathbb{C}H^n$, namely

$$(t_1 : \cdots : t_{n+1})$$

as $\mathbb{C}H^n$ is an open part of n-dimensional complex projective space. By the definition of $\mathbb{C}H^n$ we can assume $t_{n+1} = 1$.

As the compactification has to be done locally we will consider the compactification in the direction of the cusp F. Therefore we have to consider

$$\mathbb{C}H^n(F) = \mathbb{C}H^n/U(F)_{\mathbb{Z}}.$$

By standard calculations we can give an identification

$$\mathbb{C}H^n(F) \cong \mathbb{C}^* \times \mathbb{C}^{n-1}, \tag{4.18}$$

as there is an action by iax on one component while the remaining components stay the same.
For this identification we will introduce new variables α and w_i as follows:

$$\begin{aligned} t_1 &\mapsto \alpha \in \mathbb{C}, \\ t_i &\mapsto w_i \in \mathbb{C},\ 2 \leq i \leq n. \end{aligned}$$

Sometimes we will deal with the vector given by the w_i's, and therefore denote it by $\underline{w}$.
As we need an explicit description of the action of the stabiliser group restricted to Γ we will state the action of the group $N(F)$ on $\mathbb{C}H^n(F)$. The restriction is defined as $N(F)_{\mathbb{Z}} := N(F) \cap \Gamma$

Lemma 4.10. *If*

$$g = \left(\begin{array}{c|c|c} u & v & w \\ \hline 0 & X & y \\ \hline 0 & 0 & z \end{array}\right) \in N(F),$$

then g acts on $\mathbb{C}H^n$ by

$$\begin{aligned} \alpha &\mapsto \frac{1}{z}\left(\frac{\alpha}{\overline{z}} + v\underline{w} + w\right), & (4.19) \\ \underline{w} &\mapsto \frac{1}{z}\left(X\underline{w} + y\right). & (4.20) \end{aligned}$$

Proof. This easily follows from the computation

$$\left(\begin{array}{c|c|c} u & v & w \\ \hline 0 & X & y \\ \hline 0 & 0 & z \end{array}\right) \begin{pmatrix} \alpha \\ \underline{w} \\ 1 \end{pmatrix} = \begin{pmatrix} u\alpha + v\underline{w} + w \\ X\underline{w} + y \\ z \end{pmatrix} = \begin{pmatrix} \dfrac{u\alpha + v\underline{w} + w}{z} \\ \dfrac{X\underline{w} + y}{z} \\ 1 \end{pmatrix}.$$

and the property $u = \frac{1}{\overline{z}}$ given in Lemma 4.5. □

Now we will introduce the algebraic torus T from section 1.2.2. This is

$$T := U(F)_{\mathbb{C}}/U(F)_{\mathbb{Z}} \cong \mathbb{C}^*, \tag{4.21}$$

as $U(F)_{\mathbb{C}}/U(F)_{\mathbb{Z}}$ is isomorphic to $\mathbb{C}/\mathbb{Z}$.
Since we will study the singularities at the boundary in the next section we choose a variable θ on the algebraic torus T. This variable is given by

$$\theta := \exp_a(\alpha) := \begin{cases} e^{\frac{2\pi i}{ia\frac{\operatorname{lcm}(sp,rD'q)}{rD'p}\sqrt{-D}}\alpha} = e^{\frac{2\pi rD'p}{a\operatorname{lcm}(sp,rD'q)\sqrt{-D}}\alpha}, & D \equiv 2,3 \mod 4, \\ e^{\frac{2\pi i}{ia\frac{\operatorname{lcm}(sp,rD'q)}{2rD'p}\sqrt{-D}}\alpha} = e^{\frac{4\pi rD'p}{a\operatorname{lcm}(sp,rD'q)\sqrt{-D}}\alpha}, & D \equiv 1 \mod 4. \end{cases} \tag{4.22}$$

For this definition we use the same notation for r, s, p, q, D' as we did in the proof of Lemma 4.9.
One can see that this is the right choice as follows. We know that the variable θ has to be invariant under the action of $U(F)_{\mathbb{Z}}$, or more explicitly under the action

$$\alpha \mapsto \alpha + iax.$$

This is true for iax computed in the proof of Lemma 4.9.
As we have to use this exponent later again, define

$$\sigma := \begin{cases} \frac{ia\operatorname{lcm}(sp,rD'q)}{rD'p}\sqrt{-D}, & D \equiv 2,3 \mod 4, \\ \frac{ia\operatorname{lcm}(sp,rD'q)}{2rD'p}\sqrt{-D}, & D \equiv 1 \mod 4. \end{cases} \tag{4.23}$$

Using this definition we can write $iax = \sigma b$ for a $b \in \mathbb{Z}$.
Now we define $G(F)$ as $N(F)_{\mathbb{Z}}/U(F)_{\mathbb{Z}}$. Let $g \in G(F)$ and suppose g has order m, where we can assume $m > 1$. We will also write g if we think of g as an element of $N(F)$.

If we want to compactify $\Gamma\backslash\mathbb{C}H^n$ in the direction of F we have to follow the steps mentioned in 1.2.2. Thus we have to replace the torus T by the toric variety T_Σ. But as already shown in 1.2 the toric variety is simply $T_\Sigma = \mathbb{C}$.
In our situation the compactification of $\Gamma\backslash\mathbb{C}H^n$ locally means that we allow θ to be zero.
If we do this we add one point for each point over the basis $\mathbb{C}^{n-1}$ as given in (4.18). In the local situation this means that we add

$$\{0\} \times \mathbb{C}^{n-1}$$

to the boundary. This has to be divided by the action of $G(F)$, which by Proposition 1.26 extends uniquely to the boundary.

4.4 Singularities at the boundary

Having constructed the toroidal compactification and described its structure, we will now consider the singularities that can arise at the boundary. As in Chapter

3 we will show that we can choose the dimension high enough (under a restriction on D) such that the compactification has canonical singularities.
Similarly to the interior case we make use of the Reid-Tai criterion. Therefore we will make analogous assumptions as those we needed in 3.
Suppose that g fixes the boundary point $(0, \underline{w}_0)$ for an arbitrary fixed $\underline{w}_0 \in \mathbb{C}^{n-1}$. Now we are in a situation as before and can define the Reid-Tai-sum $\Sigma(g) = \sum \left\{\frac{a_i}{m}\right\}$, where ζ^{a_i} denote the eigenvalues of the action on the tangent space and ζ a primitive mth root of unity, as g is assumed to be of order m.
In the following proposition we have to make use of the units in the number field we are considering.

Remark 4.11. It is classical result in algebraic number theory that the invertible elements in the ring of integers $\mathcal{O}_{\mathbb{Q}(\sqrt{D})}$ of an imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$ are

$$\begin{cases} \langle\zeta_4\rangle, & \text{if } D = -1, \\ \langle\zeta_6\rangle, & \text{if } \ D = -3, \\ \{\pm 1\}, & \text{otherwise,} \end{cases}$$

where ζ_k denotes, as usual, a primitive kth root of unity.

Similar to the previous chapter we will first exclude quasi-reflections.

Proposition 4.12. *Let no power of g act as a quasi-reflection on the boundary point $(0, \underline{w}_0)$ and $D < D_0$. Then $\Sigma(g) \geq 1$.*

Proof. As $D < D_0$ we can assume that $z = \pm 1$, because z is invertible in $\mathcal{O}$ as $z\overline{u} = 1$ by Lemma 4.5 and Remark 4.11. Now we have to determine the action of g on the tangent space. This is for obvious reasons given by the matrix

$$\begin{pmatrix} \exp_a(\pm(v\underline{w}_0 + w)) & 0 \\ * & \pm X \end{pmatrix}. \tag{4.24}$$

We denote the order of X by m_X and investigate the decomposititon of the representation X. As before the representation decomposes into a direct sum of $\mathcal{V}_d$'s. Remember that we denote by $\mathcal{V}_d$ the irreducible representation, so it can mean V_d or $V_d{}'$ resp. $V_d{}''$, as in Definition 3.23. We have to distinguish two cases.

(1) First assume that $m_X > 2$. In this case we are in the situation of Lemma 3.24, as we are in case $D < D_0$ and the only irreducible 1-dimensional representations are V_1 and V_2. So by the lemma we get $\Sigma(g) \geq 1$.

(2) Now let $m_X = 1$ or $m_X = 2$. The action of $-1 \in \Gamma$ is trivial and so we can get $z = 1$ by replacing g by $-g$.
First assume $m_X = 1$ and hence $X = I$. As the element g fixes the boundary point $(0, \underline{w}_0)$ we get $y = 0$ from Lemma 4.10 and then by the group relations

of Lemma 4.5 we have $v = 0$ as ${}^{H}va = 0$. So the element g has to have the form

$$g = \left(\begin{array}{c|c|c} 1 & 0 & w \\ \hline 0 & I & 0 \\ \hline 0 & 0 & 1 \end{array}\right),$$

and for this reason $g \in U(F)_{\mathbb{Z}}$. This implies that, viewed in $N(F)_{\mathbb{Z}}/U(F)_{\mathbb{Z}}$, g is the identity.
Finally we have to check the case $m_X = 2$. So $g^2 \in U(F)_{\mathbb{Z}}$, and therefore we get the relations

$$\begin{aligned} v + vX &= 0, && (4.25) \\ Xy + y &= 0, \\ 2w + vy &\equiv 0 \mod \sigma, && (4.26) \end{aligned}$$

where σ is as before.
We will only consider the case $D \equiv 2, 3 \mod 4$ as the case $D \equiv 1 \mod 4$ is analogous (cf. proof of Lemma 4.9).
Define $t := v\underline{w}_0 + w$ which is the argument of the above given exponential map in the Jacobi matrix of the action on the tangent space. We want to show $2t \equiv 0 \mod \sigma\mathbb{Z}$ as this implies

$$\exp_a(t) = \pm 1.$$

We will now use $\underline{w}_0 = X\underline{w}_0 + y$ as g fixes the boundary point and the relations (4.25), (4.26). Hence we get

$$\begin{aligned} 2t = 2v\underline{w}_0 + 2w &\equiv 2v\underline{w}_0 - vy \\ &= v\underline{w}_0 + v\underline{w}_0 - vy = v\underline{w}_0 + v(\underline{w}_0 - y) \\ &= v\underline{w}_0 + vX\underline{w}_0 = v(I + X)\underline{w}_0 \\ &\equiv 0 \mod \sigma\mathbb{Z}. \end{aligned}$$

Therefore all the eigenvalues on the tangent space are ± 1, because X has order 2 and $\exp_a(t) = \pm 1$ for t as above.
So there are two possibilities: All but one eigenvalues are $+1$, so g acts as a reflection (in this case all quasi-reflections have order 2), or there are at least two eigenvalues -1 and the remaining are $+1$, so we will have $\Sigma(g) \geq 1$.

□

So if we use this proposition we can give a result for the divisors at the boundary over a boundary component.

Corollary 4.13. *At the boundary there are no divisors at the boundary over a dimension 0 cusp F that are fixed by a non-trivial element of $N(F)_{\mathbb{Z}}/U(F)_{\mathbb{Z}}$ in the case $D < D_0$.*

Proof. Each divisor at the boundary has $\theta = 0$. The only elements fixing a divisor are the quasi-reflections. The variable θ corresponds to the entry $\exp_a(\pm(v\underline{w}_0 + w))$ from the induced action on the tagent space. From the proof of Proposition 4.12 each matrix X belonging to a quasi-reflection has order greater 1. Thus no divisor $\theta = 0$ is fixed. □

Finally we have to mention quasi-reflections at the boundary. We will do this similary as for Proposition 3.34. Therefore define $\Sigma'(g)$ for $g \in G(F)$ as before and $h = g^k$ to be a quasi-reflection, where k is chosen minimal with this property.

Proposition 4.14. *Let $g \in G(F)$ be such that $h = g^k$ is a quasi-reflection. Assume that $n \geq 13$ and $D < D_0$, where $D_0 = -3$. Then $\Sigma'(g^f) \geq 1$ for every $1 \leq f < k$.*

Proof. The proof is similar to the proof of [GHS07, Proposition 2.30]. We will again study the action of h on the tangent space. If the eigenvalue not equal to 1 is $\exp_a(t)$, then X^f contributes at least 1 to $\Sigma'(g^f)$.
Now denote this unique eigenvalue of h on the tangent space by $\zeta \neq 1$. Let ν be the exceptional eigenvector of of h with the property $h(\nu) = \zeta \cdot \nu$. Assume that ν occurs in the representation $\mathcal{V}_d$, where we consider the decomposition of X as a g-module. The dimension of $\mathcal{V}_d$ has to be 1 as otherwise it would contribute another eigenvalue not equal to 1. Now we study the g-module

$$E^{\perp}_{\mathbb{Q}(\sqrt{D})}/(E_{\mathbb{Q}(\sqrt{D})} + \mathbb{Q}(\sqrt{D})\nu),$$

which is $(n-2)$-dimensional. We can refer to Theorem 3.25 as long as $D < D_0$. So if $n - 2 \geq 11$ we get $\Sigma(g) \geq 1$ and therefore $\Sigma'(g) \geq 1$. □

In the proof we use that for $D < D_0$ the only 1-dimensional representations are $\mathcal{V}_1$ and $\mathcal{V}_2$. Now we can state the main theorem that gives a bound for canonical singularities for the toroidal compactification.

Theorem 4.15. *Let $n \geq 13$ and $D < D_0$. Then the toroidal compactification $(\mathbb{C}H^n/\Gamma)^*$ of $\mathbb{C}H^n/\Gamma$ has canonical singularities. Furthermore, there are no fixed divisors in the boundary.*

Proof. This is a consequence of Theorem 3.35, Proposition 4.12, Proposition 4.14 and Corollary 4.13. □

As in section 3.5 one could ask for singularities in the case $D = -3$.

Remark 4.16. When we assume $D = -3$ most of the techniques we used will still work. But one should watch out in the proof of Proposition 4.12. There are two steps that has to be studied more intensively:

(1) In the proof we can assume $z = \pm 1$. This is no longer true as the units are now elements of $\langle \zeta_6 \rangle$.

(2) At some point we make a statement about the irreducible 1-dimensional representations. In case $D = -3$ there are some more representations to consider.

The same is true for $D = -1, -2$ (in case $D = -2$ the problem mentioned in (1) does not occur).

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10/2001-10/2006	Leibniz Universität Hannover: Studium der Mathematik mit Abschluss Diplom-Mathematiker, Thema der Diplomarbeit: Calabi-Yau 3-Varietäten mit elliptischen Faserungen über Del Pezzo-Flächen
08/1993-06/2000	Gymnasium Helene-Lange-Schule, Hannover
08/1991-06/1993	Orientierungsstufe Badenstedt, Hannover
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Berufliche Tätigkeiten

seit 12/2006	Wissenschaftlicher Mitarbeiter am Institut für Algebraische Geometrie, Leibniz Universität Hannover
10/2004-03/2006	Studentische Hilfskraft am Institut für Mathematik, Universität Hannover
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