# Dual-dual formulations for frictional contact problems in mechanics

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To my Mum.

## Abstract

This thesis deals with unilateral contact problems with Coulomb friction. The main focus of this work lies on the derivation of the dual-dual formulation for a frictional contact problem. First, we regard the complementary energy minimization problem and apply Fenchel's duality theory. The result is a saddle point formulation of dual type involving Lagrange multipliers for the governing equation, the symmetry of the stress tensor as well as the boundary conditions on the Neumann boundary and the contact boundary, respectively. For the saddle point problem an equivalent variational inequality problem is presented. Both formulations include a nondifferentiable functional arising from the frictional boundary condition. Therefore, we introduce an additional dual Lagrange multiplier denoting the friction force. This procedure yields a dual-dual formulation of a two-fold saddle point structure. For the corresponding variational inequality problem we show the unique solvability. Two different inf-sup conditions are introduced that allow an a priori error analysis of the dual-dual variational inequality problem.

To solve the problem numerically we use the Mixed Finite Element Method. We propose appropriate finite element spaces satisfying the discrete version of the inf-sup conditions. A modified nested Uzawa algorithm is used to solve the corresponding discrete system. We prove its convergence based on the discrete inf-sup conditions.

Furthermore, we present a reliable a posteriori error estimator based on a Helmholtz decomposition. Numerical experiments are performed to underline the theoretical results.

**Keywords.** Mixed Finite Element Method, dual formulation, saddle point problem, variational inequality, inf-sup condition, error estimator

# Zusammenfassung

In dieser Arbeit betrachten wir Ein-Körper-Kontaktprobleme mit Coulomb Reibung. Das Ziel dieser Arbeit ist eine Herleitung der dual-dualen Formulierung dieser Reibungskontakprobleme. Ausgehend vom Minimierungsproblem der komplementären Energie leiten wir mit Hilfe der Fenchel'schen Dualitätstheorie ein äquivalentes Sattelpunktsproblem her. Dieses beinhaltet Lagrange Multiplikatoren für die zugehörige Differentialgleichung, die Symmetrie des Spannungstensors sowie die Randbedingungen auf dem Neumannrand und dem Kontaktrand. Für das Sattelpunktsproblem geben wir ein äquivalentes Variationsungleichungsproblem an. Beide Formulierungen enthalten ein nichtdifferenzierbares Reibungsfunktional. Daher führen wir mit der Reibungskraft einen weiteren Lagrange Multiplikator ein und erhalten somit eine dual-duale Formulierung des Problems, welches eine zweifache Sattelpunktsstruktur aufweist. Für das zugehörige Variationsungleichungsproblem zeigen wir die eindeutige Lösbarkeit. Basierend auf zwei Inf-Sup-Bedingungen können wir eine a priori Analysis des dual-dualen Variationsungleichungsproblems durchführen.

Wir verwenden die Gemischte Finite Elemente Methode zur numerischen Approximation der Lösung des Problems. Hierzu führen wir geeignete Finite Element Räume ein, welche die diskrete Version der Inf-Sup-Bedingungen erfüllen. Um das diskrete System zu lösen, benutzen wir einen modifizierten, geschachtelten Uzawa Algorithmus. Wir beweisen seine Konvergenz mit Hilfe der diskreten Inf-Sup-Bedingungen.

Darüberhinaus leiten wir einen, auf einer Helmholtz-Zerlegung basierenden, a posteriori Fehlerschätzer her. Zudem stellen wir numerische Experimente vor, die unsere theoretischen Ergebnisse bestätigen.

**Schlagwörter.** Gemischte Finite Elemente Methode, duale Formulierung, Sattelpunktproblem, Variationsungleichung, Inf-Sup-Bedingung, Fehlerschätzer

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# Contents

# 1. Introduction

Contact problems in mechanics are investigated for more than hundred years. In 1881 Hertz [58] analyzed the so-called Hertz contact problem where he derives analytical representations for the shape of the contact area between two elastic spheres and determines the resulting contact pressure. The need for a more precise analysis and simulation in manufacturing and other fields of engineering science came along with better approximations of real life problems in terms of mathematical models and their numerical resolution.

The mathematical theory of elasticity, see Sokolnikoff [79], is governed by Lamé's equation. Combined with Hooke's material law we obtain problems of linear elasticity. If the stresses occurring in a body subject to extern and intern forces exceed a certain value, e.g. the yield stress, the theory of linear elasticity is no longer valid and more complex nonlinear models have to be used, e.g. models for plasticity. For an introduction to elasto-plasticity we refer to Nečas and Hlaváček [69]. The theory of plasticity is described in Han and Reddy [57] and an overview of other inelastic models can be found in Simo and Hughes [78].

The second field where nonlinearities appear are contact boundary conditions. In the simplest setting some linear elastic body is coming into contact with a rigid frictionless foundation. This is the so-called Signorini problem, see Kikuchi and Oden [63]. As the material points of the body must not pervade the rigid foundation they satisfy a nonpenetration condition, which is an inequality condition. Therefore, the theory of variational inequalities is closely connected to contact problems. In most cases contact problems are formulated in terms of variational inequality problems. Duvaut and Lions [31], Glowinski et al. [53], Kikuchi and Oden [63] and Nečas et al. [60] give an elaborate review of this topic. An abstract introduction to variational inequalities can be found in Kinderlehrer and Stampacchia [64] and Glowinski [52]. In the latter work the author distinguishes between variational inequalities of the first and of the second kind. Variational inequalities of the first kind are restricted to a convex set. For example the displacement field solving the Signorini problem is restricted to all vector fields that do not violate the nonpenetration condition. Variational inequalities of the second kind are formulated on a whole space but involve a nonlinear convex functional, for example if the rigid foundation in the Signorini problem causes friction. In this case the tangential part of the stress on the contact boundary depends on the normal part. The common model of this phenomenon is the Coulomb law of friction, where the absolute value of the tangential stress cannot exceed a multiple of the absolute value of the normal stress, denoted as the friction force. A simpler model is the law of Tresca friction. Here, the friction force is assumed to be given.

There are several other models known in literature. Oden and Martins [74] give an extensive overview. Johansson and Klarbring [62] consider additionally the influence of temperature and state a thermoelastic frictional contact model. Oden and Pires [73] propose a Coulomb friction like model with a nonlocal relation between tangential and normal stresses.

This work is limited to the case of unilateral linear elastic contact problems with Coulomb friction. Since we deal with linear elastic materials, the contact problem can also be stated as a minimization problem. The principle of the minimum potential energy states, that the displacement field, solving the contact problem, minimizes the potential energy of deformation. Similarly the corresponding stress minimizes the complementary energy, see Sokolnikoff [79]. The first one is denoted as the primal problem, whereas the second one is named dual problem. In this way the displacement field is called the primal variable and the corresponding stress tensor the dual variable. Tools from convex analysis, see Ekeland and Temam [32] and Ito and Kunisch [61], allow us to reformulate the minimization problems as saddle point problems, which on their parts are equivalent to variational inequality problems, under certain assumptions. These tools are summarized in the theory of Fenchel's duality. The introduction of a saddle point problem is attended by the usage of Lagrange multipliers. In the primal case the primal-dual problem regards the displacement field as primal variable and the friction force as dual variable. This approach is quite popular in literature. Glowinski et al. [53, 52] present an extensive analyzation of this approach. Among many others we mention Suttmeier [81] who chooses the primal-dual approach to solve grinding processes. Hild and Renard [59] and Dörsek and Melenk [30] introduce the normal stress as well as the tangential stress on the contact boundary as additional Lagrange multipliers. The latter solves the problem with an adaptive hp-Finite Element Method. Finally, Chernov, Maischak and Stephan [25] use the primal-dual formulation in combination with an hp-Mortar Boundary Element Method to solve a two-body frictional contact problem.

The main backdraw of solving the primal problem is the fact, that the stress tensor denotes a more important quantity in engineering sciences. For example the stress is a measure to determine plastic zones that may appear in a body under extern loads. As in the primal formulation the displacement field is approximated, the stress tensor has to be computed in a postprocessing which yields a further source of error. For this reason, we propose in this work an approach based on the dual problem. For the Laplace problem with unilateral frictionless contact boundary conditions Wang and Wang [84] solve a dual variational inequality problem of the first kind. A similar approach is used by Wang and Yang in [85] for a unilateral frictionless contact problem in linear elasticity. Maischak [67] presents a dual approach for a transmission problem with Signorini conditions, see also Gatica, Maischak and Stephan [49]. The probelm is solved via the coupling of FEM and BEM. For a transmission problem with friction a similar approach is presented by Maischak and Stephan in [68]. Bostan, Han and Reddy [13] derive the dual formulation for a scalar elliptic model problem involving a variational inequality of the second kind with the help of Fenchel's duality theory. Bostan and Han [12] extend this approach to some linear elastic problem with Tresca friction and homogeneous normal displacement on the contact boundary. Finally we would like to refer to Kunisch and Stadler [65] who consider the linear elastic contact problem with Coulomb friction. Using Fenchel's duality theory they derive the dual problem with the friction function as additional Lagrange multiplier for the contact problem with Tresca friction. The authors have to regularize the problem to use a semi-smooth Newton method. Finally, they use an augmented Lagrangian method to solve the original problem.

The approach within this work will include both, variational inequalities of the first and of the second kind. We also apply the theory of Fenchel's duality. But, unlike the above mentioned approaches, where the conjugate problem of the primal problem is derived, we will compute the conjugate problem of the dual problem. The first approach results in a dual minimization problem, usually involving dual variables as the stress inside the body and its normal and tangential parts on the contact boudary. In contrast, our approach leads to the primal problem with additional Lagrange multipliers, denoting the displacement on the Neumann boundary and the contact boundary. After applying the Fenchel's duality theorem, we derive a saddle point problem with the stress tensor as primary unknown. The displacement decomposed into volume and boundary parts and the rotation of the displacement are considered additionally in terms of Lagrange multipliers. Unfortunately, the corresponding equivalent variational inequality problem involves a nondifferentiable functional. To overcome this difficulty, another Lagrange multiplier, denoting the friction force, is introduced. This leads us to a problem having two-fold saddle point structure and, as the friction force denotes a dual variable, to the term of a dual-dual formulation.

The notion of dual-dual formulations was first introduced by Gatica [34] in 1999. Within this work the solvability of variational problems having dual-dual form is presented extending the theory of Babuška and Brezzi. As an application a nonlinear exterior transmission problem is considered, where the two fold structure results from the coupling of the Boundary Element Method in the exterior region with the Mixed Finite Element Method in the interior region. The work was first presented as technical report and released in parts in [33] whereas the final version [34] was published in 2002. The theory is extended to problems concerning plane hyperelasticity by Gatica and Heuer in [35] where the authors also propose suitable finite element subspaces. Finally the theory is generalized by Gatica, Heuer and Meddahi in [40]. The dual-dual formulation for a nonlinear exterior transmission problem is further investigated by Gatica and Meddahi in [43]. Moreover Gatica and Heuer present solution algorithms for dual-dual type problems in [37, 36, 38, 39].

The approximation of saddle point problems can be realized by the Mixed Finite Element Method, see Brezzi and Fortin [18] for a detailed introduction and description. The main difficulty is to find appropriate finite element spaces for which the resulting linear system is regular. Babuška [6, 7] and Brezzi [17] developed the theory of the Mixed Finite Element Method. The key is the so-called Babuška-Brezzi condition which assures unique solvability of the saddle point problem. In the discrete case it implies the regularity of the matrix in the corresponding linear system. In plane elasticity the PEERS elements introduced by Arnold, Brezzi and Douglas Jr. [5] are well suited for the triple of stress tensor, displacement field and rotation tensor. Other possible choices are listed in [18]. Our approach is based on the various contributions by Gatica et al. [45, 48, 10, 46, 19, 47, 11, 42] where the authors introduce several mixed finite element formulations for problems in plane elasticity, some of them having two-fold or even three-fold saddle point structure.

Many of the above cited references also introduce a posteriori error estimators. An introduction to the topic can be found in Verfürth [82] and Ainsworth and Oden [2]. Furthermore, Han [56] presents approaches to a posteriori error estimation via the Fenchel duality theory. One of these approaches is used in the above mentioned work by Dörsek and Melenk [30] for an estimate of the primal-dual formulation for a contact problem with friction. We will restrict ourselves to an a posteriori error estimator of residual type which is based on the works of Maischak [67], Gatica and Stephan [44], Carstensen and Dolzmann [22] and Gatica and Meddahi [50]. In [67] the author uses the Helmholtz decomposition introduced by Alonso [3] and Carstensen [21] to derive an a posteriori error estimator for an interface problem with Signorini contact.

### Outline of the work

In Chapter 2 we introduce the basic notations and some necessary definitions, particularly the Sobolev space $\mathbf{H}(\mathrm{div}, \Omega)$. The chapter finishes with a short summary of Fenchel's duality theory from convex analysis.

Chapter 3 forms the main part of this work. We introduce the boundary value problem and discuss the contact boundary conditions. After explaining the friction laws of Tresca and Coulomb in detail we give a short overview of the primal approach. In Section 3.1 we derive the dual-dual formulation of the contact problem with Tresca friction. This process is divided into several steps. After introducing the minimization problem related to the complementary energy principle we apply the theory of Fenchel's duality in Section 3.1.1. Using the Fenchel duality theorem we arrive at the saddle point problem with additional Lagrange multipliers in Section 3.1.2. The equivalence of the saddle point problem with the minimization problems of the previous section is proven in Theorem 3.10. The saddle point problem is equivalent to a variational inequality problem, which is presented in Section 3.1.3. As the problem involves the nondifferentiable functional concerning the friction force on the contact boundary a further Lagrange multiplier is introduced. In this way in Section 3.1.4 a second saddle point problem is derived, having a two-fold structure. We show the equivalence of the corresponding variational inequality problem with the variational inequality problem of the previous section in Theorem 3.17. Taking advantage of the existence and unique solvability of the minimization problems and the equivalence of the subsequent problems we show existence and unique solvability of the variational inequality problems in Section 3.1.5. Finally, in Section 3.1.6 we present two inf-sup conditions for the bilinear forms concerning the Lagrange multipliers. These conditions are necessary for the error analysis of the discrete formulation. We finish the section with the continuous dependency of the solution on the given data.

In Section 3.2 we introduce appropriate mixed finite elements for solving the variational inequality problem numerically. First we explain the setting of the discretization of the domain and define the finite element spaces to approximate the solutions of the continuous formulation. We introduce the discrete variational inequality problem in Section 3.2.1 and present some conclusions that are necessary for further observations. In Section 3.2.2 we show, that the finite element spaces satisfy the discrete versions of the inf-sup conditions. We prove two versions of the inf-sup condition for the dual Lagrange multiplier approximating the friction force. The first one is proven for a mesh-dependent norm, but we show, that it holds for a larger class of problems. On the other hand the second one gets along with a norm, that is independent of the meshsize but needs a further assumption on the given Tresca friction. The section closes with an error analysis in Section 3.2.3. We prove a Céa-type estimate and give an a priori estimate of the error under certain regularity assumptions.

Section 3.3 deals with the algorithm, that we propose for the solution of the discrete variational inequality problem. To improve readability we rewrite the problem in an algebraic form in Section 3.3.1. In Section 3.3.2 we introduce an Uzawa-type algorithm for the solution of a contact problem without friction. A similar algorithm was already proposed by Maischak in [67] for a transmission problem with Signorini boundary conditions. However, we explain the algorithm in detail, as we will need it to prove the convergence of the nested Uzawa algorithm in Section 3.3.3. This algorithm solves, in a nested loop, the variational inequality problem for the contact problem with Tresca friction. We explain, how to adjust the algorithm for the solution of the contact problem with Coulomb friction, without the need of a further loop. For the solution of the variational inequalities we rewrite the subproblems for contact and friction on the contact boundary as two minimization problems with inequality constraints, respectively. We propose an interior point method of predictor-corrector type for quadratic programs as solver for the subproblems. Finally, in Section 3.4, several numerical experiments are presented to underline the theoretical results of the previous sections. We investigate the rate of convergence and examine the sensitivity of the solution algorithm on the controlling parameters.

To perform a faster convergence of the solution algorithm we introduce an a posteriori error estimator of residual type in Chapter 4. The estimator is based on a local Helmholtz decomposition of the error of the symmetric stress tensors as proposed by Carstensen and Dolzmann [22]. We prove reliability of the estimator in Section 4.1.

In Section 4.2 we investigate the performance of the estimator on some numerical experiment.

# 2. Basic foundations

In this chapter we give a brief overview of the theoretical background concerning this work. Section 2.1 collects definitions, notations and equations that are often used within this work. Section 2.2 deals with some properties of the Sobolev space $\mathbf{H}(\mathrm{div}, \Omega)$. The results and their proofs can be found in the works of Girault and Raviart [51] and Duvaut and Lions [31]. Finally Section 2.3 states some needful results from convex analysis. The proofs can be found in the works of Ekeland and Temam [32] and Ito and Kunisch [61].

Throughout the work we will use several results from different references. Whenever it is needful to repeat the result in this work, we will use italic type.

## 2.1. Notations

Within this work we consider functions on some open bounded domain $\Omega$ with Lipschitz boundary $\Gamma := \partial\Omega$. Those functions can be scalar, vector valued or tensors of second order. In the interest of improving readability we therefore use bold letters for all vector valued functions. Small greek letters denote tensors of second order on the one hand as well as functions on some part of the boundary on the other hand. In this case the right meaning should be clear from the context. We use a similar notation for function spaces which are usually abbreviated with capital letters, e.g. for some function space $X$ containing scalar functions we use $\mathbf{X}$ for the space of those vector valued functions whose components live in $X$. A special treatment has to be done in case of tensors of second order. We do this exemplary in Section 2.2 for the spaces $H(\mathrm{div}, \Omega)$ and $\mathbf{H}(\mathrm{div}, \Omega)$, respectively.

Sometimes we will use an index notation and the Einstein notation for sums over the same index labels. We use a comma to denote a differentiation. For a function $\mathbf{u} \in \mathbf{H}^1(\Omega)$ we have for example

$$\nabla \mathbf{u} = u_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j \qquad \text{and} \qquad \mathrm{div}\, \mathbf{u} = u_{i,i} = \sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i}.$$

Here the vector product $\mathbf{e}_i \otimes \mathbf{e}_j$ of the cartesian base vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ denote the component of the tensor. In the above example $i$ is the row index and $j$ the column index of the gradient tensor $\nabla \mathbf{u}$. For tensors of second order we use

$$\sigma : \tau := \sum_{i,j=1}^{d} \sigma_{ij}\tau_{ij}$$

as the inner tensor product. Note that in general the product is defined as the multiplication of the two right most indices of the left multiplier with the first two indices of the right multiplier. As we consider contact problems with continua of linear elastic isotropic material we are faced with Hooke's law $\sigma(\mathbf{u}) := \mathbb{C} : \varepsilon(\mathbf{u})$, stating the dependence of the stress tensor $\sigma(\mathbf{u})$ and the linearized strain tensor $\varepsilon(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$ of some displacement field $\mathbf{u}$. This involves the elasticity tensor of fourth order $\mathbb{C}$ whose components are defined as

$$\mathbb{C}_{ijkl} := \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2.1}$$

using the Lamé constants $\lambda$ and $\mu$ and the Kronecker delta. Furthermore we have the following symmetry properties of the elasticity tensor (see Chapter 3 in Sokolnikoff [79] for more details)

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}. \tag{2.2}$$

The ellipticity and symmetry of $\mathbb{C}$ yields the existence of $\mathbb{C}^{-1}$ and $\alpha > 0$ such that

$$\tau : \mathbb{C}^{-1} : \tau \geq \alpha\, \tau : \tau \qquad \forall\, \tau, \tag{2.3}$$

(see Chapter 3 in Duvaut and Lions [31]). Note that the symmetry properties (2.2) are also valid for the inverse elasticity tensor $\mathbb{C}^{-1}$. We often use the decomposition of a tensor into its symmetric and antisymmetric parts. These are defined for an arbitrary tensor $\tau$ as

$$\mathrm{sym}(\tau) := \frac{1}{2}(\tau + \tau^T) \qquad \mathrm{as}(\tau) := \frac{1}{2}(\tau - \tau^T). \tag{2.4}$$

Note that the symmetric part of the gradient of any vector field $\mathbf{u} \in \mathbf{H}^1(\Omega)$ equals the linearized strain tensor of $\mathbf{u}$, i.e. $\varepsilon(\mathbf{u}) = \mathrm{sym}(\nabla\mathbf{u})$. If not explicitly identified we usually use $C$ as a generic positive constant which can change its value within computations. In elasticity problems Korn's inequality assures uniqueness of the primal problem. Since we will use the primal problem as an auxiliary problem in our work we state the result which stems from Duvaut and Lions [31, see Theorems 3.1 and 3.3 of Chapter III].

**Theorem 2.1: Korn's second inequality**
Let $\Omega$ be a bounded open set with regular boundary $\Gamma$ and let the Dirichlet boundary $\Gamma_D$ have positive measure. Let

$$\mathbf{V}_D := \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}.$$

Then there exists an $\alpha_0 > 0$ such that

$$a(\mathbf{v}, \mathbf{v}) := \int_\Omega \varepsilon(\mathbf{v}) : \mathbb{C} : \varepsilon(\mathbf{v})\, dx \geq \alpha_0 \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \qquad \forall\, \mathbf{v} \in \mathbf{V}_D. \tag{2.5}$$

Finally from the second binomial theorem we have the following inequality which will be used very often. Using $0 \leq (\varepsilon a - b)^2 = \varepsilon^2 a^2 - 2\varepsilon ab + b^2$ for two arbitrary integers $a$ and $b$ we get

$$ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2 \qquad \forall\, \varepsilon > 0. \tag{2.6}$$

## 2.2. Some properties of the Sobolev space $\mathbf{H}(\mathrm{div}, \Omega)$

If we are dealing with dual formulations in elasticity we are faced with tensors of second order. In the equilibrium equation of the mechanical system the divergence operator is acting on the stress tensor. If we further assume that the volume force on the right hand side of the equilibrium equation is at least $L^2$-integrable we are led to the space

$$\mathbf{H}(\mathrm{div}, \Omega) := \{\tau = (\tau_{ij})_{i,j=1}^d \,|\, (\tau_{ij})_{j=1}^d \in H(\mathrm{div}, \Omega) \text{ for } i = 1, \ldots, 3\},$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is an open bounded domain and

$$H(\mathrm{div}, \Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) := (L^2(\Omega))^d \,|\, \mathrm{div}\, \mathbf{v} \in L^2(\Omega)\}$$

equipped with the norm

$$\|\mathbf{v}\|_{H(\mathrm{div},\Omega)} = (\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathrm{div}\, \mathbf{v}\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Consequently $\mathbf{H}(\mathrm{div}, \Omega)$ is equipped with the norm

$$\|\tau\|_{\mathbf{H}(\mathrm{div},\Omega)} = \left( \sum_{i=1}^d \|(\tau_{ij})_{j=1}^3\|^2_{H(\mathrm{div},\Omega)} \right)^{\frac{1}{2}}.$$

From Girault and Raviart [51, see Chapter I] we have the following trace theorem concerning $H(\mathrm{div}, \Omega)$.

**Theorem 2.2:**
Let $\mathbf{n}$ denote the outer normal on the boundary $\Gamma := \partial\Omega$ of the domain $\Omega$. Then the mapping

$$\gamma_{\mathbf{n}} : \mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{n}|_\Gamma$$

defined on $(C_0^\infty(\bar{\Omega}))^d$ can be extended by continuity to a linear and continuous mapping, still denoted by $\gamma_{\mathbf{n}}$, from $H(\mathrm{div}, \Omega)$ into $H^{-\frac{1}{2}}(\Gamma)$. Moreover the range of $\gamma_{\mathbf{n}}$ is exactly $H^{-\frac{1}{2}}(\Gamma)$ and we have

$$\|\gamma_{\mathbf{n}} \mathbf{v}\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \|\mathbf{v}\|_{H(\mathrm{div},\Omega)} \quad \forall\, \mathbf{v} \in H(\mathrm{div}, \Omega).$$

In our case we are interested in the tractions on the boundary, that are $\tau \cdot \mathbf{n}|_\Gamma$. Since each row vector of a tensor $\tau \in \mathbf{H}(\mathrm{div}, \Omega)$ is in $H(\mathrm{div}, \Omega)$ we can apply the above result to get

**Theorem 2.3:**
The mapping

$$\gamma_{\mathbf{n}} : \tau \mapsto \tau \cdot \mathbf{n}|_\Gamma$$

defined on $(C_0^\infty(\bar{\Omega}))^{d\times d}$ can be extended by continuity to a linear and continuous mapping, still denoted by $\gamma_{\mathbf{n}}$, from $\mathbf{H}(\mathrm{div}, \Omega)$ into $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$. Moreover the range of $\gamma_{\mathbf{n}}$ is exactly $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and we have

$$\|\gamma_{\mathbf{n}}\tau\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \leq \|\tau\|_{\mathbf{H}(\mathrm{div},\Omega)} \quad \forall\, \tau \in \mathbf{H}(\mathrm{div}, \Omega).$$

## 2.3. Convex analysis

The following results are collected from the works of Ekeland and Temam [32] and Ito and Kunisch [61]. We use them in Chapter 3 to derive a saddle point formulation from a minimization problem and its dual.

Let $X$, $Y$ be two reflexive Banach spaces with duals $X'$, $Y'$ and let $J : X \to (-\infty, \infty]$ be a proper, lower semicontinuous (l.s.c.) and convex function. We consider the following minimization problem, denoted as the primal problem (P):

Find $\bar{x} \in X$ such that

$$J(\bar{x}) \leq J(x) \qquad \forall\, x \in X. \tag{2.7}$$

Clearly if $J$ is coercive, then there exists a minimizer $\bar{x} \in X$ of the primal problem (P). Now we define a family of perturbed problems in the following way. Let $\Phi(x, y) : X \times Y \to (-\infty, \infty]$ be proper, l.s.c. and convex with $\Phi(x, 0) = J(x)$, then the problem ($\mathrm{P}_y$) reads:

Find $\bar{x} \in X$ such that for $y \in Y$

$$\Phi(\bar{x}, y) \leq \Phi(x, y) \qquad \forall\, x \in X. \tag{2.8}$$

Obviously (P) is equal to ($\mathrm{P}_0$).

**Remark 2.4:** *If $J(x) = F(x) + G(\Lambda x)$, where $F : X \to \mathbb{R}$ and $G : Y \to (-\infty, \infty]$ are two proper, l.s.c. and convex functions and $\Lambda \in \mathcal{L}(X, Y)$, then we can set $\Phi(x, y) := F(x) + G(\Lambda x + y)$.*

**Definition 2.5:**
The functional $J^* : X' \to [-\infty, \infty]$ defined by

$$J^*(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - J(x)\}$$

is called the conjugate of $J$.

The dual problem ($P^*$) of (P) with respect to $\Phi$ is then defined by:

Find $\bar{y}^* \in Y'$ such that

$$-\Phi^*(0, \bar{y}^*) \geq -\Phi^*(0, y^*) \qquad \forall\, y^* \in Y'. \tag{2.9}$$

**Remark 2.6:** *If we consider the case of the previous remark, i.e.* $\Phi(x, y) = F(x) + G(\Lambda x + y)$, *then the conjugate* $\Phi^*(x^*, y^*)$ *of* $\Phi$ *reads*

$$\Phi^*(x^*, y^*) = F^*(x^* - \Lambda^* y^*) + G^*(y^*).$$

**Definition 2.7:**
The functional $L : X \times Y' \to [-\infty, \infty)$ defined by

$$-L(x, y^*) := \sup_{y \in Y} \{\langle y^*, y \rangle - \Phi(x, y)\} \tag{2.10}$$

is called the Lagrangian.

Now we can state the following result from [61, see Theorem 4.35].

**Theorem 2.8: Fenchel's duality theorem**
Assume that $\Phi$ is a convex, l.s.c. function that is finite at $(\bar{x}, \bar{y}^*)$. Then the following are equivalent.

- $(\bar{x}, \bar{y}^*) \in X \times Y'$ is a saddle point of $L$, i.e.

$$L(\bar{x}, y^*) \leq L(\bar{x}, \bar{y}^*) \leq L(x, \bar{y}^*) \quad \forall\, x \in X,\ y^* \in Y' \tag{2.11}$$

- $\bar{x}$ solves (P), $\bar{y}^*$ solves ($P^*$), and $\Phi(\bar{x}, 0) = -\Phi^*(0, \bar{y}^*)$.

The next result from [32, chapter VI] states equivalence between saddle point problems and variational inequality problems. It is very useful for frictional contact problems because the functional in the saddle point formulation does not need to be Gâteaux-differentiable.

**Proposition 2.9:**
Assume that $L = m + l$ with

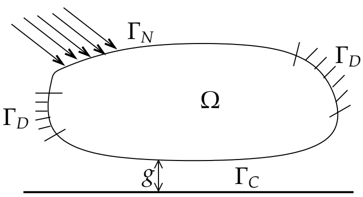
$$\begin{aligned}
&\forall\, x \in X,\ y^* \mapsto l(x, y^*) && \text{is concave and Gâteaux-differentiable,}\\
&\forall\, y^* \in Y',\ x \mapsto l(x, y^*) && \text{is convex and Gâteaux-differentiable,}\\[1ex]
&\forall\, x \in X,\ y^* \mapsto m(x, y^*) && \text{is concave,}\\
&\forall\, y^* \in Y',\ x \mapsto m(x, y^*) && \text{is convex.}
\end{aligned}$$

Then $(\bar{x}, \bar{y}^*) \in X \times Y'$ is a saddle point of $L$ if and only if

$$\begin{aligned} \langle \frac{\partial l}{\partial x}(\bar{x}, \bar{y}^*), x - \bar{x} \rangle + m(x, \bar{y}^*) - m(\bar{x}, \bar{y}^*) &\geq 0, \qquad \forall\, x \in X, \\ -\langle \frac{\partial l}{\partial y^*}(\bar{x}, \bar{y}^*), y^* - \bar{y}^* \rangle + m(\bar{x}, \bar{y}^*) - m(\bar{x}, y^*) &\geq 0, \qquad \forall\, y^* \in Y'. \end{aligned} \tag{2.12}$$

# 3. Dual-dual formulation for a contact problem with friction in 2D

In this chapter we apply the theory of Fenchel's duality from Section 2.3 in order to derive dual formulations of a contact problem with friction in 2D. Let us consider the following contact problem in 2D elasticity with Tresca friction. Assume a linear elastic body occupying the open bounded domain $\Omega \subset \mathbb{R}^2$. The Lipschitz-boundary $\Gamma := \partial\Omega$ is divided into three disjoint parts, the Dirichlet boundary $\Gamma_D$ where we assume homogeneous Dirichlet conditions, the Neumann boundary $\Gamma_N$ where a prescribed traction is acting on the body and the contact boundary $\Gamma_C$ where the body is supposed to come into contact with a rigid foundation, see Figure 3.1. To circumvent technical computations in the subsequent analysis we assume that the contact boundary $\Gamma_C$ and the Neumann boundary $\Gamma_N$ do not touch, i.e. $\overline{\Gamma_C} \cap \overline{\Gamma_N} = \emptyset$. Then the displacement vector field $\mathbf{u}(\mathbf{x})$ in each material point $\mathbf{x} \in \Omega$ of the body satisfies the following PDE: Here the stress tensor $\sigma(\mathbf{u})$ is connected to the strain




$$
\begin{aligned}
-\operatorname{div}\sigma(\mathbf{u}) &= \mathbf{f} && \text{in } \Omega,\\
\mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D,\\
\sigma(\mathbf{u})\cdot\mathbf{n} &= \mathbf{t}_0 && \text{on } \Gamma_N,\\
u_n \le g;\quad \sigma_n \le 0;\quad (u_n - g)\sigma_n &= 0 && \text{on } \Gamma_C,\\
|\boldsymbol{\sigma}_t| \le \mathcal{F};\quad \boldsymbol{\sigma}_t\cdot\mathbf{u}_t + \mathcal{F}\,|\mathbf{u}_t| &= 0 && \text{on } \Gamma_C.
\end{aligned}
\tag{3.1}
$$

Figure 3.1.: Boundary distribution

tensor $\varepsilon(\mathbf{u})$ via Hooke's law for linear elasticity, see Section 2.1. The volume body force $\mathbf{f} \in \mathbf{L}^2(\Omega)$, the prescribed traction $\mathbf{t}_0 \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$, the positive gap function $g \in H^{\frac{1}{2}}(\Gamma_C)$ and the friction function $\mathcal{F} \in L^\infty(\Gamma_C)$ are assumed to be given. On the contact boundary $\Gamma_C$ we have the decompositions $\mathbf{u} = u_n\mathbf{n} + \mathbf{u}_t$ and $\sigma\cdot\mathbf{n} = \sigma_n\mathbf{n} + \boldsymbol{\sigma}_t$ of the displacement and the traction into their normal and tangential parts, where $\mathbf{n}$ denotes the unit normal exterior to the contact boundary $\Gamma_C$ and

$$u_n = \mathbf{u}\cdot\mathbf{n}, \qquad \mathbf{u}_t = \mathbf{u} - u_n\mathbf{n}, \qquad \sigma_n = \mathbf{n}^T\cdot\sigma\cdot\mathbf{n}, \qquad \boldsymbol{\sigma}_t = \sigma\cdot\mathbf{n} - \sigma_n\mathbf{n}.$$

On the contact boundary we find some KKT-conditions for the normal part. They implicate that all material points of the body $\Omega$ may not penetrate the rigid foundation, which has positive distance $g$ to the contact boundary $\Gamma_C$. Furthermore, if some point $\mathbf{x} \in \Gamma_C$ is not in contact (i.e. $u_n(\mathbf{x}) < g(\mathbf{x})$), then the normal stress $\sigma_n(\mathbf{x})$ at this point has to be zero. Otherwise due to the nonpenetration condition the body cannot expand at those material points $\mathbf{x} \in \Gamma_C$ that are in contact (i.e. $u_n(\mathbf{x}) = g(\mathbf{x})$). Therefore the normal stress occuring at those points is always a compressive stress which leads to $\sigma_n(\mathbf{x}) \leq 0$. Finally, the equation in $(3.1)_4$ states that for all material points on the contact boundary either of the above situations is valid.

## Friction laws

The last boundary condition in the boundary value problem (3.1) state the friction law of Tresca friction. A more physical law would be the Coulomb friction law, see e.g. Kikuchi and Oden [63, Chapter 10], which reads

$$\begin{aligned} &\text{if } |\boldsymbol{\sigma}_t| < \mu_f\,|\sigma_n|, \quad \text{then } \mathbf{u}_t = 0 && \text{on } \Gamma_C, \\ &\text{if } |\boldsymbol{\sigma}_t| = \mu_f\,|\sigma_n|, \quad \text{then } \exists\, s \geq 0 : \ \mathbf{u}_t = -s\,\boldsymbol{\sigma}_t && \text{on } \Gamma_C. \end{aligned} \tag{3.2}$$

Here $\mu_f \geq 0$ is the friction coefficient, which is assumed to be uniformly Lipschitz on $\Gamma_C$. It specifies how strong the body is sticking to the rigid foundation when coming into contact at some point $\mathbf{x} \in \Gamma_C$. The second line in (3.2) is equivalent to

$$\boldsymbol{\sigma}_t \cdot \mathbf{u}_t = -\mu_f\,|\sigma_n||\mathbf{u}_t| \quad \text{on } \Gamma_C. \tag{3.3}$$

When considering real life problems the friction coefficient, being not necessarily constant is usually not known exactly since it depends on the material properties of the body $\Omega$ and the rigid foundation as well as on the roughness of both materials at each point. The last factor is a local property that changes in each material point. Nevertheless we restrict ourselves to some constant values for the friction coefficient which is sufficient for our purpose.

The Coulomb friction law indicates the direct dependence of the shear stress on the normal stress in each material point on the contact boundary. If some material point $\mathbf{x} \in \Gamma_C$ is not in contact with the rigid foundation then no shear stress (or tangential stress) appears due to $|\boldsymbol{\sigma}_t(\mathbf{x})| \leq \mu_f|\sigma_n(\mathbf{x})| = 0$. On the other hand we have to distinguish two cases. First, the material point $\mathbf{x} \in \Gamma_C$ sticks to the rigid foundation. In this case $|\boldsymbol{\sigma}_t(\mathbf{x})| < \mu_f|\sigma_n(\mathbf{x})|$ and the tangential stress is not large enough to move the material point in tangential direction. Second, the material point is moving along the rigid foundation in tangential direction. The absolute value of the tangential stress equals $\mu_f|\sigma_n(\mathbf{x})|$. In this case the tangential stress cannot increase when the normal stress is fixed at the same time. The excess energy has transformed into kinetic energy (and usually also thermal energy). Figures 3.2 and 3.3 give an example of the stick and the slip situations, respectively. The dashed, colored lines depict the deformed geometries.

However a direct treatment of contact problems with Coulomb friction seems difficult and in some cases even impossible. The first result concerning existence of a solution for a sufficiently small friction coefficient was discovered by Nečas, Jarušek and Haslinger [70]. Other references were already mentioned in the introduction. To approach the Coulomb friction law, Nečas et al. [60, see Chapter 2.5.4] propose a fixed point iteration. We give a short abstract of this approach and refer to the above references for more details.

For given $\mathcal{F}_0 \in L^\infty(\Gamma_C)$ and $\mu_f$, both positive, we compute $\mathcal{F}_{k+1} := \mu_f\, |\sigma_n^k|$ where $\sigma_n^k$ is the normal stress on $\Gamma_C$ of the solution of the contact problem (3.1) with given Tresca friction function $\mathcal{F}_k \geq 0$. We proceed until some stopping criterion is reached. The whole algorithm, in particular the part of solving problem (3.1), is explained in Section 3.3. For convenience we drop the index $k$ of the friction function $\mathcal{F}_k$. Note

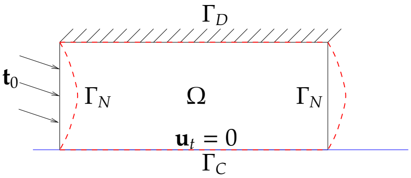



Figure 3.2.: Deformed geometry when stick occurs.

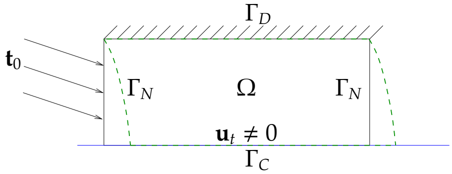



Figure 3.3.: Deformed geometry when slip occurs.

that we demand $\mathcal{F}$ to be in $L^\infty(\Gamma_C)$ and not in $H^{-\frac{1}{2}}(\Gamma_C)$ as we would expect from the definition above. The reason for this assumption will be seen later in this work.

Furthermore, we define $NC_{\mathcal{F}} := \{\mathbf{x} \in \Gamma_C : \mathcal{F} \text{ is not continuous in } \mathbf{x}\}$ and assume that the number of points $\mathbf{x} \in NC_{\mathcal{F}}$ is bounded. The motivation of this assumption is the following. If we would allow all functions $\mathcal{F} \in L^\infty(\Gamma_C)$ then we could take for example the indicator function $I_{\mathbb{R}\setminus\mathbb{Q}}$ for $\mathbf{x}$ on $\Gamma_C$

$$I_{\mathbb{R}\setminus\mathbb{Q}}(\mathbf{x}) := \begin{cases} 1, & \text{if } |\mathbf{x}| \in \mathbb{R} \setminus \mathbb{Q}, \\ 0, & \text{if } |\mathbf{x}| \in \mathbb{Q}, \end{cases}$$

which is of course in $L^\infty(\Gamma_C)$. But for the strong formulation (3.1) this would lead to an undefined contact situation on the whole contact boundary $\Gamma_C$ and we would have little prospect for success of stating theoretical or numerical results. We therefore state the following assumptions on the friction functional $\mathcal{F}$

$$\mathcal{F} \in L^2(\Gamma_C) \quad \text{and} \quad \#NC_{\mathcal{F}} < N \quad \text{for some} \quad N \in \mathbb{N}. \tag{3.4}$$

Let us also give a heuristic motivation of the above assumption. It seems natural as the diameter of some material point of a continuum is obviously bounded from below by $\varepsilon > 0$ from the physical point of view. Hence the thought is as follows. Considering the contact problem with Coulomb friction (3.2), then for all $\mathbf{x} \in \Gamma_C$ that are not in contact with the rigid foundation, the boundary conditions lead to

$u_n(\mathbf{x}) < g(\mathbf{x}) \Rightarrow \sigma_n(\mathbf{x}) = 0$ and finally $\boldsymbol{\sigma}_t(\mathbf{x}) = 0$. Now let some material point $\mathbf{x} \in \Gamma_C$ come into contact with the rigid foundation, then $u_n(\mathbf{x}) = g(\mathbf{x}) \Rightarrow \sigma_n(\mathbf{x}) \neq 0$. Without loss of generality we assume $\mathrm{dist}(\mathbf{x}, \partial\Gamma_C) > 0$ and $B_\varepsilon(\mathbf{x})|_\Gamma \subset \Gamma_C$, then $\mathbf{y}$ is in contact with the rigid foundation for all $\mathbf{y} \in B_\varepsilon(\mathbf{x})$. As the contact situation is exactly the same for all $\mathbf{y} \in B_\varepsilon(\mathbf{x})$ we have $\sigma_n(\mathbf{y})$ is continuous in $\mathbf{y}$ for all $\mathbf{y} \in B_\varepsilon(\mathbf{x})$. Finally, due to $\Gamma_C$ being bounded we can find a finite set of material points $\mathbf{x}^i \in \Gamma_C$, such that $\cup_i B_\varepsilon(\mathbf{x}^i) \supset \Gamma_C$. But this means that the number of points, where the normal stress $\sigma_n$ is not continuous, is finite. Since we expect the sequence of Tresca functions $\mathcal{F}_k$ tending to $\mu_f\,|\sigma_n|$, which satisfies assumption (3.4), it seems feasible to demand it for all elements of the sequence.

**Remark 3.1:** *Assumption* (3.4) *on the friction function $\mathcal{F}$ is necessary for the theoretical results in the next sections. In Section 3.2 we will use a further observation, concerning the support of the friction function, to handle it numerically.*

### 3.0.1. Primal formulation for contact problems with Tresca friction

Before we start with the investigation of our approach let us repeat some results on the primal formulation where the displacement field is regarded as unknown variable. We already defined the space of energetically admissible functions $\mathbf{V}_D$ in Section 2.1. Regarding the nonpenetration condition on the contact boundary we are led to the definition of the closed convex subset of $\mathbf{V}_D$

$$\mathbf{K}_g := \{\mathbf{v} \in \mathbf{V}_D : v_n \leq g \text{ a.e. on } \Gamma_C\}.$$

Defining the coercive and continuous bilinear form $a(\cdot,\cdot)$ on $\mathbf{V}_D \times \mathbf{V}_D$, the continuous linear form $L(\cdot)$ on $\mathbf{V}_D$ and the continuous but nondifferentiable functional $j(\cdot)$ on $\mathbf{V}_D$ by

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &:= \int_\Omega \varepsilon(\mathbf{u}) : \mathbb{C} : \varepsilon(\mathbf{v})\, dx, \\ L(\mathbf{v}) &:= \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx + \int_{\Gamma_N} \mathbf{t}_0 \cdot \mathbf{v}\, ds, \\ j(\mathbf{v}) &:= \int_{\Gamma_C} \mathcal{F} |\mathbf{v}_t|\, ds, \end{aligned}$$

the energy functional corresponding to the boundary value problem (3.1) reads

$$J(\mathbf{v}) := \frac{1}{2}\, a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) + j(\mathbf{v}). \tag{3.5}$$

The energy principle states that the solution of (3.1) minimizes the energy functional $J(\cdot)$ over all admissible functions in $\mathbf{K}_g$. Hence we arrive at the primal minimization problem:

Find $\mathbf{u} \in \mathbf{K}_g$ such that

$$J(\mathbf{u}) \leq J(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{K}_g. \tag{3.6}$$

Since $J(\cdot)$ is coercive on $\mathbf{K}_g$, strictly convex and l.s.c. there exists a unique solution $\mathbf{u} \in \mathbf{K}_g$ of problem (3.6), see e.g. Chapter II in Ekeland and Temam [32]. This approach was investigated for years and is well known in literature. Some few references among many others are for example the books of Duvaut and Lions [31], Glowinski et al. [53], Glowinski [52], Kikuchi and Oden [63] and Nečas et al. [60]. In the next section we will use the following well known result, see the references above, concerning the primal minimization problem.

**Lemma 3.2:**
If the solution $\mathbf{u}$ of the primal minimization problem (3.6) is smooth enough, then it is connected with the strong formulation (3.1) in the following way. Defining $\sigma(\mathbf{u}) := \mathbb{C} : \varepsilon(\mathbf{u})$ there holds

$$\begin{aligned} -\operatorname{div} \sigma(\mathbf{u}) = \mathbf{f} \quad & \text{in } \Omega, \\ \sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{t}_0 \quad & \text{on } \Gamma_N, \\ \sigma(\mathbf{u})_n \leq 0 \;\text{ and }\; |\sigma(\mathbf{u})_t| \leq \mathcal{F} \quad & \text{on } \Gamma_C, \\ \int_{\Gamma_C} \mathbf{u}_t \cdot \sigma(\mathbf{u})_t + \mathcal{F}|\mathbf{u}_t|\, ds = 0. & \end{aligned}$$

*Proof.* For completeness we show the proof. From [53] we know that the minimization problem (3.6) is equivalent to the following variational inequality problem of finding $\mathbf{u} \in \mathbf{K}_g$ s.t.

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - L(\mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \geq 0 \qquad \forall\, \mathbf{v} \in \mathbf{K}_g \tag{3.7}$$

If we choose $\mathbf{v} = \pm\boldsymbol{\phi} + \mathbf{u}$ with $\boldsymbol{\phi} \in C_0^\infty(\Omega)^2$, then $\mathbf{v} \in \mathbf{K}_g$ and we have

$$\int_\Omega \sigma(\mathbf{u}) : \varepsilon(\boldsymbol{\phi})\, dx - \int_\Omega \mathbf{f} \cdot \boldsymbol{\phi}\, dx = 0 \qquad \forall\, \boldsymbol{\phi} \in C_0^\infty(\Omega)^2.$$

Applying integration by parts it follows $-\operatorname{div} \sigma(\mathbf{u}) = \mathbf{f}$ in $\Omega$, if $\mathbf{u}$ is smooth enough. Now using

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u})\, dx = \int_{\Gamma_N \cup \Gamma_C} (\mathbf{v} - \mathbf{u}) \cdot \sigma(\mathbf{u}) \cdot \mathbf{n}\, ds$$

the variational inequality reduces to

$$\int_{\Gamma_N} (\mathbf{v} - \mathbf{u}) \cdot (\sigma(\mathbf{u}) \cdot \mathbf{n} - \mathbf{t}_0)\, ds + \int_{\Gamma_C} (v_n - u_n)\sigma(\mathbf{u})_n\, ds$$
$$+ \int_{\Gamma_C} [(\mathbf{v}_t - \mathbf{u}_t) \cdot \sigma(\mathbf{u})_t + \mathcal{F}|\mathbf{v}_t| - \mathcal{F}|\mathbf{u}_t|]\, ds \geq 0 \qquad \forall\, \mathbf{v} \in \mathbf{K}_g$$

and choosing $\mathbf{v} \in \mathbf{K}_g$ with $\mathbf{v} = \mathbf{u}$ on $\Gamma_C$ and $\mathbf{v} = \pm\boldsymbol{\phi} + \mathbf{u}$ on $\Gamma_N$ for $\boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)$ we have

$$\int_{\Gamma_N} \boldsymbol{\phi} \cdot (\sigma(\mathbf{u}) \cdot \mathbf{n} - \mathbf{t}_0)\, ds = 0 \qquad \forall\, \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N) \quad \Rightarrow \quad \sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{t}_0 \text{ on } \Gamma_N.$$

With $\mathbf{v} \in \mathbf{K}_g$, $\mathbf{v} = \mathbf{u}$ on $\Gamma_N$, $\mathbf{v}_t = \mathbf{u}_t$ and $v_n = u_n + \phi_n$ on $\Gamma_C$ for $\phi_n$ being the normal component on $\Gamma_C$ of some $\boldsymbol{\phi} \in \mathbf{K}_0$ we derive

$$\int_{\Gamma_C} \phi_n \sigma(\mathbf{u})_n\, ds \geq 0 \qquad \forall\, \boldsymbol{\phi} \in \mathbf{K}_0 \quad \Rightarrow \quad \sigma(\mathbf{u})_n \leq 0 \text{ on } \Gamma_C.$$

Taking $\mathbf{v} \in \mathbf{K}_g$ with $\mathbf{v} = \mathbf{u}$ on $\Gamma_N$ and $\mathbf{v} = \mathbf{u} + \boldsymbol{\phi}$ on $\Gamma_C$ for some $\boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_C)$ with $\phi_n = 0$, i.e. $v_n = u_n$ on $\Gamma_C$ and $\mathbf{v}_t = \mathbf{u}_t + \boldsymbol{\phi}_t$ on $\Gamma_C$, we have

$$\begin{aligned} 0 &\leq \int_{\Gamma_C} (\boldsymbol{\phi}_t \cdot \boldsymbol{\sigma}_t + \mathcal{F}|\mathbf{u}_t + \boldsymbol{\phi}_t| - \mathcal{F}|\mathbf{u}_t|)\, ds \\ &\leq \int_{\Gamma_C} (\boldsymbol{\phi}_t \cdot \boldsymbol{\sigma}_t + \mathcal{F}|\mathbf{u}_t| + \mathcal{F}|\boldsymbol{\phi}_t| - \mathcal{F}|\mathbf{u}_t|)\, ds = \int_{\Gamma_C} (\boldsymbol{\phi}_t \boldsymbol{\sigma}_t + \mathcal{F}|\boldsymbol{\phi}_t|)\, ds \qquad \forall\, \boldsymbol{\phi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_C). \end{aligned}$$

For positive $\boldsymbol{\phi}_t$ we get $\boldsymbol{\sigma}_t + \mathcal{F} \geq 0$ and for negative $\boldsymbol{\phi}_t$, $\boldsymbol{\sigma}_t - \mathcal{F} \leq 0$ on $\Gamma_C$ which reads together $|\boldsymbol{\sigma}_t| \leq \mathcal{F}$ on $\Gamma_C$. Finally, for $\mathbf{v} \in \mathbf{K}_g$ with $\mathbf{v} = \mathbf{u}$ on $\Gamma_N$, $v_n = u_n$ and $\mathbf{v}_t = 0$ and $\mathbf{v}_t = 2\mathbf{u}_t$, respectively on $\Gamma_C$ we have

$$\int_{\Gamma_C} \mathbf{u}_t \boldsymbol{\sigma}(\mathbf{u})_t + \mathcal{F}|\mathbf{u}_t|\, ds = 0 \quad \text{ on } \Gamma_C.$$

□

In the next section we investigate the complementary energy principle in order to derive dual formulations where the primary unknown variable is the stress tensor.

## 3.1. Dual-dual formulation in 2D

In this section we want to derive the dual-dual formulation of problem (3.1). To do so we have to investigate the dual minimization problem resulting from the complementary energy principle. After applying Fenchel's duality theory we can state an equivalent saddle point problem. Then in a final step we derive variational inequalities that are suitable for some numerical analysis. But first let us set up the dual minimization problem.

### 3.1.1. Dual minimization problem

Before we introduce the dual minimization problem we make the following observation. As we consider a 2D problem we can define the unit tangential vector on the boundary $\Gamma$ as $\mathbf{t} := \binom{n_2}{-n_1}$ for $\mathbf{n} = \binom{n_1}{n_2}$ being the unit outer normal to the boundary $\Gamma$. For some $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and $\tau$ its corresponding stress tensor we define

$$v_t := \mathbf{v} \cdot \mathbf{t} \qquad \text{and} \qquad \tau_t := \boldsymbol{\tau}_t \cdot \mathbf{t} = (\tau \cdot \mathbf{n} - \tau_n \mathbf{n}) \cdot \mathbf{t} = \mathbf{t} \cdot \tau \cdot \mathbf{n}.$$

Then due to $\boldsymbol{\tau}_t \cdot \mathbf{n} = 0$ and $\mathbf{v}_t \cdot \mathbf{n} = 0$ we have $|\boldsymbol{\tau}_t| = |\tau_t|$, $|\mathbf{v}_t| = |v_t|$ and

$$v_t \tau_t = (\mathbf{v} \cdot \mathbf{t})\tau_t = (v_n \mathbf{n} + \mathbf{v}_t) \cdot (\tau_t \mathbf{t}) = \mathbf{v}_t \cdot \boldsymbol{\tau}_t.$$

For this reason we can restrict the friction boundary condition $(3.1)_5$ to the scalar version

$$|\sigma_t| \leq \mathcal{F}; \quad \sigma_t u_t + \mathcal{F}\, |u_t| = 0 \qquad \text{on } \Gamma_C. \tag{3.8}$$

Similar to the primal formulation we introduce the space of energetically admissible functions

$$\mathbf{X}_s := \{\tau = (\tau_{ij})_{i,j=1,2} : \tau \in \mathbf{H}(\mathrm{div}, \Omega),\ \tau_{ij} = \tau_{ji},\ i, j = 1, 2\}. \tag{3.9}$$

In a dual problem the boundary conditions switch their roles. Hence the Dirichlet condition and the nonpenetration condition are no longer essential but natural. For reasons of readability we restrict ourselves to homogeneous Dirichlet conditions. On the other hand we have the Neumann condition as well as the inequalities on the contact boudary concerning the stress. These are the essential conditions of the dual problem and have to be built into the space of admissible functions $\mathbf{X}_s$ leading to a convex set of admissible functions. Finally, the governing partial differential equation $(3.1)_1$ is no longer of second order, when regarding $\sigma$ as primary variable. Therefore we also build this equation into the closed convex set which is defined as

$$\widetilde{\mathbf{K}} := \{\tau \in \mathbf{X}_s : -\mathrm{div}\, \tau = \mathbf{f} \text{ in } \Omega;\ \tau \cdot \mathbf{n} = \mathbf{t}_0 \text{ on } \Gamma_N;\ \tau_n \leq 0 \text{ on } \Gamma_C;\ |\tau_t| \leq \mathcal{F} \text{ on } \Gamma_C\}. \tag{3.10}$$

Since we assume the volume body forces $\mathbf{f}$ and the traction $\mathbf{t}_0$ on the Neumann boundary $\Gamma_N$ to be nonzero, we introduce Langrange multipliers for the displacement field $\mathbf{u}$ in $\Omega$ and the trace of $\mathbf{u}$ on $\Gamma_N$ to derive the dual-dual formulation. We introduce additional Lagrange multipliers denoting the tangential and normal displacement $u_t$ and $u_n$ on the contact boundary that are tested with the normal traction and the tangential traction, respectively. We have $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\mathbf{t}_0 \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$ and the trace of $\mathbf{X}_s$ on $\Gamma_C$, which is $\mathbf{H}^{-\frac{1}{2}}(\Gamma_C)$ due to Theorem 2.3. Therefore we define

$$\mathbf{Y} := \mathbf{L}^2(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\Gamma_N) \times H^{-\frac{1}{2}}(\Gamma_C) \times H^{-\frac{1}{2}}(\Gamma_C). \tag{3.11}$$

If we define the continuous bilinear form $\tilde{a}(\cdot,\cdot)$ on $\mathbf{X}_s \times \mathbf{X}_s$ and the functional $\tilde{J}(\cdot)$ on $\mathbf{X}_s$ by

$$\begin{aligned} \tilde{a}(\sigma,\tau) &:= \int_\Omega \sigma : \mathbb{C}^{-1} : \tau \, dx, \\ \tilde{J}(\tau) &:= \frac{1}{2}\,\tilde{a}(\tau,\tau) - \langle g,\, \tau_n\rangle_{\Gamma_C}, \end{aligned} \tag{3.12}$$

then $\tilde{J}(\cdot)$ is the conjugate energy functional and due to the minimum principle of the conjugate elastic potential we know that the stress tensor $\sigma(\mathbf{u})$ of the solution of (3.1) minimizes $\tilde{J}(\cdot)$ over $\widetilde{\mathbf{K}}$. This leads us to the dual minimization problem.

Find $\sigma \in \widetilde{\mathbf{K}}$ such that

$$\tilde{J}(\sigma) \le \tilde{J}(\tau) \qquad \forall\, \tau \in \widetilde{\mathbf{K}}. \tag{3.13}$$

Due to Lemma 3.2 the convex set $\widetilde{\mathbf{K}}$ is not empty. Moreover the functional $\tilde{J}(\cdot)$ is coercive on $\widetilde{\mathbf{K}}$, strictly convex and continuous and therefore we have the existence of a unique solution for the dual minimization problem (3.13), see again Ekeland and Temam [32, Proposition 1.2 in Chapter II]. The coercivity of $\tilde{J}$ on $\widetilde{\mathbf{K}}$ follows from the coercivity of $\tilde{a}(\cdot,\cdot)$ on the subspace of all divergence free tensors and the continuity of the dual product on $\Gamma_C$. From Theorem 2.3 and equation (2.3) we have

$$\begin{aligned} &\|\tau\|^2_{\mathbf{X}} = \|\tau\|^2_{(L^2(\Omega))^{2\times 2}} + \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} \qquad \forall\, \tau \in \widetilde{\mathbf{K}} \\ &\langle g, \tau_n\rangle_{\Gamma_C} \le \|g\|_{H^{\frac{1}{2}}(\Gamma_C)} \|\tau_n\|_{H^{-\frac{1}{2}}(\Gamma_C)} \le \|g\|_{H^{\frac{1}{2}}(\Gamma_C)} \|\tau\|_{\mathbf{X}} \\ \Rightarrow \quad & \tilde{J}(\tau) = \tfrac{1}{2}\tilde{a}(\tau,\tau) - \langle g,\tau\rangle \ge \tfrac{\alpha}{2}\|\tau\|^2_{\mathbf{X}} - \tfrac{\alpha}{2}\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} - \|g\|_{H^{\frac{1}{2}}(\Gamma_C)}\|\tau\|_{\mathbf{X}} \\ &\qquad = \|\tau\|_{\mathbf{X}}\left(\tfrac{\alpha}{2}\|\tau\|_{\mathbf{X}} - \|g\|_{H^{\frac{1}{2}}(\Gamma_C)}\right) - \tfrac{\alpha}{2}\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} \\ \Rightarrow \quad & \lim_{\|\tau\|_{\mathbf{X}}\to\infty} \tilde{J}(\tau) \ge \lim_{\|\tau\|_{\mathbf{X}}\to\infty} \|\tau\|_{\mathbf{X}}\left(\tfrac{\alpha}{2}\|\tau\|_{\mathbf{X}} - \|g\|_{H^{\frac{1}{2}}(\Gamma_C)}\right) - \tfrac{\alpha}{2}\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} \to \infty, \end{aligned}$$

since $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma_C)$. Furthermore, we observe for all $\tau_1 \neq \tau_2$ in $\widetilde{\mathbf{K}}$

$$\begin{aligned} &\tilde{J}(s\tau_1 + (1-s)\tau_2) - s\tilde{J}(\tau_1) - (1-s)\tilde{J}(\tau_2) \\ &= \frac{s^2}{2}\,\tilde{a}(\tau_1,\tau_1) + s(1-s)\,\tilde{a}(\tau_1,\tau_2) + \frac{(1-s)^2}{2}\,\tilde{a}(\tau_2,\tau_2) - \frac{s}{2}\,\tilde{a}(\tau_1,\tau_1) - \frac{1-s}{2}\,\tilde{a}(\tau_2,\tau_2) \\ &= \frac{s(s-1)}{2}\,\tilde{a}(\tau_1,\tau_1) + s(1-s)\,\tilde{a}(\tau_1,\tau_2) - \frac{s(1-s)}{2}\,\tilde{a}(\tau_2,\tau_2) \\ &= -s(1-s)\,\tilde{a}(\tau_1-\tau_2,\tau_1-\tau_2) < 0 \qquad \forall\, s \in (0,1), \end{aligned}$$

which means that $\tilde{J}(\cdot)$ is strictly convex on $\widetilde{\mathbf{K}}$. For the primal minimization problem (3.6) and the dual minimization problem (3.13) we can prove the following

**Theorem 3.3:**
Let $\mathbf{u} \in \mathbf{K}_g$ and $\sigma \in \widetilde{\mathbf{K}}$ be the solutions of the primal minimization problem (3.6) and the dual minimization problem (3.13), respectively. Then there holds

$$\sigma = \mathbb{C} : \varepsilon(\mathbf{u}) \quad \text{and} \quad J(\mathbf{u}) + \widetilde{J}(\sigma) = 0.$$

*Proof.* The proof follows the ideas of Nečas et al. [60, see Section 1.1.12]. See also Theorem 5.3 in Maischak [67]. Let us define $M := \{\tau \in L^2(\Omega)^{2\times 2} : \tau = \tau^T\}$ and $\mathcal{J} : \mathbf{K}_g \times M \times M \longrightarrow \mathbb{R}$

$$\mathcal{J}(\mathbf{v}, \zeta, \tau) := \frac{1}{2} \int\limits_\Omega \zeta : \mathbb{C} : \zeta \, dx + \int\limits_\Omega \tau : (\varepsilon(\mathbf{v}) - \zeta) dx - L(\mathbf{v}) + j(\mathbf{v}).$$

Obviously there holds

$$\sup_{\tau \in M} \int\limits_\Omega \tau : (\varepsilon(\mathbf{v}) - \zeta) dx = \begin{cases} 0, & \text{if } \zeta = \varepsilon(\mathbf{v}), \\ \infty, & \text{else.} \end{cases}$$

Thus the primal minimization problem (3.6) can be expressed as follows

$$\inf_{\mathbf{v} \in \mathbf{K}_g} J(\mathbf{v}) = \inf_{(\mathbf{v},\zeta) \in \mathbf{K}_g \times M} \sup_{\tau \in M} \mathcal{J}(\mathbf{v}, \zeta, \tau).$$

Defining

$$I(\tau) := \inf_{(\mathbf{v},\zeta) \in \mathbf{K}_g \times M} \mathcal{J}(\mathbf{v}, \zeta, \tau)$$

we conclude

$$I(\tau) \leq \inf_{\mathbf{v} \in \mathbf{K}_g} \mathcal{J}(\mathbf{v}, \varepsilon(\mathbf{v}), \tau) = \inf_{\mathbf{v} \in \mathbf{K}_g} J(\mathbf{v}) = J(\mathbf{u}) \qquad \forall\, \tau \in M,$$

which also holds true for the supremum

$$\sup_{\tau \in M} I(\tau) \leq J(\mathbf{u}).$$

Now if we set

$$I(\tau) = \inf_{\zeta \in M} I_1(\zeta, \tau) + \inf_{\mathbf{v} \in \mathbf{K}_g} I_2(\mathbf{v}, \tau)$$

with

$$I_1(\zeta, \tau) := \frac{1}{2} \int\limits_\Omega \zeta : \mathbb{C} : \zeta \, dx - \int\limits_\Omega \tau : \zeta \, dx \qquad \text{for } \zeta, \tau \in M,$$

$$I_2(\mathbf{v}, \tau) := \int\limits_\Omega \tau : \varepsilon(\mathbf{v}) \, dx - L(\mathbf{v}) + j(\mathbf{v}) \qquad \text{for } \mathbf{v} \in \mathbf{K}_g,\ \tau \in M,$$

then we can compute both infima and thus $I(\tau)$. The first infimum follows immediately since we are concerned with a differentiable quadratic form. It follows

$$\inf_{\zeta\in M} I_1(\zeta,\tau) = -\frac{1}{2}\int_\Omega \tau : \mathbb{C}^{-1} : \tau \, dx.$$

For the second infimum we first observe that for $g \in H^{\frac{1}{2}}(\Gamma_C)$ there exists an extension $G \in \mathbf{V}_D$ such that $G_n = g$ and $G_t = 0$ on $\Gamma_C$. Now for every $\mathbf{v} \in \mathbf{K}_g$ there exist a $\tilde{\mathbf{v}} \in \mathbf{K}_0$ with $\mathbf{v} = \tilde{\mathbf{v}} + G$ and we can write

$$I_2(\mathbf{v},\tau) = I_2(\tilde{\mathbf{v}} + G,\tau) = I_2(G,\tau) + I_2(\tilde{\mathbf{v}},\tau) \qquad \forall\, \mathbf{v} \in \mathbf{K}_g.$$

Here we used

$$j(\tilde{\mathbf{v}} + G) = \int_{\Gamma_C} \mathcal{F}|\tilde{v}_t + G_t|\, ds = \int_{\Gamma_C} \mathcal{F}|\tilde{v}_t|\, ds = j(\tilde{\mathbf{v}}) + j(G).$$

Now we are able to take the imfimum over $\mathbf{K}_0$ which reads

$$\inf_{\mathbf{v}\in\mathbf{K}_g} I_2(\mathbf{v},\tau) = I_2(G,\tau) + \inf_{\tilde{\mathbf{v}}\in\mathbf{K}_0} I_2(\tilde{\mathbf{v}},\tau).$$

Next we show that

$$\tau \in \widetilde{\mathbf{K}} \qquad \Leftrightarrow \qquad I_2(\tilde{\mathbf{v}},\tau) \geq 0 \qquad \forall\, \tilde{\mathbf{v}} \in \mathbf{K}_0. \tag{3.14}$$

To see this we first let $\tau \in \widetilde{\mathbf{K}}$ and compute

$$\begin{aligned}
I_2(\tilde{\mathbf{v}},\tau) &= \int_\Omega \tau : \varepsilon(\tilde{\mathbf{v}})\, dx - \int_\Omega \tilde{\mathbf{v}} \cdot \mathbf{f}\, dx - t(\tilde{\mathbf{v}}) + j(\tilde{\mathbf{v}}) \\
&= \int_\Omega \tau : \varepsilon(\tilde{\mathbf{v}})\, dx + \int_\Omega \tilde{\mathbf{v}} \cdot \operatorname{div}\tau\, dx - t(\tilde{\mathbf{v}}) + j(\tilde{\mathbf{v}}) = \int_{\Gamma_N\cup\Gamma_C} \tilde{\mathbf{v}} \cdot \tau \cdot \mathbf{n}\, ds - t(\tilde{\mathbf{v}}) + j(\tilde{\mathbf{v}}) \\
&= \int_{\Gamma_C} \tilde{v}_n \tau_n\, ds + \int_{\Gamma_C} \tilde{v}_t \tau_t + \mathcal{F}|\tilde{v}_t|\, ds \qquad \forall\, \tilde{\mathbf{v}} \in \mathbf{K}_0, \tau \in \widetilde{\mathbf{K}},
\end{aligned}$$

where we have used the equality constraints for $\tau$ in $\widetilde{\mathbf{K}}$ and Green's formula. As $\tilde{v}_n \leq 0$ and $\tau_n \leq 0$ for $\tilde{\mathbf{v}} \in \mathbf{K}_0$ and $\tau \in \widetilde{\mathbf{K}}$ we have $\tilde{v}_n\tau_n \geq 0$ and so the first integral is positive. For the second integral we observe

$$\tilde{v}_t\tau_t + \mathcal{F}|\tilde{v}_t| = (\tau_t + \mathcal{F}\operatorname{sign}\tilde{v}_t)\tilde{v}_t$$

and by a distinction of cases we have

1. $\tilde{v}_t < 0 \Rightarrow \operatorname{sign}\tilde{v}_t = -1 : \ \tau_t \leq |\tau_t| \leq \mathcal{F} \quad \Rightarrow \ \tau_t + \mathcal{F}\operatorname{sign}\tilde{v}_t = \tau_t - \mathcal{F} \leq 0$
2. $\tilde{v}_t > 0 \Rightarrow \operatorname{sign}\tilde{v}_t = 1 : \ \tau_t \geq -|\tau_t| \geq -\mathcal{F} \quad \Rightarrow \ \tau_t + \mathcal{F}\operatorname{sign}\tilde{v}_t = \tau_t + \mathcal{F} \geq 0,$

leading to $(\tau_t + \mathcal{F} \operatorname{sign} \tilde{v}_t)\tilde{v}_t \geq 0$. Finally, we have

$$I_2(\tilde{\mathbf{v}}, \tau) = \int\limits_{\Gamma_C} \tilde{v}_n \tau_n \, ds + \int\limits_{\Gamma_C} \tilde{v}_t \tau_t + \mathcal{F}|\tilde{v}_t| \, ds \geq 0 \qquad \forall\, \tilde{\mathbf{v}} \in \mathbf{K}_0, \tau \in \widetilde{\mathbf{K}}.$$

Now let $\tau \in M$ such that $I_2(\tilde{\mathbf{v}}, \tau) \geq 0$ for all $\tilde{\mathbf{v}} \in \mathbf{K}_0$ which means

$$\int\limits_{\Omega} \tau : \varepsilon(\tilde{\mathbf{v}}) \, dx - L(\tilde{\mathbf{v}}) + j(\tilde{\mathbf{v}}) \geq 0 \qquad \forall\, \tilde{\mathbf{v}} \in \mathbf{K}_0.$$

The terms on the boundary parts vanish if we take $\tilde{\mathbf{v}} = \pm\boldsymbol{\phi}$ with $\boldsymbol{\phi} \in C_0^\infty(\Omega)^2$ which is clearly in $\mathbf{K}_0$. Thus we have

$$\int\limits_{\Omega} (\tau : \varepsilon(\boldsymbol{\phi}) - \boldsymbol{\phi} \cdot \mathbf{f}) dx = 0 \qquad \forall\, \boldsymbol{\phi} \in C_0^\infty(\Omega)^2$$

that is $-\operatorname{div} \tau = \mathbf{f}$ in $\Omega$ in the weak sense. After integration by parts the functional reduces to

$$I_2(\tilde{\mathbf{v}}, \tau) = \int\limits_{\Gamma_N \cup \Gamma_C} \tilde{\mathbf{v}} \cdot \tau \cdot \mathbf{n} \, ds - \int\limits_{\Gamma_N} \tilde{\mathbf{v}} \cdot \mathbf{t}_0 \, ds + \int\limits_{\Gamma_C} \mathcal{F}|\tilde{v}_t| \, ds$$

and choosing $\tilde{\mathbf{v}} \in \mathbf{K}_0$ with $\tilde{\mathbf{v}} = 0$ on $\Gamma_C$ we get $\tau \cdot \mathbf{n} = \mathbf{t}_0$ on $\Gamma_N$. Next taking $\tilde{\mathbf{v}} \in \mathbf{K}_0$ with $\tilde{v}_t = 0$ on $\Gamma_C$ we have $\tau_n \leq 0$ due to $\tilde{v}_n \leq 0$ and $I_2(\tilde{\mathbf{v}}, \tau) \geq 0$. Finally, $\tilde{\mathbf{v}} \in \mathbf{K}_0$ with $\tilde{v}_n = 0$ and $\operatorname{sign} \tilde{v}_t = -\operatorname{sign} \tau_t$ on $\Gamma_C$ leads to $|\tau_t| \leq \mathcal{F}$ on $\Gamma_C$ and therefore $\tau \in \widetilde{\mathbf{K}}$ which is the desired assertion (3.14).

For $\tau \in \widetilde{\mathbf{K}}$ we just have seen that

$$I_2(\mathbf{v}, \tau) = \int\limits_{\Gamma_C} v_n \tau \, ds + \int\limits_{\Gamma_C} \tau_t v_t + \mathcal{F}|v_t| \, ds \qquad \forall\, \mathbf{v} \in \mathbf{K}_g.$$

If we take the infimum over all $\mathbf{v} \in \mathbf{K}_g$ we see with $\tau_n \leq 0$ and $v_n \leq g$ that the first integral assumes its infimum for the upper bound of $v_n$ that is

$$\inf_{\mathbf{v} \in \mathbf{K}_g} \int\limits_{\Gamma_C} v_n \tau_n \, ds = \int\limits_{\Gamma_C} g \tau_n \, ds.$$

Furthermore, we have $\tau_t v_t \geq -|\tau_t||v_t| \geq -\mathcal{F}|v_t|$, which leads

$$\int\limits_{\Gamma_C} \tau_t v_t + \mathcal{F}|v_t| \, ds \geq 0 \qquad \forall\, \mathbf{v} \in \mathbf{K}_g,$$

and so the infimum is assumed for $v_t = 0$.

On the other hand for $\tau \notin \widetilde{\mathbf{K}}$ we have seen that there exists a $\tilde{\mathbf{v}} \in \mathbf{K}_0$ with $I_2(\tilde{\mathbf{v}}, \tau) < 0$. But since $k\tilde{\mathbf{v}} \in \mathbf{K}_0$ for all $k \in \mathbb{N}$ we get

$$\lim_{k\to\infty} I_2(k\tilde{\mathbf{v}}, \tau) = -\infty$$

and therefore have

$$\inf_{\mathbf{v}\in\mathbf{K}_g} I_2(\mathbf{v}, \tau) = \begin{cases} \langle g, \tau_n \rangle_{\Gamma_C}, & \text{if } \tau \in \widetilde{\mathbf{K}}, \\ -\infty, & \text{if } \tau \notin \widetilde{\mathbf{K}}. \end{cases}$$

From the above observations $I(\tau)$ is computed to

$$I(\tau) = \begin{cases} -\frac{1}{2}\tilde{a}(\tau, \tau) + \langle g, \tau_n \rangle_{\Gamma_C}, & \text{if } \tau \in \widetilde{\mathbf{K}}, \\ -\infty, & \text{if } \tau \notin \widetilde{\mathbf{K}} \end{cases}$$

and we have

$$J(\mathbf{u}) \geq \sup_{\tau\in M} I(\tau) = \sup_{\tau\in\widetilde{\mathbf{K}}}(-\tilde{J}(\tau)) = -\inf_{\tau\in\widetilde{\mathbf{K}}} \tilde{J}(\tau) = -\tilde{J}(\sigma).$$

Next we prove that $\sigma(\mathbf{u}) = \mathbb{C} : \varepsilon(\mathbf{u})$ as defined in Lemma 3.2 minimizes $\tilde{J}(\tau)$ for all $\tau \in \widetilde{\mathbf{K}}$. Due to Lemma 3.2 we know that $\sigma(\mathbf{u}) \in \widetilde{\mathbf{K}}$. Using Green's formula we get

$$\begin{aligned}
-J(\mathbf{u}) &= -\frac{1}{2}\int_\Omega \varepsilon(\mathbf{u}) : \mathbb{C} : \varepsilon(\mathbf{u})\, dx + \int_\Omega \mathbf{f}\cdot\mathbf{u}\, dx + \int_{\Gamma_N} \mathbf{t}_0 \cdot \mathbf{u}\, ds - \int_{\Gamma_C} \mathcal{F}|u_t|\, ds \\
&= -\frac{1}{2}\int_\Omega \sigma(\mathbf{u}) : \mathbb{C}^{-1} : \sigma(\mathbf{u})\, dx + \int_\Omega \sigma(\mathbf{u}) : \mathbb{C}^{-1} : \sigma(\mathbf{u})\, dx - \int_{\Gamma_C} u_n \sigma(\mathbf{u})_n\, ds \\
&\quad - \int_{\Gamma_C} (u_t \sigma(\mathbf{u})_t + \mathcal{F}|u_t|)\, ds \\
&\geq \frac{1}{2}\int_\Omega \sigma(\mathbf{u}) : \mathbb{C}^{-1} : \sigma(\mathbf{u})\, dx - \int_{\Gamma_C} g\sigma(\mathbf{u})_n\, ds = \tilde{J}(\sigma(\mathbf{u})).
\end{aligned}$$

With $\sigma \in \widetilde{\mathbf{K}}$ being the solution of the dual minimization problem (3.13) we conclude with the above computations

$$\tilde{J}(\sigma(\mathbf{u})) \geq \tilde{J}(\sigma) \geq -J(\mathbf{u}) \geq \tilde{J}(\sigma(\mathbf{u}))$$

and since the dual minimization problem (3.13) has a unique solution we have $\sigma(\mathbf{u}) = \sigma$. Furthermore, we have seen above that $J(\mathbf{u}) = -\tilde{J}(\sigma)$ which finishes the proof. □

Now let us apply the theory of Fenchel's duality to the dual minimization problem (3.13). Our aim is to state the problem as a minimization problem over the space $\mathbf{X}_s$. The equality constraints in $\widetilde{\mathbf{K}}$ will be enforced by introducing Lagrange multipliers on $\Omega$ and $\Gamma_N$ combined with a Penalty method. For the inequality constraints we will use the indicator function. But first let us define a minimization problem on $\mathbf{X}_s$, that is equivalent to (3.13). We want to state this in the form of Remark 2.4. Therefore we define the operator $\Lambda \in \mathcal{L}(\mathbf{X}_s, \mathbf{Y})$ as follows

$$\Lambda\tau = (\Lambda_1\tau, \Lambda_2\tau, \Lambda_3\tau, \Lambda_4\tau) := (\operatorname{div}\tau, \tau \cdot \mathbf{n}, \tau_t, \tau_n). \tag{3.15}$$

Then we define the two functionals $F : \mathbf{X}_s \to \mathbb{R}$ and $G^c : \mathbf{Y} \to (-\infty, \infty]$ by

$$\begin{aligned} F(\tau) &:= \frac{1}{2}\,\tilde{a}(\tau,\tau), \qquad \tau \in \mathbf{X}_s, \\ G^c(\mathbf{v}, \boldsymbol{\psi}, \mu_t, \mu_n) &:= \frac{c}{2}\{\|\mathbf{v} + \mathbf{f}\|_0^2 + \|\boldsymbol{\psi} - \mathbf{t}_0\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)}\} + I_{\mathcal{F}}(\mu_t) + I_-^g(\mu_n), \quad (\mathbf{v}, \boldsymbol{\psi}, \mu_t, \mu_n) \in \mathbf{Y} \end{aligned}$$

where $c > 0$ denotes the penalty parameter. The indicator functions are defined by

$$I_{\mathcal{F}}(\mu_t) := \begin{cases} 0, & \text{if } |\mu_t| \le \mathcal{F}, \\ \infty, & \text{else,} \end{cases} \qquad I_-^g(\mu_n) := \begin{cases} -\langle g, \mu_n\rangle_{\Gamma_C}, & \text{if } \mu_n \le 0, \\ \infty, & \text{else.} \end{cases}$$

Now we can state the following result.

**Lemma 3.4:**
If $c > 0$ arbitrary, then the functionals $F$ and $G^c$ are proper, convex and l.s.c. on $\mathbf{X}_s$ and $\mathbf{Y}$, respectively.

*Proof.* Let $\mathbf{u} \in \mathbf{H}^1(\Omega)$ be the solution of the primal minimization problem (3.6). We define $\tau := \mathbb{C} : \varepsilon(\mathbf{u})$, which lies in $\mathbf{X}_s$ due to Lemma 3.2, and observe $F(\tau) = \frac{1}{2}a(\mathbf{u}, \mathbf{u})$. With Korn's inequality (2.5) and the continuity of $a(\cdot,\cdot)$ there exists a constant $C > 0$, such that

$$0 \le \frac{\alpha_0}{2}\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \le F(\tau) \le \frac{C}{2}\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} < \infty.$$

Due to Lemma 3.2 we have $\Lambda\sigma(\mathbf{u}) \in \mathbf{Y}$, $g\sigma(\mathbf{u})_n \le 0$ and $\|\sigma(\mathbf{u})\|_{\mathbf{H}(\mathrm{div},\Omega)} \le \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}$. Since $\mathbf{u}$ is depending continuously on the given data, its $\mathbf{H}^1(\Omega)$-norm is bounded. With

$$G^c(\Lambda\sigma(\mathbf{u})) = G^c(\operatorname{div}\sigma(\mathbf{u}), \sigma(\mathbf{u}) \cdot \mathbf{n}|_{\Gamma_N}, \sigma(\mathbf{u})_t|_{\Gamma_C}, \sigma(\mathbf{u})_n|_{\Gamma_C}) = -\langle g, \sigma(\mathbf{u})_n\rangle_{\Gamma_C}$$

we can use Theorem 2.3 to show

$$\begin{aligned} 0 \le G^c(\Lambda\sigma(\mathbf{u})) &= -\langle g, \sigma(\mathbf{u})_n\rangle_{\Gamma_C} \le \|g\|_{H^{\frac{1}{2}}(\Gamma_C)}\|\sigma(\mathbf{u})\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_C)} \\ &\le \|g\|_{H^{\frac{1}{2}}(\Gamma_C)}\|\sigma(\mathbf{u})\|_{\mathbf{H}(\mathrm{div},\Omega)} \le \|g\|_{H^{\frac{1}{2}}(\Gamma_C)}(\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}) < \infty. \end{aligned}$$

To prove convexity we first show, that any squared norm induced by a scalar product is convex. So let $V$ be a Banach space with scalar product $\langle\cdot,\cdot\rangle$ and with norm

$\|x\|_V^2 := \langle x, x\rangle$. Then for a fixed $b \in V$ we show that $\|x+b\|_V^2$ is convex for all $x \in V$. For $\alpha \in (0,1)$ and $x, y \in V$ there holds

$$\begin{aligned}
&\|\alpha x + (1-\alpha)y + b\|_V^2 - \alpha\|x+b\|_V^2 - (1-\alpha)\|y+b\|_V^2 \\
&\quad = \alpha^2\|x\|_V^2 + (1-\alpha)^2\|y\|_V^2 + 2\alpha\langle x,b\rangle + 2(1-\alpha)\langle y,b\rangle + 2\alpha(1-\alpha)\langle x,y\rangle + \|b\|_V^2 \\
&\qquad - \alpha(\|x\|_V^2 + 2\langle x,b\rangle + \|b\|_V^2) - (1-\alpha)(\|y\|_V^2 + 2\langle y,b\rangle + \|b\|_V^2) \\
&\quad = -\alpha(1-\alpha)(\|x\|_V^2 - 2\langle x,y\rangle + \|y\|_V^2) = -\alpha(1-\alpha)\|x-y\|_V^2 \le 0.
\end{aligned}$$

Due to the ellipticity of $\mathbb{C}^{-1}$ (cf. (2.3)) we have that the bilinear form $\tilde{a}(\cdot,\cdot)$ induces a norm on $[L^2(\Omega)]^{2\times 2}$ which proves the convexity of $F$.

Let $(\mathbf{v}, \psi, \mu_t, \mu_n), (\mathbf{w}, \chi, \nu_t, \nu_n) \in \mathbf{Y}$ and $\alpha \in (0,1)$. If $\mu_n > 0$ or $\nu_n > 0$ we have

$$G^c(\cdot,\cdot,\cdot,\alpha\mu_n + (1-\alpha)\nu_n) \le \infty = \alpha G^c(\cdot,\cdot,\cdot,\mu_n) + (1-\alpha)G^c(\cdot,\cdot,\cdot,\nu_n).$$

Otherwise if $\mu_n \le 0$ and $\nu_n \le 0$ we have $\alpha\mu_n + (1-\alpha)\nu_n \le 0$. In this case we have

$$\begin{aligned}
I_-^g(\alpha\mu_n + (1-\alpha)\nu_n) &= -\langle g, \alpha\mu_n + (1-\alpha)\nu_n\rangle_{\Gamma_C} \\
&= -\alpha\langle g,\mu_n\rangle_{\Gamma_C} - (1-\alpha)\langle g,\nu_n\rangle_{\Gamma_C} = \alpha I_-^g(\mu_n) + (1-\alpha)I_-^g(\nu_n).
\end{aligned}$$

The convexity for $\mu_t$ follows with the same arguments. If $|\mu_t| \le \mathcal{F}$ and $|\nu_t| \le \mathcal{F}$ it holds

$$|\alpha\mu_t + (1-\alpha)\nu_t| \le |\alpha||\mu_t| + |1-\alpha||\nu_t| \le \alpha\mathcal{F} + (1-\alpha)\mathcal{F} = \mathcal{F}.$$

The convexity for $\mathbf{v}$ and $\psi$ follows from the convexity of a squared norm and the fact that the penalty parameter $c$ is just a multiplicative constant. Therefore $G^c$ is convex for arbitrary $c > 0$.

As $F$ is continuous it is l.s.c. in particular. From Ekeland and Temam [32, see Proposition 2.3 in Chapter I] we have that $G^c$ is l.s.c. if its epigraph is closed, i.e.

$$\overline{\text{epi } G^c} = \text{epi } G^c := \{[(\mathbf{v},\psi,\mu_t,\mu_n), a] \in \mathbf{Y}\times\mathbb{R} : \; G^c(\mathbf{v},\psi,\mu_t,\mu_n) \le a\}.$$

If we define

$$\widetilde{K}_C := \{(\mathbf{v},\psi,\mu_t,\mu_n) \in \mathbf{Y} : \; \mu_n \le 0, \, |\mu_t| \le \mathcal{F}\}, \tag{3.16}$$

we clearly have epi $G^c \subset \widetilde{K}_C \times \mathbb{R}$. Now let $c > 0$ be fixed. Let us define

$$H(\mathbf{v},\psi,\mu_t,\mu_n) := \frac{c}{2}\{\|\mathbf{v}+\mathbf{f}\|_0^2 + \|\boldsymbol{\psi} - \mathbf{t}_0\|^2_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)}\} - \langle g, \mu_n\rangle_{\Gamma_C}.$$

Of course we have epi $G^c \subset$ epi $H$ and since $H$ is continuous its epigraph is closed. Furthermore, we have

$$\text{epi } G^c = \text{epi } H \cap \widetilde{K}_C \times \mathbb{R}.$$

To see this we take on the one hand $[(\mathbf{v},\psi,\mu_t,\mu_n),a] \in \text{epi } G^c$, then $(\mathbf{v},\psi,\mu_t,\mu_n) \in \widetilde{K}_C$ and we have

$$H(\mathbf{v},\psi,\mu_t,\mu_n) = G^c(\mathbf{v},\psi,\mu_t,\mu_n) \leq a \qquad \Rightarrow \qquad [(\mathbf{v},\psi,\mu_t,\mu_n),a] \in \text{epi } H.$$

On the other hand if $[(\mathbf{v},\psi,\mu_t,\mu_n),a] \in \text{epi } H \cap \widetilde{K}_C \times \mathbb{R}$, then since $|\mu_t| \leq \mathcal{F}$ and $\mu_n \leq 0$, we have

$$a \geq H(\mathbf{v},\psi,\mu_t,\mu_n) = G^c(\mathbf{v},\psi,\mu_t,\mu_n) \qquad \Rightarrow \qquad [(\mathbf{v},\psi,\mu_t,\mu_n),a] \in \text{epi } G^c.$$

But since epi $H$ and $\widetilde{K}_C \times \mathbb{R}$ are closed we have that epi $G^c$ is closed as well. If we now take the limit $c \to \infty$ we have

$$\text{epi } \lim_{c\to\infty} G^c \subset \hat{K} \times \mathbb{R},$$

with

$$\begin{aligned} \hat{K} := \{(\mathbf{v},\psi,\mu_t,\mu_n) \in \mathbf{Y} : \mathbf{v}+\mathbf{f} = 0 \text{ in } \Omega,\ \psi - \mathbf{t}_0 = 0 \text{ on } \Gamma_N, \\ \mu_n \leq 0 \text{ on } \Gamma_C,\ |\mu_t| \leq \mathcal{F} \text{ on } \Gamma_C\}. \end{aligned} \tag{3.17}$$

Note that $\hat{K}$ is a closed subset of $\mathbf{Y}$. Furthermore, we have

$$\text{epi } \lim_{c\to\infty} G^c \subset \text{epi } \hat{g}(\mathbf{v},\psi,\mu_t,\mu_n) \qquad \text{with } \hat{g}(\mathbf{v},\psi,\mu_t,\mu_n) := -\langle g,\mu_n\rangle_{\Gamma_C}.$$

Again we have that $\hat{g}$ is continuous and so its epigraph is closed. Analogously to the above case we conclude epi $\lim_{c\to\infty} G^c$ = epi $\hat{g} \cap \hat{K} \times \mathbb{R}$ and therefore epi $\lim_{c\to\infty} G^c$ is closed. □

Let us define the family of minimization problems $(\tilde{P}^c)$:

Find $\sigma \in \mathbf{X}_s$ such that for $c > 0$

$$F(\sigma) + G^c(\Lambda\sigma) \leq F(\tau) + G^c(\Lambda\tau) \qquad \forall\ \tau \in \mathbf{X}_s. \tag{3.18}$$

**Theorem 3.5:**
The family of minimization problems $(\tilde{P}^c)$ converges to the dual minimization problem (3.13), as the penalty parameter $c$ tends to infinity.

*Proof.* As we have seen in the proof of Lemma 3.4 the epigraph of $\lim_{c\to\infty} G^c$ is a subset of $\hat{K} \times \mathbb{R}$. But this means that

$$\lim_{c\to\infty} G^c(\Lambda\tau) =: G(\Lambda\tau) = \begin{cases} -\langle g,\tau_n\rangle_{\Gamma_C}, & \text{if } \Lambda\tau \in \hat{K}, \\ \infty, & \text{else.} \end{cases}$$

Again due to Lemma 3.4 the problems $(\tilde{P}^c)$ are uniquely solvable for every $c > 0$ and we have the equivalent formulation for (3.18)

$$F(\sigma) + G^c(\Lambda\sigma) = \min_{\tau\in\mathbf{X}_s} \{F(\tau) + G^c(\Lambda\tau)\}.$$

Finally, we take the limit of the minimum. We are allowed to switch the order using the fact, that the effective domains $\widetilde{K}_C$ of $G^c$ and $\hat{K}$ of $G$ defined in (3.16) and (3.17) are closed.

$$\begin{aligned}\lim_{c\to\infty}\Big(\min_{\tau\in\mathbf{X}_s}\{F(\tau)+G^c(\Lambda\tau)\}\Big) &= \min_{\tau\in\mathbf{X}_s}\Big(\lim_{c\to\infty}\{F(\tau)+G^c(\Lambda\tau)\}\Big)\\ &= \min_{\tau\in\mathbf{X}_s}(F(\tau)+G(\Lambda\tau)) = \min_{\tau\in\mathbf{X}_s,\Lambda\tau\in\hat{K}}(F(\tau)+G(\Lambda\tau)) = \min_{\tau\in\widetilde{\mathbf{K}}}\tilde{J}(\tau).\end{aligned}$$

□

If we set $\vec{\mathbf{w}} := (\mathbf{v},\boldsymbol{\psi},\mu_t,\mu_n)\in\mathbf{Y}$ and define the functional

$$\Phi^c(\tau,\vec{\mathbf{w}}) := F(\tau)+G^c(\Lambda\tau+\vec{\mathbf{w}}),$$

then we can define the perturbed problem $(\tilde{P}^c_{\vec{\mathbf{w}}})$:

Find $\sigma\in\mathbf{X}_s$ such that for $c>0$ and $\mathbf{w}\in\mathbf{Y}$

$$\Phi^c(\sigma,\mathbf{w}) \le \Phi^c(\tau,\mathbf{w}) \qquad \forall\ \tau\in\mathbf{X}_s. \tag{3.19}$$

Clearly we have $(\tilde{P}^c_{\mathbf{0}}) = \tilde{P}^c$. Now the conditions of Fenchel's duality theory hold and we are able to derive the conjugate problem of the dual minimization problem (3.13). We first have to compute the conjugate of $\Lambda$.

Let $(\mathbf{u},\boldsymbol{\varphi},\lambda_t,\lambda_n)\in\mathbf{Y}' = \mathbf{L}^2(\Omega)\times\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)\times\widetilde{H}^{\frac{1}{2}}(\Gamma_C)\times\widetilde{H}^{\frac{1}{2}}(\Gamma_C)$. If we further assume $\nabla\mathbf{u}\in\mathbf{L}^2(\Omega)$ and $\mathbf{u}|_\Gamma\in\mathbf{H}^{\frac{1}{2}}(\Gamma)$, which means $\mathbf{u}\in\mathbf{H}^1(\Omega)$, then we have

$$\begin{aligned}\langle\Lambda_1^*\mathbf{u},\tau\rangle &= \langle\mathbf{u},\Lambda_1\tau\rangle = \int_\Omega \mathbf{u}\cdot\operatorname{div}\tau\,dx = -\int_\Omega\nabla\mathbf{u}:\tau\,dx + \int_\Gamma\mathbf{u}\cdot\tau\cdot\mathbf{n}\,ds,\\ \langle\Lambda_2^*\boldsymbol{\varphi},\tau\rangle &= \int_{\Gamma_N}\boldsymbol{\varphi}\cdot\tau\cdot\mathbf{n}\,ds,\\ \langle\Lambda_3^*\lambda_n,\tau\rangle &= \int_{\Gamma_C}\lambda_n\,\mathbf{n}\cdot\tau\cdot\mathbf{n}\,ds,\\ \langle\Lambda_4^*\lambda_t,\tau\rangle &= \int_{\Gamma_C}\lambda_t\,\mathbf{t}\cdot\tau\cdot\mathbf{n}\,ds.\end{aligned}$$

Reminding Section 2.3, we have for the conjugate of $\Phi^c$

$$\Phi^{c*}(x^*,y^*) = F^*(x^*-\Lambda^*y^*)+G^{c*}(y^*).$$

The conjugate of $F$ is computed to

$$F^*(-\Lambda^*(\mathbf{u},\boldsymbol{\varphi},\lambda_t,\lambda_n)) := \sup_{\tau\in\mathbf{X}_s}\{-\langle\Lambda^*(\mathbf{u},\boldsymbol{\varphi},\lambda_t,\lambda_n),\tau\rangle - F(\tau)\}$$

$$
\begin{aligned}
&= \sup_{\tau \in \mathbf{X}_s} \Big\{ \int_\Omega \nabla \mathbf{u} : \tau \, dx - \frac{1}{2} \int_\Omega \tau : \mathbb{C}^{-1} : \tau \, dx - \int_{\Gamma_D} \mathbf{u} \cdot \tau \cdot \mathbf{n} \, ds \\
&\qquad\qquad - \int_{\Gamma_N} (\mathbf{u} + \boldsymbol{\varphi}) \cdot \tau \cdot \mathbf{n} \, ds - \int_{\Gamma_C} (u_n + \lambda_n) \tau_n \, ds - \int_{\Gamma_C} (u_t + \lambda_t) \tau_t \, ds \Big\} \\
&= \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega \nabla \mathbf{u} : \mathbb{C} : \nabla \mathbf{u} \, dx, & \text{if } \begin{array}{l} \mathbf{u} = \mathbf{0}|_{\Gamma_D},\ \mathbf{u} + \boldsymbol{\varphi} = \mathbf{0}|_{\Gamma_N}, \\ u_n + \lambda_n = 0|_{\Gamma_C},\ u_t + \lambda_t = 0|_{\Gamma_C}, \end{array} \\ \infty, & \textit{else.} \end{cases}
\end{aligned} \tag{3.20}
$$

This can be seen very easy when regarding the volume part and the boundary parts separately. Due to the ellipticity of $\mathbb{C}^{-1}$, see (2.3), the volume terms assume the supremum for $\tau = \mathbb{C} : \nabla \mathbf{u}$ at $\frac{1}{2} \int_\Omega \nabla \mathbf{u} : \mathbb{C} : \nabla \mathbf{u} \, dx$ which is positive due to Korn's inequality (2.5). Note that we also use the density of $C_0^\infty(\Omega)^{2\times 2}$ in $\mathbf{X}_s$ since we cannot expect that $\mathbb{C} : \nabla \mathbf{u} \in \mathbf{X}_s$.

For each boundary part there are two possibilities. We do this exemplary for the Neumann boundary part. The other parts follow with the same arguments. First, $\mathbf{u} + \boldsymbol{\varphi}$ does not equals zero. Then we can find some $\tau \in \mathbf{X}_s$ such that $\operatorname{sign} \tau \cdot \mathbf{n}|_{\Gamma_N} = -\operatorname{sign}(\mathbf{u} + \boldsymbol{\varphi})$ and with $k\tau \in \mathbf{X}_s$ for all $k \in \mathbb{N}$ we can take the limit of $k$ towards infinity to conclude that the supremum is infinity. Second, $\mathbf{u} + \boldsymbol{\varphi} = 0$ and the boundary part vanishes. Due to the symmetry properties (2.2) of the elastic tensor $\mathbb{C}$ we have $\nabla \mathbf{u} : \mathbb{C} : \nabla \mathbf{u} = \varepsilon(\mathbf{u}) : \mathbb{C} : \varepsilon(\mathbf{u})$ since

$$
\begin{aligned}
\nabla \mathbf{u} : \mathbb{C} &= u_{i,j} \mathbb{C}_{ijkl} \mathbf{e}_k \otimes \mathbf{e}_l = \frac{1}{2} u_{i,j} \mathbb{C}_{ijkl} \mathbf{e}_k \otimes \mathbf{e}_l + \frac{1}{2} u_{j,i} \mathbb{C}_{jikl} \mathbf{e}_k \otimes \mathbf{e}_l \\
&= \frac{1}{2} (u_{i,j} + u_{j,i}) \mathbb{C}_{ijkl} \mathbf{e}_k \otimes \mathbf{e}_l = \varepsilon(\mathbf{u}) : \mathbb{C}
\end{aligned}
$$

and the right product follows analogously with the symmetry of the last two components of $\mathbb{C}$.

The conjugate for $G^c$ is computed to

$$
\begin{aligned}
G^{c*}(\mathbf{u}, \boldsymbol{\varphi}, \lambda_t, \lambda_n) &= \sup_{\mathbf{w} \in \mathbf{Y}} \{ (\mathbf{u}, \mathbf{v})_{\mathbf{L}^2(\Omega)} + \langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_{\Gamma_N} + \langle \lambda_n, \mu_n \rangle_{\Gamma_C} + \langle \lambda_t, \mu_t \rangle_{\Gamma_C} - G^c(\vec{\mathbf{w}}) \} \\
&= \sup_{\mathbf{w} \in \mathbf{Y}} \Big\{ (\mathbf{u}, \mathbf{v})_{\mathbf{L}^2(\Omega)} - \frac{c}{2} (\mathbf{v} + \mathbf{f}, \mathbf{v} + \mathbf{f})_{\mathbf{L}^2(\Omega)} + \langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_{\Gamma_N} - \frac{c}{2} \langle \boldsymbol{\psi} - \mathbf{t}_0, \boldsymbol{\psi} - \mathbf{t}_0 \rangle_{\Gamma_N} \\
&\qquad + \langle \lambda_n, \mu_n \rangle_{\Gamma_C} - I_-^g(\mu_n) + \langle \lambda_t, \mu_t \rangle_{\Gamma_C} - I_{\mathcal{F}}(\mu_t) \Big\} \\
&= \begin{cases} \dfrac{1}{2c} (\mathbf{u}, \mathbf{u})_{\mathbf{L}^2(\Omega)} - (\mathbf{u}, \mathbf{f})_{\mathbf{L}^2(\Omega)} + \dfrac{1}{2c} \langle \boldsymbol{\varphi}, \boldsymbol{\varphi} \rangle_{\Gamma_N} + \langle \boldsymbol{\varphi}, \mathbf{t}_0 \rangle_{\Gamma_N} + j(\lambda_t), & \text{if } \lambda_n + g \geq 0, \\ \infty, & \textit{else.} \end{cases}
\end{aligned} \tag{3.21}
$$

Here the first four terms follow due to the ellipticity of the inner products of $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$. This holds true since $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$ are Hilbert spaces. The suprema are assumed for $\mathbf{v} = \frac{1}{c}\mathbf{u} - \mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{\psi} = \frac{1}{c}\boldsymbol{\varphi} + \mathbf{t}_0 \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$.

For the normal part on the contact boundary the supremum is clearly assumed for $\mu_n \le 0$. If $\lambda_n + g < 0$, then $\langle \lambda_n + g, \mu_n \rangle > 0$ for $\mu_n < 0$ and with $k\mu_n < 0$ for all $k \in \mathbb{N}$ the supremum is infinity. Otherwise, if $\lambda_n + g \ge 0$ then the dual product is negative and the supremum is assumed for $\mu_n = 0$. For the tangential part the supremum is assumed for $\mu_t = \mathcal{F}$ sign $\lambda_t$ at $j(\lambda_t)$, since $\mathcal{F} \ge 0$ and the absolute value of $\lambda_t$ is the supremum over all $\mu_t \in H^{-\frac{1}{2}}(\Gamma_C)$ with $|\mu_t| \le 1$.

Since $G = \lim_{c\to\infty} G^c$ we have

$$G^*(\mathbf{u}, \boldsymbol{\varphi}, \lambda_t, \lambda_n) = \begin{cases} -(\mathbf{u}, \mathbf{f})_0 + \langle \boldsymbol{\varphi}, \mathbf{t}_0 \rangle_{\Gamma_N} + j(\lambda_t), & \text{if } \lambda_n + g \ge 0, \\ \infty, & \textit{else}. \end{cases}$$

With $\Phi := \lim_{c\to\infty} \Phi^c$ we have

$$\Phi(\tau, \vec{\mathbf{w}}) = F(\tau) + \lim_{c\to\infty} G^c(\Lambda\tau + \vec{\mathbf{w}}) = F(\tau) + G(\Lambda\tau + \vec{\mathbf{w}})$$

and using equation (2.9) and the results from above the conjugate problem $(\tilde{P}^*)$ of the dual minimization problem (3.13) reads

$$\sup_{(\mathbf{u},\boldsymbol{\varphi},\lambda_t,\lambda_n)\in \mathbf{Y}'_L} \left\{ -\frac{1}{2} \int_\Omega \varepsilon(\mathbf{u}) : \mathbb{C} : \varepsilon(\mathbf{u})\, dx + \int_\Omega \mathbf{u} \cdot \mathbf{f}\, dx - \int_{\Gamma_N} \boldsymbol{\varphi} \cdot \mathbf{t}_0\, ds - \int_{\Gamma_C} \mathcal{F}|\lambda_t|\, ds \right\}, \quad (3.22)$$

where

$$\begin{aligned} \mathbf{Y}'_L := \Big\{ (\mathbf{u}, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{V}_D \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N) \times \widetilde{H}^{\frac{1}{2}}(\Gamma_C) \times \widetilde{H}^{\frac{1}{2}}(\Gamma_C) : \mathbf{u} + \boldsymbol{\varphi} = \mathbf{0}|_{\Gamma_N}, \\ u_n + \lambda_n = 0|_{\Gamma_C},\ u_t + \lambda_t = 0|_{\Gamma_C},\ -\lambda_n \le g \ \text{ a.e. on } \Gamma_C \Big\}. \end{aligned}$$

But this is just the primal problem (3.6) with additional Lagrange multipliers.

### 3.1.2. Saddle point formulation

Due to Lemma 3.4 the assumptions in Proposition 2.9 are satisfied and we are able to state a saddle point problem that is equivalent to the dual minimization problem (3.13). Let us first compute the Lagrange functional $L : \mathbf{X}_s \times \mathbf{Y}' \to \bar{\mathbb{R}}$. To do this, we first compute the Lagrange functional $L^c$ and then take the limit of $c$ to infinity.

From Definition 2.7 we have

$$
\begin{aligned}
-L^c(\tau;\mathbf{u},\boldsymbol{\varphi},\lambda_t,\lambda_n) &= \sup_{\vec{\mathbf{w}}\in\mathbf{Y}}\{\langle(\mathbf{u},\boldsymbol{\varphi},\lambda_t,\lambda_n),\vec{\mathbf{w}}\rangle - \Phi^c(\tau,\vec{\mathbf{w}})\}\\
&= \sup_{\vec{\mathbf{w}}\in\mathbf{Y}}\left\{(\mathbf{u},\mathbf{v})_{\mathbf{L}^2(\Omega)} + \langle\boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\Gamma_N} + \langle\lambda_n,\mu_n\rangle_{\Gamma_C} + \langle\lambda_t,\mu_t\rangle_{\Gamma_C} - F(\tau) - G^c(\Lambda\tau+\vec{\mathbf{w}})\right\}\\
&= \sup_{\vec{\mathbf{w}}\in\mathbf{Y}}\Big\{(\mathbf{u},\mathbf{v})_{\mathbf{L}^2(\Omega)} - \frac{c}{2}(\mathbf{v}+\operatorname{div}\tau+\mathbf{f},\mathbf{v}+\operatorname{div}\tau+\mathbf{f})_{\mathbf{L}^2(\Omega)} - \frac{1}{2}\int\tau:\mathbb{C}^{-1}\tau\,dx\\
&\qquad + \langle\boldsymbol{\varphi},\boldsymbol{\psi}\rangle_{\Gamma_N} - \frac{c}{2}\langle\tau\cdot\mathbf{n}+\boldsymbol{\psi}-\mathbf{t}_0,\tau\cdot\mathbf{n}+\boldsymbol{\psi}-\mathbf{t}_0\rangle_{\Gamma_N}\\
&\qquad + \langle\lambda_n,\mu_n\rangle_{\Gamma_C} - I_-^g(\tau_n+\mu_n) + \langle\lambda_t,\mu_t\rangle_{\Gamma_C} - I_{\mathcal{F}}(\tau_t+\mu_t)\Big\}\\
&= \begin{cases}
\begin{aligned}&\frac{1}{2c}(\mathbf{u},\mathbf{u})_{\mathbf{L}^2(\Omega)} - (\mathbf{u},\operatorname{div}\tau+\mathbf{f})_{\mathbf{L}^2(\Omega)} - \frac{1}{2}\tilde{a}(\tau,\tau) + \frac{1}{2c}\langle\boldsymbol{\varphi},\boldsymbol{\varphi}\rangle_{\Gamma_N}\\ &\quad -\langle\boldsymbol{\varphi},\tau\cdot\mathbf{n}-\mathbf{t}_0\rangle_{\Gamma_N} - \langle\lambda_n,\tau_n\rangle_{\Gamma_C} + j(\lambda_t) - \langle\lambda_t,\tau_t\rangle_{\Gamma_C},\end{aligned} & \text{if } \lambda_n+g\geq 0,\\
\infty, & \textit{else}.
\end{cases}
\end{aligned}
$$

The last equation follows analogously to the computations of the conjugates $F^*$ and $G^{c*}$ in the previous section. Hence the supremum on the contact boundary is assumed for $\tau_n+\mu_n=0$ and $\tau_t+\mu_t=\mathcal{F}\operatorname{sign}\lambda_t$. The other terms follow again due to the ellipticity of the inner products of $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$.

Now defining $L := \lim_{c\to\infty} L^c$ we have

$$
L(\tau;\mathbf{u},\boldsymbol{\varphi},\lambda_t,\lambda_n) := \begin{cases}
\begin{aligned}&\frac{1}{2}\,\tilde{a}(\tau,\tau) + b(\mathbf{u},\tau) + f(\mathbf{u})\\ &\quad + d_N(\boldsymbol{\varphi},\tau) - t_0(\boldsymbol{\varphi})\\ &\quad + d_{C,n}(\lambda_n,\tau) + d_{C,t}(\lambda_t,\tau) - j(\lambda_t),\end{aligned} & \text{if } \lambda_n+g\geq 0,\\
-\infty, & \text{else},
\end{cases} \tag{3.23}
$$

where the linear forms and bilinear forms are defined by

$$
\begin{aligned}
f(\mathbf{u}) &:= \int_\Omega \mathbf{u}\cdot\mathbf{f}\,dx, \qquad t_0(\boldsymbol{\varphi}) := \int_{\Gamma_N}\boldsymbol{\varphi}\cdot\mathbf{t}_0\,ds, \qquad d_N(\boldsymbol{\varphi},\tau) := \int_{\Gamma_N}\boldsymbol{\varphi}\cdot\tau\cdot\mathbf{n}\,ds,\\
d_{C,n}(\lambda_n,\tau) &:= \int_{\Gamma_C}\lambda_n\tau_n\,ds, \qquad d_{C,t}(\lambda_t,\tau) := \int_{\Gamma_C}\lambda_t\tau_t\,ds.
\end{aligned} \tag{3.24}
$$

**Remark 3.6:** *We obtain the same result for* (3.23) *if we compute the Lagrangian via* $\Phi(\tau,\mathbf{w})$.

We have the following saddle point problem:

Find $(\sigma;\mathbf{u},\boldsymbol{\varphi},\lambda_t,\lambda_n) \in \mathbf{X}_s \times \mathbf{Y}'$ such that for all $(\tau;\mathbf{v},\psi,\mu_t,\mu_n) \in \mathbf{X}_s \times \mathbf{Y}'$

$$L(\sigma;\mathbf{v},\psi,\mu_t,\mu_n) \leq L(\sigma;\mathbf{u},\boldsymbol{\varphi},\lambda_t,\lambda_n) \leq L(\tau;\mathbf{u},\boldsymbol{\varphi},\lambda_t,\lambda_n). \tag{3.25}$$

**Remark 3.7:** *Due to Lemma 3.4 and Proposition 2.8 we have that the saddle point problem* (3.25) *is equivalent to the dual minimization problem* (3.13) *and the primal problem* (3.22) *in the sense that for the solution* $(\sigma;\mathbf{u},\boldsymbol{\varphi},\lambda_t,\lambda_n)$ *of* (3.25) *we have* $\sigma \in \mathbf{X}_s$ *solves* (3.13) *and* $(\mathbf{u},\boldsymbol{\varphi},\lambda_t,\lambda_n) \in \mathbf{Y}'$ *solves* (3.22). *Since the minimization problems are both uniquely solvable this holds true for the saddle point problem.*

**Remark 3.8:** *For convenience we redefine the Lagrange multiplier $\lambda_n$ by lifting it with the gap function $g$, i.e. we set $\lambda_n := \tilde{\lambda}_n + g$ with $\tilde{\lambda}_n + g \geq 0$. In this way the gap function tested with the normal stress becomes an additional term in the Lagrange functional and we search for $\lambda_n \geq 0$. Moreover in order to treat the problem numerically we have to impose the symmetry of the tensors in form of another Lagrange multiplier to extend the problem to the space*

$$\mathbf{X} := \mathbf{H}(\mathrm{div},\Omega). \tag{3.26}$$

The fact that $\tau$ is symmetric means that the antisymmetric part $\mathrm{as}(\tau)$, as defined in (2.4), vanishes. Since the nonzero part of as(tau), i.e. $\frac{1}{2}(\tau_{12}-\tau_{21})$, is in $L^2(\Omega)$ we define

$$\widetilde{\mathbf{Y}} := \mathbf{L}^2(\Omega) \times L^2(\Omega)' \times \mathbf{H}^{-\frac{1}{2}}(\Gamma_N) \times H^{-\frac{1}{2}}(\Gamma_C) \times H^{-\frac{1}{2}}(\Gamma_C). \tag{3.27}$$

Furthermore, we define $\tilde{\Lambda} : \mathbf{X} \to \widetilde{\mathbf{Y}}$

$$\tilde{\Lambda}\tau = (\tilde{\Lambda}_1\tau,\tilde{\Lambda}_2\tau,\tilde{\Lambda}_3\tau,\tilde{\Lambda}_4\tau,\tilde{\Lambda}_5\tau) = (\mathrm{div}\,\tau,\tau_{12}-\tau_{21},\tau\cdot\mathbf{n},\tau_t,\tau_n) \tag{3.28}$$

and observe

$$\begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} : \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} = \eta(\tau_{12}-\tau_{21}) = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} : \mathrm{as}(\tau) = \eta : (\tilde{\Lambda}_2\tau) \quad \forall \eta \in L^2(\Omega),\ \tau \in \mathbf{X}.$$

If we finally define

$$\begin{aligned} \widetilde{F}(\tau) &:= \frac{1}{2}\,\tilde{a}(\tau,\tau) - \langle g,\tau_n\rangle_{\Gamma_C}, \\ \widetilde{G}^c(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n) &:= \frac{c}{2}\Big\{\|\mathbf{v}+\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)} + \langle \boldsymbol{\psi}-\mathbf{t}_0,\boldsymbol{\psi}-\mathbf{t}_0\rangle_{\Gamma_N} \\ &\qquad + I^0_-(\mu_n) + I_{\mathcal{F}}(\mu_t)\Big\}, \end{aligned} \tag{3.29}$$

for $\tau \in \mathbf{X}$ and $(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n) \in \widetilde{\mathbf{Y}}$, then the results of Lemma 3.4 and Theorem 3.5 still hold true. Furthermore, we get

$$\langle \tilde{\Lambda}_2^*\eta,\tau\rangle = \langle \eta,\tilde{\Lambda}_2\tau\rangle = \int\limits_\Omega \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} : \tau\,dx \qquad \forall \eta \in L^2(\Omega),\ \tau \in \mathbf{X}.$$

Proceeding analogously to the previous computations of the conjugates $F^*$ and $G^{c*}$ we get with (3.20)

$$\widetilde{F}^*(-\tilde{\Lambda}^*(\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n)) = \begin{cases} \frac{1}{2}a(\mathbf{u},\mathbf{u}), & \text{if } \begin{array}{l} \mathbf{u}=\mathbf{0}|_{\Gamma_D},\ \mathbf{u}+\boldsymbol{\varphi}=\mathbf{0}|_{\Gamma_N}, \\ u_t+\lambda_t=0|_{\Gamma_C},\ u_n+\lambda_n-g=0|_{\Gamma_C}, \end{array} \\ \infty, & \text{else.} \end{cases}$$

and using (3.21)

$$\widetilde{G}^{c*}(\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) = \begin{cases} \frac{1}{2c}(\mathbf{u},\mathbf{u})_0-(\mathbf{u},\mathbf{f})_0+\frac{1}{2c}\langle\boldsymbol{\varphi},\boldsymbol{\varphi}\rangle_{\Gamma_N} & \\ \qquad +\langle\boldsymbol{\varphi},\mathbf{t}_0\rangle_{\Gamma_N}+j(\lambda_t)+\frac{1}{2c}(\eta,\eta)_0, & \text{if } \lambda_n\geq 0, \\ \infty, & \text{else.} \end{cases}$$

Therefore we still get the primal problem (3.6) as the conjugate problem of the perturbed problem

Find $\sigma\in\mathbf{X}$, such that for $\widetilde{\mathbf{w}}:=(\mathbf{v},\xi,\psi,\mu_t,\mu_n)\in\widetilde{\mathbf{Y}}$

$$\widetilde{\Phi}(\sigma,\widetilde{\mathbf{w}})\leq\widetilde{\Phi}(\tau,\widetilde{\mathbf{w}}),\qquad\forall\ \tau\in\mathbf{X} \tag{3.30}$$

with $\widetilde{\Phi}:=\lim_{c\to\infty}(\widetilde{F}+\widetilde{G}^c)$. To compute the resulting Lagrange functional $\widetilde{L}:\mathbf{X}\times\widetilde{\mathbf{Y}}'\to\mathbb{R}\cup\{-\infty,\infty\}$, we make use of Remark 3.6. Defining

$$\begin{aligned}\bar{K}:=\Big\{(\mathbf{v},\xi,\psi,\mu_t,\mu_n)\in\widetilde{\mathbf{Y}}:\ \mathbf{v}+\mathbf{f}=0\ \text{ in }\Omega,\ \xi=0,\ \boldsymbol{\psi}-\mathbf{t}_0=0\ \text{ on }\Gamma_N,\\ |\mu_t|\leq\mathcal{F}\ \text{ on }\Gamma_C,\ \mu_n\leq 0\ \text{ on }\Gamma_C\Big\},\end{aligned}$$

we have

$$\widetilde{\Phi}(\tau,\widetilde{\mathbf{w}}):=\lim_{c\to\infty}(\widetilde{F}(\tau)+\widetilde{G}^c(\tilde{\Lambda}\tau+\widetilde{\mathbf{w}}))=\begin{cases}\frac{1}{2}\tilde{a}(\tau,\tau)-\langle g,\tau_n\rangle_{\Gamma_C}, & \text{if } \tilde{\Lambda}\tau+\widetilde{\mathbf{w}}\in\bar{K},\\ \infty, & \text{else.}\end{cases}$$

In this way the Lagrange functional is computed to

$$\begin{aligned}\widetilde{L}(\tau;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) &:= \sup_{\widetilde{\mathbf{w}}\in\widetilde{\mathbf{Y}}}\Big\{\langle(\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n),\widetilde{\mathbf{w}}\rangle-\widetilde{\Phi}(\tau,\widetilde{\mathbf{w}})\Big\}\\ &= \begin{cases}\frac{1}{2}\tilde{a}(\tau,\tau)-g(\tau)+b(\mathbf{u},\tau)+f(\mathbf{u}) & \\ \qquad +s(\eta,\tau)+d_N(\boldsymbol{\varphi},\tau)-t_0(\boldsymbol{\varphi}) & \text{if } \lambda_n\geq 0,\\ \qquad +d_{C,n}(\lambda_n,\tau)+d_{C,t}(\lambda_t,\tau)-j(\lambda_t), & \\ -\infty, & \text{else,}\end{cases}\end{aligned} \tag{3.31}$$

involving the new linear form and bilinear form

$$g(\tau) := \int_{\Gamma_C} g\,\tau_n\,ds \qquad s(\eta,\tau) := \int_{\Omega} \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} : \mathrm{as}(\tau)\ dx = \int_{\Omega} \eta(\tau_{12} - \tau_{21})\,dx. \quad (3.32)$$

The result is evident since the supremum is assumed for $\tilde{\Lambda}\tau + \widetilde{\mathbf{w}} \in \bar{K}$ and therefore $\operatorname{div}\tau + \mathbf{v} = -\mathbf{f}$, $\mathrm{as}(\tau) + \xi = 0$, $\tau \cdot \mathbf{n} + \boldsymbol{\psi} = \mathbf{t}_0$. The supremum concerning the last two multipliers is assumed similar to the previous computation of $L$ for $\tau_n + \mu_n = 0$ and $\tau_t + \mu_t = \mathcal{F}\,\mathrm{sign}\,\lambda_t$. We have the following saddle point problem

Find $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'$ such that for all $(\tau, \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'$

$$\widetilde{L}(\sigma; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \le \widetilde{L}(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \le \widetilde{L}(\tau; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n).$$

Note that the left inequality holds true for all $\mu_n \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ with $\mu_n \ngeq 0$. Without loss of generality we therefore restrict the Lagrange multiplier concerning the normal part on the contact boundary of the saddle point problem to

$$\mu_n \in \widetilde{H}^{\frac{1}{2}}_{+}(\Gamma_C) := \left\{\mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C) : \mu \ge 0 \ \text{a.e. on}\ \Gamma_C\right\}.$$

If we introduce

$$\widetilde{\mathbf{Y}}'_{+} := \mathbf{L}^2(\Omega) \times L^2(\Omega) \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N) \times \widetilde{H}^{\frac{1}{2}}(\Gamma_C) \times \widetilde{H}^{\frac{1}{2}}_{+}(\Gamma_C),$$

then the above saddle point problem reads

Find $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_{+}$ such that for all $(\tau, \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_{+}$

$$\widetilde{L}(\sigma; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \le \widetilde{L}(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \le \widetilde{L}(\tau; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n). \quad (3.33)$$

**Remark 3.9:** *In the case $\mathcal{F} \equiv 0$, which means that no friction occurs on the contact boundary $\Gamma_C$, a similar procedure as the one above is valid. The indicator function $I_{\mathcal{F}}(\mu_t)$ in the definitions of $G^c$ and $\widetilde{G}^c$ is exchanged by the penalizing term $\frac{c}{2}\langle\mu_t, \mu_t\rangle_{\Gamma_C}$ in order to force the tangential traction to zero on the contact boundary. This is due to $0 \le |\sigma_t| \le \mathcal{F} \equiv 0$ on $\Gamma_C$ in* (3.1). *Therefore the j-functional* (3.5) *vanishes in the Lagrange functions* (3.23) *and* (3.31)*, respectively. Remark* (3.7) *applies in a modified form.*

Now we are in the position of stating a first result concerning equivalence and uniqueness.

**Theorem 3.10:**
The saddle point problem (3.33) is equivalent to the minimization problems (3.13) and (3.6) in the following sense.

**(i)** If $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_+$ is a saddle point of (3.33), then $\sigma \in \widetilde{K}$ is the unique solution of the minimization problem (3.13). Furthermore, it holds $\sigma = \mathbb{C} : \varepsilon(\mathbf{u})$ in $\Omega$, $\eta = \frac{1}{2}((\nabla \mathbf{u})_{12} - (\nabla \mathbf{u})_{21})$ in $\Omega$, $\mathbf{u} = 0$ on $\Gamma_D$, $\mathbf{u} + \boldsymbol{\varphi} = 0$ on $\Gamma_N$, $u_t + \lambda_t = 0$ on $\Gamma_C$ and $u_n + \lambda_n = g$ on $\Gamma_C$.

**(ii)** If $\sigma \in \widetilde{K}$ is the solution of the dual minimization problem (3.13) and $\mathbf{u} \in \mathbf{K}_g$ is the unique solution of the primal minimization problem (3.6), then $(\sigma; \mathbf{u}, \frac{1}{2}((\nabla \mathbf{u})_{12} - (\nabla \mathbf{u})_{21}), -\mathbf{u}|_{\Gamma_N}, -u_t|_{\Gamma_C}, g - u_n|_{\Gamma_C}) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_+$ is a saddle point of (3.33). Since $\sigma$ and $\mathbf{u}$ are unique, the saddle point is unique as well.

*Proof.* (i) Let $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_+$ be a saddle point of (3.33). Then, since $2\lambda_n \in \widetilde{H}^{\frac{1}{2}}_+(\Gamma_C)$, we can insert $(0,0,0,0,0) \in \widetilde{\mathbf{Y}}'_+$ and $(2\mathbf{u}, 2\eta, 2\boldsymbol{\varphi}, 2\lambda_t, 2\lambda_n) \in \widetilde{\mathbf{Y}}'_+$ into the left inequality of (3.33). Noting that

$$j(2\lambda_t) = \int_{\Gamma_C} \mathcal{F}|2\lambda_t|\, ds = 2 \int_{\Gamma_C} \mathcal{F}|\lambda_t|\, ds = 2j(\lambda_t),$$

we conclude after subtracting $\widetilde{L}(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n)$

$$b(\mathbf{u}, \sigma) + f(\mathbf{u}) + s(\eta, \sigma) + d_N(\boldsymbol{\varphi}, \sigma) - t(\boldsymbol{\varphi}) + d_{C,n}(\lambda_n, \sigma) + d_{C,t}(\lambda_t, \sigma) - j(\lambda_t) = 0 \qquad (3.34)$$

and the left inequality of (3.33) reduces to

$$\begin{aligned} &b(\mathbf{v}, \sigma) + f(\mathbf{v}) + s(\xi, \sigma) + d_N(\boldsymbol{\psi}, \sigma) - t(\boldsymbol{\psi}) \\ &\quad + d_{C,n}(\mu_n, \sigma) + d_{C,t}(\mu_t, \sigma) - j(\mu_t) \le 0 \quad \forall\, (\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \in \widetilde{\mathbf{Y}}'_+. \end{aligned}$$

If we take $(\pm\tilde{\mathbf{v}}, 0, 0, 0, 0) \in \widetilde{\mathbf{Y}}'_+$ with some $\tilde{\mathbf{v}} \in \mathbf{L}^2(\Omega)$ we have

$$b(\tilde{\mathbf{v}}, \sigma) + f(\tilde{\mathbf{v}}) = \int_{\Omega} \tilde{\mathbf{v}} \cdot (\mathbf{f} + \operatorname{div} \sigma)\, dx = 0 \quad \forall\, \tilde{\mathbf{v}} \in \mathbf{L}^2(\Omega)$$

from which we deduce $-\operatorname{div} \sigma = \mathbf{f}$ in $\Omega$.

In the same way by inserting $(0, \pm\tilde{\xi}, 0, 0, 0) \in \widetilde{\mathbf{Y}}'_+$ for $\tilde{\xi} \in L^2(\Omega)$ and $(0, 0, \pm\tilde{\boldsymbol{\psi}}, 0, 0) \in \widetilde{\mathbf{Y}}'_+$ for $\tilde{\boldsymbol{\psi}} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)$, respectively we have

$$\begin{aligned} s(\tilde{\xi}, \sigma) &= \int_{\Omega} \tilde{\xi}(\sigma_{12} - \sigma_{21})\, dx = 0 && \forall\, \tilde{\xi} \in L^2(\Omega), \\ d_N(\tilde{\boldsymbol{\psi}}, \sigma) - t(\tilde{\boldsymbol{\psi}}) &= \int_{\Gamma_N} \tilde{\boldsymbol{\psi}} \cdot (\sigma \cdot \mathbf{n} - \mathbf{t}_0)\, ds = 0 && \forall\, \tilde{\boldsymbol{\psi}} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N) \end{aligned}$$

and we conclude $\sigma = \sigma^T$ in $\Omega$ and $\sigma \cdot \mathbf{n} = \mathbf{t}_0$ on $\Gamma_N$. For $(0,0,0,0,\mu_n) \in \widetilde{\mathbf{Y}}'_+$ with $\mu_n \in \widetilde{H}^{\frac{1}{2}}_+(\Gamma_C)$ we observe

$$d_{C,n}(\mu_n, \sigma) = \int\limits_{\Gamma_C} \mu_n \sigma_n \, ds \le 0 \quad \forall\ \mu_n \in \widetilde{H}^{\frac{1}{2}}_+(\Gamma_C)$$

and thus $\sigma_n \le 0$ a.e. on $\Gamma_C$. Finally, if we insert $(0,0,0,\tilde{\mu}_t,0) \in \widetilde{\mathbf{Y}}'_+$ with $\tilde{\mu}_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ and $\operatorname{sign}\tilde{\mu}_t = \operatorname{sign}\sigma_t$ we have for all $\tilde{\mu}_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ with $\operatorname{sign}\tilde{\mu}_t = \operatorname{sign}\sigma_t$

$$\begin{aligned} d_{C,t}(\tilde{\mu}_t, \sigma) - j(\tilde{\mu}_t) &= \int\limits_{\Gamma_C} (\sigma_t \tilde{\mu}_t - \mathcal{F}|\tilde{\mu}_t|)\, ds = \int\limits_{\Gamma_C} (\sigma_t \operatorname{sign}\tilde{\mu}_t - \mathcal{F})|\tilde{\mu}_t|\, ds \\ &= \int\limits_{\Gamma_C} (\sigma_t \operatorname{sign}\sigma_t - \mathcal{F})|\tilde{\mu}_t|\, ds = \int\limits_{\Gamma_C} (|\sigma_t| - \mathcal{F})|\tilde{\mu}_t|\, ds \le 0 \end{aligned}$$

and since $|\mu_t| \ge 0$ on $\Gamma_C$, we have $|\sigma_t| \le \mathcal{F}$ a.e. on $\Gamma_C$. But this simply means $\sigma \in \widetilde{K}$. Using (3.34) the right inequality of the saddle point problem reduces to

$$\widetilde{J}(\sigma) = \frac{1}{2}\tilde{a}(\sigma,\sigma) - g(\sigma) \le \begin{array}{r} \widetilde{J}(\tau) + b(\mathbf{u},\tau) + f(\mathbf{u}) + s(\eta,\tau) + d_N(\boldsymbol{\varphi},\tau) - t(\boldsymbol{\varphi}) \\ + d_{C,n}(\lambda_n,\tau) + d_{C,t}(\lambda_t,\tau) - j(\lambda_t) \end{array} \quad \forall\ \tau \in \mathbf{X}.$$

Restricting $\tau$ to the convex set $\widetilde{K}$ we have

$$b(\mathbf{u},\tau) + f(\mathbf{u}) = 0, \qquad d_N(\boldsymbol{\varphi},\tau) - t(\boldsymbol{\varphi}) = 0 \qquad \text{and} \qquad s(\eta,\tau) = 0.$$

Furthermore, with $\tau_n \le 0$, $\lambda_n \ge 0$ and $\tau_t \le |\tau_t| \le \mathcal{F}$ on $\Gamma_C$ there holds

$$d_{C,n}(\lambda_n,\tau) \le 0 \qquad \text{and} \qquad d_{C,t}(\lambda_t,\tau) - j(\lambda_t) \le 0.$$

Now the right inequality of the saddle point problem (3.33) restricted to $\widetilde{K}$ reads

$$\widetilde{J}(\sigma) \le \widetilde{J}(\tau) + d_{C,n}(\lambda_n,\tau) + d_{C,t}(\lambda_t,\tau) - j(\lambda_t) \le \widetilde{J}(\tau) \qquad \forall\ \tau \in \widetilde{K},$$

which means $\sigma \in \widetilde{K}$ is the solution of the dual minimization problem (3.13). Additionally the right inequality in (3.33) states that the tensor $\sigma \in \mathbf{X}$ minimizes the functional $\widetilde{L}(\cdot;\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n)$ in $\mathbf{X}$. The linear forms that only act on Lagrange multipliers can be regarded as constants within this minimization problem and so we have

$$\begin{aligned} &\frac{1}{2}\tilde{a}(\sigma,\sigma) - \langle g,\sigma\rangle_{\Gamma_C} + b(\mathbf{u},\sigma) + s(\eta,\sigma) + d_N(\boldsymbol{\varphi},\sigma) + d_{C,n}(\lambda_n,\sigma) + d_{C,t}(\lambda_t,\sigma) \\ \le\ &\frac{1}{2}\tilde{a}(\tau,\tau) - \langle g,\tau\rangle_{\Gamma_C} + b(\mathbf{u},\tau) + s(\eta,\tau) + d_N(\boldsymbol{\varphi},\tau) + d_{C,n}(\lambda_n,\tau) + d_{C,t}(\lambda_t,\tau) \qquad \forall\ \tau \in \mathbf{X}. \end{aligned}$$

Now $\mathbf{X}$ as a Hilbert space is evidently closed and convex and the above functional is Gâteaux differentiable with respect to $\tau$. Therefore using the theory of variational inequalities (see e.g. Kinderlehrer and Stampacchia [64]) the above minimization problem is equivalent to the following variational inequality problem of finding $\sigma \in \mathbf{X}$ such that

$$\begin{aligned} &\tilde{a}(\sigma, \tau - \sigma) - \langle g, \tau - \sigma\rangle_{\Gamma_C} + b(\mathbf{u}, \tau - \sigma) + s(\eta, \tau - \sigma) \\ &\quad + d_N(\boldsymbol{\varphi}, \tau - \sigma) + d_{C,n}(\lambda_n, \tau - \sigma) + d_{C,t}(\lambda_t, \tau - \sigma) \end{aligned} \geq 0 \qquad \forall\, \tau \in \mathbf{X}. \tag{3.35}$$

Choosing $\tau = \pm\phi + \sigma$ with $\phi \in [C_0^\infty]^{2\times 2} \cap \mathbf{X}_s$ the bilinear form $s(\cdot, \cdot)$ and the terms on the boundaries vanish and we have

$$\tilde{a}(\sigma, \phi) + b(\mathbf{u}, \phi) = \int_\Omega \sigma : \mathbb{C}^{-1} : \phi\, dx + \int_\Omega \mathbf{u} \cdot \operatorname{div} \phi\, dx = 0 \qquad \forall\, \phi \in [C_0^\infty]^{2\times 2} \cap \mathbf{X}_s.$$

Integrating by parts and using the symmetry of $\phi$ in the right integral leads to

$$\begin{aligned} \int_\Omega \mathbf{u} \cdot \operatorname{div} \phi\, dx &= -\int_\Omega \nabla\mathbf{u} : \phi\, dx = -\int_\Omega \nabla\mathbf{u} : \frac{1}{2}(\phi + \phi^T)\, dx \\ &= -\int_\Omega \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) : \phi\, dx = -\int_\Omega \varepsilon(\mathbf{u}) : \phi\, dx \end{aligned}$$

and so the above equation reads

$$\int_\Omega (\sigma : \mathbb{C}^{-1} - \varepsilon(\mathbf{u})) : \phi\, dx = 0 \qquad \forall\, \phi \in [C_0^\infty]^{2\times 2} \cap \mathbf{X}_s,$$

which means $\sigma = \mathbb{C} : \varepsilon(\mathbf{u})$ in $\Omega$. If we do not require $\phi$ to be symmetric in the above choice of $\tau$, then again integrating by parts and using $\sigma : \mathbb{C}^{-1} = \varepsilon(\mathbf{u})$ the variational inequality reduces to

$$\begin{aligned} \tilde{a}(\sigma, \phi) + b(\mathbf{u}, \phi) + s(\eta, \phi) &= \int_\Omega \varepsilon(\mathbf{u}) : \phi\, dx - \int_\Omega \nabla\mathbf{u} : \phi\, dx + \int_\Omega \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} : \phi\, dx \\ &= \int_\Omega \left[ \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} - \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^T) \right] : \phi\, dx = 0 \qquad \forall\, \phi \in [C_0^\infty]^{2\times 2} \end{aligned}$$

and therefore $\eta = \frac{1}{2}((\nabla\mathbf{u})_{12} - (\nabla\mathbf{u})_{21})$ in $\Omega$. Next we observe by choosing $\tau \in \mathbf{X}_s$ and using $\sigma : \mathbb{C}^{-1} = \varepsilon(\mathbf{u})$ we have

$$\tilde{a}(\sigma, \tau - \sigma) + b(\mathbf{u}, \tau - \sigma) = \int_\Omega \varepsilon(\mathbf{u}) : (\tau - \sigma)\, dx + \int_\Omega \mathbf{u} \operatorname{div}(\tau - \sigma)\, dx = \int_\Gamma \mathbf{u} \cdot (\tau - \sigma) \cdot \mathbf{n}\, ds.$$

If we take $\tau \in \mathbf{X}_s$ with $\tau \cdot \mathbf{n} = \sigma \cdot \mathbf{n}$ on $\Gamma_N \cup \Gamma_C$ and $\tau \cdot \mathbf{n} = \pm\boldsymbol{\psi} + \sigma \cdot \mathbf{n}$ on $\Gamma_D$ for some $\boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_D)$ we get

$$\int_{\Gamma_D} \mathbf{u} \cdot \boldsymbol{\psi}\, ds = 0 \qquad \forall\, \boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_D)$$

which leads to $\mathbf{u} = 0$ on $\Gamma_D$. In the same way choosing $\tau \in \mathbf{X}_s$ with $\tau \cdot \mathbf{n} = \sigma \cdot \mathbf{n}$ on $\Gamma_C$ and $\tau \cdot \mathbf{n} = \pm\boldsymbol{\psi} + \sigma \cdot \mathbf{n}$ on $\Gamma_N$ for some $\boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$ we get

$$\int_{\Gamma_N} (\mathbf{u} + \boldsymbol{\varphi}) \cdot \boldsymbol{\psi}\, ds = 0 \qquad \forall\ \boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$$

and so $\mathbf{u} = -\boldsymbol{\varphi}$ on $\Gamma_N$, since $\mathbf{u}|_{\Gamma_N} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)$. Next choosing $\tau \in \mathbf{X}_s$ with $\tau \cdot \mathbf{n} = \sigma \cdot \mathbf{n}$ on $\Gamma_N$, $\tau \cdot \mathbf{n} = \pm\boldsymbol{\psi} + \sigma \cdot \mathbf{n}$ on $\Gamma_C$ for some $\boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_C)$ where $\boldsymbol{\psi}_t = 0$ and $\boldsymbol{\psi}_n = \tilde{\boldsymbol{\psi}} \in H^{-\frac{1}{2}}(\Gamma_C)$ we get

$$\int_{\Gamma_C} (u_n + \lambda_n - g)\tilde{\boldsymbol{\psi}}\, ds = 0 \qquad \forall\ \tilde{\boldsymbol{\psi}} \in H^{-\frac{1}{2}}(\Gamma_C),$$

from which we deduce $u_n = g - \lambda_n$ on $\Gamma_C$. Finally, we choose $\tau \in \mathbf{X}_s$ with $\tau \cdot \mathbf{n} = \sigma \cdot \mathbf{n}$ on $\Gamma_N$, $\tau \cdot \mathbf{n} = \pm\boldsymbol{\psi} + \sigma \cdot \mathbf{n}$ on $\Gamma_C$ for some $\boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_C)$ with $\boldsymbol{\psi}_n = 0$ and $\boldsymbol{\psi}_t = \tilde{\boldsymbol{\psi}} \in H^{-\frac{1}{2}}(\Gamma_C)$ to get

$$\int_{\Gamma_C} (u_t + \lambda_t)\tilde{\boldsymbol{\psi}}\, ds = 0 \qquad \forall\ \tilde{\boldsymbol{\psi}} \in H^{-\frac{1}{2}}(\Gamma_C),$$

which states $u_t = -\lambda_t$ on $\Gamma_C$ and concludes the first assertion.

(ii) To prove the second assertion we let $\sigma \in \widetilde{\mathbf{K}}$ be the solution of the dual minimization problem (3.13) and $\mathbf{u} \in \mathbf{K}_g$ be the solution of the primal minimization problem (3.6). Then due to Theorem 3.3 we have $\sigma = \mathbb{C} : \varepsilon(\mathbf{u})$ in $\Omega$. Since $\sigma \in \widetilde{\mathbf{K}}$ we have

$$\begin{aligned} f(\mathbf{v}) + b(\mathbf{v}, \sigma) &= 0 && \forall\ \mathbf{v} \in \mathbf{L}^2(\Omega), \\ d_N(\boldsymbol{\psi}, \sigma) - t(\boldsymbol{\psi}) &= 0 && \forall\ \boldsymbol{\psi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N), \\ s(\xi, \sigma) &= 0 && \forall\ \xi \in L^2(\Omega). \end{aligned}$$

As seen in the proof of Lemma 3.2 we have that the minimization problem (3.6) is equivalent to the variational inequality problem of finding $\mathbf{u} \in \mathbf{K}_g$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - f(\mathbf{v} - \mathbf{u}) - t(\mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \geq 0 \qquad \forall\ \mathbf{v} \in \mathbf{K}_g.$$

Letting $\mathbf{v} = \mathbf{u} + \tilde{\mathbf{v}}$ with $\tilde{\mathbf{v}} \in \mathbf{K}_0$, then $\mathbf{v} \in \mathbf{K}_g$ and we have

$$\int_\Omega \sigma : \varepsilon(\tilde{\mathbf{v}})\, dx - f(\tilde{\mathbf{v}}) - t_0(\tilde{\mathbf{v}}) + j(\mathbf{u} + \tilde{\mathbf{v}}) - j(\mathbf{u}) \geq 0 \qquad \forall\ \tilde{\mathbf{v}} \in \mathbf{K}_0.$$

Using $\sigma \in \widetilde{\mathbf{K}}$ and Green's formula the above inequality reduces to

$$\int_{\Gamma_C} \tilde{v}_n \sigma_n\, ds + \int_{\Gamma_C} \tilde{v}_t \sigma_t\, ds + \int_{\Gamma_C} \mathcal{F}|u_t + \tilde{v}_t|\, ds - \int_{\Gamma_C} \mathcal{F}|u_t|\, ds \geq 0 \qquad \forall\ \tilde{\mathbf{v}} \in \mathbf{K}_0.$$

Now choosing $\tilde{\mathbf{v}} \in \mathbf{K}_0$ with $\tilde{v}_n = 0$ and $\tilde{v}_t = \pm u_t$ on $\Gamma_C$ we have using $\lambda_t = -u_t$

$$d_{C,t}(\lambda_t, \sigma) - j(\lambda_t) = -(d_{C,t}(u_t, \sigma) + j(\mathbf{u})) = -\int_{\Gamma_C} \tilde{u}_t \sigma_t \, ds - \int_{\Gamma_C} \mathcal{F}|u_t| \, ds = 0.$$

Analogously to the above argumentation we can write the dual minimization problem (3.13) as a variational inequality problem of finding $\sigma \in \widetilde{\mathbf{K}}$ such that

$$\tilde{a}(\sigma, \tau - \sigma) \geq \langle g, \tau_n - \sigma_n \rangle_{\Gamma_C} \qquad \forall\, \tau \in \widetilde{\mathbf{K}}.$$

If we now define

$$\begin{aligned}\mathcal{K}_n := \Big\{ \mathbf{v} \in \mathbf{V}_D :\ & \operatorname{div} \varepsilon(\mathbf{v}) = 0 \text{ in } \Omega,\ \varepsilon(\mathbf{v}) \cdot \mathbf{n} = 0 \text{ on } \Gamma_N, \\ & \varepsilon(\mathbf{v})_t = 0 \text{ and } \varepsilon(\mathbf{v})_n \leq -\sigma_n \text{ on } \Gamma_C \Big\},\end{aligned}$$

we have $\tau = \sigma + \varepsilon(\mathbf{v}) \in \widetilde{\mathbf{K}}$ for all $\mathbf{v} \in \mathcal{K}_n$. Integration by parts in the above inequality and using $\sigma : \mathbb{C}^{-1} = \varepsilon(\mathbf{u})$ we derive for all $\mathbf{v} \in \mathcal{K}_n$

$$\begin{aligned}\tilde{a}(\sigma, \tau - \sigma) = \tilde{a}(\sigma, \varepsilon(\mathbf{v})) &= \int_{\Omega} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx \\ &= -\int_{\Omega} \mathbf{u} \cdot \operatorname{div} \varepsilon(\mathbf{v}) \, dx + \int_{\Gamma_N \cup \Gamma_C} \mathbf{u} \cdot \varepsilon(\mathbf{v}) \cdot \mathbf{n} \, ds = \int_{\Gamma_C} u_n \, \varepsilon(\mathbf{v})_n \, ds \geq \langle g, \varepsilon(\mathbf{v})_n \rangle_{\Gamma_C}.\end{aligned}$$

Noting that $\sigma_n \leq 0 \leq -\sigma_n$ we can choose $\mathbf{v} \in \mathcal{K}_n$ with $\varepsilon(\mathbf{v})_n = \pm\sigma_n$ on $\Gamma_C$ in the above inequality and using $\lambda_n = g - u_n$ we arrive at

$$d_{C,n}(\lambda_n, \sigma_n) = -\int_{\Gamma_C} (u_n - g)\sigma_n \, ds = 0.$$

Since $\sigma \in \widetilde{\mathbf{K}}$ we observe for $\mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ and $\mu_n \in \widetilde{H}^{\frac{1}{2}}_{+}(\Gamma_C)$ with $\mathcal{F} \geq 0$

$$\left.\begin{array}{c} -\mathcal{F} \leq \sigma_t \leq \mathcal{F} \\ -|\mu_t| \leq \mu_t \leq |\mu_t| \end{array}\right\} \Longrightarrow \sigma_t \mu_t \leq \mathcal{F}|\mu_t| \quad \text{and} \quad \sigma_n \mu_n \leq 0$$

and therefore we can state the left inequality of the saddle point problem 3.33

$$\begin{aligned}\widetilde{L}(\sigma; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) &= \frac{1}{2}\tilde{a}(\sigma, \sigma) - g(\sigma) + b(\mathbf{v}, \sigma) + f(\mathbf{v}) + s(\xi, \sigma) + d_N(\boldsymbol{\psi}, \sigma) \\ &\quad + d_{C,n}(\mu_n, \sigma) + d_{C,t}(\mu_t, \sigma) - j(\mu_t) \\ &= \tilde{J}(\sigma) + \int_{\Gamma_C} (\sigma_t \mu_t - \mathcal{F}|\mu_t|) \, ds + \int_{\Gamma_C} \sigma_n \mu_n \, ds \leq \tilde{J}(\sigma) = \widetilde{L}(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n).\end{aligned}$$

To prove the right inequality we first observe for $\tau \in \mathbf{X}$

$$
\begin{aligned}
&\widetilde{L}(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) - \widetilde{L}(\tau;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n)\\
&\qquad= -\frac{1}{2}\tilde{a}(\sigma-\tau,\sigma-\tau) + \tilde{a}(\sigma,\sigma-\tau) - g(\sigma-\tau) + b(\mathbf{u},\sigma-\tau)\\
&\qquad\quad + s(\eta,\sigma-\tau) + d_N(\boldsymbol{\varphi},\sigma-\tau) + d_{C,n}(\lambda_n,\sigma-\tau) + d_{C,t}(\lambda_t,\sigma-\tau).
\end{aligned}
$$

Using $\eta = \frac{1}{2}((\nabla\mathbf{u})_{12} - (\nabla\mathbf{u})_{21})$ we have $s(\eta,\tau) = \int_\Omega (\nabla\mathbf{u} - \varepsilon(\mathbf{u})) : \tau\, dx$. Then integration by parts in the bilinear form $b(\cdot,\cdot)$ leads for all $\tau \in \mathbf{X}$ to

$$
\begin{aligned}
&\widetilde{L}(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_n,\lambda_t) - \widetilde{L}(\tau;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n)\\
&\quad= -\frac{1}{2}\tilde{a}(\sigma-\tau,\sigma-\tau) + \tilde{a}(\sigma,\sigma-\tau) - \int_\Omega \nabla\mathbf{u} : (\sigma-\tau)\,dx + \int_\Omega (\nabla\mathbf{u} - \varepsilon(\mathbf{u})) : (\sigma-\tau)\,dx\\
&\qquad + \int_{\Gamma_N} (\mathbf{u}+\boldsymbol{\varphi})\cdot(\sigma-\tau)\cdot\mathbf{n}\,ds + \int_{\Gamma_C} (u_n - g + \lambda_n)(\sigma_n - \tau_n)\,ds + \int_{\Gamma_C} (u_t + \lambda_t)(\sigma_t - \tau_t)\,ds.
\end{aligned}
$$

Finally, using the definitions for $\boldsymbol{\varphi}$, $\lambda_n$ and $\lambda_t$ the boundary integrals vanish and with $\varepsilon(\mathbf{u}) = \mathbb{C}^{-1} : \sigma$ we can show the right inequality of the saddle point problem 3.33

$$
\widetilde{L}(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_n,\lambda_t) - \widetilde{L}(\tau;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) = -\frac{1}{2}\tilde{a}(\sigma-\tau,\sigma-\tau) \leq 0 \qquad \forall\, \tau \in \mathbf{X},
$$

where the inequality is due to the ellipticity of the bilinear form $\tilde{a}(\cdot,\cdot)$. □

### 3.1.3. Variational inequalities

We want to derive a variational formulation which is equivalent the saddle point problem (3.33). Thus we have to check the assumptions of Proposition 2.9. With the restriction of $\mu_n \in \widetilde{H}^{\frac{1}{2}}_+(\Gamma_C)$ we have $\widetilde{L} : \mathbf{X} \times \widetilde{\mathbf{Y}}'_+ \to \mathbb{R}$. Of course $\mathbf{X}$ as a Hilbert space is convex, closed and not empty. Furthermore, if we take $\mu_n = -u_n|_{\Gamma_C} \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$, where $\mathbf{u}$ is the solution of the primal minimization problem (3.6), then $\widetilde{\mathbf{Y}}'_+$ is not empty. Since $H^{\frac{1}{2}}(\Gamma_C)$ is a real valued vector space and $h_s : H^{\frac{1}{2}}(\Gamma_C) \to \mathbb{R},\ \mu \mapsto (\mu + s)$ a linear form on $H^{\frac{1}{2}}(\Gamma_C)$ for $s \in H^{\frac{1}{2}}(\Gamma_C)$ the equation $h_s(\mu) = 0$ defines an affine hyperplane $\mathcal{H}$. Therefore the set $H_s := \{\mu \in H^{\frac{1}{2}}(\Gamma_C) | h_s(\mu) \geq 0\}$ defines a closed half-space bounded by $\mathcal{H}$. With $\widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ being closed we conclude that $\widetilde{H}^{\frac{1}{2}}_+(\Gamma_C) = H_0 \cap \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ is closed as well, see Ito and Kunisch [61].

**Remark 3.11:** *If we do not lift the Lagrange multiplier $\lambda_n$ by the gap function $g$ as described in Remark 3.8 we would deal with the convex set $K_{-g} := \{\mu_n \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C) : \mu_n \geq -g$ a.e. on $\Gamma_C\}$. This set is closed as well since $g \in H^{\frac{1}{2}}(\Gamma_C)$ and so we have $K_{-g} = H_g \cap \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$, i.e. $K_{-g}$ is closed.*

The convexity follows straightforward. Let us define

$$
\begin{aligned}
l : \mathbf{X} \times \widetilde{\mathbf{Y}}'_+ \to \mathbb{R}, \quad (\tau; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \ \mapsto\ & \frac{1}{2}\tilde{a}(\tau,\tau) - g(\tau) + b(\tau, \mathbf{v}) + f(\mathbf{v}) + s(\tau, \xi) \\
& + d_N(\tau, \boldsymbol{\psi}) - t_0(\boldsymbol{\psi}) + d_{C,t}(\tau, \mu_t) + d_{C,n}(\tau, \mu_n), \\
m : \mathbf{X} \times \widetilde{\mathbf{Y}}'_+ \to \mathbb{R}, \quad (\tau; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \ \mapsto\ & -j(\mu_t),
\end{aligned}
$$

then we can decompose $\widetilde{L} = l + m$. The bilinear and linear forms in $l$ are all Gâteaux-differentiable. Furthermore, all linear functionals are both, convex and concave. The bilinear form $\tilde{a}(\cdot,\cdot)$ is convex we have that $l$ is convex in $\mathbf{X}$ for all $(\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \in \widetilde{\mathbf{Y}}'_+$ and concave in $\widetilde{\mathbf{Y}}'_+$ for all $\tau \in \mathbf{X}$. Since $m$ does not depend on $\tau \in \mathbf{X}$, it is convex in $\mathbf{X}$ for all $(\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \in \widetilde{\mathbf{Y}}'_+$. Finally, the $j$-functional itsself is convex in $\widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ and therefore $m$ is concave in $\widetilde{\mathbf{Y}}'_+$ for all $\tau \in \mathbf{X}$.

Now all assumptions of Proposition 2.9 are satisfied and for $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_+$ being the saddle point of $\widetilde{L}$, we have to compute the following Gâteaux derivatives for $\tau \in \mathbf{X}$ and for $(\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \in \widetilde{\mathbf{Y}}'_+$

$$
\begin{aligned}
& \langle \frac{\partial l}{\partial \tau}(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n)\ ,\ \tau - \sigma \rangle, \\
& \langle \frac{\partial l}{\partial(\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n)}(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n)\ ,\ (\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) - (\mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \rangle.
\end{aligned}
$$

We do this exemplary for $\tau \in \mathbf{X}$. It holds

$$
\begin{aligned}
\langle \frac{\partial l}{\partial \tau}(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n)\ ,\ \tau - \sigma \rangle &= \frac{\partial}{\partial x}\left( l(\sigma + x(\tau - \sigma); \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \right)|_{x=0} \\
&= \frac{\partial}{\partial x}\Big[ \frac{1}{2}\tilde{a}(\sigma + x(\tau-\sigma), \sigma + x(\tau - \sigma)) - g(\sigma + x(\tau - \sigma)) + b(\sigma + x(\tau-\sigma), \mathbf{u}) \\
&\quad + f(\mathbf{u}) + s(\sigma + x(\tau - \sigma), \eta) + d_N(\sigma + x(\tau - \sigma), \boldsymbol{\varphi}) - t_0(\boldsymbol{\varphi}) \\
&\quad + d_{C,t}(\sigma + x(\tau - \sigma), \lambda_t) + d_{C,n}(\sigma + x(\tau - \sigma), \lambda_n) \Big]|_{x=0} \\
&= \tilde{a}(\tau - \sigma, \sigma) - g(\tau - \sigma) + b(\tau - \sigma, \mathbf{u}) + s(\tau - \sigma, \eta) \\
&\quad + d_N(\tau - \sigma, \boldsymbol{\varphi}) + d_{C,t}(\tau - \sigma, \lambda_t) + d_{C,n}(\tau - \sigma, \lambda_n).
\end{aligned}
$$

After computing the other derivatives and applying Proposition 2.9 we end up with the following variational inequality problem:

Find $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_+$ such that

$$
\begin{aligned}
&\tilde{a}(\tau-\sigma,\sigma) + b(\tau-\sigma,\mathbf{u}) + s(\tau-\sigma,\eta) + d_N(\tau-\sigma,\boldsymbol{\varphi}) \\
&\qquad\qquad + d_{C,t}(\tau-\sigma,\lambda_t) + d_{C,n}(\tau-\sigma,\lambda_n) \;\geq\; g(\tau-\sigma) && \forall\, \tau \in \mathbf{X} \\
&b(\mathbf{v}-\mathbf{u},\sigma) \;\leq\; -f(\mathbf{v}-\mathbf{u}) && \forall\, \mathbf{v} \in \mathbf{L}^2(\Omega) \\
&s(\xi-\eta,\sigma) \;\leq\; 0 && \forall\, \xi \in L^2(\Omega) \\
&d_N(\boldsymbol{\psi}-\boldsymbol{\varphi},\sigma) \;\leq\; t_0(\boldsymbol{\psi}-\boldsymbol{\varphi}) && \forall\, \boldsymbol{\psi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N) \\
&d_{C,t}(\mu_t-\lambda_t,\sigma) - j(\mu_t) + j(\lambda_t) \;\leq\; 0 && \forall\, \mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C) \\
&d_{C,n}(\mu_n-\lambda_n,\sigma) \;\leq\; 0 && \forall\, \mu_n \in \widetilde{H}^{\frac{1}{2}}_+(\Gamma_C).
\end{aligned}
$$

The first four inqualities that are valid on a whole Hilbert space can be reduced to equalities, e.g.

Taking $\tau = 0$ and $\tau = 2\sigma$ in the first inequality we have

$$
\begin{aligned}
-\{\tilde{a}(\sigma,\sigma) + b(\sigma,\mathbf{u}) + s(\sigma,\eta) + d_N(\sigma,\boldsymbol{\varphi}) + d_{C,t}(\sigma,\lambda_t) + d_{C,n}(\sigma,\lambda_n)\} \;&\geq\; -g(\sigma), \\
\tilde{a}(\sigma,\sigma) + b(\sigma,\mathbf{u}) + s(\sigma,\eta) + d_N(\sigma,\boldsymbol{\varphi}) + d_{C,t}(\sigma,\lambda_t) + d_{C,n}(\sigma,\lambda_n) \;&\geq\; g(\sigma),
\end{aligned}
\tag{3.36}
$$

which reads

$$
\tilde{a}(\sigma,\sigma) + b(\sigma,\mathbf{u}) + s(\sigma,\eta) + d_N(\sigma,\boldsymbol{\varphi}) + d_{C,t}(\sigma,\lambda_t) + d_{C,n}(\sigma,\lambda_n) = g(\sigma).
$$

Then with $\pm\,\tau \in \mathbf{X}$ we arrive at

$$
\tilde{a}(\tau,\sigma) + b(\tau,\mathbf{u}) + s(\tau,\eta) + d_N(\tau,\boldsymbol{\varphi}) + d_{C,t}(\tau,\lambda_t) + d_{C,n}(\tau,\lambda_n) = g(\tau) \quad \forall\, \tau \in \mathbf{X}.
$$

The same procedure applies for the other three inequalities mentioned above and so we finally arrive at the dual variational inequality problem:

Find $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_+$ such that

$$
\begin{aligned}
&\tilde{a}(\tau,\sigma) + \hat{B}(\tau;\mathbf{u},\eta,\boldsymbol{\varphi}) + d_{C,t}(\tau,\lambda_t) + d_{C,n}(\tau,\lambda_n) \;=\; g(\tau) && \forall\, \tau \in \mathbf{X} \\
&b(\mathbf{v},\sigma) \;=\; -f(\mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{L}^2(\Omega) \\
&s(\xi,\sigma) \;=\; 0 && \forall\, \xi \in L^2(\Omega) \\
&d_N(\boldsymbol{\psi},\sigma) \;=\; t_0(\boldsymbol{\psi}) && \forall\, \boldsymbol{\psi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N) \\
&d_{C,t}(\mu_t-\lambda_t,\sigma) - j(\mu_t) + j(\lambda_t) \;\leq\; 0 && \forall\, \mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C) \\
&d_{C,n}(\mu_n-\lambda_n,\sigma) \;\leq\; 0 && \forall\, \mu_n \in \widetilde{H}^{\frac{1}{2}}_+(\Gamma_C)
\end{aligned}
\tag{3.37}
$$

where we have used the bilinear form $\hat{B}(\tau;\mathbf{u},\eta,\boldsymbol{\varphi}) := b(\tau,\mathbf{u}) + s(\tau,\eta) + d_N(\tau,\boldsymbol{\varphi})$.

**Remark 3.12:** *Note that the computation in* (3.36) *is valid for all Lagrange multipliers as well, since setting* $\mu_t = 0$ *and* $\mu_t = 2\lambda_t$ *in* $(3.37)_5$ *and* $\mu_n = 0 \in \widetilde{H}_+^{\frac{1}{2}}(\Gamma_C)$ *and* $\mu_n = 2\lambda_n \in \widetilde{H}_+^{\frac{1}{2}}(\Gamma_C)$ *in* $(3.37)_6$ *we get for the solution* $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n)$ *of* (3.37)

$$\begin{aligned}
\tilde{a}(\sigma,\sigma) + b(\sigma,\mathbf{u}) + s(\sigma,\eta) + d_N(\sigma,\boldsymbol{\varphi}) + d_{C,t}(\sigma,\lambda_t) + d_{C,n}(\sigma,\lambda_n) &= g(\sigma) \\
b(\mathbf{u},\sigma) &= -f(\mathbf{u}) \\
s(\eta,\sigma) &= 0 \\
d_N(\boldsymbol{\varphi},\sigma) &= t_0(\boldsymbol{\varphi}) \\
d_{C,t}(\lambda_t,\sigma) - j(\lambda_t) &= 0 \\
d_{C,n}(\lambda_n,\sigma) &= 0.
\end{aligned} \tag{3.38}$$

The last two equations in (3.38) state in a weak sense the equality constraints on the contact boundary $\Gamma_C$ of the strong form (3.1).

**Remark 3.13:** *A similar procedure applies in the case where no friction occurs, i.e.* $\mathcal{F} \equiv 0$. *Following remark 3.9 and the computations above we arrive at the variational inequality problem for contact without friction:*

*Find* $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_+$ *such that*

$$\begin{aligned}
\tilde{a}(\tau,\sigma) + \hat{B}(\tau; \mathbf{u}, \eta, \boldsymbol{\varphi}) + d_{C,t}(\tau,\lambda_t) + d_{C,n}(\tau,\lambda_n) &= g(\tau) && \forall\, \tau \in \mathbf{X} \\
b(\mathbf{v},\sigma) &= -f(\mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{L}^2(\Omega) \\
s(\xi,\sigma) &= 0 && \forall\, \xi \in L^2(\Omega) \\
d_N(\boldsymbol{\psi},\sigma) &= t(\boldsymbol{\psi}) && \forall\, \boldsymbol{\psi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N) \\
d_{C,t}(\mu_t,\sigma) &= 0 && \forall\, \mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C) \\
d_{C,n}(\mu_n - \lambda_n,\sigma) &\le 0 && \forall\, \mu_n \in \widetilde{H}_+^{\frac{1}{2}}(\Gamma_C)
\end{aligned} \tag{3.39}$$

### 3.1.4. Dual-dual formulations

Since the $j$-functional in the saddle point formulation (3.33) is non-differentiable we introduce another Lagrange multiplier in order to approximate the sign of $\lambda_t$ on the contact boundary $\Gamma_C$. As the support of the integral in $j(\cdot)$ is restricted to the support of the friction function $\mathcal{F}$ we also restrict the new Lagrange multiplier to the support of $\mathcal{F}$. Let us define

$$A_C := \operatorname{supp} \mathcal{F}.$$

Then we set $\nu\lambda_t = |\lambda_t|$ a.e. on $A_C$, with

$$\nu \in \Lambda := \{\kappa \in H^{-\frac{1}{2}}(A_C) : |\kappa| \le 1, \text{ a.e. on } A_C\}$$

and define the bilinear form

$$q(\kappa, \mu) := \int_{\Gamma_C} \mathcal{F}\kappa\mu\, ds \quad \text{for } \kappa \in \Lambda,\ \mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C).$$

**Remark 3.14:** *The introduction of the additional Lagrange multiplier is done in a similar way as in the primal-dual formulation, see e.g. Glowinski [52, Chapter II, Section 5.3]. The difference is, that we choose $H^{-\frac{1}{2}}(A_C)$ instead of $L^2(\Gamma_C)$ in the definition of $\Lambda$. The reason will be seen in the proof of the inf-sup condition concerning $q(\cdot,\cdot)$ in Section 3.1.6.*

Let us consider the following saddle point formulation:

Find $(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \Lambda \times \widetilde{\mathbf{Y}}'_+$ such that

$$\begin{aligned} \hat{L}(\sigma, \nu; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \le \hat{L}(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \le \hat{L}(\tau, \kappa; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \\ \forall\ (\tau, \kappa; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \in \mathbf{X} \times \Lambda \times \widetilde{\mathbf{Y}}'_+, \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} \hat{L}(\tau, \kappa; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) := \frac{1}{2}\tilde{a}(\tau,\tau) - g(\tau) + b(\tau, \mathbf{v}) + f(\mathbf{v}) + s(\tau, \xi) \\ + d_N(\tau, \boldsymbol{\psi}) - t(\boldsymbol{\psi}) + d_{C,t}(\tau, \mu_t) - q(\kappa, \mu_t) + d_{C,n}(\tau, \mu_n). \end{aligned}$$

The bilinear form $q(\cdot,\cdot)$ is continuous and linear in both variables and therefore we can apply Proposition 2.9 with $l = \hat{L}$ and $m \equiv 0$. Analogously to the previous subsection we have the saddle point formulation (3.40) being equivalent to the following dual-dual variational inequality problem:

Find $(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \Lambda \times \widetilde{\mathbf{Y}}'_+$ such that

$$\begin{aligned} &\tilde{a}(\tau, \sigma) + \hat{B}(\tau; \mathbf{u}, \eta, \boldsymbol{\varphi}) + d_{C,t}(\tau, \lambda_t) + d_{C,n}(\tau, \lambda_n) && = g(\tau) && \forall\ \tau \in \mathbf{X} \\ &b(\mathbf{v}, \sigma) && = -f(\mathbf{v}) && \forall\ \mathbf{v} \in \mathbf{L}^2(\Omega) \\ &s(\xi, \sigma) && = \ 0 && \forall\ \xi \in L^2(\Omega) \\ &d_N(\boldsymbol{\psi}, \sigma) && = t_0(\boldsymbol{\psi}) && \forall\ \boldsymbol{\psi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N) \\ &d_{C,t}(\mu_t, \sigma) - q(\mu_t, \nu) && = \ 0 && \forall\ \mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C) \\ &d_{C,n}(\mu_n - \lambda_n, \sigma) && \le \ 0 && \forall\ \mu_n \in \widetilde{H}^{\frac{1}{2}}_+(\Gamma_C) \\ &q(\kappa - \nu, \lambda_t) && \le \ 0 && \forall\ \kappa \in \Lambda \end{aligned} \tag{3.41}$$

**Remark 3.15:** *There arise some questions from the introduction of the new Lagrange multiplier $\nu$. First we note that from* $(3.41)_7$ *we have*

$$\nu\lambda_t = |\lambda_t| \quad a.e. \ \ on\ A_C, \tag{3.42}$$

*since taking* $\kappa = \operatorname{sign}\lambda_t \in \Lambda$ *we get*

$$\int_{\Gamma_C} \mathcal{F}(|\lambda_t| - \nu\lambda_t)\, ds \leq 0$$

*in* $(3.41)_7$. *But as* $\nu \in \Lambda$ *we have* $|\lambda_t| - \nu\lambda_t \geq 0$ *and thus*

$$\int_{\Gamma_C} \mathcal{F}(|\lambda_t| - \nu\lambda_t)\, ds \geq 0 \qquad \Rightarrow \qquad \int_{\Gamma_C} \mathcal{F}(|\lambda_t| - \nu\lambda_t)\, ds = 0.$$

*Since the friction function* $\mathcal{F}$ *is positive on the contact boundary* $\Gamma_C$ *we conclude* (3.42). *If* $\mathcal{F}$ *is zero, then the inequality* $(3.41)_7$ *is of course valid, but furthermore we deduce from* $(3.41)_5$, *that* $\sigma_t = 0$ *as well, since the equation reduces to the one in the frictionless case, see Remark* (3.13).

*If* $\lambda_t = 0$ *on some part* $\Gamma_C^{st} \subset A_C$, *we are in the situation where the body is sticking on the rigid foundation. Then from* (3.42) *we have that* $\nu \in \Lambda$ *can be chosen arbitrarily. But in this case, we have from* $(3.41)_5$ *by taking* $\mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ *with* $\operatorname{supp}(\mu_t) \subset \Gamma_C^{st}$

$$0 = d_{C,t}(\mu_t, \sigma) - q(\mu_t, \nu) = \int_{\Gamma_C^{st}} \mu_t(\sigma_t - \mathcal{F}\nu)\, ds \quad \Rightarrow \quad \nu = \frac{\sigma_t}{\mathcal{F}} \ \ on\ \Gamma_C^{st}. \tag{3.43}$$

*Finally, if* $\lambda_t \neq 0$ *and* $\mathcal{F} \neq 0$, *then we conclude from* (3.42) *and* (3.43), *which is valid here as well*

$$\nu = \operatorname{sign}\lambda_t \quad \wedge \quad \nu = \frac{\sigma_t}{\mathcal{F}} \quad \Rightarrow \quad \operatorname{sign}\sigma_t = \operatorname{sign}\lambda_t = \nu \quad \wedge \quad \nu\sigma_t = |\sigma_t| = \mathcal{F}. \tag{3.44}$$

**Corollary 3.16:**
For the normal stress $\sigma_n$ and the tangential stress $\sigma_t$ on $\Gamma_C$ we conclude

$$\sigma_n \leq 0 \qquad \text{and} \qquad |\sigma_t| \leq \mathcal{F} \qquad \text{on } \Gamma_C. \tag{3.45}$$

*Proof.* The first assertion follows from $(3.41)_6$, using $(3.38)_6$ and the fact, that $\mu_n \geq 0$ on $\Gamma_C$. The second assertion follows due to Remark 3.15. □

**Theorem 3.17:**
The variational inequality problems (3.37) and (3.41) are equivalent in the following sense. If $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_+$ is a solution of (3.37), then $(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \Lambda \times \widetilde{\mathbf{Y}}'_+$ solves (3.41) with $\nu := \operatorname{sign}\lambda_t \in \Lambda$. On the other hand if $(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \Lambda \times \widetilde{\mathbf{Y}}'_+$ is a solution of (3.41), then $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n)$ solves (3.37).

*Proof.* Let $(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \Lambda \times \widetilde{\mathbf{Y}}'_+$ be the solution of (3.41). To prove that $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_+$ is a solution of (3.37), we only have to show the inequality in $(3.37)_5$ concerning the tangential displacement on the contact boundary. With $\nu \in \Lambda$ we have

$$-q(\nu, \mu_t) = -\int_{\Gamma_C} \mathcal{F} \nu \mu_t \, ds \geq -\int_{\Gamma_C} \mathcal{F} |\mu_t| \, ds = -j(\mu_t) \quad \forall \; \mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$$

and from $(3.41)_5$, (3.42) and (3.38) which is valid here as well we get

$$\begin{aligned} 0 = d_{C,t}(\mu_t, \sigma) - q(\mu_t, \nu) &= d_{C,t}(\mu_t - \lambda_t, \sigma) - q(\mu_t, \nu) + q(\lambda_t, \nu) \\ &\geq d_{C,t}(\mu_t - \lambda_t, \sigma) - j(\mu_t) + j(\lambda_t) \qquad \forall \; \mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C). \end{aligned}$$

But this is just the first inequality in (3.37).

On the other hand let $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_+$ be the solution of (3.37). For $\sigma \in \mathbf{X}$ fixed, we have $d_\sigma(\mu) := d_{C,t}(\mu, \sigma)$ is a continuous linear functional on $\widetilde{H}^{\frac{1}{2}}(\Gamma_C)$. With (3.38) and $\mu_t = \pm\, \mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ in $(3.37)_5$ we get

$$d_{C,t}(\mu, \sigma) - j(\mu) \leq 0 \wedge -d_{C,t}(\mu, \sigma) - j(\mu) \leq 0 \quad \Rightarrow \quad |d_{C,t}(\mu, \sigma)| \leq j(\mu) \quad \forall \; \mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C).$$

As $\mathcal{F} \in L^\infty(\Gamma_C) \subset L^1(\Gamma_C)$ we can define the mapping

$$\pi : \widetilde{H}^{\frac{1}{2}}(\Gamma_C) \to L^1(\Gamma_C), \quad \mu \mapsto \mathcal{F} \mu.$$

Taking into account the positivity of $\mathcal{F}$ we have

$$|d_{C,t}(\mu, \sigma)| \leq j(\mu) = \int_{\Gamma_C} \mathcal{F} |\mu| \, ds = \int_{\Gamma_C} |\mathcal{F} \mu| \, ds = \|\pi\mu\|_{L^1(\Gamma_C)} \quad \forall \; \mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C),$$

which is a seminorm on $L^1(\Gamma_C)$ and therefore sublinear. Since $\widetilde{H}^{\frac{1}{2}}(\Gamma_C) \subset L^1(\Gamma_C)$ the assumptions of the Hahn-Banach theorem, see e.g. Yosida [86, Chapter IV], are fulfilled and we have the existence of some linear functional $\tilde{d}_\sigma$ on $L^1(\Gamma_C)$ which is an extension of $d_\sigma$ such that

$$\tilde{d}_\sigma(\mu) \leq \|\pi\mu\|_{L^1(\Gamma_C)} \quad \forall \; \mu \in L^1(\Gamma_C).$$

For $\mu \in L^1(\Gamma_C)$ there exists a $\nu \in L^\infty(\Gamma_C)$, the dual of $L^1(\Gamma_C)$, with $\|\nu\|_{L^\infty(\Gamma_C)} \leq 1 \;\Rightarrow \nu \in \Lambda$ and

$$\|\mu\|_{L^1(\Gamma_C)} = \langle \mu, \nu \rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality product. This can be seen very easy from the definition of the $L^1$-norm as the supremum of the duality product over all dual functions $\kappa \in L^\infty(\Gamma_C)$

$$\|\mu\|_{L^1(\Gamma_C)} := \sup_{\|\kappa\|_{L^\infty(\Gamma_C)} \leq 1} \langle \mu, \kappa \rangle.$$

The supremum is assumed for $\nu = \operatorname{sign}\mu \in L^\infty(\Gamma_C)$. Since $\tilde{d}_\sigma$ is an extension of $d_\sigma$ we have

$$d_\sigma(\mu) = d_{C,t}(\mu,\sigma) \le \langle \nu, \pi\mu \rangle = \int\limits_{\Gamma_C} \mathcal{F}\nu\mu\, ds = q(\mu,\nu) \quad \forall\, \mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C).$$

If we take $\mu = \pm\mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ we finally arrive at

$$d_{C,t}(\mu_t,\sigma) = q(\mu_t,\nu) \quad \forall\, \mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C).$$

Taking $\mu_t = \lambda_t$ and regarding (3.38) we get

$$0 = d_{C,t}(\lambda_t,\sigma) - j(\lambda_t) \le q(\lambda_t,\nu) - j(\lambda_t) = \int\limits_{\Gamma_C} \mathcal{F}(\nu\lambda_t - |\lambda_t|)\, ds$$

and since $\nu \in \Lambda$ we have $\nu\,\lambda_t - |\lambda_t| \le 0$ from which we deduce

$$\nu\lambda_t = |\lambda_t| \text{ a.e. on } A_C.$$

Finally, we have $\nu\lambda_t = |\lambda_t| \ge \kappa\lambda_t$ for all $\kappa \in \Lambda$ leading to

$$q(\lambda_t, \kappa - \nu) \le 0 \quad \forall\, \kappa \in \Lambda$$

and thus the proof is complete. □

### 3.1.5. Existence and uniqueness results

The above derivations permit us to state existence and uniqueness results of the variational inequality problems (3.37) and (3.41).

**Theorem 3.18:**
There exists exactly one solution $(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) \in \mathbf{X} \times \widetilde{\mathbf{Y}}'_+$ of the dual variational inequality problem (3.37). There exists exactly one solution $(\sigma,\nu;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) \in \mathbf{X} \times \Lambda \times \widetilde{\mathbf{Y}}'_+$ of the dual-dual variational inequality problem (3.41).

*Proof.* In Section 3.1.3 we have seen that the dual variational inequality problem (3.37) is equivalent to the saddle point problem (3.33). Due to Theorem 3.10 we have the equivalence of the saddle point problem (3.33) with the primal minimization problem 3.6 and the dual minimization problem 3.13. Since both minimization problems are uniquely solvable we have that the saddle point problem (3.33) as well as the dual variational inequality problem (3.37) are uniquely solvable.

The second statement follows directly from Theorem 3.17 and the existence and uniqueness of the dual variational inequality problem (3.37). □

### 3.1.6. Inf-Sup conditions

In order to derive an error analysis for the variational inequality formulations of the previous section certain inf-sup conditions are required. When dealing with variational problems without inequalities, these conditions imply unique solvability of the problem. The theory was established by Babuška [6], [7] and Brezzi [17]. For existence results of two fold saddle point problems occurring in mixed methods we refer to Gatica [34] and the recent published work of Walkington and Howell [83]. The proofs go back to the theorem of Lax-Milgram and are based on the abstract result of Brezzi and Fortin [18, see Proposition 1.1 in Chapter II]. To prove the first inf-sup condition we follow very closely the works of Gatica and Wendland [45] and Babuška and Gatica [9].

If we define the bilinear form $B : \mathbf{X} \times \widetilde{\mathbf{Y}}' \to \mathbb{R}$ for $\tau \in \mathbf{X}$ and $(\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \in \widetilde{\mathbf{Y}}'$

$$B(\tau, (\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n)) := b(\mathbf{v}, \tau) + s(\xi, \tau) + d_N(\boldsymbol{\psi}, \tau) + d_{C,t}(\mu_t, \tau) + d_{C,n}(\mu_n, \tau), \tag{3.46}$$

then we can state the following

**Lemma 3.19:**
Let us assume that the boundary part $\Gamma_C$ is polygonal, then the bilinear form $B(\cdot, \cdot)$ defined in (3.46) satisfies the inf-sup condition: $\exists\ \beta_1 > 0$ such that for all $(\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \in \widetilde{\mathbf{Y}}'$

$$\sup_{0 \neq \tau \in \mathbf{X}} \frac{B(\tau, (\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n))}{\|\tau\|_{\mathbf{X}}} \geq \beta_1 \|(\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n)\|_{\widetilde{\mathbf{Y}}'}, \tag{3.47}$$

where the norm on the product space $\widetilde{\mathbf{Y}}'$ is given by

$$\|(\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n)\|_{\widetilde{\mathbf{Y}}'} := \left( \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)} + \|\boldsymbol{\psi}\|^2_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)} + \|\mu_t\|^2_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} + \|\mu_n\|^2_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \right)^{\frac{1}{2}}.$$

Before we prove this lemma, we state some needful results from Gatica and Wendland [45], where they prove an inf-sup condition for a mixed boundary value problem from elastostatics solved by the coupling of mixed FEM and BEM. The proof will have the same structure. Analogously to [45] let us define the following subspaces of $L^2(\Omega)^{2\times 2}$ and $\mathbf{X}$, respectively,

$$\mathfrak{E} := \{\varepsilon(\mathbf{v}) : \mathbf{v} \in \mathbf{V}_D\},$$
$$\mathfrak{H} := \{\tau \in \mathbf{X} : \ \frac{1}{2}(\tau + \tau^T) \in \mathfrak{E}\}.$$

Then from [45, see Lemma 4.4] we have the following

**Lemma 3.20:**
For every $\xi \in L^2(\Omega)$ there exists a unique $\tau \in \mathfrak{H}$ such that $\operatorname{div}\tau = 0$ in $\Omega$, $\tau \cdot \mathbf{n} = 0$ on $\Gamma_N \cup \Gamma_C$ and $\frac{1}{2}(\tau - \tau^T) = \begin{pmatrix} 0 & \xi \\ -\xi & 0 \end{pmatrix}$. Moreover, there exists $C > 0$ such that

$$\|\tau\|_{\mathbf{X}} \leq C\|\xi\|_{L^2(\Omega)} \quad \forall\, \xi \in L^2(\Omega).$$

Now we are in the position to prove Lemma 3.19. The proof is in parts the same as Lemma 4.5 in Gatica and Wendland [45] (see also Gatica and Babuška [9] and Gatica et al. [10],[42] for similar results). We transfer it to the present case of a frictional contact problem. For the sake of completeness we write down all steps of the proof.

*Proof.* The proof is organized in five steps. The first three steps are from Lemma 4.5 of [45]. Let $(\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \in \widetilde{\mathbf{Y}}'$ be arbitrary but fixed. According to lemma 3.20, we let $\tau(\xi)$ be the unique element in $\mathfrak{H}$ such that $\operatorname{div}\tau(\xi) = 0$ in $\Omega$, $\tau(\xi)\cdot\mathbf{n} = 0$ on $\Gamma_N \cup \Gamma_C$ and $\operatorname{as}(\tau(\xi)) = \frac{1}{2}\begin{pmatrix} 0 & \xi \\ -\xi & 0 \end{pmatrix}$ in $\Omega$. We observe that

$$\begin{aligned}\tau(\xi) : \begin{pmatrix} 0 & \xi \\ -\xi & 0 \end{pmatrix} = \tau(\xi)_{12}\xi - \tau(\xi)_{21}\xi &= \frac{1}{2}(\tau(\xi)_{12} - \tau(\xi)_{21})\xi - \frac{1}{2}(\tau(\xi)_{21} - \tau(\xi)_{12})\xi \\ &= \operatorname{as}(\tau(\xi)) : \begin{pmatrix} 0 & \xi \\ -\xi & 0 \end{pmatrix} = \xi^2.\end{aligned}$$

Thus

$$\begin{aligned}&\sup_{0\neq\tau\in\mathbf{X}} \frac{B(\tau,(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\tau\|_{\mathbf{X}}} \geq \sup_{0\neq\tau\in\mathfrak{H}} \frac{B(\tau,(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\tau\|_{\mathbf{X}}} \\ &\geq \frac{B(\tau(\xi),(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\tau(\xi)\|_{\mathbf{X}}} = \frac{\int_\Omega \tau(\xi) : \begin{pmatrix} 0 & \xi \\ -\xi & 0 \end{pmatrix} dx}{\|\tau(\xi)\|_{\mathbf{X}}} = \frac{\|\xi\|^2_{L^2(\Omega)}}{\|\tau(\xi)\|_{\mathbf{X}}} \geq C_1\|\xi\|_{L^2(\Omega)}.\end{aligned} \tag{3.48}$$

In the second step we consider $\mathbf{z} \in \mathbf{V}_D$ as the unique solution of the following boundary value problem

$$\begin{aligned}\operatorname{div}\varepsilon(\mathbf{z}) &= \mathbf{v} \text{ in } \Omega, \\ \mathbf{z} &= 0 \text{ on } \Gamma_D, \\ \varepsilon(\mathbf{z})\cdot\mathbf{n} &= 0 \text{ on } \Gamma_N \cup \Gamma_C.\end{aligned}$$

The corresponding variational problem reads

Find $\mathbf{z} \in \mathbf{V}_D$ s.t.

$$\int_\Omega \varepsilon(\mathbf{z}) : \varepsilon(\mathbf{w})\,dx = -\int_\Omega \mathbf{v}\cdot\mathbf{w}\,dx \qquad \forall\, \mathbf{w} \in \mathbf{V}_D.$$

Now the second Korn's inequality assures that the bilinear form on the left hand side of the above equation is coercive, i.e. $\int_\Omega \varepsilon(\mathbf{z}) : \varepsilon(\mathbf{w})\,dx \geq C\|\mathbf{w}\|^2_{\mathbf{H}^1(\Omega)}$ for some

$C > 0$. Furthermore, the solution is continuously depending on the right hand side, that is $\|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} \le \frac{1}{C}\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}$.

Choosing $\hat{\tau} := \varepsilon(\mathbf{z})$ we have $\operatorname{div}\hat{\tau} = \mathbf{v}$ in $\Omega$, $\hat{\tau}\cdot\mathbf{n} = 0$ on $\Gamma_N \cup \Gamma_C$, $\hat{\tau} = \hat{\tau}^T$ in $\Omega$ and

$$\|\hat{\tau}\|_X = \|\varepsilon(\mathbf{z})\|_X \le \|\varepsilon(\mathbf{z})\|_{\mathbf{L}^2(\Omega)} + \|\operatorname{div}\varepsilon(\mathbf{z})\|_{\mathbf{L}^2(\Omega)} \le \|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \le C\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}.$$

Hence, we get the following inequality

$$\begin{aligned}\sup_{0\neq\tau\in X}\frac{B(\tau,(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\tau\|_X} &\ge \frac{B(\hat{\tau},(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\hat{\tau}\|_X}\\ &= \frac{\int_\Omega \mathbf{v}\cdot\operatorname{div}\hat{\tau}\,dx}{\|\hat{\tau}\|_X} = \frac{\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)}}{\|\hat{\tau}\|_X} \ge C_2\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}.\end{aligned} \tag{3.49}$$

In the third step we consider the following boundary value problem

$$\begin{aligned} -\operatorname{div}\varepsilon(\mathbf{z}) &= 0 \text{ in } \Omega,\\ \mathbf{z} &= 0 \text{ on } \Gamma_D,\\ \varepsilon(\mathbf{z})\cdot\mathbf{n} &= 0 \text{ on } \Gamma_C,\\ \varepsilon(\mathbf{z})\cdot\mathbf{n} &= \mathfrak{g} \text{ on } \Gamma_N,\end{aligned}$$

with some $\mathfrak{g} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$. Analogously to the previous step we derive the corresponding variational problem which reads

Find $\mathbf{z} \in \mathbf{V}_D$ s.t.

$$\int_\Omega \varepsilon(\mathbf{z}) : \varepsilon(\mathbf{w})\,dx = \int_{\Gamma_N} \mathfrak{g}\cdot\mathbf{w}\,ds \qquad \forall\ \mathbf{w} \in \mathbf{V}_D.$$

Again we have a unique solution $\mathbf{z}$ with $\|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} \le C\|\mathfrak{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)}$. Taking $\bar{\tau} := \varepsilon(\mathbf{z})$ we get $\operatorname{div}\bar{\tau} = 0$ in $\Omega$, $\bar{\tau}\cdot\mathbf{n} = 0$ on $\Gamma_C$, $\bar{\tau} = \bar{\tau}^T$ in $\Omega$, $\bar{\tau}\cdot\mathbf{n} = \mathfrak{g}$ on $\Gamma_N$ and

$$\|\bar{\tau}\|_X = \|\varepsilon(\mathbf{z})\|_X = \|\varepsilon(\mathbf{z})\|_{\mathbf{L}^2(\Omega)} \le \|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} \le C\|\mathfrak{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)},$$

which leads to the inequality

$$\sup_{0\neq\tau\in X}\frac{B(\tau,(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\tau\|_X} \ge \frac{B(\bar{\tau},(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\bar{\tau}\|_X} = \frac{\int_{\Gamma_N}\boldsymbol{\psi}\cdot\bar{\tau}\cdot\mathbf{n}\,ds}{\|\bar{\tau}\|_X} \ge C\frac{\int_{\Gamma_N}\boldsymbol{\psi}\cdot\mathfrak{g}\,ds}{\|\mathfrak{g}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)}}.$$

Now the above inequality is valid for all $\mathfrak{g} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$ and therefore by taking the supremum over all $\mathfrak{g} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$ on the right hand side we arrive at

$$\sup_{0\neq\tau\in X}\frac{B(\tau,(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\tau\|_X} \ge C_3\|\boldsymbol{\psi}\|_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)}. \tag{3.50}$$

In the last two steps we consider the contact boundary $\Gamma_C$. The following boundary value problem

$$\begin{aligned} -\operatorname{div}\varepsilon(\mathbf{z}) &= 0 \text{ in } \Omega, \\ \mathbf{z} &= 0 \text{ on } \Gamma_D, \\ \varepsilon(\mathbf{z})\cdot\mathbf{n} &= 0 \text{ on } \Gamma_N, \\ \varepsilon(\mathbf{z})\cdot\mathbf{n} &= \mathfrak{g}\mathbf{n} \text{ on } \Gamma_C, \end{aligned}$$

with $\mathfrak{g} \in H^{-\frac{1}{2}}(\Gamma_C)$, leads to the variational problem

Find $\mathbf{z} \in \mathbf{V}_D$ s.t.

$$\int_\Omega \varepsilon(\mathbf{z}) : \varepsilon(\mathbf{w})\, dx = \int_{\Gamma_C} \mathfrak{g}\mathbf{n}\cdot\mathbf{w}\, ds \qquad \forall\, \mathbf{w} \in \mathbf{V}_D.$$

Since we assume the boundary to be piecewise polygonal, we can decompose the contact boundary $\overline{\Gamma_C} := \bigcup_{i=1}^{M} \overline{e_i}$, where $e_i$ denote appropriate edges on $\Gamma_C$. Let $\mathbf{n}^i$ denote the corresponding normal exterior to the edge $e_i$. For arbitrary $\varphi \in H^{-\frac{1}{2}}(\Gamma_C)$ we have

$$\varphi = \sum_{i=1}^{M} \varphi_i^* \qquad \text{with} \quad \varphi_i^*(\mathbf{x}) := \begin{cases} \varphi(\mathbf{x}), & \text{if } \mathbf{x} \in e_i \\ 0, & \text{else.} \end{cases}$$

Using this decomposition we observe the following

$$\begin{aligned} \|\varphi\mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_C)} &= \Big\|\sum_{i=1}^{M} \varphi_i^*\mathbf{n}^i\Big\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_C)} = \sup_{\mathbf{v}\in\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_C)} \frac{\langle \sum_{i=1}^{M} \varphi_i^*\mathbf{n}^i, \mathbf{v}\rangle_{\Gamma_C}}{\|\mathbf{v}\|_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_C)}} \\ &\le \sum_{i=1}^{M} \sup_{\mathbf{v}^i\in\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_C)} \frac{\langle \varphi_i^*\mathbf{n}^i, \mathbf{v}^i\rangle_{\Gamma_C}}{\|\mathbf{v}^i\|_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_C)}} = \sum_{i=1}^{M} \|\varphi_i^*\mathbf{n}^i\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_C)} = \sum_{i=1}^{M} \|\varphi_i^*\mathbf{n}^i\|_{\mathbf{H}^{-\frac{1}{2}}(e_i)}. \end{aligned}$$

For each polygon $e_i$ we have that $\mathbf{n}^i$ is continuous. Therefore we can regard the multiplication with the normal on $e_i$ as a linear, continuous mapping, i.e.

$$\begin{aligned} N^i : H^{-\frac{1}{2}}(e_i) &\longrightarrow \mathbf{H}^{-\frac{1}{2}}(e_i), \\ \varphi &\mapsto N^i\varphi := \varphi\mathbf{n}^i \qquad \forall\, \varphi \in H^{-\frac{1}{2}}(e_i). \end{aligned}$$

We have that $N^i$ is bounded which reads

$$\exists\, C > 0 : \quad \|N^i\varphi\|_{\mathbf{H}^{-\frac{1}{2}}(e_i)} \le C\|\varphi\|_{H^{-\frac{1}{2}}(e_i)}$$

leading to the estimate

$$\|\varphi\mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_C)} \leq C\sum_{i=1}^{M}\|\varphi_i^*\|_{H^{-\frac{1}{2}}(e_i)} = C\|\varphi\|_{H^{-\frac{1}{2}}(\Gamma_C)}. \tag{3.51}$$

For the unique solution $\mathbf{z}$ we thus have the estimate $\|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} \leq \tilde{C}\|\mathfrak{g}\mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_C)} \leq C\|\mathfrak{g}\|_{H^{-\frac{1}{2}}(\Gamma_C)}$ and setting $\tilde{\tau} := \varepsilon(\mathbf{z})$ we have $\operatorname{div}\tilde{\tau} = 0$ in $\Omega$, $\tilde{\tau}\cdot\mathbf{n} = 0$ on $\Gamma_N$, $\tilde{\tau} = \tilde{\tau}^T$ in $\Omega$ and $\tilde{\tau}\cdot\mathbf{n} = \mathfrak{g}\mathbf{n}$ on $\Gamma_C$. The last equation leads to

$$\begin{aligned}\tilde{\tau}_n &= \mathbf{n}\cdot\tilde{\tau}\cdot\mathbf{n} = \mathfrak{g}\mathbf{n}\cdot\mathbf{n} = \mathfrak{g},\\ \tilde{\tau}_t &= \tilde{\tau}\cdot\mathbf{n} - \tilde{\tau}_n\mathbf{n} = \mathfrak{g}\mathbf{n} - \mathfrak{g}\mathbf{n} = 0.\end{aligned}$$

Furthermore, we have the inequality

$$\|\tilde{\tau}\|_{\mathbf{X}} = \|\varepsilon(\mathbf{z})\|_{\mathbf{X}} = \|\varepsilon(\mathbf{z})\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} \leq C\|\mathfrak{g}\|_{H^{-\frac{1}{2}}(\Gamma_C)}.$$

Now we can estimate

$$\sup_{0\neq\tau\in\mathbf{X}}\frac{B(\tau,(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\tau\|_{\mathbf{X}}} \geq \frac{B(\tilde{\tau},(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\tilde{\tau}\|_{\mathbf{X}}} = \frac{\int_{\Gamma_C}\mu_n\tilde{\tau}_n\,ds}{\|\tilde{\tau}\|_{\mathbf{X}}} \geq C\frac{\int_{\Gamma_C}\mu_n\mathfrak{g}\,ds}{\|\mathfrak{g}\|_{H^{-\frac{1}{2}}(\Gamma_C)}}$$

and using the same argument as in the third step and taking the supremum over all $\mathfrak{g}\in H^{-\frac{1}{2}}(\Gamma_C)$ we arrive at

$$\sup_{0\neq\tau\in\mathbf{X}}\frac{B(\tau,(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\tau\|_{\mathbf{X}}} \geq C_4\|\mu_n\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_C)}. \tag{3.52}$$

Finally, in the last step we consider the boundary value problem

$$\begin{aligned}-\operatorname{div}\varepsilon(\mathbf{z}) &= 0 \text{ in } \Omega,\\ \mathbf{z} &= 0 \text{ on } \Gamma_D,\\ \varepsilon(\mathbf{z})\cdot\mathbf{n} &= 0 \text{ on } \Gamma_N,\\ \varepsilon(\mathbf{z})\cdot\mathbf{n} &= \mathfrak{g}\begin{pmatrix}\mathbf{n}_2\\ -\mathbf{n}_1\end{pmatrix} \text{ on } \Gamma_C,\end{aligned}$$

with $\mathfrak{g}\in H^{-\frac{1}{2}}(\Gamma_C)$ and the corresponding variational problem

Find $\mathbf{z}\in\mathbf{V}_D$ s.t.

$$\int_{\Omega}\varepsilon(\mathbf{z}):\varepsilon(\mathbf{w})\,dx = \int_{\Gamma_C}\mathfrak{g}\begin{pmatrix}\mathbf{n}_2\\ -\mathbf{n}_1\end{pmatrix}\cdot\mathbf{w}\,ds \qquad \forall\ \mathbf{w}\in\mathbf{V}_D.$$

In a same manner as in the previous step we can adapt (3.51) keeping in mind, that the multiplication with the tangent on some polygon $e_i$ is also a linear, continuous mapping. For the unique solution $\mathbf{z}$ we therefore derive the estimate

$$\|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} \leq \tilde{C}\|\mathfrak{g}\begin{pmatrix}\mathbf{n}_2\\-\mathbf{n}_1\end{pmatrix}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_C)} \leq C\|\mathfrak{g}\|_{H^{-\frac{1}{2}}(\Gamma_C)}$$

and setting $\check{\tau} := \varepsilon(\mathbf{z})$ we have $\operatorname{div}\check{\tau} = 0$ in $\Omega$, $\check{\tau}\cdot\mathbf{n} = 0$ on $\Gamma_N$, $\check{\tau} = \check{\tau}^T$ in $\Omega$ and $\check{\tau}\cdot\mathbf{n} = \mathfrak{g}\begin{pmatrix}\mathbf{n}_2\\-\mathbf{n}_1\end{pmatrix}$ on $\Gamma_C$. The last equation leads to

$$\check{\tau}_n = \mathbf{n}\cdot\check{\tau}\cdot\mathbf{n} = \mathfrak{g}\mathbf{n}\cdot\begin{pmatrix}\mathbf{n}_2\\-\mathbf{n}_1\end{pmatrix} = 0,$$
$$\check{\boldsymbol{\tau}}_t = \check{\tau}\cdot\mathbf{n} - \check{\tau}_n\mathbf{n} = \mathfrak{g}\begin{pmatrix}\mathbf{n}_2\\-\mathbf{n}_1\end{pmatrix} \qquad\Rightarrow\qquad \check{\tau}_t = \check{\boldsymbol{\tau}}_t\cdot\begin{pmatrix}\mathbf{n}_2\\-\mathbf{n}_1\end{pmatrix} = \mathfrak{g}\begin{pmatrix}\mathbf{n}_2\\-\mathbf{n}_1\end{pmatrix}\cdot\begin{pmatrix}\mathbf{n}_2\\-\mathbf{n}_1\end{pmatrix} = \mathfrak{g}.$$

We get the inequality

$$\|\check{\tau}\|_{\mathbf{X}} = \|\varepsilon(\mathbf{z})\|_{\mathbf{X}} = \|\varepsilon(\mathbf{z})\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{z}\|_{\mathbf{H}^1(\Omega)} \leq C\|\mathfrak{g}\|_{H^{-\frac{1}{2}}(\Gamma_C)},$$

and can estimate

$$\begin{aligned}\sup_{0\neq\tau\in\mathbf{X}}\frac{B(\tau,(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\tau\|_{\mathbf{X}}} &\geq \frac{B(\check{\tau},(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\check{\tau}\|_{\mathbf{X}}} = \frac{d_{C,t}(\check{\tau},\mu_t)}{\|\check{\tau}\|_{\mathbf{X}}}\\ &= \frac{\int_{\Gamma_C}\mu_t\check{\tau}_t\,ds}{\|\check{\tau}\|_{\mathbf{X}}} = \frac{\int_{\Gamma_C}\mu_t\mathfrak{g}\,ds}{\|\check{\tau}\|_{\mathbf{X}}} \geq C\frac{\int_{\Gamma_C}\mu_t\mathfrak{g}\,ds}{\|\mathfrak{g}\|_{H^{-\frac{1}{2}}(\Gamma_C)}}.\end{aligned}$$

Again using the same argument as in the third step and taking the supremum over all $\mathfrak{g}\in H^{-\frac{1}{2}}(\Gamma_C)$ we arrive at

$$\sup_{0\neq\tau\in\mathbf{X}}\frac{B(\tau,(\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n))}{\|\tau\|_{\mathbf{X}}} \geq C_5\|\mu_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}. \tag{3.53}$$

If we finally set $\beta_1 := \max\{C_1,\ldots,C_5\}$ we get the desired estimate (3.47). □

For the error analysis of the following sections we further need a second inf-sup condition concerning the bilinear form $q(\cdot,\cdot)$. Before we can prove this assertion let us define

$$I_C := \Gamma_C \setminus \overline{A_C}.$$

We make some further investigations on the friction functional $\mathcal{F}$. Let $\partial A_C$ denote the boundary of $A_C$, which is the set of all points $\mathbf{x}$ with $\mathbf{x}\in(\overline{A_C}\cap\overline{I_C})\cup\{\mathbf{x}\in\overline{A_C}:\mathcal{F}(\mathbf{x})=0\}$. Due to assumption (3.4) we have to distinguish two cases for every $\mathbf{x}\in\partial A_C$

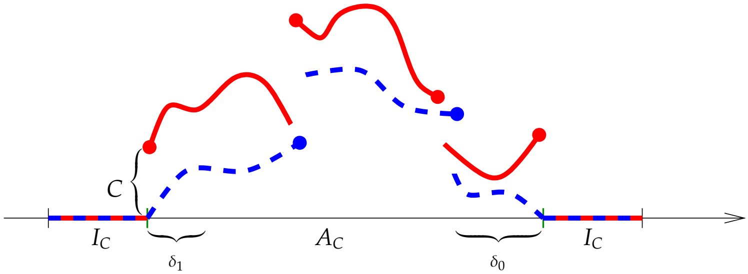



Figure 3.4.: Two cases for the friction functional $\mathcal{F}$ on $\Gamma_C$.

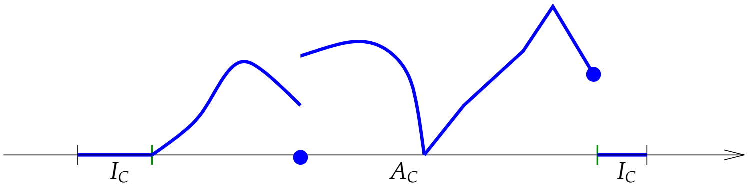



Figure 3.5.: General friction functional $\mathcal{F}$ on $\Gamma_C$.

1. $\exists\, C > 0 : \quad \mathcal{F}(\mathbf{x}) > C$, i.e. $\mathcal{F}$ has a jump in $\mathbf{x} \in NC_{\mathcal{F}}$,
2. $\mathcal{F}(\mathbf{x}) = 0$ and we can find some $\delta_0 > 0$ s.t. $\mathcal{F} \in C(B_{\delta_0}(\mathbf{x}) \cap A_C)$.

Moreover for the second case we can find for every $\mathbf{x} \in \partial A_C$ some $\delta_1 > 0$ such that $\mathcal{F}$ is strictly monotone in $B_{\delta_1}(\mathbf{x}) \cap A_C$. In Figure 3.4 we give an example of the two possible cases. Without loss of generality we can reduce all other situations to one of those above, as can be seen in Figure 3.5. We just have to cut $A_C$ appropriately into some disjoint subsets.

Before we continue with the second inf-sup condition let us adopt Lemma 3.2.2. from Chernov [24] which reads

**Lemma 3.21:**
Let $\Gamma$ be a closed Lipschitz curve with two open connected disjoint subsets $\gamma_0 \subset \Gamma$, $\gamma_1 \subset \Gamma$, $\overline{\gamma_0} \cap \overline{\gamma_1} = \emptyset$. Let also $\gamma_0^* := \Gamma \setminus \overline{\gamma_0}$, $\gamma_{01}^* := \Gamma \setminus (\overline{\gamma_0} \cup \overline{\gamma_1})$. Then for all $\phi \in H^{\frac{1}{2}}(\gamma_1)$ there exists an extension $f_\phi \in \widetilde{H}^{\frac{1}{2}}(\gamma_0^*)$ of $\phi$ onto $\gamma_0^*$, such that $f_\phi|_{\gamma_1} = \phi$ and

$$\exists\, \alpha > 0 : \quad \|\phi\|_{H^{\frac{1}{2}}(\gamma_1)} \geq \alpha \|f_\phi\|_{\widetilde{H}^{\frac{1}{2}}(\gamma_0^*)}.$$

The constant $\alpha$ depends only on $\min_i |(\gamma_{01}^*)^i|$, where $(\gamma_{01}^*)^i$ are connected components of $\gamma_{01}^*$.

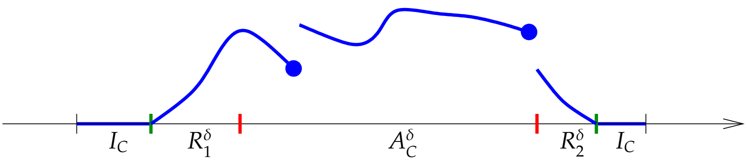



Figure 3.6.: Distribution of the contact boundary $\Gamma_C$.

Now we are in a position to state the following

**Lemma 3.22:**
The bilinear form $q(\cdot,\cdot)$ satisfies the following inf-sup condition: $\exists\ \beta_2 > 0$ such that for all $\kappa \in H^{-\frac{1}{2}}(A_C)$

$$\sup_{\mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \frac{q(\mu\,,\,\kappa)}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} \geq \beta_2 \|\kappa\|_{H^{-\frac{1}{2}}(A_C)}. \tag{3.54}$$

*Proof.* Without loss of generality we only consider the two cases mentioned above, see also Figure 3.4. Let us first consider the friction functional assuming the first case from above. Then there exists a $C > 0$ with $\min_{\mathbf{x} \in A_C} \mathcal{F}(\mathbf{x}) > C$ and we have

$$\sup_{\mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \frac{q(\mu\,,\,\kappa)}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} \geq \sup_{\mu \in \widetilde{H}^{\frac{1}{2}}(A_C)} \frac{\int_{A_C} \mathcal{F}\,\mu\,\kappa\,ds}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(A_C)}} \geq C \sup_{\mu \in \widetilde{H}^{\frac{1}{2}}(A_C)} \frac{\int_{A_C} \mu\,\kappa\,ds}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(A_C)}} = C\|\kappa\|_{H^{-\frac{1}{2}}(A_C)}.$$

In the second case we proceed as follows. First we distribute the active part of the contact boundary $A_C$ into the following sets as can be seen from Figure 3.6

$$A_C = R_1^\delta \cup A_C^\delta \cup R_2^\delta \quad \text{with } \; R_i^\delta := A_C \cap B_\delta(\mathbf{x}^i) \; \text{ and } \; A_C^\delta = A_C \setminus \bigcup_i R_i^\delta,$$

where $\mathbf{x}^i \in \partial A_C$ and $\delta = \min\{\delta_0^i, \delta_1^i\}$ for every $R_i^\delta$.

Now we can find some $C_\delta > 0$ with $\min_{\mathbf{x} \in A_C^\delta} \mathcal{F}(\mathbf{x}) > C_\delta$. If we apply Lemma 3.21 with $\gamma_0 = I_C \cup \Gamma_D \cup \Gamma_N$, $\gamma_1 = A_C^\delta$, $\gamma_0^* = A_C$ and $\gamma_{01}^* = R_1^\delta \cup R_2^\delta$, then we can find for every $\mu \in H^{\frac{1}{2}}(A_C^\delta)$ an extension $f_\mu \in \widetilde{H}^{\frac{1}{2}}(A_C)$ such that $f_\mu|_{A_C^\delta} = \mu$ and

$$\|\mu\|_{H^{\frac{1}{2}}(A_C^\delta)} \geq \alpha_\delta \|f_\mu\|_{\widetilde{H}^{\frac{1}{2}}(A_C)},$$

for some $\alpha_\delta$ depending on $\delta$. For $\kappa \in H^{-\frac{1}{2}}(A_C)$ we can find $\hat{\kappa} \in \widetilde{H}^{-\frac{1}{2}}(A_C^\delta)$ and $\bar{\kappa}_i \in H^{-\frac{1}{2}}(R_i^\delta)$ with $\kappa = \hat{\kappa}^* + \bar{\kappa}_1^* + \bar{\kappa}_2^*$ and $\hat{\kappa}^*$ and $\bar{\kappa}_i^*$ being extensions of the corresponding functions to $A_C$ by zero with equal norms, respectively. We have

$$\|\kappa\|_{H^{-\frac{1}{2}}(A_C)} = \|\hat{\kappa}^* + \bar{\kappa}_1^* + \bar{\kappa}_2^*\|_{H^{-\frac{1}{2}}(A_C)} \leq \|\hat{\kappa}^*\|_{H^{-\frac{1}{2}}(A_C)} + \|\bar{\kappa}_1^*\|_{H^{-\frac{1}{2}}(A_C)} + \|\bar{\kappa}_2^*\|_{H^{-\frac{1}{2}}(A_C)}$$
$$= \|\hat{\kappa}\|_{\widetilde{H}^{-\frac{1}{2}}(A_C^\delta)} + \|\bar{\kappa}_1\|_{H^{-\frac{1}{2}}(R_1^\delta)} + \|\bar{\kappa}_2\|_{H^{-\frac{1}{2}}(R_2^\delta)}.$$

Now we are able to conclude for all $\hat{\kappa} \in \widetilde{H}^{-\frac{1}{2}}(A_C^\delta)$

$$\sup_{\mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \frac{q(\mu\,,\,\hat{\kappa}^*)}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} \geq \sup_{\mu \in \widetilde{H}^{\frac{1}{2}}(A_C)} \frac{\int_{A_C} \mathcal{F}\mu\,\hat{\kappa}^*\,ds}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(A_C)}} \geq \sup_{\mu \in H^{\frac{1}{2}}(A_C^\delta)} \frac{\int_{A_C^\delta} \mathcal{F} f_\mu\,\hat{\kappa}\,ds}{\|f_\mu\|_{\widetilde{H}^{\frac{1}{2}}(A_C)}}$$
$$\geq \alpha_\delta \sup_{\mu \in H^{\frac{1}{2}}(A_C^\delta)} \frac{\int_{A_C^\delta} \mathcal{F}\mu\,\hat{\kappa}\,ds}{\|\mu\|_{H^{\frac{1}{2}}(A_C^\delta)}} \geq \alpha_\delta C_\delta \sup_{\mu \in H^{\frac{1}{2}}(A_C^\delta)} \frac{\int_{A_C^\delta} \mu\,\hat{\kappa}\,ds}{\|\mu\|_{H^{\frac{1}{2}}(A_C^\delta)}} = \alpha_\delta C_\delta \|\hat{\kappa}\|_{\widetilde{H}^{-\frac{1}{2}}(A_C^\delta)}. \quad (3.55)$$

As seen above $\mathcal{F}$ is continuous and strictly monotonic on $R_i^\delta$ and so there exists the inverse $\mathcal{F}^{-1}$ of $\mathcal{F}$ on $\mathbb{R}_i^\delta$ which is also continuous. Due to [61, see Chapter 9] $\mathcal{F}^{-1}$ is a factor on $\widetilde{H}^{\frac{1}{2}}(R_i^\delta)$ and therefore $\mathcal{F}^{-1}\mu \in \widetilde{H}^{\frac{1}{2}}(R_i^\delta)$ is well defined for all $\mu \in \widetilde{H}^{\frac{1}{2}}(R_i^\delta)$. Defining $\bar{\mu} := \mathcal{F}\mu$ with $\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(R_i^\delta)} = \|\mathcal{F}^{-1}\bar{\mu}\|_{\widetilde{H}^{\frac{1}{2}}(R_i^\delta)} \leq \|\mathcal{F}^{-1}\|_{L^1(R_i^\delta)}\|\bar{\mu}\|_{\widetilde{H}^{\frac{1}{2}}(R_i^\delta)}$ we get for $i = 1, 2$

$$\sup_{\mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \frac{q(\mu\,,\,\bar{\kappa}_i^*)}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} \geq \sup_{\mu \in \widetilde{H}^{\frac{1}{2}}(R_i^\delta)} \frac{\int_{R_i^\delta} \mathcal{F}\mu\,\bar{\kappa}_i^*\,ds}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(R_i^\delta)}} = \sup_{\mu \in \widetilde{H}^{\frac{1}{2}}(R_i^\delta)} \frac{\int_{R_i^\delta} \bar{\mu}\,\bar{\kappa}_i\,ds}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(R_i^\delta)}}$$
$$\geq \frac{1}{\|\mathcal{F}^{-1}\|_{L^1(R_i^\delta)}} \sup_{\bar{\mu} \in \widetilde{H}^{\frac{1}{2}}(R_i^\delta)} \frac{\int_{R_i^\delta} \bar{\mu}\,\bar{\kappa}_i\,ds}{\|\bar{\mu}\|_{\widetilde{H}^{\frac{1}{2}}(R_i^\delta)}} = \frac{1}{\|\mathcal{F}^{-1}\|_{L^1(R_i^\delta)}}\|\bar{\kappa}_i\|_{H^{-\frac{1}{2}}(R_i^\delta)}. \quad (3.56)$$

Choosing $\beta_2 = \min\{\frac{1}{\|\mathcal{F}^{-1}\|_{L^1(R_i^\delta)}}, \alpha_\delta C_\delta\}$ and adding (3.55) and (3.56) we finally arrive at

$$\sup_{\mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \frac{q(\mu\,,\,\kappa)}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} = \sup_{\mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \frac{q(\mu\,,\,\hat{\kappa}^* + \bar{\kappa}_1^* + \bar{\kappa}_2^*)}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}}$$
$$\geq \beta_2 \left( \|\hat{\kappa}\|_{\widetilde{H}^{-\frac{1}{2}}(A_C^\delta)} + \|\bar{\kappa}_1\|_{H^{-\frac{1}{2}}(R_1^\delta)} + \|\bar{\kappa}_2\|_{H^{-\frac{1}{2}}(R_2^\delta)} \right)$$
$$= \beta_2 \left( \|\hat{\kappa}^*\|_{H^{-\frac{1}{2}}(A_C)} + \|\bar{\kappa}_1^*\|_{H^{-\frac{1}{2}}(A_C)} + \|\bar{\kappa}_2^*\|_{H^{-\frac{1}{2}}(A_C)} \right)$$
$$\geq \beta_2 \|\hat{\kappa}^* + \bar{\kappa}_1^* + \bar{\kappa}_2^*\|_{H^{-\frac{1}{2}}(A_C)} = \beta_2 \|\kappa\|_{H^{-\frac{1}{2}}(A_C)}.$$

□

We have the following result concerning the continuous dependence of the solution on the given data.

**Lemma 3.23:**
There exists a constant $C > 0$ independent of the given data $\mathbf{f}, \mathbf{t}_0, g$ and $\mathcal{F}$ and a constant $C_{\mathcal{F}}$ independent of $\mathbf{f}, \mathbf{t}_0$ and $g$ such that for the solution $(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \Lambda \times \widetilde{\mathbf{Y}}$ of the variational inequality problem (3.41) there holds

$$\begin{aligned}\|\sigma\|_{\mathbf{X}} + \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|\boldsymbol{\varphi}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma_N)} + \|\lambda_t\|_{H^{\frac{1}{2}}(\Gamma_C)} + \|\lambda_n\|_{H^{\frac{1}{2}}(\Gamma_C)} \\ \leq C\left(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{t}_0\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)} + \|g\|_{H^{\frac{1}{2}}(\Gamma_C)} + \|\mathcal{F}\|_{L^\infty(\Gamma)}\right)\end{aligned}$$

and $$\|\nu\|_{H^{-\frac{1}{2}}(A_C)} \leq C_{\mathcal{F}}\left(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{t}_0\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)} + \|g\|_{H^{\frac{1}{2}}(\Gamma_C)} + \|\mathcal{F}\|_{L^\infty(\Gamma)}\right).$$

*Proof.* The first equation in the variational inequality problem (3.41) reads

$$B(\tau\,,\,(\mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n)) = -g(\tau) - \tilde{a}(\sigma, \tau) \qquad \forall\, \tau \in \mathbf{X}.$$

Thus applying the inf-sup condition of Lemma 3.19 and the continuity of $\tilde{a}(\cdot,\cdot)$ we have

$$\begin{aligned}\|\mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n\|_{\widetilde{\mathbf{Y}}'} &\leq \frac{1}{\beta_1} \sup_{\tau \in \mathbf{X}} \frac{B(\tau\,,\,(\mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n))}{\|\tau\|_{\mathbf{X}}} \\ &= \frac{1}{\beta_1} \sup_{\tau \in \mathbf{X}} \frac{-g(\tau) - \tilde{a}(\sigma, \tau)}{\|\tau\|_{\mathbf{X}}} \leq \frac{C}{\beta_1}\left\{\|g\|_{H^{\frac{1}{2}}(\Gamma_C)} + \|\sigma\|_{\mathbf{X}}\right\}.\end{aligned} \tag{3.57}$$

In the same way, using the inf-sup condition in Lemma 3.22 and the fifth equality of (3.41) we derive

$$\begin{aligned}\|\nu\|_{H^{-\frac{1}{2}}(A_C)} \leq \frac{1}{\beta_2} \sup_{\mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \frac{q(\mu_t, \nu)}{\|\mu_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} &= \frac{1}{\beta_2} \sup_{\mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \frac{d_{C,t}(\mu_t, \sigma)}{\|\mu_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} \\ &\leq \frac{1}{\beta_2}\|\sigma_t\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq \frac{1}{\beta_2}\|\sigma\|_{\mathbf{X}}.\end{aligned} \tag{3.58}$$

Now as $\sigma$ is a solution of (3.41) we have for the divergence

$$\|\operatorname{div} \sigma\|_{\mathbf{L}^2(\Omega)} = \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}$$

and using the coercivity of $\tilde{a}(\cdot,\cdot)$ on the kernel of $B(\cdot,\cdot)$ we conclude

$$\|\sigma\|_{\mathbf{X}}^2 \leq \frac{1}{\alpha}\tilde{a}(\sigma, \sigma) + \|\operatorname{div} \sigma\|_{\mathbf{L}^2(\Omega)}^2 = \frac{1}{\alpha}\left(-B(\sigma\,,\,(\mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n)) - g(\sigma)\right) + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^2.$$

Using the second to fifth equalities of the variational inequality problem (3.41) we arrive at

$$\|\sigma\|_{\mathbf{X}}^2 \leq \frac{1}{\alpha}\left(f(\mathbf{u}) - t_0(\boldsymbol{\varphi}) - q(\lambda_t, \nu) - d_{C,n}(\lambda_n, \sigma) - g(\sigma)\right) + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^2.$$

From the proof of Theorem (3.10) we have $d_{C,n}(\lambda_n, \sigma) = 0$. Furthermore, the proof of Theorem (3.17) provides $q(\lambda_t, \nu) = j(\lambda_t)$. Applying the Cauchy-Schwarz-inequality, equation (3.57) and inequality (2.6) we arrive at

$$\begin{aligned}
\|\sigma\|_{\mathbf{X}}^2 \leq{} & \frac{1}{\alpha}\Big(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{t}_0\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)}\|\boldsymbol{\varphi}\|_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)} \\
& \qquad + \|\mathcal{F}\|_{L^\infty(\Gamma_C)}\|\lambda_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} + \|g\|_{H^{\frac{1}{2}}(\Gamma_C)}\|\sigma\|_{\mathbf{X}}\Big) + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^2 \\
\leq{} & \frac{1}{\alpha}\left(\frac{C}{\beta_1}\left\{\|g\|_{H^{\frac{1}{2}}(\Gamma_C)} + \|\sigma\|_{\mathbf{X}}\right\}\left(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{t}_0\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)} + \|\mathcal{F}\|_{L^\infty(\Gamma_C)}\right)\right. \\
& \qquad \left. + \|g\|_{H^{\frac{1}{2}}(\Gamma_C)}\|\sigma\|_{\mathbf{X}}\right) + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^2 \\
\leq{} & \frac{\varepsilon}{2\alpha}\|\sigma\|_{\mathbf{X}}^2 + \frac{1}{2\varepsilon\alpha}\left(\frac{C}{\beta_1}\left(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{t}_0\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)} + \|\mathcal{F}\|_{L^\infty(\Gamma_C)}\right) + \|g\|_{H^{\frac{1}{2}}(\Gamma_C)}\right)^2 \\
& + \frac{C}{\beta_1\alpha}\|g\|_{H^{\frac{1}{2}}(\Gamma_C)}\left(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{t}_0\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)} + \|\mathcal{F}\|_{L^\infty(\Gamma_C)}\right) + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}^2.
\end{aligned}$$

Taking $\varepsilon = \alpha$ we can find some $\tilde{C} > 0$ such that

$$\|\sigma\|_{\mathbf{X}}^2 \leq \frac{\tilde{C}^2}{\alpha^2}\left\{\frac{C}{\beta_1}\left(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{t}_0\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)} + \|\mathcal{F}\|_{L^\infty(\Gamma_C)}\right) + \|g\|_{H^{\frac{1}{2}}(\Gamma_C)}\right\}^2. \tag{3.59}$$

We finish the proof by applying the square root of (3.59) to the inequalities (3.57) and (3.58). □

## 3.2. Mixed finite elements

In this section we present the discretization of the dual-dual variational inequality problem (3.41). We introduce appropriate finite element spaces such that the discrete versions of the inf-sup conditions (3.47) and (3.54) hold. Mixed Finite Element Methods have been investigated for years and we do not give an explanation of them in this work, but refer to Ciarlet [26] for an elaborate description of the standard Finite Element Method and to the work of Brezzi and Fortin [18], which is a very extensive introduction to Mixed Finite Element Methods.

Before we state the discrete dual-dual variational inequality problem, let us define the setting of the Mixed Finite Element Method, that we use to find an approximate solution of (3.1). First assume that the boundary $\Gamma$ of the considered domain $\Omega$ is piecewise polygonal. Let $\mathcal{T}_h$ be a regular, quasiuniform triangulation of $\Omega$. First assume that the boundary $\Gamma$ of the considered domain $\Omega$ is piecewise polygonal. Let $\mathcal{T}_h$ be a regular, quasiuniform triangulation of $\Omega$. The set of all edges $\mathcal{E}_h$ is decomposed into the set of all interior edges $\mathcal{E}_h^\Omega$ and the set of all boundary edges $\mathcal{E}_h^\Gamma$. Furthermore, $\mathcal{E}_h^i$ for $i \in \{D, N, C, A, I\}$ denotes the set of all boundary edges on the corresponding boundary part, i.e. $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$ for the first three letters and $A_C$

and $I_C$ for the last two letters. Note, that for all edges $e \in \mathcal{E}_h^\Omega$ there exist exactly two triangles $T_e^1, T_e^2 \in \mathcal{T}_h$ that have $e$ as a common edge. In the same way we denote the set of all vertices $\mathcal{N}_h$, decomposed into each boundary part and the interior domain by $\mathcal{N}_h^i$ for $i \in \{D, N, C, A, I, \Omega, \Gamma\}$. For some arbitrary triangle $T$ and edge $e$ we finally denote by $\mathcal{E}^T$ the set of all edges of $T$ and by $\mathcal{N}^T$ and $\mathcal{N}^e$ the set of all vertices on $T$ and $e$, respectively.

As we will outline in the proofs of the discrete inf-sup conditions (3.69) and (3.72) we need coarser mesh sizes for the finite element spaces of the Lagrange multipliers on the boundaries. In order to keep the complexity of the solution algorithms and its implementation, as simple as possible, we propose the following choice for the coarser mesh sizes $\tilde{h}$ and $\hat{h}$. Similarly to Gatica and Maischak [41] we use $\tilde{h} := 2h$ and $\hat{h} := 4h$. If the coarsening is not sufficient we use $\tilde{h} := 4h$ and $\hat{h} := 8h$ and so on. For the sets of edges with mesh sizes $\tilde{h}$ and $\hat{h}$ on each boundary part, we use the notation $\mathcal{E}_{\tilde{h}}^i$ and $\mathcal{E}_{\hat{h}}^i$ for $i \in \{D, N, C, A, I\}$, respectively. An analogous notation applies for the sets of vertices. Moreover, we assume that for all edges $\tilde{e} \in \mathcal{E}_{\tilde{h}}^i$ we have only edges $e \in \mathcal{E}_h^i$ lying entirely on $\tilde{e}$ for $i \in \{N, C\}$. We make the same assumption for all edges $\hat{e} \in \mathcal{E}_{\hat{h}}^C$, i.e. for arbitrary $\tilde{e} \in \mathcal{E}_{\tilde{h}}^i$ or $\hat{e} \in \mathcal{E}_{\hat{h}}^C$ we have for all $e \in \mathcal{E}_h^i$ and $i \in \{N, C\}$ the intersections $\tilde{e} \cap e$ and $\hat{e} \cap e$ are either empty or the whole edge $e$. The number of triangles in $\mathcal{T}_h$ is denoted by $N_{\mathcal{T}}$. Analogously $N_i$ denotes the number of edges in $\mathcal{E}_h^i$, $\tilde{N}_i$ the number of edges in $\mathcal{E}_{\tilde{h}}^i$ for $i \in \{D, N, C, A, I, \Omega\}$ and $\hat{N}_C$ the number of edges in $\mathcal{E}_{\hat{h}}^C$.

In Chapter 4 we will need the following definitions for the interpolation estimates of the Clément interpolation operator

$$\omega_T := \bigcup_{\mathcal{N}^{T'} \cap \mathcal{N}^T \neq \emptyset} T', \qquad \omega_e := \bigcup_{e \cap \mathcal{E}^{T'} \neq \emptyset} T'. \tag{3.60}$$

We are now in the position to define the discrete finite element spaces for the discretization of problem (3.41). For the triple $(\sigma, \mathbf{u}, \eta) \in \mathbf{X} \times \mathbf{L}^2(\Omega) \times L^2(\Omega)$ we use the PEERS elements that were introduced by Arnold, Brezzi and Douglas Jr. in [5]. They show, that the PEERS elements are well suited in the sense that they fulfill the discrete inf-sup condition for the bilinear form $b(\cdot,\cdot) + s(\cdot,\cdot)$. For the Lagrange multipliers on the boundaries representing the corresponding displacement fields we use continuous hat functions. This seems plausible since continuous hat functions are well suited for the space $H^1(\Omega)$ and the trace of a 2D hat function is a 1D hat function restricted to the boundary. Finally, we use piecewise constant functions to discretize $H^{-\frac{1}{2}}(A_C)$.

We need the following finite element spaces. For arbitrary $T \in \mathcal{T}_k$ and $e \in \mathcal{E}_k^j$ with

$k \in \{h, \tilde{h}, \hat{h}\}$ and $j \in \{D, N, C, A, I\}$ we define the polynomial spaces

$$\begin{aligned}
P_s^k(T) &:= \{\psi : \ \psi \text{ is a polynomial of degree at most } s \text{ on } T\}\,,\\
\mathbf{P}_s^k(T) &:= \left\{\boldsymbol{\psi} := \left(\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}\right) : \ \psi_i \in P_s^k(T) \text{ for } i = 1,2\right\},\\
P_s^k(e) &:= \{\psi : \ \psi \text{ is a polynomial of degree at most } s \text{ on } e\}\,,\\
\mathbf{P}_s^k(e) &:= \left\{\boldsymbol{\psi} := \left(\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}\right) : \ \psi_i \in P_s^k(e) \text{ for } i = 1,2\right\}
\end{aligned}$$

and the Raviart-Thomas spaces on some triangle $T \in \mathcal{T}_h$

$$\begin{aligned}
RT_0(T) &:= \{\boldsymbol{\psi} := \left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) + c\left(\begin{smallmatrix}x_1\\ x_2\end{smallmatrix}\right) \text{ on } T \in \mathcal{T}_h, \text{ for } a,b,c \in \mathbb{R}\}\,,\\
\mathbf{RT}_0(T) &:= \{\tau : \ \left(\begin{smallmatrix}\tau_{i1}\\ \tau_{i2}\end{smallmatrix}\right) \in RT_0(T) \text{ on } T, \text{ for } i = 1,2\}
\end{aligned}$$

and on $\mathcal{T}_h$

$$\begin{aligned}
RT_0 &:= \{\boldsymbol{\psi} \in H(\mathrm{div}, \Omega) : \ \boldsymbol{\psi}|_T \in RT_0(T) \ \forall\, T \in \mathcal{T}_h\}\,,\\
\mathbf{RT}_0 &:= \{\tau \in \mathbf{H}(\mathrm{div}, \Omega) : \ \tau|_T \in \mathbf{RT}_0(T) \ \forall\, T \in \mathcal{T}_h\}\,.
\end{aligned}$$

The space $RT_0$ is the space of Raviart-Thomas elements of lowest order. They were first introduced by Raviart and Thomas [76] and are suitable for the discretization of $H(\mathrm{div}, \Omega)$ since their normal component is continuous along all edges in $\mathcal{E}_h^\Omega$. In order to define the PEERS elements we further need the following function. For a $T \in \mathcal{T}_h$ we take the barycentric coordinates $\lambda_T^i$ for $i = 1, 2, 3$ and introduce the bubble function on $T$ as

$$b_T(\mathbf{x}) := \frac{\prod_{i=1}^3 \lambda_T^i(\mathbf{x})}{\prod_{i=1}^3 \lambda_T^i(p_b(T))},$$

where $p_b(T)$ is the barycenter of the triangle $T$. We define

$$\mathbf{B} := \left\{\tau : \ \tau|_T = \mathrm{Curl}\left(\begin{smallmatrix}\alpha b_T\\ \beta b_T\end{smallmatrix}\right) \ \forall\, T \in \mathcal{T}_h, \ \alpha, \beta \in \mathbb{R}\right\},$$

where

$$\mathrm{Curl}\,\boldsymbol{\psi} := \begin{pmatrix}(\mathrm{Curl}\,\psi_1)^T\\ (\mathrm{Curl}\,\psi_2)^T\end{pmatrix} \ \text{ and } \ \mathrm{Curl}\,\psi := \left(\begin{smallmatrix}\psi_{,2}\\ -\psi_{,1}\end{smallmatrix}\right),$$

according to the definition of the Curl-operator, see e.g. Carstensen and Dolzmann [22]. Now, we proceed with the definition of the following finite element spaces for the discretization of the dual-dual variational inequality problem (3.41)

$$
\begin{aligned}
\mathbf{X}_h &:= \mathbf{RT}_0 \cup \mathbf{B},\\
\mathbf{M}_h &:= \left\{\mathbf{v} :\ \mathbf{v}|_T \in \mathbf{P}_0^h(T),\ \forall\, T \in \mathcal{T}_h\right\},\\
S_h &:= \left\{\eta \in C(\Omega) :\ \eta|_T \in P_1^h(T),\ \forall\, T \in \mathcal{T}_h\right\},\\
\mathbf{N}_{\tilde{h}} &:= \left\{\boldsymbol{\psi} \in C(\Gamma_N)^2 :\ \boldsymbol{\psi}|_e \in \mathbf{P}_1^{\tilde{h}}(e), \boldsymbol{\psi} = 0|_{\partial\Gamma_N},\ \forall\, e \in \mathcal{E}_{\tilde{h}}^N\right\},\\
C_{\tilde{h}} &:= \left\{\mu \in C(\Gamma_C) :\ \mu \in P_1^{\tilde{h}}(e), \mu = 0|_{\partial\Gamma_C},\ \forall\, e \in \mathcal{E}_{\tilde{h}}^C\right\},\\
C_{\tilde{h}}^+ &:= \left\{\mu \in C_{\tilde{h}} :\ \mu(\mathbf{x}) \geq 0\ \forall\, \mathbf{x} \in \mathcal{N}_{\tilde{h}}^C\right\},\\
L_{\hat{h}} &:= \left\{\kappa :\ \kappa \in P_0^{\hat{h}}(e),\ \forall\, e \in \mathcal{E}_{\tilde{h}}^A\right\},\\
\Lambda_{\hat{h}} &:= \{\kappa \in L_{\hat{h}} :\ |\kappa| \leq 1 \text{ a.e. on } \Gamma_C\}.
\end{aligned}
\tag{3.61}
$$

Finally, we set

$$
\widetilde{\mathbf{X}}_h := \mathbf{X}_h \times \Lambda_{\hat{h}}, \qquad \mathbf{Y}_h := \mathbf{M}_h \times S_h \times \mathbf{N}_{\tilde{h}} \times C_{\tilde{h}}, \qquad \widetilde{\mathbf{Y}}_h := \mathbf{Y}_h \times C_{\tilde{h}}^+.
$$

### 3.2.1. Approximation of the dual-dual variational inequality problem

To approximate the dual-dual variational inequality problem (3.41) we propose the following solution scheme, using the above definitions of the finite element spaces:

Find $(\sigma^h, \nu^{\hat{h}}; \mathbf{u}^h, \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}) \in \widetilde{\mathbf{X}}_h \times \widetilde{\mathbf{Y}}_h$ such that

$$
\begin{aligned}
&\tilde{a}(\sigma^h, \tau) + \hat{B}(\mathbf{u}^h, \eta^h, \boldsymbol{\varphi}^{\tilde{h}}; \tau) + d_{C,t}(\lambda_t^{\tilde{h}}, \tau) + d_{C,n}(\lambda_n^{\tilde{h}}, \tau) && = g(\tau) && \forall\, \tau \in \mathbf{X}_h\\
&b(\mathbf{v}, \sigma^h) && = -f(\mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{M}_h\\
&s(\xi, \sigma^h) && = 0 && \forall\, \xi \in S_h\\
&d_N(\boldsymbol{\psi}, \sigma^h) && = t_0(\boldsymbol{\psi}) && \forall\, \boldsymbol{\psi} \in \mathbf{N}_{\tilde{h}}\\
&d_{C,t}(\mu_t, \sigma^h) - q(\mu_t, \nu^{\hat{h}}) && = 0 && \forall\, \mu_t \in C_{\tilde{h}}\\
&d_{C,n}(\mu_n - \lambda_n^{\tilde{h}}, \sigma^h) && \leq 0 && \forall\, \mu_n \in C_{\tilde{h}}^+\\
&q(\kappa - \nu^{\hat{h}}, \lambda_t^{\tilde{h}}) && \leq 0 && \forall\, \kappa \in \Lambda_{\hat{h}}
\end{aligned}
\tag{3.62}
$$

**Corollary 3.24:**
From the inequalities in (3.62) we conclude

$$d_{C,n}(\lambda_n^{\tilde{h}}, \sigma^h) = 0. \tag{3.63}$$

If furthermore $\hat{h} = \tilde{h}$, then we have

$$\nu^{\hat{h}} \lambda_t^{\tilde{h}} = |\lambda_t^{\tilde{h}}| \quad \text{on } \Gamma_C \qquad \text{yielding} \qquad q(\lambda_t^{\tilde{h}}, \nu^{\hat{h}}) = j(\lambda_t^{\tilde{h}}). \tag{3.64}$$

*Proof.* Taking $\mu_n = 0 \in C_{\tilde{h}}^+$ and $\mu_n = 2\lambda_n^{\tilde{h}} C_{\tilde{h}}^+$ in $(3.62)_6$ we conclude assertion (3.63). From the assumption $\hat{h} = \tilde{h}$ we have $\operatorname{sign}(\lambda_t^{\tilde{h}}) \in \Lambda_{\hat{h}}$. Then, (3.64) follows analogously to the argumentation in Remark 3.15. □

**Proposition 3.25:**
Analogously to the continuous case in Corollary 3.16 we conclude for the normal stress $\sigma_n^h$ on $\Gamma_C$

$$\sigma_n^h \leq 0 \qquad \text{on } \Gamma_C. \tag{3.65}$$

If furthermore $A_C \subset \Gamma_C'$, with

$$\Gamma_C' := \Gamma_C \setminus \{e \in \mathcal{E}_{\tilde{h}}^C : \bar{e} \cap \partial\Gamma_C \neq \emptyset\}, \tag{3.66}$$

then we have for the tangential stress $\sigma_t^h$ on $\Gamma_C$

$$|\sigma_t^h| \leq \mathcal{F} \qquad \text{on } \Gamma_C. \tag{3.67}$$

*Proof.* From (3.63) and $\mu_n \geq 0$ on $\Gamma_C$ we get (3.65). For the second assertion we first observe, that the restriction of $(3.62)_5$ to the inactive set $I_C$ leads to $\sigma_t^h = 0$ on $I_C$ and since $\mathcal{F}$ is positive (3.67) holds on $I_C$. Due to $\nu^{\hat{h}} \in \Lambda_{\hat{h}}$ we have

$$q(\mu_t, \nu^{\hat{h}}) = \int_{A_C} \mathcal{F}\, \mu_t \nu^{\hat{h}}\, ds \leq \int_{\Gamma_C} \mathcal{F} |\mu_t|\, ds = j(\mu_t) \qquad \forall\, \mu_t \in C_{\tilde{h}}.$$

Hence we conclude with $(3.62)_5$

$$\begin{aligned}
& d_{C,t}(\mu_t, \sigma^h) - j(\mu_t) \leq d_{C,t}(\mu_t, \sigma^h) - q(\mu_t, \nu^{\hat{h}}) = 0 && \forall\, \mu_t \in C_{\tilde{h}}, \\
\Leftrightarrow\quad & \int_{\Gamma_C} \left(\mu_t \sigma_t^h - |\mu_t| \mathcal{F}\right) ds \leq 0 && \forall\, \mu_t \in C_{\tilde{h}}, \\
\Leftrightarrow\quad & \int_{\Gamma_C} |\mu_t| \left(\sigma_t^h \operatorname{sign}(\mu_t) - \mathcal{F}\right) ds \leq 0 && \forall\, \mu_t \in C_{\tilde{h}}, \\
\Leftrightarrow\quad & \sigma_t^h \operatorname{sign}(\mu_t) - \mathcal{F} \leq 0 && \forall\, \mu_t \in C_{\tilde{h}}.
\end{aligned}$$

Choosing $\mu_t \in C_{\tilde{h}}$ such that $\mu_t|_{\Gamma'_C} = \pm 1$ we have

$$\sigma_t^h - \mathcal{F} \leq 0 \qquad \wedge \qquad -\sigma_t^h - \mathcal{F} \leq 0 \qquad \text{on } \Gamma'_C \supset A_C$$

and therefore (3.67) is also valid on $A_C$. □

**Remark 3.26:** *If no friction occurs, i.e. $\mathcal{F} \equiv 0$, we have the following discrete version of the variational inequality problem* (3.39) *for contact problems without friction:*

*Find $(\sigma^h; \mathbf{u}^h, \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}) \in \mathbf{X}_h \times \widetilde{\mathbf{Y}}_h$ such that*

$$\begin{aligned}
\tilde{a}(\sigma^h, \tau) + \hat{B}(\mathbf{u}^h, \eta^h, \boldsymbol{\varphi}^{\tilde{h}}; \tau) + d_{C,t}(\lambda_t^{\tilde{h}}, \tau) + d_{C,n}(\lambda_n^{\tilde{h}}, \tau) \quad &= g(\tau) && \forall\, \tau \in \mathbf{X}_h \\
b(\mathbf{v}, \sigma^h) \quad &= -f(\mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{M}_h \\
s(\xi, \sigma^h) \quad &= 0 && \forall\, \xi \in S_h \\
d_N(\boldsymbol{\psi}, \sigma^h) \quad &= t_0(\boldsymbol{\psi}) && \forall\, \boldsymbol{\psi} \in \mathbf{N}_{\tilde{h}} \\
d_{C,t}(\mu_t, \sigma^h) \quad &= 0 && \forall\, \mu_t \in C_{\tilde{h}} \\
d_{C,n}(\mu_n - \lambda_n^{\tilde{h}}, \sigma^h) \quad &\leq 0 && \forall\, \mu_n \in C_{\tilde{h}}^+.
\end{aligned} \tag{3.68}$$

**Remark 3.27:** *In general the subsets for the discretization of the convex sets in the variational inequalities are not conform. However this does not hold for the discrete sets $C_{\tilde{h}}^+$ and $\Lambda_{\hat{h}}$. In short, we have*

$$C_{\tilde{h}}^+ \subset \widetilde{H}_+^{\frac{1}{2}}(\Gamma_C) \qquad \textit{and} \qquad \Lambda_{\hat{h}} \subset \Lambda.$$

*This follows from the fact that we use linear functions for $C_{\tilde{h}}^+$ and demand that all functions in $C_{\tilde{h}}^+$ have to be positive in all vertices of $\mathcal{N}_{\tilde{h}}^C$. This just means that we only allow linear combinations of basis functions in $C_{\tilde{h}}^+$ with positive coefficients and as the sum of positive functions is positive again we conclude the first assertion. The second assertion is obvious by the construction of the finite element space $\Lambda_{\hat{h}}$.*

### 3.2.2. Discrete Inf-Sup conditions

In order to derive an error analysis we have to assure, that the discrete versions of the inf-sup conditions hold for the mixed finite element spaces in (3.61). For the first discrete inf-sup condition (3.69) we collect known results from literature, similar to the continuous case in Section 3.1.6. However for the sake of completeness we show a sketch of the proof and name the right reference in each step of the proof. Let us state the first discrete inf-sup condition.

**Lemma 3.28:**
The bilinear form $B$ defined in (3.46) satisfies the following discrete inf-sup condition: $\exists\, C_0,\ \beta_1\ >\ 0$, independent of $h$ and $\tilde{h}$ such that for all $h \le C_0\tilde{h}$ and $(\mathbf{v}^h, \xi^h, \boldsymbol{\psi}^{\tilde{h}}, \mu_t^{\tilde{h}}, \mu_n^{\tilde{h}}) \in \widetilde{\mathbf{Y}}_h$

$$\sup_{0\neq\tau^h\in\mathbf{X}_h} \frac{B(\tau^h, (\mathbf{v}^h, \xi^h, \boldsymbol{\psi}^{\tilde{h}}, \mu_t^{\tilde{h}}, \mu_n^{\tilde{h}}))}{\|\tau^h\|_{\mathbf{X}}} \ge \beta_1 \|(\mathbf{v}^h, \xi^h, \boldsymbol{\psi}^{\tilde{h}}, \mu_t^{\tilde{h}}, \mu_n^{\tilde{h}})\|_{\widetilde{\mathbf{Y}}'}. \tag{3.69}$$

*Proof.* The proof is divided into four steps. In [5, see Lemma 4.4], where Arnold, Brezzi and Douglas Jr. introduce the PEERS elements to discretize the triple of stress, displacement and rotation in plane elasticity, the authors show the discrete inf-sup condition for the triple of functions. For this we define

$$\hat{\mathbf{X}}_h := \left\{ \tau^h \in \mathbf{X}_h :\ \int_\Omega \operatorname{tr}(\tau^h)\, dx = 0 \right\},$$

which allows us to decompose $\mathbf{X}_h = \hat{\mathbf{X}}_h + \mathbb{R}\mathbf{I}$ and now Lemma 4.4. in [5] states

$$\exists\, C_1 > 0: \quad \sup_{0\neq\tau^h\in\hat{\mathbf{X}}_h} \frac{b(\mathbf{v}^h, \tau^h) + s(\xi^h, \tau^h)}{\|\tau^h\|_{\mathbf{X}}} \ge C_1 \|\mathbf{v}^h, \xi^h\|_{\mathbf{L}^2(\Omega)\times L^2(\Omega)} \quad \forall\, (\mathbf{v}^h, \xi^h) \in \mathbf{M}_h \times S_h.$$

Since the other terms in $B(\cdot,\cdot)$ are all bilinear forms on boundary parts we proceed analogously to Gatica and Stephan in [44, see the proof of Theorem 4.1]. That is, for all $(\mathbf{v}^h, \xi^h, \boldsymbol{\psi}^{\tilde{h}}, \mu_t^{\tilde{h}}, \mu_n^{\tilde{h}}) \in \widetilde{\mathbf{Y}}_h$ we have

$$\begin{aligned}
\sup_{0\neq\tau^h\in\mathbf{X}_h} \frac{B(\tau^h, (\mathbf{v}^h, \xi^h, \boldsymbol{\psi}^{\tilde{h}}, \mu_t^{\tilde{h}}, \mu_n^{\tilde{h}}))}{\|\tau^h\|_{\mathbf{X}}} &\ge \sup_{0\neq\tau^h\in\hat{\mathbf{X}}_h} \frac{B(\tau^h, (\mathbf{v}^h, \xi^h, 0, 0, 0))}{\|\tau^h\|_{\mathbf{X}}} \\
&= \sup_{0\neq\tau^h\in\hat{\mathbf{X}}_h} \frac{b(\mathbf{v}^h, \tau^h) + s(\xi^h, \tau^h)}{\|\tau^h\|_{\mathbf{X}}} \ge C_1 \|\mathbf{v}^h, \xi^h\|_{\mathbf{L}^2(\Omega)\times L^2(\Omega)}.
\end{aligned}$$

In the next two steps we apply Lemma 4.2. of Gatica, Márquez and Meddahi [42] and use Lemma 3.3. of Babuška and Gatica [9] to show the existence of some $\tilde{C}_0$, $C_2 > 0$ independent of $h$ and $\tilde{h}$ such that for all $h \le \tilde{C}_0\tilde{h}$ the following estimate holds

$$\sup_{0\neq\tau^h\in\mathbf{X}_h} \frac{B(\tau^h, (\mathbf{v}^h, \xi^h, \boldsymbol{\psi}^{\tilde{h}}, \mu_t^{\tilde{h}}, \mu_n^{\tilde{h}}))}{\|\tau^h\|_{\mathbf{X}}} \ge C_2 \|\boldsymbol{\psi}^{\tilde{h}}\|_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)} - \|\xi^h\|_{L^2(\Omega)} \quad \forall\, (\mathbf{v}^h, \xi^h, \boldsymbol{\psi}^{\tilde{h}}, \mu_t^{\tilde{h}}, \mu_n^{\tilde{h}}) \in \widetilde{\mathbf{Y}}_h.$$

The second norm on the right hand side arises from the fact, that the interpolation operator used within the proof of Lemma 4.2 in [42] does not necessarily preserve the symmetry of the stress tensor. Furthermore, the first norm on the right hand side is reached by applying first Lemma 3.2. of [9] and then Lemma 3.3. of [9].

Finally, since the above estimate holds also true for scalar functions on some boundary parts we can apply it to the last two Lagrange multipliers $\mu_t^{\tilde{h}}$ and $\mu_n^{\tilde{h}}$ by using the same arguments as in the continuous case in Lemma 3.19. Hence there exist some positive constants $\bar{C}_0, \hat{C}_0, C_3$ and $C_4$ such that for all $h \leq \bar{C}_0\tilde{h}$

$$\sup_{0\neq\tau^h\in\mathbf{X}_h} \frac{B(\tau^h,(\mathbf{v}^h,\xi^h,\boldsymbol{\psi}^{\tilde{h}},\mu_t^{\tilde{h}},\mu_n^{\tilde{h}}))}{\|\tau^h\|_{\mathbf{X}}} \geq C_3\|\mu_t^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} - \|\xi^h\|_{L^2(\Omega)} \quad \forall\ (\mathbf{v}^h,\xi^h,\boldsymbol{\psi}^{\tilde{h}},\mu_t^{\tilde{h}},\mu_n^{\tilde{h}}) \in \widetilde{\mathbf{Y}}_h$$

and for all $h \leq \hat{C}_0\tilde{h}$

$$\sup_{0\neq\tau^h\in\mathbf{X}_h} \frac{B(\tau^h,(\mathbf{v}^h,\xi^h,\boldsymbol{\psi}^{\tilde{h}},\mu_t^{\tilde{h}},\mu_n^{\tilde{h}}))}{\|\tau^h\|_{\mathbf{X}}} \geq C_4\|\mu_n^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} - \|\xi^h\|_{L^2(\Omega)} \quad \forall\ (\mathbf{v}^h,\xi^h,\boldsymbol{\psi}^{\tilde{h}},\mu_t^{\tilde{h}},\mu_n^{\tilde{h}}) \in \widetilde{\mathbf{Y}}_h.$$

Multiplying the last three inequalities by $\frac{C_1}{4}$ and adding up all four inequalities we arrive at the desired estimate with $C_0 := \min\{\tilde{C}_0, \bar{C}_0, \hat{C}_0\}$ and $\beta_1 := \frac{C_1\cdot\min\{1,C_2,C_3,C_4\}}{4+3C_1}$. □

Before we state the second discrete inf-sup condition we make the following observation. We want to show the inf-sup condition for each choice of $\mathcal{F}$, in particular we can assume $\mathcal{F} \equiv 1$. Therefore the active set is the whole contact boundary $A_C \equiv \Gamma_C$. Without loss of generality we can assume that $\Gamma_C$ is a line segment. In the following we show exemplary, why we have to use a coarser meshsize for $\Lambda_{\hat{h}}$.

Assume $\tilde{h} = \hat{h}$, then we can define for an arbitrary set of edges $\mathcal{E}_{\tilde{h}}^C$

$$\bar{\kappa} := \sum_{i=1}^{\tilde{N}_C}(-1)^i\chi_i(\mathbf{x}) \in \Lambda_{\hat{h}}, \qquad \text{with} \quad \chi_i(\mathbf{x}) := \begin{cases} 1, & \text{on } e_i, \\ 0, & \text{else.} \end{cases}$$

Now we observe that for all $\phi \in C_{\tilde{h}}$ we have $q(\phi,\bar{\kappa}) = 0$. To verify this we take an arbitrary $\phi \in C_{\tilde{h}}$

$$\phi(\mathbf{x}) := \sum_{i=1}^{\tilde{N}_C-1} c_i\psi_i(\mathbf{x}) \qquad \text{with } c_i \in \mathbb{R} \text{ and } \psi_i \text{ basis function in } C_{\tilde{h}}$$

and compute

$$\phi\bar{\kappa} = \left(\sum_{i=1}^{\tilde{N}_C-1} c_i\psi_i(\mathbf{x})\right)\left(\sum_{i=1}^{\tilde{N}_C}(-1)^i\chi_i(\mathbf{x})\right) = \sum_{i=1}^{\tilde{N}_C-1} c_i\left((-1)^i + (-1)^{i+1}\right) = 0.$$

Since $\mathcal{F}$ was assumed to be constant the assertion follows. In Figure 3.7 we illustrate this observation exemplary. The dashed blue lines show each hat function separately. The observation shows, that the discrete inf-sup condition does not hold for arbitrary $\mathcal{F}$, if the meshsizes for $\Lambda_{\hat{h}}$ and $C_{\tilde{h}}$ are equal.

Furthermore, we cannot regard $\mathcal{F}$ as a multiplier on $C_{\tilde{h}}$ since it can have jumps. As described in the beginning of this chapter our aim is to solve the contact problem

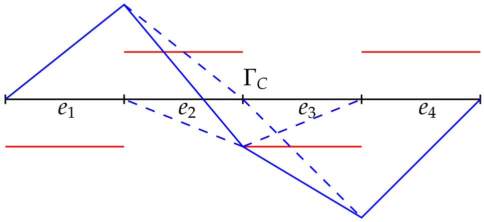



Figure 3.7.: Example for $\phi \in C_{\tilde{h}}$ (blue) and $\bar{\kappa} \in \Lambda_{\tilde{h}}$ (red) on the contact boundary $\Gamma_C$.

(3.1) with Coulomb friction (3.2). In Section 3.3 we explain the solution algorithm to solve the discrete variational inequality problem (3.62), where we update the discrete friction function after each iteration according to (3.2). This means that we define $\mathcal{F}$ in terms of the normal part of the approximated stress $\sigma^h$ on $\Gamma_C$ which is a piecewise constant function. For this reason we have to project $\mathcal{F}$ onto $L_{\hat{h}}$. This projection will be explained later in detail in Section 3.3. For our current purpose it is sufficient to know that we deal with a friction function $\mathcal{F} \in L_{\hat{h}}$ from now on. In the following we will present two different inf-sup conditions. The first one will be valid for all friction functions $\mathcal{F} \in L_{\hat{h}}$. Unfortunately it deals with a mesh dependent norm. For the second version some more restrictions on the friction function are necessary. If we know in advance, that $\operatorname{dist}(\overline{A_C}, \partial\Gamma_C) > \varepsilon > 0$, then the coarsening of the mesh described above is not required.

Let us introduce the following norms defined in Roberts and Thomas [77, see Section 6 in Chapter II]

$$
\begin{aligned}
|||\mathbf{v}|||_{H(\mathrm{div},T)} &:= \left\{ \|\mathbf{v}\|^2_{\mathbf{L}^2(T)} + \hat{h}^2 \|\mathrm{div}\,\mathbf{v}\|^2_{L^2(T)} \right\}^{\frac{1}{2}} && \forall\ \mathbf{v} \in H(\mathrm{div}, T),\ T \in \mathcal{T}_{\hat{h}}, \\
|||\kappa|||_{-\frac{1}{2},\partial T} &:= \inf_{\substack{\mathbf{v} \in H(\mathrm{div},T) \\ \mathbf{v}\cdot\mathbf{n} = \kappa|_{\partial T}}} |||\mathbf{v}|||_{H(\mathrm{div},T)} && \forall\ \kappa \in H^{-\frac{1}{2}}(\partial T),\ T \in \mathcal{T}_{\hat{h}}, \\
|||\kappa|||_{-\frac{1}{2},\partial\mathcal{T}_{\hat{h}}} &:= \left\{ \sum_{T \in \mathcal{T}_{\hat{h}}} |||\kappa|||^2_{-\frac{1}{2},\partial T} \right\}^{\frac{1}{2}} && \forall\ \kappa = (\kappa_T) \in \prod_{T \in \mathcal{T}_{\hat{h}}} H^{-\frac{1}{2}}(\partial T).
\end{aligned}
$$

We will need the inequality (18.23) defined in [77, Chapter II, Section 6]

$$
|||\kappa|||_{-\frac{1}{2},\partial\mathcal{T}_{\hat{h}}} \le C\hat{h}^{\frac{1}{2}} \|\kappa\|_{L^2(\partial\mathcal{T}_{\hat{h}})}. \tag{3.70}
$$

With the above norms we define the restriction to the boundary part as follows. For $\kappa \in L_{\hat{h}}$

$$
\|\kappa\|_{-\frac{1}{2},\hat{h}} := |||\kappa^*|||_{-\frac{1}{2},\partial\mathcal{T}_{\hat{h}}} \tag{3.71}
$$

where $\kappa^* \in \prod\limits_{T\in\mathcal{T}_{\hat{h}}} H^{-\frac{1}{2}}(\partial T)$ is the extension of $\kappa$ by zero.

Finally, we notice, that $\hat{h} = 2\tilde{h}$ is sufficient to prove the desired inf-sup condition. This will be explicated in the following lemma.

**Lemma 3.29:**
The bilinear form $q(\cdot,\cdot)$ satisfies the following discrete inf-sup condition: $\exists\, \hat{C}_0 > 1$ and $\beta_2 > 0$, where $\beta_2$ depends on $\mathcal{F}$, such that for all $\tilde{h} \geq \hat{C}_0\, \hat{h}$ and for all $\kappa \in L_{\hat{h}}$

$$\sup_{\mu\in C_{\tilde{h}}} \frac{q(\mu,\kappa)}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} \geq \beta_2\, \|\kappa\|_{-\frac{1}{2},\hat{h}}. \tag{3.72}$$

*Proof.* Let us assume $\hat{C}_0 = 2$. As written above we have $\mathcal{F} \in L_{\hat{h}}$. Therefore it holds $\mathcal{F}\,\kappa \in L_{\hat{h}}$ for all $\kappa \in L_{\hat{h}}$ with

$$\kappa = \sum_{i=1}^{\hat{N}_A} c_i \chi_i(\mathbf{x}) \qquad \text{and} \qquad \mathcal{F}\,\kappa = \sum_{i=1}^{\hat{N}_A} f_i\, c_i \chi_i(\mathbf{x}).$$

Due to the choice of $\hat{C}_0$ we can find for all edges $\hat{e}_i \in \mathcal{E}^A_{\hat{h}}$ a vertice $p_i \in \mathcal{N}^C_{\tilde{h}}$, which is the midpoint of $\hat{e}_i$. We set $\hat{I}$ as the index set of all vertices that are a midpoint of some edge $\hat{e}_i \in \mathcal{E}^A_{\hat{h}}$. For all $i \in \hat{I}$ we identify with $\hat{e}_i \in \mathcal{E}^A_{\hat{h}}$ the edge on $A_C$ and with $p_i$ the corresponding vertice which is the midpoint of $\hat{e}_i$. We need the global inverse inequality for $\mu \in C_{\tilde{h}}$, see e.g. Chapter 10.2 in Steinbach [80], which reads

$$\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \leq \tilde{C}\, \tilde{h}^{-\frac{1}{2}} \|\mu\|_{L^2(\Gamma_C)} = C\, \hat{h}^{-\frac{1}{2}} \|\mu\|_{L^2(\Gamma_C)}.$$

We define

$$\bar{\mu} := \sum_{i\in\hat{I}} \frac{c_i}{f_i} \psi_i(\mathbf{x}) \ \in C_{\tilde{h}},$$

then we have with $\min\limits_{i\in\hat{I}} f_i > C_{\mathcal{F}} > 0$

$$\begin{aligned}
\|\bar{\mu}\|^2_{L^2(\Gamma_C)} &= \int\limits_{\Gamma_C} \left(\sum_{i\in\hat{I}} \frac{c_i}{f_i}\psi_i(\mathbf{x})\right)^2 ds = \int\limits_{A_C} \left(\sum_{i\in\hat{I}} \frac{c_i}{f_i}\psi_i(\mathbf{x})\right)^2 ds = \sum_{i\in\hat{I}} \frac{c_i^2}{f_i^2} \int\limits_{\hat{e}_i} \psi_i(\mathbf{x})^2\, ds \\
&\leq \frac{1}{C^2_{\mathcal{F}}} \hat{h} \sum_{i\in\hat{I}} c_i^2 = \frac{1}{C^2_{\mathcal{F}}} \|\kappa\|^2_{L^2(A_C)}
\end{aligned}$$

and

$$q(\bar{\mu},\kappa) = \int\limits_{A_C} \left(\sum_{i=1}^{\hat{N}_A} f_i\, c_i \chi_i(\mathbf{x})\right)\left(\sum_{i\in\hat{I}} \frac{c_i}{f_i}\psi_i(\mathbf{x})\right) ds = \sum_{i\in\hat{I}} c_i^2 \int\limits_{\hat{e}_i} \psi_i(\mathbf{x})\, ds = \hat{h} \sum_{i\in\hat{I}} c_i^2 = \|\kappa\|^2_{L^2(A_C)}.$$

Now with (3.70) and the definition of the mesh depedant norm (3.71) we conclude

$$\sup_{\mu\in C_{\tilde{h}}} \frac{q(\mu,\kappa)}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} \geq \frac{q(\bar{\mu},\kappa)}{\|\bar{\mu}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} \geq C\,\hat{h}^{\frac{1}{2}}\,\frac{q(\bar{\mu},\kappa)}{\|\bar{\mu}\|_{L^2(\Gamma_C)}} \geq C_{\mathcal{F}}\,\hat{h}^{\frac{1}{2}}\,\|\kappa\|_{L^2(A_C)} \geq C_{\mathcal{F}}\,\|\kappa\|_{-\frac{1}{2},\hat{h}}.$$

Taking $\beta_2 := C_{\mathcal{F}}$ we finish the proof. □

**Remark 3.30:** *The inf-sup condition is proved in the same way, when the following mesh dependent norm is applied instead, see e.g. Braess and Dahmen [15]. For any $\kappa \in L_{\hat{h}}$ we have $\kappa \in L^2(A_C)$, therefore we set*

$$\|\kappa\|^2_{-\frac{1}{2},\hat{h}} := \hat{h}\,\|\kappa\|^2_{L^2(A_C)}.$$

As mentioned above we can prove the inf-sup condition without coarsening of the mesh. Let us demand the following additional assumption on the friction function $\mathcal{F}$

$$\exists\, \varepsilon > 0 : \qquad \operatorname{dist}(\overline{A_C}, \partial\Gamma_C) \geq \varepsilon. \tag{3.73}$$

Defining $C'_{\tilde{h}} := \left\{\mu \in C(\Gamma'_C) :\ \mu \in P_1^{\tilde{h}}(e), \forall\, e \in \mathcal{E}^C_{\tilde{h}} \cap \Gamma'_C\right\}$, where $\Gamma'_C$ is defined in (3.66), there holds for all $\tilde{h} \leq \varepsilon :\ A_C \subset \Gamma'_C$ and $C'_{\tilde{h}} \subset H^{\frac{1}{2}}(\Gamma'_C)$. According to Corollary 2.4 in Chapter I of Girault and Raviart [51] we define the $H^{-\frac{1}{2}}(A_C)$-norm as follows

$$\|\kappa\|_{H^{-\frac{1}{2}}(A_C)} := \inf_{\substack{\mathbf{q}\in H(\mathrm{div},\Omega)\\ \mathbf{q}\cdot\mathbf{n}=\kappa|_{A_C}}} \|\mathbf{q}\|_{H(\mathrm{div},\Omega)}$$

and we can prove the following inf-sup condition.

**Lemma 3.31:**
Let the friction function $\mathcal{F}$ satisfy (3.73). Then the bilinear form $q(\cdot,\cdot)$ satisfies the following inf-sup condition for $\hat{h} = \tilde{h}$: $\exists\, \beta_2 > 0$, where $\beta_2$ depends on $\mathcal{F}$, such that for all $\kappa \in L_{\tilde{h}}$

$$\sup_{\mu\in C_{\tilde{h}}} \frac{q(\mu,\kappa)}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} \geq \beta_2\,\|\kappa\|_{H^{-\frac{1}{2}}(A_C)}. \tag{3.74}$$

*Proof.* Due to the above definition of $\Gamma'_C$ and Lemma 3.21 we can find for any $\mu \in C'_{\tilde{h}}$ an extension $\mu^* \in C_{\tilde{h}}$ and $C > 0$ with

$$C\,\|\mu^*\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \leq \|\mu\|_{H^{\frac{1}{2}}(\Gamma'_C)}.$$

For some arbitrary $\kappa \in L_{\tilde{h}}$ we consider the following auxiliary problem:

Find $v \in H^1_D(\Omega) := \{v \in H^1(\Omega) : \ v = 0 \ \text{on} \ \Gamma_D\}$ such that

$$\begin{aligned} -\Delta v + v &= 0 && \text{in } \Omega, \\ v &= 0 && \text{on } \Gamma_D, \\ \frac{\partial v}{\partial \mathbf{n}} &= 0 && \text{on } \Gamma_N \cup I_C, \\ \frac{\partial v}{\partial \mathbf{n}} &= \kappa && \text{on } A_C. \end{aligned}$$

For the corresponding variational problem we define

$$V_{\tilde{h}} := \left\{ v \in H^1_D(\Omega) : \ v \in P_1^{\tilde{h}}(T), \ \forall T \in \mathcal{T}_{\tilde{h}} \right\}.$$

Then the variational problem reads:

Find $v^{\tilde{h}} \in V_{\tilde{h}}$ such that

$$\int\limits_{\Omega} \nabla v^{\tilde{h}} \cdot \nabla w \, dx + \int\limits_{\Omega} v^{\tilde{h}} \, w \, dx = \int\limits_{A_C} \kappa \, w \, ds \qquad \forall \ w \in V_{\tilde{h}}.$$

Since $\Gamma_D \neq \emptyset$ we have from the Poincaré-Friedrichs inequality that the bilinear forms in the above problem are coercive. Setting $\mathbf{p}^{\tilde{h}} := \nabla v^{\tilde{h}}$ we observe $\mathbf{p}^{\tilde{h}} \cdot \mathbf{n} = \kappa$ on $A_C$ and furthermore

$$\begin{aligned} & \operatorname{div} \mathbf{p}^{\tilde{h}} = \Delta v^{\tilde{h}} = v^{\tilde{h}} \quad \text{in } \Omega \qquad \Rightarrow \qquad \mathbf{p}^{\tilde{h}} \in H(\operatorname{div}, \Omega), \\ \text{and} \qquad & \|\mathbf{p}^{\tilde{h}}\|_{H(\operatorname{div},\Omega)} = \|v^{\tilde{h}}\|_{H^1(\Omega)}. \end{aligned}$$

From the definition of the $H^{-\frac{1}{2}}$-norm and the coercivity of the bilinear forms above we have

$$\begin{aligned} \|\kappa\|_{H^{-\frac{1}{2}}(A_C)} = \inf_{\substack{\mathbf{q} \in H(\operatorname{div},\Omega) \\ \mathbf{q}\cdot\mathbf{n} = \kappa|_{A_C}}} \|\mathbf{q}\|_{H(\operatorname{div},\Omega)} &\leq \|\mathbf{p}^{\tilde{h}}\|_{H(\operatorname{div},\Omega)} = \|v^{\tilde{h}}\|_{H^1(\Omega)} \\ &= \Big( \int\limits_{\Omega} \nabla v^{\tilde{h}} \cdot \nabla v^{\tilde{h}} \, dx + \int\limits_{\Omega} v^{\tilde{h}} \, v^{\tilde{h}} \, dx \Big)^{\frac{1}{2}} \\ &= \langle \kappa, v^{\tilde{h}} \rangle_{A_C}^{\frac{1}{2}} \leq \|\kappa\|^{\frac{1}{2}}_{H^{-\frac{1}{2}}(A_C)} \| v^{\tilde{h}}|_{A_C} \|^{\frac{1}{2}}_{H^{\frac{1}{2}}(A_C)} \end{aligned}$$

yielding

$$\|\kappa\|_{H^{-\frac{1}{2}}(A_C)} \leq \| v^{\tilde{h}}|_{A_C} \|_{H^{\frac{1}{2}}(A_C)}.$$

As $A_C \subset \Gamma'_C$ we can find some $\bar{\mu} \in C'_{\tilde{h}}$ with $\bar{\mu} = v^{\tilde{h}}|_{A_C}$. Now we have with the trace theorem and the above observations

$$\begin{aligned} \|\kappa\|_{H^{-\frac{1}{2}}(A_C)} \| v^{\tilde{h}}|_{A_C} \|_{H^{\frac{1}{2}}(A_C)} \leq \| v^{\tilde{h}}|_{A_C} \|^2_{H^{\frac{1}{2}}(A_C)} \leq \|v^{\tilde{h}}\|^2_{H^1(\Omega)} &= \int\limits_{\Omega} \nabla v^{\tilde{h}} \cdot \nabla v^{\tilde{h}} \, dx + \int\limits_{\Omega} v^{\tilde{h}} \, v^{\tilde{h}} \, dx \\ &= \langle \kappa, v^{\tilde{h}} \rangle_{A_C} = \langle \kappa, \bar{\mu} \rangle_{A_C}. \end{aligned} \tag{3.75}$$

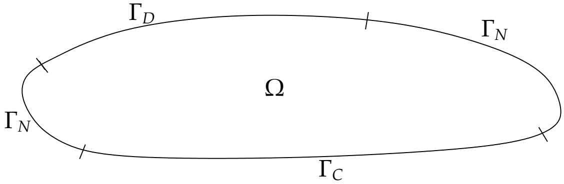



Figure 3.8.: Domain with $\operatorname{dist}(\Gamma_C, \Gamma_D) > \varepsilon > 0$.

Finally, the friction function is piecewise constant on $A_C$ and therefore we can find some $C_{\mathcal{F}} > 0$ with $\min_{\mathbf{x}\in A_C} \mathcal{F}(\mathbf{x}) > C_{\mathcal{F}}$. With (3.75) we conclude

$$\|\kappa\|_{H^{-\frac{1}{2}}(A_C)} \leq \frac{\langle \kappa, \bar{\mu} \rangle_{A_C}}{\|\bar{\mu}\|_{H^{\frac{1}{2}}(A_C)}} \leq C \frac{\langle \kappa, \bar{\mu}^* \rangle_{A_C}}{\|\bar{\mu}^*\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} \leq \frac{C}{C_{\mathcal{F}}} \sup_{\mu \in C^{\tilde{h}}} \frac{q(\mu, \kappa)}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}}$$

and thus the proof is complete. □

**Remark 3.32:** *In the following we will use the mesh dependent norm $\|\cdot\|_{-\frac{1}{2},\hat{h}}$. All results apply as well for $\|\cdot\|_{H^{-\frac{1}{2}}(A_C)}$, whenever the friction function $\mathcal{F}$ fulfills assumption* (3.73).

**Remark 3.33:** *If the contact boundary and the Dirichlet boundary have positive distance, see e.g. Figure 3.8, then we deal with $H^{\frac{1}{2}}(\Gamma_C)$ for $\lambda_t$ and $\lambda_n$. The above theory is still valid, since we can regard the Lagrange multipliers on the boundary parts as one function on $\Gamma_C \cup \Gamma_N$. In this case we decompose $\tilde{\boldsymbol{\lambda}} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N \cup \Gamma_C)$ into the corresponding parts on the boundaries, where we approximate $\tilde{\boldsymbol{\lambda}}|_{\Gamma_N}$ with some $\boldsymbol{\varphi} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)$ and take $\lambda_t$, $\lambda_n \in H^{\frac{1}{2}}(\Gamma_C)$ with $\lambda_t \mathbf{t} + \lambda_n \mathbf{n} = \tilde{\boldsymbol{\lambda}}|_{\Gamma_C}$. We approximate the Lagrange multipliers on $\Gamma_C$ with functions in*

$$\bar{C}_{\tilde{h}} := \left\{ \mu \in C(\Gamma_C) : \ \mu \in P_1^{\tilde{h}}(e), \ \forall\, e \in \mathcal{E}_{\tilde{h}}^C \right\}, \tag{3.76}$$

$$\bar{C}_{\tilde{h}}^+ := \left\{ \mu \in \bar{C}_{\tilde{h}} : \ \mu(\mathbf{x}) \geq 0 \ \forall\, \mathbf{x} \in \mathcal{N}_{\tilde{h}}^C \right\}. \tag{3.77}$$

*In this situation the inf-sup condition* (3.74) *in Lemma 3.31 with supremum over $\bar{C}_{\tilde{h}}$ is valid independently of assumption* (3.73)*. This follows trivially from the proof of the lemma.*

### 3.2.3. A priori error analysis

Collecting the continuous and discrete inf-sup conditions from above we can derive the following Céa-type estimate.

**Theorem 3.34:**
Let $(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \Lambda \times \widetilde{\mathbf{Y}}'$ and $(\sigma^h, \nu^{\hat{h}}; \mathbf{u}^h, \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}) \in \widetilde{\mathbf{X}}_h \times \widetilde{\mathbf{Y}}_h$ be the

solutions of the continuous and the discrete variational inequality problems (3.41) and (3.62), respectively. Then there exist some positive constants $C$, $C_1$ and $C_2$ such that the following a priori error estimates hold

$$\|\sigma-\sigma^h\|_X \le C\Big\{\inf_{\tau\in\mathbf{X}_h}\|\sigma-\tau\|_X+\inf_{\mathbf{v}\in\mathbf{M}_h}\|\mathbf{u}-\mathbf{v}\|_{\mathbf{L}^2(\Omega)}+\inf_{\xi\in S_h}\|\eta-\xi\|_{L^2(\Omega)}+\inf_{\psi\in\mathbf{N}_{\tilde h}}\|\boldsymbol{\varphi}-\boldsymbol{\psi}\|_{\widetilde{\mathbf{H}}^{\frac12}(\Gamma_N)}$$
$$+\inf_{\mu_t\in C_{\tilde h}}\|\lambda_t-\mu_t\|_{\widetilde{H}^{\frac12}(\Gamma_C)}+\inf_{\mu_n\in C^+_{\tilde h}}\|\lambda_n-\mu_n\|^{\frac12}_{\widetilde{H}^{\frac12}(\Gamma_C)}+\inf_{\kappa\in\Lambda_{\hat h}}\|\nu-\kappa\|^{\frac12}_{H^{-\frac12}(A_C)}\Big\}, \tag{3.78}$$

$$\|\mathbf{u}-\mathbf{u}^h\|_{\mathbf{L}^2(\Omega)}+\|\eta-\eta^h\|_{L^2(\Omega)}+\|\boldsymbol{\varphi}-\boldsymbol{\varphi}^{\tilde h}\|_{\widetilde{\mathbf{H}}^{\frac12}(\Gamma_N)}+\|\lambda_t-\lambda_t^{\tilde h}\|_{\widetilde{H}^{\frac12}(\Gamma_C)}+\|\lambda_n-\lambda_n^{\tilde h}\|_{\widetilde{H}^{\frac12}(\Gamma_C)}$$
$$\le\ C_1\Big\{\|\sigma-\sigma^h\|_X+\inf_{\mathbf{v}\in\mathbf{M}_h}\|\mathbf{u}-\mathbf{v}\|_{\mathbf{L}^2(\Omega)}+\inf_{\xi\in S_h}\|\eta-\xi\|_{L^2(\Omega)} \tag{3.79}$$
$$+\inf_{\psi\in\mathbf{N}_{\tilde h}}\|\boldsymbol{\varphi}-\boldsymbol{\psi}\|_{\widetilde{\mathbf{H}}^{\frac12}(\Gamma_N)}+\inf_{\mu_t\in C_{\tilde h}}\|\lambda_t-\mu_t\|_{\widetilde{H}^{\frac12}(\Gamma_C)}+\inf_{\mu_n\in C^+_{\tilde h}}\|\lambda_n-\mu_n\|_{\widetilde{H}^{\frac12}(\Gamma_C)}\Big\},$$

$$\|\nu-\nu^{\hat h}\|_{-\frac12,\hat h}\le C_2\Big\{\|\sigma-\sigma^h\|_X+\inf_{\kappa\in\Lambda_{\hat h}}\|\nu-\kappa\|_{L^2(A_C)}\Big\}. \tag{3.80}$$

*Proof.* We follow here the proof of Lemma 5.10 in Maischak [67]. For arbitrary $(\tau,\kappa;\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n)\in\mathbf{X}_h\times\Lambda_{\hat h}\times\mathbf{M}_h\times S_h\times\mathbf{N}_{\tilde h}\times C_{\tilde h}\times C^+_{\tilde h}$ we get the following equations from the first five equations in the variational inequality problems (3.41) and (3.62)

$$\tilde a(\sigma-\sigma^h,\tau)+B(\tau,(\mathbf{u}-\mathbf{u}^h,\eta-\eta^h,\boldsymbol{\varphi}-\boldsymbol{\varphi}^{\tilde h},\lambda_t-\lambda_t^{\tilde h},\lambda_n-\lambda_n^{\tilde h}))=0, \tag{3.81}$$

$$b(\mathbf{v},\sigma-\sigma^h)=0, \tag{3.82}$$

$$s(\xi,\sigma-\sigma^h)=0, \tag{3.83}$$

$$d_N(\boldsymbol{\psi},\sigma-\sigma^h)=0, \tag{3.84}$$

$$d_{C,t}(\mu_t,\sigma-\sigma^h)-q(\mu_t,\nu-\nu^{\hat h})=0. \tag{3.85}$$

Using the discrete inf-sup condition (3.69), equation (3.81) and the continuity of the bilinear forms involved we get

$$\beta_1\|(\mathbf{v}-\mathbf{u}^h,\xi-\eta^h,\boldsymbol{\psi}-\boldsymbol{\varphi}^{\tilde h},\mu_t-\lambda_t^{\tilde h},\mu_n-\lambda_n^{\tilde h})\|_{\widetilde{\mathbf{Y}}'}$$
$$\le\sup_{\tau\in\mathbf{X}_h}\frac{B(\tau,(\mathbf{v}-\mathbf{u}^h,\xi-\eta^h,\boldsymbol{\psi}-\boldsymbol{\varphi}^{\tilde h},\mu_t-\lambda_t^{\tilde h},\mu_n-\lambda_n^{\tilde h}))}{\|\tau\|_X}$$
$$=\sup_{\tau\in\mathbf{X}_h}\frac{\hat B(\mathbf{v}-\mathbf{u},\xi-\eta,\boldsymbol{\psi}-\boldsymbol{\varphi};\tau)+d_{C,t}(\mu_t-\lambda_t,\tau)+d_{C,n}(\mu_n-\lambda_n,\tau)-\tilde a(\sigma-\sigma^h,\tau)}{\|\tau\|_X}$$
$$\le C\Big\{\|\mathbf{u}-\mathbf{v}\|_{\mathbf{L}^2(\Omega)}+\|\eta-\xi\|_{L^2(\Omega)}+\|\boldsymbol{\varphi}-\boldsymbol{\psi}\|_{\widetilde{\mathbf{H}}^{\frac12}(\Gamma_N)}$$
$$+\|\lambda_t-\mu_t\|_{\widetilde{H}^{\frac12}(\Gamma_C)}+\|\lambda_n-\mu_n\|_{\widetilde{H}^{\frac12}(\Gamma_C)}+\|\sigma-\sigma^h\|_X\Big\}.$$

The triangle inequality applied to the left hand side concludes assertion (3.79). Analogously we use the discrete inf-sup condition (3.72), equation (3.85) and the continuity of the bilinear forms involved to estimate

$$\beta_2 \|\kappa - \nu^{\hat{h}}\|_{-\frac{1}{2},\hat{h}} \leq \sup_{\mu_t \in C_{\tilde{h}}} \frac{q(\mu_t, \kappa - \nu^{\hat{h}})}{\|\mu_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} = \sup_{\mu_t \in C_{\tilde{h}}} \frac{d_{C,t}(\mu_t, \sigma - \sigma^h) + q(\mu_t, \kappa - \nu)}{\|\mu_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}}$$
$$\leq C \left\{ \|\sigma - \sigma^h\|_{\mathbf{X}} + \|\mathcal{F}\|_{L^\infty(A_C)} \|\nu - \kappa\|_{H^{-\frac{1}{2}}(A_C)} \right\}.$$

Assertion (3.80) follows by applying the triangle inequality to the left hand side

$$\|\nu - \nu^{\hat{h}}\|_{-\frac{1}{2},\hat{h}} \leq \|\nu - \kappa\|_{-\frac{1}{2},\hat{h}} + \|\kappa - \nu^{\hat{h}}\|_{-\frac{1}{2},\hat{h}} \leq C \|\nu - \kappa\|_{L^2(A_C)} + \|\kappa - \nu^{\hat{h}}\|_{-\frac{1}{2},\hat{h}}.$$

Now let us define the bilinear form $\mathcal{A} : \ (\mathbf{X} \times H^{-\frac{1}{2}}(A_C) \times \widetilde{\mathbf{Y}}') \times (\mathbf{X} \times H^{-\frac{1}{2}}(A_C) \times \widetilde{\mathbf{Y}}') \to \mathbb{R}$ as follows

$$\mathcal{A}((\sigma, \nu, \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \ ; \ (\tau, \kappa, \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n)) := \tilde{a}(\sigma, \tau) + B(\tau, (\mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n)) + q(\mu_t, \nu)$$
$$- B(\sigma, (\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n)) - q(\lambda_t, \kappa).$$

Obviously $\mathcal{A}$ is continuous. To estimate the error in $\sigma$ we apply the coercivity of $\tilde{a}(\cdot, \cdot)$ with respect to the $\mathbf{L}^2(\Omega)$-norm and conclude

$$\begin{aligned}
\alpha \|\sigma - \sigma^h\|^2_{\mathbf{L}^2(\Omega)} &\leq \tilde{a}(\sigma - \sigma^h, \sigma - \sigma^h) \\
&= \mathcal{A}((\sigma - \sigma^h, \nu - \nu^{\hat{h}}, \mathbf{u} - \mathbf{u}^h, \eta - \eta^h, \boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t - \lambda_t^{\tilde{h}}, \lambda_n - \lambda_n^{\tilde{h}}) \ ; \\
&\qquad (\sigma - \sigma^h, \nu - \nu^{\hat{h}}, \mathbf{u} - \mathbf{u}^h, \eta - \eta^h, \boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t - \lambda_t^{\tilde{h}}, \lambda_n - \lambda_n^{\tilde{h}})) \\
&= \mathcal{A}((\sigma - \sigma^h, \nu - \nu^{\hat{h}}, \mathbf{u} - \mathbf{u}^h, \eta - \eta^h, \boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t - \lambda_t^{\tilde{h}}, \lambda_n - \lambda_n^{\tilde{h}}) \ ; \\
&\qquad (\sigma - \tau, \nu - \kappa, \mathbf{u} - \mathbf{v}, \eta - \xi, \boldsymbol{\varphi} - \boldsymbol{\psi}, \lambda_t - \mu_t, \lambda_n - \mu_n)) \\
&\quad + \mathcal{A}((\sigma - \sigma^h, \nu - \nu^{\hat{h}}, \mathbf{u} - \mathbf{u}^h, \eta - \eta^h, \boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t - \lambda_t^{\tilde{h}}, \lambda_n - \lambda_n^{\tilde{h}}) \ ; \\
&\qquad (\tau - \sigma^h, \kappa - \nu^{\hat{h}}, \mathbf{v} - \mathbf{u}^h, \xi - \eta^h, \boldsymbol{\psi} - \boldsymbol{\varphi}^{\tilde{h}}, \mu_t - \lambda_t^{\tilde{h}}, \mu_n - \lambda_n^{\tilde{h}})).
\end{aligned} \tag{3.86}$$

From the commuting diagram property, see Section 6 in Chapter II of Roberts and Thomas [77], we know that $\operatorname{div} \tau \in \mathbf{M}_h$ for all $\tau \in \mathbf{X}_h$. Furthermore, we obtain from (3.82) and the Cauchy-Schwarz inequality

$$\begin{aligned}
\|\operatorname{div}(\sigma - \sigma^h)\|^2_{\mathbf{L}^2(\Omega)} &= \left(\operatorname{div}(\sigma - \sigma^h) \, , \, \operatorname{div}(\sigma - \sigma^h)\right)_{\mathbf{L}^2(\Omega)} = \left(\operatorname{div}(\sigma - \sigma^h) \, , \, \operatorname{div}(\sigma - \tau)\right)_{\mathbf{L}^2(\Omega)} \\
&\leq \|\operatorname{div}(\sigma - \sigma^h)\|_{\mathbf{L}^2(\Omega)} \|\operatorname{div}(\sigma - \tau)\|_{\mathbf{L}^2(\Omega)}
\end{aligned}$$

$$\Rightarrow \quad \|\operatorname{div}(\sigma - \sigma^h)\|_{\mathbf{L}^2(\Omega)} \leq \|\operatorname{div}(\sigma - \tau)\|_{\mathbf{L}^2(\Omega)} \qquad \forall \ \tau \in \mathbf{X}_h.$$

But this means that it is sufficient to estimate the term on the left hand side in (3.78) with respect to the $\mathbf{L}^2$-norm. Applying the continuity of the bilinear form $\mathcal{A}$ and inequality (2.6) for some $\varepsilon > 0$, we conclude for the first term in (3.86)

$$\begin{aligned}
&\mathcal{A}((\sigma-\sigma^h, \nu-\nu^{\hat{h}}, \mathbf{u}-\mathbf{u}^h, \eta-\eta^h, \boldsymbol{\varphi}-\boldsymbol{\varphi}^{\tilde{h}}, \lambda_t-\lambda_t^{\tilde{h}}, \lambda_n-\lambda_n^{\tilde{h}})\,;\\
&\qquad (\sigma-\tau, \nu-\kappa, \mathbf{u}-\mathbf{v}, \eta-\xi, \boldsymbol{\varphi}-\boldsymbol{\psi}, \lambda_t-\mu_t, \lambda_n-\mu_n))\\
&\leq C\,\|(\sigma-\sigma^h, \nu-\nu^{\hat{h}}, \mathbf{u}-\mathbf{u}^h, \eta-\eta^h, \boldsymbol{\varphi}-\boldsymbol{\varphi}^{\tilde{h}}, \lambda_t-\lambda_t^{\tilde{h}}, \lambda_n-\lambda_n^{\tilde{h}})\|_{\mathbf{X}\times H^{-\frac{1}{2}}(A_C)\times\widetilde{\mathbf{Y}}'}\\
&\qquad \cdot \|(\sigma-\tau, \nu-\kappa, \mathbf{u}-\mathbf{v}, \eta-\xi, \boldsymbol{\varphi}-\boldsymbol{\psi}, \lambda_t-\mu_t, \lambda_n-\mu_n)\|_{\mathbf{X}\times H^{-\frac{1}{2}}(A_C)\times\widetilde{\mathbf{Y}}'}\\
&\leq \frac{\varepsilon C}{2}\,\|(\sigma-\sigma^h, \nu-\nu^{\hat{h}}, \mathbf{u}-\mathbf{u}^h, \eta-\eta^h, \boldsymbol{\varphi}-\boldsymbol{\varphi}^{\tilde{h}}, \lambda_t-\lambda_t^{\tilde{h}}, \lambda_n-\lambda_n^{\tilde{h}})\|^2_{\mathbf{X}\times H^{-\frac{1}{2}}(A_C)\times\widetilde{\mathbf{Y}}'}\\
&\qquad + \frac{C}{2\varepsilon}\,\|(\sigma-\tau, \nu-\kappa, \mathbf{u}-\mathbf{v}, \eta-\xi, \boldsymbol{\varphi}-\boldsymbol{\psi}, \lambda_t-\mu_t, \lambda_n-\mu_n)\|^2_{\mathbf{X}\times H^{-\frac{1}{2}}(A_C)\times\widetilde{\mathbf{Y}}'}\\
&\leq \frac{\varepsilon C}{2}\Bigg((1+C_1+C_2)\|\sigma-\sigma^h\|^2_{\mathbf{X}} + C_2 \inf_{\kappa\in\Lambda_{\hat{h}}}\|\nu-\kappa\|^2_{H^{-\frac{1}{2}}(A_C)} + C_1\Bigg\{\inf_{\mathbf{v}\in\mathbf{M}_h}\|\mathbf{u}-\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)}\\
&\qquad\qquad + \inf_{\xi\in S_h}\|\eta-\xi\|^2_{L^2(\Omega)} + \inf_{\psi\in\mathbf{N}_{\tilde{h}}}\|\boldsymbol{\varphi}-\boldsymbol{\psi}\|^2_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)}\\
&\qquad\qquad + \inf_{\mu_t\in C_{\tilde{h}}}\|\lambda_t-\mu_t\|^2_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} + \inf_{\mu_n\in C^+_{\tilde{h}}}\|\lambda_n-\mu_n\|^2_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}\Bigg\}\Bigg) \qquad (3.87)\\
&\quad + \frac{C}{2\varepsilon}\,\|(\sigma-\tau, \nu-\kappa, \mathbf{u}-\mathbf{v}, \eta-\xi, \boldsymbol{\varphi}-\boldsymbol{\psi}, \lambda_t-\mu_t, \lambda_n-\mu_n)\|^2_{\mathbf{X}\times H^{-\frac{1}{2}}(A_C)\times\widetilde{\mathbf{Y}}'}
\end{aligned}$$

where we used (3.79) and (3.80) in the last inequality. Applying the definition of $\mathcal{A}$ the second term in (3.86) reads

$$\begin{aligned}
\tilde{a}(\sigma-\sigma^h, \tau-\sigma^h) &+ B(\tau-\sigma^h, (\mathbf{u}-\mathbf{u}^h, \eta-\eta^h, \boldsymbol{\varphi}-\boldsymbol{\varphi}^{\tilde{h}}, \lambda_t-\lambda_t^{\tilde{h}}, \lambda_n-\lambda_n^{\tilde{h}}))\\
&- B(\sigma-\sigma^h, (\mathbf{v}-\mathbf{u}^h, \xi-\eta^h, \boldsymbol{\psi}-\boldsymbol{\varphi}^{\tilde{h}}, \mu_t-\lambda_t^{\tilde{h}}, \mu_n-\lambda_n^{\tilde{h}}))\\
&+ q(\mu_t-\lambda_t^{\tilde{h}}, \nu-\nu^{\hat{h}}) - q(\lambda_t-\lambda_t^{\tilde{h}}, \kappa-\nu^{\hat{h}}).
\end{aligned}$$

Since $\tau-\sigma^h\in\mathbf{X}_h$ the first two terms vanish due to (3.81). Moreover using equations (3.82)-(3.85) in the third and fourth term the above formula reduces to

$$-d_{C,n}(\mu_n-\lambda_n^{\tilde{h}}, \sigma-\sigma^h) - q(\lambda_t-\lambda_t^{\tilde{h}}, \kappa-\nu^{\hat{h}}). \qquad (3.88)$$

As we have chosen conform subsets for $\lambda_n^{\tilde{h}}$ and $\nu^{\hat{h}}$, see Remark 3.27, we have $C^+_{\tilde{h}}\subset\widetilde{H}^{\frac{1}{2}}_+(\Gamma_C)$ and $\Lambda_{\hat{h}}\subset\Lambda$. Now using the inequalities in the discrete and continuous variational inequality problems (3.62) and (3.41), respectively, we have due to $\lambda_n^{\tilde{h}}\in\widetilde{H}^{\frac{1}{2}}_+(\Gamma_C)$ and the continuity of $d_{C,n}(\cdot,\cdot)$

$$-d_{C,n}(\mu_n-\lambda_n^{\tilde{h}}, \sigma-\sigma^h) \leq d_{C,n}(\lambda_n^{\tilde{h}}-\mu_n, \sigma) \leq d_{C,n}(\lambda_n-\mu_n, \sigma) \leq C\|\sigma\|_{\mathbf{X}}\|\lambda_n-\mu_n\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}.$$

In the same way we can estimate the second term in (3.88) due to $\nu^{\hat{h}} \in \Lambda$ and the continuity of $q(\cdot,\cdot)$

$$-q(\lambda_t - \lambda_t^{\tilde{h}}, \kappa - \nu^{\hat{h}}) \leq q(\lambda_t, \nu^{\hat{h}} - \kappa) \leq q(\lambda_t, \nu - \kappa) \leq \|\mathcal{F}\|_{L^\infty(A_C)} \|\lambda_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \|\nu - \kappa\|_{H^{-\frac{1}{2}}(A_C)}.$$

Using Lemma 3.23 we bound $\|\sigma\|_{\mathbf{X}}$ and $\|\lambda_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}$. Choosing $\varepsilon = \frac{\alpha}{C(1+C_1+C_2)} > 0$ in (3.87) finishes the proof. □

Before closing this section with an a priori result concerning the theoretical rate of convergence of the solution scheme (3.62), we list some approximation properties of the finite element spaces in (3.61). The results can be found in Chapter 4-6 of Babuška and Aziz [8], in Chapter IV of Roberts and Thomas [77] and in Arnold, Brezzi and Douglas jr. [5], see also Section 5.1.5 in Maischak [67] for a comprehensive list.

For all $\tau \in H^1(\Omega)^{2\times 2}$ with $\operatorname{div} \tau \in \mathbf{H}(\operatorname{div}, \Omega)$, there exists $\tau^h \in \mathbf{X}_h$ and $C > 0$ such that

$$\|\tau - \tau^h\|_{\mathbf{X}} \leq C\, h \left\{ \|\tau\|_{H^1(\Omega)^{2\times 2}} + \|\operatorname{div} \tau\|_{\mathbf{H}^1(\Omega)} \right\}. \tag{3.89}$$

For all $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and $\xi \in H^1(\Omega)$, there exists $\mathbf{v}^h \in \mathbf{M}_h$, $\xi^h \in S_h$ and $C > 0$ such that

$$\|\mathbf{v} - \mathbf{v}^h\|_{\mathbf{L}^2(\Omega)} \leq C\, h\, \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \qquad \text{and} \qquad \|\xi - \xi^h\|_{L^2(\Omega)} \leq C\, h\, \|\xi\|_{H^1(\Omega)}. \tag{3.90}$$

For all $\boldsymbol{\psi} \in \widetilde{\mathbf{H}}^{\frac{3}{2}}(\Gamma_N) \cap \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)$ and $\mu \in \widetilde{H}^{\frac{3}{2}}(\Gamma_C) \cap \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$, there exists $\boldsymbol{\psi}^{\tilde{h}} \in \mathbf{N}_{\tilde{h}}$, $\mu^{\tilde{h}} \in C_{\tilde{h}}$ and $C > 0$ such that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}^{\tilde{h}}\|_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)} \leq C\, \tilde{h}\, \|\boldsymbol{\psi}\|_{\widetilde{\mathbf{H}}^{\frac{3}{2}}(\Gamma_N)} \qquad \text{and} \qquad \|\mu - \mu^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \leq C\, \tilde{h}\, \|\mu\|_{\widetilde{H}^{\frac{3}{2}}(\Gamma_C)}. \tag{3.91}$$

For all $\kappa \in L^2(A_C)$, there exists $\kappa^{\hat{h}} \in L_{\hat{h}}$ and $C > 0$ such that

$$\|\kappa - \kappa^{\hat{h}}\|_{H^{-\frac{1}{2}}(A_C)} \leq C\, \hat{h}^{\frac{1}{2}}\, \|\kappa\|_{L^2(A_C)}. \tag{3.92}$$

We have the following a priori result of the mixed finite element scheme.

**Theorem 3.35:**
Let $(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \Lambda \times \widetilde{\mathbf{Y}}'$ and $(\sigma^h, \nu^{\hat{h}}; \mathbf{u}^h, \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}) \in \widetilde{\mathbf{X}}_h \times \widetilde{\mathbf{Y}}_h$ be the solutions of the continuous and the discrete variational inequality problems (3.41) and (3.62), respectively. Assume that $\sigma \in H^1(\Omega)^{2\times 2}$ with $\operatorname{div} \sigma \in \mathbf{H}(\operatorname{div}, \Omega)$, $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $\eta \in H^1(\Omega)$, $\boldsymbol{\varphi} \in \widetilde{\mathbf{H}}^{\frac{3}{2}}(\Gamma_N)$ and $\lambda_t$, $\lambda_n \in \widetilde{H}^{\frac{3}{2}}(\Gamma_C)$. Then there exists a constant $C > 0$, independent of $h$, $\tilde{h}$ and $\hat{h}$, such that

$$\begin{aligned} &\|\sigma - \sigma^h\|_{\mathbf{X}} + \|\nu - \nu^{\hat{h}}\|_{-\frac{1}{2},\hat{h}} + \|\mathbf{u} - \mathbf{u}^h\|_{\mathbf{L}^2(\Omega)} + \|\eta - \eta^h\|_{L^2(\Omega)} \\ &\qquad + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}\|_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)} + \|\lambda_t - \lambda_t^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} + \|\lambda_n - \lambda_n^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \\ &\qquad \leq C\, h \left\{ \|\sigma\|_{H^1(\Omega)^{2\times 2}} + \|\operatorname{div} \sigma\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\eta\|_{H^1(\Omega)} \right\} \\ &\qquad\quad + C\, \tilde{h} \left\{ \|\boldsymbol{\varphi}\|_{\widetilde{\mathbf{H}}^{\frac{3}{2}}(\Gamma_N)} + \|\lambda_t\|_{\widetilde{H}^{\frac{3}{2}}(\Gamma_C)} \right\} + C\, \tilde{h}^{\frac{1}{2}}\, \|\lambda_n\|_{\widetilde{H}^{\frac{3}{2}}(\Gamma_C)} + C\, \hat{h}^{\frac{1}{4}} \|\nu\|_{L^2(A_C)}. \end{aligned} \tag{3.93}$$

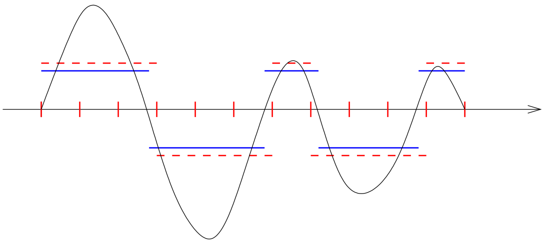


Figure 3.9.: Continuous function (black) with sign (blue) and its approximation (dashed red).

*Proof.* The assertion follows from the Céa-type estimate in Theorem 3.34 and the regularity assumptions on the continuous solution and the approximation properties (3.89)-(3.92). □

We can improve the expected rate of convergence in Theorem 3.35 using a heuristic observation concerning the approximation of the sign of $\lambda_t$ by $\nu^{\hat{h}}$. For this we will need Lemma 3.3.2 in Chernov [24] which we adapt to our situation

**Lemma 3.36:**
Let $\psi \in L^2(A_C)$ and $\Pi_h : L^2(A_C) \to L_{\hat{h}}$ be the $L^2$-projection operator. Then there holds

$$\|\psi - \Pi_{\hat{h}}\psi\|_{H^{-\frac{1}{2}}(A_C)} \leq C\,\hat{h}^{\frac{1}{2}}\|\psi\|_{L^2(A_C)}.$$

In particular, there holds

$$\|\psi - \Pi_{\hat{h}}\psi\|_{H^{-\frac{1}{2}}(A_C)} \leq C\,\hat{h}^{\frac{1}{2}}\|\psi - \Pi_{\hat{h}}\psi\|_{L^2(A_C)}.$$

**Remark 3.37: A heuristic approximation result** *Since $\nu^{\hat{h}}$ approximates the sign of a continuous function $\lambda_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ with piecewise constants we conclude, that the approximation is exact on those edges $\hat{e} \in \mathcal{E}^A_{\hat{h}}$ where $\lambda_t$ does not change its sign, i.e. $\lambda_t$ has no root on $\hat{e}$. We have visualized this observation exemplary in Figure 3.9. For a better view the sign and its approximation are not superposing.*

*We conclude, if $\lambda_t$ has a finite number of roots $N_r$, then the error of the approximation $\nu^{\hat{h}}$ in the $L^2$-norm is*

$$\|\nu^{\hat{h}} - \nu\|^2_{L^2(A_C)} \leq C^2\,\hat{h}\,\|\nu\|^2_{L^2(A_C)}, \qquad \text{with } C \leq 2\,N_r^{\frac{1}{2}}.$$

*Using the above estimate and Lemma 3.36 we have*

$$\|\nu - \Pi_{\hat{h}}\nu\|_{H^{-\frac{1}{2}}(A_C)} \leq C\,\hat{h}^{\frac{1}{2}}\,\|\nu - \Pi_{\hat{h}}\nu\|_{L^2(A_C)} \leq C\,\hat{h}^{\frac{1}{2}}\,\|\nu^{\hat{h}} - \nu\|_{L^2(A_C)} \leq C(N_r)\,\hat{h}\,\|\nu\|_{L^2(A_C)}. \quad (3.94)$$

## 3.3. Numerical Algorithms

In this section we will propose an algorithm to solve the discrete variational inequality problem (3.62). The algorithm is a nested Uzawa-type algorithm. We will prove its convergence with the help of the inf-sup conditions of Sections 3.1.6 and 3.2.2. This proof will be done in three steps. First, we introduce the terms, projections and mappings appearing in the algorithm. We show that the assumptions for the solvers, that we use inside the algorithm, are satisfied. Second, we show the convergence of the Uzawa-type algorithm that we propose for the variational inequality problem (3.39) for contact problems without friction. Finally, in the third step we prove convergence of the nested Uzawa algorithm. Let us first explain the algebraic form of the discrete formulations with the help of an abstract framework. For more details on the Finite Element Method we refer to Ciarlet [26]. In the following we will use 0 as a generic zero denoting the scalar integer, a null vector or the null matrix. The right dimension will always be clear from context.

Let $U, V$ be two arbitrary finite dimensional vector spaces with basis $\{\phi_i\}_{i=1}^n$ and $\{\psi_j\}_{j=1}^m$, respectively. Let $m(\cdot,\cdot)$ be a bilinear form on $U \times V$ and $h(\cdot)$ a linear form on $V$. Then the abstract variational problem reads

Find $u \in U$ such that

$$m(u, v) = h(v) \qquad \forall\, v \in V.$$

Since the vector spaces all have a finite dimension we can set $\sum_{i=1}^n c_i\phi_i =: u$ and thus for each base function $\psi_j$ of $V$ it holds

$$m(u, \psi_j) = h(\psi_j) \quad \Rightarrow \quad \sum_{i=1}^n c_i\, m(\phi_i, \psi_j) = h(\psi_j).$$

Without loss of generality we identify the solution $u$ with the vector of coefficients $u$ with components $u_i = c_i$. Defining the matrix $M \in \mathbb{R}^{m\times n}$, $M_{ji} := m(\phi_i, \psi_j)$ and the vector $h \in \mathbb{R}^m$, $h_j := h(\psi_j)$ we arrive at the algebraic form of the above abstract variational problem

$$Mu = h.$$

### 3.3.1. Algebraic dual-dual formulation

Using the concept described above we identify the following matrices with their corresponding bilinear forms defined in (3.12), (3.24) and (3.32)

$$\begin{aligned}
A \;\text{ with }\; A_{ji} &:= \tilde{a}(\phi_i, \phi_j) \quad \text{for basis functions } \phi_i, \phi_j \in \mathbf{X}_h, \\
\tilde{B} \;\text{ with }\; \tilde{B}_{ji} &:= b(\phi_i, \psi_j) \quad \text{for basis functions } \phi_i \in M_h,\ \psi_j \in \mathbf{X}_h, \\
S \;\text{ with }\; S_{ji} &:= s(\phi_i, \psi_j) \quad \text{for basis functions } \phi_i \in S_h,\ \psi_j \in \mathbf{X}_h, \\
N \;\text{ with }\; N_{ji} &:= d_N(\phi_i, \psi_j) \quad \text{for basis functions } \phi_i \in N_{\tilde{h}},\ \psi_j \in \mathbf{X}_h, \\
C_t \;\text{ with }\; (C_t)_{ji} &:= d_{C,t}(\phi_i, \psi_j) \quad \text{for basis functions } \phi_i \in C_{\tilde{h}},\ \psi_j \in \mathbf{X}_h, \\
C_n \;\text{ with }\; (C_n)_{ji} &:= d_{C,n}(\phi_i, \psi_j) \quad \text{for basis functions } \phi_i \in C_{\tilde{h}},\ \psi_j \in \mathbf{X}_h, \\
Q_{\mathcal{F}} \;\text{ with }\; (Q_{\mathcal{F}})_{ji} &:= q(\phi_i, \psi_j) \quad \text{for basis functions } \phi_i \in C_{\tilde{h}},\ \psi_j \in \Lambda_{\hat{h}}.
\end{aligned} \tag{3.95}$$

Furthermore, we define the following vectors corresponding to the linear forms in (3.24) and (3.32) and used on the right hand side of the discrete variational inequality problem (3.62)

$$\begin{aligned}
g \;\text{ with }\; g_i &:= g(\phi_i) \quad \text{for basis functions } \phi_i \in \mathbf{X}_h, \\
f \;\text{ with }\; f_i &:= f(\phi_i) \quad \text{for basis functions } \phi_i \in M_h, \\
t \;\text{ with }\; t_i &:= t_0(\phi_i) \quad \text{for basis functions } \phi_i \in N_{\tilde{h}}.
\end{aligned} \tag{3.96}$$

To improve readability we define the matrices $\bar{B}$ and $B$ and the vectors $\bar{b}$ and $b$ as follows

$$\bar{B} := \begin{pmatrix} \tilde{B} & S & N \end{pmatrix}, \qquad B := \begin{pmatrix} \tilde{B} & S & N & C_t \end{pmatrix}, \qquad \bar{b} := \begin{pmatrix} -f \\ 0 \\ t \end{pmatrix} \qquad b := \begin{pmatrix} -f \\ 0 \\ t \\ 0 \end{pmatrix}.$$

Defining $N := \dim(\widetilde{\mathbf{X}}_h \times \widetilde{\mathbf{Y}}_h)$, $N_C := \dim(C_{\tilde{h}})$ and $N_A := \dim(L_{\hat{h}})$, the algebraic form of the solution scheme (3.62) reads

Find $(\sigma, \nu, \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbb{R}^N$ such that

$$\begin{pmatrix} A & \bar{B} & C_t & C_n & 0 \\ \bar{B}^T & 0 & 0 & 0 & 0 \\ C_t^T & 0 & 0 & 0 & Q_{\mathcal{F}} \end{pmatrix} \begin{pmatrix} \sigma \\ (\mathbf{u}, \eta, \boldsymbol{\varphi}) \\ \lambda_t \\ \lambda_n \\ \nu \end{pmatrix} = \begin{pmatrix} g \\ \bar{b} \\ 0 \end{pmatrix} \tag{3.97}$$

$$\begin{aligned}
&(\mu_n - \lambda_n)^T C_n^T \sigma \leq 0 \quad \text{with} \quad (\lambda_n)_i \geq 0 && \forall\, \mu_n \in \mathbb{R}^{N_C},\ (\mu_n)_i \geq 0, \\
&(\kappa - \nu)^T Q_{\mathcal{F}}^T \lambda_t \leq 0 \quad \text{with} \quad |\nu_i| \leq 1 && \forall\, \kappa \in \mathbb{R}^{N_A},\ |\kappa_i| \leq 1.
\end{aligned}$$

### 3.3.2. Uzawa algorithm for contact problems without friction

Following the approach of Maischak [67, see Section 5.3] we introduce the surjective projection $P^+_{\tilde{h}} : C_{\tilde{h}} \to C^+_{\tilde{h}}$ uniquely defined by

$$\left(P^+_{\tilde{h}}\mu - \mu\,,\, \chi - P^+_{\tilde{h}}\mu\right)_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \geq 0 \qquad \forall\, \chi \in C^+_{\tilde{h}}, \tag{3.98}$$

where $\mu \in C_{\tilde{h}}$ and we use the inner product of $\widetilde{H}^{\frac{1}{2}}(\Gamma_C)$. The surjectivity follows from the fact that for all $\mu \in C^+_{\tilde{h}}$ we have $P^+_{\tilde{h}}\mu = \mu$ and therefore the left term in the inner product is zero. The projection is contractive. We only have to show this for $\mu \in C_{\tilde{h}} \setminus C^+_{\tilde{h}}$ since for $\mu \in C^+_{\tilde{h}}$ we have $P^+_{\tilde{h}}\mu = \mu$. Let $\mu \in C_{\tilde{h}} \setminus C^+_{\tilde{h}}$, then from (3.98) with $\chi = 0 \in C^+_{\tilde{h}}$ and using (2.6) with $\varepsilon = 1$ it follows

$$\left(P^+_{\tilde{h}}\mu,\, P^+_{\tilde{h}}\mu\right)_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \leq \left(\mu\,,\, P^+_{\tilde{h}}\mu\right)_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \leq \frac{1}{2}\,(\mu,\,\mu)_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} + \frac{1}{2}\left(P^+_{\tilde{h}}\mu\,,\, P^+_{\tilde{h}}\mu\right)_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}$$

and hence $\|P^+_{\tilde{h}}\| \leq 1$. Moreover we define the linear mapping $\Phi : \mathbf{X} \to C_{\tilde{h}} \subset \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ by

$$d_{C,n}(\mu,\tau) = (\mu\,,\,\Phi(\tau))_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \qquad \forall\, \mu \in C_{\tilde{h}}, \tag{3.99}$$

with $\tau \in \mathbf{X}$. Due to Theorem 2.3 we can adapt the argumentation of Maischak in [67, Section 5.3] to show the continuity of $\Phi$. For $\tau \in \mathbf{X}$ we have

$$\begin{aligned}\|\Phi(\tau)\|^2_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} &= (\Phi(\tau)\,,\,\Phi(\tau))_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} = d_{C,n}(\Phi(\tau),\tau) = \langle\Phi(\tau),\tau_n\rangle_{\Gamma_C} \\ &\leq \|\Phi(\tau)\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}\|\tau_n\|_{H^{-\frac{1}{2}}(\Gamma_C)} \leq \|\Phi(\tau)\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}\|\tau\|_{\mathbf{X}}.\end{aligned}$$

We will need another surjective and contractive projection operator $P^\Lambda_{\hat{h}} : L_{\hat{h}} \to \Lambda_{\hat{h}}$ which is uniquely defined by

$$\left(P^\Lambda_{\hat{h}}\kappa - \kappa\,,\, \psi - P^\Lambda_{\hat{h}}\kappa\right)_{L^2(A_C)} \geq 0 \qquad \forall\, \psi \in \Lambda_{\hat{h}}, \tag{3.100}$$

for $\kappa \in L_{\hat{h}}$. Here we use the inner product of $L^2(A_C)$. The surjectivity and contraction are proven in the same way as for $P^+_{\tilde{h}}$. Furthermore, let the mapping $\Phi^\Lambda : \widetilde{H}^{\frac{1}{2}}(\Gamma_C) \to L_{\hat{h}} \subset L^2(A_C)$ be defined by

$$q(\mu,\kappa) = \left(\Phi^\Lambda\mu,\,\kappa\right)_{L^2(A_C)} \geq 0 \qquad \forall\, \kappa \in L_{\hat{h}}, \tag{3.101}$$

with $\mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$. The continuity follows due to

$$\begin{aligned}\|\Phi^\Lambda(\mu)\|^2_{L^2(A_C)} &= \left(\Phi^\Lambda(\mu)\,,\,\Phi^\Lambda(\mu)\right)_{L^2(A_C)} = q(\mu,\Phi^\Lambda(\mu)) = \int_{A_C} \mathcal{F}\,\Phi^\Lambda(\mu)\,\mu\,ds \\ &\leq \|\mathcal{F}\|_{L^\infty(A_C)}\,\|\Phi^\Lambda(\mu)\|_{L^2(A_C)}\,\|\mu\|_{L^2(A_C)} \leq \|\mathcal{F}\|_{L^\infty(A_C)}\,\|\Phi^\Lambda(\mu)\|_{L^2(A_C)}\,\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}.\end{aligned} \tag{3.102}$$

Before we present the algorithm, solving the frictional contact problem, we first state an algorithm for the case where no friction occurs. In this case our aim is to solve the solution scheme (3.68) defined in Remark 3.26. The corresponding algebraic form reads

Find $(\sigma, \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbb{R}^{N-N_A}$

$$\begin{pmatrix} A & B & C_n \\ & & \\ B^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ (\mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t) \\ \lambda_n \end{pmatrix} = \begin{pmatrix} g \\ \\ b \end{pmatrix} \tag{3.103}$$
$$(\mu_n - \lambda_n)^T C_n^T \sigma \leq 0 \quad \text{with} \quad (\lambda_n)_i \geq 0 \qquad \forall\, \mu_n \in \mathbb{R}^{N_C},\ (\mu_n)_i \geq 0.$$

The following algorithm is used in many variants for different kinds of variational inequalities. For a contact problem without friction a similar algorithm is proposed by Wang and Wang in [84], where the authors investigate a dual formulation of a unilateral beaming problem in linear elasticity. Glowinski, Lions and Trémolières [53, see Section 4 in Chapter 2] give several examples of solution algorithms for problem schemes having saddle point structure.

We adapt the Uzawa-type algorithm of Maischak [67, see Algorithm 5.1], which then reads

**Algorithm 3.38:**

**1** Choose feasible $\lambda_n^0 \in C_{\tilde{h}}^+$, $(\sigma^0; \mathbf{u}^0, \eta^0, \boldsymbol{\varphi}^0, \lambda_t^0) := \mathbf{0}$, $\rho \in (0, 2\alpha)$ and $\varepsilon \geq 0$,

**2** **for** $i = 1, 2, \ldots$

**3** Solve

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \sigma^i \\ \mathbf{u}^i, \eta^i, \boldsymbol{\varphi}^i, \lambda_t^i \end{pmatrix} = \begin{pmatrix} g - C_n \lambda_n^{i-1} \\ b \end{pmatrix},$$

**4** Set $\lambda_n^i := P_{\tilde{h}}^+(\lambda_n^{i-1} + \rho \Phi(\sigma^i))$,

**5**

STOP if $$\frac{\|(\sigma^i - \sigma^{i-1}, \mathbf{u}^i - \mathbf{u}^{i-1}, \eta^i - \eta^{i-1}, \boldsymbol{\varphi}^i - \boldsymbol{\varphi}^{i-1}, \lambda_t^i - \lambda_t^{i-1}, \lambda_n^i - \lambda_n^{i-1})\|}{\|\sigma^i; \mathbf{u}^i, \eta^i, \boldsymbol{\varphi}^i, \lambda_t^i, \lambda_n^i\|} \leq \varepsilon,$$

Else go to **2**,

**end for**

**Theorem 3.39:**
The Uzawa algorithm 3.38 converges for arbitrary initial value $\lambda_n^0 \in C_{\tilde{h}}^+$ towards the discrete solution of the dual variational inequality problem (3.39) for $0 < \rho < 2\alpha$, where $\alpha$ denotes the ellipticity constant in (2.3).

*Proof.* The proof follows analogously to the proof of Theorem 5.12 in Maischak [67]. Furthermore, it has the same structure as the proof of Theorem 3.42. Therefore we only give a sketch of the proof. The details can be derived analogously to the proof of Theorem 3.42.

First, we show that the approximate solution $\lambda_n^{\tilde{h}}$ satisfies

$$\lambda_n^{\tilde{h}} = P_{\tilde{h}}^+(\lambda_n^{\tilde{h}} + \rho\Phi(\sigma^h)).$$

Then we use the definition of the projection $P_{\tilde{h}}^+$ and the mapping $\Phi$ to show, that the sequence of errors $\|\lambda_n^i - \lambda_n^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}$ is monotonically decreasing, if we choose $0 < \rho < 2\alpha$. This leads to the convergence of $\sigma^i$ to $\sigma^h$ in the $\mathbf{H}(\mathrm{div}, \Omega)$-norm. Finally, we use the discrete inf-sup condition in Lemma 3.28 to prove the convergence of the Lagrange multipliers. □

**Remark 3.40:** *Due to the discrete inf-sup condition* (3.69) *the matrix in step* **3** *of algorithm 3.38 is regular. For completeness we repeat Remark 5.10 in Maischak [67]:*

*The operators $P_{\tilde{h}}^+$ and $\Phi$ are defined with respect to the scalar product of $\widetilde{H}^{\frac{1}{2}}(\Gamma_C)$, which is not practical from the computational point of view. Fortunately, inspection of the proof of Theorem 3.39 shows, that it is sufficient that the norm induced by the scalar product used in the algorithm is equivalent to the $\widetilde{H}^{\frac{1}{2}}(\Gamma_C)$-norm. Therefore we can use the bilinear form $\langle W\cdot, \cdot\rangle$ instead of the scalar product $(\cdot,\cdot)_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}$. Then, we have to solve: Find $P_{\tilde{h}}^+\mu \in C_{\tilde{h}}^+$ such that*

$$\langle WP_{\tilde{h}}^+\mu, \chi - P_{\tilde{h}}^+\mu\rangle \geq \langle W\mu, \chi - P_{\tilde{h}}^+\mu\rangle \qquad \forall\, \chi \in C_{\tilde{h}}^+, \tag{3.104}$$

*and find $\Phi(\tau) \in C_{\tilde{h}}$ such that*

$$\langle W\Phi(\tau), \mu\rangle = d_{C,n}(\mu, \tau) \qquad \forall\, \mu \in C_{\tilde{h}}. \tag{3.105}$$

*Both systems are small compared with the total size of the problem, because they are only defined on the contact boundary $\Gamma_C$. Applying* (3.105) *to* (3.104) *we obtain for $\mu = \lambda_n^{i-1} + \rho\Phi(\sigma^i)$,*

$$\langle W\mu, \chi - P_{\tilde{h}}^+\mu\rangle = \langle W\lambda_n^{i-1}, \chi - P_{\tilde{h}}^+\mu\rangle + \rho d_{C,n}(\chi - P_{\tilde{h}}^+\mu, \sigma^i),$$

*and hence, the explicit solution of* (3.105) *is avoided.*

*In our case the operator W denotes the hypersingular operator for the Lamé problem of plane elasticity, see e.g. Gwinner and Stephan [55] or Nédélec [71].*

For step **4** in algorithm 3.38 we observe the following. In the algebraic form the scalar product coincide with the euklidean product. Furthermore, we identify with the matrix $W$ the corresponding hypersingular operator which is positive semidefinite, see Section 6.7 in Steinbach [80]. From Remark 3.40 we know, that we have to solve the problem of finding $\lambda_n^i := P_{\tilde{h}}^+ \mu$, such that

$$\langle W\lambda_n^i, \chi - \lambda_n^i \rangle \geq \langle W\lambda_n^{i-1}, \chi - \lambda_n^i \rangle + \rho d_{C,n}(\chi - \lambda_n^i, \sigma^i) \qquad \forall\, \chi \in C_{\tilde{h}}^+.$$

Defining $b^i := W\lambda_n^{i-1} + \rho C_n^T \sigma^i$ the above problem reads: Find $\lambda_n^i \in \mathbb{R}^{N_C}$ with $|(\lambda_n^i)_j| \geq 0$, such that

$$-(\lambda_n^i)^T W \lambda_n^i - (\chi - \lambda_n^i)^T b^i \geq -\chi^T W \lambda_n^i \qquad \forall\, \chi \in \mathbb{R}^{N_C}, |\chi_j| \geq 0. \tag{3.106}$$

Furthermore, due to $W$ being positive semidefinite we have

$$\tfrac{1}{2}\chi^T W \chi + \tfrac{1}{2}(\lambda_n^i)^T W \lambda_n^i \geq \chi^T W \lambda_n^i \qquad \forall\, \chi \in \mathbb{R}^{N_C}. \tag{3.107}$$

With the inequalities (3.106) and (3.107) we deduce

$$\begin{aligned}
& \tfrac{1}{2}\chi^T W \chi + \tfrac{1}{2}(\lambda_n^i)^T W \lambda_n^i - (\lambda_n^i)^T W \lambda_n^i - (\chi - \lambda_n^i)^T b^i \geq 0 && \forall\, \chi \in \mathbb{R}^{N_C}, |\chi_j| \geq 0,\\
\Leftrightarrow \quad & \tfrac{1}{2}\chi^T W \chi - \chi^T b^i \geq \tfrac{1}{2}(\lambda_n^i)^T W \lambda_n^i - (\lambda_n^i)^T b^i && \forall\, \chi \in \mathbb{R}^{N_C}, |\chi_j| \geq 0,\\
\Leftrightarrow \quad & \lambda_n^i = \underset{\chi \in \mathbb{R}^{N_C}, |\chi_j| \geq 0}{\operatorname{argmin}} \left\{ \tfrac{1}{2}\chi^T W \chi - \chi^T b^i \right\}.
\end{aligned}$$

For this reason we execute step **4** in Algorithm 3.38 by solving a convex quadratic program. This will be explained in the following subsection.

**Abstract minimization problem with inequality constraints**

We consider the following minimization problem

$$\begin{gathered}
\min_{x \in \mathbb{R}^n} \; q(x) \\
\text{subject to } \; G \cdot x \geq c,
\end{gathered}$$

where $q(x) := \frac{1}{2} x^T \cdot A \cdot x + x^T \cdot b$ with $A \in \mathbb{R}^{n \times n}$ symmetric and positive semidefinite $b \in \mathbb{R}^n$, $c \in \mathbb{R}^m$ and $G \in \mathbb{R}^{m \times n}$. This kind of optimization problem is called convex quadratic program since the objective function $q(x)$ is quadratic with symmetric positive semidefinite Hessian matrix $A$ and linear inequality constraints, see Nocedal and Wright [72, Chapter 16]. There exist numerous algorithms for solving convex quadratic problems and we refer to the extensive collections of Boyd and Vandenberghe [14] and Nocedal and Wright [72]. We will restrict ourselve to the usage of a predictor-corrector algorithm which is of interior-point method type and well suited for large problems. The algorithm can be found in Chapter 16.6 of the above mentioned reference [72]. It will be used for solving the projection $\lambda_n^i := P_{\tilde{h}}^+(\lambda_n^{i-1} + \rho\Phi(\sigma^i))$ in step **4** of algorithm 3.38 and the projections within algorithm 3.41, which is the matter of the following section.

### 3.3.3. Nested Uzawa algorithm for frictional contact problems

We propose an algorithm for solving the solution scheme (3.62). It consists of two nested algorithms of the Uzawa-type as presented in algorithm 3.38. With the matrices defined in (3.95), the vectors defined in (3.96) and the projections defined in (3.98) and (3.100) the algorithm reads

**Algorithm 3.41:**

**1** Choose feasible $\hat{\lambda}_n^0 \in C_{\tilde{h}}^+$ and $\nu^0 \in \Lambda_{\hat{h}}$, $(\hat{\sigma}^0; \hat{\mathbf{u}}^0, \hat{\eta}^0, \hat{\boldsymbol{\varphi}}^0, \hat{\lambda}_t^0) := \mathbf{0}$,

$\rho_1 \in (0, 2\alpha)$, $\rho_2 \in \left(0, \frac{2\alpha\beta_1^2}{C^2\|\mathcal{F}\|^2_{L^2(A_C)}}\right)$ and $\varepsilon_1, \varepsilon_2 \geq 0$,

**2** **for** $i = 1, 2, \ldots$

**3** Set $(\sigma^0; \mathbf{u}^0, \eta^0, \boldsymbol{\varphi}^0, \lambda_t^0, \lambda_n^0) := (\hat{\sigma}^{i-1}; \hat{\mathbf{u}}^{i-1}, \hat{\eta}^{i-1}, \hat{\boldsymbol{\varphi}}^{i-1}, \hat{\lambda}_t^{i-1}, \hat{\lambda}_n^{i-1})$,

$b^i := b + \left(0, 0, 0, Q_{\mathcal{F}}^T \nu^{i-1}\right)^T$,

**4** **for** $j = 1, 2, \ldots$

**5** Solve
$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \sigma^j \\ \mathbf{u}^j, \eta^j, \boldsymbol{\varphi}^j, \lambda_t^j \end{pmatrix} = \begin{pmatrix} g - C_n \lambda_n^{j-1} \\ b^i \end{pmatrix},$$

**6** Set $\lambda_n^j := P_{\tilde{h}}^+(\lambda_n^{j-1} + \rho_1 \Phi(\sigma^j))$,

**7**

$$\text{If} \quad \frac{\|(\sigma^j - \sigma^{j-1}, \mathbf{u}^j - \mathbf{u}^{j-1}, \eta^j - \eta^{j-1}, \boldsymbol{\varphi}^j - \boldsymbol{\varphi}^{j-1}, \lambda_t^j - \lambda_t^{j-1}, \lambda_n^j - \lambda_n^{j-1})\|}{\|\sigma^j; \mathbf{u}^j, \eta^j, \boldsymbol{\varphi}^j, \lambda_t^j, \lambda_n^j\|} \leq \varepsilon_1,$$

set $(\hat{\sigma}^i; \hat{\mathbf{u}}^i, \hat{\eta}^i, \hat{\boldsymbol{\varphi}}^i, \hat{\lambda}_t^i, \hat{\lambda}_n^i) := (\sigma^j; \mathbf{u}^j, \eta^j, \boldsymbol{\varphi}^j, \lambda_t^j, \lambda_n^j)$ and go to **8**,

Else go to **3**,

**end for**

**8** Set $\nu^i := P_{\hat{h}}^\Lambda(\nu^{i-1} + \rho_2 \Phi^\Lambda \hat{\lambda}_t^i)$,

**9**

$$\text{STOP if} \quad \frac{\|\nu^i - \nu^{i-1}\|}{\|\nu^i\|} \leq \varepsilon_2,$$

Else go to **2**,

**end for**

**Theorem 3.42:**
The nested Uzawa algorithm 3.41 converges for arbitrary initial values $\hat{\lambda}_n^0 \in C_{\tilde{h}}^+$ and $\nu^0 \in \Lambda_{\hat{h}}$ towards the discrete solution of the dual-dual variational inequality problem (3.62) for $0 < \rho_1 < 2\alpha$ and $\rho_2 \in (0, \frac{2\alpha\beta_1^2}{C^2\|\mathcal{F}\|^2_{L^\infty(A_C)}})$, where the constants $\alpha$, $C$ and $\beta_1$ denote the ellipticity constant in (2.3), the constant from the continuity of $\tilde{a}(\cdot,\cdot)$ and the constant in the discrete inf-sup condition (3.69), respectively.

*Proof.* The proof is constructed analogously to the proof of Theorem 5.12 in Maischak [67]. First of all we notice, that the inner loop in algorithm 3.41, i.e. step **4** to step **7**, converges for all $0 < \rho_1 < 2\alpha$, every $\nu^{i-1}$ on the right hand side and arbitrary initial value $\lambda_n^0 \in C_{\tilde{h}}^+$ towards the solution of the following discrete problem

Find $(\hat{\sigma}^i; \hat{\mathbf{u}}^i, \hat{\eta}^i, \hat{\boldsymbol{\varphi}}^i, \hat{\lambda}_t^i, \hat{\lambda}_n^i) \in \mathbf{X}_h \times \widetilde{\mathbf{Y}}_h$ such that

$$
\begin{aligned}
&\tilde{a}(\hat{\sigma}^i,\tau) + \hat{B}(\hat{\mathbf{u}}^i, \hat{\eta}^i, \hat{\boldsymbol{\varphi}}^i; \tau) + d_{C,t}(\hat{\lambda}_t^i, \tau) + d_{C,n}(\hat{\lambda}_n^i, \tau) && = g(\tau) && \forall\, \tau \in \mathbf{X}_h \\
&b(\mathbf{v}, \hat{\sigma}^i) && = -f(\mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{M}_h \\
&s(\xi, \hat{\sigma}^i) && = \quad 0 && \forall\, \xi \in S_h \\
&d_N(\boldsymbol{\psi}, \hat{\sigma}^i) && = \quad t_0(\boldsymbol{\psi}) && \forall\, \boldsymbol{\psi} \in \mathbf{N}_{\tilde{h}} \\
&d_{C,t}(\mu_t, \hat{\sigma}^i) && = q(\mu_t, \nu^{i-1}) && \forall\, \mu_t \in C_{\tilde{h}} \\
&d_{C,n}(\mu_n - \hat{\lambda}_n^i, \hat{\sigma}^i) && \leq \quad 0 && \forall\, \mu_n \in C_{\tilde{h}}^+ .
\end{aligned}
\tag{3.108}
$$

The convergence is obvious since problem (3.108) corresponds to a contact problem with prescribed nonzero tangential traction on the contact boundary. As the solution is depending continuously on the given data, see Lemma 3.23, the convergence follows from Theorem 3.39.

Let $(\sigma^h, \nu^{\hat{h}}; \mathbf{u}^h, \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}) \in \widetilde{\mathbf{X}}_h \times \widetilde{\mathbf{Y}}_h$ denote the solution of the discrete dual-dual variational inequality problem (3.62). Then, using $\rho_2 > 0$, $\nu^{\hat{h}} \in \Lambda_{\hat{h}}$ and $(3.62)_7$ we have for all $\kappa \in \Lambda_{\hat{h}}$

$$
\begin{aligned}
\left(\rho_2\, \Phi^\Lambda \lambda_t^{\tilde{h}} + \nu^{\hat{h}}, \nu^{\hat{h}} - \kappa\right)_{L^2(A_C)} &= \rho_2\, q(\lambda_t^{\tilde{h}}, \nu^{\hat{h}} - \kappa) + \left(\nu^{\hat{h}}, \nu^{\hat{h}} - \kappa\right)_{L^2(A_C)} \\
&\geq \left(\nu^{\hat{h}}, \nu^{\hat{h}} - \kappa\right)_{L^2(A_C)} .
\end{aligned}
\tag{3.109}
$$

Since $\Lambda_{\hat{h}}$ is convex we have for any $\gamma \in (0,1)$

$$
\chi = \gamma\, \nu^{\hat{h}} + (1-\gamma)\, P_{\hat{h}}^\Lambda(\rho_2\, \Phi^\Lambda \lambda_t^{\tilde{h}} + \nu^{\hat{h}}) \in \Lambda_{\hat{h}}.
$$

Inserting $\chi$ into (3.100) and using (3.109) we deduce

$$\Big(P_{\hat{h}}^{\Lambda}(\rho_2\,\Phi^{\Lambda}\lambda_t^{\tilde{h}}+\nu^{\hat{h}})-(\rho_2\,\Phi^{\Lambda}\lambda_t^{\tilde{h}}+\nu^{\hat{h}}),\chi-P_{\hat{h}}^{\Lambda}(\rho_2\,\Phi^{\Lambda}\lambda_t^{\tilde{h}}+\nu^{\hat{h}})\Big)_{L^2(A_C)}\geq 0 \qquad \forall\,\chi\in\Lambda_{\hat{h}}\,,$$

$$\begin{aligned}
&\gamma\,\Big(P_{\hat{h}}^{\Lambda}(\rho_2\,\Phi^{\Lambda}\lambda_t^{\tilde{h}}+\nu^{\hat{h}}),\nu^{\hat{h}}-P_{\hat{h}}^{\Lambda}(\rho_2\,\Phi^{\Lambda}\lambda_t^{\tilde{h}}+\nu^{\hat{h}})\Big)_{L^2(A_C)}\\
\Leftrightarrow\qquad &\geq\gamma\,\Big(\rho_2\,\Phi^{\Lambda}\lambda_t^{\tilde{h}}+\nu^{\hat{h}},\nu^{\hat{h}}-P_{\hat{h}}^{\Lambda}(\rho_2\,\Phi^{\Lambda}\lambda_t^{\tilde{h}}+\nu^{\hat{h}})\Big)_{L^2(A_C)}\\
&\geq\gamma\Big(\nu^{\hat{h}},\nu^{\hat{h}}-P_{\hat{h}}^{\Lambda}(\rho_2\,\Phi^{\Lambda}\lambda_t^{\tilde{h}}+\nu^{\hat{h}})\Big)_{L^2(A_C)}\,,\\
\Leftrightarrow\qquad & -\gamma\,\|P_{\hat{h}}^{\Lambda}(\rho_2\,\Phi^{\Lambda}\lambda_t^{\tilde{h}}+\nu^{\hat{h}})-\nu^{\hat{h}}\|^2_{L^2(A_C)}\geq 0 \qquad \forall\,\gamma\in(0,1)\,,\\
\Leftrightarrow\qquad & P_{\hat{h}}^{\Lambda}(\rho_2\,\Phi^{\Lambda}\lambda_t^{\tilde{h}}+\nu^{\hat{h}})=\nu^{\hat{h}}.
\end{aligned}\tag{3.110}$$

Using the fact that the projection operator $P_{\hat{h}}^{\Lambda}$ is contractive we have with (3.102) and (3.110) for any iterate $\nu^i$

$$\begin{aligned}
\|\nu^i-\nu^{\hat{h}}\|^2_{L^2(A_C)} &= \|P_{\hat{h}}^{\Lambda}(\rho_2\,\Phi^{\Lambda}\hat{\lambda}_t^i+\nu^{i-1})-P_{\hat{h}}^{\Lambda}(\rho_2\,\Phi^{\Lambda}\lambda_t^{\tilde{h}}+\nu^{\hat{h}})\|^2_{L^2(A_C)}\\
&\leq\|\rho_2\,\Phi^{\Lambda}(\hat{\lambda}_t^i-\lambda_t^{\tilde{h}})+\nu^{i-1}-\nu^{\hat{h}}\,\|^2_{L^2(A_C)}\\
&\leq\rho_2^2\,\|\mathcal{F}\|^2_{L^\infty(A_C)}\|\hat{\lambda}_t^i-\lambda_t^{\tilde{h}}\|^2_{L^2(A_C)}+2\rho_2\,\Big(\Phi^{\Lambda}(\hat{\lambda}_t^i-\lambda_t^{\tilde{h}}),\nu^{i-1}-\nu^{\hat{h}}\Big)_{L^2(A_C)}+\|\nu^{i-1}-\nu^{\hat{h}}\|^2_{L^2(A_C)}\\
&\leq\rho_2^2\,\|\mathcal{F}\|^2_{L^\infty(A_C)}\|\hat{\lambda}_t^i-\lambda_t^{\tilde{h}}\|^2_{\widetilde{H}^{\frac12}(\Gamma_C)}+2\rho_2\,q(\hat{\lambda}_t^i-\lambda_t^{\tilde{h}},\,\nu^{i-1}-\nu^{\hat{h}})+\|\nu^{i-1}-\nu^{\hat{h}}\|^2_{L^2(A_C)}\,,
\end{aligned}\tag{3.111}$$

where we have used the Cauchy-Schwarz inequality. Subtracting the first five equations in (3.62) from the first five in (3.108) we get

$$\begin{aligned}
&\tilde{a}(\hat{\sigma}^i-\sigma^h,\tau)+\hat{B}(\hat{\mathbf{u}}^i-\mathbf{u}^h,\hat{\eta}^i-\eta^h,\hat{\boldsymbol{\varphi}}^i-\boldsymbol{\varphi}^{\tilde{h}};\tau)\\
&\qquad\qquad+d_{C,t}(\hat{\lambda}_t^i-\lambda_t^{\tilde{h}},\tau)+d_{C,n}(\hat{\lambda}_n^i-\lambda_n^{\tilde{h}},\tau) &&= 0 &&\forall\,\tau\in\mathbf{X}_h\,,\\
&b(\mathbf{v},\hat{\sigma}^i-\sigma^h) &&= 0 &&\forall\,\mathbf{v}\in\mathbf{M}_h\,,\\
&s(\xi,\hat{\sigma}^i-\sigma^h) &&= 0 &&\forall\,\xi\in S_h\,,\\
&d_N(\boldsymbol{\psi},\hat{\sigma}^i-\sigma^h) &&= 0 &&\forall\,\boldsymbol{\psi}\in\mathbf{N}_{\tilde{h}}\,,\\
&d_{C,t}(\mu_t,\hat{\sigma}^i-\sigma^h) &&= q(\mu_t,\nu^{i-1}-\nu^{\hat{h}}) &&\forall\,\mu_t\in C_{\tilde{h}}.
\end{aligned}\tag{3.112}$$

From the inequalities concerning the normal part on the boundary in (3.62) and (3.108) we conclude

$$-d_{C,n}(\hat{\lambda}_n^i-\lambda_n^{\tilde{h}},\hat{\sigma}^i-\sigma^h)=d_{C,n}(\lambda_n^{\tilde{h}}-\hat{\lambda}_n^i,\hat{\sigma}^i)+d_{C,n}(\hat{\lambda}_n^i-\lambda_n^{\tilde{h}},\sigma^h)\leq 0. \tag{3.113}$$

From the commuting diagram property, see Section 6 in Chapter II of Roberts and Thomas [77], we know that $\operatorname{div}(\hat{\sigma}^i - \sigma^h) \in \mathbf{M}_h$. Therefore, taking $\mathbf{v} = \operatorname{div}(\hat{\sigma}^i - \sigma^h)$ in the second equation of (3.112) we have

$$0 = b(\operatorname{div}(\hat{\sigma}^i - \sigma^h), \hat{\sigma}^i - \sigma^h) = \int_\Omega \operatorname{div}(\hat{\sigma}^i - \sigma^h)^2 \, dx = \|\operatorname{div}(\hat{\sigma}^i - \sigma^h)\|^2_{\mathbf{L}^2(\Omega)}. \tag{3.114}$$

Choosing $\mu_t := \hat{\lambda}^i_t - \lambda^{\tilde{h}}_t$ in the fifth equation of (3.112) and $\tau := \hat{\sigma}^i - \sigma^h$ in the first one we can use the coercivity of $\tilde{a}(\cdot,\cdot)$ and equations (3.112), (3.113) and (3.114) to estimate

$$\begin{aligned} q(\hat{\lambda}^i_t - \lambda^{\tilde{h}}_t, \nu^{i-1} - \nu^{\hat{h}}) &= -\tilde{a}(\hat{\sigma}^i - \sigma^h, \hat{\sigma}^i - \sigma^h) - d_{C,n}(\hat{\lambda}^i_n - \lambda^{\tilde{h}}_n, \hat{\sigma}^i - \sigma^h) \\ &\le -\alpha \|\hat{\sigma}^i - \sigma^h\|^2_{L^2(\Omega)^{2\times 2}} = -\alpha \|\hat{\sigma}^i - \sigma^h\|^2_{\mathbf{X}}. \end{aligned} \tag{3.115}$$

Furthermore, from the discrete inf-sup condition (3.69), the first equation in (3.112) and the continuity of $\tilde{a}(\cdot,\cdot)$ we get

$$\begin{aligned} \|\hat{\lambda}^i_t - \lambda^{\tilde{h}}_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} &\le \|(\hat{\mathbf{u}}^i - \mathbf{u}^h, \hat{\eta}^i - \eta^h, \hat{\boldsymbol{\varphi}}^i - \boldsymbol{\varphi}^{\tilde{h}}, \hat{\lambda}^i_t - \lambda^{\tilde{h}}_t, \hat{\lambda}^i_n - \lambda^{\tilde{h}}_n)\|_{\widetilde{\mathbf{Y}}}, \\ &\le \frac{1}{\beta_1} \sup_{0 \ne \tau^h \in \mathbf{X}_h} \frac{B(\tau^h, (\hat{\mathbf{u}}^i - \mathbf{u}^h, \hat{\eta}^i - \eta^h, \hat{\boldsymbol{\varphi}}^i - \boldsymbol{\varphi}^{\tilde{h}}, \hat{\lambda}^i_t - \lambda^{\tilde{h}}_t, \hat{\lambda}^i_n - \lambda^{\tilde{h}}_n))}{\|\tau^h\|_{\mathbf{X}}} \\ &= \frac{1}{\beta_1} \sup_{0 \ne \tau^h \in \mathbf{X}_h} \frac{\tilde{a}(\sigma^h - \hat{\sigma}^i, \tau^h)}{\|\tau^h\|_{\mathbf{X}}} \le \frac{C}{\beta_1} \|\sigma^h - \hat{\sigma}^i\|_{\mathbf{X}}, \end{aligned} \tag{3.116}$$

where $\beta_1$ is the constant in the inf-sup condition and $C$ the constant from the continuity of $\tilde{a}(\cdot,\cdot)$. Inserting (3.115) and (3.116) in (3.111) we have

$$\begin{aligned} \|\nu^i - \nu^{\hat{h}}\|^2_{L^2(A_C)} &\le \rho_2^2 \|\mathcal{F}\|^2_{L^\infty(A_C)} \frac{C^2}{\beta_1^2} \|\sigma^h - \hat{\sigma}^i\|^2_{\mathbf{X}} - 2\rho_2\, \alpha \|\hat{\sigma}^i - \sigma^h\|^2_{\mathbf{X}} + \|\nu^{i-1} - \nu^{\hat{h}}\|^2_{L^2(A_C)} \\ &= \left(\rho_2^2 \frac{C^2 \|\mathcal{F}\|^2_{L^\infty(A_C)}}{\beta_1^2} - 2\rho_2\, \alpha\right) \|\hat{\sigma}^i - \sigma^h\|^2_{\mathbf{X}} + \|\nu^{i-1} - \nu^{\hat{h}}\|^2_{L^2(A_C)}. \end{aligned} \tag{3.117}$$

For $\rho_2 \in (0, \frac{2\alpha\beta_1^2}{C^2 \|\mathcal{F}\|^2_{L^\infty(A_C)}})$ the sequence $\|\nu^i - \nu^{\hat{h}}\|^2_{L^2(A_C)}$ is monotonically decreasing. Moreover it is bounded from below by zero and therefore convergent. Reordering the terms in (3.117) and taking the limit $i \to \infty$ we conclude

$$\begin{aligned} 0 &\le \lim_{i\to\infty} \left(\rho_2^2 \frac{C^2 \|\mathcal{F}\|^2_{L^\infty(A_C)}}{\beta_1^2} - 2\rho_2\, \alpha\right) \|\hat{\sigma}^i - \sigma^h\|^2_{\mathbf{X}} \\ &\le \lim_{i\to\infty} \left(\|\nu^{i-1} - \nu^{\hat{h}}\|^2_{L^2(A_C)} - \|\nu^i - \nu^{\hat{h}}\|^2_{L^2(A_C)}\right) = 0, \\ &\Rightarrow \quad \lim_{i\to\infty} \|\hat{\sigma}^i - \sigma^h\|^2_{\mathbf{X}} = 0. \end{aligned}$$

Using again the discrete inf-sup condition (3.69), the first equation in (3.112) and the continuity of $\tilde{a}(\cdot,\cdot)$ we can show the convergence of the Lagrange multipliers in $\widetilde{\mathbf{Y}}_h$

$$\begin{aligned}
&\|(\hat{\mathbf{u}}^i - \mathbf{u}^h, \hat{\eta}^i - \eta^h, \hat{\boldsymbol{\varphi}}^i - \boldsymbol{\varphi}^{\tilde{h}}, \hat{\lambda}_t^i - \lambda_t^{\tilde{h}}, \hat{\lambda}_n^i - \lambda_n^{\tilde{h}})\|_{\widetilde{\mathbf{Y}}'} \\
&\qquad \le \frac{1}{\beta_1} \sup_{0\neq\tau^h\in\mathbf{X}_h} \frac{B(\tau^h, (\hat{\mathbf{u}}^i - \mathbf{u}^h, \hat{\eta}^i - \eta^h, \hat{\boldsymbol{\varphi}}^i - \boldsymbol{\varphi}^{\tilde{h}}, \hat{\lambda}_t^i - \lambda_t^{\tilde{h}}, \hat{\lambda}_n^i - \lambda_n^{\tilde{h}}))}{\|\tau^h\|_{\mathbf{X}}} \\
&\qquad = \frac{1}{\beta_1} \sup_{0\neq\tau^h\in\mathbf{X}_h} \frac{\tilde{a}(\sigma^h - \hat{\sigma}^i, \tau^h)}{\|\tau^h\|_{\mathbf{X}}} \le \frac{C}{\beta_1}\|\sigma^h - \hat{\sigma}^i\|_{\mathbf{X}}.
\end{aligned}$$

Finally, the discrete inf-sup condition (3.72), the fifth equation in (3.112) and the continuity of the bilinear form $d_{C,t}(\cdot,\cdot)$ leads to

$$\|\nu^{i-1} - \nu^{\hat{h}}\|_{-\frac{1}{2},\hat{h}} \le \frac{1}{\beta_2} \sup_{0\neq\mu_t\in C_{\tilde{h}}} \frac{q(\mu_t, \nu^{i-1} - \nu^{\hat{h}})}{\|\mu_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} = \frac{1}{\beta_2} \sup_{0\neq\mu_t\in C_{\tilde{h}}} \frac{d_{C,t}(\mu_t, \hat{\sigma}^i - \sigma^h)}{\|\mu_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} \le \frac{\tilde{C}}{\beta_1}\|\sigma^h - \hat{\sigma}^i\|_{\mathbf{X}},$$

which finishes the proof. □

**Remark 3.43:** *We can change the order of the loops in Algorithm 3.41. We just have to exchange step* **6** *with step* **8** *and adjust step* **7** *and step* **9**. *The convergence of this algorithm follows analogously to the proof of Theorem 3.42 without changing the bounds for the parameters $\rho_1$ and $\rho_2$.*

**Remark 3.44:** *For the projection in step* **8** *of Algorithm 3.41 we can proceed analogously to Remark 3.40 and use the bilinear form $\langle M\cdot,\cdot\rangle$ instead of the scalar product $(\cdot,\cdot)_{L^2(A_C)}$. Here $M$ denotes the mass matrix on $L_{\hat{h}}$. Then, we have to solve:*

*Find $P^{\Lambda}_{\hat{h}}\kappa \in \Lambda_{\hat{h}}$ such that*

$$\langle M P^{\Lambda}_{\hat{h}}\kappa, \psi - P^{\Lambda}_{\hat{h}}\kappa\rangle \ge \langle M\kappa, \psi - P^{\Lambda}_{\hat{h}}\kappa\rangle \qquad \forall\, \psi \in \Lambda_{\hat{h}}, \tag{3.118}$$

*and find $\Phi^{\Lambda}(\mu) \in L_{\hat{h}}$ such that*

$$\langle M\,\Phi^{\Lambda}(\mu), \kappa\rangle = q(\mu,\kappa) \qquad \forall\, \kappa \in L_{\hat{h}}. \tag{3.119}$$

*Inserting $\kappa = \nu^{i-1} + \rho_2\Phi^{\Lambda}(\hat{\lambda}_t^i)$ into* (3.118) *and using* (3.119) *we have to find $P^{\Lambda}_{\hat{h}}\kappa \in \Lambda_{\hat{h}}$ such that*

$$\langle M P^{\Lambda}_{\hat{h}}\kappa, \psi - P^{\Lambda}_{\hat{h}}\kappa\rangle \ge \langle M\nu^{i-1}, \psi - P^{\Lambda}_{\hat{h}}\kappa\rangle + \rho_2\, q(\hat{\lambda}_t^i, \psi - P^{\Lambda}_{\hat{h}}\kappa) \qquad \forall\, \psi \in \Lambda_{\hat{h}}. \tag{3.120}$$

Defining $\nu^i := P^{\Lambda}_{\hat{h}}\kappa$ and $b^i := M\nu^{i-1} + \rho_2\, Q_{\mathcal{F}}\hat{\lambda}_t^i$ the above problem reads: Find $\nu^i \in \mathbb{R}^{N_A}$ with $|(\nu^i)_j| \le 1$, such that

$$-(\nu^i)^T M\nu^i - (\psi - \nu^i)^T b^i \ge -\psi^T M\nu^i \qquad \forall\, \psi \in \mathbb{R}^{N_A}, |\psi_j| \le 1. \tag{3.121}$$

Furthermore, due to $M$ being positive definite we have

$$\tfrac{1}{2}\psi^T M\psi + \tfrac{1}{2}(v^i)^T M v^i \geq \psi^T M v^i \qquad \forall\, \psi \in \mathbb{R}^{N_A}. \tag{3.122}$$

Adding inequalities (3.121) and (3.122) we deduce

$$\begin{aligned}
& \tfrac{1}{2}\psi^T M\psi + \tfrac{1}{2}(v^i)^T M v^i - (v^i)^T M v^i - (\psi - v^i)^T b^i \geq 0 && \forall\, \psi \in \mathbb{R}^{N_A}, |\psi_j| \leq 1,\\
\Leftrightarrow\quad & \tfrac{1}{2}\psi^T M\psi - \psi^T b^i \geq \tfrac{1}{2}(v^i)^T M v^i - (v^i)^T b^i && \forall\, \psi \in \mathbb{R}^{N_A}, |\psi_j| \leq 1,\\
\Leftrightarrow\quad & v^i = \operatorname*{argmin}_{\psi\in\mathbb{R}^{N_A}, |\psi_j|\leq 1} \left\{\tfrac{1}{2}\psi^T M\psi - \psi^T b^i\right\}.
\end{aligned}$$

For this reason we can execute step **8** in Algorithm 3.41 by solving a convex quadratic program.

**Remark 3.45:** *Since our aim is to solve a contact problem with Coulomb friction we can modify Algorithm 3.41 in order to avoid a third loop within the algorithm. For this purpose we replace step* **8** *and* **9** *with the following*

**8′** *Compute* $\mathcal{F}^i := \mu_f |\hat{\sigma}_n^i|_{\Gamma_C}|$

**9′**

$$\text{STOP if} \qquad \frac{\|\mathcal{F}^i - \mathcal{F}^{i-1}\|}{\|\mathcal{F}^i\|} \leq \varepsilon_2,$$

*Else go to* **10′**,

**10′** *Set* $v^i := P^{\Lambda}_{\hat{h}}(v^{i-1} + \rho_2 \Phi^{\Lambda} \hat{\lambda}^i_t)$ *and go to* **2**,

*We have to approximate the friction function* $\mathcal{F} \in L_{\hat{h}}$ *in terms of the normal stress* $\sigma_n^h$ *on* $A_C$. *If we define by* $\hat{n}$ *the number of edges* $e \in \mathcal{E}_h^A$ *lying on some edge* $\hat{e} \in \mathcal{E}_{\hat{h}}^A$ *we define*

$$\mathcal{F}(\sigma^h)|_{\hat{e}} := \frac{\mu_f}{\hat{n}} \sum_{e\cap\hat{e}\neq\emptyset} |\,\sigma_n^h|_e\,| \qquad \forall\, \hat{e} \in \mathcal{E}_{\hat{h}}^A. \tag{3.123}$$

*Then, we use* $\mathcal{F}^0(\hat{\sigma}^0)$ *as initial friction function. In this case the bound for the second parameter* $\rho_2$ *may change within the algorithm. To overcome a possible failure of the algorithm in an iteration step at later date, the parameter* $\rho_2$ *can be coupled with the norm of the friction functional* $\mathcal{F}$. *Then the parameter is changing in each step. However, in our numerical experiments this process did not significantly improve the stability or even the speed of convergence of the algorithm.*

## 3.4. Numerical Experiments

In this section we present numerical experiments which underline the theoretical results of the previous section. The example is performed for the investigation of the two parameters $\rho_1$ and $\rho_2$ in Algorithm 3.41. Furthermore, we show, that the inf-sup condition (3.74) does not hold for some special case.

The results where computed on a cluster with 5 nodes à 8 cores with 2.93 GHz and 48GB. Each node uses two Intel Nehalem X5570 processors. All finite element matrices and right hand side vectors as well as the solver for the convex quadratic problem and the executing scripts are implemented in Matlab. The hypersingular operator for the scalar product $\langle W\cdot,\cdot\rangle$ as described in Remark 3.40 is computed by the software package MaiProgs, see Maischak [66]. Furthermore, the initial triangulations and the uniform refinement for the h-version is also done in MaiProgs. Since the solver for step **3** in Algorithm 3.38 and step **5** in Algorithm 3.41 is not our concern we use Matlab's internal LU-decomposition of the corresponding matrix with all possible optimizations, e.g. taking advantage of the symmetry of the matrix. This approach has not significantly affected the total solution time of the algorithm, e.g. in the first experiment on the finest triangulation level with about 4 000 000 unknowns the optimal solution time was 164 minutes, see Table 3.6, compared to 9 minutes of preprocessing for the LU-decomposition of the matrix. Of course we have to mention, that the factor between the solution time and the time for the LU-decomposition also depends on the storage of the CPU, the sparsity of the matrix and other criteria. Nevertheless it is sufficient for our purpose.

**Example 3.46:** *Let us consider the domain* $\Omega := [-4,4]\times[-1,1]$ *with boundary* $\Gamma$ *divided into the Dirichlet part* $\Gamma_D := \{-4,4\}\times[-1,1]$*, the Neumann part* $\Gamma_N := [-4,4]\times\{1\}$ *and the contact part* $\Gamma_C := [-4,4]\times\{-1\}$*. We choose Young's modulus* $E := 200\,000$ *and Poisson's ratio* $\nu := 0.25$ *which leads to the Lamé coefficients* $\lambda = \mu = 80\,000$*. The friction coefficient* $\mu_f$ *takes different values that will be specified in each example. The volume body force is set to zero, furthermore we assume, that the body is subject to the boundary traction*

$$\mathbf{t}_0 = \begin{cases} \begin{pmatrix} 0 \\ -800\,(1-\frac{x^2}{2}+\frac{x^4}{16}) \end{pmatrix}, & \textit{for } x \in [-2,2], \\ 0, & \textit{else}, \end{cases}$$

*on the Neumann boundary* $\Gamma_N$*. The body is fixed at the Dirichlet boundary* $\Gamma_D$*. On the contact boundary we assume the body* $\Omega$ *to come into contact with a rigid foundation which has positive distance* $g$ *to* $\Omega$*. In Figure 3.10 we show the domain and the distribution of the boundary as well as two examples of the gap function* $g$*. The red line denotes a constant gap function, whereas the blue line corresponds to some arbitrary gap function, not necessarily constant. We will solve the discrete scheme* (3.62) *for different gap functions, especially the case* $g \equiv 0$*. In this case assumption* (3.73) *does not hold anymore and we have to take* $\hat{h} = 2\tilde{h}$ *in the definition of the finite element space* $L_{\hat{h}}$*.*

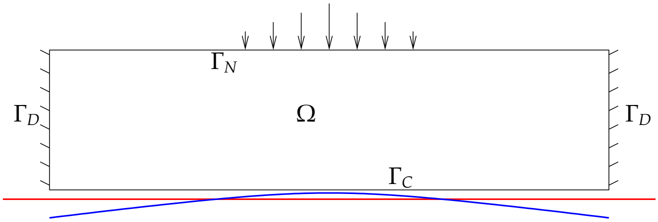



Figure 3.10.: Geometry, boundary parts and rigid foundations.

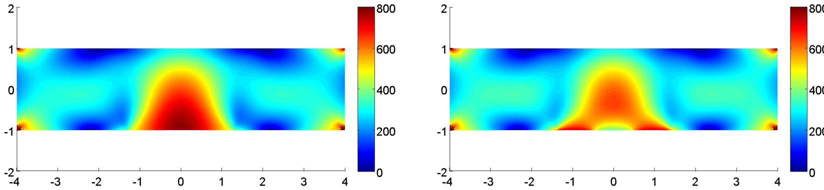
*We take eight different triangulations of the domain* $\Omega$ *and the boundary parts* $\Gamma_N$ *and* $\Gamma_C$*, where the last seven results from the previous triangulation by halving the meshsize, respectively. The corresponding degrees of freedom (Dof) of the functions in the different finite element spaces and the mesh size are shown in Table 3.1. On the boundary we have the mesh size* $\tilde{h} = \sqrt{2}h$ *The degrees of freedom for* $v^{\hat{h}} \in \Lambda_{\hat{h}}$ *are shown for the case* $\hat{h} = 2\tilde{h}$*, if we consider* $\hat{h} = \tilde{h}$*, the values in the last column have to be doubled.*

| mesh size $h$ | Dof($\mathbf{X}_h$) | Dof ($\mathbf{M}_h$) | Dof ($S_h$) | Dof($\mathbf{N}_{\tilde{h}}$) | Dof($C_{\tilde{h}}$) | Dof ($\Lambda_{\hat{h}}$) |
|---|---|---|---|---|---|---|
| 1.4142 | 180 | 64 | 27 | 6 | 3 | 2 |
| 0.7071 | 680 | 256 | 85 | 14 | 7 | 4 |
| 0.3536 | 2 640 | 1 024 | 297 | 30 | 15 | 8 |
| 0.1768 | 10 400 | 4 096 | 1 105 | 62 | 31 | 16 |
| 0.0884 | 41 280 | 16 384 | 4 257 | 126 | 63 | 32 |
| 0.0442 | 164 480 | 65 536 | 16 705 | 254 | 127 | 64 |
| 0.0221 | 656 640 | 262 144 | 66 177 | 510 | 255 | 128 |
| 0.0110 | 2 624 000 | 1 048 576 | 263 425 | 1 022 | 511 | 256 |

Table 3.1.: Degrees of freedom and mesh sizes for the first numerical example.

*In order to study the influence of friction on the solution we solved the described problem for friction coefficients* $\mu_f = 0$ *and* $\mu_f = 0.5$*. In the first case we have solved the discrete scheme* (3.39) *using Algorithm 3.38 with* $\varepsilon = 10^{-8}$*, whereas in the second case we used Algorithm 3.41 with* $\varepsilon_1 = 10^{-8}$ *and* $\varepsilon_2 = 10^{-6}$ *to solve the discrete scheme* (3.62)*. In both cases the gap function is* $g = 0.01$*. We use a mesh size of* $\hat{h} = 2\tilde{h}$ *but remark, that the solution algorithm would also converge for* $\hat{h} = \tilde{h}$*. In the following figures the solutions for both schemes are displayed. On the left side we show the solutions of the contact problem without friction and on the right side the one for the contact problem with friction.*

*In Figures 3.11(a) and 3.11(b) the von Mises equivalent stress for plane strain is illustrated. Using the theory of Nečas and Hlaváček [69, Section 10.2] and the yield criterion of Han and Reddy [57, Section 3.3] we can derive the following equation for the von Mises equivalent*




(a) Contact problem without friction, i.e. $\mu_f = 0$. (b) Frictional contact problem with $\mu_f = 0.5$.

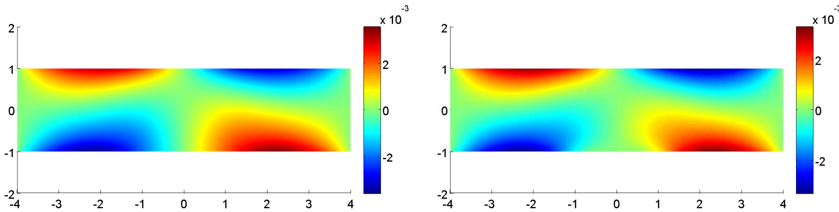
Figure 3.11.: Von Mises equivalent stress in $\Omega$

*stress for plane strain*

$$\sigma_0 := \left((\nu^2 - \nu + 1)[\sigma_{11}^2 + \sigma_{22}^2] + (2\nu^2 - 2\nu - 1)\sigma_{11}\sigma_{22} + 3\sigma_{12}\right)^{\frac{1}{2}},$$

*where $\nu$ denotes Poisson's ratio. Note, that we have used $\frac{3}{2}(\sigma_{12}^h + \sigma_{21}^h)$ instead of $3\sigma_{12}$ in the above equation, as the approximated stress tensor does not have to be symmetric. In engineering science the von Mises equivalent stress denotes a yield criterion to predict plastic zones, which occur in those areas, where the value of the equivalent stress exceeds some given limit.*

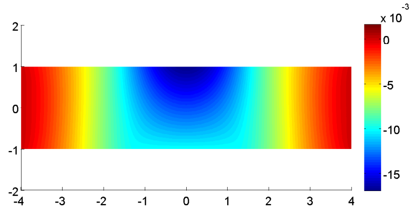
*The singularities of the equivalent stress in the corners of the domain are due to the change of the boundary conditions and will be neglected in the following discussion. We can see from the figures, that the presence of friction leads to a smaller maximal value of the equivalent stress. Furthermore, the zone of the maximal value has moved to the regions where the body is sliding on the rigid foundation.*



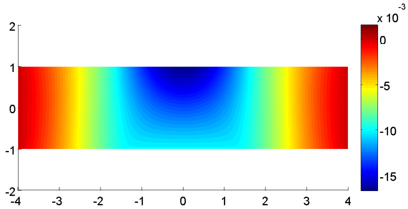

(a) Contact problem without friction, i.e. $\mu_f = 0$. (b) Frictional contact problem with $\mu_f = 0.5$.

Figure 3.12.: Displacement $u_1^h(\mathbf{x})$ in $\Omega$.

*Figures 3.12(a) to 3.13(b) show the two components $u_1^h$ and $u_2^h$ of the displacement fields for both cases. As we would expect, the only difference between the frictionless case and the case where friction occurs can be observed at the contact boundary for the first component $u_1^h$. This is the direction of the tangential displacement on $\Gamma_C$. As displayed more precisely in*



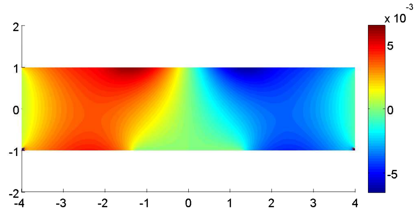

(a) Contact problem without friction, i.e. $\mu_f = 0$.



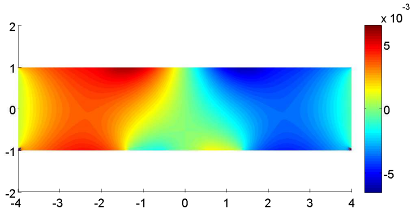

(b) Frictional contact problem with $\mu_f = 0.5$.

Figure 3.13.: Displacement $u_2^h(\mathbf{x})$ in $\Omega$.

*Figure 3.19(b), there exists a sticking zone around $x_1 \in (-0.5, 0.5)$ on the contact boundary for the frictional case.*




(a) Contact problem without friction, i.e. $\mu_f = 0$.

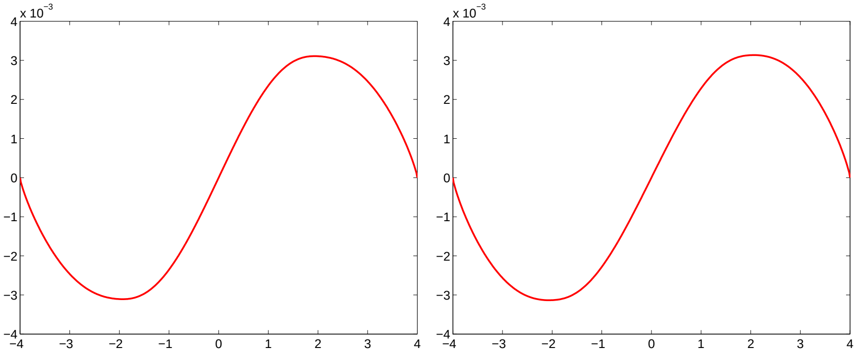



(b) Frictional contact problem with $\mu_f = 0.5$.

Figure 3.14.: Rotation $\eta^h$ in $\Omega$.

*Figures 3.14(a) and 3.14(b) show the rotational part $\eta^h$ of the gradient of the displacement field. The only remarkable difference between the two cases can be observed on the active part $A_C$ of the contact boundary , i.e. the part around $x_1 \in (-1.3, 1.3)$ on $\Gamma_C$, where the body comes into contact. In the frictionless case, the values of the rotation are nearly zero. This is obvious, since here the change of the normal displacement $\lambda_n^{\tilde{h}}$, corresponding to $u_2^h$, in $x_1$-direction is zero, cf. Figure 3.17(a). Furthermore, we observe in Figure 3.12(a), that the change of the first component $u_1^h$, in $x_2$-direction is also zero on this part of the contact boundary. The level set, illustrated by the colors, is nearly orthogonal to the contact boundary. This is not the case, when friction occurs. Therefore we observe local extrema in the zone where the body starts to slide, compare Figure 3.19(b). Another characteristic is, that in both Figures 3.14(a) and 3.14(b), the extrema occur on the boundaries.*

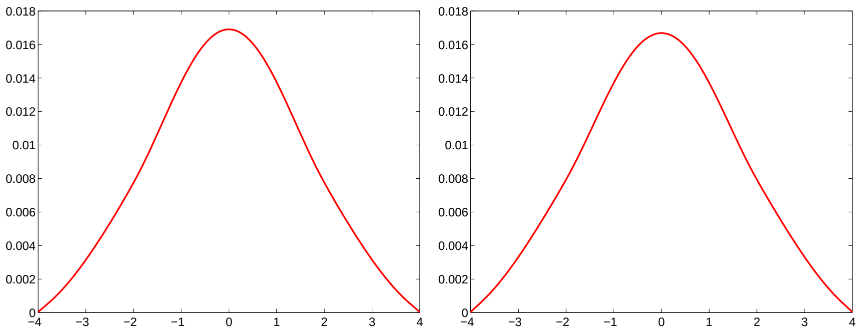
*Figures 3.15(a) to 3.16(b) show the components $\varphi_1^{\tilde{h}}$ and $\varphi_2^{\tilde{h}}$ of the displacement field on the Neumann boundary. There is no difference between the frictionless and the frictional case, as the traction $\mathbf{t}_0$ is the same in both cases. Note, that according to Theorem 3.10 the signs of the values of $\boldsymbol{\varphi}^{\tilde{h}}$ are opposite to the signs of the values of $\mathbf{u}^h|_{\Gamma_N}$.*




(a) Contact problem without friction, i.e. $\mu_f = 0$.

(b) Frictional contact problem with $\mu_f = 0.5$.

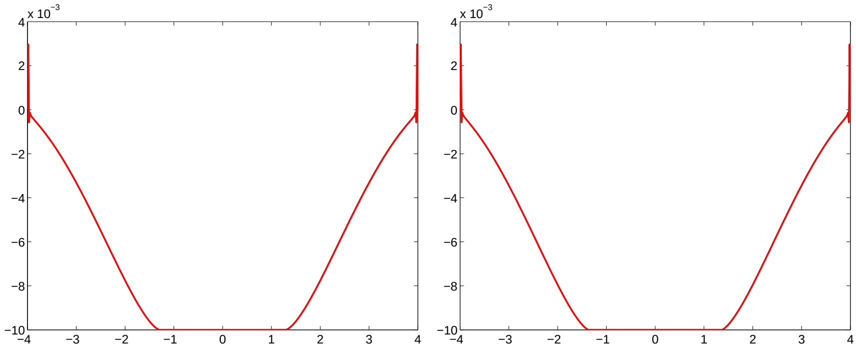
Figure 3.15.: Displacement $\boldsymbol{\varphi}_1^h(\mathbf{x})$ on $\Gamma_N$..




(a) Contact problem without friction, i.e. $\mu_f = 0$.

(b) Frictional contact problem with $\mu_f = 0.5$.

Figure 3.16.: Displacement $\boldsymbol{\varphi}_2^h(\mathbf{x})$ on $\Gamma_N$.

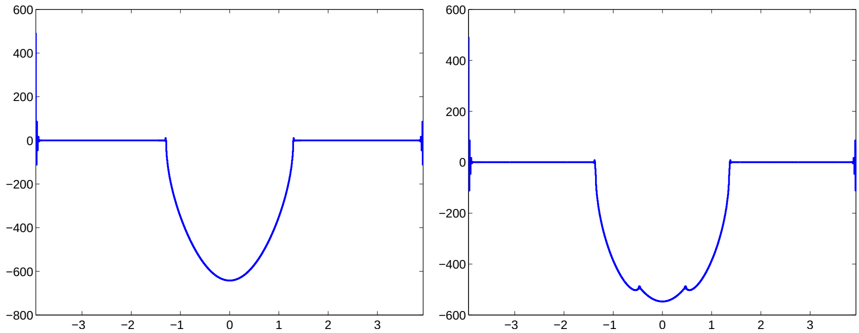



(a) Contact problem without friction, i.e. $\mu_f = 0$.

(b) Frictional contact problem with $\mu_f = 0.5$.

Figure 3.17.: Normal displacement $\lambda_n^{\tilde{h}}$ on $\Gamma_C$.

*At first glance the normal displacements* $\lambda_n^{\tilde{h}} - g$ *on the contact boundary in Figures 3.17(a) and 3.17(b) seem to be equal. However, a comparison of the vectors of coefficients for the two approximated solutions on the finest level shows, that in the frictional case there are ten more vertices in contact then in the case where no friction occurs. In the first case there are 165 vertices in contact, whereas in the second case we have 175 vertices in contact. From Table 3.1 we therefore deduce, that the contact zone in the second case is* 0.156 *larger than in the first case. Again we observe the result of Theorem 3.10, namely the sign of* $\lambda_n^{\tilde{h}} - g$ *is opposite to the sign of the normal displacement on* $\Gamma_C$ *which is* $-u_2^h|_{\Gamma_C}$.



(a) Contact problem without friction, i.e. $\mu_f = 0$.

(b) Frictional contact problem with $\mu_f = 0.5$.

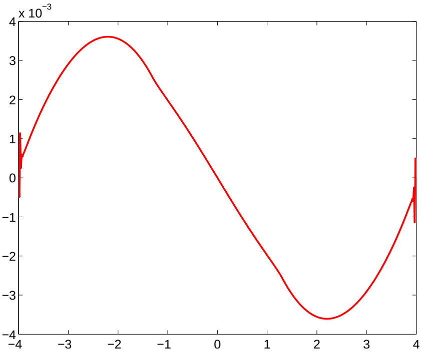
Figure 3.18.: Normal stress $\sigma_n^h$ on $\Gamma_C$.

*The maximal absolute value of the normal stress* $\sigma_n^h$ *on the contact boundary is larger in the case where no friction occurs although the contact zone is larger in the case with friction, see*

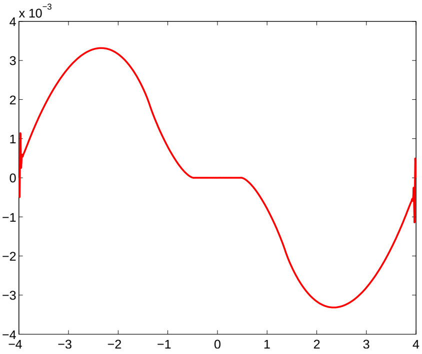
*Figures 3.18(a) and 3.18(b). The reason is, that the normal stress in the latter case exhibits a kink at those points, where the body changes from sliding to sticking and vice versa. On the other hand the absolute value of the total force $F_{abs}$ acting on the active part of $\Gamma_C$ is larger in the frictional case, where the total force is defined by*

$$F^h_{abs} := \int_{A_C} |\sigma^h_n| + |\sigma^h_t|\, ds.$$

*Using this equation we get a total force of* 1226 *in the frictionless case compared to a total force of* 1606 *in the frictional case. This is due to the presence of friction forces in the second case.*



(a) Contact problem without friction, i.e. $\mu_f = 0$.



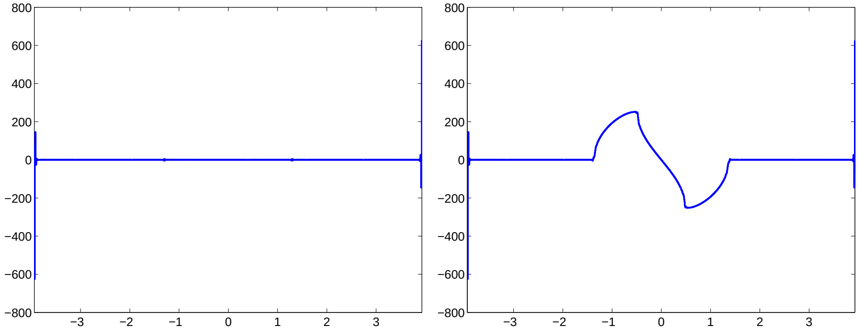
(b) Frictional contact problem with $\mu_f = 0.5$.

Figure 3.19.: Tangential displacement $\lambda^{\tilde{h}}_t$ on $\Gamma_C$.

*As shown in Figure 3.19(b), the tangential displacement $\lambda^{\tilde{h}}_t$ on the contact boundary possesses a sticking zone for $x_1 \in (-0.5, 0.5)$ in the case where friction occurs. Due to the Coulomb friction law* (3.2) *the tangential displacement is nonzero, if the tangential stress fulfills $|\sigma_t| = \mu_f|\sigma_n|$. This is visualized in Figure 3.21(a), where the ratio $\frac{|\sigma^h_t|}{|\sigma^h_n|}$ on the active part of the contact boundary is mapped. Note, that the value does not exceed the friction coefficient $\mu_f = 0.5$.*

*The sign of the tangential displacement $\lambda^{\tilde{h}}_t$ equals the sign of the tangential stress $\sigma^h_t$ which corresponds to the Coulomb friction law* (3.2)*, if we keep in mind the assertion of Theorem 3.10, i.e. $\lambda_t = -u_t|_{\Gamma_C}$. The tangential stress is pictured in Figures 3.20(a) and 3.20(b). In the frictionless case, the body is not kept from free sliding and therefore the tangential stress equals zero. In the second case friction forces occur that act in tangential direction.*

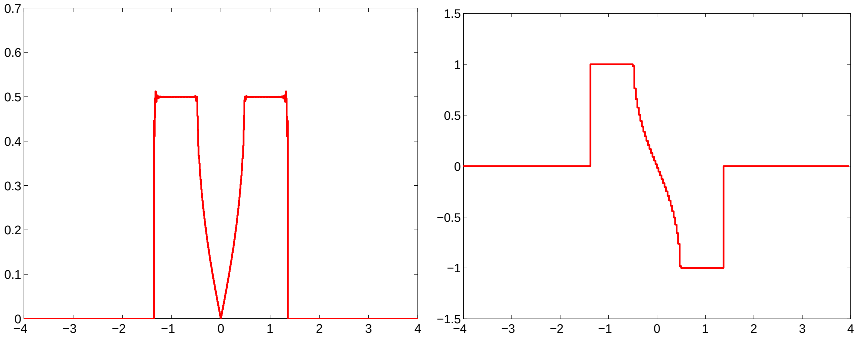
*Finally, we show in Figure 3.21(b) the sign $\nu^{\hat{h}}$ of the tangential displacement $\lambda^{\tilde{h}}_t$ on the active part of the boundary in the case of frictional contact. We remark, that the values are not*




(a) Contact problem without friction, i.e. $\mu_f = 0$.

(b) Frictional contact problem with $\mu_f = 0.5$.

Figure 3.20.: Tangential stress $\sigma_t^h$ on $\Gamma_C$.




(a) Contact problem without friction, i.e. $\mu_f = 0$.

(b) Frictional contact problem with $\mu_f = 0.5$.

Figure 3.21.: Sign $\nu^{\hat{h}}$ of $\lambda_t^{\tilde{h}}$ on $A_C$.

*zero in the zone where sticking occurs and* $\lambda_t^{\tilde{h}}$ *equals zero. The value is connected to the tangential stress by*

$$\nu^{\hat{h}}(\mathbf{x}) = \frac{\sigma_t^h(\mathbf{x})}{\mu_f |\sigma_n^h(\mathbf{x})|},$$

*on the whole active part* $A_C$ *regardless of whether sticking or sliding occurs. This was already pointed out in Remark 3.15.*

| $\|\sigma^h\|_X$ | $\|\mathbf{u}^h\|_{\mathbf{L}^2(\Omega)}$ | $\|\eta^h\|_{L^2(\Omega)}$ | $\|\boldsymbol{\varphi}^{\tilde{h}}\|_{\mathbf{L}^2(\Gamma_N)}$ | $\|\lambda_t^{\tilde{h}}\|_{L^2(\Gamma_C)}$ | $\|\lambda_n^{\tilde{h}}\|_{L^2(\Gamma_C)}$ | $\|\nu^{\hat{h}}\|_{L^2(A_C)}$ |
|---|---|---|---|---|---|---|
| 2617 | 3.399e-02 | 1.1797e-02 | 2.8114e-02 | 4.404e-03 | 6.208e-03 | 2.828 |
| 2489 | 3.534e-02 | 1.4050e-02 | 2.9210e-02 | 5.811e-03 | 1.166e-02 | 2.000 |
| 2481 | 3.550e-02 | 1.4573e-02 | 2.9198e-02 | 6.440e-03 | 1.286e-02 | 1.900 |
| 2488 | 3.550e-02 | 1.4588e-02 | 2.9161e-02 | 6.376e-03 | 1.323e-02 | 1.471 |
| 2497 | 3.543e-02 | 1.4571e-02 | 2.9111e-02 | 6.269e-03 | 1.340e-02 | 1.479 |
| 2504 | 3.541e-02 | 1.4566e-02 | 2.9094e-02 | 6.204e-03 | 1.349e-02 | 1.395 |
| 2508 | 3.539e-02 | 1.4565e-02 | 2.9084e-02 | 6.168e-03 | 1.354e-02 | 1.397 |
| 2510 | 3.538e-02 | 1.4566e-02 | 2.9076e-02 | 6.151e-03 | 1.357e-02 | 1.398 |

Table 3.2.: Norms of the solutions for the frictional contact problem with $\mu_f = 0.5$.

| $\|\sigma^h\|_X$ | $\|\mathbf{u}^h\|_{\mathbf{L}^2(\Omega)}$ | $\|\eta^h\|_{L^2(\Omega)}$ | $\|\boldsymbol{\varphi}^{\tilde{h}}\|_{\mathbf{L}^2(\Gamma_N)}$ | $\|\lambda_t^{\tilde{h}}\|_{L^2(\Gamma_C)}$ | $\|\lambda_n^{\tilde{h}}\|_{L^2(\Gamma_C)}$ |
|---|---|---|---|---|---|
| 26143 | 3.4595e-02 | 1.2200e-02 | 2.8466e-02 | 6.4989e-03 | 6.0165e-03 |
| 24659 | 3.5116e-02 | 1.4620e-02 | 2.9051e-02 | 7.6865e-03 | 1.2287e-02 |
| 24714 | 3.5604e-02 | 1.4896e-02 | 2.9269e-02 | 7.5524e-03 | 1.3070e-02 |
| 24775 | 3.5456e-02 | 1.4841e-02 | 2.9139e-02 | 7.3521e-03 | 1.3449e-02 |
| 24876 | 3.5445e-02 | 1.4827e-02 | 2.9126e-02 | 7.2336e-03 | 1.3590e-02 |
| 24940 | 3.5419e-02 | 1.4820e-02 | 2.9108e-02 | 7.1663e-03 | 1.3671e-02 |
| 24979 | 3.5399e-02 | 1.4819e-02 | 2.9097e-02 | 7.1323e-03 | 1.3720e-02 |
| 25003 | 3.5387e-02 | 1.4820e-02 | 2.9090e-02 | 7.1159e-03 | 1.3752e-02 |

Table 3.3.: Norms of the solutions for the contact problem without friction, i.e. $\mu_f = 0$.

*In Tables 3.2 and 3.3 we present the norms of the approximated solutions for each iteration step. For the Lagrange multipliers on the boundary parts we have computed the* $L^2$*-norms. We observe an asymptotic behaviour for all functions. In table 3.4 we have computed the total error and the rate of convergence. Since we do not know the exact solution of the problem, we use Aitken's* $\Delta^2$ *extrapolation process for the sequence of the norms of the approximated solutions, see e.g. Press et.al. [75, Section 5.1]. In this way we estimate the norms of the exact solutions of* (3.41). *Then we can estimate the error for an approximation in some iteration step i. For example if* $n_\sigma$ *is the extrapolated norm of the solution* $\sigma$*, then the error*

| $e_{tot}$ | $\alpha$ | $e_{tot}$ | $\alpha$ |
|---|---|---|---|
| 1.2121e+00 | - | 1.359e+00 | - |
| 6.7037e-01 | 0.904 | 6.815e-01 | 1.060 |
| 5.4220e-01 | 0.316 | 4.915e-01 | 0.487 |
| 3.8997e-01 | 0.483 | 4.023e-01 | 0.294 |
| 2.9375e-01 | 0.412 | 2.908e-01 | 0.472 |
| 2.1630e-01 | 0.443 | 2.063e-01 | 0.498 |
| 1.5780e-01 | 0.456 | 1.379e-01 | 0.582 |
| 1.1232e-01 | 0.491 | 9.985e-02 | 0.467 |

Table 3.4.: Total error and rate of convergence for the contact problem with $\mu_f = 0$ (left) and $\mu_f = 0.5$ (right).

*for the approximation $\sigma^i$ is estimated to*

$$e_i(\sigma) := \left(|\, \|\sigma^i\|_{\mathbf{X}}^2 - n_\sigma^2|\right)^{\frac{1}{2}}.$$

*In this way the total error for some iteration step i with mesh size $h_i$ is defined by*

$$e_{tot}^i := \left(e_i(\sigma)^2 + e_i(\mathbf{u})^2 + e_i(\eta)^2 + e_i(\boldsymbol{\varphi})^2 + e_i(\lambda_t)^2 + e_i(\lambda_n)^2 + e_i(\nu)^2\right)^{\frac{1}{2}}. \tag{3.124}$$

*If we do not have friction, then we drop the last term in the above definition. The rate of convergence is computed as follows. Since the number of degrees of freedom N is connected with the mesh size h by $N \approx \frac{1}{h^2}$ we have for the rate of convergence $\alpha$ of the error e,*

$$\|e\| \approx C h^\alpha \approx C N^{-\frac{\alpha}{2}}.$$

*Assuming that the constant does not change within two iterations of the h-version for the solution scheme* (3.41) *we conclude*

$$\alpha = -2 \frac{\log(\frac{\|e_i\|}{\|e_{i+1}\|})}{\log(\frac{N_i}{N_{i+1}})}.$$

*In Figure 3.22 we plot the total errors from Tables 3.2 and 3.3 against the corresponding number of degrees of freedom. The slope of $-\frac{1}{5}$ corresponds to a rate of convergence of* $\alpha = \frac{1}{\sqrt{5}} \approx 0.45$.

*In Tables 3.5 and 3.6 we present the number of iterations and the solution time (in seconds) of the algorithms for solving the two discussed problems. Here we have abbreviated the iteration step by It., the number of iterations for Algorithm 3.38 by #It, the number of iterations for the outer loop in Algorithm 3.41 by $\#It_{out}$ and the sum of all inner iterations by $\#It_{in}$. Furthermore, we have studied the sensitivity of both algorithms on the choice of the parameters $\rho, \rho_1$ and $\rho_2$. For the above problem we have identified the optimal parameters*

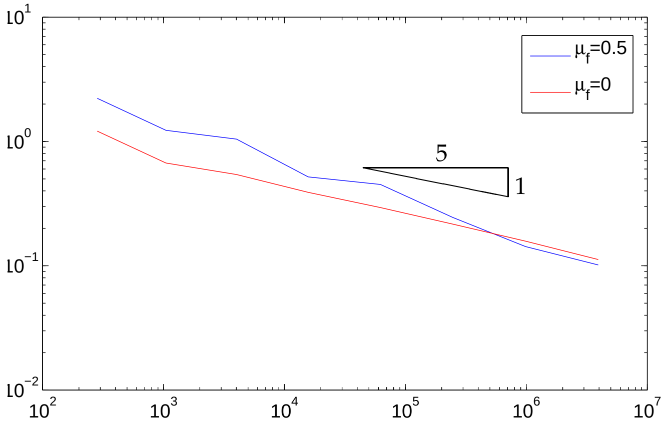
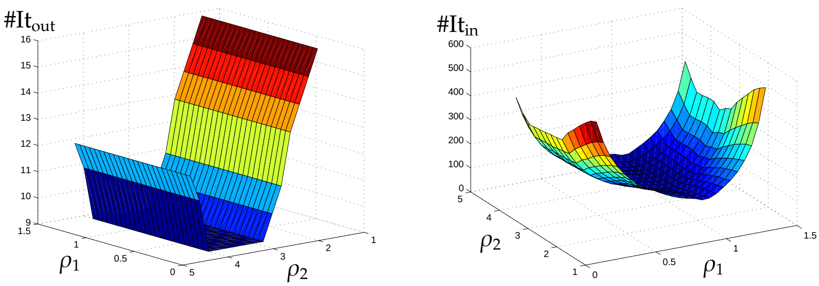



Figure 3.22.: Convergence of the frictionless and frictional contact problems in Example 3.46.




(a) Number of outer iterations.

(b) Total number of inner iterations.

Figure 3.23.: Iteration numbers for different pairs of $\rho_1$ and $\rho_2$.

| It. | #It | Sol. time | $\rho^{\text{opt}}$ | #It | Sol. time |
|---|---|---|---|---|---|
| 1 | 14 | 0.06 | 1.3 | 5 | 0.03 |
| 2 | 15 | 0.1 | 1.2 | 12 | 0.08 |
| 3 | 14 | 0.2 | 1.02 | 13 | 0.16 |
| 4 | 18 | 0.5 | 0.95 | 16 | 0.4 |
| 5 | 24 | 1.9 | 0.9 | 17 | 1.4 |
| 6 | 30 | 12.7 | 0.85 | 18 | 7.3 |
| 7 | 38 | 84.6 | 0.82 | 19 | 39 |
| 8 | 48 | 909 | 0.8 | 20 | 364 |

Table 3.5.: Number of iterations and solution times for the contact problem without friction.

| $\#\text{It}_{\text{out}}$ | $\#\text{It}_{\text{in}}$ | Sol. time | $\rho_1^{\text{opt}}$ | $\rho_2^{\text{opt}}$ | $\#\text{It}_{\text{out}}$ | $\#\text{It}_{\text{in}}$ | Sol. time |
|---|---|---|---|---|---|---|---|
| 10 | 86 | 0.4 | 1.3 | 3.5 | 7 | 27 | 0.15 |
| 7 | 78 | 0.6 | 1.225 | 3.0 | 7 | 55 | 0.4 |
| 18 | 140 | 1.6 | 1.05 | 1.85 | 10 | 74 | 0.9 |
| 20 | 141 | 3.9 | 0.97 | 3.9 | 8 | 71 | 2.1 |
| 49 | 301 | 30.6 | 0.93 | 2.2 | 24 | 153 | 15.3 |
| 84 | 589 | 282 | 0.9 | 2.0 | 48 | 252 | 120 |
| 134 | 964 | 2737 | 0.9 | 1.8 | 83 | 384 | 1094 |
| 198 | 1051 | 19433 | 0.89 | 1.75 | 135 | 558 | 9820 |

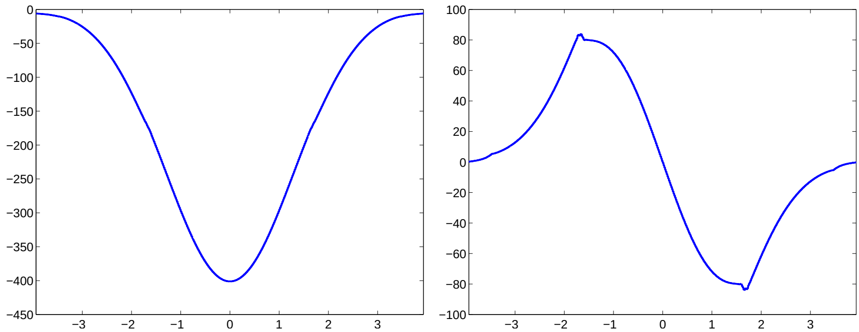
Table 3.6.: Number of iterations and solution times for the frictional contact problem.

*in each iteration of the h-version. The time for solving the problem is halved on average for both algorithms, when choosing the optimal parameters.*

*The optimal choice for the parameter $\rho_1$ is always around 1 and decrease with a smaller mesh size. The second parameter is more sensitive. This is due to the restriction of $\rho_2$, which depends on four different values, see Theorem 3.42, whereas the first parameter $\rho_1$ is only depending on the constant for the ellipticity of $\mathbb{C}^{-1}$, see* (2.3).

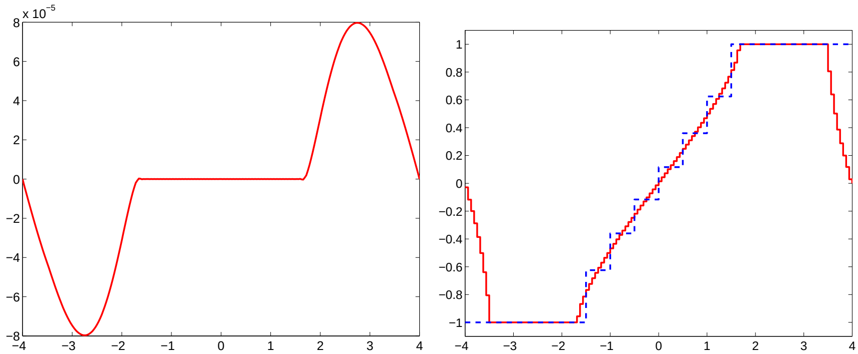
*For the frictional case we studied the dependence of the number of iterations on the choice of the parameters more precisely. In Figures 3.23(a) and 3.23(b) we plot the number of iterations for the outer and inner loop against the pair of parameters $(\rho_1, \rho_2)$ for the fourth iteration step in the h-version. The color illustrates the optimal choice of parameters. Here, blue means a small number of iterations and red a large one.*

*If we choose the gap function $g$ to be zero, then assumption* (3.73) *does not hold any longer, since the active part of the contact boundary is $\Gamma_C$. In this case the algorithm is not converging, when we take $\hat{h} = \tilde{h}$, because the discrete inf-sup condition for the bilinear form $q(\cdot,\cdot)$ does not hold. But as we proved in Lemma 3.29 the inf-sup condition is valid, when choosing $\hat{h} = 2\tilde{h}$. We have computed the above described problem with $g = 0$ and friction coefficient $\mu_f = 0.5$*




(a) Normal stress $\sigma_n^h$ on $\Gamma_C$. (b) Tangential stress $\sigma_t^h$ on $\Gamma_C$.

Figure 3.24.: Stresses on $\Gamma_C$ with gap function $g = 0$ using $\hat{h} = 2\tilde{h}$.

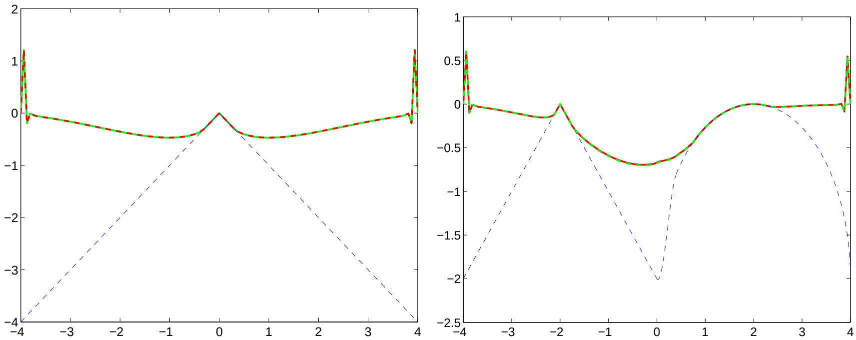



(a) Tangential displacement $\lambda_t^{\tilde{h}}$ on $\Gamma_C$. (b) Sign $\nu^{\hat{h}}$ of $\lambda_t^{\tilde{h}}$ on $A_C$.

Figure 3.25.: Tagential displacement and corresponding sign on $\Gamma_C$ with gap function $g = 0$ using $\hat{h} = 2\tilde{h}$.

*using a coarser mesh for $\nu^{\hat{h}}$. In Figures 3.24(a) to 3.25(b) we show the normal and tangential stress on the contact boundary, the tangential displacement on the contact boundary and the corresponding sign. The first observation we make is, that the singularities at the end points have vanished. The kink in the function of the tangential stress is due to the change from slip to stick and vice versa. In Figure 3.25(b) we notice, that the red colored sign function $\nu^{\hat{h}}$ of the tangential displacement $\lambda_t^{\tilde{h}}$ is tending to zero at the end points of the contact boundary. This is due to the fact, that we have chosen $\varepsilon_2 = 10^{-6}$ in Algorithm 3.41. In Figure 3.24(a) we see, that the normal stress is tending very fast to zero at the end points of $\Gamma_C$. For this reason the friction functional is small here and the convergence criterion in Algorithm 3.41 is satisfied before a good approximation is assumed for $\nu^{\hat{h}}$. Therefore, we have decreased the bound $\varepsilon_2$ to $10^{-8}$ resulting in the dashed blue function in Figure 3.25(b), which represents the approximated sign $\nu^{\hat{h}}$ for a coarser triangulation.*

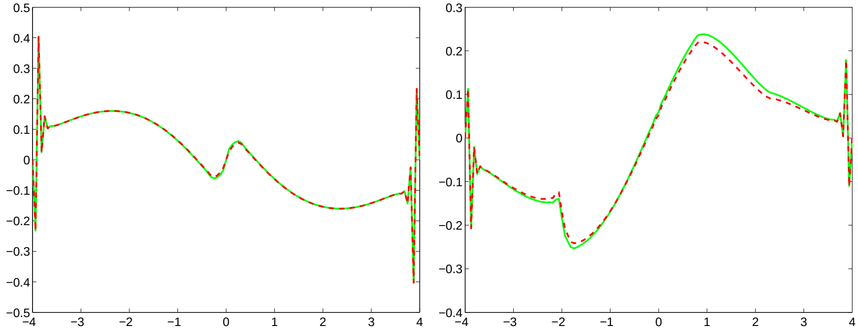
*The inf-sup condition in Lemma 3.29 is only valid for a mesh dependent norm. For this reason the constant within the condition is decreasing with the mesh size and therefore the parameter $\rho_2$ has to be chosen very small. Furthermore, as we have seen above, we have to decrease the parameter $\varepsilon_2$ to get a satisfactory approximation of $\nu$. This leads to a large number of iterations and thus to a large solution time. For the blue dashed line in Figure 3.24(a), which corresponds to $\nu^{\hat{h}}$ in the fourth iteration of the h-version, the algorithm took* 22 872 *outer loops and all in all* 45 743 *inner iterations. Compared to 3.9 seconds as we see e.g. in the fourth row of Table 3.6 the solution time of 3471 seconds is very large in this case.*




(a) Example with gap function $g = |x_1|$ (b) Example with gap function defined in (3.125)

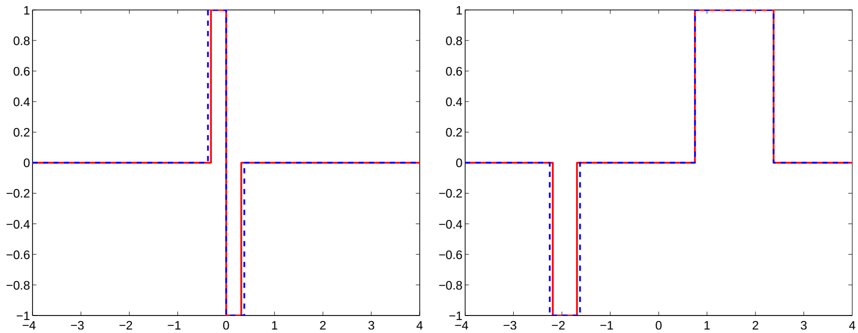
Figure 3.26.: Normal displacements $\lambda_n^{\tilde{h}}$ for $\hat{h} = \tilde{h}$ (dashed red) and $\hat{h} = 2\tilde{h}$ (green) and rigid foundation (dashed blue).

*To show, that the algorithm is converging to the same solutions independently of the choice of $\hat{h}$, if Assumption* (3.73) *is fulfilled, we have computed two examples with $\hat{h} = \tilde{h}$ and $\hat{h} = 2\tilde{h}$. In both examples the friction coefficient is $\mu_f = 0.5$. In Figures 3.26(a) to 3.28(b) we show the normal and tangential displacements and the corresponding signs. On the left hand side we have the gap function $g = |x_1|$. In Figure 3.26(a) the corresponding rigid foundation is*




(a) Example with gap function $g = |x_1|$ (b) Example with gap function defined in (3.125)

Figure 3.27.: Tangential displacements $\lambda_t^{\tilde{h}}$ for $\hat{h} = \tilde{h}$ (dashed red) and $\hat{h} = 2\tilde{h}$ (green).




(a) Example with gap function $g = |x_1|$ (b) Example with gap function defined in (3.125)

Figure 3.28.: Sign $\nu^{\hat{h}}$ of $\lambda_t^{\tilde{h}}$ for $\hat{h} = \tilde{h}$ (red) and $\hat{h} = 2\tilde{h}$ (dashed blue).

*shown as a dashed blue line. One observes, that the normal displacement in Figure 3.26(a) and the tangential displacement in Figure 3.27(a) fit very well for the two choices of* $\hat{h}$*. The same is with the signs in Figure 3.28(a), where the different width of the active set* $A_C$ *is due to the different length of the edges. In the second example the gap function is*

$$g = \begin{cases} |x_1 + 2|, & \text{if } x_1 \leq 0, \\ 2 + x_1 - \dfrac{145\, x_1^2}{6} + \dfrac{425\, x_1^3}{12}, & \text{if } x_1 \in (0, 0.4), \\ 2 - \sqrt{4 - (x_1 - 2)^2}, & \text{if } x_1 \geq 0.4. \end{cases} \tag{3.125}$$

*Here the tangential displacements in Figure 3.27(b) are a bit different, which is due to the complexity of the gap function and the chosen parameter* $\varepsilon_2 = 10^{-6}$ *for the convergence criterion in step* **9** *of Algorithm 3.41.*

# 4. Adaptive methods for the dual-dual contact problem with friction

This chapter deals with an adaptive algorithm for the discrete variational inequality problem (3.62) in Section 3.2. The algorithm is based on an a posteriori error estimator which is derived by following the approaches of Maischak [67, Chapter 5.2], Gatica and Meddahi [50] and Gatica, Gatica and Stephan in [46].

## 4.1. A posteriori error estimate

In this section we will derive an a posteriori error estimator for the dual-dual contact problem of Chapter 3. To do so let $(\sigma, \nu, \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \Lambda \times \mathbf{Y}$ and $(\sigma^h, \nu^{\hat{h}}, \mathbf{u}^h, \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}) \in \mathbf{X}_h \times \Lambda_{\hat{h}} \times \mathbf{Y}_h$ be the solutions of the continuous and discrete variational inequality problems (3.41) and (3.62), respectively. We use the notations of Section 3.4 and assume here, that the boundary parts of the domain are polygonal. Furthermore, let the assumption (3.73) be fulfilled. In this case we are allowed to choose $\hat{h} = \tilde{h}$, see Lemma 3.31, which is assumed here as well.

For an arbitrary $T \in \mathcal{T}_h$ we consider the following local auxiliary problem of finding $\mathbf{z}_T \in \mathbf{H}^1(T)$ such that

$$
\begin{aligned}
-\operatorname{div} \mathbb{C} : \varepsilon(\mathbf{z}_T) &= \mathbf{f} + \operatorname{div}(\sigma^h - \operatorname{as}(\sigma^{\mathrm{h}})) && \text{in } T\,, \\
\mathbf{z}_T &= 0 && \text{on } \partial T \cap \mathcal{E}_h^D\,, \\
\mathbb{C} : \varepsilon(\mathbf{z}_T) \cdot \mathbf{n} &= \mathbf{t}_0 - (\sigma^h - \operatorname{as}(\sigma^{\mathrm{h}})) \cdot \mathbf{n} && \text{on } \partial T \cap \mathcal{E}_h^N\,, \\
\mathbb{C} : \varepsilon(\mathbf{z}_T) \cdot \mathbf{n} &= \operatorname{as}(\sigma^{\mathrm{h}}) \cdot \mathbf{n} && \text{on } \partial T \cap \mathcal{E}_h^C\,, \\
\mathbb{C} : \varepsilon(\mathbf{z}_T) \cdot \mathbf{n} &= 0 && \text{on } \partial T \cap \mathcal{E}_h^\Omega\,,
\end{aligned}
\tag{4.1}
$$

with $\operatorname{as}(\tau)$ denoting the antisymmetric part of some tensor $\tau$, i.e. $\operatorname{as}(\tau) := \frac{1}{2}(\tau - \tau^T)$. If we set $\sigma^* \in \mathbf{X}$ with $\sigma^*|_T := \mathbb{C} : \varepsilon(\mathbf{z}_T)\ \forall\, T \in \mathcal{T}_h$ and $\tilde{\sigma} := \sigma^* + \sigma^h - \operatorname{as}(\sigma^{\mathrm{h}})$, then we observe $\operatorname{div}(\sigma - \tilde{\sigma}) = 0$ and $\operatorname{as}(\sigma - \tilde{\sigma}) = 0$. Using the auxiliary problem (4.1) we can prove the following lemma.

**Lemma 4.1:**
There exists a constant $C > 0$, such that

$$\begin{aligned}\alpha \,\|\sigma - \tilde{\sigma}\|_{\mathbf{X}}^2 \leq &- \tilde{a}(\sigma^h, \sigma - \tilde{\sigma} - \tau) - s(\eta^h, \sigma - \tilde{\sigma} - \tau) - d_N(\boldsymbol{\varphi}^{\tilde{h}}, \sigma - \tilde{\sigma} - \tau) \\ &- d_{C,t}(\lambda_t^{\tilde{h}}, \sigma - \tilde{\sigma} - \tau) + g(\sigma - \tilde{\sigma} - \tau) - d_{C,n}(\lambda_n^{\tilde{h}}, \sigma - \tilde{\sigma} - \tau) \\ &+ C\left\{\|\mathbf{f} + \operatorname{div}\sigma^h\|_{\mathbf{L}^2(\Omega)} + \|\mathrm{as}(\sigma^{\mathrm{h}})\|_{\mathbf{X}} + \|\mathbf{t}_0 - \sigma^h \cdot \mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)}\right\} \|\sigma - \tilde{\sigma}\|_{\mathbf{X}}\,,\end{aligned} \tag{4.2}$$

for all $\tau \in \mathbf{X}_h$, with $\operatorname{div}\tau = 0$.

*Proof.* The auxiliary problem (4.1) is uniquely solvable and depends continuously on the given data. Using $\int_\Omega \cdot\, dx = \sum_{T\in\mathcal{T}_h} \int_T \cdot\, dx$ and Theorem 2.3 we have with the triangle inequality

$$\begin{aligned}\|\sigma^*\|_{\mathbf{X}} &\leq C \sum_{T\in\mathcal{T}_h} \Big\{\|\mathbf{f} + \operatorname{div}(\sigma^h - \mathrm{as}(\sigma^{\mathrm{h}}))\|_{L^2(T)} + \|\mathbf{t}_0 - (\sigma^h - \mathrm{as}(\sigma^{\mathrm{h}}))\cdot\mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(e_T\cap\mathcal{E}_h^N)} \\ &\qquad\qquad + \|\mathrm{as}(\sigma^{\mathrm{h}})\cdot\mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(e_T\cap\mathcal{E}_h^C)}\Big\} \\ &\leq C \sum_{T\in\mathcal{T}_h} \Big\{\|\mathbf{f} + \operatorname{div}\sigma^h\|_{L^2(T)} + \|\operatorname{div}\mathrm{as}(\sigma^{\mathrm{h}})\|_{L^2(T)} + \|\mathbf{t}_0 - \sigma^h\cdot\mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(e_T\cap\mathcal{E}_h^N)} \\ &\qquad\qquad + \|\mathrm{as}(\sigma^{\mathrm{h}})\cdot\mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(e_T\cap\mathcal{E}_h^{CN})}\Big\} \\ &= C\left\{\|\mathbf{f} + \operatorname{div}\sigma^h\|_{L^2(\Omega)} + \|\mathrm{as}(\sigma^{\mathrm{h}})\|_{\mathbf{X}} + \|\mathbf{t}_0 - \sigma^h\cdot\mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)}\right\},\end{aligned} \tag{4.3}$$

where $\mathcal{E}_h^{CN} = \mathcal{E}_h^C \cup \mathcal{E}_h^N$. Using the coercivity of $\tilde{a}(\cdot,\cdot)$ and $\operatorname{div}(\sigma - \tilde{\sigma}) = 0$ we have

$$\begin{aligned}\alpha\|\sigma - \tilde{\sigma}\|_{\mathbf{X}}^2 = \alpha\|\sigma - \tilde{\sigma}\|_{\mathbf{L}^2(\Omega)^{2\times 2}}^2 &\leq \tilde{a}(\sigma - \tilde{\sigma}, \sigma - \tilde{\sigma}) \\ &= \tilde{a}(\sigma - \sigma^h, \sigma - \tilde{\sigma}) - \tilde{a}(\sigma^* - \mathrm{as}(\sigma^{\mathrm{h}}), \sigma - \tilde{\sigma}).\end{aligned} \tag{4.4}$$

The second term in (4.4) can be estimated with the continuity of the bilinear form $\tilde{a}(\cdot,\cdot)$, the triangle inequality and (4.3)

$$\begin{aligned}\tilde{a}(\sigma^* - \mathrm{as}(\sigma^{\mathrm{h}}), \sigma - \tilde{\sigma}) &\leq C\,\|\sigma^* - \mathrm{as}(\sigma^{\mathrm{h}})\|_{\mathbf{X}}\|\sigma - \tilde{\sigma}\|_{\mathbf{X}} \\ &\leq C\left\{\|\sigma^*\|_{\mathbf{X}} + \|\mathrm{as}(\sigma^{\mathrm{h}})\|_{\mathbf{X}}\right\}\|\sigma - \tilde{\sigma}\|_{\mathbf{X}} \\ &\leq C\left\{\|\mathbf{f} + \operatorname{div}\sigma^h\|_{L^2(\Omega)} + \|\mathrm{as}(\sigma^{\mathrm{h}})\|_{\mathbf{X}} + \|\mathbf{t}_0 - \sigma^h\cdot\mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)}\right\}\|\sigma - \tilde{\sigma}\|_{\mathbf{X}}.\end{aligned} \tag{4.5}$$

For the first term in (4.4) we take some tensor $\tau \in \mathbf{X}_h$ with $\operatorname{div}\tau = 0$. Then from $(3.41)_1$ and $(3.62)_1$ we conclude

$$\begin{aligned}\tilde{a}(\sigma - \sigma^h, \tau) &= -B(\tau; \mathbf{u} - \mathbf{u}^h, \eta - \eta^h, \boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t - \lambda_t^{\tilde{h}}, \lambda_n - \lambda_n^{\tilde{h}}) \\ &= -s(\eta - \eta^h, \tau) - d_N(\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}, \tau) - d_{C,t}(\lambda_t - \lambda_t^{\tilde{h}}, \tau) - d_{C,n}(\lambda_n - \lambda_n^{\tilde{h}}, \tau).\end{aligned} \tag{4.6}$$

Moreover, due to

$$\operatorname{div}(\sigma - \tilde{\sigma} - \tau) = \operatorname{div}(\sigma - \tilde{\sigma}) - \operatorname{div}\tau = 0 \quad \text{in}\Omega$$

and using equation $(3.41)_1$ we have

$$\begin{aligned}
\tilde{a}(\sigma, \sigma - \tilde{\sigma} - \tau) &= g(\sigma - \tilde{\sigma} - \tau) - B(\sigma - \tilde{\sigma} - \tau; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \\
&= g(\sigma - \tilde{\sigma} - \tau) - s(\eta, \sigma - \tilde{\sigma} - \tau) - d_N(\boldsymbol{\varphi}, \sigma - \tilde{\sigma} - \tau) \\
&\qquad - d_{C,t}(\lambda_t, \sigma - \tilde{\sigma} - \tau) - d_{C,n}(\lambda_n, \sigma - \tilde{\sigma} - \tau).
\end{aligned} \tag{4.7}$$

Using (4.6) and (4.7) we can rewrite the first term on the right hand side of equation (4.4) as

$$\begin{aligned}
\tilde{a}(\sigma - \sigma^h, \sigma - \tilde{\sigma}) &= \tilde{a}(\sigma, \sigma - \tilde{\sigma} - \tau) + \tilde{a}(\sigma - \sigma^h, \tau) - \tilde{a}(\sigma^h, \sigma - \tilde{\sigma} - \tau) \\
&= \tilde{a}(\sigma, \sigma - \tilde{\sigma} - \tau) - s(\eta - \eta^h, \tau) - d_N(\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}, \tau) \\
&\quad - d_{C,t}(\lambda_t - \lambda_t^{\tilde{h}}, \tau) - d_{C,n}(\lambda_n - \lambda_n^{\tilde{h}}, \tau) - \tilde{a}(\sigma^h, \sigma - \tilde{\sigma} - \tau) \\
&= -\tilde{a}(\sigma^h, \sigma - \tilde{\sigma} - \tau) - s(\eta, \sigma - \tilde{\sigma} - \tau) - s(\eta - \eta^h, \tau) \\
&\quad - d_N(\boldsymbol{\varphi}, \sigma - \tilde{\sigma} - \tau) - d_N(\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}, \tau) \\
&\quad - d_{C,t}(\lambda_t, \sigma - \tilde{\sigma} - \tau) - d_{C,t}(\lambda_t - \lambda_t^{\tilde{h}}, \tau) \\
&\quad - g(\sigma - \tilde{\sigma} - \tau) - d_{C,n}(\lambda_n, \sigma - \tilde{\sigma} - \tau) - d_{C,n}(\lambda_n - \lambda_n^{\tilde{h}}, \tau).
\end{aligned} \tag{4.8}$$

Since $\operatorname{as}(\sigma - \tilde{\sigma}) = 0$, the terms in (4.8), concerning the rotational parts of the displacement, read

$$\begin{aligned}
-s(\eta, \sigma - \tilde{\sigma} - \tau) - s(\eta - \eta^h, \tau) &= -s(\eta^h, \sigma - \tilde{\sigma} - \tau) - s(\eta - \eta^h, \sigma - \tilde{\sigma}) \\
&= -s(\eta^h, \sigma - \tilde{\sigma} - \tau).
\end{aligned} \tag{4.9}$$

For the normal component of $\sigma - \tilde{\sigma}$ on the boundary we observe

$$\begin{aligned}
(\sigma - \tilde{\sigma}) \cdot \mathbf{n} &= \mathbf{t}_0 - (\mathbf{t}_0 - (\sigma^h - \operatorname{as}(\sigma^{\mathrm{h}})) \cdot \mathbf{n}) - \sigma^h \cdot \mathbf{n} + \operatorname{as}(\sigma^{\mathrm{h}}) \cdot \mathbf{n} = 0 \quad &&\text{on } \Gamma_N, \\
(\sigma - \tilde{\sigma}) \cdot \mathbf{n} &= \sigma \cdot \mathbf{n} - \operatorname{as}(\sigma^{\mathrm{h}}) \cdot \mathbf{n} - \sigma^h \cdot \mathbf{n} + \operatorname{as}(\sigma^{\mathrm{h}}) \cdot \mathbf{n} = \sigma \cdot \mathbf{n} - \sigma^h \cdot \mathbf{n} \quad &&\text{on } \Gamma_C.
\end{aligned}$$

By further reordering the boundary parts in equation (4.8) and applying the above observations, we obtain

$$\begin{aligned}
-d_N(\boldsymbol{\varphi}, \sigma - \tilde{\sigma} - \tau) - d_N(\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}, \tau) &= -d_N(\boldsymbol{\varphi}^{\tilde{h}}, \sigma - \tilde{\sigma} - \tau) - d_N(\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}, \sigma - \tilde{\sigma}) \\
&= -d_N(\boldsymbol{\varphi}^{\tilde{h}}, \sigma - \tilde{\sigma} - \tau),
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
-d_{C,t}(\lambda_t, \sigma - \tilde{\sigma} - \tau) - d_{C,t}(\lambda_t - \lambda_t^{\tilde{h}}, \tau) &= -d_{C,t}(\lambda_t^{\tilde{h}}, \sigma - \tilde{\sigma} - \tau) - d_{C,t}(\lambda_t - \lambda_t^{\tilde{h}}, \sigma - \tilde{\sigma}) \\
&= -d_{C,t}(\lambda_t^{\tilde{h}}, \sigma - \tilde{\sigma} - \tau) - d_{C,t}(\lambda_t - \lambda_t^{\tilde{h}}, \sigma - \sigma^h),
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
-d_{C,n}(\lambda_n, \sigma - \tilde{\sigma} - \tau) - d_{C,n}(\lambda_n - \lambda_n^{\tilde{h}}, \tau) &= -d_{C,n}(\lambda_n^{\tilde{h}}, \sigma - \tilde{\sigma} - \tau) - d_{C,n}(\lambda_n - \lambda_n^{\tilde{h}}, \sigma - \tilde{\sigma}) \\
&= -d_{C,n}(\lambda_n^{\tilde{h}}, \sigma - \tilde{\sigma} - \tau) - d_{C,n}(\lambda_n - \lambda_n^{\tilde{h}}, \sigma - \sigma^h).
\end{aligned} \tag{4.12}$$

The second term in (4.11) can be bounded by zero. To see this we insert $\mu_t = \lambda_t$ in $(3.41)_5$ and use equations (3.42) and (3.45) to get

$$-d_{C,t}(\lambda_t - \lambda_t^{\tilde{h}}, \sigma) = -j(\lambda_t) + d_{C,t}(\lambda_t^{\tilde{h}}, \sigma) \leq -j(\lambda_t) + j(\lambda_t^{\tilde{h}}).$$

Analogously we take in the discrete case $\mu_t = \lambda_t^{\tilde{h}}$ in $(3.62)_5$ and use (3.64) and (3.67) to conclude

$$d_{C,t}(\lambda_t - \lambda_t^{\tilde{h}}, \sigma^h) = d_{C,t}(\lambda_t, \sigma^h) - j(\lambda_t^{\tilde{h}}) \leq j(\lambda_t) - j(\lambda_t^{\tilde{h}}).$$

The sum of the two inequalities above leads us to the desired estimate. Finally, the second term in (4.12) is also bounded by zero. Here we use equations $(3.38)_6$, (3.45), (3.63), (3.65) and the fact, that $\lambda_n \geq 0$ and $\lambda_n^{\tilde{h}} \geq 0$ on $\Gamma_C$ yield

$$-d_{C,n}(\lambda_n - \lambda_n^{\tilde{h}}, \sigma - \sigma^h) = d_{C,n}(\lambda_n^{\tilde{h}}, \sigma) + d_{C,n}(\lambda_n, \sigma^h) \leq 0.$$

Collecting the estimates (4.5) and (4.8) - (4.12) in (4.4) we finish the proof. □

To estimate the error of $\sigma^h$ we will need the Clément interpolation operator $\mathcal{I}_h$ : $\mathbf{H}^1(\Omega) \to \mathbf{V}_h$, see Clément [27]. Here we have used the standard finite element space

$$\mathbf{V}_h := \left\{ \mathbf{v} \in \mathbf{V}_D : \ \mathbf{v} \in \mathbf{P}_1^h(T), \ \forall T \in \mathcal{T}_h \right\}. \tag{4.13}$$

Let $h_T$ and $h_e$ denote the diameter of $\omega_T$ and $\omega_e$, defined in (3.60). Then, for arbitrary triangle $T \in \mathcal{T}_h$ and edge $e \in \mathcal{E}_h$ the Clément operator satisfies the following interpolation estimates for $\mathbf{v} \in \mathbf{H}^1(\Omega)$

$$\begin{aligned} ||\mathbf{v} - \mathcal{I}_h \mathbf{v}||_{\mathbf{L}^2(T)} &\leq C_1 \, h_T ||\mathbf{v}||_{\mathbf{H}^1(\omega_T)}, \\ ||\mathbf{v} - \mathcal{I}_h \mathbf{v}||_{\mathbf{L}^2(e)} &\leq C_2 \, h_e^{\frac{1}{2}} ||\mathbf{v}||_{\mathbf{H}^1(\omega_e)}, \end{aligned}$$

where the positive constants $C_1$ and $C_2$ are independent of the meshsize $h$. For some vector $\mathbf{v}$ and some tensor $\tau$ we define the following curl-operators, see e.g. Carstensen and Dolzmann [22],

$$\operatorname{curl} \mathbf{v} := v_{2,1} - v_{1,2} \qquad \text{and} \qquad \operatorname{curl} \tau := \begin{pmatrix} \tau_{12,1} - \tau_{11,2} \\ \tau_{22,1} - \tau_{21,2} \end{pmatrix}.$$

In the following we will often use $\eta^h$ as asymmetric tensor. Since we have

$$s(\eta^h, \tau^h) = \int_\Omega \eta^h \Big( (\tau^h)_{12} - (\tau^h)_{21} \Big) dx = \int_\Omega \begin{pmatrix} 0 & \eta^h \\ -\eta^h & 0 \end{pmatrix} : \tau^h \, dx,$$

we identify $\eta^h$ with the tensor $\begin{pmatrix} 0 & \eta^h \\ -\eta^h & 0 \end{pmatrix}$ to improve readability.

**Theorem 4.2:**
There exists a constant $C > 0$, independent of the mesh size $h$, such that

$$\|\sigma - \sigma^h\|_{\mathbf{X}} \leq C\left\{\sum_{T\in\mathcal{T}_h} \eta_{\sigma,T} + \|\mathbf{t}_0 - \sigma^h \cdot \mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)}\right\}, \tag{4.14}$$

where the local estimator $\eta_{\sigma,T}$ is defined for every $T \in \mathcal{T}_h$ by

$$\begin{aligned}
\eta_{\sigma,T}^2 :=\ & \|\mathbf{f} + \operatorname{div}\sigma^h\|^2_{\mathbf{L}^2(T)} + \|\mathrm{as}(\sigma^{\mathrm{h}})\|^2_{\mathbf{H}(\mathrm{div},T)} + h_T^2\,\|\mathrm{curl}(\mathbb{C}^{-1} : \sigma^h + \eta^h)\|_{\mathbf{L}^2(T)} \\
& + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^\Omega} h_e\,\|\,[(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot\mathbf{t}]\,\|^2_{\mathbf{L}^2(e)} + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^D} h_e\,\|(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot\mathbf{t}\,\|^2_{\mathbf{L}^2(e)} \\
& + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^N} h_e\,\|(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot\mathbf{t} + \frac{\partial\boldsymbol{\varphi}^{\tilde{h}}}{\partial\mathbf{t}}\|^2_{\mathbf{L}^2(e)} \\
& + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^C} h_e\,\|(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot\mathbf{t} + \frac{\partial\lambda_t^{\tilde{h}}}{\partial\mathbf{t}}\mathbf{t} + \frac{\partial(\lambda_n^{\tilde{h}} - g)}{\partial\mathbf{t}}\mathbf{n}\|^2_{\mathbf{L}^2(e)}.
\end{aligned} \tag{4.15}$$

Here, $[\cdot]$ denotes the jump across an interior edge $e \in \mathcal{E}_h^\Omega$, if $e$ is not an interior edge, then we just take the value inside the brackets.

*Proof.* We follow the works of Gatica et al. [11] and Maischak [67, see Theorem 5.10] and make use of the Helmholtz decomposition of $\sigma - \sigma^h + \mathrm{as}(\sigma^{\mathrm{h}})$. With the definition of $\tilde{\sigma}$ we have $\operatorname{div}(\sigma - \tilde{\sigma}) = 0$. Since $\Omega$ is connected we have the existence of some stream function $\mathbf{s} \in \mathbf{H}^1(\Omega)$, with $\int_\Omega s_i\,dx = 0$ for $i = 1, 2$, such that

$$\sigma - \tilde{\sigma} = \mathrm{Curl}\,\mathbf{s},$$

where the Curl-operator was already defined in the definition of the PEERS elements in Section 3.2. Let us define the Clément interpolant $\mathbf{s}^h := \mathcal{I}_h\mathbf{s}$. Then, obviously $\operatorname{div}\mathrm{Curl}\,\mathbf{s}^h = 0$ and we can choose $\tau = \mathrm{Curl}\,\mathbf{s}^h$ in Lemma 4.1 to estimate

$$\begin{aligned}
A :=\ & -\tilde{a}(\sigma^h, \sigma - \tilde{\sigma} - \tau) - s(\eta^h, \sigma - \tilde{\sigma} - \tau) - d_N(\boldsymbol{\varphi}^{\tilde{h}}, \sigma - \tilde{\sigma} - \tau) - d_{C,t}(\lambda_t^{\tilde{h}}, \sigma - \tilde{\sigma} - \tau) \\
& + g(\sigma - \tilde{\sigma} - \tau) - d_{C,n}(\lambda_n^{\tilde{h}}, \sigma - \tilde{\sigma} - \tau) \\
=\ & -\sum_{T\in\mathcal{T}_h}\int_T (\mathbb{C}^{-1} : \sigma^h + \eta^h) : \mathrm{Curl}(\mathbf{s} - \mathbf{s}^h)\,dx - \sum_{e\in\mathcal{E}_h^N}\int_e \boldsymbol{\varphi}^{\tilde{h}}\cdot\mathrm{Curl}(\mathbf{s} - \mathbf{s}^h)\cdot\mathbf{n}\,ds \\
& - \sum_{e\in\mathcal{E}_h^C}\int_e (\lambda_t^{\tilde{h}}\mathbf{t} + (\lambda_n^{\tilde{h}} - g)\mathbf{n})\cdot\mathrm{Curl}(\mathbf{s} - \mathbf{s}^h)\cdot\mathbf{n}\,ds \\
=\ & \sum_{T\in\mathcal{T}_h}\Big\{\int_T \mathrm{curl}(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot(\mathbf{s} - \mathbf{s}^h)\,dx + \sum_{e\in\mathcal{E}^T}\int_e [(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot\mathbf{t}]\cdot(\mathbf{s} - \mathbf{s}^h)\,ds\Big\}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{e\in\mathcal{E}_h^N} \int_e \boldsymbol{\varphi}^{\tilde{h}} \cdot \frac{\partial(\mathbf{s}-\mathbf{s}^h)}{\partial\mathbf{t}}\, ds - \sum_{e\in\mathcal{E}_h^C} \int_e (\lambda_t^{\tilde{h}}\mathbf{t} + (\lambda_n^{\tilde{h}} - g)\mathbf{n}) \cdot \frac{\partial(\mathbf{s}-\mathbf{s}^h)}{\partial\mathbf{t}}\, ds \\
= &\sum_{T\in\mathcal{T}_h} \Big\{ \int_T \operatorname{curl}(\mathbb{C}^{-1} : \sigma^h + \eta^h) \cdot (\mathbf{s}-\mathbf{s}^h)\, dx + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^\Omega} \int_e [(\mathbb{C}^{-1} : \sigma^h + \eta^h) \cdot \mathbf{t}] \cdot (\mathbf{s}-\mathbf{s}^h)\, ds \\
&+ \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^D} \int_e \mathbf{t} \cdot (\mathbb{C}^{-1} : \sigma^h + \eta^h) \cdot (\mathbf{s}-\mathbf{s}^h)\, ds \qquad (4.16) \\
&+ \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^N} \int_e \Big( \mathbf{t} \cdot (\mathbb{C}^{-1} : \sigma^h + \eta^h) + \frac{\partial\boldsymbol{\varphi}^{\tilde{h}}}{\partial\mathbf{t}} \Big) \cdot (\mathbf{s}-\mathbf{s}^h)\, ds \\
&+ \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^C} \int_e \Big( \mathbf{t} \cdot (\mathbb{C}^{-1} : \sigma^h + \eta^h) + \frac{\partial\lambda_t^{\tilde{h}}}{\partial\mathbf{t}}\mathbf{t} + \frac{\partial(\lambda_n^{\tilde{h}} - g)}{\partial\mathbf{t}}\mathbf{n} \Big) \cdot (\mathbf{s}-\mathbf{s}^h)\, ds \Big\}.
\end{aligned}
$$

Here we applied integration by parts on triangles and edges and the relation

$$
\operatorname{Curl} s_i \cdot \mathbf{n} = \begin{pmatrix} \frac{\partial s}{\partial x_2} \\ -\frac{\partial s}{\partial x_1} \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial s}{\partial x_2} \\ -\frac{\partial s}{\partial x_1} \end{pmatrix} \cdot \begin{pmatrix} -t_2 \\ t_1 \end{pmatrix} = -\frac{\partial s_i}{\partial\mathbf{t}}, \qquad \text{for } i = 1,2
$$

leading to

$$
\operatorname{Curl} \mathbf{s} \cdot \mathbf{n} = \begin{pmatrix} \operatorname{Curl} s_1 \cdot \mathbf{n} \\ \operatorname{Curl} s_2 \cdot \mathbf{n} \end{pmatrix} = -\frac{\partial\mathbf{s}}{\partial\mathbf{t}}.
$$

Furthermore, since the unit normal and tangential vectors are constant on every edge, we have used in (4.16)

$$
\frac{\partial\lambda_t^{\tilde{h}}\mathbf{t}}{\partial\mathbf{t}} = \begin{pmatrix} t_1\, \nabla\lambda_t^{\tilde{h}} \cdot \mathbf{t} \\ t_2\, \nabla\lambda_t^{\tilde{h}} \cdot \mathbf{t} \end{pmatrix} = \frac{\partial\lambda_t^{\tilde{h}}}{\partial\mathbf{t}}\mathbf{t} \qquad \text{and} \qquad \frac{\partial(\lambda_n^{\tilde{h}} - g)\mathbf{n}}{\partial\mathbf{t}} = \begin{pmatrix} n_1\, \nabla(\lambda_n^{\tilde{h}} - g) \cdot \mathbf{t} \\ n_2\, \nabla(\lambda_n^{\tilde{h}} - g) \cdot \mathbf{t} \end{pmatrix} = \frac{\partial(\lambda_n^{\tilde{h}} - g)}{\partial\mathbf{t}}\mathbf{n}.
$$

Using Cauchy-Schwarz inequality in (4.16) we conclude with the interpolation estimates for the Clément operator

$$
\begin{aligned}
A \le &\sum_{T\in\mathcal{T}_h} \Big\{ \|\operatorname{curl}(\mathbb{C}^{-1} : \sigma^h + \eta^h)\|_{\mathbf{L}^2(T)} \|\mathbf{s}-\mathbf{s}^h\|_{\mathbf{L}^2(T)} \\
&+ \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^\Omega} \|\, [(\mathbb{C}^{-1} : \sigma^h + \eta^h) \cdot \mathbf{t}]\, \|_{\mathbf{L}^2(e)} \|\mathbf{s}-\mathbf{s}^h\|_{\mathbf{L}^2(e)} \\
&+ \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^D} \|(\mathbb{C}^{-1} : \sigma^h + \eta^h) \cdot \mathbf{t}\|_{\mathbf{L}^2(e)} \|\mathbf{s}-\mathbf{s}^h\|_{\mathbf{L}^2(e)} \\
&+ \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^N} \|(\mathbb{C}^{-1} : \sigma^h + \eta^h) \cdot \mathbf{t} + \frac{\partial\boldsymbol{\varphi}^{\tilde{h}}}{\partial\mathbf{t}}\|_{\mathbf{L}^2(e)} \|\mathbf{s}-\mathbf{s}^h\|_{\mathbf{L}^2(e)}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^C} \|(\mathbb{C}^{-1}:\sigma^h+\eta^h)\cdot\mathbf{t}+\frac{\partial\lambda_t^{\tilde h}}{\partial\mathbf{t}}\mathbf{t} \\
&\quad + \frac{\partial(\lambda_n^{\tilde h}-g)}{\partial\mathbf{t}}\mathbf{n}\|_{\mathbf{L}^2(e)}\,\|\mathbf{s}-\mathbf{s}^h\|_{\mathbf{L}^2(e)}\Big\} \\
&\le C\sum_{T\in\mathcal{T}_h}\Big\{h_T\,\|\mathrm{curl}(\mathbb{C}^{-1}:\sigma^h+\eta^h)\|_{\mathbf{L}^2(T)}\,\|\mathbf{s}\|_{\mathbf{H}^1(\omega_T)} \\
&\quad + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^\Omega} h_e^{\frac12}\|\,[(\mathbb{C}^{-1}:\sigma^h+\eta^h)\cdot\mathbf{t}]\,\|_{\mathbf{L}^2(e)}\,\|\mathbf{s}\|_{\mathbf{H}^1(\omega_e)} \\
&\quad + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^D} h_e^{\frac12}\|(\mathbb{C}^{-1}:\sigma^h+\eta^h)\cdot\mathbf{t}\|_{\mathbf{L}^2(e)}\,\|\mathbf{s}\|_{\mathbf{H}^1(\omega_e)} \\
&\quad + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^N} h_e^{\frac12}\|(\mathbb{C}^{-1}:\sigma^h+\eta^h)\cdot\mathbf{t}+\frac{\partial\boldsymbol{\varphi}^{\tilde h}}{\partial\mathbf{t}}\|_{\mathbf{L}^2(e)}\,\|\mathbf{s}\|_{\mathbf{H}^1(\omega_e)} \\
&\quad + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^C} h_e^{\frac12}\|(\mathbb{C}^{-1}:\sigma^h+\eta^h)\cdot\mathbf{t}+\frac{\partial\lambda_t^{\tilde h}}{\partial\mathbf{t}}\mathbf{t}+\frac{\partial(\lambda_n^{\tilde h}-g)}{\partial\mathbf{t}}\mathbf{n}\|_{\mathbf{L}^2(e)}\,\|\mathbf{s}\|_{\mathbf{H}^1(\omega_e)}\Big\} \\
&\le C\sum_{T\in\mathcal{T}_h}\Big\{h_T\,\|\mathrm{curl}(\mathbb{C}^{-1}:\sigma^h+\eta^h)\|_{\mathbf{L}^2(T)} + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^\Omega} h_e^{\frac12}\|\,[(\mathbb{C}^{-1}:\sigma^h+\eta^h)\cdot\mathbf{t}]\,\|_{\mathbf{L}^2(e)} \\
&\quad + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^D} h_e^{\frac12}\|(\mathbb{C}^{-1}:\sigma^h+\eta^h)\cdot\mathbf{t}\|_{\mathbf{L}^2(e)} + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^N} h_e^{\frac12}\|(\mathbb{C}^{-1}:\sigma^h+\eta^h)\cdot\mathbf{t}+\frac{\partial\boldsymbol{\varphi}^{\tilde h}}{\partial\mathbf{t}}\|_{\mathbf{L}^2(e)} \\
&\quad + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^C} h_e^{\frac12}\|(\mathbb{C}^{-1}:\sigma^h+\eta^h)\cdot\mathbf{t}+\frac{\partial\lambda_t^{\tilde h}}{\partial\mathbf{t}}\mathbf{t}+\frac{\partial(\lambda_n^{\tilde h}-g)}{\partial\mathbf{t}}\mathbf{n}\|_{\mathbf{L}^2(e)}\Big\}\,\|\mathbf{s}\|_{\mathbf{H}^1(\Omega)}. \qquad (4.17)
\end{aligned}
$$

For the stream function $\mathbf{s}$ we observe

$$
\begin{aligned}
&\nabla\mathbf{s}:\nabla\mathbf{s} = s_{1,1}^2+s_{1,2}^2+s_{2,1}^2+s_{2,2}^2 = \begin{pmatrix} s_{1,2} & -s_{1,1} \\ s_{2,2} & -s_{2,1}\end{pmatrix}:\begin{pmatrix} s_{1,2} & -s_{1,1} \\ s_{2,2} & -s_{2,1}\end{pmatrix} = \mathrm{Curl}\,\mathbf{s}:\mathrm{Curl}\,\mathbf{s} \\
\Rightarrow\quad &\|\nabla\mathbf{s}\|_{L^2(\Omega)^{2\times2}} = \|\mathrm{Curl}\,\mathbf{s}\|_{L^2(\Omega)^{2\times2}} = \|\sigma-\tilde\sigma\|_{L^2(\Omega)^{2\times2}} = \|\sigma-\tilde\sigma\|_{\mathbf{X}}. \qquad (4.18)
\end{aligned}
$$

Furthermore, applying Lemma 3.3 in Großmann and Roos [54] we get

$$
\|\mathbf{s}\|_{\mathbf{L}^2(\Omega)}^2 = \sum_{i=1}^2\|s_i\|_{\mathbf{L}^2(\Omega)}^2 \le C\Big\{\sum_{i=1}^2|s_i|_{H^1(\Omega)}^2+\Big(\int_\Omega s_i\,dx\Big)^2\Big\} = C\sum_{i=1}^2|s_i|_{H^1(\Omega)}^2 = C|\mathbf{s}|_{\mathbf{H}^1(\Omega)}^2 \qquad (4.19)
$$

and therefore the equivalence of the $\mathbf{H}^1$-norm and the $\mathbf{H}^1$-seminorm. Therefore, using (4.18) and (4.19) in (4.17) we conclude

$$\begin{aligned}
A \leq C \sum_{T\in\mathcal{T}_h} \Big\{ & h_T \,\|\mathrm{curl}(\mathbb{C}^{-1} : \sigma^h + \eta^h)\|_{\mathbf{L}^2(T)} + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^{\Omega}} h_e^{\frac{1}{2}} \|\, [(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot \mathbf{t}]\, \|_{\mathbf{L}^2(e)} \\
& + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^{D}} h_e^{\frac{1}{2}} \|(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot \mathbf{t}\|_{\mathbf{L}^2(e)} + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^{N}} h_e^{\frac{1}{2}} \|(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot \mathbf{t} + \frac{\partial \boldsymbol{\varphi}^{\tilde{h}}}{\partial \mathbf{t}}\|_{\mathbf{L}^2(e)} \\
& + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^{C}} h_e^{\frac{1}{2}} \|(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot \mathbf{t} + \frac{\partial \lambda_t^{\tilde{h}}}{\partial \mathbf{t}}\mathbf{t} + \frac{\partial (\lambda_n^{\tilde{h}} - g)}{\partial \mathbf{t}}\mathbf{n}\|_{\mathbf{L}^2(e)} \Big\} \|\nabla \mathbf{s}\|_{\mathbf{L}^2(\Omega)} \\
\leq C \sum_{T\in\mathcal{T}_h} \Big\{ & h_T \,\|\mathrm{curl}(\mathbb{C}^{-1} : \sigma^h + \eta^h)\|_{\mathbf{L}^2(T)} + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^{\Omega}} h_e^{\frac{1}{2}} \|\, [(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot \mathbf{t}]\, \|_{\mathbf{L}^2(e)} \\
& + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^{D}} h_e^{\frac{1}{2}} \|(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot \mathbf{t}\|_{\mathbf{L}^2(e)} + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^{N}} h_e^{\frac{1}{2}} \|(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot \mathbf{t} + \frac{\partial \boldsymbol{\varphi}^{\tilde{h}}}{\partial \mathbf{t}}\|_{\mathbf{L}^2(e)} \\
& + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^{C}} h_e^{\frac{1}{2}} \|(\mathbb{C}^{-1} : \sigma^h + \eta^h)\cdot \mathbf{t} + \frac{\partial \lambda_t^{\tilde{h}}}{\partial \mathbf{t}}\mathbf{t} + \frac{\partial (\lambda_n^{\tilde{h}} - g)}{\partial \mathbf{t}}\mathbf{n}\|_{\mathbf{L}^2(e)} \Big\} \|\sigma - \tilde{\sigma}\|_{\mathbf{X}}. \qquad (4.20)
\end{aligned}$$

With (4.20) the assertion in Lemma 4.1 reads

$$\|\sigma - \tilde{\sigma}\|_{\mathbf{X}} \leq C \sum_{T\in\mathcal{T}_h} \eta_{\sigma,T} + \|\mathbf{t}_0 - \sigma^h \cdot \mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)}.$$

We finish the proof by using the triangle inequality, (4.3) and Lemma 4.1. □

**Remark 4.3:** *As we mentioned in the proof of Theorem 4.2 the representation of $\sigma - \tilde{\sigma}$ in terms of the Curl of some stream function corresponds indeed to the Helmholtz decomposition of $\sigma - \sigma^h + \mathrm{as}(\sigma^{\mathrm{h}})$ on each triangle. In view of the definition of $\tilde{\sigma}$ we have on each tringle $T \in \mathcal{T}_h$*

$$\begin{aligned}
& \sigma - \tilde{\sigma} = \sigma - \sigma^* - \sigma^h + \mathrm{as}(\sigma^{\mathrm{h}}) = \sigma - \mathbb{C} : \varepsilon(\mathbf{z}_T) - \sigma^h + \mathrm{as}(\sigma^{\mathrm{h}}) \\
\Rightarrow \quad & \sigma - \sigma^h + \mathrm{as}(\sigma^{\mathrm{h}}) = \mathbb{C} : \varepsilon(\mathbf{z}_T) + \mathrm{Curl}\,\mathbf{s} \qquad \forall\, T \in \mathcal{T}_h.
\end{aligned}$$

*But this is just the Helmholtz decomposition of the symmetric tensor $\sigma - \sigma^h + \mathrm{as}(\sigma^{\mathrm{h}})$, as it was introduced by Carstensen and Dolzmann in [22], see also Carstensen et al. [23].*

The estimates of the errors of the Lagrange multipliers is splitted into three parts.

**Lemma 4.4:**
There exists some $C > 0$, such that

$$\begin{aligned}\|\eta - \eta^h\|_{L^2(\Omega)} + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}\|_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)} + \|\lambda_t - \lambda_t^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} + \|\lambda_n - \lambda_n^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \\ \leq C\left\{\sum_{T\in\mathcal{T}_h} \eta_{\sigma,T} + \|\mathbf{t}_0 - \sigma^h \cdot \mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)}\right\}.\end{aligned} \tag{4.21}$$

*Proof.* From the proof of Lemma 3.19 we have for $\tau \in \mathbf{X}$ with $\operatorname{div}\tau = 0$ the continuous inf-sup condition for $\hat{B}(\tau; \xi, \boldsymbol{\psi}, \mu_t, \mu_n) := s(\xi, \tau) + d_N(\boldsymbol{\psi}, \tau) + d_{C,t}(\mu_t, \tau) + d_{C,n}(\mu_n, \tau)$

$$\begin{aligned}\|\eta - \eta^h\|_{L^2(\Omega)} + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}\|_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)} + \|\lambda_t - \lambda_t^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} + \|\lambda_n - \lambda_n^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \\ \leq \sup_{0\neq\tau\in\mathbf{X}} \frac{\hat{B}(\tau; \eta - \eta^h, \boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t - \lambda_t^{\tilde{h}}, \lambda_n - \lambda_n^{\tilde{h}})}{\|\tau\|_{\mathbf{X}}}.\end{aligned} \tag{4.22}$$

As $\operatorname{div}\tau = 0$ equation $(3.41)_1$ reads

$$\hat{B}(\tau; \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) = g(\tau) - \tilde{a}(\sigma, \tau). \tag{4.23}$$

Furthermore, for some $\tau^h \in \mathbf{X}_h$ with $\operatorname{div}\tau^h = 0$ equation $(3.62)_1$ reads

$$\hat{B}(\tau^h; \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}) = g(\tau^h) - \tilde{a}(\sigma^h, \tau^h). \tag{4.24}$$

Therefore, by adding zero we conclude with (4.23) and (4.24)

$$\begin{aligned}\hat{B}(\tau;&\eta - \eta^h, \boldsymbol{\varphi} - \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t - \lambda_t^{\tilde{h}}, \lambda_n - \lambda_n^{\tilde{h}}) \\ &= \hat{B}(\tau; \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) - \hat{B}(\tau - \tau^h; \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}) - \hat{B}(\tau^h; \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}) \\ &= g(\tau) - \tilde{a}(\sigma, \tau) - \hat{B}(\tau - \tau^h; \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}) - g(\tau^h) + \tilde{a}(\sigma^h, \tau^h) \\ &= -\tilde{a}(\sigma - \sigma^h, \tau) + g(\tau - \tau^h) - \tilde{a}(\sigma^h, \tau - \tau^h) - \hat{B}(\tau - \tau^h; \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}).\end{aligned} \tag{4.25}$$

The first term in (4.25) can be estimated using the continuity of the bilinear form $\tilde{a}(\cdot, \cdot)$

$$-\tilde{a}(\sigma - \sigma^h, \tau) \leq C\,\|\sigma - \sigma^h\|_{\mathbf{X}}\,\|\tau\|_{\mathbf{X}}. \tag{4.26}$$

For the last three terms we use the argumentation in the proof of Theorem 4.2. Since $\tau$ has divergence zero we have the existence of some stream function $\mathbf{s}(\tau) \in \mathbf{H}^1(\Omega)$ with $\int_\Omega s(\tau)_i\,dx = 0$ for $i = 1, 2$, such that $\tau = \operatorname{Curl}\mathbf{s}(\tau)$. Choosing $\mathbf{s}^h(\tau)$ as the Clément interpolant of $\mathbf{s}(\tau)$, we have $\tau^h := \operatorname{Curl}\mathbf{s}^h(\tau) \in \mathbf{X}_h$ with $\operatorname{div}\tau^h = 0$ and we can proceed in exactly the same way as in the proof of Theorem 4.2 using $\tau$ instead of $\sigma - \tilde{\sigma}$, see equations (4.16)-(4.20). The corresponding final result reads

$$
\begin{aligned}
& g(\tau - \tau^h) - \tilde{a}(\sigma^h, \tau - \tau^h) - \hat{B}(\tau - \tau^h; \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}) \\
& \leq C \sum_{T \in \mathcal{T}_h} \Big\{ h_T \, \|\mathrm{curl}(\mathbb{C}^{-1} : \sigma^h + \eta^h)\|_{\mathbf{L}^2(T)} + \sum_{e \in \mathcal{E}^T \cap \mathcal{E}_h^\Omega} h_e^{\frac{1}{2}} \| \, [(\mathbb{C}^{-1} : \sigma^h + \eta^h) \cdot \mathbf{t}] \, \|_{\mathbf{L}^2(e)} \\
& \quad + \sum_{e \in \mathcal{E}^T \cap \mathcal{E}_h^D} h_e^{\frac{1}{2}} \|(\mathbb{C}^{-1} : \sigma^h + \eta^h) \cdot \mathbf{t}\|_{\mathbf{L}^2(e)} + \sum_{e \in \mathcal{E}^T \cap \mathcal{E}_h^N} h_e^{\frac{1}{2}} \|(\mathbb{C}^{-1} : \sigma^h + \eta^h) \cdot \mathbf{t} + \frac{\partial \boldsymbol{\varphi}^{\tilde{h}}}{\partial \mathbf{t}} \|_{\mathbf{L}^2(e)} \qquad (4.27) \\
& \quad + \sum_{e \in \mathcal{E}^T \cap \mathcal{E}_h^C} h_e^{\frac{1}{2}} \|(\mathbb{C}^{-1} : \sigma^h + \eta^h) \cdot \mathbf{t} + \frac{\partial \lambda_t^{\tilde{h}}}{\partial \mathbf{t}} \mathbf{t} + \frac{\partial (\lambda_n^{\tilde{h}} - g)}{\partial \mathbf{t}} \mathbf{n} \|_{\mathbf{L}^2(e)} \Big\} \, \|\tau\|_{\mathbf{X}}.
\end{aligned}
$$

Using (4.25), (4.26) and (4.27) in (4.22) the proof is complete. □

Before we estimate the error of the displacement $\mathbf{u}^h$, let us define

$$
\mathbf{M}_h^\Gamma := \Big\{ \boldsymbol{\psi} \in \mathbf{L}^2(\Gamma) : \ \boldsymbol{\psi}|_e \in \mathbf{P}_0^h(e) \quad \forall \, e \in \mathcal{E}_h^\Gamma \Big\}. \qquad (4.28)
$$

For arbitrary $\mathbf{v}^h \in \mathbf{M}_h$ we have $\mathbf{v} = \sum_{T \in \mathcal{T}_h} \Big\{ c_1^T \begin{pmatrix} \chi_T \\ 0 \end{pmatrix} + c_2^T \begin{pmatrix} 0 \\ \chi_T \end{pmatrix} \Big\}$ and we define the following projection $P_0^{h,\Gamma} : \mathbf{M}_h \to \mathbf{M}_h^\Gamma$

$$
\mathbf{v}_\Gamma^h := P_0^{h,\Gamma} \mathbf{v}^h = \sum_{e \in \mathcal{E}_h^\Gamma} \Big\{ c_1^{T_e} \begin{pmatrix} \chi_e \\ 0 \end{pmatrix} + c_2^{T_e} \begin{pmatrix} 0 \\ \chi_e \end{pmatrix} \Big\}. \qquad (4.29)
$$

**Lemma 4.5:**
There exists some $C > 0$, such that

$$
\|\mathbf{u} - \mathbf{u}^h\|_{\mathbf{L}^2(\Omega)} \leq C \Big\{ \sum_{T \in \mathcal{T}_h} \Big( \eta_{\sigma,T} + \eta_{\mathbf{u},T} \Big) + \|\mathbf{t}_0 - \sigma^h \cdot \mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)} \Big\}, \qquad (4.30)
$$

where the local estimator $\eta_{\mathbf{u},T}$ is defined for every $T \in \mathcal{T}_h$ by

$$
\begin{aligned}
\eta_{\mathbf{u},T}^2 := & \sum_{T \in \mathcal{T}_h} h_T^2 \|\mathbb{C}^{-1} : \sigma^h + \eta^h\|_{L^2(T)^{2\times 2}} + \sum_{e \in \mathcal{E}^T \cap \mathcal{E}_h^N} h_e \, \|\boldsymbol{\varphi}^{\tilde{h}} + \mathbf{u}_\Gamma^h\|_{\mathbf{L}^2(e)}^2 \qquad (4.31) \\
& + \sum_{e \in \mathcal{E}^T \cap \mathcal{E}_h^D} h_e \, \|\mathbf{u}_\Gamma^h\|_{\mathbf{L}^2(e)}^2 + \sum_{e \in \mathcal{E}^T \cap \mathcal{E}_h^C} h_e \, \|\lambda_t^{\tilde{h}} \mathbf{t} + (\lambda_n^{\tilde{h}} - g)\mathbf{n} + \mathbf{u}_\Gamma^h\|_{\mathbf{L}^2(e)}^2.
\end{aligned}
$$

*Proof.* We proceed analogously to Maischak [67, Theorem 5.11]. Hence, we want to use the equilibrium interpolation operator $E_h : H^1(\Omega)^{2\times 2} \cap \mathbf{X} \to \mathbf{X}_h$ as defined in

Roberts and Thomas [77, see Chapter II, Section 6]. For $\tau \in H^1(\Omega)^{2\times 2} \cap \mathbf{X}$ we have for every $T \in \mathcal{T}_h$ the following estimate

$$\begin{aligned} \|\tau - E_h\tau\|_{L^2(T)^{2\times 2}} &\leq C\, h_T\, |\tau|_{H^1(T)^{2\times 2}}\,, \\ \|\mathrm{div}(\tau - E_h\tau)\|_{\mathbf{L}^2(T)} &\leq C\, \|\mathrm{div}\,\tau\|_{\mathbf{L}^2(T)}. \end{aligned} \tag{4.32}$$

Using this estimate demands a higher regularity than $\mathbf{X}$. Therefore, we apply the same approach as Maischak in [67]. We let $\tau \in \mathbf{X}$ be arbitrary but fixed. Defining the convex domain $\Omega' \supset \Omega$, we assume $\mathbf{z} \in \mathbf{H}^1(\Omega')$ with $\int_{\Omega'} z_i\, dx = 0$, for $i = 1, 2$ being the unique solution of

$$\begin{aligned} \mathrm{div}\,\mathbb{C} : \varepsilon(\mathbf{z}) &= \begin{cases} \mathrm{div}\,\tau & \text{in } \Omega, \\ -\frac{1}{\mathrm{meas}\,\Omega'\setminus\Omega} \int\limits_\Omega \mathrm{div}\,\tau\, dx & \text{in } \Omega' \setminus \Omega, \end{cases} \\ \mathbb{C} : \varepsilon(\mathbf{z}) \cdot \mathbf{n} &= 0 \ \text{ on } \partial\Omega'. \end{aligned} \tag{4.33}$$

Here we have used

$$\int\limits_\Omega \mathrm{div}\,\tau\, dx = \begin{pmatrix} \int_\Omega (\mathrm{div}\,\tau)_1\, dx \\ \int_\Omega (\mathrm{div}\,\tau)_2\, dx \end{pmatrix}.$$

The condition $\mathbf{z} \in \mathbf{H}^1(\Omega')$ with $\int_{\Omega'} z_i\, dx = 0$, for $i = 1, 2$ assures the validity of Korn's inequality, see e.g. Brenner and Scott [16, Chapter 9]. Furthermore, the given data fulfills $\int_{\Omega'} f_i'\, dx = 0$ for $i = 1, 2$ with

$$f_i' := \begin{cases} (\mathrm{div}\,\tau)_i & \text{in } \Omega, \\ -\frac{1}{\mathrm{meas}\,\Omega'\setminus\Omega} \int\limits_\Omega (\mathrm{div}\,\tau)_i\, dx & \text{in } \Omega'\setminus\Omega, \end{cases}$$

which is a necessary and sufficient condition for the unique solvability of the auxiliary problem (4.33). Using the regularity result in Chapter 9 of [16] we conclude $\mathbf{z} \in \mathbf{H}^2(\Omega')$ and $\|\mathbf{z}\|_{\mathbf{H}^2(\Omega')} \leq C(\Omega')\|\mathrm{div}\,\tau\|_{L^2(\Omega)^{2\times 2}}$. Defining $r_\tau := \mathbb{C} : \varepsilon(\mathbf{z})|_\Omega$ we have $r_\tau \in H^1(\Omega)^{2\times 2}$ and $\mathrm{div}\, r_\tau = \mathrm{div}\,\tau$ in $\Omega$. Moreover, we have

$$\|r_\tau\|_{H^1(\Omega)^{2\times 2}} \leq C\, \|\mathbf{z}\|_{\mathbf{H}^2(\Omega)} \leq C\, \|\mathbf{z}\|_{\mathbf{H}^2(\Omega')} \leq C(\Omega')\, \|\mathrm{div}\,\tau\|_{L^2(\Omega)^{2\times 2}}.$$

Using the commuting diagram property, see Roberts and Thomas [77, Section 6 in Chapter II], we know that

$$\mathrm{div}(E_h\tau) = P_h^0(\mathrm{div}\,\tau) \qquad \forall\ \tau \in H^1(\Omega)^{2\times 2}, \tag{4.34}$$

where $P_h^0$ is the orthogonal projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{M}_h$. Therefore, we deduce for $\mathbf{v}^h \in \mathbf{M}_h$

$$\int\limits_\Omega \mathbf{v}^h \cdot \mathrm{div}\,\tau\, dx = \int\limits_\Omega \mathbf{v}^h \cdot P_h^0(\mathrm{div}\,\tau)\, dx = \int\limits_\Omega \mathbf{v}^h \cdot \mathrm{div}(E_h\tau)\, dx \qquad \forall\ \tau \in H^1(\Omega)^{2\times 2}. \tag{4.35}$$

From the proof of Lemma 3.19 we conclude the continuous inf-sup condition of $b(\cdot,\cdot)$. Thus, we have

$$\|\mathbf{u}-\mathbf{u}^h\|_{\mathbf{L}^2(\Omega)} \leq C \sup_{0\neq\tau\in\mathbf{X}} \frac{b(\mathbf{u}-\mathbf{u}^h)}{\|\tau\|_{\mathbf{X}}}. \tag{4.36}$$

Using (4.35) and the first equation of the continuous and discrete variational inequality problems (3.41) and (3.62) we have

$$\begin{aligned}
b(\mathbf{u}-\mathbf{u}^h,\tau) &= \int_\Omega (\mathbf{u}-\mathbf{u}^h)\cdot \operatorname{div} r_\tau \, dx = b(\mathbf{u}, r_\tau) - \int_\Omega \mathbf{u}^h \cdot \operatorname{div}(E_h r_\tau)\, dx \\
&= g(r_\tau) - \tilde{a}(\sigma, r_\tau) - \hat{B}(r_\tau;\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) - b(\mathbf{u}^h, E_h r_\tau) \\
&= g(r_\tau) - \tilde{a}(\sigma, r_\tau) - \hat{B}(r_\tau;\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) - g(E_h r_\tau) + \tilde{a}(\sigma^h, E_h r_\tau) \\
&\quad + \hat{B}(E_h r_\tau;\eta^h,\boldsymbol{\varphi}^{\tilde{h}},\lambda_t^{\tilde{h}},\lambda_n^{\tilde{h}}) \\
&= -\tilde{a}(\sigma-\sigma^h, r_\tau) - \hat{B}(r_\tau;\eta-\eta^h,\boldsymbol{\varphi}-\boldsymbol{\varphi}^{\tilde{h}},\lambda_t-\lambda_t^{\tilde{h}},\lambda_n-\lambda_n^{\tilde{h}}) \\
&\quad + g(r_\tau - E_h r_\tau) - \tilde{a}(\sigma^h, r_\tau - E_h r_\tau) - \hat{B}(r_\tau - E_h r_\tau;\eta^h,\boldsymbol{\varphi}^{\tilde{h}},\lambda_t^{\tilde{h}},\lambda_n^{\tilde{h}}),
\end{aligned} \tag{4.37}$$

where we have extended by zero in the last step. The first two terms in (4.37) can be estimated by using the continuity of the bilinear forms and Theorem 2.3

$$\begin{aligned}
-\tilde{a}(\sigma-\sigma^h, r_\tau) &- \hat{B}(r_\tau;\eta-\eta^h,\boldsymbol{\varphi}-\boldsymbol{\varphi}^{\tilde{h}},\lambda_t-\lambda_t^{\tilde{h}},\lambda_n-\lambda_n^{\tilde{h}}) \\
&\leq C\Big\{\|\sigma-\sigma^h\|_{\mathbf{X}} + \|\eta-\eta^h\|_{L^2(\Omega)} + \|\boldsymbol{\varphi}-\boldsymbol{\varphi}^{\tilde{h}}\|_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)} \\
&\qquad + \|\lambda_t-\lambda_t^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} + \|\lambda_n-\lambda_n^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}\Big\}\|r_\tau\|_{\mathbf{X}} \\
&\leq C\Big\{\|\sigma-\sigma^h\|_{\mathbf{X}} + \|\eta-\eta^h\|_{L^2(\Omega)} + \|\boldsymbol{\varphi}-\boldsymbol{\varphi}^{\tilde{h}}\|_{\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)} \\
&\qquad + \|\lambda_t-\lambda_t^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} + \|\lambda_n-\lambda_n^{\tilde{h}}\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}\Big\}\|\tau\|_{\mathbf{X}}.
\end{aligned} \tag{4.38}$$

The last three terms will be splitted into parts and we first regard the bilinear forms on the domain. Using Cauchy-Schwarz inequality and (4.32) we have

$$\begin{aligned}
-\tilde{a}(\sigma^h, r_\tau - E_h r_\tau) - s(r_\tau - E_h r_\tau, \eta^h) &= -\int_\Omega (\mathbb{C}^{-1}:\sigma^h + \eta^h):(r_\tau - E_h r_\tau)\, dx \\
&= \sum_{T\in\mathcal{T}_h} \int_T (\mathbb{C}^{-1}:\sigma^h + \eta^h):(E_h r_\tau - r_\tau)\, dx \\
&\leq \sum_{T\in\mathcal{T}_h} \|\mathbb{C}^{-1}:\sigma^h + \eta^h\|_{L^2(T)^{2\times 2}} \|E_h r_\tau - r_\tau\|_{L^2(T)^{2\times 2}} \\
&\leq C \sum_{T\in\mathcal{T}_h} h_T \|\mathbb{C}^{-1}:\sigma^h + \eta^h\|_{L^2(T)^{2\times 2}} \|r_\tau\|_{H^1(T)^{2\times 2}}
\end{aligned}$$

$$\leq C\,\|\tau\|_{\mathbf{X}} \sum_{T\in\mathcal{T}_h} h_T\,\|\mathbb{C}^{-1} : \sigma^h + \eta^h\|_{L^2(T)^{2\times 2}}\,. \tag{4.39}$$

We need the following trace inequality, see Agmon [1, Theorem 3.10] and Arnold [4, eq. (2.5)]

$$\|\frac{\partial \mathbf{v}}{\partial \mathbf{n}}\|^2_{\mathbf{L}^2(e)} \leq C\left\{h_e^{-1}\,|\mathbf{v}|^2_{\mathbf{H}^1(T_e)} + h_e\,|\mathbf{v}|^2_{\mathbf{H}^2(T_e)}\right\} \qquad \forall\, \mathbf{v}\in\mathbf{H}^2(\Omega),\ \forall e\in\mathcal{E}_h^\Gamma. \tag{4.40}$$

From the boundedness and ellipticity of Hooke's tensor we deduce for $e\in\mathcal{E}_h^\Gamma$ by using (4.40), (4.32), (4.34) and the definition of $r_\tau$

$$\begin{aligned}\|(E_h r_\tau - r_\tau)\cdot\mathbf{n}\|_{\mathbf{L}^2(e)} &\leq C\,h_e^{\frac{1}{2}}\left\{h_{T_e}^{-1}\,\|E_h r_\tau - r_\tau\|_{L^2(T_e)^{2\times 2}} + \|\mathrm{div}(E_h r_\tau - r_\tau)\|_{\mathbf{L}^2(T_e)}\right\}\\ &\leq C\,h_e^{\frac{1}{2}}\,\|\tau\|_{\mathbf{X}}.\end{aligned} \tag{4.41}$$

Using the following property of the equilibrium interpolation operator

$$\int_e (r_\tau - E_h r_\tau)\cdot\mathbf{n} = 0, \qquad \forall\, e\in\mathcal{E}_h^\Gamma,$$

see Roberts and Thomas [77], we can add $0 = \int_\Gamma \mathbf{u}_\Gamma^h\cdot(E_h r_\tau - r_\tau)\cdot\mathbf{n}\,ds$ to the boundary terms in (4.37), where $\mathbf{u}_\Gamma^h := P_0^{h,\Gamma}\mathbf{u}^h$ is defined in (4.29). Moreover, using the Cauchy-Schwarz inequality and (4.41) in (4.37) we get

$$\begin{aligned}
&-d_N(\boldsymbol{\varphi}^{\tilde h}, r_\tau - E_h r_\tau) - d_{C,t}(\lambda_t^{\tilde h}, r_\tau - E_h r_\tau) - d_{C,n}(\lambda_n^{\tilde h}, r_\tau - E_h r_\tau) + g(r_\tau - E_h r_\tau)\\
&\quad= \sum_{e\in\mathcal{E}_h^N}\int_e (\boldsymbol{\varphi}^{\tilde h} + \mathbf{u}_\Gamma^h)\cdot(E_h r_\tau - r_\tau)\cdot\mathbf{n}\,ds + \sum_{e\in\mathcal{E}_h^D}\int_e \mathbf{u}_\Gamma^h\cdot(E_h r_\tau - r_\tau)\cdot\mathbf{n}\,ds\\
&\qquad+ \sum_{e\in\mathcal{E}_h^C}\int_e (\lambda_t^{\tilde h}\mathbf{t} + (\lambda_n^{\tilde h} - g)\mathbf{n} + \mathbf{u}_\Gamma^h)\cdot(E_h r_\tau - r_\tau)\cdot\mathbf{n}\,ds\\
&\quad\leq C\sum_{e\in\mathcal{E}_h^N}\|\boldsymbol{\varphi}^{\tilde h} + \mathbf{u}_\Gamma^h\|_{\mathbf{L}^2(e)}\,\|(E_h r_\tau - r_\tau)\cdot\mathbf{n}\|_{\mathbf{L}^2(e)} + \sum_{e\in\mathcal{E}_h^D}\|\mathbf{u}_\Gamma^h\|_{\mathbf{L}^2(e)}\,\|(E_h r_\tau - r_\tau)\cdot\mathbf{n}\|_{\mathbf{L}^2(e)}\\
&\qquad+ \sum_{e\in\mathcal{E}_h^C}\|\lambda_t^{\tilde h}\mathbf{t} + (\lambda_n^{\tilde h} - g)\mathbf{n} + \mathbf{u}_\Gamma^h)\|_{\mathbf{L}^2(e)}\,\|(E_h r_\tau - r_\tau)\cdot\mathbf{n}\|_{\mathbf{L}^2(e)}\\
&\quad\leq C\,\|\tau\|_{\mathbf{X}}\Big[\sum_{e\in\mathcal{E}_h^N} h_e^{\frac{1}{2}}\,\|\boldsymbol{\varphi}^{\tilde h} + \mathbf{u}_\Gamma^h\|_{\mathbf{L}^2(e)}\\
&\qquad\qquad+ \sum_{e\in\mathcal{E}_h^D} h_e^{\frac{1}{2}}\,\|\mathbf{u}_\Gamma^h\|_{\mathbf{L}^2(e)} \sum_{e\in\mathcal{E}_h^C} h_e^{\frac{1}{2}}\,\|\lambda_t^{\tilde h}\mathbf{t} + (\lambda_n^{\tilde h} - g)\mathbf{n} + \mathbf{u}_\Gamma^h)\|_{\mathbf{L}^2(e)}\Big].
\end{aligned} \tag{4.42}$$

Using (4.37), (4.38), (4.39) and (4.42) in (4.36) we finish the proof. □

Finally, there is one Lagrange multiplier left to estimate.

**Lemma 4.6:**
Exists some $C > 0$, such that

$$\|\nu - \nu^{\hat{h}}\|_{H^{-\frac{1}{2}}(A_C)} \leq C \left\{ \sum_{T \in \mathcal{T}_h} \Big( \eta_{\sigma,T} + \sum_{e \in \mathcal{E}^T \cap \mathcal{E}_h^A} \tilde{h}_e^{\frac{1}{2}} \|\sigma_t^h - \mathcal{F}\nu^{\hat{h}}\|_{L^2(e)} \Big) + \|\mathbf{t}_0 - \sigma^h \cdot \mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)} \right\}. \quad (4.43)$$

*Proof.* From the continuous inf-sup condition (3.54) we have

$$\|\nu - \nu^{\hat{h}}\|_{H^{-\frac{1}{2}}(A_C)} \leq C \sup_{0 \neq \mu_t \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \frac{q(\mu_t, \nu - \nu^{\hat{h}})}{\|\mu_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}}. \quad (4.44)$$

Using the fifth equality in (3.41) and adding zero in terms of $(3.62)_5$ we have for $\mu_t^{\tilde{h}} \in C_{\tilde{h}}$

$$\begin{aligned} q(\mu_t, \nu - \nu^{\hat{h}}) &= d_{C,t}(\mu_t, \sigma) - q(\mu_t - \mu_t^{\tilde{h}}, \nu^{\hat{h}}) - d_{C,t}(\mu_t^{\tilde{h}}, \sigma^h) \\ &= d_{C,t}(\mu_t, \sigma - \sigma^h) - q(\mu_t - \mu_t^{\tilde{h}}, \nu^{\hat{h}}) + d_{C,t}(\mu_t - \mu_t^{\tilde{h}}, \sigma^h). \end{aligned} \quad (4.45)$$

Applying the continuity of the bilinear form in the first term of (4.45) we have

$$d_{C,t}(\mu_t, \sigma - \sigma^h) \leq C \|\mu_t\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)} \|\sigma - \sigma^h\|_{\mathbf{X}}. \quad (4.46)$$

Choosing $\mu_t^{\tilde{h}} := Q_{\tilde{h}} \mu_t$ as the $L^2$-projection of $\mu_t$ onto $C_{\tilde{h}}$, see e.g. Steinbach [80, Chapter 10], the following estimate holds

$$\|\mu_t - Q_{\tilde{h}} \mu_t\|_{L^2(e)} \leq C \tilde{h}_e^{\frac{1}{2}} \|\mu_t\|_{H^{\frac{1}{2}}(e)}. \quad (4.47)$$

We conclude for the last two terms in (4.46) by applying the Cauchy-Schwarz inequality and (4.47)

$$\begin{aligned} -q(\mu_t - \mu_t^{\tilde{h}}, \nu^{\hat{h}}) + d_{C,t}(\mu_t - \mu_t^{\tilde{h}}, \sigma^h) &= \int_{A_C} (\mu_t - \mu_t^{\tilde{h}})(\sigma_t^h - \mathcal{F}\nu^{\hat{h}})\, ds \\ &= \sum_{e \in \mathcal{E}_{\tilde{h}}^A} \int_e (\mu_t - \mu_t^{\tilde{h}})(\sigma_t^h - \mathcal{F}\nu^{\hat{h}})\, ds \\ &\leq \sum_{e \in \mathcal{E}_{\tilde{h}}^A} \|\mu_t - \mu_t^{\tilde{h}}\|_{L^2(e)} \|\sigma_t^h - \mathcal{F}\nu^{\hat{h}}\|_{L^2(e)} \\ &\leq C \|\mu_t\|_{H^{\frac{1}{2}}(\Gamma_C)} \sum_{e \in \mathcal{E}_h^A} \tilde{h}_e^{\frac{1}{2}} \|\sigma_t^h - \mathcal{F}\nu^{\hat{h}}\|_{L^2(e)}. \end{aligned} \quad (4.48)$$

Applying (4.45), (4.46), (4.48) and the equivalence of the $H^{\frac{1}{2}}$-norm and the $\widetilde{H}^{\frac{1}{2}}$-norm to (4.44) we finish the proof. □

**Remark 4.7:** *If we demand a higher regularity on the Neumann traction* $\mathbf{t}_0$*, say* $\mathbf{t}_0 \in \mathbf{L}^2(\Gamma_N)$*, we can estimate the nonlocal term* $\|\mathbf{t}_0 - \sigma^h \cdot \mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_N)}$ *by the same argumentation as in the proof of Theorem 6.3 in Gatica, Gatica, Stephan [46]. Here the authors use Theorem 2 in Carstensen [20], taking advantage of the* $L^2$*-orthogonality of* $\mathbf{t}_0 - \sigma^h \cdot \mathbf{n}$ *on* $C_{\tilde{h}}$*.*

Let us define

$$\tilde{C}_N := \max\left\{ \frac{h_{e_1}}{h_{e_2}} \,:\, \forall\, e_1, e_2 \in \mathcal{E}_{\tilde{h}}^N \text{ with } \bar{e}_1 \cap \bar{e}_2 \neq \emptyset \right\}.$$

We can state the following result for the total error of the approximate solution of the discrete variational inequality problem (3.62).

**Theorem 4.8:**
In addition to the hypotheses of this Section assume that $\mathbf{t}_0 \in \mathbf{L}^2(\Gamma_N)$. Then there exists $C > 0$, independent of the meshsizes $h$ and $\tilde{h}$, such that

$$\|(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) - (\sigma^h, \nu^{\hat{h}}; \mathbf{u}^h, \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}})\|_{\mathbf{X}\times H^{-\frac{1}{2}}(A_C)\times \widetilde{\mathbf{Y}}'} \leq C \sum_{T\in\mathcal{T}_h} \eta_{tot,T}, \qquad (4.49)$$

where the total local error indicator $\eta_{tot,T}$ is defined by

$$\eta_{tot,T}^2 := \eta_{\sigma,T}^2 + \eta_{\mathbf{u},T}^2 + \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^A} \tilde{h}_e \,\|\sigma_t^h - \mathcal{F}\nu^{\hat{h}}\|_{L^2(e)}^2 + \log(1+\tilde{C}_N) \sum_{e\in\mathcal{E}^T\cap\mathcal{E}_h^N} \tilde{h}_e \,\|\sigma^h\cdot\mathbf{n} - \mathbf{t}_0\|_{L^2(e)}^2 \quad (4.50)$$

and we have used the definitions of $\eta_{\sigma,T}$ and $\eta_{\mathbf{u},T}$ in (4.15) and (4.31).

*Proof.* As we have mentioned in Remark 4.7 the fourth equation in (3.62) states $L^2$-orthogonality of $\sigma^h \cdot \mathbf{n} - \mathbf{t}_0$ on $C_{\tilde{h}}$, i.e.

$$0 = d_N(\boldsymbol{\psi}, \sigma^h) - t_0(\boldsymbol{\psi}) = \langle \boldsymbol{\psi}, \sigma^h \cdot \mathbf{n} - \mathbf{t}_0 \rangle_{\mathbf{L}^2(\Gamma_N)} \qquad \forall\, \boldsymbol{\psi} \in C_{\tilde{h}}.$$

Therefore, the assumptions of Theorem 2 in Carstensen [20] are fulfilled and we have

$$\|\mathbf{t}_0 - \sigma^h \cdot \mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(e)} \leq \log(1+\tilde{C}_N)\, \tilde{h}_e^{\frac{1}{2}}\, \|\sigma^h \cdot \mathbf{n} - \mathbf{t}_0\|_{L^2(e)} \qquad \forall\, e \in \mathcal{E}_h^N. \qquad (4.51)$$

Using (4.51) in the results of Theorem 4.2 and Lemmas 4.4, 4.5 and 4.6 and applying the triangle inequality we finish the proof. □

## 4.2. Numerical experiment using adaptive algorithms

In this section we perform the same numerical experiment as described in Example 3.46 but this time we use an adaptive algorithm based on the a posteriori error estimator derived in Section 4.1. In the first case we use a refinement algorithm, where a triangle $T \in \mathcal{T}_h$ is refined, if the corresponding total error indicator $\eta_{tot,T}$ is greater or equal a factor $\theta \in (0,1)$ times the maximal local error indicator

$$\eta_{max} := \max_{T \in \mathcal{T}_h} \eta_{tot,T}.$$

In Table 4.1 we present the norms of the approximating solutions and the corresponding degrees of freedom for the adaptive algorithm with $\theta = 0.5$. If we compare the norms in Table 4.1 with the norms in Table 3.2 we see, that we are still close to the pre-asymptotic case for $\sigma^h$. Too few triangles were refined during the algorithm, see Figures 4.1-4.8 although we have run the algorithm for 28 steps. This is caused by the large differences in the scales of the error indicators. There are three indicators having main influence on $\eta_{tot,T}$. Namely, the indicator corresponding to the Neumann boundary condition

$$\eta_{N,T}^2 := \sum_{e \in \mathcal{E}^T \cap \mathcal{E}_h^N} \tilde{h}_e \, \|\sigma^h \cdot \mathbf{n} - \mathbf{t}_0\|_{L^2(e)}^2,$$

the indicator corresponding to the friction force

$$\eta_{\mathcal{F},T}^2 := \sum_{e \in \mathcal{E}^T \cap \mathcal{E}_h^C} \tilde{h}_e \, \|\sigma_t^h - \mathcal{F}\, \nu^{\hat{h}}\|_{L^2(e)}^2$$

and finally, the indicator corresponding to the symmetry of the stress tensor

$$\eta_{as,T} := \|\mathrm{as}(\sigma^{\mathrm{h}})\|_{L^2(T)}.$$

The three indicators all include the stress tensor which has large coefficients in the solution vector due to the large Lamé coefficients. Therefore, the only triangles that are refined are close to the corners of the domain, on the Neumann boundary $\Gamma_N$ and on the active part of the contact boundary $A_C$.

In Section 3.4 we have observed singularities in the corners of the domain. As the boundary traction $\mathbf{t}_0$ is zero at the end points of the Neumann boundary, this leads to large values of the indicators $\eta_{N,T}$ for triangles $T \in \mathcal{T}_h$ near the corners of the Neumann boundary having an edge on $\Gamma_N$. This can be seen in Figures 4.1-4.8. The great influence of $\eta_{\mathcal{F},T}$ on the refinement is due to the different mesh sizes for $\sigma^h$ and $\nu^{\hat{h}}$ and the approximation error for the friction functional as defined in (3.123). Finally, the maximal value for the indicator $\eta_{as,T}$ is even growing with a smaller mesh size. This can be seen in Table 4.2 in the third column. But even if we use the $L^2$-norm as proposed by Gatica, Gatica and Stephan in [46] or Carstensen and Dolzmann in [22] the maximal value stays almost constant, as can be seen in the first

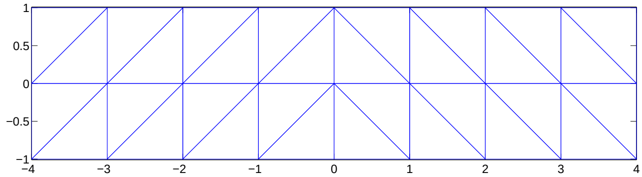



Figure 4.1.: Initial triangulation of Ω for the adaptive algorithm.

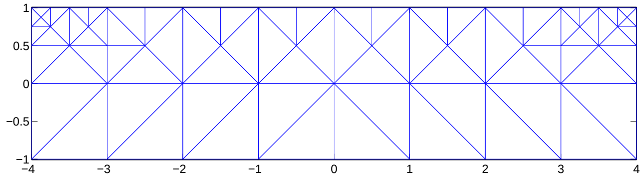



Figure 4.2.: Triangulation after 4 steps of the adaptive algorithm.

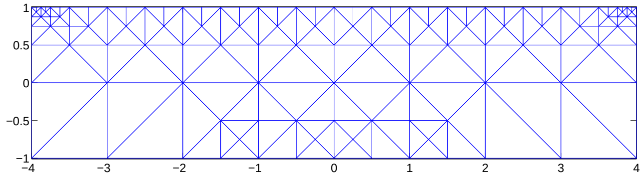



Figure 4.3.: Triangulation after 8 steps of the adaptive algorithm.

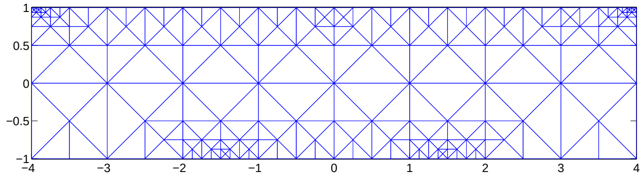



Figure 4.4.: Triangulation after 12 steps of the adaptive algorithm.

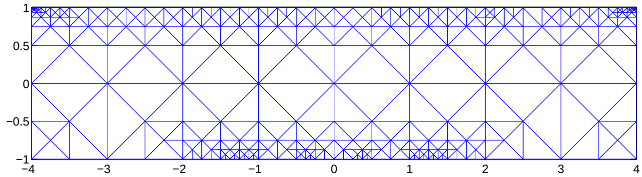



Figure 4.5.: Triangulation after 16 steps of the adaptive algorithm.

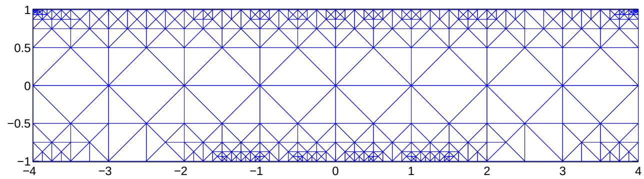



Figure 4.6.: Triangulation after 20 steps of the adaptive algorithm.

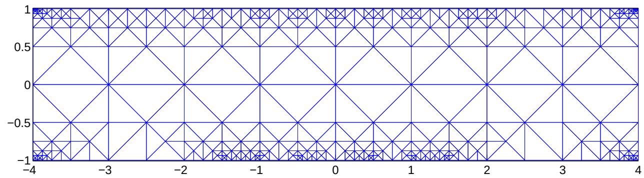



Figure 4.7.: Triangulation after 24 steps of the adaptive algorithm.

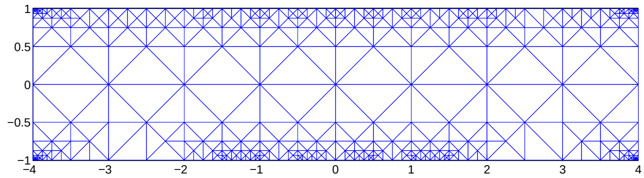



Figure 4.8.: Triangulation after 28 steps of the adaptive algorithm.

column of Table 4.2. The value of the singularity in the corners is increased, if the mesh size is decreased. Since the indicator $\eta_{as,T}$ is not weighted with the mesh size we deduce the above mentioned behaviour.

Here, we observe a backdraw of using PEERS elements for the discretization of the stress tensor, the displacement field and the rotation tensor. The maximal value

$$\eta_{as}^{max} := \max_{T \in \mathcal{T}_h} \eta_{as,T}$$

is always reached in the two lower corners of the domain, where the Dirichlet boundary and the contact boundary touch. We assume, that the PEERS elements are not suitable for such situations, since the symmetry of the stress tensor is not approximated well there.

| $\|\sigma^h\|_{\mathbf{X}}$ | $\|\mathbf{u}^h\|_{\mathbf{L}^2(\Omega)}$ | $\|\eta^h\|_{L^2(\Omega)}$ | $\|\boldsymbol{\varphi}^{\tilde{h}}\|_{\mathbf{L}^2(\Gamma_N)}$ | $\|\lambda_t^{\tilde{h}}\|_{L^2(\Gamma_C)}$ | $\|\lambda_n^{\tilde{h}}\|_{L^2(\Gamma_C)}$ | Dof |
|---|---|---|---|---|---|---|
| 2617 | 3.399e-02 | 1.180e-02 | 2.811e-02 | 4.404e-03 | 6.208e-03 | 317 |
| 2593 | 3.524e-02 | 1.229e-02 | 2.864e-02 | 4.218e-03 | 5.234e-03 | 683 |
| 2498 | 3.463e-02 | 1.253e-02 | 2.829e-02 | 5.453e-03 | 7.926e-03 | 1513 |
| 2502 | 3.583e-02 | 1.315e-02 | 2.916e-02 | 5.765e-03 | 7.775e-03 | 2388 |
| 2424 | 3.549e-02 | 1.425e-02 | 2.920e-02 | 6.767e-03 | 1.193e-02 | 3988 |
| 2646 | 3.555e-02 | 1.486e-02 | 2.931e-02 | 6.735e-03 | 1.314e-02 | 4978 |
| 2661 | 3.549e-02 | 1.488e-02 | 2.929e-02 | 6.419e-03 | 1.348e-02 | 5278 |
| 2669 | 3.546e-02 | 1.489e-02 | 2.927e-02 | 6.317e-03 | 1.354e-02 | 5578 |

Table 4.1.: Norms of the solutions and degrees of freedom for the frictional contact problem with $\mu_f = 0.5$.

| $\eta_{as}^{max}$ with $L^2$-norm | $q_{as}$ | $\eta_{as}^{max}$ with $H(\mathrm{div}, \Omega)$-norm | $q_{as}$ |
|---|---|---|---|
| 2.2600 | 0.792 | 18.1975 | 0.722 |
| 3.2031 | 0.272 | 22.7789 | 0.471 |
| 2.2388 | 0.210 | 40.6292 | 0.294 |
| 3.2002 | 0.132 | 45.0999 | 0.169 |
| 2.2293 | 0.101 | 82.4388 | 0.145 |
| 3.1956 | 0.068 | 89.9081 | 0.091 |
| 2.2243 | 0.050 | 165.4811 | 0.080 |
| 3.1950 | 0.035 | 179.7075 | 0.043 |

Table 4.2.: Maximal indicators $\eta_{as}^{max}$ and corresponding quotient of triangles with $\eta_{as,T} \geq 0.1\eta_{as}^{max}$ .

However, if we compare the quotient $q_{as}$ of the number of triangles, where $\eta_{as,T} \geq 0.1\eta_{as}^{max}$ we see in the second and fourth column of Table 4.2, that this quotient is tending to zero with decreasing mesh size. Therefore, we propose a second

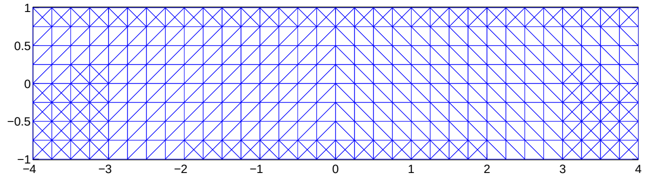



Figure 4.9.: Triangulation after 1 step of the adaptive algorithm.

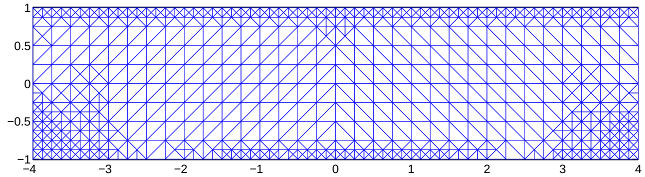



Figure 4.10.: Triangulation after 3 steps of the adaptive algorithm.

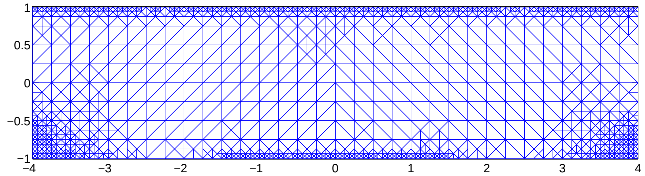



Figure 4.11.: Triangulation after 5 steps of the adaptive algorithm.

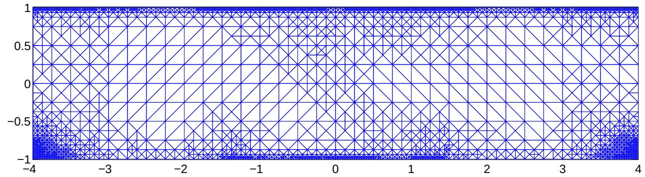



Figure 4.12.: Triangulation after 7 steps of the adaptive algorithm.

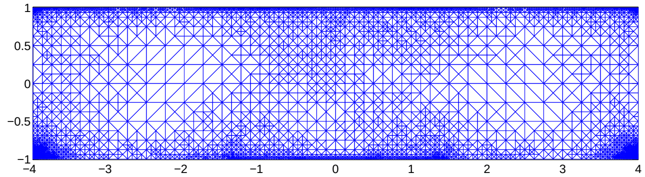



Figure 4.13.: Triangulation after 9 steps of the adaptive algorithm.

| $\|\sigma^h\|_{\mathbf{X}}$ | $\|\mathbf{u}^h\|_{\mathbf{L}^2(\Omega)}$ | $\|\eta^h\|_{L^2(\Omega)}$ | $\|\boldsymbol{\varphi}^{\tilde{h}}\|_{\mathbf{L}^2(\Gamma_N)}$ | $\|\lambda_t^{\tilde{h}}\|_{L^2(\Gamma_C)}$ | $\|\lambda_n^{\tilde{h}}\|_{L^2(\Gamma_C)}$ |
|---|---|---|---|---|---|
| 2478 | 3.549e-02 | 1.461e-02 | 2.920e-02 | 6.544e-03 | 1.290e-02 |
| 2456 | 3.547e-02 | 1.466e-02 | 2.918e-02 | 6.435e-03 | 1.335e-02 |
| 2463 | 3.541e-02 | 1.464e-02 | 2.914e-02 | 6.300e-03 | 1.347e-02 |
| 2469 | 3.539e-02 | 1.463e-02 | 2.912e-02 | 6.225e-03 | 1.352e-02 |
| 2473 | 3.539e-02 | 1.462e-02 | 2.910e-02 | 6.186e-03 | 1.355e-02 |

Table 4.3.: Norms of the solutions for the frictional contact problem with $\mu_f = 0.5$.

| $e_{tot}$ | Dof | $\eta_{tot}/e_{tot}$ |
|---|---|---|
| 4.955e-01 | 5297 | 0.648 |
| 4.030e-01 | 9492 | 0.803 |
| 2.905e-01 | 17262 | 1.111 |
| 2.077e-01 | 32296 | 1.550 |
| 1.481e-01 | 60989 | 2.171 |

Table 4.4.: Norms of the solutions, total error and degrees of freedom for the frictional contact problem with $\mu_f = 0.5$.

adaptive algorithm, where $\theta$ denotes the percentage of triangles that are refined in each step. The triangles are sorted according to their indicators and then, $\theta$ percent of the triangles with largest indicators are refined. In Figures 4.9-4.13 we show the sequence of refinements for the second adaptive algorithm with $\theta = 0.2$. Although we start with a finer triangulation, the algorithm also refines triangles inside the domain.

The norms of the approximated solutions are shown in Table 4.3. For the second algorithm we also show the total error as defined in (3.124) and the effectivity index $\frac{\eta_{tot}}{e_{tot}}$, with

$$\eta_{tot} := \sum_{T \in \mathcal{T}_h} \eta_{tot,T}$$

in Table 4.4. The effectivity index is not satisfactory, which is a consequence of the increasing maximal indicator $\eta_{as}^{max}$ as already described above. Nevertheless, the refinement algorithm behaves sensible as the local refinements are observed in critical areas, e.g. the active part $A_C$ of the contact boundary.

# A. Numerical methods for process-oriented structures in metal chipping

This work was made possible by the German Research Foundation (DFG) who supported the author within the priority program SPP 1180 *Prediction and Manipulation of Interactions between Structure and Process*. The project focusses on the phenomenon of tool extraction. In some industrial applications the extraction of a milling cutter out of thermal shrink-fit chucks during high speed milling processes was reported.

To identify the factors causing the extraction we develop a mathematical model of the structure and its interaction with the process. Namely, we choose a finite element model of the tool and the tool holder, i.e. $\Omega_1$-$\Omega_6$ in Figure A.1 and impose on one hand the interaction of the tool holder with the dynamic characteristics of the spindle in terms of inhomogeneous Dirichlet conditions at the interface of tool holder and spindle $\Gamma_D := \overline{\Omega}_6 \cap \overline{\Omega}_7$. On the other hand the interactions of the tool with the work piece are incorporated as Neumann conditions on the cutters of the tool $\Gamma_N^c(t) := \overline{\Omega}_0 \cap \overline{\Omega}_1$. In the following we define $\Omega_a$ as the domain of the tool and $\Omega_b$ as the domain of the tool holder.

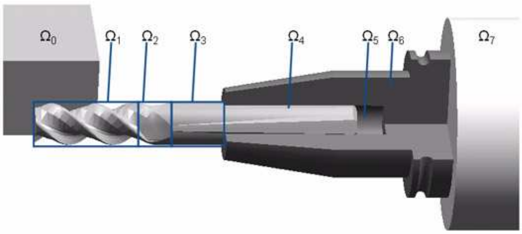


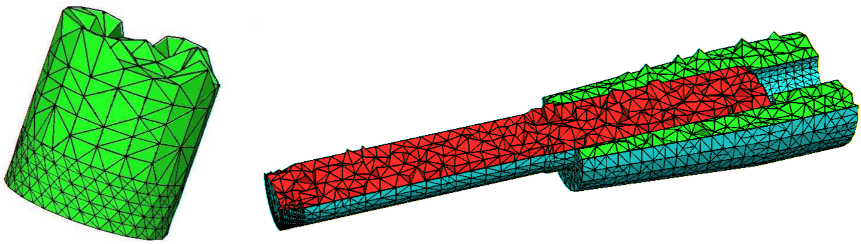

Figure A.1.: System of workpiece $\Omega_0$, tool $\Omega_1$-$\Omega_4$, tool holder $\Omega_6$ and spindle $\Omega_7$.

Due to the complex structure of the domains and the large interface between $\Omega_a$ and $\Omega_b$ a transient model involving two body contact conditions is not feasible. Since the tool and the tool holder are sticking to each other after the shrink process we neglect the contact conditions and consider $\Omega := \overline{\Omega}_a \cup \overline{\Omega}_b$ as an inhomogeneous compound with corresponding material parameters depending on $\mathbf{x} \in \Omega$. Using this approach we have to solve the following boundary and intial value problem of the elastic wave.

Find $\mathbf{u}(\mathbf{x},t) : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\begin{aligned}
\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) - \operatorname{div}\sigma(\mathbf{u}(\mathbf{x},t)) &= \mathbf{f}(\mathbf{x},t) && \forall \mathbf{x} \in \Omega,\ t \in \mathbb{R}_+ \\
\sigma(\mathbf{u}(\mathbf{x},t)) \cdot \mathbf{n} &= \mathbf{t}(\mathbf{x},t) && \forall \mathbf{x} \in \Gamma_N,\ t \in \mathbb{R}_+ \\
\mathbf{u}(\mathbf{x},t) &= \tilde{\mathbf{u}}(\mathbf{x},t) && \forall \mathbf{x} \in \Gamma_D,\ t \in \mathbb{R}_+ \\
\mathbf{u}(\mathbf{x},0) = \mathbf{U}(\mathbf{x}) \ \text{ and } \ \dot{\mathbf{u}}(\mathbf{x},0) &= \dot{\mathbf{U}}(\mathbf{x}) && \forall \mathbf{x} \in \Omega.
\end{aligned}$$

Here the volume body force $\mathbf{f}$ contains the contribution of the shrink process which is approximated via heat strain. To discretize the problem we use a 3D FE-model of the tool and the tool holder, see Figures A.2(a) and A.2(b). For the time discretization we apply the discontinuous Galerkin method on the system of first order ordinary differential equations involving the displacement and the velocity as unknowns. We use piecewise linear functions in space and piecewise quadratic functions in time. As the system is rotating during the whole process we reformulated the problem in rotational coordinates.



(a) Magnification of the tool tip.

(b) System of tool and tool holder.

Figure A.2.: FE-model of tool and tool holder.

The given displacement $\tilde{\mathbf{u}}$ on the Dirichlet boundary $\Gamma_D$ results from the vibrations of the whole system. For this reason a vibration model of the spindle is used to determine $\tilde{\mathbf{u}}$ at each time step. The cutting forces computed with a geometrical cutting force model imply the boundary traction $\mathbf{t}$. For more details, see [29] and [28]. Both models, vibration model and cutting force model, are coupled with the FE-model of tool and tool holder in the following way. For given displacement $\tilde{\mathbf{u}}(\mathbf{x},0)$ and traction $\mathbf{t}(\mathbf{x},0)$ we solve the elastic wave equation on the tool and the tool holder at each time step $t_k$. The resulting displacement field $\mathbf{u}(\mathbf{x},t_k)$ is used in the vibration model and the cutting force model to compute $\tilde{\mathbf{u}}(\mathbf{x},t_k)$ and $\mathbf{t}(\mathbf{x},t_k)$, respectively. This data denotes the new boundary data for the FE-model of tool and tool holder in the next time step $t_{k+1}$.

The phenomenon of tool extraction is caused by the change of the contact situation from stick to slip at the interface of the tool and the tool holder. Therefore, we observe the tangential and normal pressure at this interface during the process, see Figure A.3. A significant change of the quotient of normal pressure and tangential pressure at some time step would then be a hint for a possible tool extraction.

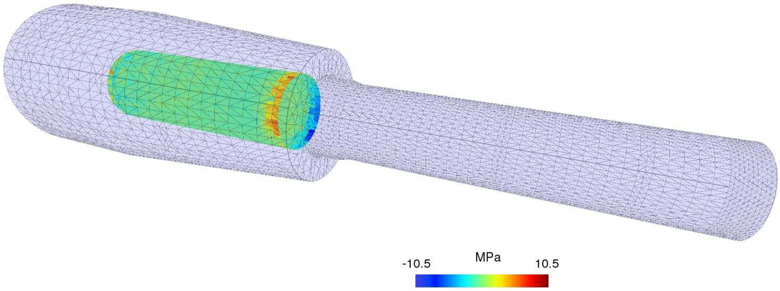



Figure A.3.: Change of normal pressure at the interface of tool and tool holder.

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