Moduli of Spin Curves

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Zusammenfassung

Diese Arbeit widmet sich der Bestimmung der Singularitäten des groben Modulraumes $\overline{S}_g$ der Spinkurven über $\mathbb{C}$. Das genaue Verständnis der Singularitäten führt zu dem Resultat, dass plurikanonische Formen auf der offenen Teilmenge der glatten Punkte holomorph zu einer Desingularisierung liften. Der von M. Cornalba in [Cor89] konstruierte Modulraum $\overline{S}_g$ kompaktifiziert den groben Modulraum S_g der glatten Spinkurven, dies sind Paare (C, L) bestehend aus einer glatten Kurve C arithmetischen Geschlechts $g \geq 2$ und einer Thetacharakteristik L, d.h. einem Geradenbündel L auf C mit der Eigenschaft, dass $L^{\otimes 2}$ isomorph ist zum kanonischen Bündel ω_C. Diese Kompaktifizierung verträgt sich mit der Deligne-Mumford-Kompaktifizierung $\overline{M}_g$ mittels stabiler Kurven des groben Modulraumes M_g der glatten Kurven arithmetischen Geschlechts g [DM69]. Insbesondere gibt es einen natürlichen Morphismus $\pi : \overline{S}_g \to \overline{M}_g$, der dem Modulpunkt einer Spinkurve den Modulpunkt der zugrunde liegenden Kurve zuordnet. π ist eine endliche Abbildung vom Grad 2^{2g}.

Im Vordergrund steht die lokale (analytische) Struktur des Modulraumes $\overline{S}_g$. Ähnlich wie $\overline{M}_g$ verhält sich $\overline{S}_g$ am Modulpunkt einer Spinkurve lokal isomorph zu einem Quotienten V/G eines $3g-3$-dimensionalen Vektorraumes V nach einer endlichen Gruppe G. Diese entspricht im Wesentlichen der Automorphismengruppe der betrachteten Spinkurve. Eine genaue Analyse der auftretenden Quotienten ermöglicht mit Hilfe des Reid-Tai-Kriteriums eine Beschreibung derjenigen Punkte in $\overline{S}_g$, an denen kanonische bzw. nicht-kanonische Singularitäten auftreten. Letztere bilden eine Teilmenge der Kodimension 2 in $\overline{S}_g$. Außerdem wird der Ort $\overline{S}_g^{\text{reg}} \subset \overline{S}_g$ der glatten Punkte in $\overline{S}_g$ bestimmt. Der Morphismus $\pi : \overline{S}_g \to \overline{M}_g$ spielt hierbei eine entscheidende Rolle, da er Zusammenhänge zwischen den gut verstandenen Singularitäten des Modulraumes $\overline{M}_g$ [HM82] und denen von $\overline{S}_g$ herstellt. Dabei ergibt sich auch eine Beschreibung des Verzweigungsverhaltens der endlichen Abbildung π.

Diese lokalen Resultate erlauben es zu beweisen, dass sich alle plurikanonischen Formen auf $\overline{S}_g^{\text{reg}}$ - d.h. Schnitte in $\Gamma(\overline{S}_g^{\text{reg}}, \mathcal{O}_{\overline{S}_g}(kK_{\overline{S}_g}))$ - holomorph zu einer Desingularisierung $\widetilde{S}_g$ von $\overline{S}_g$ fortsetzen lassen. Auch hier wird entscheidend auf das entsprechende Resultat von J. Harris und D. Mumford für $\overline{M}_g$ zurückgegriffen.

Schlagworte: Modulraum, Spinkurve, Singularitäten, plurikanonische Formen

Abstract

This thesis determines the singularities of the coarse moduli space $\overline{S}_g$ over $\mathbb{C}$ of spin curves. The precise description of the singularities yields the result that pluricanonical forms on the smooth locus of $\overline{S}_g$ lift holomorphically to a desingularisation. The moduli space $\overline{S}_g$ constructed by M. Cornalba in [Cor89] compactifies the coarse moduli space S_g of smooth spin curves. These are pairs (C, L) of a smooth curve of (arithmetic) genus $g \geq 2$ and a theta characteristic L on C, i.e. a line bundle L on C such that $L^{\otimes 2}$ is isomorphic to the canonical bundle ω_C. This compactification is compatible with the Deligne-Mumford compactification $\overline{M}_g$ of the coarse moduli space M_g of smooth curves of genus g via stable curves [DM69]. In particular there exists a natural morphism $\pi : \overline{S}_g \to \overline{M}_g$ which sends the moduli point of a spin curve to the moduli point of the underlying curve. π is a finite map of degree 2^{2g}.

This thesis focuses on the local (analytic) structure of the moduli space $\overline{S}_g$. As in the case of $\overline{M}_g$ an analytic neighbourhood of the moduli point of a spin curve in $\overline{S}_g$ is isomorphic to the quotient V/G of a $3g-3$-dimensional vector space V with respect to a finite group G. This group is essentially the automorphism group of the spin curve under consideration. A careful analysis of the occurring quotients gives a description of the locus of canonical singularities of $\overline{S}_g$ with the help of the Reid-Tai criterion. This locus has codimension 2 in $\overline{S}_g$. Moreover, the smooth locus $\overline{S}_g^{\text{reg}} \subset \overline{S}_g$ is determined. The morphism π plays an important role in these calculations, since it establishes a connection between the well understood singularities of $\overline{M}_g$ [HM82] and those of $\overline{S}_g$. In order to understand this connection the ramification of the finite map π is described.

These local results are used to prove that all pluricanonical forms on $\overline{S}_g^{\text{reg}}$, i.e. sections in $\Gamma(\overline{S}_g^{\text{reg}}, \mathcal{O}_{\overline{S}_g}(kK_{\overline{S}_g}))$, extend holomorphically to a desingularisation $\widetilde{S}_g$ of $\overline{S}_g$. An important ingredient is the analogous result for $\overline{M}_g$ by J. Harris and D. Mumford.

Keywords: moduli space, spin curve, singularities, pluricanonical forms

Contents

Chapter 1

Introduction

This thesis investigates the geometry of the coarse moduli space S_g of pairs of a smooth curve C of genus $g \geq 2$ (over $\mathbb{C}$) together with a theta characteristic L, i.e. a line bundle such that $L^{\otimes 2}$ is isomorphic to the canonical bundle ω_C. Theta characteristics are a classical subject, for example in the guise of bitangents of a smooth plane quartic. There is a natural forgetful morphism π from S_g to M_g, the moduli space of smooth curves, by sending the moduli point $[(C, L)] \in S_g$ of a pair (C, L) to the moduli point $[C] \in M_g$ of the curve. Since a smooth curve of genus g has exactly 2^{2g} non-isomorphic theta characteristics, π is finite of degree 2^{2g}, see for example the articles of D. Mumford [Mum71] and M. Atiyah [Ati71]. Moreover, every smooth curve C has $2^{g-1}(2^g + 1)$ even theta characteristics, i.e. $\dim H^0(C, L)$ is even, and $2^{g-1}(2^g - 1)$ odd theta characteristics. In the articles mentioned the authors prove that even and odd theta characteristics do not mix. This means, that in a family of curves with theta characteristics over a connected base scheme the dimension of the space of sections of the theta characteristic is constant modulo 2, showing that S_g is the disjoint union of the moduli spaces S_g^+ and S_g^- of curves with even resp. odd theta characteristics, which can be proven to be the irreducible components of S_g.

As in the case of M_g it is natural to ask for a geometrically meaningful compactification $\overline{S}_g$ of S_g. In the case of curves such a compactification was given by P. Deligne and D. Mumford in [DM69] in terms of stable curves; where curves having only nodes as singularities and finite automorphism groups are included into the moduli problem to give the projective moduli space $\overline{M}_g$ which contains M_g as a dense open subset. In his article [Cor89] M. Cornalba gave a compactification $\overline{S}_g$ of S_g which is compatible with the forgetful morphism $\pi : S_g \to M_g$ and the Deligne-Mumford compactification $\overline{M}_g$, i.e. $\overline{S}_g$ fits into the commutative diagram

$$\begin{array}{ccc} S_g & \hookrightarrow & \overline{S}_g \\ \downarrow \pi & & \downarrow \\ M_g & \hookrightarrow & \overline{M}_g \end{array}$$

and the map $\overline{S}_g \to \overline{M}_g$ is again a forgetful morphism which is finite of degree 2^{2g}. Points in $\overline{S}_g$ correspond to spin curves, i.e. triples (X, L, b) of a so called quasi-stable curve X, a line bundle L on X and a homomorphism b from $L^{\otimes 2}$ to a "modification" of the canonical bundle ω_X which is nearly an isomorphism. The homomorphism is needed to get a separated moduli space. In the smooth case it just amounts to choosing a specific isomorphism $L^{\otimes 2} \xrightarrow{\cong} \omega_C$. T. Jarvis gave other geometrically meaningful compactifications of S_g compatible with $\overline{M}_g$, where boundary points correspond to triples $(C, \mathcal{E}, b)$, where C is a stable curve and $\mathcal{E}$ a rank one torsion free sheaf. The resulting moduli spaces are isomorphic in the case considered in this thesis. It is therefore only a question of taste whether one prefers to work with torsion free sheaves on stable curves or with line bundles on curves in the slightly bigger class of quasi-stable curves. In this thesis Cornalba's description via quasistable curves and line bundles will be used.

Chapter 2 will provide the necessary background on stable curves and spin curves. In particular the vague definitions given above will be made precise and important properties will be explained.

Having understood the objects Chapter 3 turns to the question of the existence of coarse moduli spaces of stable curves and spin curves. In addition the morphism $\overline{S}_g \to \overline{M}_g$ and its fibres will be studied, since this is a very good tool to transfer results known for $\overline{M}_g$ to $\overline{S}_g$. Afterwards the local structure of the moduli spaces will be described with the help of deformation theory.

Chapter 4 begins with a short introduction into canonical singularities. Section 4.2 explains and sharpens the analysis of singularities of $\overline{M}_g$ by J. Harris and D. Mumford in [HM82]. Section 4.3 is dedicated to a careful study of the singularities of $\overline{S}_g$. In particular the locus of canonical singularities is precisely described.

A variety X with only canonical singularities has the important property that all global pluricanonical forms, i.e. sections of $\omega_X^{\otimes k}$ over the smooth locus X^{reg}, extend to every desingularisation $\widetilde{X}$. This lifting result implies that in order to determine the Kodaira dimension of X it is enough to consider the rate of growth of the space of global pluricanonical forms on X, implying that one does not have to concern oneself with the desingularisation $\widetilde{X}$. Despite the fact that $\overline{M}_g$ does have some non-canonical singularities Harris and Mumford proved that the lifting result in fact holds for $\overline{M}_g$, $g \geq 4$, and used this to show that $\overline{M}_g$ is of general type for $g \geq 24$ (see also the subsequent article [EH87] of J. Harris with D. Eisenbud). The lifting result is explained in Section 5.2. and transferred to $\overline{S}_g$ with the help of the description of the non-canonical locus of $\overline{S}_g$ and the map π. In Section 5.1 some information on the Picard groups of $\overline{M}_g$ and $\overline{S}_g$ are given, in particular the canonical divisors of the irreducible components of $\overline{S}_g$ are computed.

Conventions

A *curve* will always mean a complete reduced algebraic curve. In particular a curve is not necessarily irreducible. The genus of a curve C is always the arithmetic genus, i.e. $g = g(C) = 1 - \chi(\mathcal{O}_C)$. In case C is connected this is just $h^1(C, \mathcal{O}_C)$ and if C is smooth this coincides with the geometric genus $h^0(C, \omega_C)$ by Serre duality. All schemes are schemes over the complex numbers $\mathbb{C}$.

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Chapter 2

Spin curves

2.1 Stable curves

The notion of stable curves was introduced by Deligne and Mumford in their article [DM69] in order to prove the irreducibility of the moduli space of curves in any characteristic. Another reference is the book [HM98] of Harris and Morrison which gives an introduction to the topic.

Definition 2.1.1. (i) A *nodal curve of genus g* is a curve C with only nodes, i.e. ordinary double points, as singularities and (arithmetic) genus $1-\chi(\mathcal{O}_C) = g$.
(ii) An *n-pointed nodal curve of genus g (with ordering)* is a nodal curve C of genus g together with an ordered tuple $(Q_1, \ldots, Q_n)$ of distinct smooth points $Q_k \in C$. The *automorphism group* of $(C; Q_1, \ldots, Q_n)$ is

$$\mathrm{Aut}(C; Q_1, \ldots, Q_n) = \{\sigma \in \mathrm{Aut}(C) | \sigma(Q_k) = Q_k \ \forall k\}.$$

(iii) A *stable curve of genus g* is a connected nodal curve C of genus g, whose automorphism group $\mathrm{Aut}\, C$ is finite.
(iv) An *n-pointed stable curve of genus g (with ordering)* is an n-pointed nodal curve $(C; Q_1 \ldots, Q_n)$ of genus g, which is connected and whose automorphism group $\mathrm{Aut}(C; Q_1, \ldots, Q_n)$ is finite.
(v) A *family of curves* over a scheme Z is a proper flat morphism $f : \mathcal{C} \to Z$ such that every geometric fibre $C_z = \mathcal{C} \times_Z \{z\})$ is a curve.
(vi) A *family of stable curves* over a scheme Z is a family $f : \mathcal{C} \to Z$ of curves such that every geometric fibre C_z is stable.
(vii) A *family of n-pointed stable curves* over Z is a family $f : \mathcal{C} \to Z$ of curves together with n sections $s_1, \ldots, s_n$ of f such that every $(C_z; s_1(z), \ldots, s_n(z))$ is an n-pointed stable curve.

Notation 2.1.2. The notation $(C; \{Q_1, \ldots, Q_n\})$ is used for an *n-pointed nodal curve (without ordering)*, i.e. if the ordering of the points is disregarded. The automorphism group is then

$$\mathrm{Aut}(C;\{Q_1,\dots,Q_n\}) = \{\sigma \in \mathrm{Aut}(C) | \sigma(\{Q_k\}) = \{Q_k\}\},$$

i.e. permutations of the marked points are allowed. $(C;\{Q_1,\dots,Q_n\})$ is *stable* if its automorphism group is finite.

Many properties of a (pointed) nodal curve can already be seen by the "combinatorics" of the curve, which are summarized in a labeled graph.

Definition 2.1.3. Let $(C;Q_1,\dots,Q_n)$ resp. $(C;\{Q_1,\dots,Q_n\})$ be a pointed nodal curve. The *dual graph* $\Gamma = \Gamma(C;Q_1,\dots,Q_n)$ resp. $\Gamma = \Gamma(C;\{Q_1,\dots,Q_n\})$ of the pointed curve is defined as follows.

(i) The *set of vertices* $V(\Gamma)$ consists of one vertex $v(C_j)$ for every irreducible component C_j of C.
(ii) The *set of edges* $E(\Gamma)$ consists of one edge $e(P_i)$ for every node P_i of C and this edge joins the two vertices corresponding to the two irreducible components of C meeting at the node. In case the two branches at the node are in the same component, the edge is a loop.
(iii) The *set of markings* $H(\Gamma)$ consists of one halfedge $h(Q_k)$ for every marked point Q_k, i.e. an edge whose one end is incident to a vertex while the other end is loose. The halfedge $h(Q_k)$ is incident to the vertex corresponding to the component on which Q_k lies.
(iv) The *labelling* $g : V(\Gamma) \to \mathbb{Z}$ where every vertex $v(C_j)$ is labeled with the genus $g(v(C_j)) = g(C_j^\nu)$ of the normalisation C_j^ν of the component C_j.

For any graph Γ there are the following notions.

Definition 2.1.4. (i) A *1-chain of* Γ (with values in $\mathbb{F}_2$) is an element of the $\mathbb{F}_2$-vector space

$$C_1(\Gamma,\mathbb{F}_2) = \left\{ \sum_{e\in E(\Gamma)} \lambda_e e \,\middle|\, \lambda_e \in \mathbb{F}_2 \right\}$$

and a *0-chain of* Γ (with values in $\mathbb{F}_2$) is an element of the $\mathbb{F}_2$-vector space

$$C_0(\Gamma,\mathbb{F}_2) = \left\{ \sum_{v\in V(\Gamma)} \lambda_v v \,\middle|\, \lambda_v \in \mathbb{F}_2 \right\}.$$

The *boundary map* ∂ is the linear map $\partial = \partial_1 : C_1(\Gamma,\mathbb{F}_2) \to C_0(\Gamma,\mathbb{F}_2)$ determined by $e \mapsto v_1 + v_2$ if v_1 and v_2 are the vertices incident to the edge e. This gives a complex

$$C_\bullet : \quad 0 \xrightarrow{\partial_2} C_1(\Gamma,\mathbb{F}_2) \xrightarrow{\partial_1} C_0(\Gamma,\mathbb{F}_2) \xrightarrow{\partial_0} 0.$$

$Z_1(\Gamma,\mathbb{F}_2) = \ker \partial_1$ denotes the vector space of *1-cycles of* Γ.

(ii) The *ith Betti number* $b_i(\Gamma)$ of the graph Γ is the dimension of the ith homology group $H_i(C_\bullet) = \ker \partial_i / \operatorname{im} \partial_{i+1}$ of the complex $C_\bullet$.

Remark 2.1.5. The first homology group of the complex $C_\bullet$ is just the space of 1-cycles, i.e. $b_1(\Gamma) = \dim Z_1(\Gamma, \mathbb{F}_2)$. By induction on the number of elements in $E(\Gamma)$ one can show that

$$b_1(\Gamma) = \#E(\Gamma) - \#V(\Gamma) + \#CC(\Gamma)$$

where $CC(\Gamma) = \ker \partial_0 / \operatorname{im} \partial_1 = H_0(C_\bullet)$ is the set of connected components of Γ. In case $\Gamma = \Gamma(C)$ for a connected nodal curve C, the graph Γ is also connected, i.e. $\#CC(\Gamma) = 1$, and $b_1(\Gamma) = \#E(\Gamma) - \#V(\Gamma) + 1$. The first Betti number counts the number of independent 1-cycles. For example if a set V of vertices is given and Γ is a connected graph with $V(\Gamma) = V$ which has the minimal number of edges, then $\#E(\Gamma) = \#V(\Gamma) - 1$, $b_1(\Gamma) = 0$ and Γ is called a *tree*.

This gives an important class of nodal curves:

Definition 2.1.6. A nodal curve C is of *compact type* iff its dual graph $\Gamma = \Gamma(C)$ is a *tree*, i.e. $b_1(\Gamma) = 0$ and Γ is connected.

Automorphisms of (labeled) graphs are compatible triples of permutations, more precisely:

Definition 2.1.7. Let C be a nodal curve and $Q_1, \dots, Q_n$ distinct smooth points.

(i) An *automorphism of the labeled graph* $\Gamma = \Gamma(C; \{Q_1, \dots, Q_n\})$ is a triple $(\sigma_V, \sigma_E, \sigma_H)$ where $\sigma_\star$, $\star \in \{V, E, H\}$, is a permutation of $\star(\Gamma)$ such that σ_V is compatible with the labels and with σ_E and σ_H. σ_V is *compatible with the labels* if $g(v) = g(\sigma_V(v))$ for every vertex v. It is *compatible with* σ_E if for every edge e which is incident to the vertices v_1 and v_2 the image $\sigma_E(e)$ is incident to $\sigma_V(v_1)$ and $\sigma_V(v_2)$. σ_V is *compatible with* σ_H if for every halfedge h which is incident to the vertex v the image $\sigma_H(h)$ is incident to $\sigma_V(v)$.
(ii) An *automorphism of the labeled graph* $\Gamma = \Gamma(C; Q_1, \dots, Q_n)$ is an automorphism $(\sigma_V, \sigma_E, \mathrm{id}_H) \in \operatorname{Aut}(\Gamma(C; \{Q_1, \dots, Q_n\}))$, i.e. every marking $h \in H(\Gamma)$ is fixed.

Remark 2.1.8. Let C be a nodal curve and $Q_1, \dots, Q_n$ distinct smooth points. It is clear that the automorphism group of the dual graph $\Gamma = \Gamma(C; Q_1, \dots, Q_n)$ resp. $\Gamma = \Gamma(C; \{Q_1, \dots, Q_n\})$ is finite. Every automorphism σ of the pointed curve $(C; Q_1, \dots, Q_n)$ resp. $(C; \{Q_1, \dots, Q_n\})$ induces a permutation of the nodes, a permutation of the components and a permutation of the markings (in case the markings are considered as an ordered tuple, the last permutation is just the identity). This gives a triple of permutations $(\sigma_V, \sigma_E, \sigma_H)$ of $V(\Gamma)$, $E(\Gamma)$ and $H(\Gamma)$ respectively, where σ_V is compatible with the genus labels and with σ_E and σ_H. Therefore every automorphism of the pointed curve gives an automorphism of the dual graph. The induced automorphism of the dual graph contains all "combinatorial" information of σ.

Proposition 2.1.9. [HM98, Section 3.A.] *Let C be a connected nodal curve of genus g with dual graph Γ. Then the genus formula*

$$g = \sum_j g(C_j^\nu) + b_1(\Gamma)$$

holds.

Remark 2.1.10. For a nodal curve, which is not necessarily connected, the genus is

$$g = \sum_j g(C_j^\nu) + b_1(\Gamma) + 1 - \#CC(\Gamma).$$

This means for example that the genus of the disjoint union of two $\mathbb{P}^1$'s is -1, which is also clear from the definition $g = 1 - \chi(\mathcal{O}_C) = 1 - 2 + 0 = -1$.

One has the following geometric characterisation of stable curves.

Proposition 2.1.11. [HM98, p. 47] *Let C be a connected nodal curve and $\nu : C^\nu = \coprod_j C_j^\nu \to C$ its normalisation. Let $Q_1, \ldots, Q_n$ be distinct smooth points on C. $(C; Q_1, \ldots, Q_n)$ is stable iff every rational C_j^ν contains at least three points lying over a node or a marked point and every elliptic C_j^ν contains at least one such point.*

Notation 2.1.12. Let $P_i^\pm$ be the two preimages of P_i under ν and denote the unique preimage of Q_k also by Q_k. The short hand $(C_j^\nu; \{Q_k, P_i^\pm\})$ will be used for the pointed curve (without ordering) $(C_j^\nu; \{Q_1, \ldots, Q_n, P_1^\pm, \ldots, P_m^\pm\} \cap C_j^\nu)$. Fixing a (non-canonical) ordering of $\{Q_1, \ldots, Q_n, P_1^\pm, \ldots, P_m^\pm\} \cap C_j^\nu$ for every component C_j^ν gives a pointed curve (with ordering) which is denoted by $(C_j^\nu; Q_k, P_i^\pm)$. Both pointed curves are called the *pointed normalisation of the component C_j in C*.

Proof of Proposition 2.1.11. Claim: $\mathrm{Aut}(C; Q_1, \ldots, Q_n)$ is finite if and only if $\mathrm{Aut}(C_j^\nu; Q_k, P_i^\pm)$ is finite for every component C_j.

"$\Rightarrow$" Let C_j be a component, σ_j^ν an automorphism of $(C_j^\nu; Q_k, P_i^\pm)$ and set $\sigma_{j'}^\nu = \mathrm{id}_{C_{j'}^\nu}$ for all other components $C_{j'}$. The collection of these automorphisms gives an automorphism σ^ν of $(C^\nu; Q_k, P_i^\pm)$. Since every $P_i^\pm$ and every Q_k is fixed, σ^ν induces an automorphism σ of C which fixes every Q_k. Therefore if there would exist a component C_j such that $\mathrm{Aut}(C_j^\nu; Q_k, P_i^\pm)$ is infinite, the group $\mathrm{Aut}(C; Q_1, \ldots, Q_n)$ would also be infinite.

"$\Leftarrow$" Suppose that $\mathrm{Aut}(C_j^\nu; Q_k, P_i^\pm)$ is finite for every component C_j. Let Γ be the dual graph of $(C; Q_1, \ldots, Q_n)$. Γ has only finitely many automorphisms, let $(\sigma_V, \sigma_E, \mathrm{id}_H)$ be one. Let $v = v(C_j)$ be a vertex and $\sigma_V(v) = v(C_{j'})$ its image. Order the halfedges and edges incident to the two vertices compatibly, i.e. if $(R_1, \ldots, R_l)$ is the ordering of $\{Q_k, P_i^\pm\} \cap C_j^\nu$ then $(\sigma_\star(R_1), \ldots, \sigma_\star(R_l)) =:$

$(R'_1, \ldots, R'_l)$ is an ordering of $\{Q_k, P_i^{\pm}\} \cap C_{j'}^{\nu}$, here $\sigma_\star$ denotes the appropriate permutation σ_E or id_H.

Either the set of isomorphisms $\mathrm{Isom}((C_j^\nu; R_1, \ldots, R_l), (C_{j'}^\nu; R'_1, \ldots, R'_l))$ of the two smooth pointed curves is empty or it has the same number of elements as $\mathrm{Aut}(C_j^\nu; R_1, \ldots, R_l)$. Since if σ_j^ν is such an isomorphism, then

$$\begin{aligned} \mathrm{Isom}((C_j^\nu; R_1, \ldots, R_l), (C_{j'}^\nu; R'_1, \ldots, R'_l)) &\xleftrightarrow{1:1} \mathrm{Aut}(C_j^\nu; R_1, \ldots, R_l) \\ \sigma_j^\nu \circ \varphi_j &\longleftarrow\!\shortmid \varphi_j \\ (\sigma_j^\nu)' &\longmapsto (\sigma_j^\nu)' \circ (\sigma_j^\nu)^{-1} \end{aligned}$$

If for the given automorphism $(\sigma_V, \sigma_E, \mathrm{id}_H)$ of the graph there exists a component C_j such that this set of isomorphisms is empty, $(\sigma_V, \sigma_E, \mathrm{id}_H)$ is not induced by an automorphism of the curve (see Remark 2.1.8). If for $(\sigma_V, \sigma_E, \mathrm{id}_H)$ there exists an isomorphism σ_j^ν for every component C_j there are only finitely many choices for the collection $\sigma^\nu = \{\sigma_j^\nu\}_{C_j^\nu}$, which is an automorphism of $(C^\nu; Q_1, \ldots, Q_n)$ such that for every i the automorphism σ^ν maps $\{P_i^\pm\}$ to $\{P_{i'}^\pm\}$ for an appropriate i'. Hence every such σ^ν induces a unique automorphism of $\mathrm{Aut}(C; Q_1, \ldots, Q_n)$. Therefore the pointed curve $(C; Q_1, \ldots, Q_n)$ is stable and the claim is proven.

In order to prove the proposition three cases have to be considered. If the component C_j^ν has genus at least two, the automorphism group $\mathrm{Aut}\, C_j^\nu$ is finite and $\mathrm{Aut}(C_j^\nu; R_1, \ldots, R_l)$ being a subgroup is also finite for every choice of points R_k. If the component C_j^ν is elliptic resp. rational $\mathrm{Aut}(C_j^\nu; Q_k, P_i^\pm)$ is finite iff there is at least one resp. three marked points. □

Remark 2.1.13. A pointed curve $(C; \{Q_1, \ldots, Q_n\})$ is stable iff $(C; Q_1, \ldots, Q_n)$ is stable. This follows easily from the above proof, the only difference is that non-trivial permutations σ_H of the markings in the dual graph are allowed. But this gives only finitely many additional choices.

The dualizing sheaf of a nodal curve is a very important tool which in many situations especially in Serre duality plays the role of the bundle Ω_C of differentials of a smooth curve C.

Definition 2.1.14. Let X be a scheme over $\mathbb{C}$. A *dualizing sheaf* for X is a coherent sheaf ω_X together with an isomorphism $H^{\dim X}(X, \omega_X) \cong \mathbb{C}$ such that for all coherent sheaves $\mathcal{F}$ on X the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^{\dim X}(X, \mathcal{F}) \longrightarrow H^{\dim X}(X, \omega_X) \cong \mathbb{C}$$

gives an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \cong H^{\dim X}(X, \mathcal{F})^*,$$

where * denotes the dual vector space.

A dualizing sheaf, if it exists, is unique. In [Har77, Section III.7.] Hartshorne constructs the dualizing sheaf for projective schemes over an algebraicly closed field and in case X is Cohen-Macaulay and equidimensional he proves Serre duality, i.e.

$$\operatorname{Ext}^i(\mathcal{F}, \omega_X) \xrightarrow{\cong} H^{\dim X - i}(X, \mathcal{F})^*$$

for $i \geq 0$ and all coherent sheaves $\mathcal{F}$ on X, where $\operatorname{Ext}^i(\mathcal{F}, \cdot)$, $i \geq 0$, are the right derived functors of $\operatorname{Hom}(\mathcal{F}, \omega)$. If X is a local complete intersection in $\mathbb{P}^r$ with ideal sheaf $\mathcal{I}$, then $\omega_X = \omega_{\mathbb{P}^r} \otimes \wedge^r(\mathcal{I}/\mathcal{I}^2)^*$ which is invertible. In case X is smooth this invertible sheaf is just the canonical sheaf $\wedge^{\dim X}\Omega_X$ of $\dim X$-forms on X.

Since a nodal curve C is a local complete intersection, there is a dualizing sheaf ω_C which makes Serre duality work and ω_C is invertible. It can be described very concretely via the normalisation $\nu : C^\nu \to C$ of C [HM98, Section 3.A.]. For an open $U \subset C$ $\omega_C(U)$ is the space of rational one-forms η on $\nu^{-1}(U) \subset C^\nu$ having at worst simple poles at the preimages $P_i^\pm$ of every node $P_i \in U \cap \operatorname{sing} C$ and such that

$$\operatorname{res}_{P_i^+} \eta + \operatorname{res}_{P_i^-} \eta = 0. \tag{res}$$

ω_C has degree $2g - 2$ and in case the curve C is connected $\dim H^0(C, \omega_C) = g$. Proposition 2.1.11 then translates into: A connected nodal curve C is stable iff ω_C is ample, i.e. $\nu^*\omega_{C|C_j} = \omega_{C_j^\nu}(\sum_{Q \in \{P_i^\pm\} \cap C_j^\nu} Q)$ is ample for every irreducible component. Moreover, for every stable curve C of genus $g \geq 2$ and every $n \geq 3$ the line bundle $\omega_C^{\otimes n}$ is very ample ([DM69, Theorem 1.2.]).

There is also a relative version of the dualizing sheaf, i.e. for a morphism $f : X \to Y$ a sheaf $\omega_{X/Y}$ which replaces the bundle $\wedge^{\dim X}\Omega_{X/Y}$ of relative $\dim X$-forms on X. The general theory can be found in [Har66]. For a family $f : \mathcal{C} \to Z$ of nodal curves $\omega_{\mathcal{C}/Z}$ can be defined as the sheaf of rational relative differentials, i.e. rational sections of the relative cotangent bundle $\operatorname{coker}(df : f^*\Omega_Z \to \Omega_{\mathcal{C}})$, satisfying the residue condition (res). Therefore $\omega_{\mathcal{C}/Z}$ restricts to the dualizing sheaf $\omega_{\mathcal{C}_z}$ on the fibre $\mathcal{C}_z$. Moreover, one can show that $R^1 f_*\omega_{\mathcal{C}/Z} = \mathcal{O}_Z$, where $R^1 f_*$ is the first higher direct image functor of f_*, i.e. the first right derived functor of f_* (see [HM98, 3.A.]).

The relative dualizing sheaf is compatible with base change, i.e. if

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\overline{\varrho}} & \mathcal{C} \\ {\scriptstyle f'}\downarrow & \ulcorner & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{\varrho} & Z \end{array}$$

is a fibre square, then $\overline{\varrho}^*\omega_{\mathcal{C}/Z} = \omega_{\mathcal{C}'/Z'}$.

Let $f : \mathcal{C} \to Z$ be a family of stable curves. From the facts that $R^1 f_*\omega_{\mathcal{C}/Z}$ is the trivial bundle on Z and $H^1(\mathcal{C}_z, \omega_{\mathcal{C}_z}^{\otimes n}) = 0$ for every $n \geq 2$ and every $z \in Z$ it follows by the relative version of Serre duality that $f_*\omega_{\mathcal{C}/Z}^{\otimes n}$ is locally free of rank

$(2n-1)(g-1)$ for $n \geq 2$. The rank is just the dimension of $H^0(C, \omega_C^{\otimes n})$ for $n \geq 2$ and any stable curve C.

From now on the letter C is reserved to denote only stable curves.

Definition 2.1.15. Let X be a connected nodal curve of genus $g \geq 2$ and $(X_j^\nu; \{P_i^\pm\})$ the pointed normalisation of any irreducible component X_j of X.

(i) X is *semistable* iff every rational component X_j^ν contains at least two marked points and every elliptic component X_j^ν contains at least one marked point.
(ii) A rational X_j^ν of a semistable curve X is called an *exceptional component* or a *rational ladder* if it contains exactly two marked points.
(iii) The *non-exceptional subcurve* $\widetilde{X}$ of a semistable curve X is the closure of the complement of all exceptional components, i.e.

$$\widetilde{X} = \overline{X \setminus \bigcup_{\substack{E \text{ is} \\ \text{exc.} \\ \text{comp.}}} E}$$

(iv) A semistable curve X is *quasistable* if the intersection of any two exceptional components is empty.
(v) The *stable model* of a quasistable curve X is the (unique) stable curve C obtained by contracting every exceptional component of X to a point. The blow down map $\beta : X \to C$ will also be called *stable model* of X.

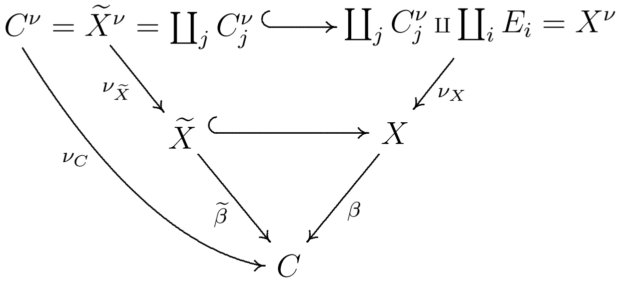
Remark 2.1.16. Let X be a semistable curve of genus $g \geq 2$ and $\beta : X \to C$ be the contraction of every exceptional component of X to a point. Then C is a stable curve.

Assume C is not a curve. This would mean, that every component of X is exceptional and C is a point. Since X is connected this would imply that X is a cycle of exceptional components, i.e. all irreducible components of X are exceptional and there is an ordering $X_1, \dots, X_l$ of these, such that the two marked points on X_j^ν map to the unique nodes P_j resp. P_{j+1} of X where X_j meets X_{j-1} resp. X_{j+1}. Here the indices are understood modulo l and if $l = 1$ this means that the two marked points map to the same node. Since exceptional components are rational by definition, the genus formula shows that X has genus 1, giving a contradiction to $g \geq 2$. Therefore C is a curve.

After contracting every exceptional component of X to a point the pointed normalisation of the resulting curve C has only rational components with at least three marked points, elliptic components with at least one marked point and components with higher genus, hence C is stable by Proposition 2.1.11.

Remark 2.1.17. The most important class of curves will be the class of quasistable curves. For these the following facts are fundamental.

(i) For a fixed stable curve C there is a $1:1$-correspondence between quasistable curves with stable model C and subsets N of $\operatorname{sing} C$. Given a quasistable curve X with stable model $\beta : X \to C$ the corresponding subset N contains exactly those nodes of C which are images of exceptional components E of X. On the other hand if N is a subset of $\operatorname{sing} C$ and $\beta : X \to C$ the blow up of C at N, the resulting curve X is quasistable.
(ii) Using the above notation the restriction of β to the non-exceptional subcurve $\widetilde{X} = \overline{X \setminus \bigcup_i E_i}$, the union taken over all exceptional components E_i, is the partial normalisation of C at N. Therefore denoting by $\nu_\star : \star^\nu \to \star$ the normalisation of $\star \in \{C, X, \widetilde{X}\}$ gives the following commutative diagram



If C_j is an irreducible component of C there exists a unique irreducible component of $\widetilde{X}$ which is mapped onto C_j by $\widetilde{\beta}$. By abuse of notation this component is also denoted by C_j. The image of C_j in X is then a *non-exceptional component* and is still denoted by C_j.

In the following lemma the relations between the set of isomorphisms between two quasistable curves and their stable models are studied.

Proposition 2.1.18. *Let C (resp. C') be a stable curve of genus $g \geq 2$, $N \subset \operatorname{sing} C$ (resp. $N' \subset \operatorname{sing} C'$) a set of nodes and $\beta : X \to C$ (resp. $\beta' : X' \to C'$) the blow up of C at N (resp. of C' at N'). In this situation either* $\operatorname{Isom}(X, X')$ *and* $\operatorname{Isom}((C; N), (C'; N'))$ *are both empty or there is a surjective map*

$$\varphi : \operatorname{Isom}(X, X') \longrightarrow \operatorname{Isom}((C; N), (C'; N'))$$

with fibres $(\mathbb{C}^*)^l$ where $l = \#N = \#N'$ and

$$\operatorname{Isom}((C; N), (C'; N')) = \{\sigma \in \operatorname{Isom}(C, C') | \sigma(N) = N'\}$$

is the set of isomorphisms between C and C' inducing a bijection between N and N'.

Proof. Consider the map

$$\varphi : \operatorname{Isom}(X, X') \longrightarrow \operatorname{Isom}((C; N), (C'; N'))$$
$$\sigma \longmapsto \sigma_C$$

where σ_C is the unique isomorphism such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X' \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta'} \\ C & \xrightarrow{\sigma_C} & C' \end{array}$$

φ is well defined since the isomorphism σ gives a bijection of the sets of exceptional components of X and X' and therefore the induced isomorphism σ_C gives a bijection of N and N'.

Claim: φ is surjective. Let $\sigma_C \in \mathrm{Isom}((C;N),(C';N'))$ and define $\widetilde{\sigma} : \widetilde{X} \to \widetilde{X}'$ to be the unique morphism such that

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\sigma}} & \widetilde{X}' \\ \downarrow{\scriptstyle\widetilde{\beta}} & & \downarrow{\scriptstyle\widetilde{\beta}'} \\ C & \xrightarrow{\sigma_C} & C' \end{array}$$

commutes, where $\widetilde{X} \subset X$ and $\widetilde{X}' \subset X'$ are the non-exceptional subcurves. Then $\widetilde{\sigma}$ is an isomorphism and if $N \ni P_i \overset{\sigma_C}{\longmapsto} P_i' \in N'$ then $\widetilde{\sigma}(\{P_i^{\pm}\}) = \{P_i'^{\pm}\}$, where $\{P_i^{\pm}\} = \widetilde{\beta}^{-1}(P_i)$ and $\{P_i'^{\pm}\} = \widetilde{\beta'}^{-1}(P_i')$. $\widetilde{\sigma}$ has to be extended to X. Denote by $E_i = \beta^{-1}(P_i)$ and $E_i' = \beta'^{-1}(P_i')$ the exceptional components of X resp. X' over these nodes. Then E_i meets $\widetilde{X}$ exactly in the points $P_i^{\pm}$. E_i and E_i' are rational and one can choose identifications $E_i \cong \mathbb{P}^1$ and $E_i' \cong \mathbb{P}^1$ such that $P_i^+ = 0$, $P_i^- = \infty$, $\widetilde{\sigma}(P_i^+) = 0$ and $\widetilde{\sigma}(P_i^-) = \infty$. This shows that choosing an extension to E_i is the same as choosing an automorphism of $(\mathbb{P}^1; 0, \infty)$ which is the same as choosing a non-zero complex number. Therefore either both sets of isomorphisms are empty or the fibre $\varphi^{-1}(\sigma_C)$ is $(\mathbb{C}^*)^l$ where l is the number of exceptional components of X, which equals the number of elements in N. Since the curves X and X' are isomorphic in the last case, X' has the same number of exceptional components. □

Definition 2.1.19. A *family of quasistable curves of genus* $g \geq 2$ is a proper flat morphism $f : \mathcal{X} \to Z$ such that every geometric fibre of f is a quasistable curve of genus g. The *stable model* of a family $f : \mathcal{X} \to Z$ of quasistable curves is the family $f_{\mathcal{C}} : \mathcal{C} \to Z$ of the fibrewise stable models, i.e.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\beta} & \mathcal{C} \\ {\scriptstyle f}\searrow & & \swarrow{\scriptstyle f_{\mathcal{C}}} \\ & Z & \end{array}$$

and for every $z \in Z$ the restriction $\mathcal{X}_z = \mathcal{X} \times_Z \{z\} \to \mathcal{C}_z = \mathcal{C} \times_Z \{z\}$ of β is the stable model of the quasistable curve $\mathcal{X}_z$.

2.2 Spin curves

The main references for this section are the articles [Cor89, CC03, CCC04] by Cornalba at al. In particular his definition of spin curves will be used. Jarvis gave different definitions in his articles [Jar98, Jar00] via rank 1 torsion free sheaves on stable curves. These will not be used, but it should be mentioned, that in the cases considered in this thesis the definitions are equivalent.

Definition 2.2.1. A *spin curve of genus* $g \geq 2$ is a triple (X, L, b), where X is a quasistable curve of genus g with stable model $\beta : X \to C$, L a line bundle on X and $b : L^{\otimes 2} \to \beta^* \omega_C$ a homomorphism, such that the restriction $L_{|E}$ to any exceptional component E is isomorphic to $\mathcal{O}_E(1)$ and the restriction of b to the non-exceptional subcurve $\widetilde{X} = \overline{X \setminus \bigcup_E E}$ induces an isomorphism

$$\widetilde{b} : L^{\otimes 2}_{|\widetilde{X}} \xrightarrow{\cong} \omega_{\widetilde{X}} \left(\hookrightarrow (\beta^* \omega_C)_{|\widetilde{X}} \right).$$

The curve X is the *support* of the spin curve (X, L, b), the pair (L, b) a *spin structure* on X. A spin curve (X, L, b) is *smooth* if its support is.

Definition 2.2.2. The *parity* of a spin curve (X, L, b) is defined as $h^0(X, L) \bmod 2 \in \mathbb{F}_2$. The spin curve is *even* resp. *odd* if its parity is even resp. odd.

Example 2.2.3. A smooth spin curve is a triple of a smooth curve C, a theta characteristic L on C and an isomorphism $b : L^{\otimes 2} \xrightarrow{\cong} \omega_C$. For every theta characteristic the isomorphism b is unique only up to a non-zero scalar, but it will be shown later that these different choices give isomorphic spin curves.

Remark 2.2.4. Let (X, L, b) be a spin curve, $\beta : X \to C$ the stable model of X and $N \subset \operatorname{sing} C$ the set of nodes that are blown up in β. The nodes in N are called *exceptional nodes*, those in $\Delta = \operatorname{sing} C \setminus N$ *non-exceptional*. Let $\widetilde{\beta}$ be the restriction of β to the non-exceptional subcurve $\widetilde{X}$, i.e. $\widetilde{\beta} : \widetilde{X} \to C$ is the partial normalisation of C at the set N, then

$$(\beta^* \omega_C)_{|\widetilde{X}} = \widetilde{\beta}^* \omega_C.$$

On the other hand $\omega_{\widetilde{X}} = \widetilde{\beta}^* \omega_C(-D)$, where D is the pull back of N via $\widetilde{\beta}$ considered as a divisor on $\widetilde{X}$. Therefore

$$\begin{array}{ccc} L^{\otimes 2}_{|\widetilde{X}} & \xrightarrow{\;b_{|\widetilde{X}}\;} & (\beta^* \omega_C)_{|\widetilde{X}} = \widetilde{\beta}^* \omega_C \\ & {\scriptstyle \cong} \searrow^{\widetilde{b}} \qquad\qquad \nearrow & \\ & \omega_{\widetilde{X}} = \widetilde{\beta}^* \omega_C(-D) & \end{array}$$

commutes, where the inclusion is an inclusion of sheaves. On every exceptional component E of X the line bundle $L^{\otimes 2}$ has degree 2, $L^{\otimes 2}_{|E} \cong (\mathcal{O}_E(1))^{\otimes 2}$, while the degree of $\beta^* \omega_C$ is zero, hence the homomorphism b must vanish on E.

Definition 2.2.5. A *family of spin curves of genus* $g \geq 2$ over a scheme Z is a triple $(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B})$ where $f : \mathcal{X} \to Z$ is a family of quasistable curves of genus g, $\mathcal{L} \in \operatorname{Pic} \mathcal{X}$ is a line bundle and $\mathcal{B} : \mathcal{L}^{\otimes 2} \to \beta^* \omega_{\mathcal{C}/Z}$ is a homomorphism, where $\beta : \mathcal{X} \to \mathcal{C}$ is the stable model and $\omega_{\mathcal{C}/Z}$ is the relative dualizing sheaf of $f_{\mathcal{C}} : \mathcal{C} \to Z$, such that for every closed $z \in Z$ the restriction $\left(\mathcal{X}_z, \mathcal{L}_{|\mathcal{X}_z}, \mathcal{B}_{|\mathcal{X}_z}\right)$ to the fibre $\mathcal{X}_z$ is a spin curve.

Definition 2.2.6. (i) Let $(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B})$ and $(f' : \mathcal{X}' \to Z, \mathcal{L}', \mathcal{B}')$ be families of spin curves over Z. An *isomorphism* between the two is a pair (σ, γ) where $\sigma : \mathcal{X} \to \mathcal{X}'$ and $\gamma : \sigma^* \mathcal{L}' \to \mathcal{L}$ are isomorphisms over Z such that the diagram

$$\begin{array}{ccc} (\sigma^* \mathcal{L}')^{\otimes 2} & \xrightarrow{\gamma^{\otimes 2}} & \mathcal{L}^{\otimes 2} \\ \downarrow{\scriptstyle \sigma^* \mathcal{B}'} & & \downarrow{\scriptstyle \mathcal{B}} \\ \sigma^* \beta'^* \omega_{\mathcal{C}'/Z} & \xrightarrow{\delta} & \beta^* \omega_{\mathcal{C}/Z} \end{array}$$

commutes, where $\beta : \mathcal{X} \to \mathcal{C}$ and $\beta' : \mathcal{X}' \to \mathcal{C}'$ are the stable models and δ is the canonical isomorphism, i.e. δ^{-1} is induced by the canonical isomorphism $\omega_{\mathcal{C}/Z} \to \sigma_{\mathcal{C}}^* \omega_{\mathcal{C}'/Z}$ obtained by pulling back forms via $\sigma_{\mathcal{C}} : \mathcal{C} \xrightarrow{\cong} \mathcal{C}'$ induced by σ. Observe that because of the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\sigma} & \mathcal{X} \\ \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} \\ \mathcal{C}' & \xrightarrow{\sigma_{\mathcal{C}}} & \mathcal{C} \end{array}$$

the two pull backs $\sigma^* \beta'^* \omega_{\mathcal{C}'/Z}$ and $\beta^* \sigma_{\mathcal{C}}^* \omega_{\mathcal{C}'/Z}$ coincide. The set of all isomorphisms between the two spin curves is denoted

$$\operatorname{Isom}\left((f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B}), (f' : \mathcal{X}' \to Z, \mathcal{L}', \mathcal{B}')\right).$$

(ii) Let $(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B})$ and $(f' : \mathcal{X}' \to Z, \mathcal{L}', \mathcal{B}')$ be two families of spin curves with the same stable model $\mathcal{C} \to Z$. An isomorphism $(\sigma, \gamma) : (f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B}) \to (f' : \mathcal{X}' \to Z, \mathcal{L}', \mathcal{B}')$ is an *inessential isomorphism* if σ is an isomorphism over $\mathcal{C}$, i.e. there is a commutative diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\sigma} & \mathcal{X} \\ & {\scriptstyle \beta'} \searrow \quad \swarrow {\scriptstyle \beta} & \\ & \mathcal{C} & \end{array}$$

over Z. The set of all inessential isomorphisms will be denoted by

$$\operatorname{Isom}_0\left((f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B}), (f' : \mathcal{X}' \to Z, \mathcal{L}', \mathcal{B}')\right).$$

For a family of spin curves $(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B})$ the group of automorphisms resp. inessential automorphisms of the family is denoted by $\operatorname{Aut}(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B})$ resp. $\operatorname{Aut}_0(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B})$.

Remark 2.2.7. Let (X, L, b) be any spin curve of genus $g \geq 2$, then its automorphism group $\operatorname{Aut}(X, L, b)$ contains the two automorphisms $(\mathrm{id}_X, \pm \mathrm{id}_L)$.

Definition 2.2.8. Denote by $\overline{S}_C$ the set of equivalence classes of spin curves (X, L, b) with stable model C, i.e. C is the stable model of the support X, with respect to isomorphisms of spin curves. $\overline{S}^0_C$ denotes the set of equivalence classes of spin curves (X, L, b) with stable model C with respect to inessential isomorphisms of spin curves.

Example 2.2.9. Let C be a smooth curve of genus $g \geq 2$. Then $\overline{S}_C$ is the set of isomorphism classes of spin structures on C. It is well known (see e.g. [ACGH85]) that on a smooth curve there exist exactly 2^{2g} non-isomorphic theta characteristics.

Claim: Let L and L' be theta characteristics on C and $b : L^{\otimes 2} \to \omega_C$ and $b' : L'^{\otimes 2} \to \omega_C$ isomorphisms. Then (C, L, b) is isomorphic to (C, L', b') iff there exists an automorphism $\sigma \in \operatorname{Aut} C$ such that $\sigma^* L' \cong L$.

"$\Rightarrow$" is obvious from the definition. For "$\Leftarrow$" let σ be such an automorphism and $\varphi : \sigma^* L' \xrightarrow{\cong} L$, then there exists a unique $\eta \in \mathbb{C}^*$ such that

$$\begin{array}{ccc} \sigma^* L'^{\otimes 2} & \xrightarrow{\eta\varphi^{\otimes 2}} & L^{\otimes 2} \\ \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle b} \\ \sigma^* \omega_C & \xrightarrow{\delta} & \omega_C \end{array}$$

commutes. Let λ be one of the two square roots of η, then $\gamma = \lambda\varphi : \sigma^* L' \xrightarrow{\cong} L$ gives an isomorphism (σ, γ) of the two spin curves. These two choices are the only lifts of σ to the spin curves. In particular this shows that the two spin curves (X, L, b) and (X, L, b') are isomorphic, i.e. choosing different isomorphisms between $L^{\otimes 2}$ and ω_C gives isomorphic spin curves. This also shows that for a smooth spin curve (C, L, b) the group of inessential automorphisms is $\operatorname{Aut}_0(C, L, b) = \{(\mathrm{id}_C, \mathrm{id}_L), (\mathrm{id}_C, -\mathrm{id}_L)\}$. Notice that though different b give isomorphic spin curves it is important to include b into the data, since if one would not fix the isomorphism b, i.e. consider only pairs (C, L) of a curve and a theta characteristic, the automorphism group would be infinite.

Another consequence is that $\overline{S}^0_C$ contains exactly 2^{2g} points and every point is determined by the choice of a theta characteristic. Here only isomorphisms with $\sigma = \mathrm{id}_C$ are considered and therefore there exists an inessential isomorphism between (C, L, b) and (C, L', b') iff $L' \cong L$.

It is obvious that for a general smooth curve C, meaning that $\operatorname{Aut} C = \{\mathrm{id}_C\}$, all automorphisms of a spin curve with support C are inessential. Hence in this case $\overline{S}_C = \overline{S}^0_C$. But if the curve C does have (non-trivial) automorphisms it is possible that $\overline{S}_C \neq \overline{S}^0_C$, i.e. the notion of isomorphism of spin curves is coarser than that of isomorphism of line bundles.

Example 2.2.10. Examples for a smooth curve C with $\overline{S}_C \neq \overline{S}^0_C$ are special hyperelliptic curves. For hyperelliptic curves the theta characteristics may be easily described in terms of Weierstraß points (see e.g. [ACGH85]). Let C be a hyperelliptic curve of genus g with Weierstraß points $P_1, \ldots, P_{2g+2}$, $|D|$ the hyperelliptic linear series and $f : C \to \mathbb{P}^1$ the corresponding $2:1$-morphism. Then every theta characteristic is isomorphic to $\mathcal{O}(E)$ for some

$$E = mD + P_{i_1} + \cdots + P_{i_{g-1-2m}}$$

with $-1 \leq m \leq \frac{g-1}{2}$ and the P_{i_j} are distinct. This representation is unique if $m \geq 0$, while if $m = -1$ there is a single relation

$$-D + P_{i_1} + \cdots + P_{i_{g+1}} \sim -D + P_{j_1} + \cdots + P_{j_{g+1}} \qquad (*)$$

if $\{i_1 \ldots, i_{g+1}, j_1, \ldots, j_{g+1}\} = \{1, \ldots, 2g+2\}$. Furthermore $h^0(C, \mathcal{O}(E)) = m+1$.

The general hyperelliptic curve has exactly two automorphisms, the identity and the hyperelliptic involution ι which fixes the Weierstraß points and the divisor D. Therefore for every theta characteristic $L = \mathcal{O}(E)$ one has $\iota^* L \cong L$ and the spin curve (C, L, b) has exactly four automorphisms, the two inessential ones $(\mathrm{id}_C, \pm\,\mathrm{id}_L)$ and two that lift ι, i.e. there are two isomorphisms $\gamma : \iota^* L \to L$ which are compatible with b, i.e. such that

$$\begin{array}{ccc} \iota^* L^{\otimes 2} & \xrightarrow{\gamma^{\otimes 2}} & L^{\otimes 2} \\ \downarrow{\scriptstyle \iota^* b} & & \downarrow{\scriptstyle b} \\ \iota^* \omega_C & \xrightarrow{\delta} & \omega_C \end{array}$$

commutes. Hence for the general hyperelliptic curve there are still 2^{2g} non-isomorphic spin curves with C as support and $\overline{S}_C = \overline{S}^0_C$.

Now let C be a hyperelliptic curve of genus 2, such that the branch points $x_k = f(P_k)$ are in special position, i.e. $\mathrm{Aut}(\mathbb{P}^1; \{x_1, \ldots, x_6\}) \neq \{\mathrm{id}\}$. For a concrete example let the x_k be the 6th roots of unity, $x_k = e^{2\pi\sqrt{-1}k/6}$, $k = 1, \ldots, 6$. A straightforward calculation shows that $\mathrm{Aut}(\mathbb{P}^1; \{x_1, \ldots, x_6\})$ is generated by the rotation $x \mapsto e^{2\pi\sqrt{-1}/6} x$ and the inversion $x \mapsto \frac{1}{x}$. The induced permutations of the x_k are (in cycle notation)

$$(x_1, x_2, x_3, x_4, x_5, x_6)$$

for the rotation and

$$(x_1, x_5)(x_2, x_4)(x_3)(x_6)$$

for the inversion, i.e. the six points are reflected with respect to the diameter of the unit disc through $x_6 = 1$ and $x_3 = -1$. The automorphism group of the hyperelliptic curve C is then a $\mathbb{Z}_2$-extension of $\mathrm{Aut}(\mathbb{P}^1; \{x_1, \ldots, x_6\})$, i.e.

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Aut}\, C \longrightarrow \mathrm{Aut}(\mathbb{P}^1; \{x_1, \ldots, x_6\}) \longrightarrow 0$$
$$\sigma \longmapsto \sigma_{\mathbb{P}^1}$$

is exact (see [ACGH85]), here $\mathbb{Z}_2 = \langle \iota \rangle$ and $\sigma_{\mathbb{P}^1}$ is the unique automorphism given by the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\sigma} & C \\ \downarrow f & & \downarrow f \\ \mathbb{P}^1 & \xrightarrow{\sigma_{\mathbb{P}^1}} & \mathbb{P}^1 \end{array}$$

Since two smooth spin curves (C, L, b) and (C, L', b') are isomorphic iff there exists a $\sigma \in \mathrm{Aut}\, C$ such that $\sigma^* L' \cong L$, it is enough to understand the action of $\mathrm{Aut}\, C$ on the set of theta characteristics, i.e. $\sigma^* L = \mathcal{O}(\sigma^* E)$ for a theta characteristic $L = \mathcal{O}(E)$ and $\sigma \in \mathrm{Aut}\, C$. The class of the divisor D is invariant under all automorphisms, therefore it is enough to understand which permutations of the Weierstraß points are induced by automorphisms. The group of all possible permutations is generated by

$$(P_1, P_2, P_3, P_4, P_5, P_6)$$

and

$$(P_1, P_5)(P_2, P_4)(P_3)(P_6).$$

$\overline{S}^0_C$ consists of the $16 = 2^{2 \cdot 2}$ points $(C, \mathcal{O}(E), b_E)$ where E is a divisor in the following list.

$m = -1$	$m = 0$
$-D + P_1 + P_2 + P_4$	P_1
$-D + P_1 + P_2 + P_5$	P_2
$-D + P_1 + P_3 + P_4$	P_3
$-D + P_1 + P_3 + P_6$	P_4
$-D + P_1 + P_4 + P_5$	P_5
$-D + P_1 + P_4 + P_6$	P_6
$-D + P_1 + P_2 + P_3$	
$-D + P_1 + P_2 + P_6$	
$-D + P_1 + P_5 + P_6$	
$-D + P_1 + P_3 + P_5$	

Claim: There are four isomorphism classes of spin curves with support C, every class corresponding to a box in the above list.

If σ is any automorphism and E a divisor of the form $-D + P_{i_1} + P_{i_2} + P_{i_3}$ then $\sigma^* E$ is also of this form, i.e. $\sigma^* E = -D + P_{j_1} + P_{j_2} + P_{j_3}$ for the appropriate j_k's. The same is true for divisors of the form P_i. Since $\mathcal{O}(-D + P_{i_1} + P_{i_2} + P_{i_3}) \not\cong$

$\mathcal{O}(P_i)$ (mind that the P_{i_j} are distinct), divisors in different columns cannot give isomorphic spin curves. In the right hand column all divisors give isomorphic spin curves, since if σ is an automorphism realising the cyclic permutation of the P_i then $\sigma^* P_i = P_{i-1}$, where the index is considered modulo 6. In the left hand column a careful analysis of the action of the group of all possible permutations of the Weierstraß points on the divisors taking into account the relation $(*)$ gives the three isomorphism classes in the above list. For example iterating the cyclic permutation gives

$$\begin{aligned} -D + P_1 + P_2 + P_3 &\xmapsto{\sigma^*} -D + P_6 + P_1 + P_2 \\ &\xmapsto{\sigma^*} -D + P_5 + P_6 + P_1 \\ &\xmapsto{\sigma^*} -D + P_4 + P_5 + P_6 \sim -D + P_1 + P_2 + P_3 \end{aligned}$$

while the permutation $(P_1, P_5)(P_2, P_4)(P_3)(P_6)$ acts as

$$\begin{aligned} -D + P_1 + P_2 + P_3 &\longleftrightarrow -D + P_5 + P_4 + P_3 \sim -D + P_1 + P_2 + P_6 \\ -D + P_5 + P_6 + P_1 &\longleftrightarrow -D + P_1 + P_6 + P_5 \end{aligned}$$

Therefore the three spin curves

$$\begin{aligned} &(C, \mathcal{O}(-D + P_1 + P_2 + P_3), b_{-D+P_1+P_2+P_3}) \\ &(C, \mathcal{O}(-D + P_1 + P_2 + P_6), b_{-D+P_1+P_2+P_6}) \\ &(C, \mathcal{O}(-D + P_1 + P_5 + P_6), b_{-D+P_1+P_5+P_6}) \end{aligned}$$

are isomorphic to each other but not isomorphic to any other spin curve in the list. In their article [KS06] Kallel and Sjerve analyse the action of the automorphism group of a smooth curve on the set of its theta characteristics.

With this example in mind the next goal is to describe $\overline{S}_C^0$ and $\overline{S}_C$ for any stable curve C of genus at least 2. In order to do this the statements of Examples 2.2.9 and 2.2.10 have to be generalised to stable curves, in particular a criterion to determine which spin curves are isomorphic is needed and the group of inessential automorphisms of a spin curve as well as the action of $\operatorname{Aut} C$ on $\overline{S}_C^0$ must be studied.

Let (X, L, b) and (X', L', b') be two spin curves of genus $g \geq 2$ with stable models $\beta : X \to C$ and $\beta' : X' \to C'$. Denote by $N \subset \operatorname{sing} C$ and $N' \subset \operatorname{sing} C'$ the sets of exceptional nodes. Remember the map $\varphi : \operatorname{Isom}(X, X') \to \operatorname{Isom}((C; N), (C'; N'))$, $\sigma \mapsto \sigma_C$ from Proposition 2.1.18, where σ_C is the induced isomorphism of the stable models. Consider the concatenation

$$\begin{array}{rcccl} \psi : \operatorname{Isom}((X, L, b), (X', L', b')) & \longrightarrow & \operatorname{Isom}(X, X') & \xrightarrow{\varphi} & \operatorname{Isom}((C; N), (C'; N')) \\ (\sigma, \gamma) & \longmapsto & \sigma & \longmapsto & \sigma_C \end{array}$$

Remember also that for every $\sigma_C \in \mathrm{Isom}((C;N),(C';N'))$ there exists a unique $\widetilde{\sigma} \in \mathrm{Isom}(\widetilde{X},\widetilde{X}')$ such that $\widetilde{\beta}' \circ \widetilde{\sigma} = \sigma_C \circ \widetilde{\beta}$ where $\widetilde{\beta} : \widetilde{X} \to C$ (resp. $\widetilde{\beta}' : \widetilde{X}' \to C'$) is the partial normalisation of C at N (resp. of C' at N').

Proposition 2.2.11. *With this notation*

$$\mathrm{im}\,\psi = \left\{ \sigma_C \in \mathrm{Isom}\left((C;N),(C';N')\right) \middle| \widetilde{\sigma}^*\widetilde{L}' \cong \widetilde{L} \right\},$$

where $\widetilde{L} = L_{|\widetilde{X}}$ *resp.* $\widetilde{L}' = L'_{|\widetilde{X}'}$*, and* $\psi : \mathrm{Isom}((X,L,b),(X',L',b')) \to \mathrm{im}\,\psi$ *is a* $2^c : 1$*-map where* $c = \#CC(\Gamma(\widetilde{X}))$ *is the number of connected components of* $\widetilde{X}$*.*

Proof. "$\subset$" If (σ,γ) is an isomorphism of the given spin curves $\gamma : \sigma^* L' \to L$ is an isomorphism and its restriction to the non-exceptional subcurve $\widetilde{X}$ gives an isomorphism $\widetilde{\gamma} : \widetilde{\sigma}^*\widetilde{L} = \sigma^*_{|\widetilde{X}} L'_{|\widetilde{X}'} \to L_{|\widetilde{X}} = \widetilde{L}$.

Proof of "$\supset$" Let $\sigma_C \in \mathrm{Isom}\left((C;N),(C';N')\right)$ and $\widetilde{\varphi} : \widetilde{\sigma}^*\widetilde{L}' \xrightarrow{\cong} \widetilde{L}$. It is possible that the isomorphism $\widetilde{\varphi}$ is incompatible with the homomorphisms b and b'. An isomorphism $\widetilde{\gamma} : \widetilde{\sigma}^*\widetilde{L}' \xrightarrow{\cong} \widetilde{L}$, such that the diagram of isomorphisms on $\widetilde{X}$

$$\begin{array}{ccc} \widetilde{\sigma}^*\widetilde{L}'^{\otimes 2} & \xrightarrow{\widetilde{\gamma}^{\otimes 2}} & \widetilde{L}^{\otimes 2} \\ \downarrow{\scriptstyle \widetilde{\sigma}^*\widetilde{b}'} & & \downarrow{\scriptstyle \widetilde{b}} \\ \widetilde{\sigma}^*\omega_{\widetilde{X}'} & \xrightarrow{\delta} & \omega_{\widetilde{X}} \end{array}$$

commutes, has to be defined. The curve $\widetilde{X}$ consists of $c = \#CC(\Gamma(\widetilde{X}))$ connected components and it is enough to construct the isomorphism $\widetilde{\gamma}$ on every connected component $\widetilde{X}_j$. All isomorphisms between $\left(\widetilde{\sigma}^*\widetilde{L}'^{\otimes 2}\right)_{|\widetilde{X}_j}$ and $\widetilde{L}^{\otimes 2}_{|\widetilde{X}_j}$ are scalar multiples of $\widetilde{\varphi}^{\otimes 2}_{|\widetilde{X}_j}$. Therefore there exits a unique scalar $\eta_j \in \mathbb{C}^*$ such that

$$\begin{array}{ccc} \left(\widetilde{\sigma}^*\widetilde{L}'^{\otimes 2}\right)_{|\widetilde{X}_j} & \xrightarrow{\eta_j \widetilde{\varphi}^{\otimes 2}_{|\widetilde{X}_j}} & \left(\widetilde{L}^{\otimes 2}\right)_{|\widetilde{X}_j} \\ \downarrow{\scriptstyle \left(\widetilde{\sigma}^*\widetilde{b}'\right)_{|\widetilde{X}_j}} & & \downarrow{\scriptstyle \widetilde{b}_{|\widetilde{X}_j}} \\ \left(\widetilde{\sigma}^*\omega_{\widetilde{X}'}\right)_{|\widetilde{X}_j} & \xrightarrow{\delta_{|\widetilde{X}_j}} & \left(\omega_{\widetilde{X}}\right)_{|\widetilde{X}_j} \end{array}$$

commutes. Let λ_j be one of the two roots of η_j, then setting $\widetilde{\gamma}_{|\widetilde{X}_j} = \pm\lambda_j \widetilde{\varphi}_{|\widetilde{X}_j}$ gives the desired isomorphism. Therefore $\widetilde{\gamma}$ is determined by the choice of a root for every connected component $\widetilde{X}_j$. There are 2^c different choices.

Claim: The extension of the isomorphisms $\widetilde{\sigma}$ and $\widetilde{\gamma}$ on $\widetilde{X}$ to the whole curve X is unique. As in Proposition 2.1.18 let $E_i = \beta^{-1}(P_i)$, $P_i \in N$, be an exceptional component, $P_i^{\pm} = \widetilde{\beta}^{-1}(P_i)$ the two nodes of X on E_i and $P_i' = \sigma_C(P_i)$, $E_i' =$

$\beta'^{-1}(P_i')$ the corresponding exceptional component of X' with nodes $P_i'^+ = \widetilde{\sigma}(P_i^+)$ and $P_i'^- = \widetilde{\sigma}(P_i^-)$. E_i (resp. E_i') can be identified with $\mathbb{P}^1$ such that $P_i^+ = 0$ and $P_i^- = \infty$ (resp. $P_i'^+ = 0$ and $P_i'^- = \infty$). Since the line bundle L restricted to E_i is $\mathcal{O}_{E_i}(1)$ there is gluing data

$$\widetilde{L}_{P_i^+} \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^1}(1)_0 \quad \text{and} \quad \widetilde{L}_{P_i^-} \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^1}(1)_\infty$$

as well as

$$\widetilde{L}'_{P_i'^+} \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^1}(1)_0 \quad \text{and} \quad \widetilde{L}'_{P_i'^-} \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^1}(1)_\infty.$$

With these identifications and gluing data fixed the extension $\sigma_{|E_i}$ of $\widetilde{\sigma}$ to E_i is an automorphism of $\mathbb{P}^1$ fixing 0 and ∞. Moreover, the extension $\gamma_{|E_i}$ of $\widetilde{\gamma}$ to E_i is an isomorphism $\sigma^*_{|E_i}\mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(1)$ such that the following two diagrams

$$\begin{array}{ccc} \left(\widetilde{\sigma}^*\widetilde{L}'\right)_{P_i^+} & \longrightarrow & \left(\sigma^*_{|E_i}\mathcal{O}_{\mathbb{P}^1}(1)\right)_0 \\ \downarrow \widetilde{\gamma}_{P_i^+} & & \downarrow (\gamma_{|E_i})_0 \\ \widetilde{L}_{P_i^+} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(1)_0 \end{array} \qquad \begin{array}{ccc} \left(\widetilde{\sigma}^*\widetilde{L}'\right)_{P_i^-} & \longrightarrow & \left(\sigma^*_{|E_i}\mathcal{O}_{\mathbb{P}^1}(1)\right)_\infty \\ \downarrow \widetilde{\gamma}_{P_i^-} & & \downarrow (\gamma_{|E_i})_\infty \\ \widetilde{L}_{P_i^-} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(1)_\infty \end{array}$$

commute, where the left hand vertical map in either diagram is induced by $\widetilde{\gamma}$. The claim is now, that there exist a unique automorphism of $\mathbb{P}^1$ fixing 0 and ∞ and a unique isomorphism $\sigma^*_{|E_i}\mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(1)$ which coincides with the right hand vertical maps in the two diagrams.

For this problem it is convenient to work with the geometric bundles $\mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(1))$ and $\mathbb{V}(\sigma^*_{|E_i}\mathcal{O}_{\mathbb{P}^1}(1))$ (see [CCC04, Lemma 2.3.2]). Let V be a complex vector space of dimension two with basis v_0, v_∞, set $\mathbb{P}^1 = \mathbb{P}(V)$, $0 = [v_0]$ and $\infty = [v_\infty]$. Since $\sigma_{|E_i}$ fixes 0 and ∞ it has to be induced by an automorphism σ_V of V given by $v_0 \mapsto a_0 v_0$ and $v_\infty \mapsto a_\infty v_\infty$ with appropriate non-zero scalars a_0 and a_∞, then $\sigma_{|E_i}$ is multiplication by $\frac{a_\infty}{a_0}$. The points of the geometric line bundle $\mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(1))$ can be discribed as pairs (v, φ) where $\varphi : \mathbb{C}v \longrightarrow \mathbb{C}$ is an element in the dual of the line in V spanned by $v \in V$. A point in the fibre $\mathbb{V}\left(\sigma^*_{|E_i}\mathcal{O}_{\mathbb{P}^1}(1)\right)_{[v]} \cong \mathbb{V}\left(\mathcal{O}_{\mathbb{P}^1}(1)\right)_{[\sigma_V(v)]}$ is of the form $(v, \widetilde{\varphi})$ where $\widetilde{\varphi} : \mathbb{C}\cdot\sigma_V(v) \to \mathbb{C}$. σ_V induces an isomorphism $\mathbb{V}(\sigma^*_{|E_i}\mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(1))$ mapping $(v, \widetilde{\varphi})$ to $(v, \widetilde{\varphi}\circ\sigma_V)$ where $\widetilde{\varphi}\circ\sigma_V : \mathbb{C}v \underset{\cong}{\xrightarrow{\sigma_V}} \mathbb{C}\cdot\sigma_V(v) \xrightarrow{\widetilde{\varphi}} \mathbb{C}$. A short calculation shows that the action on the fibre over 0 (resp. over ∞) is multiplication by a_0 (resp. a_∞). Since every isomorphism pair $(\sigma_{|E_i}, \gamma_{|E_i})$ with $\sigma_{|E_i} \in \mathrm{Aut}(\mathbb{P}^1; 0, \infty)$ and $\gamma_{|E_i} : \sigma^*_{|E_i}\mathcal{O}_{\mathbb{P}^1} \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^1}$ arises in this way and the action on the fibre over 0 and ∞ is given by the above two diagrams there is a unique choice of a_0 and a_∞ giving unique extensions $\sigma_{|E_i}$ and $\gamma_{|E_i}$.

Therefore if σ_C has the property $\widetilde{\sigma}^*\widetilde{L}' \cong \widetilde{L}$ there are exactly $2^{\#CC(\Gamma(\widetilde{X}))}$ isomorphisms (σ, γ) mapped to σ_C under ψ. □

For the automorphism group of a spin curve this means the following.

Corollary 2.2.12. *Let (X, L, b) be a spin curve of genus $g \geq 2$ with stable model $\beta : X \to C$ and $N \subset \operatorname{sing} C$ the set of exceptional nodes. Then the following sequence of groups is exact.*

$$0 \longrightarrow \mathbb{Z}_2^{CC(\Gamma(\widetilde{X}))} \longrightarrow \operatorname{Aut}(X, L, b) \xrightarrow{\psi} \left\{ \sigma_C \in \operatorname{Aut}(C; N) \middle| \widetilde{\sigma}^* \widetilde{L} \cong \widetilde{L} \right\} \longrightarrow 0$$

where the image of $(\gamma_1, \ldots, \gamma_c) \in \mathbb{Z}_2^{CC(\Gamma(\widetilde{X}))}$ is the unique automorphism $(\sigma, \gamma) \in \operatorname{Aut}(X, L, b)$ such that $\widetilde{\sigma} = \operatorname{id}_{\widetilde{X}}$ and $\widetilde{\gamma} : \widetilde{\sigma}^* \widetilde{L} = \widetilde{L} \to \widetilde{L}$ is $(-1)^{\gamma_j} \operatorname{id} : \widetilde{L}_{|\widetilde{X}_j} \to \widetilde{L}_{|\widetilde{X}_j}$ on the jth connected component $\widetilde{X}_j$ of $\widetilde{X}$. The image of $\mathbb{Z}_2^{CC(\Gamma(\widetilde{X}))}$ in $\operatorname{Aut}(X, L, b)$ is exactly $\operatorname{Aut}_0(X, L, b)$.

Notation 2.2.13. The image in $\operatorname{Aut}_0(X, L, b)$ of $(\gamma_1, \ldots, \gamma_c) \in \mathbb{Z}_2^{CC(\Gamma(\widetilde{X}))}$ is denoted by $(\sigma, (\gamma_1, \ldots, \gamma_c)) = (\sigma, (\gamma_j)_j)$.

The group of inessential automorphisms depends only on the combinatorial data of the support X captured in the following graph.

Definition 2.2.14. For a quasistable curve X of genus $g \geq 2$ the graph $\Sigma(X)$ has a vertex for every connected component of the non-exceptional subcurve $\widetilde{X}$, i.e. $V(\Sigma(X)) = CC(\Gamma(\widetilde{X}))$, and an edge for every exceptional component. Such an edge connects the vertices corresponding to the connected components that the exceptional component meets.

Definition 2.2.15. Let (X, L, b) be a spin curve with stable model C and exceptional nodes $N \subset \operatorname{sing} C$. *The automorphism $\sigma_C \in \operatorname{Aut} C$ lifts to* (X, L, b) if $\sigma_C(N) = N$ and $\widetilde{\sigma}^* \widetilde{L} \cong \widetilde{L}$. Any automorphism $(\sigma, \gamma) \in \operatorname{Aut}(X, L, b)$ which is mapped to σ_C by ψ is a *lift* of σ_C.

Remark 2.2.16. Note that the above sequence does not split in general, i.e. $\operatorname{Aut}(X, L, b)$ is not the direct product of the groups on the left and the right hand side. The group on the right hand side acts on $\mathbb{Z}_2^{V(\Sigma(X))}$ by conjugation of any lift $(\sigma, \gamma) \in \operatorname{Aut}(X, L, b)$ of σ_C. To be precise consider $(\sigma', (\gamma_j)_j)$ given by $(\gamma_j)_j \in \mathbb{Z}_2^{V(\Sigma(X))}$, then $(\sigma, \gamma) \circ (\sigma', (\gamma_j)_j) \circ (\sigma, \gamma)^{-1}$ is again an inessential automorphism, since the automorphism of the support is $\sigma \circ \sigma' \circ \sigma^{-1}$ which is the identity on the non-exceptional subcurve $\widetilde{X}$ since σ' is. Determining the isomorphism on the spin structure of the concatenation is a little bit more complicated. The inverse of $\gamma : \sigma^* L \to L$ is an isomorphism between L and $\sigma^* L$ therefore $(\sigma^{-1})^* \gamma^{-1}$ is an isomorphism $(\sigma^{-1})^* L \to L$, this gives $(\sigma, \gamma)^{-1} = (\sigma^{-1}, (\sigma^{-1})^* \gamma^{-1})$. Hence

$$\begin{aligned} \left(\sigma \circ \sigma' \circ \sigma^{-1}\right)^* L = \left(\sigma^{-1}\right)^* \left((\sigma')^* (\sigma^* L)\right) &\xrightarrow{\left(\sigma^{-1}\right)^* \left((\sigma')^* \gamma\right)} \left(\sigma^{-1}\right)^* \left((\sigma')^* L\right) \\ &\xrightarrow{\left(\sigma^{-1}\right)^* (\gamma_j)_j} \left(\sigma^{-1}\right)^* L \xrightarrow{\left(\sigma^{-1}\right)^* \gamma^{-1}} L \end{aligned}$$

and

$$(\sigma,\gamma)\circ\left(\sigma',(\gamma_j)_j\right)\circ(\sigma,\gamma)^{-1}$$
$$=\left(\sigma\circ\sigma'\circ\sigma^{-1},\left(\sigma^{-1}\right)^*\gamma^{-1}\circ\left(\sigma^{-1}\right)^*(\gamma_j)_j\circ\left(\sigma^{-1}\right)^*\left((\sigma')^*\gamma\right)\right)$$

Denote by $\sigma^{-1}_{V(\Sigma(X))}$ the automorphism of the vertices $V(\Sigma(X))$ of $\Sigma(X)$ induced by σ^{-1}. Now let P be any point in the connected component $\widetilde{X}_j$ of $\widetilde{X}$ and $\widetilde{X}_{j'}$ the connected component of $\widetilde{X}$ in which $\sigma^{-1}(P)$ lies, i.e. $v(\widetilde{X}_j)\mapsto v(\widetilde{X}_{j'})$ under $\sigma^{-1}_{V(\Sigma(X))}$. The induced map on the stalks over P is

$$L_P=\left(\left(\sigma\circ\sigma'\circ\sigma^{-1}\right)^*L\right)_P\longrightarrow L_{(\sigma'\circ\sigma^{-1})(P)}=L_{\sigma^{-1}(P)}\xrightarrow{(-1)^{\gamma_{j'}}\,\mathrm{id}}L_{\sigma^{-1}(P)}\longrightarrow L_P$$

since $\sigma'_{|\widetilde{X}}=\mathrm{id}_{\widetilde{X}}$. Since the first and the third map are inverse to each other, this means that the concatenation is multiplication with $(-1)^{\gamma_{j'}}$ in the fibres over the connected component $\widetilde{X}_j$, i.e. it is the inessential automorphism given by $\sigma^{-1}_{V(\Sigma(X))}(\gamma_j)_j\in\mathbb{Z}_2^{V(\Sigma(X))}$, the permutation of $(\gamma_j)_j$ given by $\sigma^{-1}_{V(\Sigma(X))}$. This action is independent of the specific lift (σ,γ) chosen and in general non-trivial.

The description of the group of inessential automorphisms in the corollary is essential for the following description of $\overline{S}^0_C$ for any stable curve C of genus at least 2 (see [CC03]).

Definition 2.2.17. Let C be a stable curve of genus at least 2 and $\Delta\subset\operatorname{sing}C$. Consider the preimage of Δ under the normalisation $\nu_C:C^\nu\to C$ as a divisor and denote for every irreducible component C_j of C by Δ_j the divisor $\nu_C^{-1}(\Delta)\cap C_j$. Then the set Δ is *even* if for every C_j the divisor Δ_j has even degree.

Remark 2.2.18. Note that if $\widetilde{\beta}:\widetilde{X}_\Delta\to C$ is the partial normalisation of C at $N=\operatorname{sing}C\setminus\Delta$ then Δ is the set of nodes of $\widetilde{X}_\Delta$. Δ is even iff $\omega_{\widetilde{X}_\Delta}$ has even degree on every component C_j. Since if $\nu_{\widetilde{X}_\Delta}:C^\nu\to\widetilde{X}_\Delta$ is the normalisation of $\widetilde{X}_\Delta$, then $(\nu^*_{\widetilde{X}_\Delta}\omega_{\widetilde{X}_\Delta})_{|C^\nu_j}=(\omega_{C^\nu}(\nu^{-1}_{\widetilde{X}_\Delta}(\Delta))_{|C^\nu_j}=\omega_{C^\nu_j}(\Delta_j)$ and this line bundle has even degree iff Δ_j has even degree.

Proposition 2.2.19. [CC03, Proposition 5] *Let C be a stable curve of genus $g\geq 2$. Then $\overline{S}^0_C$ consists of*

$$2^{2p}\cdot\left(\sum_{\substack{\Delta\subset\operatorname{sing}C\\ \text{even}}}2^{b_1\left(\Gamma\left(\widetilde{X}_\Delta\right)\right)}\right)$$

points, where $p=\sum_j g(C^\nu_j)$. In particular the blow up $\beta:X_\Delta\to C$ at $\operatorname{sing}C\setminus\Delta$ is the support of a spin curve iff Δ is even.

Proof. Suppose X is the support of a spin curve, say (X, L, b), with stable model C. Consider the normalisation $\nu_X : X^\nu \to X$ and the pull back $L^\nu = \nu_X^* L$. The homomorphism b induces an isomorphism $\widetilde{b} : L^{\otimes 2}_{|\widetilde{X}} \xrightarrow{\cong} \omega_{\widetilde{X}}$. Let C_j be an irreducible component of C, i.e. a non-exceptional component of X and C_j^ν its normalisation. The pull back of $\widetilde{b}$ via the normalisation $\nu_{\widetilde{X}} : C^\nu \to \widetilde{X}$ restricted to C_j^ν gives an isomorphism

$$b_j : (L^\nu_{|C_j^\nu})^{\otimes 2} \xrightarrow{\cong} \left(\nu_{\widetilde{X}}^* \omega_{\widetilde{X}}\right)_{|C_j^\nu} = \omega_{C_j^\nu}(\Delta_j) \, .$$

Since the line bundle on the left hand side is a square, its degree is even, therefore the degree of $\omega_{C_j^\nu}(\Delta_j)$ is even. The degree of $\omega_{C_j^\nu}$ is $2g(C_j^\nu) - 2$, hence the degree of Δ_j must be even. This shows that the support of a spin curve must be the blow up $\beta : X = X_\Delta \to C$ at the complement $N = \operatorname{sing} C \setminus \Delta$ of an even set $\Delta \subset \operatorname{sing} C$.

Now let $\Delta \subset \operatorname{sing} C$ be even, $\beta : X \to C$ the blow up at N and ν_X the normalisation of X. First of all a line bundle L^ν on X^ν has to be defined. On every exceptional component E of X set $L^\nu_{|E} = \mathcal{O}_E(1)$. Let C_j be an irreducible non-exceptional component of X and C_j^ν its normalisation. Since Δ is even the degree of Δ_j is even. Therefore the line bundle $\omega_{C_j^\nu}(\Delta_j)$ has even degree and it is a classical result, that it has $2^{2g(C_j^\nu)}$ non-isomorphic square roots, i.e. line bundles $L^\nu_{|C_j^\nu}$ such that

$$(L^\nu_{|C_j^\nu})^{\otimes 2} \cong \omega_{C_j^\nu}(\Delta_j).$$

For every C_j let $L^\nu_{|C_j^\nu}$ be such a root and $b_j : (L^\nu_{|C_j^\nu})^{\otimes 2} \xrightarrow{\cong} \omega_{C_j^\nu}(\Delta_j)$ a fixed isomorphism.

Secondly gluing data for L^ν over the nodes of X are needed, such that the resulting line bundle L on X admits a homomorphism b, such that (L, b) is a spin structure. This gluing data will be given in two steps, considering in the first step only the non-exceptional nodes and in the second the nodes lying on an exceptional component. For the normalisation $\nu_{\widetilde{X}} : \widetilde{X}^\nu = C^\nu \to \widetilde{X}$ there is the following exact sequence

$$1 \longrightarrow (\mathbb{C}^*)^{b_1\left(\Gamma(\widetilde{X})\right)} \longrightarrow \operatorname{Pic}\left(\widetilde{X}\right) \xrightarrow{\nu_{\widetilde{X}}^*} \operatorname{Pic}\left(\widetilde{X}^\nu\right) \longrightarrow 0,$$

where $\nu_{\widetilde{X}}^*$ is the pull back of line bundles and $b_1\big(\Gamma(\widetilde{X})\big)$ is the first Betti number of the dual graph $\Gamma(\widetilde{X})$ of the curve $\widetilde{X}$. The sequence says that there are $(\mathbb{C}^*)^{b_1(\Gamma(\widetilde{X}))}$ different gluings of $L^\nu_{|\widetilde{X}^\nu}$ over the nodes of $\widetilde{X}$, i.e. non-isomorphic line bundles $\widetilde{L}$ on $\widetilde{X}$ pulling back to $L^\nu_{|\widetilde{X}^\nu}$. But not all of these gluings give line bundles satisfying

$\widetilde{L}^{\otimes 2} \cong \omega_{\widetilde{X}}$. The canonical gluing of $\omega_{\widetilde{X}^\nu}\big(\sum_j \Delta_j\big)$ such that the resulting line bundle on $\widetilde{X}$ is $\omega_{\widetilde{X}}$, is given by the requirement

$$\operatorname{res}_{P+} \eta + \operatorname{res}_{P-} \eta = 0$$

for every node P of $\widetilde{X}$ (see (res) on page 9). With this gluing fixed there are $2^{b_1(\Gamma(\widetilde{X}))}$ different gluings of $L^\nu_{|\widetilde{X}}$ such that the isomorphisms $b_j : (L^\nu_{|C_j^\nu})^{\otimes 2} \to \omega_{C_j^\nu}(\Delta_j)$ glue to an isomorphism $\widetilde{b} : \widetilde{L}^{\otimes 2} \to \omega_{\widetilde{X}}$, where $\widetilde{L}$ is the resulting line bundle on $\widetilde{X}$. This is the case because the following diagram has to commute for every node P of $\widetilde{X}$

$$\begin{array}{ccc}
\Big(\omega_{C^\nu_{j+}}(\Delta_{j+})\Big)_{P+} & \xrightarrow{\text{gluing of } \omega_{\widetilde{X}} \text{ at } P} & \Big(\omega_{C^\nu_{j-}}(\Delta_{j-})\Big)_{P-} \\
\Big\uparrow \scriptstyle (b_{j+})_{P+} & & \scriptstyle (b_{j-})_{P-} \Big\uparrow \\
\Big(L^\nu_{|C^\nu_{j+}}\Big)^{\otimes 2}_{P+} & \xrightarrow[\text{gluing of } \widetilde{L}^{\otimes 2} \text{ at } P]{} & \Big(L^\nu_{|C^\nu_{j-}}\Big)^{\otimes 2}_{P-}
\end{array}$$

where C_{j+} and C_{j-} are the irreducible components of $\widetilde{X}$ such that $P^+ \in C^\nu_{j+}$ and $P^- \in C^\nu_{j-}$. Therefore the square of the gluing is fixed for every node.

Since the homomorphism b of a spin curve vanishes over the exceptional components there are no restrictions for the gluings over the nodes on exceptional components. This means that fixing any gluing of the line bundle $\widetilde{L}$ and the $\mathcal{O}_E(1)$, $E \subset X$ exceptional, over the nodes on the exceptional components gives a line bundle L on X which is the line bundle of a spin curve. The homomorphism b is zero over the exceptional components and it coincides with

$$L^{\otimes 2}_{|\widetilde{X}} = \widetilde{L}^{\otimes 2} \xrightarrow{\widetilde{b}} \omega_{\widetilde{X}} = \widetilde{\beta}^* \omega_C(-D) \hookrightarrow \widetilde{\beta}^* \omega_C = (\beta^* \omega_C)_{|\widetilde{X}}$$

over $\widetilde{X}$, where D is the divisor on $\widetilde{X}$ given by the preimages of the exceptional nodes under the partial normalisation $\widetilde{\beta} : \widetilde{X} \to C$. Now let (X, L, b) and (X, L', b') be two spin curves which differ only in the choice of the gluing over the nodes on exceptional components. Then $\widetilde{L'} = \widetilde{L}$ and Proposition 2.2.11 shows that the two spin curves are isomorphic. Therefore given any even subset $\Delta \subset \operatorname{sing} C$ there are

$$\Big(\prod_j 2^{2g(C_j^\nu)}\Big) \cdot 2^{b_1\left(\Gamma\left(\widetilde{X}_\Delta\right)\right)}$$

non-isomorphic spin curves with non-exceptional subcurve $\widetilde{X}_\Delta$, i.e. the support X_Δ is the blow up of C at N. □

Example 2.2.20. Let C be a stable reducible curve of genus g with one node P and irreducible components C_1 and C_2 of genera i and $g - i$ respectively. The subset $\Delta = \{P\}$ of $\operatorname{sing} C$ is not even, since the two divisors Δ_1 and Δ_2 have both degree one. The only even subset is therefore $\Delta = \emptyset$, since then Δ_j is the zero divisor for $j = 1, 2$. Let $\beta : X \to C$ be the blow up of P and E the exceptional component of X. The equivalence class of a spin curve (X, L, b) in $\overline{S}^0_C$ is then determined by choosing a theta characteristic L^ν_j on C^ν_j for $j = 1, 2$. There are $2^{2g(C^\nu_1)} \cdot 2^{2g(C^\nu_2)} = 2^{2g}$ choices for the pair (L^ν_1, L^ν_2). In order to fix (X, L, b) one has to choose isomorphisms $b_j : L^{\nu\otimes 2}_j \to \omega_{C^\nu_j}$ and gluings of the theta characteristics and $\mathcal{O}_E(1)$ over the two nodes of X but here different choices give the same point in $\overline{S}^0_C$. Hence in this case $\overline{S}^0_C$ consists of 2^{2g} points.

The same argument works for all curves of compact type. For a stable curve C of compact type with smooth irreducible components C_j of genera $g(C_j)$ a point in $\overline{S}^0_C$ is determined by choosing a theta characteristic L^ν_j on $C_j = C^\nu_j$. This gives $\prod_j 2^{2g(C_j)} = 2^{2g}$ choices. A representative of such a point is then (X, L, b), where $\beta : X \to C$ is the blow up at $\operatorname{sing} C$, i.e. $\Delta = \emptyset$ and L is any line bundle on X that restricts to L^ν_j on C_j and to $\mathcal{O}_E(1)$ on any exceptional component E.

Now let C be a stable irreducible curve of genus g with one node P, i.e. the normalisation $C^\nu_1 = C^\nu$ is a smooth curve of genus $g-1$ with two marked points P^+ and P^-. In this case there are two even subsets of $\operatorname{sing} C$. If $\Delta = \emptyset$ and $\beta : X \to C$ the blow up of P then Δ_1 is the zero divisor and one has to choose a theta characteristic L^ν_1 on C^ν_1. There are $2^{2(g-1)}$ choices for it. Different choices of an isomorphism $b_1 : L^{\nu\otimes 2}_1 \to \omega_{C^\nu_1}$ and the gluings of L^ν_1 and $\mathcal{O}_E(1)$ over the two nodes of X give the same point in $\overline{S}^0_C$.

On the other hand if $\Delta = \{P\}$, then $X = C$ and one has to choose a square root L^ν_1 of the line bundle $\omega_{C^\nu_1}(P^+ + P^-)$, there are also $2^{2(g-1)}$ choices for it. But here the gluing of L^ν_1 over the node P has to be fixed. After choosing any isomorphism $b_1 : L^{\nu\otimes 2}_1 \to \omega_{C^\nu_1}(P^+ + P^-)$ (which does not give different equivalence classes) there are exactly two different choices for a gluing which is compatible with b_1. All together this gives exactly $2^{2(g-1)} + 2 \cdot 2^{2(g-1)} = 3 \cdot 2^{2(g-1)}$ points in $\overline{S}^0_C$.

The automorphism group of a stable curve C of genus $g \geq 2$ acts on $\overline{S}^0_C$ in the following way.

Definition 2.2.21. Let C be a stable curve of genus $g \geq 2$, $[(X, L, b)] \in \overline{S}^0_C$ and $\sigma_C \in \operatorname{Aut} C$. As usual $N \subset \operatorname{sing} C$ denotes the set of exceptional nodes of X. The *pull back of* $[(X, L, b)]$ via σ_C is $\sigma^*_C[(X, L, b)] := [(X', L', b')] \in \overline{S}^0_C$, where (X', L', b') is the following spin curve. Let $\beta' : X' \to C$ be the blow up at $N' := \sigma^{-1}_C(N)$ and choose any lift $\sigma : X' \xrightarrow{\cong} X$ of σ_C (see 2.1.18). Then $L' := \sigma^* L$ and b' is the concatenation

$$L'^{\otimes 2} = \sigma^* L^{\otimes 2} \xrightarrow{\sigma^* b} \sigma^* \beta^* \omega_C = \beta'^* \sigma^*_C \omega_C \cong \beta'^* \omega_C.$$

Note that

$$\begin{array}{ccc} X' & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \beta} \\ C & \xrightarrow{\sigma_C} & C \end{array}$$

commutes and σ_C induces a canonical isomorphism $\omega_C \cong \sigma_C^* \omega_C$ by pull back of forms.

Remark 2.2.22. The pull back $\sigma_C^*[(X, L, b)]$ is well defined. If $\sigma' : X' \xrightarrow{\cong} X$ is another lift of σ_C, denote by (X', L'', b'') the corresponding spin curve. Then the restrictions $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ of σ and σ' to the non-exceptional subcurve $\widetilde{X}'$ are equal. Therefore $\widetilde{L}' = \widetilde{\sigma}^*(L_{|\widetilde{X}}) = \widetilde{\sigma}'^*(L_{|\widetilde{X}}) = \widetilde{L}''$ and by Proposition 2.2.11 there exists an inessential isomorphism $(X', L', b') \cong (X', L'', b'')$, i.e. $[(X', L', b')] = [(X', L'', b'')] \in \overline{S}_C^0$.

Remark 2.2.23. For a fixed stable curve C of genus $g \geq 2$ the set $\overline{S}_C$ of spin curves with stable model C (modulo isomorphism of spin curves) is the quotient $\overline{S}_C^0 / \operatorname{Aut} C$.

Example 2.2.24. Let C be a stable reducible curve with one node P and two smooth irreducible components C_1 and C_2 of genera $g-1$ and 1 respectively. Suppose that $(C_1; P^+)$ has no automorphisms, i.e. $\operatorname{Aut}(C_1; P^+) = \{\mathrm{id}_{C_1}\}$ and choose P^- as the origin of the elliptic curve C_2, where $P^{\pm}$ are the preimages of the node under the normalisation $\nu_C : C^\nu = C_1 \amalg C_2 \to C$. Note that in this situation C_1 has genus at least 2, since if the genus was zero the curve would not be stable and if the genus was one, the pointed curve $(C_1; P^+)$ would have at least two automorphisms, the identity and the elliptic involution with respect to P^+. Example 2.2.20 shows that $\overline{S}_C^0$ consists of 2^{2g} points, where every point is uniquely given by the choice of theta characteristics L_1^ν and L_2^ν on C_1 and C_2 respectively.

The automorphism group of C is essentially that of $(C_2; P^-)$. Any automorphism σ of C must fix the node P, since it is the only node. Hence σ either fixes both components or interchanges them. If σ would interchange C_1 and C_2, the restriction $\sigma_{|C_1} : C_1 \to C_2$ would give an isomorphism of the two components, but this is not possible since the genera are different. Therefore σ induces automorphisms σ_j of C_j fixing P^+ resp. P^-. Hence $\sigma_1 = \mathrm{id}_{C_1}$ and $\sigma_2 \in \operatorname{Aut}(C_2; P^-)$. On the other hand if an automorphism $\sigma_2 \in \operatorname{Aut}(C_2; P^-)$ is given it can be glued to the identity on C_2^ν to give an isomorphism σ of C.

In order to understand the action of $\operatorname{Aut} C$ on $\overline{S}_C^0$ it is enough to understand the pull backs of the two theta characteristics L_1^ν and L_2^ν via σ_1 resp. σ_2. Since $\sigma_1 = \mathrm{id}_{C_1}$ the action of σ_1 on the set of L_q^ν's is trivial. C_2 is an elliptic curve, hence $\omega_{C_2} = \mathcal{O}_{C_2}$ and the four theta characteristics are given by the four 2-torsion points of C_2, i.e. the trivial bundle $\mathcal{O}_{C_2}$ and the three bundles $\mathcal{O}_{C_2}(Q_i - P^-)$, where the Q_i are the three non-trivial 2-torsion points.

Every elliptic curve has the elliptic involution ι as an automorphism. ι fixes the 2-torsion points, therefore $\iota^* L_2^\nu = L_2^\nu$. Hence if C_2 is a general elliptic curve, i.e. if $\operatorname{Aut}(C_2; P^-) = \{\mathrm{id}, \iota\}$, the action of $\operatorname{Aut} C$ on $\overline{S}_C^0$ is trivial and $\overline{S}_C = \overline{S}_C^0$. If C_2 is a special elliptic curve either the j-invariant is 0 and $\operatorname{Aut}(C_2; P^-) \cong \mathbb{Z}_6$ or the j-invariant is 1728 and $\operatorname{Aut}(C_2; P^-) \cong \mathbb{Z}_4$. C_2 can be identified with $\mathbb{C}/\Lambda$ such that P^- is identified with $0 \in \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} + \mathbb{Z}\varrho^2$, $\varrho = e^{2\pi\sqrt{-1}/6}$, resp. $\Lambda = \mathbb{Z} + \mathbb{Z}\sqrt{-1}$. The 2-torsion points are then 0, $Q_1 = \frac{1}{2}$, $Q_2 = \frac{1}{2}\varrho^2$ and $Q_3 = \frac{1}{2} + \frac{1}{2}\varrho^2$ resp. 0, $Q_1 = \frac{1}{2}$, $Q_2 = \frac{1}{2}\sqrt{-1}$ and $Q_3 = \frac{1}{2} + \frac{1}{2}\sqrt{-1}$. The automorphism group is generated by ϱ resp. $\sqrt{-1}$, where the automorphism ϱ resp. $\sqrt{-1}$ is induced by multiplication with ϱ resp. $\sqrt{-1}$ on $\mathbb{C}$. The elliptic involution ι in this notation is -1, i.e. ϱ^3 resp. $\sqrt{-1}^2$. The action is now easy to compute, the automorphism ϱ maps

$$\begin{aligned} 0 &\longmapsto 0 \\ Q_1 = \tfrac{1}{2} &\longmapsto \tfrac{1}{2}\varrho = \tfrac{1}{2}(\varrho^2 + 1) = Q_3 \\ Q_2 = \tfrac{1}{2}\varrho^2 &\longmapsto \tfrac{1}{2}\varrho^3 = -\tfrac{1}{2} = Q_1 \\ Q_3 = \tfrac{1}{2} + \tfrac{1}{2}\varrho^2 &\longmapsto \tfrac{1}{2}\varrho + \tfrac{1}{2}\varrho^3 = \tfrac{1}{2}\varrho^2 = Q_2, \end{aligned}$$

while $\sqrt{-1}$ maps

$$\begin{aligned} 0 &\longmapsto 0 \\ Q_1 = \tfrac{1}{2} &\longmapsto \tfrac{1}{2}\sqrt{-1} = Q_2 \\ Q_2 = \tfrac{1}{2}\sqrt{-1} &\longmapsto -\tfrac{1}{2} = Q_1 \\ Q_3 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{-1} &\longmapsto -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-1} = Q_3. \end{aligned}$$

Therefore in case C_2 has j-invariant not equal to 0 or 1728 two pairs of theta characteristics (L_1^ν, L_2^ν) and $({L'_1}^\nu, {L'_2}^\nu)$ give isomorphic spin curves iff $L_1^\nu \cong {L'_1}^\nu$ and $L_2^\nu \cong {L'_2}^\nu$. In case C_2 has j-invariant 0 two pairs of theta characteristics (L_1^ν, L_2^ν) and $({L'_1}^\nu, {L'_2}^\nu)$ give isomorphic spin curves iff $L_1^\nu \cong {L'_1}^\nu$ and either L_2^ν and ${L'_2}^\nu$ are both non-trivial theta characteristics, i.e. $L_2^\nu, {L'_2}^\nu \in \{\mathcal{O}_{C_2}(Q_i - P^-) | i = 1, 2, 3\}$ or L_2^ν and ${L'_2}^\nu$ are both trivial theta characteristics, i.e. $L_2^\nu = {L'_2}^\nu = \mathcal{O}_{C_2}$. In case C_2 has j-invariant 1728 two pairs of theta characteristics (L_1^ν, L_2^ν) and $({L'_1}^\nu, {L'_2}^\nu)$ give isomorphic spin curves iff $L_1^\nu \cong {L'_1}^\nu$ and either $L_2^\nu = {L'_2}^\nu = \mathcal{O}_{C_2}$ or $L_2^\nu = {L'_2}^\nu = \mathcal{O}_{C_2}(Q_3 - P^-)$ or $L_2^\nu, {L'_2}^\nu \in \{\mathcal{O}_{C_2}(Q_i - P^-) | i = 1, 2\}$. This shows that $\overline{S}_C$ consists of $2^{2(g-1)} \cdot 2^2 = 2^{2g}$ points if C_2 is a general elliptic curve, of $2^{2(g-1)} \cdot 2 = 2^{2g-1}$ points if $j(C_2) = 0$ and of $2^{2(g-1)} \cdot 3$ points if $j(C_2) = 1728$.

Chapter 3

Moduli spaces

After introducing the objects under consideration this chapter will give the background on their moduli spaces. Firstly the moduli space of smooth and stable curves will be described. Secondly the moduli space of spin curves will be introduced and the sets $\overline{S}_C$ and $\overline{S}_C^0$ will be given a scheme structure. An important tool to understand the local structure of the moduli spaces is deformation theory which will be described in the second section of this chapter.

3.1 Moduli spaces

In this section the moduli functors of stable curves and spin curves will be defined and the corresponding coarse moduli spaces will be described.

3.1.1 Stable curves

Definition 3.1.1. Let $\mathcal{S}ch$ be the category of schemes over $\mathbb{C}$ and $\mathcal{S}ets$ the category of sets. For $g \geq 2$ the *moduli functor of stable curves of genus g* is the following contravariant functor.

$$\begin{aligned} \overline{\mathcal{M}}_g : \qquad \mathcal{S}ch &\longrightarrow \mathcal{S}ets \\ Z &\longmapsto \overline{\mathcal{M}}_g(Z) = \left.\begin{Bmatrix} \mathcal{C} \to Z \text{ family of stable} \\ \text{curves of genus } g \text{ over } Z \end{Bmatrix}\right/ \cong \\ Z \xrightarrow{\varrho} Z' &\longmapsto \overline{\mathcal{M}}_g(\varrho) : \ \overline{\mathcal{M}}_g(Z') \to \overline{\mathcal{M}}_g(Z) \\ & \qquad\qquad [\mathcal{C}' \to Z'] \mapsto [\mathcal{C}' \times_{Z'} Z \to Z] \end{aligned}$$

where $[\cdot]$ denotes the equivalence class. This functor is well defined, since the fibre product is unique up to unique isomorphism. Restricting everything to smooth curves gives the *moduli functor $\mathcal{M}_g$ of smooth curves of genus g.*

Definition 3.1.2. A contravariant functor $F : \mathcal{S}ch \to \mathcal{S}ets$ is *representable (in the category $\mathcal{S}ch$)* if there exists a scheme S in $\mathcal{S}ch$ and an isomorphism of functors between F and the functor of points of S, i.e.

$$\begin{aligned} \operatorname{Hom}(\cdot, S): \qquad \mathcal{S}ch &\longrightarrow \mathcal{S}ets \\ Z &\longmapsto \operatorname{Hom}(Z, S) \\ Z \xrightarrow{\varrho} Z' &\longmapsto \operatorname{Hom}(Z', S) \xrightarrow{\varrho^*} \operatorname{Hom}(Z, S) \\ &\qquad\qquad \varphi \mapsto \varphi \circ \varrho \end{aligned}$$

The scheme S is then called a *fine moduli space for the functor F.*

Unfortunately, the moduli functors of stable resp. smooth curves are not representable. Suppose that there existed a fine moduli space S for the moduli functor of smooth curves. Then there existed a bijection between the set of families of smooth curves $\mathcal{M}_g(Z)$ over a scheme Z and $\operatorname{Hom}(Z, S)$. But it is possible to construct non-trivial isotrivial families of smooth curves, i.e. a family $\mathcal{C} \to Z$ of smooth curves such that all fibres are isomorphic to some smooth curve C, but $\mathcal{C}$ is not birational to the product $C \times Z$. The different families $\mathcal{C} \to Z$ and $C \times Z \to Z$ both induce the constant morphism $Z \to S$, $z \mapsto [C]$, where $[C] \in S$ is the image of the homomorphism $pt \to S$ induced by the zero-dimensional family $C \to pt$. Therefore the existence of non-trivial isotrivial families contradicts the existence of a fine moduli space.

The construction of such a family uses a non-trivial automorphism of the curve C. Restricting the moduli functor $\mathcal{M}_g$ of smooth curves to smooth curves with trivial automorphism group gives a functor $\mathcal{M}_g^0$ which is representable (see [HM98, p. 37]).

Remark 3.1.3. One possibility to get around the non-representability is to enlarge the category, in which the functors should be representable, in such a way, that the objects of this larger category still enjoy geometric properties. This can be done in the category of stacks, see for example the original paper by Deligne and Mumford [DM69] or the appendix of [Vis89] for an introduction to stacks. The idea is to include the automorphisms of the objects and to consider the following functor from the category $\mathcal{S}ch$ into the category $\mathcal{G}roup$ of grupoids, i.e. categories whose only morphisms are isomorphisms. The functor sends a scheme Z to the category whose objects are families of stable curves over Z and whose morphisms are isomorphisms of such families over Z. This functor gives the moduli stack of stable curves, which is a smooth Deligne-Mumford stack and contains the moduli stack of smooth curves as a dense open substack.

Another way to deal with the non-existence of a fine moduli scheme, which will be followed here, is to weaken the notion of representability.

Definition 3.1.4. Let $F : \mathcal{S}ch \to \mathcal{S}ets$ be a contravariant functor. A scheme S and a natural transformation Ψ between the functor F and the functor of points $\mathrm{Hom}(\cdot, S)$ are a *coarse moduli space for the functor F* if the map $\Psi_{\operatorname{Spec}\mathbb{C}} : F(\operatorname{Spec}\mathbb{C}) \to \mathrm{Hom}(\operatorname{Spec}\mathbb{C}, S)$ is a bijection in $\mathcal{S}ets$ and (S, Ψ) is universal in the sense that given a scheme S' and a natural transformation Ψ' from F to $\mathrm{Hom}(\cdot, S')$, there is a unique morphism $S \to S'$ such that the associated natural transformation $\Phi : \mathrm{Hom}(\cdot, S) \to \mathrm{Hom}(\cdot, S')$ commutes with Ψ and Ψ', i.e. $\Phi \circ \Psi = \Psi'$.

For the functor $\overline{\mathcal{M}}_g$ of stable curves of genus g a coarse moduli space $\overline{M}_g$ has been constructed by means of geometric invariant theory (GIT).

Theorem 3.1.5. [DM69, Gie82] *For $g \geq 2$ there exists a coarse moduli scheme $\overline{M}_g$ resp. M_g of stable resp. smooth curves of genus g. $\overline{M}_g$ is irreducible and contains M_g as a dense open subscheme.*

Remark 3.1.6. The theoretical background of GIT can be found in [MFK94]. There the theory is applied to the moduli functor of smooth curves in order to give the coarse moduli space M_g of smooth curves of genus g. The construction of $\overline{M}_g$ is given for example in [Gie82] and also explained in [HM98]. The idea is to embed a stable curve C by the very ample line bundle $\omega_C^{\otimes n}$, $n \geq 3$ into the projective space $\mathbb{P}^r$, $r + 1 = (2n - 1)(g - 1)$, which amounts to choosing a basis of $H^0\left(C, \omega_C^{\otimes n}\right)$. The Hilbert polynomial of this embedding is then $P_g(m) = \deg \omega_C^{\otimes n} \cdot m - (g - 1) = (2nm - 1)(g - 1)$. This embedding can be generalized to families of stable curves via the relative dualizing sheaf (see page 9).

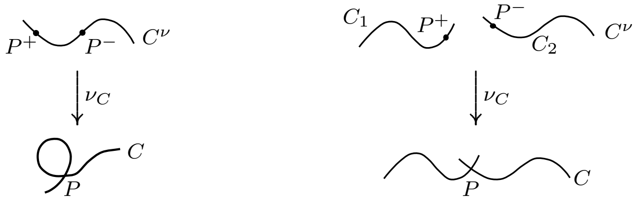
There exists a subscheme $\mathcal{K} \subset \mathrm{Hilb}_{\mathbb{P}^r}^{P_g}$ of the Hilbert scheme of curves in $\mathbb{P}^r$ with Hilbert polynomial P_g parametrising all "n-canonically embedded" stable curves. This is a fine moduli space and the projective linear group scheme $\mathrm{PGL}(r+1)$ acts on it by coordinate change of the ambient space $\mathbb{P}^r$. The stabilizer of this action at a point $[C \hookrightarrow \mathbb{P}^r]$ is just the automorphism group of the abstract curve C. In this situtation GIT guarantees the existence of a quotient scheme $\mathcal{K}/\mathrm{PGL}(r+1)$. Informally speaking quotienting out the action of $\mathrm{PGL}(r+1)$ amounts to quotienting out the choice of a basis of $H^0\left(C, \omega_C^{\otimes n}\right)$. The quotient $\mathcal{K}/\mathrm{PGL}(r+1)$ is then the sought for coarse moduli space $\overline{M}_g$ (see also Remark 3.2.6).

It should also be mentioned that the irreduciblity of M_g (over $\mathbb{C}$) is classically known, for example a proof already appears in [EC18]. Deligne and Mumford proved the result in any characteristic in [DM69].

Theorem 3.1.7. [Knu83, Theorem 6.1.] *$\overline{M}_g$ is a normal projective variety.*

The boundary of $\overline{M}_g$, i.e. $\partial\overline{M}_g = \overline{M}_g \setminus M_g$, is already described in [DM69].

Proposition 3.1.8. *$\partial\overline{M}_g$ is a divisor with normal crossings in $\overline{M}_g$. It has $[g/2]+1$ irreducible components $\Delta_0, \Delta_1, \dots, \Delta_{[g/2]}$, where Δ_0 is the closure in $\overline{M}_g$ of the set of points corresponding to irreducible stable curves with one node, and Δ_i, $i \geq 1$, is the closure in $\overline{M}_g$ of the set of points corresponding to reducible stable curves with two irreducible components of genera i and $g-i$ respectively joined at one node.*



general $[C] \in \Delta_0$
$g(C^\nu) = g-1$

general $[C] \in \Delta_i$
$g(C_1) = i$, $g(C_2) = g-i$

Definition 3.1.9. Let C be a stable curve of genus $g \geq 2$, $P \in \operatorname{sing} C$ a node and $\nu_P : \widetilde{C} \to C$ the partial normalisation of C at P. P is called a *disconnecting node* if $\widetilde{C}$ is disconnected. If $\widetilde{C}$ is the disjoined union of two curves $\widetilde{C}_1$ and $\widetilde{C}_2$ of genus i and $g-i$ respectively, P is a *node of type i*. P is called a *non-disconnecting node* or *node of type 0* if $\widetilde{C}$ is connected. If in addition the two preimages of P lie in the same irreducible component of $\widetilde{C}$, P is an *irreducible node*.

P irreducible, $g(\widetilde{C}) = g-1$

P node of type $i = g(\widetilde{C}_1)$

Remark 3.1.10. A stable curve C has a node of type i for some $i = 0, \dots, \left[\frac{g}{2}\right]$ iff $[C] \in \Delta_i$.

Remark 3.1.11. Let $g \geq 3$. The general point $[C]$ of the divisor Δ_1 is a reducible curve with two smooth irreducible components of genera $g-1$ and 1, which are general, joined at one node P. The automorphism group of C is then $\{\mathrm{id}, \iota\}$ where ι is the identity on the genus $g-1$ component and the elliptic involution on the other. The general point of any other boundary divisor Δ_0, $\Delta_2, \dots, \Delta_{[g/2]}$ does not have non-trivial automorphisms.

3.1.2 Spin curves

The same questions now arise for spin curves.

Definition 3.1.12. For $g \geq 2$ the *moduli functor of spin curves of genus* g is the following contravariant functor.

$$\begin{aligned}
\overline{\mathcal{S}}_g : \quad & \mathcal{S}ch \longrightarrow \mathcal{S}ets \\
& Z \longmapsto \overline{\mathcal{S}}_g(Z) = \left\{ \begin{matrix} (f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B}) \text{ family of} \\ \text{spin curves of genus } g \text{ over } Z \end{matrix} \right\} \Big/ \cong \\
Z \xrightarrow{\varrho} Z' \longmapsto \overline{\mathcal{S}}_g(\varrho) : \quad & \overline{\mathcal{S}}_g(Z') \to \overline{\mathcal{S}}_g(Z) \\
& [(\mathcal{X}' \to Z', \mathcal{L}', \mathcal{B}')] \mapsto \varrho^*[(\mathcal{X}' \to Z', \mathcal{L}', \mathcal{B}')]
\end{aligned}$$

here $\varrho^*[(\mathcal{X}' \to Z', \mathcal{L}', \mathcal{B}')] = [(\mathcal{X} \to Z, \mathcal{L}, \mathcal{B})]$ where

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\overline{\varrho}} & \mathcal{X}' \\
\downarrow & \ulcorner & \downarrow \\
Z & \xrightarrow{\varrho} & Z'
\end{array}$$

is a fibre square, $\mathcal{L} = \overline{\varrho}^* \mathcal{L}'$ and $\mathcal{B}$ is the concatenation

$$\mathcal{L}^{\otimes 2} = \overline{\varrho}^* \mathcal{L}'^{\otimes 2} \xrightarrow{\overline{\varrho}^* \mathcal{B}'} \overline{\varrho}^* (\beta'^* \omega_{\mathcal{C}'/Z'}) = \beta^* (\overline{\varrho}_{\mathcal{C}}^* \omega_{\mathcal{C}'/Z'}) \to \beta^* \omega_{\mathcal{C}/Z}$$

with $\beta : \mathcal{X} \to \mathcal{C}$ and $\beta' : \mathcal{X}' \to \mathcal{C}'$ the stable models. Restricting everything to families of smooth curves gives the *moduli functor* $\mathcal{S}_g$ *of smooth spin curves of genus* g.

Theorem 3.1.13. [Cor89, Proposition 5.2.] *There exists a coarse moduli scheme* $\overline{S}_g$ *for the moduli functor* $\overline{\mathcal{S}}_g$*.* $\overline{S}_g$ *is a normal, projective variety and contains* S_g*, the coarse moduli space of smooth spin curves, as an open dense subvariety.*

Moreover, Cornalba shows, that the forgetful map

$$\begin{aligned}
\pi : \overline{S}_g &\longrightarrow \overline{M}_g \\
[(X, L, b)] &\longmapsto [C],
\end{aligned}$$

which sends the moduli point of a spin curve to the moduli point of the stable model of its support, is a finite morphism.

It is a classical result that even and odd theta characteristics do not mix, i.e. that in a family of smooth spin curves over a connected base the parity of the spin curve is constant (see [Ati71, Mum71]). This result extends to arbitrary spin curves, actually there is the following theorem.

Theorem 3.1.14. [Cor89, Lemma 6.3.] *$\overline{S}_g$ consists of two connected components $\overline{S}_g^+$ and $\overline{S}_g^-$, which are coarse moduli spaces for even resp. odd spin curves. $\overline{S}_g^+$ and $\overline{S}_g^-$ are irreducible.*

Cornalba also describes the boundary $\partial\overline{S}_g = \overline{S}_g \setminus S_g$, see also Example 2.2.20.

Proposition 3.1.15. [Cor89, p. 585] *The boundary $\partial\overline{S}_g$ of $\overline{S}_g$ is a divisor with normal crossings. The irreducible components of $\partial\overline{S}_g$ are described below by their general point $[(X, L, b)]$, as always C denotes the stable model of X, $\nu : X^\nu \to X$ denotes the normalisation and for any component C_j of C L_j^ν denotes the restriction of $\nu^* L$ to C_j^ν.*

A_0^+: *$[C] \in \Delta_0$ general, $X = C$ and $h^0(X, L)$ is even,*

A_0^-: *$[C] \in \Delta_0$ general, $X = C$ and $h^0(X, L)$ is odd,*

B_0^+: *$[C] \in \Delta_0$ general, $\beta : X \to C$ is the blow up at the unique node of C and $h^0(X, L)$ is even,*

B_0^-: *$[C] \in \Delta_0$ general, $\beta : X \to C$ is the blow up at the unique node of C and $h^0(X, L)$ is odd,*

A_i^+, $i = 1, \ldots, [\frac{g}{2}]$: *$[C] \in \Delta_i$ general with irreducible components C_1 and C_2, $\beta : X \to C$ is the blow up at the unique node, $h^0(C_1, L_1^\nu)$ and $h^0(C_2, L_2^\nu)$ are even,*

B_i^+, $i = 1, \ldots, [\frac{g}{2}]$: *$[C] \in \Delta_i$ general with irreducible components C_1 and C_2, $\beta : X \to C$ is the blow up at the unique node, $h^0(C_1, L_1^\nu)$ and $h^0(C_2, L_2^\nu)$ are odd,*

A_i^-, $i = 1, \ldots, [\frac{g}{2}]$: *$[C] \in \Delta_i$ general with irreducible components C_1 and C_2, $\beta : X \to C$ is the blow up at the unique node, $h^0(C_1, L_1^\nu)$ is even and $h^0(C_2, L_2^\nu)$ is odd,*

B_i^-, $i = 1, \ldots, [\frac{g-1}{2}]$: *$[C] \in \Delta_i$ general with irreducible components C_1 and C_2, $\beta : X \to C$ is the blow up at the unique node, $h^0(C_1, L_1^\nu)$ is odd and $h^0(C_2, L_2^\nu)$ is even.*

Remark 3.1.16. The general curve in Δ_0 has trivial automorphism group, therefore the general point $[(X, L, b)]$ in $A_0^\pm$ and $B_0^\pm$ has only inessential automorphisms. Corollary 2.2.12 shows that

$$\mathrm{Aut}(X, L, b) = \mathrm{Aut}_0(X, L, b) = \{(\mathrm{id}_X, \pm\,\mathrm{id}_L)\}$$

since in all cases the non-exceptional subcurve $\widetilde{X}$ is connected. For $i \geq 2$ the situation for the general point $[(X, L, b)]$ in $A_i^\pm$ and $B_i^\pm$ is similar. The stable model C has trivial automorphism group but the non-exceptional subcurve has two connected components. Therefore the automorphism group is

$\mathrm{Aut}(X, L, b) = \mathrm{Aut}_0(X, L, b) = \{(\mathrm{id}_X, \pm\,\mathrm{id}_L), (\sigma, \pm\gamma)\}$, where σ is the identity off the exceptional component and γ is multiplication with 1 in the fibres over the first non-exceptional component and with -1 in those over the second component. The situation for the general point $[(X, L, b)]$ in $A_1^{\pm}$ and $B_1^{\pm}$ is a little bit more complicated, since the stable model C has the non-trivial automorphism ι, which is the identity on the genus $g-1$ component and the elliptic involution on the elliptic tail. But Example 2.2.24 shows that ι lifts to the spin structure L and Corollary 2.2.12 then proves that $\mathrm{Aut}(X, L, b)$ contains exactly four lifts of ι and four of the identity.

3.1.3 The moduli spaces $\overline{S}_C$ and $\overline{S}_C^0$

On pages 15ff. $\overline{S}_C$ and $\overline{S}_C^0$ were introduced and studied as sets of isomorphism classes of spin curves with a fixed stable model C (with respect to isomorphisms resp. inessential isomorphisms). In this subsection a scheme structure will be defined on $\overline{S}_C$ and $\overline{S}_C^0$ in such a way, that they are the underlying sets of moduli spaces.

Definition 3.1.17. Let C be a stable curve of genus $g \geq 2$. The functors $\overline{\mathcal{S}}_C^0$ and $\overline{\mathcal{S}}_C$ from the category $\mathcal{S}ch$ of schemes to the category $\mathcal{S}ets$ of sets are defined as follows.

$$\overline{\mathcal{S}}_C^0 : \qquad \mathcal{S}ch \longrightarrow \mathcal{S}ets$$

$$Z \longmapsto \overline{\mathcal{S}}_C^0(Z) = \left\{ \begin{matrix} (f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B}) \text{ family} \\ \text{of spin curves over } Z \text{ with} \\ \text{stable model } C \times Z \to Z \end{matrix} \right\} \Big/ \begin{matrix} \text{inessential} \\ \text{isomorphism} \end{matrix}$$

$$Z \xrightarrow{\varrho} Z' \longmapsto \overline{\mathcal{S}}_C^0(\varrho) : \qquad \overline{\mathcal{S}}_C^0(Z') \to \overline{\mathcal{S}}_C^0(Z)$$

$$[(\mathcal{X}' \to Z', \mathcal{L}', \mathcal{B}')] \mapsto \varrho^*[(\mathcal{X}' \to Z', \mathcal{L}', \mathcal{B}')]$$

Replacing “inessential isomorphism” by “isomorphism” gives the functor $\overline{\mathcal{S}}_C$.

Theorem 3.1.18. [CCC04, Theorem 2.4.1.] *Given a stable curve C of genus $g \geq 2$ there exists a coarse moduli scheme $\overline{S}_C^0$ of the functor $\overline{\mathcal{S}}_C^0$.*

Remark 3.1.19. The underlying set of the moduli scheme $\overline{S}_C^0$ is the set described in Proposition 2.2.19. The scheme structure on the set $\overline{S}_C^0$ which makes it the coarse moduli space of $\overline{\mathcal{S}}_C^0$ will be described in the next section.

The functor $\overline{\mathcal{S}}_C$ is a subfunctor of $\overline{\mathcal{S}}_g$ by restricting everything to families of spin curves over some base Z which have the trivial family $C \times Z \to Z$ as stable model. Therefore the fibre $\overline{S}_g \times_{\overline{M}_g} [C]$ of $\pi : \overline{S}_g \to \overline{M}_g$ over $[C]$ is the coarse moduli space $\overline{S}_C$ of $\overline{\mathcal{S}}_C$. The scheme structure of $\overline{S}_C$ will also be discussed more detailed in the next section.

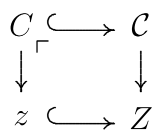
3.2 Deformation theory

This section provides the necessary background on deformation theory of stable curves and spin curves. This is important for understanding the local structure of the moduli spaces, which will also be described. A general reference for deformation theory of local complete intersections is Vistoli's article [Vis97]. Some information on the deformation theory of curves and stable curves can be found in Looijenga's minicourse [Loo00] and also in [HM98].

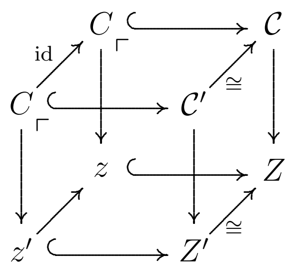
3.2.1 Stable curves

Definition 3.2.1. Let C be a stable curve and (Z, z) any pointed scheme.

(i) A *deformation of C over (Z, z)* is a fibre square



where $\mathcal{C} \to Z$ is a family of stable curves over Z.
(ii) Two deformations of C are *equivalent* if there exists an isomorphism of fibre squares which is the identity on C, i.e. there is a diagram



(iii) An *infinitesimal deformation (or first-order deformation) of C* is a deformation of C over the dual numbers $(\operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2), 0)$, where 0 is the unique closed point in $\operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)$. Denote by $\mathbb{T}^1_C$ the set of equivalence classes of infinitesimal deformations of C.

The following facts about infinitesimal deformations of stable curves are important, they can be found in [DM69] and [Bar89]. Let C be a fixed stable curve of genus $g \geq 2$ and Ω_C its sheaf of Kähler differentials, which is an easy instance of the so called "cotangent complex" [LS67]. For a smooth curve C the dualizing sheaf ω_C is isomorphic to Ω_C. But at a node P of a stable curve C with local equation $xy = 0$ the dualizing sheaf ω_C is locally free while Ω_C is generated by dx and dy modulo the relation $xdy + ydx = 0$. An *extension* of Ω_C by $\mathcal{O}_C$ is an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{E} \longrightarrow \Omega_C \longrightarrow 0$$

and two such sequences are *isomorphic* if there is an isomorphism of the sequences which is the identity on $\mathcal{O}_C$ and Ω_C. Isomorphism classes of such extensions are classified by $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$, where $\mathrm{Ext}^i(\Omega_C, \cdot)$, $i \geq 0$, denote the right derived functors of $\mathrm{Hom}(\Omega_C, \cdot)$ (see [Har77, Chapter III.6.]).

Proposition 3.2.2. [DM69, p. 79ff] *For a stable curve C of genus $g \geq 2$*

(i) $\mathbb{T}^1_C \cong \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$
(ii) $\mathrm{Ext}^2(\Omega_C, \mathcal{O}_C) = 0$, *i.e. "all obstructions of lifting deformations vanish",*
(iii) $\mathrm{Ext}^0(\Omega_C, \mathcal{O}_C) = 0$, *i.e. there exists no nonzero, everywhere regular vector field on C.*

Remark 3.2.3. The Proposition implies that there exists a "universal formal deformation" of any stable curve over $\mathrm{Spec}\,\mathbb{C}[[t_1, \ldots, t_N]]$, where N is the dimension of $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$, see [Vis97] for an introduction.

In order to understand $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$ consider its local to global spectral sequence (see [God73, Corollaire 7.3.3.])

$$\begin{aligned} 0 \longrightarrow H^1\left(C, \mathcal{H}om(\Omega_C, \mathcal{O}_C)\right) \longrightarrow \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \longrightarrow H^0\left(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)\right) \\ \longrightarrow H^2\left(C, \mathcal{H}om(\Omega_C, \mathcal{O}_C)\right) \longrightarrow \mathrm{Ext}^2(\Omega_C, \mathcal{O}_C), \end{aligned}$$

where $\mathcal{E}xt^i(\Omega_C, \cdot)$, $i \geq 0$, are the right derived functors of the sheaf $\mathcal{H}om(\Omega_C, \cdot)$. This simplifies to the short exact sequence

$$0 \longrightarrow H^1\left(C, \Theta_C\right) \longrightarrow \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \longrightarrow H^0\left(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)\right) \longrightarrow 0$$

since C is one-dimensional, where $\Theta_C = \mathcal{H}om(\Omega_C, \mathcal{O}_C)$ denotes the sheaf of derivations of C. This sheaf can be identified with the push forward via the normalisation $\nu_C : C^\nu \to C$ of the sheaf $\Theta_{C^\nu}(-D) = T_{C^\nu}(-D)$ on C^ν, where $D = \sum_i (P_i^+ + P_i^-)$ is the divisor of preimages of the nodes P_i of C. Denoting the restriction of D to an irreducible component C_j^ν by D_j gives

$$\begin{aligned} H^1\left(C, \Theta_C\right) &\cong H^1\left(C^\nu, \Theta_{C^\nu}(-D)\right) \\ &\cong \bigoplus_j H^1(C_j^\nu, \Theta_{C_j^\nu}(-D_j)) \\ &\cong \bigoplus_j H^0(C_j^\nu, \Omega_{C_j^\nu}^{\otimes 2}(D_j))^*, \end{aligned}$$

where the third identification is Serre duality, * denotes the dual vector space; notice that C_j^ν is smooth, hence $\omega_{C_j^\nu} = \Omega_{C_j^\nu} = \mathcal{H}om(\Theta_{C_j^\nu}, \mathcal{O}_{C_j^\nu})$. The jth summand is the first order deformation space of the smooth pointed curve $(C_j^\nu, \{P_i^\pm\})$ (see [HM98, p. 94]), where a deformation of the pointed curve $(C_j^\nu, \{P_i^\pm\})$ is a deformation of C_j^ν together with a section of the family for every $P_i^\pm \in C_j^\nu$, whose pull back to C_j^ν is $P_i^\pm$. Therefore an element in $H^1(C, \Theta_C)$ is a collection of a first order deformation for every $(C_j^\nu, \{P_i^\pm\})$. Identifying for every i the images of the sections corresponding to P_i^+ and P_i^- gives a first order deformation of the stable curve C, in which the combinatorial type, i.e. the dual graph, is preserved. The dimension of $H^1(C_j^\nu, \Theta_{C_j^\nu}(-D_j))$ can be calculated with Riemann-Roch as

$$\dim H^0(C_j^\nu, 2K_{C_j^\nu} + D_j) = 3g(C_j^\nu) - 3 + \deg D_j$$

where $\deg D_j$ is just the number of marked points on C_j^ν and hence one has

$$\dim H^1(C, \Theta_C) = 3\sum_j (g(C_j^\nu) - 1) + 2 \cdot \#\operatorname{sing} C,$$

since $\sum_j \deg D_j = \deg \sum_i (P_i^+ + P_i^-) = 2 \cdot \#\operatorname{sing} C$.

The quotient $H^0\left(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)\right)$ in the short exact sequence is the disjoint union of the stalks $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_{P_i}$ at the nodes P_i, since $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$ is a skyscraper sheaf supported at the nodes. If $xy = 0$ is a local equation of C at the node P_i the stalk $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_{P_i} \cong \mathcal{E}xt^1(\Omega_{\{xy=0\}}, \mathcal{O}_{\{xy=0\}})_0$ fits into the following exact sequence

$$0 \to \Theta_{\{xy=0\}} \to T_{\mathbb{A}^2_{x,y}|\{xy=0\}} \to N_{\{xy=0\}/\mathbb{A}^2_{x,y}} \to \mathcal{E}xt^1(\Omega_{\{xy=0\}}, \mathcal{O}_{\{xy=0\}})_0 \to 0,$$

where $T_{\mathbb{A}^2_{x,y}}$ is the tangent space and $N_{\{xy=0\}/\mathbb{A}^2_{x,y}}$ the normal sheaf, and a local description (see [Bar89]) shows that $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_{P_i} \cong \mathbb{C}[x,y]/(xy, x, y) \cong \mathbb{C}$, which is just the universal deformation space of the node $xy = 0$, i.e. if t is a coordinate of $\mathbb{C}$, then $xy - t = 0$ is an equation of the universal deformation space of the node (see [HM98, p. 97]).

Therefore the dimension of $\operatorname{Ext}^1(\Omega_C, \mathcal{O}_C)$ is

$$\begin{aligned}
\dim \operatorname{Ext}^1(\Omega_C, \mathcal{O}_C) &= \dim H^1(C, \Theta_C) + \bigoplus_{P_i \in \operatorname{sing} C} \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_{P_i} \\
&= 3\sum_j (g(C_j^\nu) - 1) + 2 \cdot \#\operatorname{sing} C + \#\operatorname{sing} C \\
&= 3\Bigg(\sum_j g(C_j^\nu) - \#V(\Gamma(C)) + \#E(\Gamma(C))\Bigg) - 3 \\
&= 3g - 3,
\end{aligned}$$

infinitesimal deformations lying in the kernel $H^1(C, \Theta_C)$ preserve the combinatorial type of the curve and those mapping to a non-zero element in $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)_{P_i}$ "smooth" the node P_i.

Coming back to arbitrary deformations, every deformation of the stable curve C, say

$$\begin{array}{ccc} C & \hookrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ z & \hookrightarrow & Z \end{array}$$

defines a canonical morphism

$$T_z Z \longrightarrow \mathbb{T}^1_C$$

by sending a tangent vector $\operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2) \to Z$ at z to the (class of the) infinitesimal deformation given by the pull back of the family via the tangent vector.

It would be very convenient to have a *universal deformation* of C, i.e. a deformation over some pointed scheme (Z, z) such that any other deformation over any pointed scheme (Z', z') is the pull back of the universal one via a unique morphism $Z' \to Z$. Unfortunately, the presence of non-trivial automorphisms prevents the existence of such a universal deformation.

Nevertheless there exists a so called versal deformation of any stable curve.

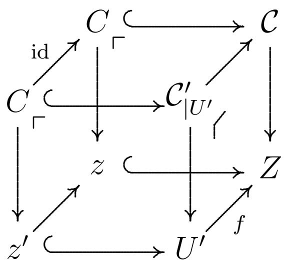
Definition 3.2.4. A deformation

$$\begin{array}{ccc} C & \hookrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ z & \hookrightarrow & Z \end{array}$$

of a stable curve C is *versal*, if any other deformation

$$\begin{array}{ccc} C & \hookrightarrow & \mathcal{C}' \\ \downarrow & & \downarrow \\ z' & \hookrightarrow & Z' \end{array}$$

is analytically isomorphic in a sufficiently small neighbourhood U' of z' to the pull back of the versal deformation via a map $f : U' \to Z$, i.e. there is the following diagram



If in addition the map f is always unique the versal deformation is called a *local universal deformation*.

Theorem 3.2.5. [HM98, p. 102ff.] *Let C be a stable curve of genus $g \geq 2$. Then there exists a local universal deformation of C over a smooth pointed scheme (B, b_0). Any two local universal deformations are locally isomorphic, i.e. the germ of the local universal deformation is unique. The canonical map $T_{b_0}B \to \mathbb{T}^1_C$ is an isomorphism.*

Remark 3.2.6. Assuming the existence of the local universal deformation the last claim follows immediately, since every deformation, in particular every infinitesimal deformation is the pull back of the local universal deformation via a unique map.

The local universal deformation can be constructed with the help of the universal family on the Hilbert scheme (see [HM98, p.102ff]). Let $n \geq 3$, $d = \deg \omega_C^{\otimes n} = 2n(g-1)$, $r + 1 = \dim H^0\left(C, \omega_C^{\otimes n}\right) = d + 1 - g = (2n-1)(g-1)$ and denote by $\mathrm{Hilb}^{P_g}_{\mathbb{P}^r}$ the Hilbert scheme of curves in $\mathbb{P}^r$ with Hilbert polynomial $P_g(m) = dm - (g-1)$. Let

$$\begin{array}{ccc} \mathcal{C}' & \hookrightarrow & \mathbb{P}^r \times \mathrm{Hilb}^{P_g}_{\mathbb{P}^r} \\ & \searrow & \downarrow \\ & & \mathrm{Hilb}^{P_g}_{\mathbb{P}^r} \end{array}$$

be the universal family with universal line bundle $\mathcal{O}_{\mathcal{C}'}(1)$. The restriction to the open subset $\mathcal{U} \subset \mathrm{Hilb}^{P_g}_{\mathbb{P}^r}$ of nodal curves has a relative dualizing sheaf $\omega_{\mathcal{C}'/\mathcal{U}}$ and one considers the closed subscheme $\mathcal{K} \subset \mathcal{U}$ where $\mathcal{O}_{\mathcal{C}'}(1)$ and $\omega^{\otimes n}_{\mathcal{C}'/\mathcal{U}}$ coincide, i.e. $\mathcal{K}$ is the locus of n-canonically embedded curves. Since a curve is stable iff ω_C is ample and by definition points in $\mathcal{K}$ have very ample $\omega_C^{\otimes n}$, $\mathcal{K}$ is the locus of n-canonically embedded stable curves. It can be shown that this locus is smooth of dimension $3g - 3 + r^2 + 2r$ and the group $\mathrm{PGL}(r+1)$ operates on it by the restriction of

$$\begin{aligned} \mathrm{PGL}(r+1) \times \mathrm{Hilb}^{P_g}_{\mathbb{P}^r} &\longrightarrow \mathrm{Hilb}^{P_g}_{\mathbb{P}^r} \\ (\alpha, [C \hookrightarrow \mathbb{P}^r]) &\longmapsto [\alpha(C) \hookrightarrow \mathbb{P}^r] \end{aligned}$$

The choice of an abstract stable curve depends on $3g - 3$ moduli, the choice of a basis for $H^0\left(C, \omega_C^{\otimes n}\right)$ modulo scalars on $r^2 + 2r$.

An automorphism σ_C induces a canonical isomorphism between $\sigma_C^* \omega_C^{\otimes n}$ and $\omega_C^{\otimes n}$, which for a fixed basis of $H^0\left(C, \omega_C^{\otimes n}\right)$ in turn gives an element $\alpha \in \mathrm{PGL}(r+1)$ (induced by the base change in $H^0\left(C, \omega_C^{\otimes n}\right)$) such that $\alpha \cdot [C \hookrightarrow \mathbb{P}^r] = [C \hookrightarrow \mathbb{P}^r]$. This construction gives the identification $\mathrm{Aut}\, C = \mathrm{stab}_{\mathrm{PGL}(r+1)}[C \hookrightarrow \mathbb{P}^r]$. It is possible to choose an analytic neighbourhood $B \subset \mathcal{K}$ of $[C \hookrightarrow \mathbb{P}^r]$ such that B is $\mathrm{Aut}\, C$-invariant, meets the $\mathrm{PGL}(r+1)$-orbits of its points transversely and the stabilizer of any point is a subgroup of $\mathrm{Aut}\, C$. The restriction of the universal family to B (forgetting the embedding) is then the local universal deformation of C. After embedding the Hilbert scheme into some projective space, the neighbourhood B can be chosen in such a way, that the action of $\mathrm{PGL}(r+1)$ on B is linear. Since deformations of stable curves are unobstructed, B is smooth, i.e. the germ is just the germ of $\mathbb{C}^{3g-3} = T_{b_0}B$ at 0 and the linear action of $\operatorname{Aut} C$ on B is a linear action on $T_{b_0}B$.

Let σ_C be an automorphism of C and consider the action $\alpha \in \mathrm{PGL}(r+1)$ induced by σ_C. A point $b \in B$ near b_0 is a fixed point of α iff $\alpha \in \mathrm{stab}_{\mathrm{PGL}(r+1)}\, b = \mathrm{stab}_{\mathrm{PGL}(r+1)}[\mathcal{C}_b \hookrightarrow \mathbb{P}^r]$, where $\mathcal{C}_b \hookrightarrow \mathbb{P}^r$ is the fibre of the universal family of the Hilbert scheme at b. This in turn implies that α is also the action of an automorphism $\sigma_{\mathcal{C}_b} \in \operatorname{Aut} \mathcal{C}_b$, i.e. "the automorphism σ_C deforms to the nearby curve $\mathcal{C}_b$".

Comparing the GIT-construction of the quotient $\overline{M}_g = \mathcal{K}/\mathrm{PGL}(r+1)$ with the construction of the local universal deformation and the action of the automorphism group $\operatorname{Aut} C$ on it shows that locally (analytically) at $[C]$ the space $\overline{M}_g$ is the quotient $T_{b_0}B/\operatorname{Aut} C$, in particular this proves:

Proposition 3.2.7. *$\overline{M}_g$ has only quotient singularities and is normal.*

Definition 3.2.8. A point s in a scheme S is a *quotient singularity* if it is locally analytically isomorphic to $0 \in V/G$, where V is an appropriate finite-dimensional $\mathbb{C}$-vector space and $G \subset \mathrm{GL}(V)$ a finite subgroup.

In order to understand which quotient singularities appear, it is useful to choose appropriate coordinates on $T_{b_0}B \cong \mathbb{T}^1_C \cong \operatorname{Ext}^1(\Omega_C, \mathcal{O}_C)$.

Proposition 3.2.9. *Let C be a stable curve of genus $g \geq 2$ with $\{P_1, \ldots, P_n\} = \operatorname{sing} C$. It is possible to choose coordinates $t_1, \ldots, t_{3g-3}$ of $T_{b_0}B$ such that $t_i = 0$ is the locus where the node P_i is preserved and the remaining coordinates $t_{n+1}, \ldots, t_{3g-3}$ are compatible with*

$$H^1(C, \Theta_C) \cong \bigoplus_j H^0(C_j^\nu, \Omega_{C_j^\nu}^{\otimes 2}(D_j))^*,$$

i.e. for every irreducible component C_j of C there is an $I \subset \{n+1, \ldots, 3g-3\}$ such that $\{t_i = 0 | i \notin I\}$ is the image of $H^0(C_j^\nu, \Omega_{C_j^\nu}^{\otimes 2}(D_j))^*$ under the above isomorphism and t_i, $i \in I$, are coordinates of this subspace.

Notation 3.2.10. Fix coordinates as in the proposition. For a node P_i of C, the one-dimensional subspace of $\mathbb{C}_t^{3g-3}$ given by the $3g-4$ equations $t_j = 0$, $j \neq i$, is denoted by $W_t(P_i)$. This space has coordinate t_i and corresponds to *smoothing the node* P_i, i.e. it is a (non-canonical) embedding of the universal deformation space of the node into $T_{b_0}B$. Let $(C_j^\nu, \{P_i^\pm\})$ be the pointed normalisation of the irreducible component C_j of C. $W_t(C_j) \subset \mathbb{C}_t^{3g-3}$ denotes the image of $H^0(C_j^\nu, \Omega_{C_j^\nu}^{\otimes 2}(D_j))^*$ in $\mathbb{C}_t^{3g-3}$. It is $d_j = (3g(C_j^\nu) - 3 + \deg D_j)$-dimensional. This subspace corresponds to deformations of the curve C that preserve the combinatorial type and the pointed normalisations of all irreducible components but C_j.

With these notations

$$\mathbb{C}_t^{3g-3} = \bigoplus_{P_i \in \operatorname{sing} C} W_t(P_i) \oplus \bigoplus_{C_j \subset C \text{ irr.}} W_t(C_j).$$

Definition 3.2.11. The representation of $\operatorname{Aut} C$ in $\operatorname{GL}(\mathbb{C}_t^{3g-3})$ induced by a choice of coordinates as in Proposition 3.2.9 is denoted

$$\begin{aligned} M : \operatorname{Aut} C &\longrightarrow \operatorname{GL}(\mathbb{C}_t^{3g-3}) \\ \sigma_C &\longmapsto M(\sigma_C). \end{aligned}$$

Remark 3.2.12. For $g \geq 3$ the automorphism group of a general smooth curve is trivial. If $M(\sigma_C) = \mathbb{I}$ for an automorphism σ_C of a stable curve C of genus g, σ_C is the identity. This is the case since some nearby curve has trivial automorphism group and $M(\sigma_C) = \mathbb{I}$ means that σ_C deforms to *every* nearby curve in the local universal deformation of C. Therefore M is injective. In case $g = 2$ the general smooth hyperelliptic curve has the hyperelliptic involution ι and the identity as automorphism. Therefore the hyperelliptic involution deforms to every nearby curve and in this case the kernel of M is $\{\mathrm{id}, \iota\}$. For $g = 1$ the kernel is $\{\mathrm{id}, \iota\}$, where now ι is the elliptic involution.

Proposition 3.2.13. *Let $\sigma_C \in \operatorname{Aut} C$ be an automorphism of the stable curve C. If P is a node and C_j an irreducible component of C, the action of σ_C on $\mathbb{C}_t^{3g-3}$ induces isomorphisms*

$$W_t(P) \longrightarrow W_t(\sigma_C(P)) \qquad\qquad W_t(C_j) \longrightarrow W_t(\sigma_C(C_j)).$$

Corollary 3.2.14. *Denote by σ_E resp. σ_V the automorphism on the set $E = E(\Gamma(C))$ of nodes resp. on the set $V = V(\Gamma(C))$ of irreducible components of C induced by an automorphism $\sigma_C \in \operatorname{Aut} C$. Then the matrix $M(\sigma_C)$ is of the following form*

$$M(\sigma_C) = \begin{pmatrix} M_E \mathbb{E}_{\sigma_E} & \mathbf{0} \\ \mathbf{0} & M_V \mathbb{E}_{\sigma_V} \end{pmatrix},$$

where $\mathbb{E}_{\sigma_E} = (\delta_{i,\sigma_E(k)})_{1 \leq i,k \leq \#E}$ is the permutation matrix of σ_E, M_E is a diagonal matrix of the same size, $\mathbb{E}_{\sigma_V}$ is the following generalized permutation matrix of σ_V

$$\mathbb{E}_{\sigma_V} = \left(\delta_{j,\sigma_V(k)} \mathbb{E}_{d_j}\right)_{1 \leq j,k \leq \#V}$$

where $\mathbb{E}_{d_j}$ is the identity matrix with $d_j = \dim W_t(C_j)$ rows and M_V is an appropriate block diagonal matrix, i.e. the blocks on the diagonal are of the sizes $d_j \times d_j$, $j = 1, \ldots, \#V$.

Remark 3.2.15. If $\operatorname{Aut} C$ is cyclic or more generally if a cyclic subgroup $\langle \sigma_C \rangle \subset \operatorname{Aut} C$ is given, the coordinates of $\mathbb{C}_t^{3g-3}$ can be ordered in such a way, that $\mathbb{E}_{\sigma_E}$ and $\mathbb{E}_{\sigma_V}$ are block diagonal matrices whose blocks are cyclic permutation matrices. In general this is not possible.

Proof of Proposition 3.2.13. Let P_i be a node of C and $P_{i'} = \sigma_C(P_i)$ its image under the automorphism. The coordinate t_i of $W_t(P_i)$ gives the universal deformation $xy = t_i$ of the local equation $xy = 0$ of C at P_i. Let $x'y' = 0$ be a local equation of C at $P_{i'}$ such that σ_C maps $\{x, y\}$ to $\{\mu_1 x', \mu_2 y'\}$ for appropriate scalars μ_k. Then $t_{i'} = x'y'$ is the universal deformation of the node $P_{i'}$. The action of σ_C on t_i is then

$$t_i = xy \longmapsto \mu_1\mu_2 x'y' = \mu_1\mu_2 t_{i'}$$

and gives an isomorphism $W_t(P_i) \to W_t(P_{i'})$.

Now let C_j be an irreducible component of C and $C_{j'} = \sigma_C(C_j)$ its image. Bear in mind that there is the induced isomorphism $\sigma^\nu_{C|C^\nu_j} : (C^\nu_j, \{P^\pm_i\}) \to (C^\nu_{j'}, \{P^\pm_i\})$ of the pointed normalisations, in particular $d_j = \dim W_t(C_j) = \dim W_t(C_{j'}) = d_{j'}$. A point in $W_t(C_j)$ corresponds to a deformation of $(C^\nu_j, \{P^\pm_i\})$ (preserving all nodes of C and all the other components), its image under the action of the automorphism σ_C is just the via $\sigma^\nu_{C|C^\nu_j}$ induced deformation of $(C^\nu_{j'}, \{P^\pm_i\})$ (preserving all nodes of C and all the other components), which is a point in $W_t(C_{j'})$. This is clearly an isomorphism. Of course the isomorphism $\sigma^\nu_{C|C^\nu_j}$ gives an identification of the spaces $H^0(C^\nu_j, \Omega^{\otimes 2}_{C^\nu_j}(D_j))^*$ and $H^0(C^\nu_{j'}, \Omega^{\otimes 2}_{C^\nu_{j'}}(D_{j'}))^*$, but since the coordinates $t_1, \ldots, t_{3g-3}$ are (in general) not adapted to the automorphism σ_C, the isomorphism $W_t(C_j) \to W_t(C_{j'})$ is (in general) not the identity. □

Proof of Corollary 3.2.14. This follows directly from Proposition 3.2.13. The entries of M_E are just the numbers $\mu_1\mu_2$ of the above proof, where P_i runs through all nodes. The block matrices on the diagonal of M_V are the matrices representing the isomorphisms $W_t(C_j) \to W_t(C_{j'})$, where C_j runs through all irreducible components. □

Remark 3.2.16. Let $\sigma_C \in \operatorname{Aut} C$ be an automorphism of C and consider the corresponding matrix $M(\sigma_C)$. Let $\mathcal{C}_b$ be a fibre of $\mathcal{C} \to \mathbb{C}^{3g-3}_t$ near C. Then σ_C deforms to an automorphism $\sigma_{\mathcal{C}_b} \in \operatorname{Aut} \mathcal{C}_b$ iff b is fixed by $M(\sigma_C)$.

3.2.2 Spin curves

The local structure of $\overline{S}_g$ is very similar to that of $\overline{M}_g$. It will described in this section as well as the local universal deformation of a fixed spin curve. Let (X, L, b) be a spin curve of genus $g \geq 2$.

Definition 3.2.17. (i) A *deformation of* (X, L, b) *over a pointed scheme* (Z, z) is a family $(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B})$ of spin curves together with an isomorphism between (X, L, b) and the central fibre $\left(\mathcal{X}_z, \mathcal{L}_{|\mathcal{X}_z}, \mathcal{B}_{|\mathcal{X}_z}\right)$ of the family.

(ii) A *local universal family of* (X, L, b) is a deformation of (X, L, b) over some pointed scheme, such that locally analytically any other deformation is the pull back of the local universal one via a unique map.

In his article [Cor89] Cornalba constructs the local universal deformation of a spin curve (X, L, b) in the following way (see also [CCC04]). First of all consider the stable model C of the support X of the spin curve. As explained in the last section, C has a local universal deformation $\mathcal{C}'$ over (B, b_0) and there exist coordinates $t_1, \ldots, t_{3g-3}$ of the tangent space $T_{b_0}B$ such that for any node $P_i \in \operatorname{sing} C$ the locus where the node P_i persists is given by $t_i = 0$. Locally analytically B at b_0 is isomorphic to its tangent space $T_{b_0}B$ at 0. Therefore the family $\mathcal{C}' \to B$ can be considered as a family $\mathcal{C}' \to \mathbb{C}^{3g-3}_t$. Order the first $\# \operatorname{sing} C$ coordinates in such a way, that the coordinates $t_1, \ldots, t_{\#N}$ correspond to the exceptional nodes $P_i \in N$. For these exceptional nodes let $s'_i : \{t_i = 0\} \to \mathcal{C}'$ be the map, which sends a point in $\{t_i = 0\}$ to the node in the fibre over the point to which P_i deforms. Define new coordinates $\tau_1, \ldots, \tau_{3g-3}$ and a map $\mathbb{C}^{3g-3}_\tau \to \mathbb{C}^{3g-3}_t$ by setting

$$\tau_i^2 = t_i, \text{ for } i = 1, \ldots, \#N \quad \text{and} \quad \tau_i = t_i, \text{ for } i = \#N + 1, \ldots, 3g - 3.$$

The family $\mathcal{C} \to \mathbb{C}^{3g-3}_\tau$ given by the fibre square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}' \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{C}^{3g-3}_\tau & \longrightarrow & \mathbb{C}^{3g-3}_t \end{array}$$

will be the stable model of the family of spin curves wanted. Denote by $s_i : \{\tau_i = 0\} \to \mathcal{C}$ the pull back of the maps s'_i and consider the blow up $\beta : \mathcal{X} \to \mathcal{C}$ of $\mathcal{C}$ in the images of the s_i, $i = 1, \ldots, \#N$, with exceptional divisors $\mathcal{E}_i$. The resulting family $\mathcal{X} \to \mathbb{C}^{3g-3}_\tau$ is then a family of quasistable curves with central fibre $\mathcal{X}_0$ isomorphic to X. In [Cor89, p. 563] Cornalba shows (see also [CCC04, p. 17]) that there exists an inessential isomorphism $X \to \mathcal{X}_0$ (i.e. one compatible with the embedding $C \to \mathcal{C}'$) such that the homomorphism $\mathcal{B}_0 : \mathcal{L}_0{}^{\otimes 2} \to \left(\beta^* \omega_{\mathcal{C}/\mathbb{C}^{3g-3}_\tau}\right)_{|\mathcal{X}_0}$ over the central fibre induced by $b : L^{\otimes 2} \to \beta^* \omega_C$ agrees with the restriction to $\mathcal{X}_0$ of the natural inclusion $\beta^* \omega_{\mathcal{C}/\mathbb{C}^{3g-3}_\tau}(-\sum_i \mathcal{E}_i) \hookrightarrow \beta^* \omega_{\mathcal{C}/\mathbb{C}^{3g-3}_\tau}$. In this situation $(\mathcal{L}_0, \mathcal{B}_0)$ can be extended to a spin structure $(\mathcal{L}, \mathcal{B})$ on the family $\mathcal{X} \to \mathbb{C}^{3g-3}_\tau$ in a small neighbourhood of $0 \in \mathbb{C}^{3g-3}_\tau$. The central fibre $(\mathcal{X}_0, \mathcal{L}_0, \mathcal{B}_0)$ is isomorphic to (X, L, b), where the isomorphism of the supports is the above mentioned inessential isomorphism $X \to \mathcal{X}_0$. Fix any such isomorphism between the spin curves $(\mathcal{X}_0, \mathcal{L}_0, \mathcal{B}_0)$ and (X, L, b). By abuse of notation this family is denoted by $(\mathcal{X} \to \mathbb{C}^{3g-3}_\tau, \mathcal{L}, \mathcal{B})$, however, the family is only defined over a neighbourhood of $0 \in \mathbb{C}^{3g-3}_\tau$.

Theorem 3.2.18. [Cor89, Proposition 4.6.], [CCC04, Proposition 3.3.2.] *The family $(\mathcal{X} \to \mathbb{C}_\tau^{3g-3}, \mathcal{L}, \mathcal{B})$ together with the identification of (X, L, b) with the central fibre is the local universal deformation of (X, L, b).*

As for the stable curves the automorphism group $\operatorname{Aut}(X, L, b)$ of the spin curve (X, L, b) acts on the deformation space $\mathbb{C}_\tau^{3g-3}$.

Proposition 3.2.19. [Cor89] *The automorphism group of a spin curve (X, L, b) of genus $g \geq 2$ acts linearly on the local universal deformation space $\mathbb{C}_\tau^{3g-3}$ of (X, L, b) and this action is compatible with the homomorphism $\operatorname{Aut}(X, L, b) \to \operatorname{Aut} C$ mapping an automorphism of the spin curve to the induced automorphism of the stable model C and the morphism $\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_t^{3g-3}$ to the local universal deformation space $\mathbb{C}_t^{3g-3}$ of C.*

Denote the representation of $\operatorname{Aut}(X, L, b)$ by

$$\begin{aligned} M : \operatorname{Aut}(X, L, b) &\longrightarrow \operatorname{GL}(\mathbb{C}_\tau^{3g-3}) \\ (\sigma, \gamma) &\longmapsto M(\sigma, \gamma). \end{aligned}$$

Remark 3.2.20. Let $g \geq 3$ and $(\sigma, \gamma) \in \operatorname{Aut}(X, L, b)$ for a spin curve (X, L, b) of genus g with $M(\sigma, \gamma) = \mathbb{I}$. Since $\{(\operatorname{id}_X, \pm \operatorname{id}_L)\}$ is contained in $\operatorname{Aut}(X, L, b)$ for any spin curve and the general smooth spin curve has exactly these two automorphisms, $\sigma = \operatorname{id}_X$ and $\gamma = \pm \operatorname{id}_L$. Hence the kernel of M is $\{(\operatorname{id}_X, \pm \operatorname{id}_L)\}$.

Then there is the following commutative diagram

$$\begin{array}{ccccc} \mathbb{C}_\tau^{3g-3} & \longrightarrow & \mathbb{C}_\tau^{3g-3}/M(\operatorname{Aut}(X, L, b)) & \xhookrightarrow{\text{loc.}} & \overline{S}_g \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}_t^{3g-3} & \longrightarrow & \mathbb{C}_t^{3g-3}/M(\operatorname{Aut} C) & \xhookrightarrow{\text{loc.}} & \overline{M}_g \end{array}$$

Again an automorphism $(\sigma, \gamma) \in \operatorname{Aut}(X, L, b)$ deforms to a nearby spin curve (X', L', b') in the local universal deformation $(\mathcal{X} \to \mathbb{C}_\tau^{3g-3}, \mathcal{L}, \mathcal{B})$ iff $M(\sigma, \gamma)$ fixes the point in $\mathbb{C}_\tau^{3g-3}$ over which (X', L', b') lies.

Notation 3.2.21. Let P_i be a node of C and $C_j \subset C$ an irreducible component of C or equivalently a non-exceptional component of X. Then $W_\tau(P_i) \subset \mathbb{C}_\tau^{3g-3}$ resp. $W_\tau(C_j) \subset \mathbb{C}_\tau^{3g-3}$ denotes the preimage of $W_t(P_i)$ resp. $W_t(C_j)$ via $\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_t^{3g-3}$. If P_i is an exceptional node for the spin curve (X, L, b), i.e. P_i is blown up in $X \to C$, the corresponding coordinates t_i and τ_i fulfil $\tau_i^2 = t_i$, i.e. $W_\tau(P_i) \to W_t(P_i)$ is a $2:1$-cover. If P_i is not blown up in $X \to C$ $\tau_i = t_i$ and $W_\tau(P_i) \xrightarrow{\cong} W_t(P_i)$. For a component C_j the coordinates $t_{k_1}, \ldots, t_{k_{d_j}}$ of $W_t(C_j)$ also satisfy $t_i = \tau_i$ and therefore $W_t(C_j) \xrightarrow{\cong} W_\tau(C_j)$.

Proposition 3.2.22. *Let $(\sigma,\gamma)\in \operatorname{Aut}(X,L,b)$ be an automorphism of the spin curve (X,L,b) and σ_C the induced automorphism of the stable model C. The matrix $M(\sigma,\gamma)$ is of the form*

$$M(\sigma,\gamma)=\begin{pmatrix} M_N\mathbb{E}_{\sigma_N} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & M_\Delta\mathbb{E}_{\sigma_\Delta} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & M_V\mathbb{E}_{\sigma_V}\end{pmatrix},$$

where $N\subset \operatorname{sing} C$ is the set of exceptional nodes, $\Delta=\operatorname{sing} C\setminus N$ that of non-exceptional nodes, σ_N and σ_Δ are the permutations of nodes induced by σ_C and σ_V is the permutation of irreducible components of C. Furthermore the $\mathbb{E}_{\sigma_\star}$ are the corresponding (generalized) permutation matrices and the $M_\star$ are (block-)diagonal matrices of the appropriate forms. Moreover, the matrix of σ_C is then

$$M(\sigma_C)=\begin{pmatrix} M_N^2\mathbb{E}_{\sigma_N} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & M_\Delta\mathbb{E}_{\sigma_\Delta} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & M_V\mathbb{E}_{\sigma_V}\end{pmatrix}.$$

Proof. The induced automorphism σ_C maps exceptional nodes to exceptional nodes as well as non-exceptional to non-exceptional. Therefore the permutation σ_E of the nodes $E=E(\Gamma(C))$ decomposes into a permutation σ_N of the exceptional nodes and a permutation σ_Δ of the non-exceptional nodes. This proves the block form of the two matrices. Since the matrices are compatible with the map $\mathbb{C}^{3g-3}_\tau\to\mathbb{C}^{3g-3}_t$ the assertion about the matrices $M_\star$ follows. □

Remark 3.2.23. Fix an automorphism $(\sigma,\gamma)\in\operatorname{Aut}(X,L,b)$ of the spin curve (X,L,b). The permutations σ_N, σ_Δ and σ_V in the above proposition can be decomposed into cycles, say $\sigma_{N,i}$, $\sigma_{\Delta,i}$ and $\sigma_{V,i}$. After permuting the coordinates $\tau_1,\dots,\tau_{3g-3}$ the permutation matrices $\mathbb{E}_{\sigma_N}$ and $\mathbb{E}_{\sigma_\Delta}$ as well as the generalized permutation matrix $\mathbb{E}_{\sigma_V}$ break up into block diagonal form with exactly one block for each cycle, i.e.

$$M(\sigma,\gamma)=\left(\begin{array}{c|c|c} M_{N,1}\mathbb{E}_{\sigma_{N,1}} \quad & & \\ \qquad\ddots & & \\ \hline & M_{\Delta,1}\mathbb{E}_{\sigma_{\Delta,1}} \quad & \\ & \qquad\ddots & \\ \hline & & M_{V,1}\mathbb{E}_{\sigma_{V,1}}\quad \\ & & \qquad\ddots \end{array}\right)$$

where the $M_{N,i}$ and $M_{\Delta,i}$ are diagonal matrices and

$$\mathbb{E}_{\star,i}=\begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1\\ 1 & 0 & \cdots & 0\end{pmatrix}$$

for $\star \in \{N, \Delta\}$ is a square matrix with $\operatorname{ord} \sigma_{\star,i}$ rows. Furthermore if $\sigma_{V,i} = (v_1\ v_2\ \ldots\ v_{\operatorname{ord} \sigma_{V,i}})$ is a cycle in σ_V the v_j correspond to irreducible components C_j of C, such that $\sigma_C(C_j) = C_{j+1}$ (set $v_{\operatorname{ord} \sigma_{V,i}} + 1 = 1$). Therefore the dimensions of the corresponding $W_\tau(C_j)$ are all equal, denote this dimension by d_i. Then

$$\mathbb{E}_{V,i} = \begin{pmatrix} \mathbf{0} & \mathbb{I} & & \\ \vdots & & \ddots & \\ \mathbf{0} & & & \mathbb{I} \\ \mathbb{I} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}$$

is a generalized cyclic permutation matrix, where $\mathbb{I}$ is the identity matrix with d_i rows. The matrix $M_{V,i}$ is a block diagonal matrix with $\operatorname{ord} \sigma_{V,i}$ blocks of the size $d_i \times d_i$ on the diagonal.

In order to abbreviate notation the cycles $\sigma_{N,1}, \ldots, \sigma_{\Delta,1}, \ldots, \sigma_{V,1}, \ldots$ in this order will be denoted by $\pi_1, \ldots, \pi_l$ and the corresponding matrix $M_{\star,i}$ will be denoted M_j if $\sigma_{\star,i} = \pi_j$. Therefore

$$M(\sigma, \gamma) = \begin{pmatrix} M_1\mathbb{E}_{\pi_1} & & \\ & \ddots & \\ & & M_l\mathbb{E}_{\pi_l} \end{pmatrix}$$

and such a block $M_i\mathbb{E}_{\pi_i}$ *corresponds to exceptional nodes, non-exceptional nodes resp. irreducible components* if π_i is a cycle of σ_N, σ_Δ resp. σ_V.

The choosen permutation of the coordinates $\tau_1, \ldots, \tau_{3g-3}$ depends strongly on the fixed automorphism (σ, γ). In addition this choice is not even unique for a fixed (σ, γ).

Remark 3.2.24. Proposition 2.2.19 stated that the underlying set of the coarse moduli scheme $\overline{S}^0_C$ of spin curves with stable model C modulo inessential automorphisms consists of

$$2^{2p} \cdot \left(\sum_{\substack{\Delta \subset \operatorname{sing} C \\ \text{even}}} 2^{b_1\left(\Gamma\left(\widetilde{X}_\Delta\right)\right)} \right)$$

points, where $p = \sum_j g(C^\nu_j)$ and $\beta : X_\Delta \to C$ is the blow up at $\operatorname{sing} C \setminus \Delta$, $\Delta \subset \operatorname{sing} C$ even. Now the scheme structure of $\overline{S}^0_C$ as given in the proof of Theorem 2.4.1. in [CCC04] shall be described. Let (X, L, b) be a spin curve with support $X = X_\Delta$ and $\mathbb{C}^{3g-3}_\tau$ its local universal deformation space. Remember the graph $\Sigma(X)$ with vertices $V(\Sigma(X)) = CC(\Gamma(\widetilde{X}))$ corresponding to connected components of the non-exceptional subcurve $\widetilde{X}$ and edges $E(\Sigma(X)) = N$ corresponding to the exceptional components. Here the edge $e(E_i)$ corresponding to the exceptional component E_i is incident to the vertices $v(\widetilde{X}_j)$ and $v(\widetilde{X}_{j'})$ if $\widetilde{X}_j$

and $\widetilde{X}_{j'}$ are the components met by E_i. Therefore the map $\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_t^{3g-3}$ from the local universal deformation of (X, L, b) to that of C is given by $\tau_i^2 = t_i$ for $i = 1, \ldots, \#N$ and $\tau_i = t_i$ else. To put it differently $\mathbb{C}_t^{3g-3}$ is the quotient of $\mathbb{C}_\tau^{3g-3}$ by the abelian group $\mathbb{D}_N$ of diagonal matrices of the following form

$$\left(\begin{array}{ccc|c} \pm 1 & & & \\ & \ddots & & \\ & & \pm 1 & \\ \hline & & & \mathbb{I} \end{array}\right)$$

where the upper left block has size $\#N \times \#N$.

The action of the group $\mathrm{Aut}_0(X, L, b)$ of inessential automorphisms of (X, L, b) on $\mathbb{C}_\tau^{3g-3}$ can be described with the help of the graph $\Sigma(X)$ as follows. Remember that an inessential automorphism (σ, γ) corresponds to an element $(\gamma_j) \in \mathbb{Z}_2^{CC(\Gamma(\widetilde{X}))} = \mathbb{Z}_2^{V(\Sigma(X))}$ (Corollary 2.2.12). By definition the induced automorphism on the stable model C is id_C. Therefore by Proposition 3.2.22 $M(\sigma, \gamma)$ has the form

$$M(\sigma,\gamma) = \begin{pmatrix} M_N \mathbb{E}_{\mathrm{id}_N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{E}_{\mathrm{id}_\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{E}_{\mathrm{id}_V} \end{pmatrix} = \begin{pmatrix} M_N & \\ & \mathbb{I} \end{pmatrix}$$

where $\mathbb{I}$ denotes the identity matrix and $M_N = M_N(\sigma, \gamma)$ is a diagonal matrix whose square is the identity. Hence the quotient $\mathbb{C}_\tau^{3g-3}/M(\mathrm{Aut}_0(X, L, b))$ can be written as $\mathbb{C}_\tau^{\#N}/M_N(\mathrm{Aut}_0(X, L, b)) \times \mathbb{C}_\tau^{3g-3-\#N}$, where

$$M_N(\mathrm{Aut}_0(X, L, b)) = \{M_N(\sigma, \gamma) | (\sigma, \gamma) \in \mathrm{Aut}_0(X, L, b)\}.$$

Obviously $M(\mathrm{Aut}_0(X, L, b))$ is a normal subgroup of $\mathbb{D}_N$ and the map $\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_t^{3g-3}$ factors as

$$\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_\tau^{3g-3}/M(\mathrm{Aut}_0(X, L, b)) \to \mathbb{C}_\tau^{3g-3}/\mathbb{D}_N = \mathbb{C}_t^{3g-3}.$$

Caporaso, Casagrande and Cornalba prove that the connected component of $\overline{S}_C^0$ corresponding to the point $[(X, L, b)] \in \overline{S}_C^0$ is isomorphic to the fibre of $\mathbb{C}_\tau^{3g-3}/M(\mathrm{Aut}_0(X, L, b)) \to \mathbb{C}_t^{3g-3}$ over 0 (see [CCC04, Theorem 2.4.1.]). This gives the scheme structure of $\overline{S}_C^0$ at $[(X, L, b)] \in \overline{S}_C^0$.

In order to analyse the action of $M_N(\mathrm{Aut}_0(X, L, b))$ on the first $\#N$ coordinates consider a nearby spin curve (X', L', b') over a point in $W_\tau(P_i)$, where $P_i \in N$ is an exceptional node. The graph $\Sigma(X')$ is then obtained from $\Sigma(X)$ by contracting the edge $e(E_i)$. Therefore the inessential automorphism (σ, γ) deforms to an automorphism (σ', γ') of (X', L', b') (which is necessarily inessential again) iff $(\gamma_j) \in \mathbb{Z}_2^{\Sigma(X)}$ descends well defined to an element in $\mathbb{Z}_2^{\Sigma(X')}$ iff $\gamma_j = \gamma_{j'}$, where $\widetilde{X}_j$ and $\widetilde{X}'_j$ are the two connected components met by E_i. This shows that the ith entry of M_N is $(-1)^{\overline{\gamma}_i}$, where $\overline{\gamma}_i \in \mathbb{Z}_2$ is 0 in case $\gamma_j = \gamma_{j'}$ and 1 else.

Thinking of (γ_j) resp. $(\overline{\gamma}_i)$ as a labeling of the vertices resp. the edges of $\Sigma(X)$ gives an injective map $\mathbb{Z}_2^{V(\Sigma(X))} \to \mathbb{Z}_2^{E(\Sigma(X))}$, mapping (γ_j) to $(\overline{\gamma}_i)$, whose image consists of all elements which are "compatible with different paths", i.e. if v and v' are two vertices in $\Sigma(X)$ and $e_{i_1}, \dots, e_{i_k}$ and $e_{i'_1} \dots, e_{i'_l}$ are two paths of edges in $\Sigma(X)$ connecting v and v', then the sums of labels $\sum_j e_{i_j}$ and $\sum_j e_{i'_j}$ are equal. After fixing a spanning tree $T(X)$ of $\Sigma(X)$, i.e. a subgraph with $V(T(X)) = V(\Sigma(X))$ which is a tree, a labeling of $E(\Sigma(X))$ which is compatible with paths is uniquely determined by labeling all edges $E(T(X))$ of the tree. The coordinates $\tau_1, \dots, \tau_{\#N}$ can be ordered such that the first $\#E(T(X)) = \#V(\Sigma(X)) - 1$ correspond to the edges of the spanning tree. Then giving an inessential automorphism is the same as choosing $\big(\overline{\gamma}_1, \dots, \overline{\gamma}_{\#E(T(X))}\big) \in \mathbb{Z}_2^{E(T(X))}$, while for $i = \#E(T(X)) + 1, \dots, \#N$ the label $\overline{\gamma}_i = \overline{\gamma}_i\big(\overline{\gamma}_1, \dots, \overline{\gamma}_{\#E(T(X))}\big)$ is a function in these: Let e_i, $i = \#E(T(X)) + 1, \dots, \#N$, be an edge incident to the vertices v and v' (where $v = v'$ if e_i is a loop). Since $T(X)$ is a spanning tree there exists a unique path $e_{i'_1}, \dots, e_{i'_l}$ of edges in $E(T(X))$ connecting v to v' (in case $v = v'$ it is the empty path). Then $\overline{\gamma}_i\big(\overline{\gamma}_1, \dots, \overline{\gamma}_{\#E(T(X))}\big) = \sum_{k=1}^{l} \overline{\gamma}_{i'_k}$, in particular if e_i is a loop $\overline{\gamma}_i\big(\overline{\gamma}_1, \dots, \overline{\gamma}_{\#E(T(X))}\big) = 0$.

Proposition 3.2.25. [CC03, Proposition 5] and [CCC04, Lemma 4.1.1.] *$\overline{S}_C^0$ is a 0-dimensional scheme supported on*

$$2^{2p} \cdot \left(\sum_{\substack{\Delta \subset \operatorname{sing} C \\ even}} 2^{b_1\left(\Gamma\left(\tilde{X}_\Delta\right)\right)} \right)$$

points, where $p = \sum_j g(C_j^\nu)$ and $\beta : X_\Delta \to C$ is the blow up at $\operatorname{sing} C \setminus \Delta$, $\Delta \subset \operatorname{sing} C$ even. A point $[(X_\Delta, L, b)] \in \overline{S}_C^0$ has multiplicity $2^{b_1(\Sigma(X_\Delta))}$ and the length of $\overline{S}_C^0$ is 2^{2g}.

Proof. The description of the action of the group of inessential automorphisms on the local universal deformation space of a spin curve (X_Δ, L, b) with stable model C directly gives the multiplicity of the point $[(X_\Delta, L, b)] \in \overline{S}_C^0$. It is just the length of the fibre of the map $\mathbb{C}_\tau^{3g-3}/M(\mathrm{Aut}_0(X_\Delta, L, b)) \to \mathbb{C}_t^{3g-3}$ over 0, which is the *order of ramification* of this map at $0 \in \mathbb{C}_\tau^{3g-3}/M(\mathrm{Aut}_0(X_\Delta, L, b))$. The analysis of the action of $M(\mathrm{Aut}_0(X_\Delta, L, b))$ gives that the $\#E(T(X_\Delta))$ coordinates corresponding to edges of the spanning tree $T(X_\Delta)$ do not contribute to the ramification as well as the last $3g - 3 - \#N$ coordinates do not contribute. But the remaining $\#N - \#E(T(X_\Delta))$ coordinates each contribute 2, i.e. the order of ramification is $2^{\#N - \#E(T(X_\Delta))}$. Because of $\#N - \#E(T(X_\Delta)) = \#N - \#V(T(X_\Delta)) + 1 = \#E(\Sigma(X_\Delta)) - \#V(\Sigma(X_\Delta)) + 1 = b_1(\Sigma(X_\Delta))$ the claim on the multiplicity follows.

The only thing not yet proved is the assertion about the length of $\overline{S}_C^0$. The first Betti number of the graph $\Sigma(X_\Delta)$, which is connected, since X_Δ is, can be rewritten as

$$\begin{aligned} b_1(\Sigma(X_\Delta)) &= \#E(\Sigma(X_\Delta)) - \#V(\Sigma(X_\Delta)) + 1 \\ &= \#N - \#CC(\Gamma(\widetilde{X}_\Delta)) + 1 \\ &= \big(\#\operatorname{sing} C - \#\Delta\big) - \underbrace{\big(\#V(\Gamma(C)) - \#V(\Gamma(\widetilde{X}_\Delta))\big)}_{=0} \\ &\qquad + 1 - \#CC(\Gamma(\widetilde{X}_\Delta)) \\ &= \#E(\Gamma(C)) - \#E(\Gamma(\widetilde{X}_\Delta)) - \#V(\Gamma(C)) + \#V(\Gamma(\widetilde{X}_\Delta)) \\ &\qquad + 1 - \#CC(\Gamma(\widetilde{X}_\Delta)) \\ &= b_1(\Gamma(C)) - b_1(\Gamma(\widetilde{X}_\Delta)) \\ &= b_1(\Gamma(X_\Delta)) - b_1(\Gamma(\widetilde{X}_\Delta)) \end{aligned}$$

Therefore the length of $\overline{S}^0_C$, being the sum over the multiplicities of all points, is

$$\begin{aligned} \operatorname{length} \overline{S}^0_C &= \sum_{[(X,L,b)] \in \overline{S}^0_C} \operatorname{mult}[(X,L,b)] \\ &= 2^{2p} \cdot \left(\sum_{\substack{\Delta \subset \operatorname{sing} C \\ \text{even}}} 2^{b_1\left(\Gamma\left(\widetilde{X}_\Delta\right)\right)} \cdot \underbrace{2^{b_1\left(\Gamma\left(X_\Delta\right)\right) - b_1\left(\Gamma\left(\widetilde{X}_\Delta\right)\right)}}_{=\operatorname{mult}[(X_\Delta,L,b)]} \right) \\ &= 2^{2p} \cdot \left(\sum_{\substack{\Delta \subset \operatorname{sing} C \\ \text{even}}} 2^{b_1(\Gamma(C))} \right) \\ &= 2^{2p} \cdot 2^{b_1(\Gamma(C))} \cdot 2^{b_1(\Gamma(C))} \\ &= 2^{2g} \end{aligned}$$

Here the last step is just the genus formula $g = \sum_j g(C^\nu_j) + b_1(\Gamma(C)) = p + b_1(\Gamma(C))$. The second to last step uses the fact that there are $2^{b_1(\Gamma(C))}$ even subsets $\Delta \subset \operatorname{sing} C$. This is true since any subset $\Delta \subset \operatorname{sing} C$ induces a 1-chain $\sum_{P \in \Delta} e(P) \in C^1(\Gamma(C), \mathbb{F}_2)$ in $\Gamma(C)$. Then Δ is even iff the associated 1-chain is a 1-cycle. By definition of the first Betti number as the dimension of the space of 1-cycles there are $2^{b_1(\Gamma(C))}$ 1-cycles. □

Corollary 3.2.26. *$\overline{S}^0_C$ is reduced iff C is of compact type, i.e.* $b_1(\Gamma(C)) = 0$.

Example 3.2.27. Let $[C]$ be a general point in Δ_0 and (X, L, b) a spin curve with stable model $\beta : X \to C$. In case β is the identity, i.e. the even subset Δ corresponding to X is $\operatorname{sing} C = \{P\}$, the map $\mathbb{C}^{3g-3}_\tau \to \mathbb{C}^{3g-3}_t$ is the identity, since there are no exceptional components. The graph $\Sigma(X)$ is just one vertex with no edges and $\operatorname{Aut}_0(X, L, b) = \mathbb{Z}_2^{V(\Sigma(X))} = \mathbb{Z}_2$, i.e. the only inessential automorphisms are $(\mathrm{id}_C, \mathrm{id}_L)$ and $(\mathrm{id}_C, -\mathrm{id}_L)$. These are inessential automorphisms of every spin curve, therefore they deform to all nearby spin curves, the action of $\mathrm{Aut}_0(X, L, b)$ on $\mathbb{C}_\tau^{3g-3}$ is trivial and $[(X, L, b)] \in \overline{S}_C^0$ has multiplicity one.

The only other possible choice for Δ is the empty set, i.e. $\beta : X \to C$ is the blow up of C at the unique node P. In this case the map $\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_t^{3g-3}$ is given by $\tau_1^2 = t_1$ and $\tau_i = t_i$, $i = 2, \dots, 3g-3$. The graph $\Sigma(X)$ has one vertex and one edge, which is a loop at the vertex. The group of inessential automorphisms is still $\mathbb{Z}_2 = \{(\mathrm{id}_X, \pm\mathrm{id}_L)\}$, every inessential automorphism deforms to all nearby curves, therefore the action of $\mathrm{Aut}_0(X, L, b)$ on $\mathbb{C}_\tau^{3g-3}$ is trivial, but the multiplicity of $[(X, L, b)] \in \overline{S}_C^0$ is two since $\tau_1^2 = t_1$.

As a last example choose $[C] \in \Delta_k$, $1 \leq k \leq \left[\frac{g}{2}\right]$ general and a spin curve (X, L, b) with stable model C. Then $\beta : X \to C$ is the blow up at the unique node P of C. The map between the local universal deformation spaces is given by $\tau_1^2 = t_1$ and $\tau_i = t_i$, $i = 2, \dots, 3g-3$. The graph $\Sigma(X)$ has two vertices and an edge joining the two vertices and $\mathrm{Aut}_0(X, L, b) = \{(\mathrm{id}_X, \pm\mathrm{id}_L), (\sigma, \pm(1, -1))\}$. Here σ is the unique automorphism of X lifting the identity on C, such that the automorphisms $L_1^\nu \to L_1^\nu$, given by multiplication with ± 1 in every fibre, and $L_2^\nu \to L_2^\nu$, given by multiplication with ∓ 1 in every fibre, can be extended to an automorphism $\gamma : \sigma^* L \to L$ (see the proof of Proposition 2.2.11). As before the automorphisms $(\mathrm{id}_X, \pm\mathrm{id}_L)$ deform to all nearby spin curves, i.e. $M(\mathrm{id}_X, \pm\mathrm{id}_L) = \mathbb{I}$. The automorphisms $(\sigma, \pm(1, -1))$ deform to all nearby spin curves with the same combinatorial type, i.e. to all nearby spin curves over the locus $\{\tau_1 = 0\}$, therefore

$$M(\sigma, \pm(1, -1)) = \begin{pmatrix} M_N & \\ & \mathbb{I} \end{pmatrix}$$

and $M_N = (\pm 1)$ is a 1×1-matrix with entry 1 or -1. But since the automorphism $(\sigma, \pm(1, -1))$ cannot deform to a smooth nearby spin curve, i.e. one over a point in $W_\tau(P)$, this last entry cannot be 1. Hence

$$M(\sigma, \pm(1, -1)) = \begin{pmatrix} -1 & \\ & \mathbb{I} \end{pmatrix}$$

and the map $\mathbb{C}_\tau^{3g-3}/M(\mathrm{Aut}_0(X, L, b)) \to \mathbb{C}_t^{3g-3}$ is an isomorphism.

Remember that $\overline{S}_C$ is the fibre of the morphism $\pi : \overline{S}_g \to \overline{M}_g$ over the point $[C]$. The local description of the moduli spaces $\overline{S}_g$ and $\overline{M}_g$ gives the following description of the scheme structure of $\overline{S}_C$ at a point $[(X, L, b)] \in \overline{S}_C$.

Corollary 3.2.28. *Let (X, L, b) be a spin curve of genus $g \geq 2$ with stable model $\beta : X \to C$. Then the connected component of $\overline{S}_C$ corresponding to the spin curve (X, L, b) is the fibre of $\mathbb{C}_\tau^{3g-3}/M(\mathrm{Aut}(X, L, b)) \to \mathbb{C}_t^{3g-3}/M(\mathrm{Aut}\, C)$ over 0. $[(X, L, b)] \in \overline{S}_C$ has multiplicity one iff C is of compact type and every automorphism $\sigma_C \in \mathrm{Aut}\, C$ lifts to an automorphism $(\sigma, \gamma) \in \mathrm{Aut}(X, L, b)$.*

Corollary 3.2.29. *Let C be a stable curve. $\overline{S}_C$ is reduced iff C is of compact type and every automorphism $\sigma_C \in \operatorname{Aut} C$ lifts to every spin curve (X, L, b) with stable model C.*

Remark 3.2.30. Since $\overline{S}_C$ is the fibre of the forgetful morphism $\pi : \overline{S}_g \to \overline{M}_g$ over the point $[C] \in \overline{M}_g$ the corollary says that π is unramified over C iff C is of compact type and every automorphism $\sigma_C \in \operatorname{Aut} C$ lifts to every spin curve (X, L, b) with stable model C.

Example 3.2.31. The easiest example of a stable curve C with reduced $\overline{S}_C$ is of course a smooth curve without automorphisms. A first non-trivial example appeared already in Example 2.2.10. If C is a general hyperelliptic curve of genus $g \geq 2$, its automorphism group is $\operatorname{Aut} C = \{\mathrm{id}_C, \iota\}$, where ι is the hyperelliptic involution. Both automorphisms lift to every theta characteristic L on C. $\mathbb{C}^{3g-3}_\tau = \mathbb{C}^{3g-3}_t$ and $M(\sigma, \gamma) = M(\sigma_C)$ for all automorphisms $(\sigma, \gamma) \in \operatorname{Aut}(C, L, b)$. The action of $(\mathrm{id}_C, \pm\mathrm{id}_L)$ and id_C are trivial, while the action of $(\iota, \pm\gamma)$ and ι, where $\pm\gamma$ are the two isomorphisms $\iota^* L \to L$ compatible with the maps to the canonical bundle, are

$$M(\iota, \pm\gamma) = M(\iota) = \begin{pmatrix} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

where the number of $+1$'s in the matrix is $2g-1$, the dimension of the locus of hyperelliptic curves in M_g. Therefore $\mathbb{C}^{3g-3}_\tau / M(\operatorname{Aut}(C, L, b)) = \mathbb{C}^{3g-3}_t / M(\operatorname{Aut} C)$ for every theta characteristic L.

An example with C a singular curve can be given by curves with an elliptic tail, see Example 2.2.24. Let $[C] \in \Delta_1$ be a general point, i.e. C has two smooth irreducible components C_1 and C_2 of genera $g-1$ and 1 respectively joined at a node P, C_2 is then called an *elliptic tail*. Since the curve is general, the automorphism group of the pointed curve (C_1, P^+) is trivial and that of (C_2, P^-) is $\{\mathrm{id}_{C_2}, \iota\}$, where ι now denotes the elliptic involution fixing P^-. Then $\operatorname{Aut} C = \{\mathrm{id}_C, \iota\}$ (extend ι by the identity on C_1) and ι acts as

$$M(\iota) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

on $\mathbb{C}^{3g-3}_t$, since ι deforms to all nearby stable curves with elliptic tails, i.e. fixes the locus $t_1 = 0$, but not to the smoothings of P.

Example 2.2.24 showed that ι lifts to every spin curve (X, L, b) with support the blow up $\beta : X \to C$ at P, in particular $\iota^* L_2^\nu \cong L_2^\nu$ where $L_j^\nu = \nu^* L_{|C_j}$ and $\nu : C_1 \amalg C_2 = C^\nu \to C$ is the normalisation. Corollary 2.2.12 in this situation gives the following exact sequence

$$0 \longrightarrow \mathbb{Z}_2^{V(\Sigma(X))} \longrightarrow \operatorname{Aut}(X, L, b) \xrightarrow{\psi} \underbrace{\left\{\sigma_C \in \operatorname{Aut}(C; P) \,\middle|\, \widetilde{\sigma}^* \widetilde{L} \cong \widetilde{L}\right\}}_{= \operatorname{Aut} C} \longrightarrow 0,$$

where $\widetilde{L} = L_{|\widetilde{X}}$ is just the pair (L_1^ν, L_2^ν) of line bundles, $\widetilde{\sigma}$ is the unique lift of σ_C to $\widetilde{X}$ and $\Sigma(X)$ is a graph with two vertices joined by an edge, i.e. id_C and ι each have four lifts to automorphisms in $\operatorname{Aut}(X, L, b)$. Those of id_C are canonically given as $(\mathrm{id}_X, \pm(1,1))$ and $(\sigma, \pm(1,-1))$. The lifts of ι are the following. Let $\pm\gamma_{|C_2} : \iota^* L_2^\nu \to L_2^\nu$ be the two unique isomorphisms compatible with the homomorphisms to the canonical bundle, on C_1 the isomorphisms are $\pm \mathrm{id}_{L_1^\nu} : \mathrm{id}_{C_1}^* L_1^\nu \to L_1^\nu$. Therefore there are four isomorphisms $\widetilde{\gamma} : \iota^* \widetilde{L} \to \widetilde{L}$ compatible with the canonical bundle, namely $\widetilde{\gamma} \in \{\pm(\mathrm{id}_{L_1^\nu}, \gamma_{|C_2}), \pm(\mathrm{id}_{L_1^\nu}, -\gamma_{|C_2})\}$. Let ι_1 resp. ι_2 be the unique automorphism of X such that $\pm(\mathrm{id}_{L_1^\nu}, \gamma_{|C_2})$ resp. $\pm(\mathrm{id}_{L_1^\nu}, -\gamma_{|C_2})$ can be extended to an isomorphism $\pm\gamma_1 : \iota^* L \to L$ resp. $\pm\gamma_2 : \iota^* L \to L$. With this notation

$$\operatorname{Aut}(X, L, b) = \{(\mathrm{id}_X, \pm(1,1)), (\sigma, \pm(1,-1)), (\iota_1, \pm\gamma_1), (\iota_2, \pm\gamma_2)\}.$$

The actions of the inessential automorphisms on $\mathbb{C}_\tau^{3g-3}$ are $M(\mathrm{id}_X, \pm(1,1)) = \mathbb{I}$ and

$$M(\sigma, \pm(1,-1)) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

The lifts of ι act as $\tau_k = t_k \mapsto t_k = \tau_k$ for $k = 2, \ldots, 3g-3$, and since ι acts as $t_1 \mapsto -t_1$ and $\tau_1^2 = t_1$ the lifts have to map $\tau_1 \mapsto \pm i\tau_1$, where $i^2 = -1$, i.e.

$$M(\iota_1, \pm\gamma_1) = \begin{pmatrix} i & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \quad \text{and} \quad M(\iota_2, \pm\gamma_2) = \begin{pmatrix} -i & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

(It might be necessary to interchange $(\iota_1, \pm\gamma_1)$ and $(\iota_2, \pm\gamma_2)$.) The only interesting coordinates are therefore t_1 and τ_1, where $\tau_1^2 = t_1$, the action of $\operatorname{Aut} C$ is generated by $t_1 \mapsto -t_1$, while the action of $\operatorname{Aut}(X, L, b)$ is generated by $\tau_1 \mapsto i\tau_1$. These two quotients are naturally isomorphic to $\mathbb{C}_{t^2}^1$ and therefore $\mathbb{C}_\tau^{3g-3}/M(\operatorname{Aut}(X, L, b)) \cong \mathbb{C}_t^{3g-3}/M(\operatorname{Aut} C) \cong \mathbb{C}_u^{3g-3}$ where the u-coordinates are given by $u_1 = t_1^2 = \tau_1^4$ and $u_k = t_k = \tau_k$ for $k = 2, \ldots, 3g-3$. This shows that in this case $\overline{S}_C$ is reduced.

From the above description of the fibre $\overline{S}_C$ of $\pi : \overline{S}_g \to \overline{M}_g$ over a point $[C]$ the ramification divisor of the map π can easily be deduced. The *ramification divisor* of π is defined in the following way. Consider the restriction of π to the smooth loci of $\overline{S}_g$ and $\overline{M}_g$, i.e. let $\overline{M}_g^{\text{reg}} \subset \overline{M}_g$ resp. $\overline{S}_g^{\text{reg}} \subset \overline{S}_g$ be the smooth locus and restrict π to $\overline{S}_g^{\text{reg}} \cap \pi^{-1}(\overline{M}_g^{\text{reg}})$ ($\overline{S}_g^{\text{reg}}$ will be analysed in Theorem 4.3.27). The complement of this locus has codimension at least two since both moduli spaces are normal. Then the ramification divisor of the restricted map is a finite $\mathbb{Z}$-linear combination of irreducible subvarieties of $\overline{S}_g^{\text{reg}} \cap \pi^{-1}(\overline{M}_g^{\text{reg}})$ of codimension 1 such that if e_P is the order of ramification of π at a general point P in such a subvariety D, then $e_P - 1$ is the coefficient of D in the ramification divisor. Replacing every D in this linear combination by its closure in $\overline{S}_g$ gives the ramification divisor of π. This is well defined since the complement of $\overline{S}_g^{\text{reg}} \cap \pi^{-1}(\overline{M}_g^{\text{reg}})$ in $\overline{S}_g$ has codimension at least two.

Corollary 3.2.32. *The ramification divisor of* $\pi : \overline{S}_g \to \overline{M}_g$ *for* $g \geq 4$ *is* $B_0^+ + B_0^-$.

Proof. Since the locus $\{[C] \in M_g | \operatorname{Aut} C \neq \{\mathrm{id}_C\}\}$ has codimension $g-2$ (see for example [HM98]) and a smooth curve is of compact type the restriction $\pi : S_g \to M_g$ has no ramification divisor. Therefore the ramification divisor is a linear combination of the irreducible divisors A_i^+, A_i^-, B_i^+ and B_i^- (see Proposition 3.1.15). Hence it is enough to calculate the ramification of π at the general point $[(X, L, b)]$ of these divisors. For $i \geq 2$ the stable model C of the support of the general spin curve in $A_i^\pm$ and $B_i^\pm$ is of compact type and has trivial automorphism group, therefore Corollary 3.2.29 yields that $\overline{S}_C$ is reduced. Hence π is not ramified at $[(X, L, b)]$. For $i = 1$ by Example 2.2.24 every automorphism of the stable model C lifts to (X, L, b). Since C is of compact type $\overline{S}_C$ is reduced and π not ramified at $[(X, L, b)]$. Let $[(X, L, b)] \in A_0^\pm$ be the general point. By definition $X = C$, i.e. there is no exceptional component. Hence $\mathbb{C}_\tau^{3g-3} = \mathbb{C}_t^{3g-3}$. The respective automorphism groups are $\operatorname{Aut} C = \{\mathrm{id}_C\}$ and $\operatorname{Aut}(X, L, b) = \{(\mathrm{id}_C, \pm\,\mathrm{id}_L)\}$ and act both trivially. Therefore π is locally $\mathbb{C}_\tau^{3g-3} = \mathbb{C}_\tau^{3g-3}/M(\operatorname{Aut}(X, L, b)) \to \mathbb{C}_t^{3g-3}/M(\operatorname{Aut} C) = \mathbb{C}_t^{3g-3}$, i.e. the identity, and π is not ramified at $[(X, L, b)]$.

Now let $[(X, L, b)]$ be the general point of B_0^+ or B_0^-. Then $\beta : X \to C$ is the blow up at the only node of C, which is irreducible. The map $\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_t^{3g-3}$ is given by $\tau_1^2 = t_1$ and $\tau_i = t_i$ else. The automorphism group of C is trivial, hence $\mathbb{C}_t^{3g-3}/M(\operatorname{Aut} C) = \mathbb{C}_t^{3g-3}$. The automorphism group of (X, L, b) is $\{(\mathrm{id}_X, \pm\,\mathrm{id}_L)\}$ which acts trivially, i.e. locally π is $\mathbb{C}_\tau^{3g-3} = \mathbb{C}_\tau^{3g-3}/M(\operatorname{Aut}(X, L, b)) \to \mathbb{C}_t^{3g-3}$. This map is simply ramified along the locus $\tau_1 = 0$ which is the local equation of $B_0^\pm$. Therefore the ramification divisor is $B_0^+ + B_0^-$. □

Chapter 4

Singularities

This chapter is dedicated to the main result, the locus of canonical singularities of the moduli space of spin curves. In the first section the background on canonical singularities is given. Afterwards the singularities of $\overline{M}_g$ as analysed in the article [HM82] by Harris and Mumford are described. In the third section the same analysis is performed for $\overline{S}_g$.

4.1 Canonical singularities

In this section X denotes a normal, quasi projective variety of dimension n over $\mathbb{C}$. Then X has a dualizing sheaf ω_X, which can be constructed as $j_*(\Omega^n_{X^{\mathrm{reg}}})$ where $j : X^{\mathrm{reg}} \hookrightarrow X$ is the inclusion of the smooth locus X^{reg} of X, and there exists a Weil divisor K_X on X such that $\omega_X = \mathcal{O}_X(K_X)$, see for example [Rei87].

Definition 4.1.1. X has *canonical singularities* if

(i) for some integer $r \geq 1$ the Weil divisor rK_X is Cartier and
(ii) if $f : \widetilde{X} \to X$ is a desingularisation of X and $\{E_i\}$ the family of all exceptional prime divisors of f, then

$$rK_{\widetilde{X}} = f^*(rK_X) + \sum a_i E_i$$

with all $a_i \geq 0$.

A point $x \in X$ is a *canonical singularity* if there exists a neighbourhood of x which has canonical singularities.

For a quotient singularity $0 \in V/G = X$ where $V = \mathbb{C}^m$ for some m and $G \subset \mathrm{GL}(V)$ is a finite group, there are some criteria to determine whether 0 is a canonical singularity. Let ω be a G-invariant pluricanonical form on V, i.e. $\omega \in$

$H^0\left(V, \mathcal{O}_V\left(kK_V\right)\right)^G$ for some integer k. Then X has canonical singularities if and only if every such ω lifts holomorphically to one (hence every) desingularisation $\widetilde{X}$, i.e. ω considered as a meromorphic form on $\widetilde{X}$ does not have poles on any exceptional divisor. Denote by $X_M = V/\langle M\rangle$ the quotient by the cyclic subgroup generated by a matrix $M \in G$ and by $\widetilde{X}_M$ a desingularisation of X_M.

Proposition 4.1.2. *A G-invariant pluricanonical form ω on V extends to $\widetilde{X}$ if and only if ω extends to $\widetilde{X}_M$ for every $M \in G$.*

Proof. [Tai82, Proposition 3.1.] □

If for some $M \in G$ the quotient X_M has canonical singularities that means that every $\langle M\rangle$-invariant pluricanonical form extends to $\widetilde{X}_M$. Since every G-invariant form is invariant under $\langle M\rangle$ this shows

Proposition 4.1.3. *If for every $M \in G$, X_M has canonical singularities then X has canonical singularities.*

In general this is not an equivalence:

Example 4.1.4. Consider the group $G \subset \mathrm{GL}\left(\mathbb{C}^2\right)$ generated by

$$\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$$

where $\zeta = e^{2\pi i/3}$. The quotient $\mathbb{C}^2/G$ is smooth

$$\begin{aligned} \mathbb{C}^2/G &\xrightarrow{\cong} \mathbb{C}^2 \\ [(x_1, x_2)] &\longmapsto \left(x_1^3, x_2^3\right) \end{aligned}$$

where $[x]$ denotes the equivalence class of x. But the quotient by the cyclic subgroup generated by

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}$$

is the cone over the twisted cubic curve, and has a non-canonical singularity at 0 (see [Rei87, Example (1.9)(2)]).

But this phenomenon is due to the existence of so called quasi reflections in G.

Definition 4.1.5. An invertible $(m \times m)$-matrix of finite order is a *quasi reflection* if 1 is an eigenvalue of multiplicity exactly $m-1$, in particular the identity matrix is not a quasi reflection.

Proposition 4.1.6. *If there are no quasi reflections in G the quotient X has canonical singularities iff X_M does for every $M \in G$.*

Proof. [Rei80, Remark 3.2.] □

On the other hand the quotient of V by a group H generated by quasi reflections is always smooth, because there is an isomorphism $V/H \cong W = \mathbb{C}^m$ (see [Pri67]). In general one has

Proposition 4.1.7. *Let $G \subset \mathrm{GL}(V)$ be any finite group and denote by H the subgroup generated by quasi reflections. Then H is a normal subgroup of G and there exists $W = \mathbb{C}^m$ and a finite group $K \subset \mathrm{GL}(W)$ containing no quasi reflections such that*

$$\begin{array}{ccccc} V & \longrightarrow & V/H & \xrightarrow{\cong} & W \\ \downarrow & & \downarrow & & \downarrow \\ V/G & \xrightarrow{\cong} & (V/H)\,/\,(G/H) & \xrightarrow{\cong} & W/K \end{array}$$

commutes.

Proof. [Pri67, Proposition 6] □

Remark 4.1.8. Let $G \subset \mathrm{GL}(V)$ be a finite subgroup and $H \subset G$ the normal subgroup generated by quasi reflections. Assume there is a morphism $\varphi : V \to W$, where $W = \mathbb{C}^m$, inducing an isomorphism $\overline{\varphi} : V/H \to W$ and a group homomorphism $\psi : G \to K$, $K \subset \mathrm{GL}(W)$ a finite subgroup, which induces an isomorphism $\overline{\psi} : G/H \to K$. Assume furthermore, that the actions of G on V and K on W are compatible, i.e.

$$\begin{array}{ccc} G \times V & \xrightarrow{(\psi,\varphi)} & K \times W \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi} & W \end{array}$$

commutes. Then it is easy to see that $V/G \cong W/K$. Furthermore K contains no quasi reflections: Assume there is a quasi reflection $k \in K$, i.e. the fixed point locus $\mathrm{Fix}(k)$ has codimension one in W. Let $\psi^{-1}(k) = gH \subset G$. Then a short calculation yields

$$\varphi^{-1}(\mathrm{Fix}(k)) = \bigcup_{h \in H} \mathrm{Fix}(gh)$$

and this locus has codimension one in V. Since H is finite there exists an $h \in H$ such that $\mathrm{Fix}(gh)$ has codimension one. In particular gh is a quasi reflection and $gH = H$. But this implies that $k = \mathrm{id}$, which gives a contradiction to k being a quasi reflection.

The Reid-Tai criterion gives a necessary and sufficient condition for a group without quasi reflections to have canonical singularities.

Definition 4.1.9. Let $M \in \mathrm{GL}(V)$ be an element of finite order $n = \operatorname{ord} M$, ζ a primitive nth root of 1 and

$$M \sim \begin{pmatrix} \zeta^{a_1} & & \\ & \ddots & \\ & & \zeta^{a_m} \end{pmatrix}$$

for appropriate $0 \leq a_i < n$. Then

$$\frac{1}{n} \sum_{j=1}^{m} a_j$$

is called the *Reid-Tai sum* of M with respect to the root ζ.

Remark 4.1.10. Since the group $G \subset \mathrm{GL}(V)$ considered is always finite, every element $M \in G$ has finite order and is therefore diagonalisable.

Theorem 4.1.11 (Reid-Tai criterion)**.** [Tai82, Theorem 3.3.], [Rei80, Theorem 3.1.] *Let $G \subset \mathrm{GL}(V)$ be a finite group without quasi reflections. The quotient V/G has canonical singularities if and only if for every $M \in G$, $M \neq \mathbb{I}$, and every primitive* $\operatorname{ord} M$*th root ζ of* 1 *the Reid-Tai sum fulfils the Reid-Tai inequality*

$$\frac{1}{n} \sum_{j=1}^{m} a_j \geq 1.$$

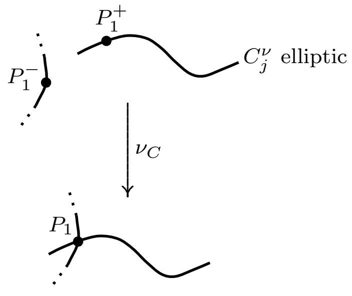
4.2 Singularities of $\overline{M}_g$

In their article [HM82] Harris and Mumford analysed the singularities of $\overline{M}_g$. Their results will be summarized in this section.

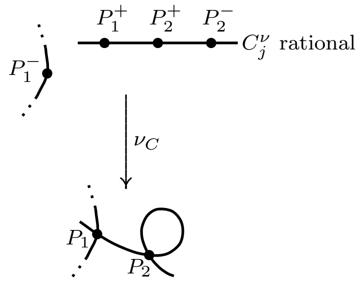
Definition 4.2.1. Let C be a stable curve of genus $g \geq 2$.

(i) An irreducible component C_j of C is an *elliptic tail* if C_j has genus 1 and meets the rest of the curve in exactly one point P, i.e. $(C_j^\nu, \{P_i^\pm\})$ is a smooth elliptic curve with exactly one marked point (*smooth elliptic tail*) or $(C_j^\nu, \{P_i^\pm\})$ is rational with exactly three marked points, where two are mapped to the same node (*singular elliptic tail*). The node P is called the *elliptic tail node corresponding to C_j.*

(ii) An automorphism σ_C of C is an *elliptic tail automorphism of order $n \geq 2$* if C has an elliptic tail C_j such that σ_C is the identity on $\overline{C \setminus C_j}$ and has order n on C_j.



smooth elliptic tail



singular elliptic tail

Remark 4.2.2. Let C_j be an elliptic tail, P the corresponding elliptic tail node and σ_C an elliptic tail automorphism of order $n \geq 2$, such that $\varphi_j = \sigma_{C|C_j} \neq \mathrm{id}$ and $\sigma_{C|\overline{C \setminus C_j}} = \mathrm{id}$. Then P is a fixed point of σ_C and $(C_j^\nu, \{P_i^\pm\}, \varphi_j)$ is of one of the following types.

(i) $(C_j^\nu, \{P_i^\pm\})$ is rational with three marked points, say P_1^+ and $P_2^\pm$. φ_j is the unique automorphism fixing P_1^+ and interchanging P_2^+ and P_2^-, otherwise all three points would be fixed and φ_j would be the identity. Therefore $n = \operatorname{ord} \varphi_j = \operatorname{ord} \sigma_C = 2$.
(ii) $(C_j^\nu, \{P_i^\pm\})$ is any elliptic curve with one marked point, say P_1^+, and φ_j is the elliptic involution with respect to P_1^+, $n = 2$.
(iii) $(C_j^\nu, \{P_i^\pm\})$ is an elliptic curve with j-invariant 1728 and one marked point, say P_1^+, there is an isomorphism $C_j^\nu \cong \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z}i + \mathbb{Z}$ such that φ_j is one of the two order 4 automorphisms of $\mathbb{C}/\Lambda$ and P_1^+ is a fixed point of φ_j, in particular $n = 4$.
(iv) $(C_j^\nu, \{P_i^\pm\})$ is an elliptic curve with j-invariant 0 and one marked point, say P_1^+, there is an isomorphism $C_j^\nu \cong \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z}e^{2\pi i/3} + \mathbb{Z}$ such that φ_j is one of the two order 3 automorphisms of $\mathbb{C}/\Lambda$ and P_1^+ is a fixed point of φ_j, in particular $n = 3$.
(v) $(C_j^\nu, \{P_i^\pm\})$ is an elliptic curve with j-invariant 0 and one marked point, say P_1^+, there is an isomorphism $C_j^\nu \cong \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z}e^{2\pi i/3} + \mathbb{Z}$ such that φ_j is one of the two order 6 automorphisms of $\mathbb{C}/\Lambda$ and P_1^+ is a fixed point of φ_j, in particular $n = 6$.

Theorem 4.2.3. *Let C be a stable curve of genus $g \geq 4$ and $\sigma_C \in \operatorname{Aut} C$ an automorphism. Let n be the order of the matrix $M(\sigma_C)$, ζ any primitive nth root of 1 and*

$$M(\sigma_C) \sim \begin{pmatrix} \zeta^{a_1} & & \\ & \ddots & \\ & & \zeta^{a_{3g-3}} \end{pmatrix}$$

where $0 \leq a_j < n$*. Then either* $\sigma_C = \mathrm{id}_C$ *or*

$$\frac{1}{n}\sum_{j=1}^{3g-3} a_j \geq 1$$

or σ_C *is an elliptic tail automorphism of order* $2, 4, 3$ *or* 6*.*

Proof. [HM82, Theorem 2] □

Corollary 4.2.4. *For* $g \geq 4$ $\overline{M}_g \setminus \Delta_1$ *has canonical singularities.*

Proposition 4.2.5. *In the notation of the last proposition let* σ_C *be an elliptic tail automorphism. Let* t_1 *be the coordinate corresponding to the elliptic tail node* P_1 *and* C_j *the elliptic tail. If* C_j *is smooth, let* t_2 *be the coordinate of* $W_t(C_j)$*. If* C_j *is a singular elliptic, tail let* t_2 *be the coordinate corresponding to the node* P_2 *of* C_j*. Then*

$$M(\sigma_C) = \begin{cases} \begin{pmatrix} -1 & & \\ & 1 & \\ & & \mathbb{I} \end{pmatrix} & n = 2 \\ \begin{pmatrix} \pm i & & \\ & -1 & \\ & & \mathbb{I} \end{pmatrix} & n = 4 \\ \begin{pmatrix} \varrho^2 & & \\ & \varrho^4 & \\ & & \mathbb{I} \end{pmatrix} \text{ or } \begin{pmatrix} \varrho^4 & & \\ & \varrho^2 & \\ & & \mathbb{I} \end{pmatrix} & n = 3 \\ \begin{pmatrix} \varrho^5 & & \\ & \varrho^4 & \\ & & \mathbb{I} \end{pmatrix} \text{ or } \begin{pmatrix} \varrho & & \\ & \varrho^2 & \\ & & \mathbb{I} \end{pmatrix} & n = 6 \end{cases}$$

where $\varrho = e^{2\pi i/6}$*.*

Proof. Consider the singular case, i.e. C_j^ν is rational with three marked points P_1^+, $P_2^\pm$. Then P_1^+ is a fixed point of $\varphi_j = \sigma^\nu_{C|C_j^\nu}$, while $P_2^\pm$ are interchanged by φ_j. t_1 and t_2 are the coordinates corresponding to the nodes P_1 and P_2. Since $\dim W_t(C_j) = 3\cdot 0 - 3 + 3 = 0$ all other coordinates t_i, $i \neq 1, 2$, correspond to nodes or components where σ_C is the identity, i.e. $t_i \mapsto t_i$. Let $xy = 0$ be a local equation for C at P_1 such that x is a coordinate on C_j. Since φ_j is multiplication by -1 on C_j^ν one has $x \mapsto -x$. Since σ_C is the identity on $\overline{C \setminus C_j}$ $y \mapsto y$, i.e. $t_1 = xy \mapsto -xy = -t_1$. Now let $xy = 0$ be a local equation for C at P_2 such that $x \mapsto y$, which is possible since P_2^+ and P_2^- are interchanged. Since the order of σ_C is two σ_C must map $y \mapsto x$. Hence $t_2 = xy \mapsto yx = t_2$ and

$$M(\sigma_C) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \mathbb{I} \end{pmatrix}.$$

In the smooth case the dimension of $W_t(C_j)$ is $3 \cdot 1 - 3 + 1 = 1$, since C_j is a smooth elliptic tail. Denote by P_1^+ the preimage of P_1 in the normalisation C_j^ν. There is an isomorphism $(C_j^\nu, P_1^+) \cong (\mathbb{C}/\Lambda_{z_0}, 0)$ where $\Lambda_{z_0} = \mathbb{Z}z_0 + \mathbb{Z}$ for a suitable z_0 in the upper half plane

$$\mathbb{H}_1 = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}.$$

If $n = 4$ resp. $n = 3, 6$ one can choose $z_0 = i$ resp. $z_0 = \varrho^2$ with $\varrho = e^{2\pi i/6}$. $\mathbb{H}_1$ is the local universal deformation space of (C_j^ν, P_1^+) and hence t_2 is a coordinate of the tangent space $T_{z_0}\mathbb{H}_1$. The aim is to calculate the action of the automorphism $\varphi_j : \mathbb{C}/\Lambda_{z_0} \to \mathbb{C}/\Lambda_{z_0}$ on $T_{z_0}\mathbb{H}_1$. φ_j is induced by the analytic representation $A : \mathbb{C} \to \mathbb{C}$, $w \mapsto Aw$, where

$$A = \begin{cases} -1 & n = 2 \\ \pm i & n = 4 \\ \varrho^2 \text{ or } \varrho^4 & n = 3 \\ \varrho \text{ or } \varrho^5 & n = 6 \end{cases}$$

see for example [BL04]. The rational representation, i.e. the restriction of A to the lattice Λ_{z_0}, is given (with respect to the $\mathbb{Z}$-basis $(z_0, 1)$) by the following matrix $R \in \mathrm{SL}_2(\mathbb{Z})$.

$$R = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & A = -1 \\ \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & A = \pm i \\ \pm\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} & A = \pm\varrho \\ \pm\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} & A = \pm\varrho^2 \end{cases}$$

For example if $A = \varrho$ then $z_0 = \varrho^2$ and since ϱ is a primitive 6th root of 1, i.e. $\varrho^3 = -1$, $\varrho^2 + 1 = \varrho$,

$$\begin{aligned} \varrho^2 &\xmapsto{\cdot\varrho} \varrho^3 = 0 \cdot \varrho^2 + (-1) \cdot 1 \\ 1 &\xmapsto{\cdot\varrho} \varrho \;\; = 1 \cdot \varrho^2 + 1 \cdot 1 \end{aligned}$$

and therefore $R = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$.

Now let $\pi : \mathcal{X} \to \mathbb{H}_1$ be the family of elliptic curves over $\mathbb{H}_1$, which is locally at every point $z \in \mathbb{H}_1$ the local universal deformation of the fibre $\pi^{-1}(z) = \mathbb{C}/\Lambda_z$, i.e. $\mathcal{X} = \mathbb{C} \times \mathbb{H}_1/\mathbb{Z}^2$, where an element $(m, n) \in \mathbb{Z}^2$ acts as $(w, z) \mapsto (w + mz + n, z)$

on $(w, z) \in \mathbb{C} \times \mathbb{H}_1$ (see [BL04, Section 8.7.]). The automorphism φ_j of the fibre $\pi^{-1}(z_0) = \mathbb{C}/\Lambda_{z_0}$ has to be extended to $\mathcal{X}$. The extension φ_j maps fibres to fibres, hence for every $z \in \mathbb{H}_1$ there exists $z' =: \varphi_j(z) \in \mathbb{H}_1$ such that

$$\varphi_j : \pi^{-1}(z) = \mathbb{C}/\Lambda_z \xrightarrow{\cong} \pi^{-1}(z') = \mathbb{C}/\Lambda_{z'}.$$

The analytic representation of this map is multiplication by A and hence the rational representation is R, i.e.

$$\begin{aligned} \Lambda_z &\longrightarrow \Lambda_{z'} \\ z &\longmapsto Az = az' + b \\ 1 &\longmapsto A = cz' + d \end{aligned}$$

implying

$$z = \frac{Az}{A} = \frac{az' + b}{cz' + d} =: {}^tR(z').$$

This means that $\varphi_j(z) = z' = {}^tR^{-1}(z) = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)(z) = \frac{dz-b}{-cz+a}$, i.e. the induced action on $\mathbb{H}_1$ is

$$\begin{aligned} \varphi_j : \mathbb{H}_1 &\longrightarrow \mathbb{H}_1 \\ z &\longmapsto {}^tR^{-1}(z) = \frac{dz - b}{-cz + a} =: f(z). \end{aligned}$$

In the different cases ${}^tR^{-1}$ is

$${}^tR^{-1} = \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & A = -1 \\ \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & A = \pm i \\ \pm\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} & A = \pm\varrho \\ \pm\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} & A = \pm\varrho^2 \end{cases}$$

and

$$f(z) = \begin{cases} z & A = -1 \\ -\frac{1}{z} & A = \pm i \\ -\frac{z+1}{z} & A = \pm\varrho \\ -\frac{1}{z+1} & A = \pm\varrho^2 \end{cases}$$

The action on the tangent space $T_{z_0}\mathbb{H}_1$ is then $t_2 \mapsto f'(z_0)t_2$ with

$$f'(z_0) = \begin{cases} f'(z)_{|z=z_0} = 1 & A = -1 \\ \left(\frac{1}{z^2}\right)_{|z=i} = -1 & A = \pm i \\ \left(\frac{1}{z^2}\right)_{|z=\varrho^2} = \varrho^2 & A = \pm\varrho \\ \left(\frac{1}{(z+1)^2}\right)_{|z=\varrho^2} = \varrho^4 & A = \pm\varrho^2 \end{cases}$$

In order to calculate the action on t_1 let $xy = 0$ be a local equation for C at P_1, such that x is a coordinate of the elliptic tail C_j at P_1. Then $t_1 = xy$. Since $\sigma_{C|\overline{C \setminus C_j}} = \mathrm{id}$ the action on the coordinate of $\overline{C \setminus C_j}$ at P_1 is trivial, i.e. $y \mapsto y$. On C_j the automorphism σ_C is given by multiplication with A, therefore $x \mapsto Ax$ and $t_1 = xy \mapsto Axy = At_1$, i.e.

$$t_1 \mapsto \begin{cases} -t_1 & A = -1 \\ \pm i t_1 & A = \pm i \\ \varrho t_1 & A = \varrho \\ \varrho^2 t_1 & A = \varrho^2 \\ \varrho^4 t_1 & A = \varrho^4 \\ \varrho^5 t_1 & A = \varrho^5 \end{cases}$$

This shows that the action of $M(\sigma_C)$ on $\mathbb{C}^2_{t_1,t_2}$ is

$$M(\sigma_C)_{|\mathbb{C}^2_{t_1,t_2}} = \begin{cases} \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} & A = -1 \\ \begin{pmatrix} \pm i & \\ & -1 \end{pmatrix} & A = \pm i \\ \begin{pmatrix} \varrho & \\ & \varrho^2 \end{pmatrix} & A = \varrho \\ \begin{pmatrix} \varrho^2 & \\ & \varrho^4 \end{pmatrix} & A = \varrho^2 \\ \begin{pmatrix} \varrho^4 & \\ & \varrho^2 \end{pmatrix} & A = \varrho^4 \\ \begin{pmatrix} \varrho^5 & \\ & \varrho^4 \end{pmatrix} & A = \varrho^5 \end{cases}$$

All other coordinates t_i, $i \neq 1, 2$, correspond to nodes resp. components where σ_C is the identity, i.e. $t_i \mapsto t_i$. □

Corollary 4.2.6. *Let $g \geq 4$ and $\sigma_C \in \operatorname{Aut} C$ an automorphism of a stable curve of genus g. If $M(\sigma_C)$ is a quasi reflection, then σ_C is an elliptic tail automorphism of order 2.*

Proof. A quasi reflection does not fulfil the Reid-Tai inequality, therefore by Theorem 4.2.3 the non-trivial elliptic tail automorphisms are the only possibilities. For these the last Proposition shows that the only quasi reflection is the reflection induced by the elliptic tail automorphism of order 2. □

Proposition 4.2.7. *Let $g \geq 4$ and $[C] \in \overline{M}_g$. Then $[C]$ is a non-canonical singularity iff C has an elliptic tail with j-invariant 0.*

Idea of proof. Fix a stable curve of genus g and consider $M(\operatorname{Aut} C) \subset \mathrm{GL}(\mathbb{C}_t^{3g-3})$. The subgroup generated by quasi reflections is

$$H(\operatorname{Aut} C) = \left\langle \begin{pmatrix} \mathbb{I} & & \\ & -1 & \\ & & \mathbb{I} \end{pmatrix} \leftarrow i \;\middle|\; t_i \text{ corresponds to the elliptic tail node } P_i \right\rangle$$

Proposition 4.1.7 gives an isomorphism $\mathbb{C}_t^{3g-3}/H(\operatorname{Aut} C) \cong \mathbb{C}_u^{3g-3}$ with new coordinates $t_i^2 = u_i$ if t_i corresponds to an elliptic tail node P_i, say if $i = 1, \ldots, l$, and $t_i = u_i$ else. Furthermore there is a group $K = K(\operatorname{Aut} C) \subset \mathrm{GL}(\mathbb{C}_u^{3g-3})$ without quasi reflections such that $\mathbb{C}_t^{3g-3}/M(\operatorname{Aut} C) \cong \mathbb{C}_u^{3g-3}/K(\operatorname{Aut} C)$. The matrix $K(\sigma_C)$ is obtained from $M(\sigma_C)$ in the following way. The automorphism σ_C cannot map elliptic tail nodes to other nodes nor the other way around, i.e.

$$M(\sigma_C) = \begin{pmatrix} M_{E_1}\mathbb{E}_{\sigma_{E_1}} & & \\ & M_{E_2}\mathbb{E}_{\sigma_{E_2}} & \\ & & M_V\mathbb{E}_{\sigma_V} \end{pmatrix},$$

where $E_1 \subset E(\Gamma(C))$ is the set of elliptic tail nodes, E_2 its complement and $V = V(\Gamma(C))$ the set of irreducible components. Then

$$K(\sigma_C) = \begin{pmatrix} M_{E_1}^2\mathbb{E}_{\sigma_{E_1}} & & \\ & M_{E_2}\mathbb{E}_{\sigma_{E_2}} & \\ & & M_V\mathbb{E}_{\sigma_V} \end{pmatrix} = \begin{pmatrix} K_{E_1}\mathbb{E}_{\sigma_{E_1}} & & \\ & K_{E_2}\mathbb{E}_{\sigma_{E_2}} & \\ & & K_V\mathbb{E}_{\sigma_V} \end{pmatrix}.$$

With this construction $[C] \in \overline{M}_g$ is non-canonical iff there exists an automorphism $\sigma_C \in \operatorname{Aut} C$ such that $K(\sigma_C) \neq \mathbb{I}$ and there exists an $\operatorname{ord} K(\sigma_C)$th root ζ of 1 such that the Reid-Tai sum of $K(\sigma_C)$ with respect to ζ is smaller than 1.

In [HM82] Harris and Mumford analyse the action of an automorphism such that $M(\sigma_C)$ does not fulfil the Reid-Tai inequality. But they do not consider the action of the group modulo quasireflections. In any case their analysis applies nearly word for word to the matrix $K(\sigma_C)$. The major change appears at the end of the argument, where everything is already reduced to the study of elliptic tail automorphisms. Their argument shows that if $K(\sigma_C)$ does not fulfil the Reid-Tai inequality (for some ζ) then σ_C is an elliptic tail automorphism. From Proposition 4.2.5 it follows that in case $\operatorname{ord}\sigma_C = 2$

$$K(\sigma_C) = \begin{pmatrix} (-1)^2 & & \\ & 1 & \\ & & \mathbb{I} \end{pmatrix},$$

if $\operatorname{ord}\sigma_C = 4$ then

$$K(\sigma_C) = \begin{pmatrix} (\pm i)^2 & & \\ & -1 & \\ & & \mathbb{I} \end{pmatrix},$$

if the order of σ_C is 3

$$K(\sigma_C) = \begin{pmatrix} (\varrho^2)^2 & & \\ & \varrho^4 & \\ & & \mathbb{I} \end{pmatrix} \text{ or } K(\sigma_C) = \begin{pmatrix} (\varrho^4)^2 & & \\ & \varrho^2 & \\ & & \mathbb{I} \end{pmatrix}$$

and finally if the order is 6

$$K(\sigma_C) = \begin{pmatrix} (\varrho^5)^2 & & \\ & \varrho^4 & \\ & & \mathbb{I} \end{pmatrix} \text{ or } K(\sigma_C) = \begin{pmatrix} (\varrho)^2 & & \\ & \varrho^2 & \\ & & \mathbb{I} \end{pmatrix}.$$

In the first case $K(\sigma_C)$ is just the identity, which by the way proves that the general point of Δ_1 is smooth. In the second case the Reid-Tai sum is $\frac{1}{2}+\frac{1}{2}=1$, contradicting the assumption, that the Reid-Tai inequality is not fulfilled. In the other cases the nonzero eigenvalues are ϱ^2, ϱ^2 or ϱ^4, ϱ^4, which gives a Reid-Tai sum of $\frac{1}{3}+\frac{1}{3}$ with respect to the third root ϱ^2 of 1 resp. a Reid-Tai sum of $\frac{1}{3}+\frac{1}{3}$ with respect to the third root ϱ^4 of 1. This shows that if $[C] \in \overline{M}_g$ is non-canonical then it has an elliptic tail with j-invariant 0. The other direction, i.e. "if C has an elliptic tail with j-invariant 0, then $[C]$ is non-canonical" is clear, since then in the group $K(\operatorname{Aut} C)$ there is the matrix $K(\sigma_C)$ of an elliptic tail automorphism of order 3, which does not fulfil the Reid-Tai inequality. Since the group $K(\operatorname{Aut} C)$ does not contain quasi reflections the quotient by $K(\operatorname{Aut} C)$ is non-canonical. □

Corollary 4.2.8. *Let $g \geq 4$. Then $[C] \in \overline{M}_g$ is in the smooth locus $\overline{M}_g^{reg}$ iff* $\operatorname{Aut} C$ *is generated by elliptic tail automorphisms of order* 2.

4.3 Singularities of $\overline{S}_g$

In this section the singularities of $\overline{S}_g$ will be analysed in case $g \geq 4$. Fix a $g \geq 4$ and a spin curve (X, L, b) of genus g.

4.3.1 Quasi reflections in $M(\operatorname{Aut}(X, L, b))$

In order to be able to apply the Reid-Tai criterion, the group divided out by must not contain quasi reflections. Unfortunately, in general the group $M(\operatorname{Aut}(X, L, b))$ contains many quasi reflections. Therefore Proposition 4.1.7 will be used. In this section the subgroup $H(\operatorname{Aut}(X, L, b)) \subset M(\operatorname{Aut}(X, L, b))$ generated by quasi reflections and an isomorphism $\mathbb{C}_\tau^{3g-3}/H(\operatorname{Aut}(X, L, b)) \xrightarrow{\cong} \mathbb{C}_u^{3g-3}$ will be determined. Afterwards the group $K(\operatorname{Aut}(X, L, b)) \subset \mathrm{GL}(\mathbb{C}_u^{3g-3})$ such that

$$\mathbb{C}_\tau^{3g-3}/M(\operatorname{Aut}(X, L, b)) \cong \mathbb{C}_u^{3g-3}/K(\operatorname{Aut}(X, L, b))$$

will be described.

Let $\beta : X \to C$ be the stable model of the fixed spin curve (X, L, b), $(\sigma, \gamma) \in \mathrm{Aut}(X, L, b)$ an automorphism and σ_C the induced automorphism on C.

Proposition 4.3.1. *If $M(\sigma, \gamma) \in M(\mathrm{Aut}(X, L, b))$ is a quasi reflection then either (σ, γ) is an inessential automorphism or σ_C is an elliptic tail automorphism of order* 2.

Proof. Let $M(\sigma, \gamma)$ be a quasi reflection, i.e.

$$M(\sigma, \gamma) \sim \begin{pmatrix} \xi & \\ & \mathbb{I} \end{pmatrix}$$

with $\xi \neq 1$. Choose an ordering of the coordinates $\tau_1, \ldots, \tau_{3g-3}$ as in Remark 3.2.23, i.e. such that

$$M(\sigma, \gamma) = \begin{pmatrix} M_1\mathbb{E}_{\pi_1} & & \\ & \ddots & \\ & & M_l\mathbb{E}_{\pi_l} \end{pmatrix}$$

where the π_i are cyclic permutations of either exceptional nodes, non-exceptional nodes or irreducible components of C. Since $M(\sigma, \gamma)$ is a quasi reflection there exists exactly one block $M_i\mathbb{E}_{\pi_i} \neq \mathbb{I}$, which has to diagonalise as

$$M_i\mathbb{E}_{\pi_i} \sim \begin{pmatrix} \xi & \\ & \mathbb{I} \end{pmatrix}.$$

Suppose this block corresponds to coordinates of some $W_\tau(C_j)$'s, i.e. π_i is a cycle of the permutation σ_V of irreducible components $V = V(\Gamma(C))$. Since all other blocks are identity matrices (especially those corresponding to the exceptional nodes), Proposition 3.2.22 yields $M(\sigma_C) = M(\sigma, \gamma)$, but then $M(\sigma_C)$ is also a quasi reflection. This contradicts Corollary 4.2.6, which implies that all quasi reflections in $M(\mathrm{Aut}\, C)$ fix all the subspaces $W_t(C_j) = W_\tau(C_j)$. Hence the block $M_i\mathbb{E}_{\pi_i}$ corresponds to coordinates of some $W_\tau(P)$'s, i.e. π_i is a cycle of the permutation σ_N or σ_Δ, say

$$M_i\mathbb{E}_{\pi_i} = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_m \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

where $m = \mathrm{ord}\, \pi_i$ and $\alpha_j \in \mathbb{C}^*$. It is an easy exercise that the characteristic polynomial of $M_i\mathbb{E}_{\pi_i}$ is

$$\chi(\lambda) = \lambda^m - \prod_{j=1}^{m} \alpha_j.$$

But that means that this block has as eigenvalues the m different mth roots of $\prod \alpha_j$, being a quasi reflection at the same time implies that either $m = 1$ and $M_i\mathbb{E}_{\pi_i} = (\xi)$ or $m = 2$, $\prod \alpha_j = \alpha_1\alpha_2 = 1$ and

$$M_i\mathbb{E}_{\pi_i} = \begin{pmatrix} 0 & \alpha_1 \\ \frac{1}{\alpha_1} & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose the block is a 2×2-matrix on coordinates τ_1 and τ_2 such that $\tau_i = t_i$, then again $M(\sigma_C) = M(\sigma, \gamma)$ giving a contradiction to Corollary 4.2.6, since all the quasi reflections in $M(\operatorname{Aut} C)$ are already diagonalised in the chosen coordinates. If the coordinates satisfy $\tau_i^2 = t_i$, the matrix $M(\sigma_C)$ has exactly one non-trivial block which is

$$M_i^2 E_{\pi_i} = \begin{pmatrix} 0 & \alpha_1^2 \\ \frac{1}{\alpha_1^2} & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So this case is also impossible.

Hence the block must be a 1×1-matrix, say on the coordinate τ_1. $\tau_1 = t_1$ would imply that $M(\sigma_C) = M(\sigma, \gamma)$ and since this is a quasi reflection t_1 is the coordinate corresponding to the node P connecting an elliptic tail to the rest of the curve. Such a node must be exceptional because of degree reasons, if not the set Δ of non-exceptional nodes would not be even. But for an exceptional node $\tau_1^2 = t_1$, hence $\tau_1 = t_1$ is not possible. Therefore $\tau_1^2 = t_1$

$$M(\sigma, \gamma) = \begin{pmatrix} \xi & \\ & \mathbb{I} \end{pmatrix} \qquad \text{and} \qquad M(\sigma_C) = \begin{pmatrix} \xi^2 & \\ & \mathbb{I} \end{pmatrix}.$$

Proposition 4.2.5 implies that either $\xi^2 = 1$ or $\xi^2 = -1$. In the first case σ_C is the identity hence (σ, γ) inessential, in the second it is an elliptic tail automorphism of order 2. □

Remark 4.3.2. From the description of the action of inessential automorphisms in Remark 3.2.24 it follows that an inessential automorphism (σ, γ) induces a quasi reflection

$$M(\sigma, \gamma) = \begin{pmatrix} -1 & \\ & \mathbb{I} \end{pmatrix}$$

in the coordinate τ_1 iff t_1 corresponds to a disconnecting node P, i.e. the normalisation of C at P has two connected components, and γ is multiplication by 1 over one of the components and by -1 on the other component.

Moreover, if (σ, γ) is an automorphism such that σ_C is an elliptic tail automorphism of order 2, then

$$M(\sigma, \gamma) = \begin{pmatrix} \pm i & \\ & \mathbb{I} \end{pmatrix}$$

where τ_1 is the coordinate corresponding to the exceptional component connecting the elliptic tail on which σ acts non-trivially to the rest of the curve.

Corollary 4.3.3. *The subgroup $H = H(\mathrm{Aut}(X,L,b)) \subset M(\mathrm{Aut}(X,L,b))$ generated by the quasi reflections $M(\sigma,\gamma) \in M(\mathrm{Aut}(X,L,b))$ is*

$$H = \left\langle \begin{pmatrix} \mathbb{I} & & \\ & \xi_j & \\ & & \mathbb{I} \end{pmatrix} \middle| \, j = 1, \ldots, 3g-3 \right\rangle$$

where ξ_j is the (j,j)th entry of the matrix and $\xi_j = i$ if τ_j corresponds to an elliptic tail node P_j, $\xi_j = -1$ if τ_j corresponds to a disconnecting node, which is not an elliptic tail node and $\xi_j = 1$ in all other cases, i.e. if τ_j corresponds to a non-disconnecting node or to some component C_j.

Remark 4.3.4. The set of nodes $\operatorname{sing} C$ of the stable model C of the fixed spin curve (X,L,b) decomposes into the following four disjoint sets.

$$\begin{aligned}
T = T(X,L,b) &= \left\{ P \in \operatorname{sing} C \;\middle|\; P \text{ is an elliptic tail node} \right\} \\
D = D(X,L,b) &= \left\{ P \in \operatorname{sing} C \;\middle|\; \begin{matrix} P \text{ is disconnecting but} \\ \text{not an elliptic tail node} \end{matrix} \right\} \\
\overline{N} = \overline{N}(X,L,b) &= \left\{ P \in \operatorname{sing} C \;\middle|\; \begin{matrix} P \text{ is exceptional for } (X,L,b) \\ \text{but not disconnecting} \end{matrix} \right\} \\
\Delta = \Delta(X,L,b) &= \left\{ P \in \operatorname{sing} C \;\middle|\; P \text{ is non-exceptional for } (X,L,b) \right\}
\end{aligned}$$

The union $T \cup D$ is the set of all disconnecting nodes, since every elliptic tail node is necessarily disconnecting. The union $T \cup D \cup \overline{N}$ is the set N of exceptional nodes, since a disconnecting node is necessarily exceptional for (X,L,b), otherwise the set $\operatorname{sing} C \setminus N$ of non-exceptional nodes would not be even. Every automorphism σ_C of the stable curve C fixes the set T of elliptic tail nodes and the set D. And if σ_C is induced by an automorphism (σ,γ) it also fixes the sets $\overline{N}$ and Δ.

Therefore the coordinates $\tau_1, \ldots, \tau_{3g-3}$ can be ordered such that the first $\#T$ coordinates correspond to elliptic tail nodes, the next $\#D$ to the remaining disconnecting nodes, the next $\#\overline{N}$ to the remaining exceptional nodes, the next $\#\Delta$ to the non-exceptional nodes and the remaining $3g-3-\#\operatorname{sing} C$ coordinates correspond to the irreducible components $V = V(\Gamma(C))$ of C. In these coordinates *every* $M(\sigma,\gamma) \in M(\mathrm{Aut}(X,L,b))$ has the following form

$$M(\sigma,\gamma) = \begin{pmatrix} M_T \mathbb{E}_{\sigma_T} & & & & \\ & M_D \mathbb{E}_{\sigma_D} & & & \\ & & M_{\overline{N}} \mathbb{E}_{\sigma_{\overline{N}}} & & \\ & & & M_\Delta \mathbb{E}_{\sigma_\Delta} & \\ & & & & M_V \mathbb{E}_{\sigma_V} \end{pmatrix}$$

where M_V is block diagonal, $\mathbb{E}_{\sigma_V}$ is the generalized permutation matrix given by the permutation σ_V of the components and any other $\sigma_\star$ is the permutation of the set $\star \in \{T, D, \overline{N}, \Delta\}$ given by σ_C, while $M_\star$ is a diagonal matrix. In these coordinates $H(\mathrm{Aut}(X, L, b))$ is the set of diagonal matrices

$$\begin{pmatrix} \mathbb{D}_T &&&& \\ & \mathbb{D}_D &&& \\ && \mathbb{I} && \\ &&& \mathbb{I} & \\ &&&& \mathbb{I} \end{pmatrix}$$

where $\operatorname{ord}\mathbb{D}_T$ divides 4 and $\operatorname{ord}\mathbb{D}_D$ divides 2.

Fix an ordering of the coordinates as in the above remark, then

$$\begin{aligned} \varphi : \mathbb{C}_\tau^{3g-3}/H(\mathrm{Aut}(X, L, b)) &\longrightarrow \mathbb{C}_u^{3g-3} \\ [(\tau_1, \ldots, \tau_{3g-3})] &\longmapsto \left(\tau_1^{\varepsilon_1}, \ldots, \tau_{3g-3}^{\varepsilon_{3g-3}}\right) \end{aligned}$$

where $\varepsilon_j = \operatorname{ord}\xi_j \in \{1, 2, 4\}$ is an isomorphism. The quotient map $\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_u^{3g-3}$ is given by $u_i = \tau_i^4$ if $i = 1, \ldots, \#T$, i.e. τ_i corresponds to an elliptic tail node, $u_i = \tau_i^2$ if $i = \#T + 1, \ldots, \#T + \#D$, i.e. τ_i corresponds to a disconnecting node, which is not an elliptic tail node and $u_i = \tau_i$ otherwise.

Notation 4.3.5. Denote by $W_u(P) \subset \mathbb{C}_u^{3g-3}$ the image of $W_\tau(P) \subset \mathbb{C}_\tau^{3g-3}$ corresponding to the node $P \in \operatorname{sing} C$ and by $W_u(C_j) \subset \mathbb{C}_u^{3g-3}$ the image of $W_\tau(C_j) \subset \mathbb{C}_\tau^{3g-3}$ corresponding to an irreducible component C_j of C under the quotient map $\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_u^{3g-3}$.

Definition 4.3.6. Let (σ, γ) be an automorphism of (X, L, b) with matrix

$$M(\sigma,\gamma) = \begin{pmatrix} M_T\mathbb{E}_{\sigma_T} &&&& \\ & M_D\mathbb{E}_{\sigma_D} &&& \\ && M_{\overline{N}}\mathbb{E}_{\sigma_{\overline{N}}} && \\ &&& M_\Delta\mathbb{E}_{\sigma_\Delta} & \\ &&&& M_V\mathbb{E}_{\sigma_V} \end{pmatrix}$$

then $K(\sigma, \gamma) \in \mathrm{GL}(\mathbb{C}_u^{3g-3})$ is defined as

$$K(\sigma,\gamma) = \begin{pmatrix} M_T^4\mathbb{E}_{\sigma_T} &&&& \\ & M_D^2\mathbb{E}_{\sigma_D} &&& \\ && M_{\overline{N}}\mathbb{E}_{\sigma_{\overline{N}}} && \\ &&& M_\Delta\mathbb{E}_{\sigma_\Delta} & \\ &&&& M_V\mathbb{E}_{\sigma_V} \end{pmatrix}$$

Define $K_T = M_T^4$, $K_D = M_D^2$ and $K_\star = M_\star$ for $\star \in \{\overline{N}, \Delta, V\}$.

A straightforward calculation shows that the map $M(\mathrm{Aut}(X, L, b)) \to \mathrm{GL}(\mathbb{C}_u^{3g-3})$, $M(\sigma, \gamma) \mapsto K(\sigma, \gamma)$ is a homomorphism, hence $K(\mathrm{Aut}(X, L, b)) \subset \mathrm{GL}(\mathbb{C}_u^{3g-3})$ is a finite subgroup. The homomorphism $\psi : M(\mathrm{Aut}(X, L, b)) \to K(\mathrm{Aut}(X, L, b))$ induces an isomorphism between the quotient $M(\mathrm{Aut}(X, L, b))/H(\mathrm{Aut}(X, L, b))$ and $K(\mathrm{Aut}(X, L, b))$, since by construction $H(\mathrm{Aut}(X, L, b))$ is the kernel of ψ.

Proposition 4.3.7. *$K(\mathrm{Aut}(X,L,b))$ contains no quasi reflections and*

$$\begin{aligned} \mathbb{C}_\tau^{3g-3}/M(\mathrm{Aut}(X,L,b)) &\longrightarrow \mathbb{C}_u^{3g-3}/K(\mathrm{Aut}(X,L,b)) \\ [(\tau_1,\dots,\tau_{3g-3})] &\longmapsto \left[\left(\tau_1^{\varepsilon_1},\dots,\tau_{3g-3}^{\varepsilon_{3g-3}}\right)\right] \end{aligned}$$

is an isomorphism.

Proof. The action of $M(\mathrm{Aut}(X,L,b))$ on $\mathbb{C}_\tau^{3g-3}$ and that of $K(\mathrm{Aut}(X,L,b))$ on $\mathbb{C}_u^{3g-3}$ are compatible by construction. Therefore the maps $\varphi : \mathbb{C}_\tau^{3g-3} \to \mathbb{C}_u^{3g-3}$ and $\psi : M(\mathrm{Aut}(X,L,b)) \to K(\mathrm{Aut}(X,L,b))$ induce an isomorphism

$$\mathbb{C}_\tau^{3g-3}/M(\mathrm{Aut}(X,L,b)) \xrightarrow{\cong} \mathbb{C}_u^{3g-3}/K(\mathrm{Aut}(X,L,b))$$

and $K(\mathrm{Aut}(X,L,b))$ contains no quasi reflections (see Remark 4.1.8). □

Remark 4.3.8. For an automorphism $(\sigma,\gamma) \in \mathrm{Aut}(X,L,b)$ the important matrices are the following.

$$M(\sigma,\gamma) = \begin{pmatrix} M_T\mathbb{E}_{\sigma_T} &&&& \\ & M_D\mathbb{E}_{\sigma_D} &&& \\ && M_{\overline{N}}\mathbb{E}_{\sigma_{\overline{N}}} && \\ &&& M_\Delta\mathbb{E}_{\sigma_\Delta} & \\ &&&& M_V\mathbb{E}_{\sigma_V} \end{pmatrix}$$

$$K(\sigma,\gamma) = \begin{pmatrix} M_T^4\mathbb{E}_{\sigma_T} &&&& \\ & M_D^2\mathbb{E}_{\sigma_D} &&& \\ && M_{\overline{N}}\mathbb{E}_{\sigma_{\overline{N}}} && \\ &&& M_\Delta\mathbb{E}_{\sigma_\Delta} & \\ &&&& M_V\mathbb{E}_{\sigma_V} \end{pmatrix}$$

$$M(\sigma_C) = \begin{pmatrix} M_T^2\mathbb{E}_{\sigma_T} &&&& \\ & M_D^2\mathbb{E}_{\sigma_D} &&& \\ && M_{\overline{N}}^2\mathbb{E}_{\sigma_{\overline{N}}} && \\ &&& M_\Delta\mathbb{E}_{\sigma_\Delta} & \\ &&&& M_V\mathbb{E}_{\sigma_V} \end{pmatrix}$$

Hence knowing $M(\sigma_C)$ gives everything of $K(\sigma,\gamma)$ but the middle block, which depends on choosing square roots of the diagonal elements of the diagonal matrix of $M(\sigma_C)$ in the third block. Which roots are to take depends on the isomorphism γ.

4.3.2 Singularities of $\overline{S}_g$

Theorem 4.3.9. *Let (X, L, b) be a spin curve of genus $g \geq 4$ with stable model $\beta : X \to C$ and normalisation $\nu : X^\nu \to X$. Then $[(X, L, b)] \in \overline{S}_g$ is a non-canonical singularity iff C has an elliptic tail C_j with j-invariant 0 and L is trivial on C_j, i.e. $\nu^* L_{|C_j^\nu} = \mathcal{O}_{C_j^\nu}$.*

Remark 4.3.10. That $\overline{S}_g$ at such a point has a non-canonical singularity follows easily from what has already been said. Let (X, L, b) be a spin curve of genus $g \geq 4$ such that the stable model C has an elliptic tail C_j with j-invariant 0 over which L is trivial. Let σ_C be an automorphism which is the identity on $\overline{C \setminus C_j}$ and has order 6 on C_j. Proposition 4.2.5 then says, that there exists a primitive sixth root ζ_6 of 1 such that

$$M(\sigma_C) = \begin{pmatrix} \zeta_6 & & \\ & \mathbb{I} & \\ & & \zeta_6^2 \end{pmatrix}$$

where the first entry corresponds to the elliptic tail node and the last to the elliptic tail itself.

All automorphisms of the elliptic curve C_j^ν lift to the trivial theta characteristic $\mathcal{O}_{C_j^\nu}$. In order to give a specific lift of σ_C choose $\gamma_{|C_j} : (\sigma_{C|C_j})^* \mathcal{O}_{C_j} \to \mathcal{O}_{C_j}$ (for example the canonical map given as the inverse of the pull back of functions via $\sigma_{C|C_j}$) and let $\gamma_{|\overline{X \setminus C_j}} : \mathrm{id}^*_{\overline{X \setminus C_j}} L_{|\overline{X \setminus C_j}} \to L_{|\overline{X \setminus C_j}}$ be the identity. These give an isomorphism $\widetilde{\gamma} : \widetilde{\sigma}^* \widetilde{L} \to \widetilde{L}$ and there exists a unique extension $(\sigma, \gamma) \in \mathrm{Aut}(X, L, b)$.

The last remark then yields

$$K(\sigma, \gamma) = \left(\begin{array}{c|c|c|c|c} \begin{smallmatrix} \zeta_6^2 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{smallmatrix} & & & & \\ \hline & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & & & \\ \hline & & \begin{smallmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{smallmatrix} & & \\ \hline & & & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & \\ \hline & & & & \begin{smallmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \zeta_6^2 \end{smallmatrix} \end{array}\right)$$

Since (σ, γ) is the identity away from the elliptic tail and the incident exceptional component, this automorphism deforms to every nearby spin curve in which the elliptic tail, the theta characteristic over it and the elliptic tail node are unaltered, i.e. to every spin curve over the locus $\tau_1 = 0 = \tau_{3g-3}$. Therefore all the ± 1's in

$K(\sigma, \gamma)$ are in fact 1's and the order of $K(\sigma, \gamma)$ is three. The Reid-Tai sum of this matrix with respect to the primitive third root ζ_6^2 of 1 is only $\frac{2}{3}$. Since $K(\mathrm{Aut}(X, L, b))$ contains no quasi reflections but a non-trivial element which does not fulfil the Reid-Tai inequality, the quotient $\mathbb{C}_u^{3g-3}/K(\mathrm{Aut}(X, L, b))$ has a non-canonical singularity at 0. Therefore $[(X, L, b)] \in \overline{S}_g$ is non-canonical. This shows that Theorem 4.3.9 is the best one can hope for.

An analogous calculation for an automorphism σ_C which has order three on the elliptic tail and is the identity on the complement yields

$$M(\sigma_C) = \begin{pmatrix} \zeta_3^2 & & \\ & \mathbb{I} & \\ & & \zeta_3 \end{pmatrix}$$

for an appropriate primitive third root ζ_3 of 1. There exists $(\sigma, \gamma) \in \mathrm{Aut}(X, L, b)$ lifting σ_C with

$$K(\sigma, \gamma) = \begin{pmatrix} \zeta_3 & & \\ & \mathbb{I} & \\ & & \zeta_3 \end{pmatrix}$$

which has only Reid-Tai sum $\frac{2}{3}$.

The proof of the "only if" part of Theorem 4.3.9 will be given in the remainder of this section. Let (X, L, b) be a spin curve of genus $g \geq 4$ such that $[(X, L, b)] \in \overline{S}_g$ is a non-canonical singularity. Since $\overline{S}_g$ is locally at $[(X, L, b)]$ isomorphic to $\mathbb{C}_u^{3g-3}/K(\mathrm{Aut}(X, L, b))$ and $K(\mathrm{Aut}(X, L, b))$ contains no quasi reflections, the Reid-Tai criterion yields the existence of an element $K \in K(\mathrm{Aut}(X, L, b))$, $K \neq \mathbb{I}$ and a primitive ord$K$th root ζ of 1 such that the Reid-Tai sum of K with respect to ζ is smaller than 1 (and of course bigger than 0).

Let $(\sigma, \gamma) \in \mathrm{Aut}(X, L, b)$ be an automorphism such that $K(\sigma, \gamma) = K$ and denote by n the order of $K(\sigma, \gamma)$. For the above mentioned primitive nth root ζ of 1 write

$$K(\sigma, \gamma) \sim \begin{pmatrix} \zeta^{a_1} & & \\ & \ddots & \\ & & \zeta^{a_{3g-3}} \end{pmatrix}$$

where $0 \leq a_i < n$. Then by the assumption that $K(\sigma, \gamma)$ is "responsible" for the non-canonicity of $\overline{S}_g$ at $[(X, L, b)]$ the Reid-Tai sum of $K(\sigma, \gamma)$ with respect to ζ satisfies

$$0 < \frac{1}{n} \sum_{j=1}^{3g-3} a_j < 1.$$

The idea of the proof is to fix such a pair $((X, L, b), (\sigma, \gamma))$ and analyse which properties of the spin curve are necessary for the existence of such an automorphism. In the first step the action of the induced automorphism σ_C on the set of nodes of the stable model C of X will be investigated. In particular it will be proved in Corollary 4.3.14 that one may assume that the pair $((X, L, b), (\sigma, \gamma))$ is *singularity reduced* (see Definition 4.3.13). Here the basic idea is that under certain conditions it is possible to deform the spin curve (X, L, b) to a nearby curve $(X, L, b)'$ in such a way that the point $[(X, L, b)'] \in \overline{S}_g$ is a "more general" point in the locus of non-canonical singularities near $[(X, L, b)]$, in particular (σ, γ) deforms to an automorphism $(\sigma, \gamma)'$ of the nearby curve and $K(\sigma, \gamma)$ and $K(\sigma, \gamma)'$ have the same eigenvalues. Another important property of the deformation is that the spin curve $(X, L, b)'$ has an elliptic tail with j-invariant 0 and trivial theta characteristic on the tail if and only if (X, L, b) has an elliptic tail with these properties. Afterwards it will be proved in several steps, that if the fixed pair $((X, L, b), (\sigma, \gamma))$ is singularity reduced then (X, L, b) has an elliptic tail with j-invariant 0 and trivial theta characteristic on the tail. Since this property is preserved in the reduction step to a singularity reduced pair this proves the theorem.

Order the coordinates τ_i of the local universal deformation space $\mathbb{C}^{3g-3}_\tau$ (and the t_i and u_i accordingly) in such a way that the three matrices $K(\sigma, \gamma)$, $M(\sigma, \gamma)$ and $M(\sigma_C)$ have the form described in Remark 4.3.8, i.e. the first coordinates correspond to elliptic tail nodes $T = T(X, L, b)$, the next to disconnecting nodes $D = D(X, L, b)$, which are not elliptic tail nodes, the next to non-disconnecting exceptional nodes $\overline{N} = \overline{N}(X, L, b)$, the next to non-exceptional nodes Δ and the remaining ones correspond to the irreducible components $V = V(\Gamma(C))$ of C, the stable model of X. Refine this ordering according to Remark 3.2.23, i.e. such that

$$M(\sigma, \gamma) = \begin{pmatrix} M_1\mathbb{E}_{\pi_1} & & \\ & \ddots & \\ & & M_l\mathbb{E}_{\pi_l} \end{pmatrix}$$

where the π_i are cycles, $\mathbb{E}_{\pi_i}$ the appropriate (generalized) permutation matrices and M_i (block-)diagonal matrices. Set $\varepsilon_i = 4$ if π_i is a cycle of the permutation σ_T of elliptic tail nodes, $\varepsilon_i = 2$ if π_i is a cycle of the permutation σ_D of disconnecting (not elliptic tail) nodes and in all other cases set $\varepsilon_i = 1$. Then

$$K(\sigma, \gamma) = \begin{pmatrix} K_1\mathbb{E}_{\pi_1} & & \\ & \ddots & \\ & & K_l\mathbb{E}_{\pi_l} \end{pmatrix} = \begin{pmatrix} M_1^{\varepsilon_1}\mathbb{E}_{\pi_1} & & \\ & \ddots & \\ & & M_l^{\varepsilon_l}\mathbb{E}_{\pi_l} \end{pmatrix}$$

Remark 4.3.11. The spin curve (X, L, b) cannot be smooth. Since if it were, all three matrices $M(\sigma, \gamma)$, $K(\sigma, \gamma)$ and $M(\sigma_C)$ were equal and $\mathbb{C}^{3g-3}_t = \mathbb{C}^{3g-3}_\tau = \mathbb{C}^{3g-3}_u$. Since, as a special case of Theorem 4.2.3, every automorphism of a smooth curve of genus $g \geq 4$ is either the identity or fulfils the Reid-Tai inequality (see also the Proposition on page 28 of [HM82]), this gives a contradiction. Hence

S_g has only canonical singularities and the spin curve (X, L, b) leading to a non-canonical singularity is not smooth.

As a first step some of the singularities of the curve can be smoothed in such a way, that the automorphism deforms to the new curve.

Proposition 4.3.12. *Let $M\mathbb{E}_\pi$ be a block of $M(\sigma, \gamma)$ that corresponds to nodes $P_1, \ldots, P_m$, where m is the order of the cycle π, i.e.*

$$M\mathbb{E}_\pi = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_m \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Suppose that $\det M = 1$ then there exists a deformation $(X, L, b)'$ of (X, L, b) in which the singularities $P_1, \ldots, P_m$ disappear such that the automorphism $(\sigma, \gamma) \in \mathrm{Aut}(X, L, b)$ deforms to an automorphism $(\sigma, \gamma)' \in \mathrm{Aut}(X, L, b)'$. More formally this means that there exists a one-dimensional subspace of $\mathbb{C}^{3g-3}_\tau$, which is fixed pointwise by $M(\sigma, \gamma)$, $(X, L, b)'$ is the fibre of the local universal family over a non-zero point in this subspace and the singularities $P_1, \ldots, P_m$ are smoothed, i.e. $\tau_i \neq 0$ for $i = 1, \ldots, m$ at this point.

Proof. Set $B := M\mathbb{E}_\pi$. $1 = \det M = \prod_{i=1}^m \alpha_i$ implies

$$B^m = \prod_{i=1}^m \alpha_i \cdot \mathbb{I} = \mathbb{I}.$$

The fact that the block B corresponds to the nodes $P_1, \ldots, P_m$ means that $B = M(\sigma, \gamma)_{|W}$ where

$$W = \bigoplus_{i=1}^m W_\tau(P_i) \subset \mathbb{C}^{3g-3}_\tau.$$

Fix a non-zero $w_0 \in W_\tau(P_1) \subset W$ and consider

$$w = \sum_{i=0}^{m-1} B^i w_0,$$

since $B(W) = W$ this is an element of W and it satisfies

$$Bw = \sum_{i=0}^{m-1} B^{i+1} w_0 = B^m w_0 + \sum_{i=1}^{m-1} B^i w_0 = w.$$

Considering w as an element in $\mathbb{C}^{3g-3}_\tau$ it is a fixed point of $M(\sigma, \gamma)$, hence $\mathbb{C}w$ is fixed pointwise. Let $(X, L, b)'$ be a nearby spin curve over $\mathbb{C}w \setminus \{0\}$. Then the automorphism (σ, γ) deforms to $(\sigma, \gamma)' \in \operatorname{Aut}(X, L, b)'$. Since $w_0 \neq 0$ the term of w in every summand $W_\tau(P_i)$, $i = 1, \dots, m$, is non-zero, hence these nodes are smoothed in the direction w. □

Definition 4.3.13. A pair $((X, L, b), (\sigma, \gamma))$, $(\sigma, \gamma) \in \operatorname{Aut}(X, L, b)$, is *singularity reduced* if for every block $M_i \mathbb{E}_{\pi_i}$ corresponding to non-disconnecting nodes, i.e. π_i is a cycle of $\sigma_{\overline{N}}$ or σ_Δ, $\det M \neq 1$.

Corollary 4.3.14. *Without loss of generality one may assume that the fixed pair $((X, L, b), (\sigma, \gamma))$ is singularity reduced, i.e. there exists a nearby curve $(X, L, b)'$ such that the automorphism (σ, γ) deforms to $(\sigma, \gamma)' \in \operatorname{Aut}(X, L, b)'$, the pair $((X, L, b)', (\sigma, \gamma)')$ is singularity reduced, the matrices $K(\sigma, \gamma)$ and $K(\sigma, \gamma)'$ have the same eigenvalues and (X, L, b) has an elliptic tail with j-invariant 0 and trivial theta characteristic on the tail iff $(X, L, b)'$ has such an elliptic tail.*

Proof. Suppose there is a block $M_i \mathbb{E}_{\pi_i}$ corresponding to non-disconnecting nodes in $M(\sigma, \gamma)$ with $\det M = 1$. Then the spin curve-automorphism pair can be deformed as in the above proposition. Since the eigenvalues of the matrix $M(\sigma, \gamma)$ vary continuously under this deformation and are roots of 1 they must be constant. Hence $M(\sigma, \gamma)$ and $M(\sigma, \gamma)'$ have the same eigenvalues.

For the eigenvalues of $K(\sigma, \gamma)$ and $K(\sigma, \gamma)'$ this implies the following. The condition that the nodes are non-disconnecting translates into $\varepsilon_i = 1$, i.e. $K_i \mathbb{E}_{\pi_i} = M_i \mathbb{E}_{\pi_i}$. The smoothing of (some of) the non-disconnecting nodes in the deformation does not change the combinatorics of the elliptic tail nodes and disconnecting nodes, hence the subgroups $H = H(\operatorname{Aut}(X, L, b)) \subset M(\operatorname{Aut}(X, L, b))$ and $H' = H(\operatorname{Aut}(X, L, b)') \subset M(\operatorname{Aut}(X, L, b)')$ generated by quasi reflections coincide.

Consider the indicated three blocks of $K(\sigma, \gamma)$ corresponding to the three cases where ε_i is 4, 2 or 1

$$K(\sigma,\gamma) = \left(\begin{array}{c|c|ccc}
K_{T,1}\mathbb{E}_{\sigma_{T,1}} \ddots & & & & \\
\hline
 & K_{D,1}\mathbb{E}_{\sigma_{D,1}} \ddots & & & \\
\hline
 & & K_{\overline{N},1}\mathbb{E}_{\sigma_{\overline{N},1}} \ddots & & \\
 & & & K_{\Delta,1}\mathbb{E}_{\sigma_{\Delta,1}} \ddots & \\
 & & & & K_{V,1}\mathbb{E}_{\sigma_{V,1}} \ddots
\end{array}\right)$$

Then $M(\sigma, \gamma)$ has the same block form. Deforming the automorphism to the spin curve $(X, L, b)'$ smoothes only non-disconnecting nodes, i.e. it affects only the third block of $M(\sigma, \gamma)$. Therefore the first two blocks of $M(\sigma, \gamma)$ and $M(\sigma, \gamma)'$ are the same implying together with $H = H'$ that also the first two blocks of $K(\sigma, \gamma)$ and $K(\sigma, \gamma)'$ are equal. The third blocks of $M(\sigma, \gamma)$ and $M(\sigma, \gamma)'$ might be different but in any case they have the same eigenvalues. Since the third block corresponds to the case $\varepsilon_i = 1$ the third block of $K(\sigma,\gamma)$ resp. $K(\sigma,\gamma)'$ is the same as the third of $M(\sigma,\gamma)$ resp. $M(\sigma,\gamma)'$.

This proves that $K(\sigma,\gamma)$ and $K(\sigma,\gamma)'$ have the same eigenvalues. Moreover, the nearby spin curve $(X,L,b)'$ has an elliptic tail with j-invariant 0 with trivial theta characteristic on the tail if and only if (X,L,b) has such an elliptic tail, since only non-disconnecting nodes are smoothed, in particular the elliptic tail node is disconnecting hence preserved and the elliptic tail is also preserved. Therefore one may replace (X,L,b) and (σ,γ) by $(X,L,b)'$ and $(\sigma,\gamma)'$. The latter are singularity reduced. $\square$

From now on the fixed pair $((X,L,b),(\sigma,\gamma))$, such that $[(X,L,b)] \in \overline{S}_g$ is a non-canonical singularity and $K(\sigma,\gamma)$ does not fulfil the Reid-Tai inequaltity, is assumed to be singularity reduced.

Proposition 4.3.15. *The induced automorphism σ_C on the stable model C either fixes every node but two which are interchanged or fixes every node.*

Proof. Let $P_1,\dots,P_m$ be nodes which are interchanged cyclicly by σ_C and $B := K\mathbb{E}_\pi$ the corresponding block of $K(\sigma,\gamma)$, i.e. $W = W_u(P_1)\oplus\cdots\oplus W_u(P_m)$ and $B = K(\sigma,\gamma)_{|W}$. Since $m = \operatorname{ord}\mathbb{E}_\pi$ this number divides the order $n = \operatorname{ord} K(\sigma,\gamma)$

$$\begin{aligned}
&\Longrightarrow (KE_\pi)^m = \det K\cdot\mathbb{I}\\
&\Longrightarrow \mathbb{I} = (KE_\pi)^n = ((KE_\pi)^m)^{\frac{n}{m}} = (\det K)^{\frac{n}{m}}\cdot\mathbb{I}\\
&\Longrightarrow 1 = (\det K)^{\frac{n}{m}}
\end{aligned}$$

Remember that ζ is a primitive nth root of 1 such that $K(\sigma,\gamma)$ does not fulfil the Reid-Tai inequality with respect to this root. There exists an integer $0 \le l < \frac{n}{m}$ such that

$$\det K = \zeta^{lm}.$$

But the characteristic polynomial of B is $\chi(\lambda) = \lambda^m - \det K = \lambda^m - \zeta^{lm}$. Hence the eigenvalues of B are $\zeta^{l+j\frac{n}{m}}$ for $j = 0,\dots,m-1$ and the corresponding part of the Reid-Tai sum is

$$\frac{1}{n}\sum_{j=0}^{m-1}\left(l + j\frac{n}{m}\right) = \frac{ml}{n} + \frac{m-1}{2}.$$

This information gives the inequality

$$1 > \frac{1}{n}\sum_{j=1}^{3g-3} a_j \ge \frac{ml}{n} + \frac{m-1}{2} \ge \frac{m-1}{2},$$

hence m is 1 or 2 and σ_C can only interchange pairs of nodes or fix nodes.

Suppose that σ_C interchanges two pairs of nodes, say $P_1 \mapsto P_2 \mapsto P_1$ and $P_3 \mapsto P_4 \mapsto P_3$. Then every pair contributes $\frac{1}{2}$ to the Reid-Tai sum, hence

$$1 > \frac{1}{n} \sum_{j=1}^{3g-3} a_j \geq \underbrace{\frac{2-1}{2}}_{W_u(P_1) \oplus W_u(P_2)} + \underbrace{\frac{2-1}{2}}_{W_u(P_3) \oplus W_u(P_4)} \geq 1$$

but this is false, therefore there can be at most one pair of nodes which are interchanged, all other nodes are fixed by σ_C. □

For the block structure of the matrices $M(\sigma_C)$, $M(\sigma, \gamma)$ and $K(\sigma, \gamma)$ this proposition implies that the submatrix made out of the four blocks corresponding to nodes, i.e. to T, D, $\overline{N}$ and Δ, is either a diagonal matrix, in case all nodes are fixed by σ_C, or there is one 2×2-block for the interchanged pair of nodes and the remainder is a diagonal matrix. In the next step the fifth block, i.e. the one corresponding to irreducible components of C or equivalently non-exceptional components of X, shall be investigated.

Proposition 4.3.16. *Every non-exceptional component C_j of X is fixed by σ, i.e. $\sigma(C_j) = C_j$.*

Proof. Let C_j, $\sigma(C_j), \ldots, \sigma^{m-1}(C_j)$ be different non-exceptional components and $\sigma^m(C_j) = C_j$. Assume that C_j is not fixed, i.e. $m \geq 2$. Then

$$W = \bigoplus_{i=0}^{m-1} W_u(\sigma^i(C_j)) = \bigoplus_{i=0}^{m-1} W_\tau(\sigma^i(C_j)) = \bigoplus_{i=0}^{m-1} W_t(\sigma^i(C_j))$$

corresponds to deformations of the pointed normalisations of the components $\sigma^i(C_j)$ and $B := K(\sigma, \gamma)_{|W} = K\mathbb{E}_\pi$ where $\mathbb{E}_\pi$ a generalized permutation matrix, i.e.

$$\mathbb{E}_\pi = \begin{pmatrix} \mathbf{0} & \mathbb{I} & & \\ \vdots & & \ddots & \\ \mathbf{0} & & & \mathbb{I} \\ \mathbb{I} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}$$

where $\mathbb{I}$ is the identity matrix of size $d \times d$, $d = \dim W_t(C_j)$ and K is block diagonal, i.e.

$$\begin{pmatrix} K_1 & & \\ & \ddots & \\ & & K_m \end{pmatrix}.$$

for $d \times d$-matrices K_i. Then m is the number of blocks in one row, hence $m|n = \operatorname{ord} K(\sigma, \gamma)$. Because of

$$B^m = \begin{pmatrix} \prod_{i=1}^m K_i & & \\ & \ddots & \\ & & \prod_{i=1}^m K_i \end{pmatrix}$$

B^m restricts to the endomorphism $\prod_{i=1}^m K_i$ of $W_u(C_j)$ and since n is the order of $K(\sigma, \gamma)$

$$\left(B^m_{|W_u(C_j)}\right)^{\frac{n}{m}} = \mathbb{I}.$$

Therefore the eigenvalues of $B^m_{|W_u(C_j)}$ are $\frac{n}{m}$th roots of 1, say

$$\zeta^{l_1 m}, \ldots, \zeta^{l_d m}$$

where $0 \le l_i < \frac{n}{m}$ with eigenvectors $w_1, \ldots, w_d \in W_u(C_j) \subset W$, i.e. $B^m w_i = \zeta^{l_i m} w_i$. Define

$$w_i(\mu) = \sum_{k=0}^{m-1} \zeta^{-k\left(l_i + \mu\frac{n}{m}\right)} B^k w_i$$

for $i = 1, \ldots, d$ and $\mu = 0, \ldots, m-1$. Then B maps $w_i(\mu)$ as follows

$$\begin{aligned} Bw_i(\mu) &= \sum_{k=0}^{m-1} \zeta^{-k\left(l_i+\mu\frac{n}{m}\right)} B^{k+1} w_i \\ &= \zeta^{-(m-1)\left(l_i+\mu\frac{n}{m}\right)} B^m w_i + \sum_{k=1}^{m-1} \zeta^{-(k-1)\left(l_i+\mu\frac{n}{m}\right)} B^k w_i \\ &= \zeta^{l_i+\mu\frac{n}{m}} \left(\zeta^{-m\left(l_i+\mu\frac{n}{m}\right)} \cdot \zeta^{l_i m} w_i + \sum_{k=1}^{m-1} \zeta^{-k\left(l_i+\mu\frac{n}{m}\right)} B^k w_i \right) \\ &= \zeta^{l_i+\mu\frac{n}{m}} w_i(\mu). \end{aligned}$$

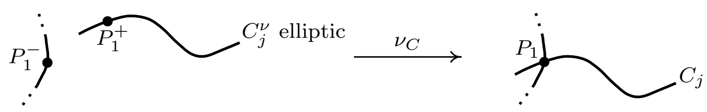
Therefore the eigenvalues of B are $\zeta^{l_i+\mu\frac{n}{m}}$ for $i = 1, \ldots, d$, $\mu = 0, \ldots, m-1$.

$$\Longrightarrow 1 > \frac{1}{n}\sum_{j=1}^{3g-3} a_j \ge \frac{1}{n}\sum_{i=1}^{d}\sum_{\mu=0}^{m-1}\left(l_i + \mu\frac{n}{m}\right) = \frac{d(m-1)}{2} + \frac{m}{n}\sum_{i=1}^{d} l_i \ge \frac{d(m-1)}{2}.$$

This shows that either $d = 0$ and $m \ge 2$ is arbitrary or $d = 1$ and $m = 2$. Since every non-exceptional component is stable $d = 3g(C_j^\nu) - 3 + \#(\text{marked points})$ and if $d = 0$ the normalisation C_j^ν is rational with exactly 3 marked points, if $d = 1$ either C_j^ν is rational with exactly 4 marked points or C_j^ν is a smooth elliptic curve with exactly 1 marked point.

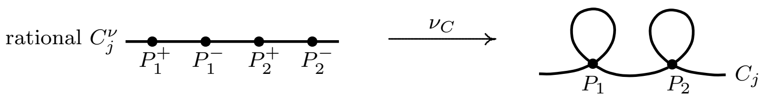
There are six possibilities for $C_j \subset C$:

(i) C_j^ν elliptic, 1 marked point, C_j is a smooth elliptic tail



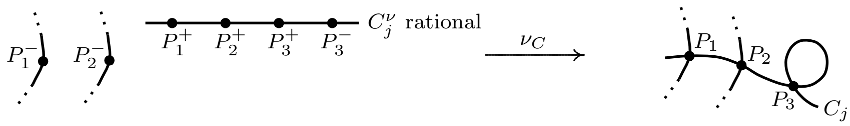
and $d = 1$, $m = 2$.

(ii) C_j^ν rational, 4 marked points mapping to 2 irreducible nodes



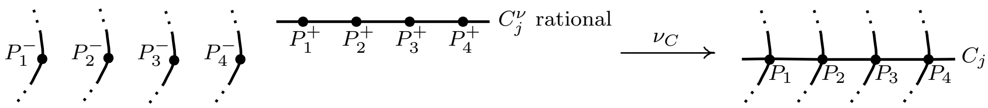
and $d = 1$, $m = 2$.

(iii) C_j^ν rational, 4 marked points mapping to 1 irreducible node and 2 non-irreducible nodes



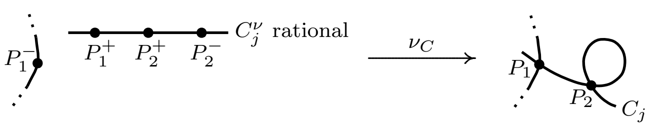
and $d = 1$, $m = 2$.

(iv) C_j^ν rational, 4 marked points mapping to 4 non-irreducible nodes



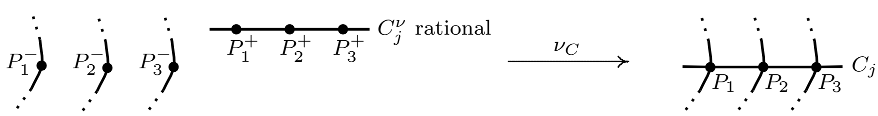
and $d = 1$, $m = 2$.

(v) C_j^ν rational, 3 marked points mapping to 1 irreducible and 1 disconnecting node, i.e. C_j is a singular elliptic tail



and $d = 0$, $m \geq 2$.

(vi) C_j^ν rational, 3 marked points mapping to 3 non-irreducible nodes

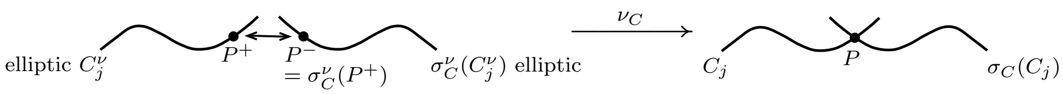


and $d = 0$, $m \geq 2$.

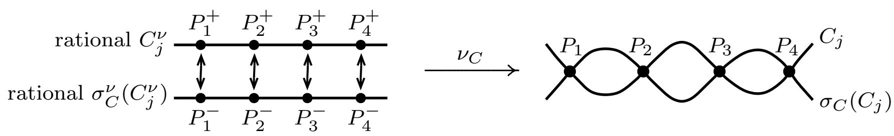
The case (ii) is not possible because then $2 = g \geq 4$. Consider the remaining cases with $d = 1$, i.e. (i), (iii) and (iv). Proposition 4.3.15 shows that at most one pair of nodes may be interchanged. Suppose that this is the case, i.e. $P_1 \mapsto P_2 \mapsto P_1$. Then $K(\sigma, \gamma)_{|W_u(P_1) \oplus W_u(P_2)}$ contributes at least $\frac{1}{2}$ to the Reid-Tai sum. But then

$$1 > \frac{1}{n} \sum_{j=1}^{3g-3} a_j \geq \underbrace{\frac{1}{2}}_{\oplus W_u(P_i)} + \underbrace{\frac{1}{2}}_{\mathrm{W}},$$

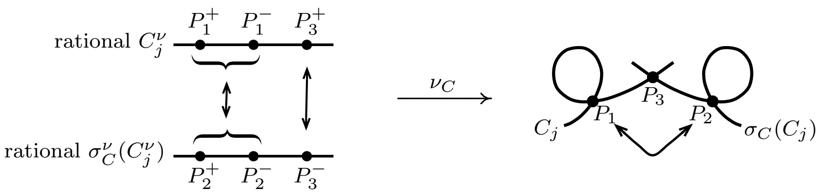
giving a contradiction. Hence every node must be fixed by σ_C. In case (i) the elliptic tail node P must be fixed, then $\sigma_C(C_j)$ is the second component through P and C has genus $2 = g \geq 4$



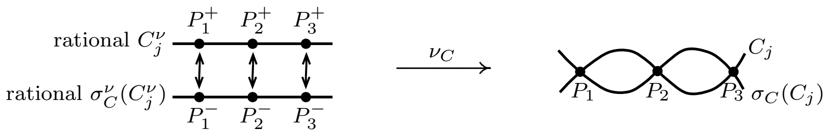
In case (iii) the irreducible node is fixed but the component on which it lies is moved. This is not possible. In case (iv) $m = 2$ hence the four non-irreducible nodes can only be fixed if $\sigma_C(C_j)$ is the second component through each of the four. Then C has only these two rational components and hence genus 3.



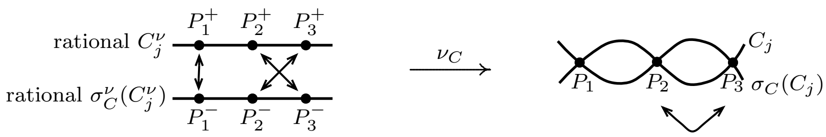
This excludes the case $d = 1$. In case (v) denote the irreducible node on C_j by P_1, since $\sigma_C(C_j) \neq C_j$ the node P_1 must move, i.e. $\sigma_C(P_1) = P_2 \neq P_1$. But then Proposition 4.3.15 shows that $P_1 \mapsto P_2 \mapsto P_1$ and all other nodes are fixed. Denoting by P_3 the disconnecting node on C_j P_3 must be fixed, hence $\sigma_C(C_j)$ is the second component through P_3. But then C has genus 2.



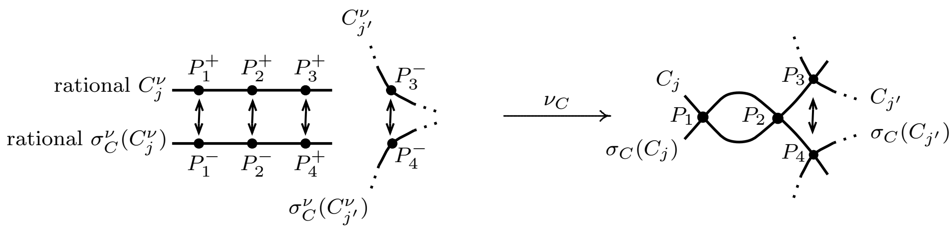
Hence only the last case remains. Since there are three nodes P_1, P_2 and P_3 on C_j Proposition 4.3.15 implies that at least one of these must be fixed. If all are fixed $\sigma_C(C_j)$ is the second component through all of them, hence C has genus 2.



Now assume exactly one, say P_1, is fixed, then the other two must be interchanged, i.e. $P_2 \mapsto P_3 \mapsto P_2$. Then $\sigma_C(C_j)$ must be the second component through all three nodes. Again C has only genus 2.

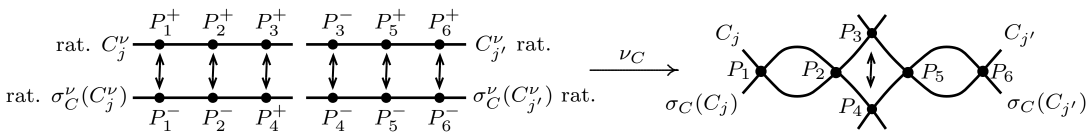


Hence exactly two of the nodes on C_j, say P_1 and P_2, are fixed and $P_3 \mapsto P_4 \mapsto P_3$ where $P_3 \neq P_4 \in \sigma_C(C_j)$. Since P_1 and P_2 are fixed $\sigma_C(C_j)$ must be the second component through these nodes.

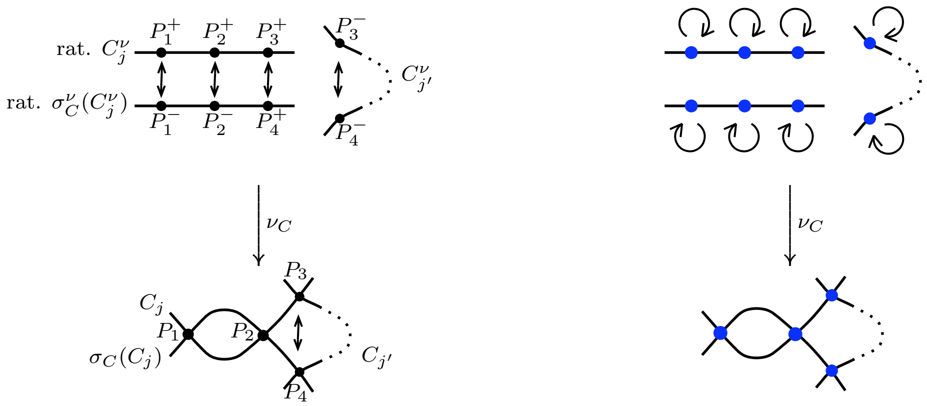


But P_3 is not fixed hence the branches at P_3 are C_j and $C_{j'} \neq C_j, \sigma_C(C_j)$, then the branches at $P_4 = \sigma_C(P_3)$ are $\sigma_C(C_j)$ and $\sigma_C(C_{j'}) \neq C_j, \sigma_C(C_j)$. Now either the component $C_{j'}$ is fixed, i.e. $\sigma_C(C_{j'}) = C_{j'}$ or it is not fixed.

In the last case $C_{j'}$ and $\sigma_C(C_{j'})$ must also be as in (vi), i.e the normalisations are rational with three marked points mapping to three non-irreducible nodes. One on $C_{j'}$ is P_3 and one on $\sigma_C(C_{j'})$ is P_4. But the remaining nodes must be fixed, hence there are only two additional nodes where the branches at each are $C_{j'}$ and $\sigma_C(C_{j'})$. The curve has only genus 3.



Hence the component $C_{j'}$ must be fixed and it is the second branch through P_3 and P_4. Consider the restriction $\varphi = \sigma_{C|C_j \cup \sigma_C(C_j)}$, then φ^2 fixes the two components and all the nodes, hence all the marked points in the pointed normalisations.



The left hand side shows the action of the automorphisms σ_C^ν resp. σ_C and the right hand side the squares of these automorphisms, where a blue dot stands for a fixed node, whose branches are also fixed by the automorphism under consideration, or its preimages, which are fixed points.

Since there are 3 marked points on the normalisations of each of the two rational components, φ^2 induces the identity on the pointed normalisations, hence φ^2 is the identity. Let x and y be local coordinates of C at the node P_1. Then σ_C must interchange the coordinates but its square fixes them. Hence the coordinates are interchanged, i.e. $x \mapsto y \mapsto x$. Let t_1 be the coordinate corresponding to P_1, then $xy = t_1$ is the deformation of the node and $t_1 = xy \mapsto yx = t_1$. For the action in the u-coordinates this means the following. Since P_1 is not a disconnecting node $u_1 = \tau_1$, where either $\tau_1^2 = t_1$, in case P_1 is exceptional, or $\tau_1 = t_1$ else. In the last case $M(\sigma, \gamma)$ has a 1×1-block $M\mathbb{E}_\pi = (1)$ but this is impossible since (σ, γ) is singularity reduced. Hence P_1 must be an exceptional node and $\tau_1 \mapsto \pm\tau_1$. Again $\tau_1 \mapsto \tau_1$ is not possible since (σ, γ) is singularity reduced. Hence $\tau_1 \mapsto -\tau_1 = \zeta^{\frac{n}{2}}\tau_1$ giving a contradiction

$$1 > \frac{1}{n}\sum_{j=1}^{3g-3} a_j \geq \underbrace{\frac{1}{n}\cdot\frac{n}{2}}_{P_1} + \underbrace{\frac{1}{2}}_{P_3 \leftrightarrow P_4} = 1.$$

Therefore all the cases where a non-exceptional component is moved are excluded. □

Remark 4.3.17. Let $K_i\mathbb{E}_{\pi_i}$ be a block in

$$K(\sigma, \gamma) = \begin{pmatrix} K_1\mathbb{E}_{\pi_1} & & \\ & \ddots & \\ & & K_l\mathbb{E}_{\pi_l} \end{pmatrix}$$

corresponding to deformations of components, i.e. $K_i\mathbb{E}_{\pi_i} = K(\sigma,\gamma)_{|W}$ and W is the direct sum of appropriate $W_u(C_j)$'s. Proposition 4.3.16 implies that $W = W_u(C_j)$ for some non-exceptional component C_j, i.e. $K_i\mathbb{E}_{\pi_i} = K_i$. Therefore the block in $M(\sigma_C)$, $M(\sigma,\gamma)$ and $K(\sigma,\gamma)$ corresponding to components is just a block diagonal matrix with one block for every non-exceptional component. In other words in the above form of $K(\sigma,\gamma)$ either all π_i are cycles of order 1 or all but one are cycles of order 1 and the special one has order two and corresponds to interchanging a pair of nodes.

Let σ^ν be the induced automorphism on the normalisation X^ν, C_j any non-exceptional component of X and C_j^ν its normalisation. Since σ fixes C_j the restriction $\varphi_j = \sigma^\nu_{|C_j^\nu}$ is an automorphism of C_j^ν.

Proposition 4.3.18. *The only possibilities for the pair (C_j^ν, φ_j) are the following.*

(i) $\varphi_j = \mathrm{id}_{C_j^\nu}$*, any* C_j^ν

(ii) C_j^ν *is rational and* $\operatorname{ord}\varphi_j = 2,4$

(iii) C_j^ν *is elliptic and* $\operatorname{ord}\varphi_j = 2,4,3,6$

(iv) C_j^ν *is hyperelliptic of genus* 2 *and* φ_j *is the hyperelliptic involution*

(v) C_j^ν *is hyperelliptic of genus* 3 *and* φ_j *is the hyperelliptic involution*

(vi) C_j^ν *is bielliptic of genus* 2*, i.e. it is a double cover of an elliptic curve and has genus* 2*,* φ_j *is the associated involution.*

Proof. Let $B := K(\sigma,\gamma)_{|W_u(C_j)}$, since C_j is fixed B is an endomorphism of $W_u(C_j)$. Since for coordinates corresponding to components $t_i = \tau_i = u_i$, the spaces $W_u(C_j)$ and $W_t(C_j)$ coincide and the corresponding blocks of $K(\sigma,\gamma)$ and $M(\sigma_C)$ are equal, i.e. $B = M(\sigma_C)_{|W_t(C_j)}$. $W_t(C_j)$ is the deformation space of the pointed normalisation $(C_j^\nu, \{P_i^\pm\})$ of the component C_j and has dimension $3g(C_j^\nu) - 3 + \#(\text{marked points})$. Consider C_j^ν without the marked points $\{P_i^\pm\} \cap C_j^\nu$, the deformation space of C_j^ν is then a subspace of $W_t(C_j)$ of dimension $3g(C_j^\nu) - 3$ if $g(C_j^\nu) \geq 2$ and of dimension 1 resp. 0 if $g(C_j^\nu) = 1$ resp. $g(C_j^\nu) = 0$. φ_j acts linearly on this subspace and in this way induces a submatrix of B. Therefore the eigenvalues of the action of φ_j are also eigenvalues of B. Since $\frac{1}{n}\sum_{j=1}^{3g-3} a_j < 1$ the proposition on page 28 of [HM82] shows that the pair (C_j^ν, φ_j) is of one of the following types:

(i) $\varphi_j = \mathrm{id}_{C_j^\nu}$, any C_j^ν

(ii) C_j^ν is rational

(iii) C_j^ν is elliptic

(iv) C_j^ν is hyperelliptic of genus 2 and φ_j is the hyperelliptic involution

(v) C_j^ν is hyperelliptic of genus 3 and φ_j is the hyperelliptic involution

(vi) C_j^ν is bielliptic of genus 2, φ_j is the associated involution.

The only things missing are the assertions on the order of φ_j in the cases (ii) and (iii). Since σ_C fixes all singularities or interchanges a pair of singularities and fixes the rest, σ_C^2 has to fix all nodes. But then σ_C^4 fixes all nodes and all branches at all nodes. For the normalisation this means that φ_j^4 fixes all marked points on C_j^ν.

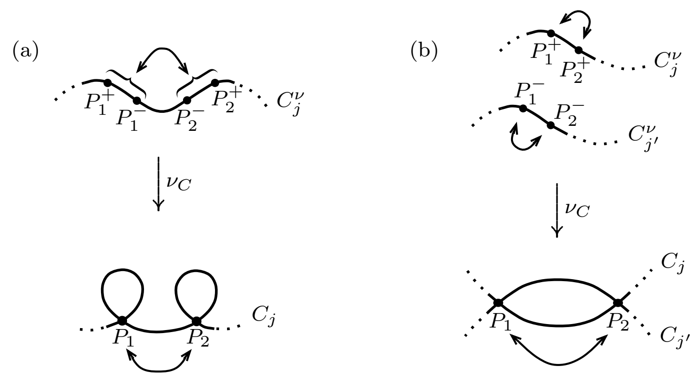
In case (ii) C_j^ν is rational and has at least three marked points, hence φ_j^4 is the identity and the order of φ_j divides 4. The case $\operatorname{ord}\varphi_j = 1$ is covered by case (i), therefore the only remaining possibilities are 2 and 4.

In case (iii) C_j^ν is elliptic and φ_j is either a translation or has a fixed point. In the first case φ_j^4 is still a translation and has at least one fixed point since C_j^ν has at least one marked point. Hence φ_j^4 is the identity and $\operatorname{ord}\varphi_j = 1$, 2 or 4. In case φ_j is not a translation it has a fixed point and the order must be 1, 2, 4, 3 or 6. □

This proposition gives enough information on the curve C to exclude the possibility that σ_C interchanges a pair of nodes.

Proposition 4.3.19. *σ_C fixes all nodes.*

Proof. Suppose σ_C does not fix all nodes, then there exists exactly one pair of nodes of C, say P_1 and P_2, which are interchanged. Since every irreducible component of C is fixed by σ_C there are only two possibilities:



Denote by $P_i^\pm$ the preimages of the two nodes under the normalisation of C. In case (a) all four $P_i^\pm$ lie on one component C_j^ν, in case (b) they lie on two components C_j^ν and $C_{j'}^\nu$ such that for each of the two nodes one preimage, say P_i^+ lies on C_j^ν and the other P_i^- on $C_{j'}^\nu$.

Denote by t_i the coordinate corresponding to the node P_i. Since the nodes are interchanged by σ_C and σ_C is induced by $\sigma \in \operatorname{Aut} X$ either both nodes are blown up by $\beta : X \to C$ or none, i.e. either $\tau_i = t_i$ for $i = 1, 2$ or $\tau_i^2 = t_i$ for $i = 1, 2$. In case (a) as well as in case (b) the nodes are not disconnecting, hence $u_i = \tau_i$,

$$B := K(\sigma,\gamma)_{|W_u} = M(\sigma,\gamma)_{|W_\tau} = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix}$$

and

$$B_C := M(\sigma_C)_{|W_t} = \begin{cases} \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix} & \text{if } \tau_i = t_i \\ \begin{pmatrix} 0 & \alpha_1^2 \\ \alpha_2^2 & 0 \end{pmatrix} & \text{if } \tau_i^2 = t_i \end{cases}$$

where $W_\star = W_\star(P_1) \oplus W_\star(P_2)$ for $\star = u, \tau, t$. In case $\tau_i = t_i$ the order of B is $\operatorname{ord} B_C$, in case $\tau_i^2 = t_i$ it is either $\operatorname{ord} B_C$ or $2 \operatorname{ord} B_C$.

In cases (a) and (b) $\varphi_j = \sigma^\nu_{|C_j^\nu}$ cannot be the identity, hence by the last proposition (C_j^ν, φ_j) must be one of the following cases

(ii) C_j^ν is rational and $\operatorname{ord} \varphi_j = 2, 4$

(iii) C_j^ν is elliptic and $\operatorname{ord} \varphi_j = 2, 4, 3, 6$

(iv) C_j^ν is hyperelliptic of genus 2 and φ_j is the hyperelliptic involution

(v) C_j^ν is hyperelliptic of genus 3 and φ_j is the hyperelliptic involution

(vi) C_j^ν is bielliptic of genus 2 and φ_j is the associated involution.

Therefore the order of φ_j is 2, 4 or 6, since an order 3 automorphism cannot induce the two order 2 nodes. The same is true for $(C_{j'}^\nu, \varphi_{j'})$ where $\varphi_{j'} = \sigma^\nu_{|C_{j'}^\nu}$.

In case (a) denote by n_j the order of φ_j. Obviously $\varphi_j^{n_j}$ fixes neighbourhoods of P_1 resp. P_2 pointwise, i.e. $B_C^{n_j} = \mathbb{I}$ and hence $\operatorname{ord} B_C$ divides n_j. The eigenvalues of B can be calculated as in the proof of Proposition 4.3.15. This gives $(\alpha_1\alpha_2)^{\frac{\operatorname{ord} B}{2}} = 1$ and since the automorphism (σ, γ) is assumed to be singularity reduced, $\alpha_1\alpha_2 \neq 1$, hence there exists an integer $1 \leq l < \frac{\operatorname{ord} B}{2}$ such that $\alpha_1\alpha_2 = \zeta^{l\frac{2n}{\operatorname{ord} B}}$ and the eigenvalues of B are $\zeta^{l\frac{n}{\operatorname{ord} B}}$ and $\zeta^{\frac{n}{2}+l\frac{n}{\operatorname{ord} B}}$. Therefore they give a contribution to the Reid-Tai sum of $\frac{1}{2} + \frac{2l}{\operatorname{ord} B}$.

In case $\operatorname{ord} B = \operatorname{ord} B_C$ this implies

$$1 > \frac{1}{n}\sum_{j=1}^{3g-3} a_j \geq \frac{1}{2} + \frac{2l}{\operatorname{ord} B} \geq \frac{1}{2} + \frac{2l}{n_j}$$
$$\Longrightarrow n_j > 4l \geq 4$$

hence $n_j = \operatorname{ord}\varphi_j = 6$ and C_j^ν is elliptic. But then $K(\sigma,\gamma)_{|W_u(C_j)} = M(\sigma_C)_{|W_t(C_j)}$ and the eigenvalues of this block contribute at least $\frac{1}{3}$. To see this consider the action of the order 6 automorphism φ_j on the one-dimensional deformation space of C_j^ν which has order three since the third power is the elliptic involution, which acts trivially on the deformation space (see also [HM82, page 37]). This gives the contradiction

$$1 > \frac{1}{n}\sum_{j=1}^{3g-3} a_j \geq \underbrace{\frac{1}{2}+\frac{2}{6}}_{P_1 \text{ and } P_2} + \frac{1}{3} = \frac{7}{6}.$$

Therefore $\operatorname{ord} B = 2 \operatorname{ord} B_C$ and

$$1 > \frac{1}{n}\sum_{j=1}^{3g-3} a_j \geq \frac{1}{2} + \frac{2l}{\operatorname{ord} B} \geq \frac{1}{2} + \frac{2l}{2n_j}$$
$$\Longrightarrow n_j > 2l \geq 2$$

hence $n_j = \operatorname{ord}\varphi_j = 4$ or 6. If $n_j = 6$

$$1 > \frac{1}{n}\sum_{j=1}^{3g-3} a_j \geq \underbrace{\frac{1}{2}+\frac{2}{12}}_{P_1 \text{ and } P_2} + \frac{1}{3} = 1$$

as above. Now let $n_j = 4$, then C_j^ν is rational or elliptic with at least four marked points, therefore the dimension of $W_t(C_j)$ is at least one. The order of $K(\sigma,\gamma)_{|W_u(C_j)} = M(\sigma_C)_{|W_t(C_j)}$ has to divide $n_j = 4$, hence its contribution to the Reid-Tai sum is $\frac{\kappa}{4}$ for some non-negative integer κ.

$$\Longrightarrow 1 > \frac{1}{n}\sum_{j=1}^{3g-3} a_j \geq \underbrace{\frac{1}{2}+\frac{2}{8}}_{P_1 \text{ and } P_2} + \frac{\kappa}{4} = \frac{3}{4} + \frac{\kappa}{4}$$

Therefore $\kappa = 0$ and the matrix $M(\sigma_C)_{|W_t(C_j)}$ must be trivial. But this would mean that the order 4 automorphism φ_j deforms to every deformation of the pointed curve $(C_j^\nu, \{P_i^\pm\})$. But since the general elliptic curve has no order 4 automorphism the elliptic case is excluded. If C_j^ν is rational, φ_j interchanges $\{P_1^\pm\}$ and $\{P_2^\pm\}$. The general rational curve with four marked points, i.e. one where the four points are in general position, does not have an automorphism that realizes this permutation, hence φ_j cannot deform to every deformation of $(C_j^\nu, \{P_i^\pm\})$. Therefore case (a) is impossible.

In case (b) denote by n_j resp. $n_{j'}$ the order of φ_j resp. $\varphi_{j'}$. Then $\overline{n} = \operatorname{lcm}(n_j, n_{j'})$ is the order of the restriction $\sigma^\nu_{|C_j^\nu \cup C_{j'}^\nu}$ and $B_C^{\overline{n}} = \mathbb{I}$. As above B contributes

$\frac{1}{2} + \frac{2l}{\operatorname{ord} B}$, $l \geq 1$. In case $\operatorname{ord} B = \operatorname{ord} B_C$ this implies

$$\begin{aligned} 1 > \frac{1}{n}\sum_{j=1}^{3g-3} a_j \geq \frac{1}{2} + \frac{2l}{\operatorname{ord} B} \geq \frac{1}{2} + \frac{2}{\overline{n}} \\ \Longrightarrow \overline{n} > 4. \end{aligned}$$

$\operatorname{ord} B = \operatorname{ord} B_C$ divides $\overline{n}$, hence $\overline{n} > 4$. Since n_j is 2, 4 or 6 as well as $n_{j'}$, their lowest common multiple $\overline{n}$ can only be strictly bigger than 4 if one of them, say n_j, equals 6, and $\overline{n} = 6$ or 12. But then as above the action on the deformation space of $(C_j^\nu, \{P_i^\pm\})$ contributes at least $\frac{1}{3}$ giving the contradiction

$$1 > \underbrace{\frac{1}{2} + \frac{2}{\overline{n}}}_{P_1 \text{ and } P_2} + \frac{1}{3} \geq \frac{1}{2} + \frac{2}{12} + \frac{1}{3} = 1.$$

Therefore $\operatorname{ord} B = 2 \operatorname{ord} B_C$ and

$$\begin{aligned} 1 > \frac{1}{n}\sum_{j=1}^{3g-3} a_j \geq \frac{1}{2} + \frac{2l}{2\operatorname{ord} B_C} \geq \frac{1}{2} + \frac{1}{\overline{n}} \\ \Longrightarrow \overline{n} > 2 \\ \Longrightarrow \overline{n} = 4, 6, 12. \end{aligned}$$

In case $\overline{n} = 4$ wlog $n_j = 4$ and $n_{j'} = 2$ or 4. Since C_j^ν contains only one preimage of P_1 and only one of P_2, say P_1^+ and P_2^+, φ_j must interchange P_1^+ and P_2^+. Since an order 4 automorphism of $\mathbb{P}^1$ does not have points of order exactly 2, C_j^ν must be elliptic with j-invariant 1728 and (after identifying C_j^ν with $\mathbb{C}/\mathbb{Z}i + \mathbb{Z}$) the two points P_1^+ and P_2^+ are the two 2-torsion points interchanged by the order four automorphism φ_j. But then φ_j does not deform to every deformation of the pointed curve $(C_j^\nu, \{P_i^\pm\})$, hence the action on the deformation space of $(C_j^\nu, \{P_i^\pm\})$ gives at least $\frac{1}{4}$ and this yields the contradiction

$$1 > \underbrace{\frac{1}{2} + \frac{1}{4}}_{P_1 \text{ and } P_2} + \frac{1}{4} = 1$$

Now let $\overline{n} = 6$, then wlog $n_j = 6$ and $n_{j'} = 2$ or 6. The curve C_j^ν is elliptic and its deformation space gives a contribution of $\frac{1}{3}$, hence

$$1 > \underbrace{\frac{1}{2} + \frac{1}{6}}_{P_1 \text{ and } P_2} + \frac{1}{3} = 1.$$

Therefore $\overline{n} = 12$ is the only case left hence wlog $n_j = 6$ and $n_{j'} = 4$, where C_j^ν and $C_{j'}^\nu$ are both elliptic. But then

$$1 > \underbrace{\frac{1}{2} + \frac{1}{12}}_{P_1 \text{ and } P_2} + \underbrace{\frac{1}{3}}_{W_u(C_j)} + \underbrace{\frac{1}{4}}_{W_u(C_{j'})} = \frac{14}{12}$$

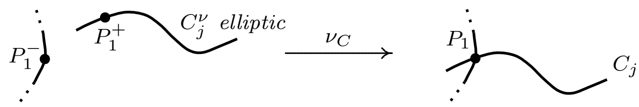
gives a contradiction. Therefore all cases where a pair of nodes is interchanged are excluded. □

This shows that all the cycles π_i have order one and $K(\sigma,\gamma)$ is a block diagonal matrix with one 1×1-block for every node and one $d_j \times d_j$-block for every non-exceptional component of X, where $d_j = \dim W_u(C_j)$. As a next step the action of the induced automorphism on the normalisation of a non-exceptional component shall be analysed in detail.

Proposition 4.3.20. *Let $(C_j^\nu, \{P_i^\pm\})$ be the pointed normalisation of the non-exceptional component C_j of X and denote by φ_j the restriction $\sigma^\nu_{|C_j^\nu}$ of the induced automorphism σ^ν on the normalisation. Then $(C_j^\nu, \{P_i^\pm\}, \varphi_j)$ is of one of the following types and the contribution to the Reid-Tai sum of the eigenvalues of $K(\sigma,\gamma)$ on the deformation space $W_t(C_j) = W_u(C_j)$ of the pointed curve $(C_j^\nu, \{P_i^\pm\})$ is at least w_j.*

Identity: $\varphi_j = \mathrm{id}$, *any* $(C_j^\nu, \{P_i^\pm\})$, $w_j = 0$.

Elliptic tail: C_j^ν is elliptic and $(C_j^\nu, \{P_i^\pm\})$ has exactly one marked point P_1^+, i.e. $C_j \subset C$ looks like



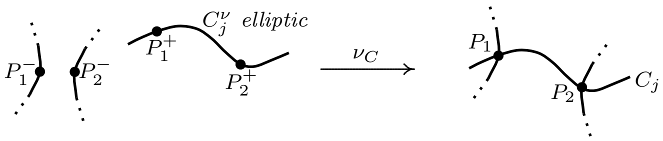
order 2: C_j^ν is any elliptic curve, there is an identification $C_j^\nu \cong \mathbb{C}/\mathbb{Z}_{z_0} + \mathbb{Z}$ for an appropriate $z_0 \in \mathbb{H}_1$, such that φ_j is the elliptic involution and P_1^+ is a 2-torsion point. Then $w_j = 0$.

order 4: C_j^ν is an elliptic curve with j-invariant 1728, there is an identification $C_j^\nu \cong \mathbb{C}/\mathbb{Z}i + \mathbb{Z}$ such that φ_j is one of the two order 4 automorphisms and P_1^+ is one of the two fixed points of φ_j. Then $w_j = \frac{1}{2}$.

order 3: C_j^ν is an elliptic curve with j-invariant 0, there is an identification $C_j^\nu \cong \mathbb{C}/\mathbb{Z}\varrho^2 + \mathbb{Z}$, $\varrho = e^{2\pi i/6}$, such that φ_j is one of the two order 3 automorphisms and P_1^+ is one of the three fixed points of φ_j. Then $w_j = \frac{1}{3}$.

order 6: C_j^ν is an elliptic curve with j-invariant 0, there is an identification $C_j^\nu \cong \mathbb{C}/\mathbb{Z}\varrho^2 + \mathbb{Z}$, $\varrho = e^{2\pi i/6}$, such that φ_j is one of the two order 6 automorphisms and P_1^+ is the fixed point of φ_j. Then $w_j = \frac{1}{3}$.

Elliptic ladder: C_j^ν is elliptic and there are exactly two marked points P_1^+ and P_2^+ on $(C_j^\nu, \{P_i^\pm\})$, which map to different singularities and hence are fixed by φ_j, i.e. $C_j \subset C$ looks like



order 2: C_j^ν *is any elliptic curve, there is an identification* $C_j^\nu \cong \mathbb{C}/\mathbb{Z}_{z_0} + \mathbb{Z}$ *for an appropriate* $z_0 \in \mathbb{H}_1$, *such that* φ_j *is the elliptic involution and the* P_i^+ *are 2-torsion points. Then* $w_j = \frac{1}{2}$.

order 4: C_j^ν *is an elliptic curve with* j*-invariant* 1728, *there is an identification* $C_j^\nu \cong \mathbb{C}/\mathbb{Z}i + \mathbb{Z}$ *such that* φ_j *is one of the two order 4 automorphisms and the* P_i^+ *are the two fixed points of* φ_j. *Then* $w_j = \frac{3}{4}$.

order 3: C_j^ν *is an elliptic curve with* j*-invariant* 0, *there is an identification* $C_j^\nu \cong \mathbb{C}/\mathbb{Z}\varrho^2 + \mathbb{Z}$, $\varrho = e^{2\pi i/6}$, *such that* φ_j *is one of the two order 3 automorphisms and the* P_i^+ *are fixed points of* φ_j. *Then* $w_j = \frac{2}{3}$.

Hyperelliptic tail: C_j^ν *has genus 2,* φ_j *is the hyperelliptic involution and there is exactly one marked point* P_1^+ *on* $(C_j^\nu, \{P_i^\pm\})$, *which is one of the six Weierstraß points of* C_j^ν *and* $w_j = \frac{1}{2}$.

Proof. Consider the deformation space $W_t(C_j) \subset \mathbb{C}_t^{3g-3}$ of the pointed curve $(C_j^\nu, \{P_i^\pm\})$ and the restriction $B = M(\sigma_C)_{|W_t(C_j)}$. Every coordinate t_i of $W_t(C_j)$ has $t_i = \tau_i = u_i$ therefore $B = K(\sigma, \gamma)_{|W_u(C_j)}$. Since φ_j is an automorphism of C_j^ν that induces a permutation of the marked points $P_i^\pm \in C_j^\nu$, B is an endomorphism of $W_t(C_j)$ and diagonalises as

$$B \sim \begin{pmatrix} \zeta^{b_1} & & \\ & \ddots & \\ & & \zeta^{b_d} \end{pmatrix}$$

where $0 \leq b_k < n = \operatorname{ord} K(\sigma, \gamma)$ and d is the dimension of $W_t(C_j)$, i.e. $d = 3g(C_j^\nu) - 3 + \#\{P_i^\pm\} \cap C_j^\nu$.

Proposition 4.3.18 implies that (C_j^ν, φ_j) is of one of the types

(i) $\varphi_j = \mathrm{id}_{C_j^\nu}$, any C_j^ν

(ii) C_j^ν is rational and $\operatorname{ord} \varphi_j = 2, 4$

(iii) C_j^ν is elliptic and $\operatorname{ord} \varphi_j = 2, 4, 3, 6$

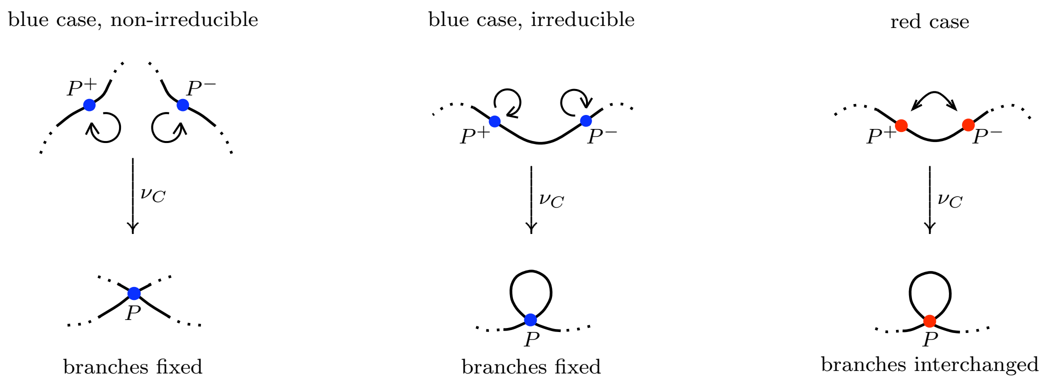
(iv) C_j^ν is hyperelliptic of genus 2 and φ_j is the hyperelliptic involution

(v) C_j^ν is hyperelliptic of genus 3 and φ_j is the hyperelliptic involution

(vi) C_j^ν is bielliptic of genus 2 and φ_j is the associated involution.

Consider the marked points $P_i^\pm$ on C_j^ν. Let P be a node of C and $P^\pm$ the two preimages of P under the normalisation $\nu_C : C^\nu \to C$. Since σ_C fixes all nodes (Proposition 4.3.19) either both preimages $P^\pm$ are fixed by σ_C^ν (blue case) or they are interchanged (red case). In addition all components of C are fixed by σ_C (Proposition 4.3.16), hence the second case is only possible if P^+ and P^- lie in the same component.

In C that looks like



Now consider the cases (i) to (vi).

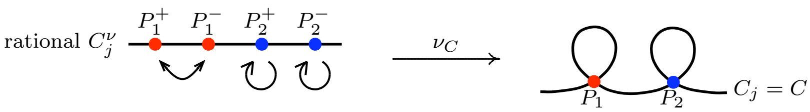
Case (i): φ_j is the identity. Therefore B is also the identity and the corresponding part of the Reid-Tai sum is 0.

Case (ii): C_j^ν is rational and $\operatorname{ord}\varphi_j = 2$ or 4.

Subcase $\operatorname{ord}\varphi_j = 4$*:* An automorphism of $\mathbb{P}^1$ of order 4 has two fixed points and no points of order 2, i.e. all points have order 1 or 4. Therefore the red case is not possible and all marked points $P_i^\pm$ on C_j^ν must be fixed by φ_j. Since there are only two fixed points on C_j^ν this would imply that the curve C is not stable. Hence this case is excluded.

Subcase $\operatorname{ord}\varphi_j = 2$*:* φ_j has two fixed points and all other points have order 2. Denote by l_b the number of marked points on C_j^ν, that are fixed (blue case), and by l_r the number of marked points having order 2 (red case). Then $l_b \in \{0, 1, 2\}$ and l_r is even. Since C is stable and C_j^ν rational, there are at least three marked points on C_j^ν, i.e. $l_b + l_r \geq 3$, hence $l_r \geq 2$ and there is at least one node, say P_1 of red type, i.e. P_1 is fixed by σ_C but the branches are interchanged. Let $xy = 0$ be a local equation for C at P_1 and t_1 the coordinate corresponding to P_1. Since the order of φ_j is two, σ_C interchanges the coordinates, i.e. $x \mapsto y \mapsto x$ but then $t_1 = xy \mapsto yx = t_1$. Now either $t_1 = \tau_1$ or $t_1 = \tau_1^2$, implying in any case $\tau_1 \mapsto \pm\tau_1$. The node P_1 is non-disconnecting and since (σ, γ) is singularity reduced, the (1×1)-block of $M(\sigma,\gamma)$ corresponding to P_1 is non-trivial. Therefore P_1 must be exceptional, i.e. $t_1 = \tau_1^2$, and $\tau_1 \mapsto -\tau_1$, hence -1 is the eigenvalue of $K(\sigma,\gamma)_{|W_u(P_1)} = M(\sigma,\gamma)_{|W_\tau(P_1)}$ giving a contribution of $\frac{1}{2}$ to the Reid-Tai sum. Since this sum is smaller than 1 there can be at most one node of this type. Therefore P_1 is the only node on C_j of red type and $l_r = 2$. $l_r + l_b \geq 3$ and $l_b \in \{0,1,2\}$ implies $l_b = 1$ or 2.

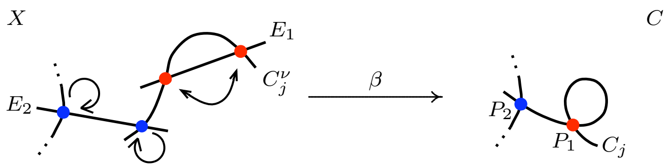
Consider firstly the case $l_b = 2$, i.e. there are two marked points on C_j^ν that are fixed by φ_j. These two are either of irreducible or non-irreducible type. If they are irreducible C has genus 2.



Therefore the two fixed marked points are not of irreducible type. Since there are exactly four marked points the dimension d of $W_t(C_j)$ is $3\cdot 0 - 3 + 4 = 1$. Let t_j be the coordinate of $W_t(C_j)$, then $t_j \mapsto \pm t_j$, because $\operatorname{ord}\varphi_j = 2$. $t_j \mapsto t_j$ would mean that φ_j deforms to every deformation of $(C_j^\nu, \{P_i^\pm\})$. But the general rational curve with four marked point does not have an automorphism of order 2. Therefore $t_j \mapsto -t_j$, giving a contribution of $\frac{1}{2}$ to the Reid-Tai sum.

$$\Longrightarrow 1 > \frac{1}{n}\sum_{j=1}^{3g-3} a_j \geq \underbrace{\frac{1}{2}}_{W_u(P_1)} + \underbrace{\frac{1}{2}}_{W_u(C_j)} = 1$$

Hence t_b must be 1, i.e. there is only one marked point, which is fixed by φ_j. Then C_j is a singular elliptic tail of C and the irreducible node P_1 on C_j is exceptional, i.e.

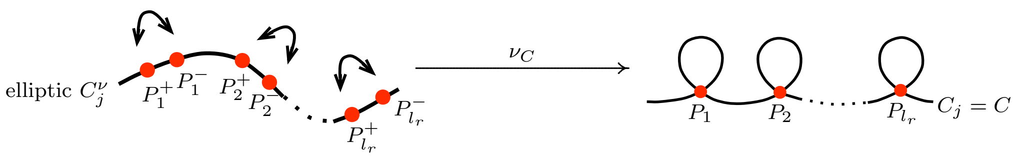


Consider the one-dimensional deformation of (X, L, b) in which the exceptional component E_1 disappears, i.e. the restriction of the local universal deformation

$(\mathcal{X} \to \mathbb{C}_\tau^{3g-3}, \mathcal{L}, \mathcal{B})$ to $W_\tau(P_1)$ and the corresponding deformation of C, i.e. the restriction of $\mathcal{C} \to \mathbb{C}_t^{3g-3}$ to $W_t(P_1)$. Let $(X, L, b)'$ be a nearby curve with stable model C', i.e. $(X, L, b)'$ is the fibre over a point in $W_\tau(P_1)$ which is mapped by $\mathbb{C}_\tau^{3g-3} \to \mathbb{C}_t^{3g-3}$ to the point in $W_t(P_1)$ over which C' is the fibre. The singular elliptic tail C_j deforms to a smooth elliptic tail C_j'. Since the action of σ_C sends t_1 to t_1, σ_C deforms to an automorphism $\sigma_{C'}$ of the curve C'. The restriction of $\sigma_{C'}$ to the elliptic tail C_j' has the same order as that of σ_C to C_j, i.e. it is the elliptic involution. The automorphism (σ, γ) of (X, L, b) then deforms to an automorphism $(\sigma, \gamma)'$ of $(X, L, b)'$ in the following way. σ' is the same as σ on the complement of $C_j^\nu \cup E_1 \cup E_2$ while on the elliptic tail C_j' it is the elliptic involution. Since the elliptic involution lifts to every theta characteristic of the elliptic curve the restriction of γ to C_j^ν deforms to an isomorphism $\sigma'^*_{|C_j'} L'^\nu_j \to L'^\nu_j$ while on the complement γ' is just γ. This extends to the automorphism $(\sigma, \gamma)'$. This shows that the action of (σ, γ) on τ_1 is trivial, which give a contradiction to the singularity reducedness of (σ, γ), which as explained above implies $\tau_1 \mapsto -\tau_1$. Therefor case (ii) is excluded.

Case (iii): C_j^ν is elliptic, say $C_j^\nu \cong \mathbb{C}/\mathbb{Z}z_0 + \mathbb{Z}$, $z_0 \in \mathbb{H}_1$. Then φ_j is a translation or has a fixed point, and if it has a fixed point its order is 2, 4, 3 or 6.

Subcase translation: Since C is stable, there must be at least one marked point on C_j^ν. But the translation φ_j has no fixed points, hence all the marked points must be of the red type, i.e. the marked points appear in pairs of order 2 points. In order to have order 2 points the automorphism φ_j must have order 2, i.e. it is the translation by one of the three non-trivial 2-torsion points.



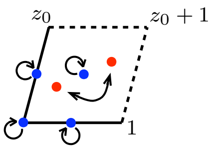
Since C is connected and has genus at least 4, there must be at least three nodes on C, denote them by P_i, $i = 1, \dots, l_r$ and let t_i the corresponding coordinate. Since φ_j has order 2 and σ_C interchanges the branches at each node, σ_C acts by

$$x \mapsto y \mapsto x$$

where $xy = 0$ is a local equation for C at P_i. Therefore $t_i \mapsto t_i$ and $\tau_i \mapsto \pm\tau_i$. Since (σ, γ) is singularity reduced and P_i non-disconnecting, $\tau_i \mapsto \tau_i$ is not possible and every node P_i contributes $\frac{1}{2}$ to the Reid-Tai sum.

$$\Longrightarrow 1 > \frac{1}{n} \sum_{j=1}^{3g-3} a_j \geq \#\{P_i\} \cdot \frac{1}{2} \geq \frac{3}{2}.$$

Subcase $\operatorname{ord}\varphi_j = 2$: φ_j is the elliptic involution and has exactly four fixed points, the remaining points are of order 2, i.e. there can be up to four marked points of the blue type and any (even) number of the red type. In the fundamental parallelogram of the elliptic curve $\mathbb{C}/\mathbb{Z}z_0 + \mathbb{Z} \cong C_j^\nu$ the picture is the following.



If there exists a node of the red type, say P_1, i.e. φ_j interchanges the two preimages $P_1^\pm$, the action on the corresponding coordinate is $t_1 \mapsto t_1$ and (as above) $\tau_1 \mapsto -\tau_1$, giving a contribution of $\frac{1}{2}$ to the Reid-Tai sum. Since the sum is smaller than 1, P_1 is the only node of this type. If there were not a node of the blue type, the curve would have genus 2, therefore there is at least one marked point, which is fixed by φ_j. There exists a one-dimensional deformation of $(C_j^\nu, \{P_i^\pm\})$ such that φ_j does not deform in this family: Fix the curve C_j^ν and all marked points but P_1^-, which is deformed to Q. In this deformation the elliptic involution does not interchange the points P_1^+ and Q, hence the involution does not deform to the deformed pointed curve. This implies that the action on $W_t(C_j)$ is not trivial and since the order of φ_j is two B must have -1 as an eigenvalue.

$$\Longrightarrow 1 > \frac{1}{n}\sum_{j=1}^{3g-3} a_j \geq \underbrace{\frac{1}{2}}_{W_u(P_1)} + \underbrace{\frac{1}{2}}_{W_u(C_j)} = 1.$$

Therefore there is no node of the red type, all marked points are fixed points of the involution, i.e. 2-torsion points, and there are 1 up to 4 of these. The dimension d is $3\cdot 1 - 3 + \#(\text{marked points}) = 1, 2, 3$ or 4. The deformation space of the elliptic curve with one marked point is one-dimensional and the elliptic involution deforms to every deformation, giving an eigenvalue 1 for the matrix B. Choosing one of the marked points as the origin of the elliptic curve, i.e. sending this point to $0 \in \mathbb{C}/\mathbb{Z}z_0 + \mathbb{Z}$, determines the set of 2-torsion points, hence the remaining marked points, which are 2-torsion points, cannot be deformed without loosing the involution as an automorphism of the pointed curve, i.e. 1 is eigenvalue of multiplicity 1. Since the order of φ_j is 2 the remaining eigenvalues must be -1, i.e.

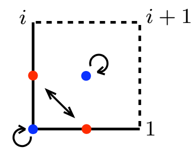
$$B \sim \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$$

giving a contribution of $(d-1)\cdot\frac{1}{2}$ to the Reid-Tai sum. Hence if d is three or four

$$1 > \frac{1}{n} \sum_{j=1}^{3g-3} a_j \geq (d-1) \cdot \frac{1}{2} \geq 1$$

Therefore there are one or two marked points, which are fixed points of the involution. If there is only one point $(C_j^\nu, \{P_i^\pm\}, \varphi_j)$ is the elliptic tail case of order 2 and the contribution of B to the Reid-Tai sum is $w_j = 0$. In case of two marked points, they could either map to one node (irreducible case), giving a genus 2 curve hence a contradiction, or to two different nodes, giving the elliptic ladder case of order 2 and a contribution to the Reid-Tai sum of $w_j = (2-1) \cdot \frac{1}{2} = \frac{1}{2}$.

Subcase ord $\varphi_j = 4$: C_j^ν is elliptic with j-invariant 1728. There is an identification $C_j^\nu \cong \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z}i + \mathbb{Z}$, and φ_j is one of the two order 4 automorphisms. Then φ_j has two fixed points and one pair of order 2 points, all other points are of order 4. In the fundamental parallelogram this looks as follows.



Hence the number of marked points is at least 1 and at most 4. If there is only one marked point, it must be fixed by φ_j. The automorphism φ_j does not deform to a deformation of the pointed elliptic curve, but its square, the elliptic involution does, hence $B = (-1)$, giving a contribution of $w_j = \frac{1}{2}$ to the Reid-Tai sum. This is the elliptic tail case of order 4. If there are two marked points, there are two possibilities, either they map to the same node or they are fixed points and map to different nodes (blue case, non-irreducible). In the first case C has genus 2, therefore only the second case is possible, which is the elliptic ladder case of order 4. The contribution w_j to the Reid-Tai sum is at least $\frac{3}{4}$, since φ_j^2, the elliptic involution, acts as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on $W_t(C_j)$, which is the same as B^2, hence

$$B \sim \begin{pmatrix} -1 & 0 \\ 0 & \xi \end{pmatrix}$$

where ξ is a 4th root of 1, but $\xi \neq 1$. B cannot have 1 as an eigenvalue, because the order four automorphism does not deform to any deformation of the pointed curve.)

If there are three resp. four marked points on C_j^ν an analogous argument shows that φ_j^2 acts as the (3×3)- resp. (4×4)-matrix

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$$

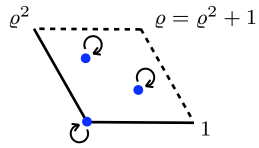
$$\Longrightarrow B = \begin{pmatrix} -1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_d \end{pmatrix}$$

with appropriate 4th roots ξ_i of 1, $\xi_i \neq 1$. The contribution to the Reid-Tai sum is then

$$\geq \frac{1}{2} + (d-1) \cdot \frac{1}{4} \geq 1,$$

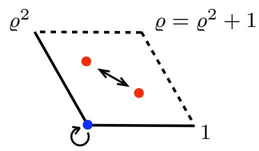
hence these cases are excluded.

Subcase $\operatorname{ord} \varphi_j = 3$: C_j^ν is elliptic with j-invariant 0. Fix an isomorphism $C_j^\nu \cong \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z}\varrho^2 + \mathbb{Z}$, where $\varrho = e^{2\pi i/6}$. φ_j is one of the two order 3 automorphisms, which have three fixed points, all other points have order 3. In the fundamental parallelogram the picture is the following.



The red case does not appear and the number of marked points is 1, 2 or 3. If it is 1, this is the elliptic tail case of order 3. The contribution w_j to the Reid-Tai sum is at least $\frac{1}{3}$, since $d = 1$ and the order 3 automorphism does not deform to other elliptic curves, hence B is not trivial. If there are two marked points, they must map to two different nodes, because else C has only genus 2, therefore this is the elliptic ladder case of order 2. The contribution w_j to the Reid-Tai sum is at least $\frac{2}{3}$, because $d = 2$ and φ_j does not deform to any deformation of $(C_j^\nu, \{P_i^\pm\})$. If there are three marked points the dimension d is 3 and since φ_j still does not deform to any deformation of $(C_j^\nu, \{P_i^\pm\})$ the contribution to the Reid-Tai sum is at least $\frac{3}{3} = 1$, hence this case is excluded.

Subcase $\operatorname{ord} \varphi_j = 6$: C_j^ν is elliptic with j-invariant 0. Fix an identification $C_j^\nu \cong \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z}\varrho^2 + \mathbb{Z}$, where $\varrho = e^{2\pi i/6}$. φ_j is one of the two order 6 automorphisms, which have one fixed point and one pair of order 2 points.



Hence the number of marked points is 1, 2 or 3. If there is only one marked point this the elliptic tail case of order 6 and $d = 1$. Since φ_j^3 is the elliptic involution, which deforms to every deformation of the pointed elliptic curve, the only eigenvalue of B is a third root of 1, since the order 6 automorphisms do not deform to deformations of $(C_j^\nu, \{P_i^\pm\})$, this eigenvalue is not 1, hence gives a contribution to the Reid-Tai sum of at least $\frac{1}{3}$. If there are two marked points, they must be the two order 2 points, but then C has only genus 2, hence this case is not possible. If there are three marked points, one is a fixed point, the other two are interchanged, the dimension d is 3 and φ_j^3, the elliptic involution, acts as

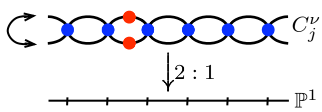
$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

This is the case, since the elliptic involution deforms to a deformation of the pointed curve if and only if the third marked point is the image of the second under the involution; therefore B^3, the matrix induced by φ_j^3, has 1 as an eigenvalue of multiplicity 2, the third eigenvalue must be -1 out of degree reasons. For the matrix B this implies that

$$B = \begin{pmatrix} \xi_3 & & \\ & \xi_3' & \\ & & \xi_6 \end{pmatrix},$$

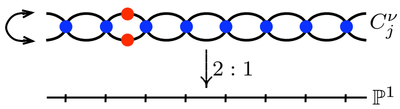
where ξ_3 and ξ_3' are primitive 3rd roots and ξ_6 a primitive 6th root. The contribution to the Reid-Tai sum is at least $\frac{5}{6}$. But there is also a contribution of the irreducible node P_1 on C_j. Let $xy = 0$ be a local equation of C at P_1, such that σ_C acts as $x \mapsto y$ (this is possible since σ_C interchanges the branches at P_1). Then $y \mapsto \alpha \cdot x$ for a primitive 3rd root α of 1 and $t_1 = xy \mapsto y \cdot \alpha x = \alpha t_1$. Since the node P_1 is non-disconnecting $u_1 = \tau_1$ and as always $t_1 = \tau_1$ or $t_1 = \tau_1^2$. Then $\tau_1 \mapsto \widetilde{\alpha}\tau_1$, where $\widetilde{\alpha} = \alpha$ in case $t_1 = \tau_1$ or $\widetilde{\alpha}$ is a square root of α in case $t_1 = \tau_1^2$. In any case $\widetilde{\alpha}$ is a primitive 3rd root or a primitive 6th root, giving a contribution of at least $\frac{1}{6}$ to the Reid-Tai sum. Together with the $\frac{5}{6}$ of the matrix B, the Reid-Tai sum is at least 1, hence this case is excluded.

Case (iv): C_j^ν has genus 2 and φ_j is the hyperelliptic involution, which has 6 fixed points, the Weierstraß points, the remaining points are of order 2. The orbits under the hyperelliptic involution are exactly the fibres of the associated degree 2 map



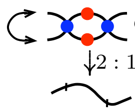
Suppose there is at least one pair of order 2 points in the marked points, i.e. C has an irreducible node P_1 on C_j and σ_C interchanges the branches. Let $xy = 0$ be a local equation at P_1 then σ_C interchanges the two coordinates and induces $t_1 \mapsto t_1$, i.e. $\tau_1 \mapsto \pm\tau_1$. Since P_1 is non-disconnecting and (σ, γ) singularity reduced it must be $\tau_1 \mapsto -\tau_1$, i.e. the node gives a contribution of $\frac{1}{2}$. Therefore P_1 is the only node of this type. At least one Weierstraß point must be a marked point, since otherwise, C has only genus 3. Denote by l_b the number of marked Weierstraß points, then $1 \le l_b \le 6$, there are $2 + l_b$ marked points and the dimension $d = 3 \cdot 2 - 3 + (2 + l_b) = 5 + l_b$. If the curve C_j^ν is deformed to $(C_j^\nu)'$ (three degrees of freedom) and the point P_1^+ to $(P_1^+)'$ (one additional degree of freedom) there is a unique choice of l_b Weierstraß points of $(C_j^\nu)'$ such that the hyperelliptic involution deforms to this new pointed curve. If the image of $(P_1^+)'$ under the involution is added to the marked points, the involution will still deform to this pointed curve, i.e. there is a four-dimensional deformation of $(C_j^\nu, \{P_i^\pm\})$ to which the involution deforms and the multiplicity of the eigenvalue 1 of the matrix B is 4. The remaining eigenvalues must be -1 giving a contribution of $(d-4) \cdot \frac{1}{2} = (1 + l_b) \cdot \frac{1}{2} \ge 1$ to the Reid-Tai sum. Therefore all marked points must be Weierstraß points, say there are l_b of them, $1 \le l_b \le 6$. Then $d = 3 + l_b$. The curve C_j^ν can be deformed (three degrees of freedom) but in order to deform the involution to the pointed curve the marked points must be Weierstraß points, therefore the multiplicity of the eigenvalue 1 is 3 and the remaining eigenvalues are -1, giving a contribution of $(d-3) \cdot \frac{1}{2} = l_b \cdot \frac{1}{2}$. As long as $l_b \ge 2$ the Reid-Tai sum is ≥ 1 and if $l_b = 1$ this is the hyperelliptic tail case and the contribution to the Reid-Tai sum is $\frac{1}{2}$.

Case (v): C_j^ν has genus 3 and φ_j is the hyperelliptic involution, which has 8 fixed points, the Weierstraß points, the remaining points are of order 2. The orbits under the hyperelliptic involution are again the fibres of the associated degree 2 map



Suppose there is a node of the red type, say P_1, as above $x \mapsto y \mapsto x$ for local coordinates of C at P_1 and $t_1 \mapsto t_1$, $\tau_1 \mapsto \pm\tau_1$. Since (σ, γ) is singularity reduced, it must be $\tau_1 \mapsto -\tau_1$ and P_1 is the only node of this type. The hyperelliptic locus is a divisor in M_3 therefore there exists a deformation of C_j^ν to which the hyperelliptic involution does not deform, giving at least one non-trivial eigenvalue, which must contribute $\frac{1}{2}$, together with the contribution of the node, the Reid-Tai sum is at least 1. Therefore the marked points are all Weierstraß points and $d = 3 \cdot 3 - 3 + l_b = 6 + l_b$ where l_b is the number of marked points and $1 \le l_b \le 8$. There is a five-dimensional deformation of C_j^ν to which the hyperelliptic involution deforms, but then the marked points are fixed. Therefore the multiplicity of the eigenvalue 1 is 5 and that of the eigenvalue -1 is $6 + l_b - 5 = 1 + l_b \geq 2$, giving a contribution of at least 1 to the Reid-Tai sum.

Case (vi): C_j^ν is bielliptic of genus 2 and φ_j is the associated involution. The Hurwitz formula implies that the covering map from C_j^ν to the elliptic curve has two ramification points, hence φ_j has two fixed points, the remaining points have order 2. The double cover looks like



If P_1 is a node of the red type, it gives a contribution of $\frac{1}{2}$. Since the locus of double coverings of elliptic curves in M_2 is two-dimensional, the involution does not deform to every genus 2 curve, hence -1 is an eigenvalue and the Reid-Tai sum is at least 1. Therefore all marked points are fixed points, the number of marked points l_b is 1 or 2 and $d = 3 \cdot 2 - 3 + l_b = 3 + l_b = 4$ or 5. There is a two-dimensional deformation of C_j^ν such that the involution deforms to every deformation, but then the marked points are fixed. Hence the multiplicity of the eigenvalue 1 is 2, but then the multiplicity of the eigenvalue -1 is $3 + l_b - 2 = 1 + l_b \geq 2$ giving a contribution of at least 1 to the Reid-Tai sum. Therefore this last case is also excluded. □

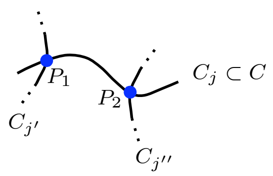
Proposition 4.3.21. *The hyperelliptic tail case is impossible.*

Proof. Let C_j be a hyperelliptic tail, i.e. $(C_j^\nu, \{P_i^\pm\})$ has genus 2, one marked point say P_1^+ and φ_j is the hyperelliptic involution. The action of $K(\sigma, \gamma)$ on $W_u(C_j)$ contributes $\frac{1}{2}$. Denote by $C_{j'}$ the irreducible component of C that meets C_j at the node P_1. Then $(C_{j'}^\nu, \{P_i^\pm\}, \varphi_{j'})$ must be of one of the types in Proposition 4.3.20.

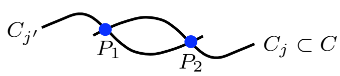
If the second component is a hyperelliptic tail or an elliptic ladder the action of $K(\sigma, \gamma)$ on $W_u(C_{j'})$ gives at least $\frac{1}{2}$, hence the two components together give at least 1. If the second component is an elliptic tail the whole curve has only genus 3. Therefore the only case not excluded is the identity case, i.e. $\varphi_{j'} = \mathrm{id}$. Consider the action on $W_u(P_1)$. Let $xy = 0$ be a local equation for C at P_1. Then (wlog) $x \mapsto -x$ (on C_j) and $y \mapsto y$ (on $C_{j'}$). Hence $t_1 \mapsto -t_1$. P_1 is a disconnecting node but not an elliptic tail node, hence $\tau_1^2 = t_1$ and $u_1 = \tau_1^2$. Therefore $u_1 \mapsto -u_1$ contributing $\frac{1}{2}$ and together with the action on $W_u(C_j)$ this gives at least 1. Therefore the hyperelliptic tail case is not possible. □

Proposition 4.3.22. *The elliptic ladder cases are impossible.*

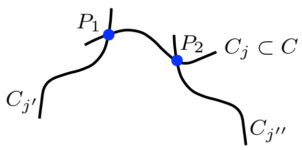
Proof. Let C_j be an elliptic ladder, i.e. $(C_j^\nu, \{P_i^\pm\})$ has genus 1, two marked points, say P_1^+ and P_2^+, which are fixed points of φ_j and $\operatorname{ord}\varphi_j = 2$, 4 or 3. Denote by $C_{j'}$ resp. $C_{j''}$ the components meeting C_j at P_1 resp P_2. They must be of one of the types of Proposition 4.3.20, but not hyperelliptic tails, i.e. they are identity components, elliptic tails or elliptic ladders. It is possible that $C_{j'} = C_{j''}$, then $C_{j'}$ is an identity component or an elliptic ladder.



Suppose $C_{j'}$ is an elliptic ladder. If $C_{j'} = C_{j''}$ the whole curve has only genus 3.



If $C_{j'} \neq C_{j''}$ the contribution of the action on $W_u(C_j)$ and $W_u(C_{j'})$ is at least 1, since every elliptic ladder gives at least $\frac{1}{2}$. Hence $C_{j'}$ and $C_{j''}$ are either elliptic tails or identity components and if they are both elliptic tails, the genus of the curve is only 3.

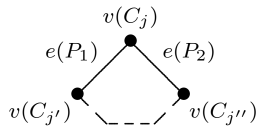


Therefore one of the two components, say $C_{j'}$, is an identity component. Consider the action on $W_u(P_1)$. Let $xy = 0$ be a local equation for C at P_1. Then (wlog) $x \mapsto x$, since $\varphi_{j'}$ is the identity, and

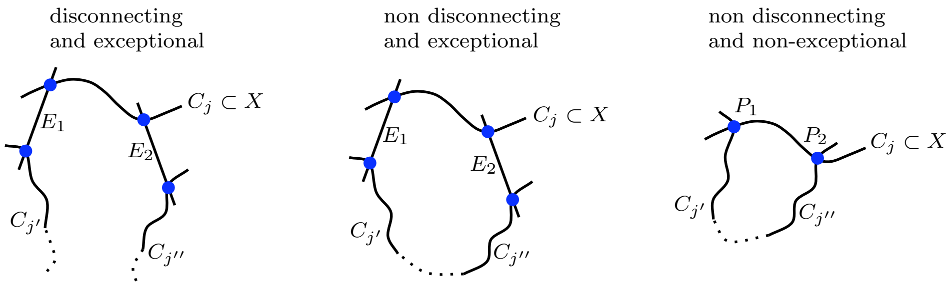
$$y \mapsto \alpha y = \begin{cases} -y & \text{if } \operatorname{ord}\varphi_j = 2 \\ \pm iy & \text{if } \operatorname{ord}\varphi_j = 4 \\ \zeta_3 y & \text{if } \operatorname{ord}\varphi_j = 3 \end{cases}$$

on C_j, where ζ_3 is an appropriate primitive 3rd root. Therefore $t_1 = xy \mapsto \alpha t_1$. The node P_1 is either disconnecting and exceptional ($u_1 = \tau_1^2$ and $\tau_1^2 = t_1$) or non-disconnecting and exceptional ($u_1 = \tau_1$ and $\tau_1^2 = t_1$) or non-disconnecting and non-exceptional ($u_1 = \tau_1$ and $\tau_1 = t_1$).

Claim: The node P_2 has the same type as P_1. If P_1 is exceptional resp. non-exceptional P_2 must be exceptional resp. non-exceptional out of degree reasons. Now consider the dual graph $\Gamma(C)$ of C. P_1 is non-disconnecting if and only if removing the edge $e(P_1)$ corresponding to P_1 in $\Gamma(C)$ leads to a connected graph. This is the case if and only if there exists a cycle of edges in $\Gamma(C)$ starting with $e(P_1)$. Since P_2 is the only other node on C_j the vertex $v(C_j)$ corresponding to C_j has exactly two edges, namely $e(P_1)$ and $e(P_2)$.



Therefore every cycle of edges starting with $e(P_1)$ must contain $e(P_2)$. Hence the edge $e(P_1)$ is non-disconnecting if and only if the edge $e(P_2)$ is, and the same is true for the nodes P_1 and P_2. Then $C_j \subset X$ looks like one of the following cases.



In the first picture $\overline{X \setminus C_j}$ consists of two connected components and $C_{j'} \neq C_{j''}$. In the last two pictures the dotted line indicates the fact that $\overline{X \setminus C_j}$ is connected; here $C_{j'} = C_{j''}$ and $C_{j'} \neq C_{j''}$ is possible.

This shows that either $u_1 = t_1$ and $u_2 = t_2$ (if P_1 and P_2 are both disconnecting and exceptional or both non-disconnecting and non-exceptional) or $u_1^2 = t_1$ and $u_2^2 = t_2$ (if P_1 and P_2 are both non-disconnecting and exceptional).

In the first case $u_1 \mapsto \alpha u_1$ and summing up the contributions of $W_u(P_1)$ and $W_u(C_j)$ (in this order) gives at least

$$1 = \begin{cases} \frac{1}{2} + \frac{1}{2} & \text{if } \operatorname{ord} \varphi_j = 2 \\ \frac{1}{4} + \frac{3}{4} & \text{if } \operatorname{ord} \varphi_j = 4 \\ \frac{1}{3} + \frac{2}{3} & \text{if } \operatorname{ord} \varphi_j = 3 \end{cases}$$

This calculation is also valid in case $C_{j'} = C_{j''}$, which is only possible if P_1 and P_2 are non-disconnecting and non-exceptional.

In the second case, i.e. $u_1^2 = t_1$ and P_1 and P_2 are non-disconnecting and exceptional, the action must be $u_1 \mapsto \widetilde{\alpha} u_1$ for a suitable root $\widetilde{\alpha}$ of α, in particular

$\widetilde{\alpha} \neq 1$. $\widetilde{\alpha}$ is a (not necessarily primitive) root of 1 of order

$$\operatorname{ord}\widetilde{\alpha} = \begin{cases} 4 & \text{if } \operatorname{ord}\varphi_j = 2 \\ 8 & \text{if } \operatorname{ord}\varphi_j = 4 \\ 6 & \text{if } \operatorname{ord}\varphi_j = 3 \end{cases}$$

Since $\widetilde{\alpha}$ is not 1, this gives a contribution of at least $\frac{1}{4}$, $\frac{1}{8}$ resp. $\frac{1}{6}$.

If $C_{j''}$ were an elliptic tail, the node P_2 would have to be disconnecting. Hence $C_{j''}$ is also an identity component or $C_{j'} = C_{j''}$. In both cases the action on u_2 has the same order as that on u_1 and there is an additional contribution of at least $\frac{1}{4}$, $\frac{1}{8}$ resp. $\frac{1}{6}$. Summing up the contributions on $W_u(C_j)$, $W_u(P_1)$ and $W_u(P_2)$ (in this order) gives at least

$$1 = \begin{cases} \frac{1}{2} + \frac{1}{4} + \frac{1}{4} & \text{if } \operatorname{ord}\varphi_j = 2 \\ \frac{3}{4} + \frac{1}{8} + \frac{1}{8} & \text{if } \operatorname{ord}\varphi_j = 4 \\ \frac{2}{3} + \frac{1}{6} + \frac{1}{6} & \text{if } \operatorname{ord}\varphi_j = 3 \end{cases}$$

Therefore all the elliptic ladder cases are excluded. □

Proposition 4.3.23. *The elliptic tail case of order 4 is impossible.*

Proof. Let C_j be an elliptic tail, $\operatorname{ord}\varphi_j = 4$, P_1 the node on C_j with corresponding coordinate t_1 and t_2 the coordinate of the one-dimensional deformation space $W_t(C_j)$. The node P_1 is an elliptic tail node

$$\Longrightarrow \tau_1^2 = t_1 \text{ and } u_1 = \tau_1^4 = t_1^2.$$

Since the cases of hyperelliptic tails and elliptic ladders are already excluded, the second component $C_{j'}$ through P_1 must be an elliptic tail or an identity component and if it is an elliptic tail, the curve C has only genus 2. Hence $C_{j'}$ is an identity component and the action of $M(\sigma_C)$ on $W_t(P_1) \oplus W_t(C_j)$ is the following

$$M(\sigma_C)_{|W_t(P_1)\oplus W_t(C_j)} = \begin{pmatrix} \pm i & \\ & -1 \end{pmatrix}$$

In order to get the action of $K(\sigma, \gamma)$ on the corresponding subspace of $\mathbb{C}_u^{3g-3}$ the upper left entry of the matrix has to be squared, since $u_1 = t_1^2$.

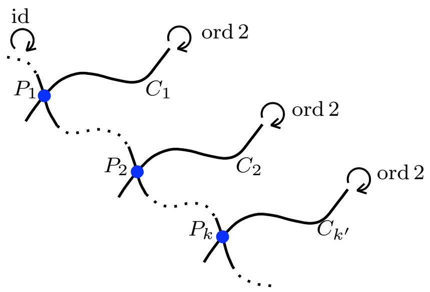
$$\Longrightarrow K(\sigma, \gamma)_{|W_u(P_1)\oplus W_u(C_j)} = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$$

The contribution to the Reid-Tai sum is $\frac{1}{2} + \frac{1}{2} = 1$ with respect to every primitive 4th root. Hence these cases are excluded. □

Denote by C_j, $j = 1, \ldots, k$, all elliptic tails of C on which σ_C acts non-trivially. Denote by P_j, $j = 1, \ldots, k$, the elliptic tail node on C_j and choose the preimage P_j^+ of P_j on C_j^ν under the normalisation as the base point of the elliptic curve C_j^ν. The last proposition implies that the automorphism $\varphi_j = \sigma^\nu_{C|C_j^\nu}$, $j = 1, \ldots, k$, is of order 2, 3 or 6. On all other components of C σ_C is trivial. Denote by $L_j^\nu = \nu^* L_{|C_j^\nu}$, $j = 1, \ldots, k$, the theta characteristic of (X, L, b) over C_j^ν.

Proposition 4.3.24. *If for all C_j, $j = 1, \ldots, k$, with j-invariant 0 the theta characteristic L_j^ν is non-trivial, i.e. L_j^ν is a theta characteristic corresponding to a non-trivial 2-torsion point the matrix $K(\sigma, \gamma)$ has order 2.*

Proof. Consider an elliptic tail C_j with j-invariant 0. The assumption that the theta characteristic L_j^ν on C_j is non-trivial implies that the automorphisms of order 3 and 6 of C_j do not fix L_j^ν, i.e. these automorphisms do not lift to the spin structure. Hence this requirement translates into the fact, that the automorphisms φ_j for the elliptic tails C_j, $j = 1, \ldots, k$, with j-invariant 0 are elliptic involutions. Therefore C consists of identity components and elliptic tails C_j, $j = 1, \ldots, k'$ of order 2, where $C_{k+1}, \ldots, C_{k'}$ are the remaining elliptic tails on which σ_C acts non-trivially.



Denote by t_j, $j = 1, \ldots, k'$, the coordinate corresponding to the elliptic tail node P_j connecting C_j to the rest of the curve. Then

$$M(\sigma_C)_{|W_t(P_j) \oplus W_t(C_j)} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

and, since P_j is an elliptic tail node, $u_j = \tau_j^4 = t_j^2$, and

$$K(\sigma, \gamma)_{|W_u(P_j) \oplus W_u(C_j)} = \begin{pmatrix} (-1)^2 & \\ & 1 \end{pmatrix} = \mathbb{I}.$$

Let $C_{j'}$ be any component of C that is not an elliptic tail of order 2. Then σ_C is the identity on $C_{j'}$ and $M(\sigma_C)_{|W_t(C_{j'})} = \mathbb{I}$. Since the t-, τ- and u-coordinates corresponding to components coincide, this gives

$$K(\sigma, \gamma)_{|W_u(C_{j'})} = M(\sigma_C)_{|W_t(C_{j'})} = \mathbb{I}.$$

Claim: C has only exceptional nodes. Assume to the contrary that there is a non-exceptional node P_m, then P_m cannot be disconnecting out of degree reasons, in particular P_m is not an elliptic tail node, and $\tau_m = t_m$ (non exceptional) and $u_m = \tau_m$ (non disconnecting). Since P_m is not an elliptic tail node, the components meeting at P_m must be identity components. This means that near P_m the automorphism σ_C acts as the identity and $t_m \mapsto t_m$. But then $\tau_m \mapsto \tau_m$ which is not possible, since (σ, γ) is singularity reduced. Therefore the only nodes that appear are exceptional.

Now let $P_m \neq P_j$, $j = 1, \dots, k'$ be a node. As above σ_C is the identity near P_m and $t_m \mapsto t_m$. Since P_m has to be exceptional $\tau_m^2 = t_m$ and $\tau_m \mapsto \pm\tau_m$, where the $+$-sign is only possible, if the node is disconnecting, since (σ, γ) is singularity reduced. There are three possibilities for P_m. Either it is an elliptic tail node, where σ_C is the identity on the elliptic tail, i.e. $u_m = \tau_m^4$, or it is disconnecting but not an elliptic tail node, i.e. $u_m = \tau_m^2$, or it is non-disconnecting, i.e. $u_m = \tau_m$. In the first two cases $u_m \mapsto u_m$, hence $K(\sigma, \gamma)_{|W_u(P_m)} = (1)$. In the third case $\tau_m \mapsto -\tau_m$ since (σ, γ) is singularity reduced, hence also $u_m \mapsto -u_m$ and $K(\sigma, \gamma)_{|W_u(P_m)} = (-1)$.

Remembering that T is the set of elliptic tail nodes, $T \cup D$ the set of disconnecting nodes, $T \cup D \cup \overline{N} = N$ the set of exceptional nodes, which is here the set of all nodes, i.e. Δ, the set of non-exceptional nodes, is empty and putting everything together shows that

$$K(\sigma, \gamma) = \begin{pmatrix} K_T \mathbb{E}_{\mathrm{id}_T} & & & \\ & K_D \mathbb{E}_{\mathrm{id}_D} & & \\ & & K_{\overline{N}} \mathbb{E}_{\mathrm{id}_{\overline{N}}} & \\ & & & K_V \mathbb{E}_{\mathrm{id}_V} \end{pmatrix} = \begin{pmatrix} \mathbb{I} & & & \\ & \mathbb{I} & & \\ & & -\mathbb{I} & \\ & & & \mathbb{I} \end{pmatrix}$$

Since $K(\sigma, \gamma) \neq \mathbb{I}$ by assumption, the order of $K(\sigma, \gamma)$ is two. □

Since there are no quasi reflections in $K(\mathrm{Aut}(X, L, b))$ the order-2-matrix $K(\sigma, \gamma)$ of the proposition has -1 as an eigenvalue of multiplicity at least 2, hence the Reid-Tai sum is at least 1 and the case considered in the proposition is excluded.

Corollary 4.3.25. *The case where C has only identity components and elliptic tails of order 2 is impossible.*

In conclusion all cases but the elliptic tail cases of order 3 and 6 are excluded. This together with the calculations in Remark 4.3.10 shows that for a singularity reduced pair $((X, L, b), (\sigma, \gamma))$ the matrix $K(\sigma, \gamma)$ does not fulfil the Reid-Tai inequality for some primitive root ζ if and only if the induced automorphism σ_C on the stable model is an elliptic tail automorphism of order 3 or 6. Which in turn implies that the theta characteristic on the elliptic tail in X is trivial.

If $[(X,L,b)] \in \overline{S}_g$ is non-canonical with $K(\sigma,\gamma) \in K(\mathrm{Aut}(X,L,b))$ not fulfilling the Reid-Tai inequality (for some root) and $((X,L,b),(\sigma,\gamma))$ is not singularity reduced Corollary 4.3.14 states that smoothing an appropriate set of non-disconnecting nodes yields a nearby spin curve $(X,L,b)'$ such that (σ,γ) deforms to an automorphism $(\sigma,\gamma)'$ of $(X,L,b)'$, $K(\sigma,\gamma)$ and $K(\sigma,\gamma)'$ have the same eigenvalues and $((X,L,b)',(\sigma,\gamma)')$ is singularity reduced. As the matrix $K(\sigma,\gamma)$ does not fulfil the Reid-Tai inequality the matrix $K(\sigma,\gamma)'$ cannot fulfil it either. Hence $\sigma'_{C'}$ is an elliptic tail automorphism of order 3 or 6 and the theta characteristic of $(X,L,b)'$ on the corresponding elliptic tail is trivial. But since deforming (X,L,b) to $(X,L,b)'$ only involves smoothing some non-disconnecting nodes (X,L,b) also has an elliptic tail C_j with j-invariant 0 and trivial theta characteristic on the tail. Moreover, σ_C is an elliptic tail automorphism with respect to C_j of the same order as $\sigma'_{C'}$. Therefore the assertion is also true if the pair under consideration is not singularity reduced. This ends the proof of Theorem 4.3.9.

Remark 4.3.26. A general point $[(X,L,b)] \in \overline{S}_g$ in the locus of non-canonical singularities has exactly one elliptic tail C_2 with j-invariant 0 and trivial theta characteristic on C_2. The closure of the complement of C_2 in C is a general smooth curve C_1 of genus $g-1$ with trivial automorphism group. For such a spin curve, the group $K(\sigma,\gamma)$ is generated by the matrix

$$\begin{pmatrix} \zeta_3 & & \\ & \zeta_3 & \\ & & \mathbb{I} \end{pmatrix}$$

where ζ_3 is a primitive third root of 1. But there are loci (of higher codimension) where more complicated non-canonical singularities occur, for example if the curve C_1 has non-trivial automorphisms or the curve has more elliptic tails with j-invariant 0 and trivial theta characteristics on the tails.

The local description of $\overline{S}_g$ at a point $[(X,L,b)]$ as $\mathbb{C}^{3g-3}_\tau/M(\mathrm{Aut}(X,L,b))$ also allows a description of the locus of smooth points of $\overline{S}_g$. Remember the graph $\Sigma(X)$ with vertices $V(\Sigma(X)) = \{\text{connected components of } \widetilde{X}\}$ and edges $E(\Sigma(X)) = \{\text{exceptional components of } X\}$. Since $\widetilde{X}$ is the partial normalisation of the stable model C at all exceptional nodes, $\Sigma(X)$ can be constructed from the dual graph $\Gamma(C)$ by contracting all edges corresponding to non-exceptional nodes. $\Sigma(X)$ is called *tree-like* if removing all loops in $E(\Sigma(X))$ leads to a tree.

Theorem 4.3.27. *Let (X,L,b) be a spin curve of genus $g \geq 4$ with stable model C. The point $[(X,L,b)] \in \overline{S}_g$ is smooth iff*

(i) $\Sigma(X)$ is tree-like and
(ii) the subgroup $\{\sigma_C \in \mathrm{Aut}\, C | \sigma_C$ lifts to $(X,L,b)\}$ is generated by elliptic tail automorphisms of order 2.

Proof. First step: The two conditions are sufficient. Let (X, L, b) be a spin curve with stable model C such that the graph $\Sigma(X)$ is tree-like and the subgroup of liftable automorphisms of C is generated by elliptic tail automorphisms of order 2. Recall the subgroup $H(\operatorname{Aut}(X, L, b)) \subset M(\operatorname{Aut}(X, L, b))$ generated by quasi reflections described in Corollary 4.3.3. Since $[(X, L, b)] \in \overline{S}_g$ is smooth iff $M(\operatorname{Aut}(X, L, b))$ is generated by quasireflections, one has to show that for every automorphism $(\sigma, \gamma) \in \operatorname{Aut}(X, L, b)$ the matrix $M(\sigma, \gamma)$ is contained in $H(\operatorname{Aut}(X, L, b))$.

Let (σ, γ) be an automorphism of (X, L, b) with induced automorphism σ_C of the stable model, in particular σ_C lifts to (X, L, b). Therefore σ_C can be written as a composition of elliptic tail automorphisms of order 2, i.e. there exist elliptic tails $C_1, \dots, C_k$ of C such that $\sigma_C = \iota_1 \circ \dots \circ \iota_k$, where ι_i is the elliptic tail automorphism of order 2 with respect to C_i.

Denote by P_i the node of C connecting C_i to the rest of the curve. Out of degree reasons $P_1, \dots, P_k$ are exceptional nodes. In case C_i is a smooth elliptic tail $L_i^\nu = \nu_X^* L_{|C_i^\nu}$ corresponds to a two-torsion point of C_i^ν, where $\nu_X : X^\nu \to X$ is the normalisation, hence $(\iota^\nu_{i\,|C_i^\nu})^* L_i^\nu \cong L_i^\nu$. In case C_i is a singular elliptic tail, denote by Q_i the node contained in C_i. If the node Q_i is exceptional $L_i^\nu \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ and $(\iota^\nu_{i\,|C_i^\nu})^* L_i^\nu \cong L_i^\nu$. If the node Q_i is non-exceptional $L_i^\nu = \mathcal{O}_{\mathbb{P}^1}$, $(\iota^\nu_{i\,|C_i^\nu})^* L_i^\nu \cong L_i^\nu$ and also $(\iota_{i|C_i})^* L_{|C_i} \cong L_{|C_i}$ since ι_i fixes Q_i. In all cases ι_i fixes all nodes of C, $\iota_{i|\overline{C \setminus C_i}}$ is the identity and $\widetilde{\iota_i}^* \widetilde{L} \cong \widetilde{L}$. Therefore every ι_i lifts to the spin curve (X, L, b).

Abusing notation denote by $(\iota_i, \gamma^{(i)}) \in \operatorname{Aut}(X, L, b)$, $i = 1, \dots, k$, any lift of $\iota_i \in \operatorname{Aut} C$ and consider the concatenation $(\sigma', \gamma') = (\sigma, \gamma) \circ (\iota_1, \gamma^{(1)}) \circ \dots \circ (\iota_k, \gamma^{(k)})$. By construction $\sigma' = \sigma \circ \iota_1 \circ \dots \circ \iota_k$ in $\operatorname{Aut} X$ is the identity on *every* non-exceptional component C_j of C. Therefore (σ', γ') is an inessential automorphism.

By Remark 4.3.2 the $M(\iota_i, \gamma^{(i)})$ are quasi reflections of order 4. Hence one has that $M(\sigma, \gamma) \in H(\operatorname{Aut}(X, L, b))$ iff $(\sigma', \gamma') \in H(\operatorname{Aut}(X, L, b))$ and it is enough to prove, that for every inessential automorphism (σ', γ') the matrix $M(\sigma', \gamma')$ lies in $H(\operatorname{Aut}(X, L, b))$.

Now let $(\sigma, \gamma) \in \operatorname{Aut}(X, L, b)$ be an inessential automorphism and denote by $P_1, \dots, P_k \in \operatorname{sing} C$ the exceptional nodes. By Remark 3.2.24

$$M(\sigma, \gamma) = \begin{pmatrix} (-1)^{\overline{\gamma}_1} & & & \\ & \ddots & & \\ & & (-1)^{\overline{\gamma}_k} & \\ & & & \mathbb{I} \end{pmatrix}$$

where $\overline{\gamma}_i \in \mathbb{Z}_2$ is 1 iff τ_i corresponds to an exceptional component of X connecting two connected components $\widetilde{X}_j$ and $\widetilde{X}_{j'}$ of the non-exceptional subcurve $\widetilde{X}$ such that $\gamma_j \neq \gamma_{j'} \in \mathbb{Z}_2$, i.e. γ is multiplication with $(-1)^{\gamma_j}$ in fibres of L over $\widetilde{X}_j$ and with $(-1)^{\gamma_{j'}}$ over $\widetilde{X}_{j'}$. In particular $\overline{\gamma}_i = 0$ in case τ_i corresponds to an exceptional component which gives a loop in $\Sigma(X)$.

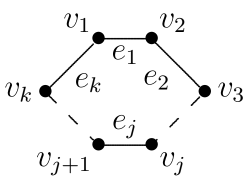
For every $i \in \{1, \ldots, k\}$ let $(\sigma_i, \gamma^{(i)}) \in \mathrm{Aut}_0(X, L, b)$ be defined as follows. If $\overline{\gamma}_i = 0$ let $(\sigma_i, \gamma^{(i)}) = (\mathrm{id}_X, \mathrm{id}_L)$. If $\overline{\gamma}_i = 1$ the exceptional component E_i corresponding to τ_i does not give a loop in $\Sigma(X)$. Every edge in $\Sigma(X)$ is either a loop or disconnecting since $\Sigma(X)$ is tree-like, hence the edge $e(E_i)$ is disconnecting in $\Sigma(X)$. Denote by $X^{(1)}$ and $X^{(2)}$ the two connected components of $\overline{X \setminus E_i}$. Then $(\sigma_i, \gamma^{(i)})$ is the unique inessential automorphism which is multiplication with 1 in the fibres of L over $X^{(1)}$ and with -1 over $X^{(2)}$ and

$$M(\sigma_i, \gamma^{(i)}) = \begin{pmatrix} \mathbb{I} & & \\ & -1 & \\ & & \mathbb{I} \end{pmatrix} \leftarrow i\text{th row}$$

is a quasi reflection. Therefore $M(\sigma, \gamma) = M(\sigma_1, \gamma^{(1)}) \cdots M(\sigma_k, \gamma^{(k)})$ and $M(\sigma, \gamma)$ lies in $H(\mathrm{Aut}(X, L, b))$. Hence the two conditions of the theorem are sufficient.

Second step: The two conditions are necessary. Let $[(X, L, b)]$ be a smooth point of $\overline{S}_g$, therefore, $M(\mathrm{Aut}(X, L, b))$ is generated by quasi reflections, i.e. $M(\mathrm{Aut}(X, L, b)) = H(\mathrm{Aut}(X, L, b))$. Let (σ, γ) be an automorphism of (X, L, b) with induced automorphism σ_C on the stable model C and write $M(\sigma, \gamma)$ as a product of quasi reflections $M(\sigma_1, \gamma^{(1)}) \cdots M(\sigma_k, \gamma^{(k)})$. By Proposition 4.3.1 the induced automorphism of every $(\sigma_i, \gamma^{(i)})$ on C is either the identity or an elliptic tail automorphism of order 2. Therefore σ_C is a composition of elliptic tail automorphisms of order 2 and condition (ii) is necessary.

Now assume that $\Sigma(X)$ is not tree-like. This implies that there exists a cycle $e_1, \ldots, e_k$ of edges in $\Sigma(X)$ such that no e_i is a loop. Denote by $v_1, \ldots, v_k$ the vertices of this cycle, i.e. the edge e_j connects the vertices v_j and v_{j+1}, where indices are considered modulo k. Considering a smaller cycle without loops if necessary one may assume that the v_j are pairwise distinct.



Denote by E_j the exceptional component corresponding to e_j and by $\widetilde{X}_j$ the connected component of $\widetilde{X}$ corresponding to v_j. Consider the inessential automorphism (σ, γ) of (X, L, b) which is multiplication with -1 in fibres of L over the component $\widetilde{X}_1$ and multiplication with 1 over all other connected components of $\widetilde{X}$. By Remark 3.2.24 $M(\sigma, \gamma)$ is a diagonal matrix with (j, j)th entry -1 if τ_j corresponds to an exceptional component, which gives an edge in $\Sigma(X)$ that starts at the vertex v_1 and ends at any vertex $v \neq v_1$, and (j,j)th entry 1 in all other cases. In particular the entries corresponding to the edges e_1 and e_k are -1.

Recall from Corollary 4.3.3 that $H(\mathrm{Aut}(X,L,b))$ is generated by the matrices

$$\begin{pmatrix} \mathbb{I} & & \\ & \xi_j & \\ & & \mathbb{I} \end{pmatrix} \leftarrow j\text{th row}$$

where $\xi_j = i$ if τ_j corresponds to an elliptic tail node P_j of the stable model C, $\xi_j = -1$ if τ_j corresponds to a disconnecting node, which is not an elliptic tail node and $\xi_j = 1$ in all other cases, i.e. if τ_j corresponds to a non-disconnecting node or to some component C_j. In particular let τ_1 be the coordinate corresponding to the non-disconnecting edge e_1. Then the $(1,1)$th entry of every matrix in $H(\mathrm{Aut}(X,L,b))$ is 1. Therefore $M(\sigma,\gamma)$, which has $(1,1)$th entry -1 is not an element of $H(\mathrm{Aut}(X,L,b))$ and $M(\mathrm{Aut}(X,L,b))$ is not generated by quasireflections. This proves that in order for $[(X,L,b)] \in \overline{S}_g$ to be a smooth point, $\Sigma(X)$ is necessarily tree-like. □

Corollary 4.3.28. *For $g \geq 4$ the image of the singular locus $\mathrm{sing}\,\overline{S}_g$ under the forgetful morphism $\pi : \overline{S}_g \to \overline{M}_g$ is*

$$\pi(\mathrm{sing}\,\overline{S}_g) = \mathrm{sing}\,\overline{M}_g \cup \left\{[C] \in \overline{M}_g | \Gamma(C) \text{ is not tree-like}\right\}.$$

Proof. "$\subset$" For $g \geq 4$ let $[(X,L,b)]$ be a singular point of $\overline{S}_g$. By Theorem 4.3.27 either $\Sigma(X)$ is not tree-like or there exists an automorphism $\sigma_C \in \mathrm{Aut}\,C$ which lifts to (X,L,b) and is not a composition of elliptic tail automorphisms of order 2. In the second case $\mathrm{Aut}\,C$ is not generated by elliptic tail automorphisms of order 2, hence by Corollary 4.2.8 $[C]$ is a singular point of $\overline{M}_g$. In the first case $\Sigma(X)$ is not tree-like. Since $\Sigma(X)$ comes from $\Gamma(C)$ by contracting the subset of edges corresponding to non-exceptional nodes of C, $\Gamma(C)$ cannot be tree-like either.

"$\supset$" *First step: If C is a stable curve such that $\Gamma(C)$ is not tree-like, then $[C] \in \pi(\mathrm{sing}\,\overline{S}_g)$.* Let C be a stable curve of genus $g \geq 4$ such that the dual graph $\Gamma(C)$ is not tree-like. The set $\Delta = \emptyset \subset \mathrm{sing}\,C$ is an even subset. This implies that there exists a spin curve (X,L,b) such that $\beta : X \to C$ is the blow up of C at $N = \mathrm{sing}\,C \setminus \Delta = \mathrm{sing}\,C$. Since the graph $\Sigma(X)$ can be constructed from $\Gamma(C)$ by contracting all edges corresponding to non-exceptional nodes, i.e. nodes in $\Delta = \emptyset$, the graphs $\Sigma(X)$ and $\Gamma(C)$ coincide. In particular $\Sigma(X)$ is not tree-like, $[(X,L,b)] \in \overline{S}_g$ is a singular point and $[C] = \pi([(X,L,b)]) \in \pi(\mathrm{sing}\,\overline{S}_g)$.

Second step: If C is a smooth curve and $[C] \in \mathrm{sing}\,M_g$, then $[C] \in \pi(\mathrm{sing}\,\overline{S}_g)$. Let C be a smooth curve of genus $g \geq 4$ such that $[C]$ is a singular point of M_g.

Then there exists a non-trivial automorphism $\sigma_C \in \operatorname{Aut} C$. M. Atiyah proved in his article [Ati71] that there exists a theta characteristic L on C which is fixed by σ_C, i.e. $\sigma_C^* L \cong L$. Therefore σ_C lifts to the spin curve (C, L, b), $[(C, L, b)]$ is a singular point of S_g and $[C] \in \pi(\operatorname{sing} \overline{S}_g)$.

Third step: If C is a singular stable curve such that the dual graph $\Gamma(C)$ is tree-like and $[C] \in \overline{M}_g$ is a singular point, then $[C] \in \pi(\operatorname{sing} \overline{S}_g)$. Let C be such a curve. By Corollary 4.2.8 there exists an automorphism $\sigma_C \in \operatorname{Aut} C$ which is not a product of elliptic tail automorphisms of order 2. If one can find a spin curve (X, L, b) with stable model C such that σ_C lifts to (X, L, b) the point $[(X, L, b)] \in \overline{S}_g$ is singular and $[C] \in \pi(\operatorname{sing} \overline{S}_g)$.

Let $\Delta = \emptyset \subset \operatorname{sing} C$ and $\beta : X \to C$ the blow up at $N = \operatorname{sing} C \setminus \Delta = \operatorname{sing} C$. The non-exceptional subcurve $\widetilde{X}$ is the normalization of C, in particular $\widetilde{X}$ is a disjoint union $\coprod_j C_j^\nu$ of smooth curves, where the union is taken over all irreducible components C_j of C and $C_j^\nu \to C_j$ is the normalisation. The automorphism σ_C induces an automorphism $\widetilde{\sigma}$ on $\widetilde{X}$. The aim is to define a line bundle $\widetilde{L}$ on $\widetilde{X}$ such that $\widetilde{\sigma}^* \widetilde{L} \cong \widetilde{L}$. Consider a component $C_{i_0}^\nu$ of $\widetilde{X}$ and let m be the smallest number such that $C_{i_0}^\nu, C_{i_1}^\nu = \widetilde{\sigma}(C_{i_0}^\nu), \dots, C_{i_{m-1}}^\nu = \widetilde{\sigma}^{m-1}(C_{i_0}^\nu)$ are distinct and $\widetilde{\sigma}^m(C_{i_0}^\nu) = C_{i_0}^\nu$. By Atiyah's result there exists a theta characteristic $\widetilde{L}_{i_0}$ on $C_{i_0}^\nu$ which is fixed by $\left(\widetilde{\sigma}_{|C_{i_0}^\nu}\right)^m$. Fix an isomorphism $b_{i_0} : \widetilde{L}_{i_0}^{\otimes 2} \xrightarrow{\cong} \omega_{C_{i_0}^\nu}$ and let $(\widetilde{L}_{i_j}, b_{i_j})$ for $j = 1, \dots, m-1$ be the appropriate pull back of $(\widetilde{L}_{i_0}, b_{i_0})$ to $C_{i_j}^\nu$, i.e.

$$\widetilde{L}_{i_j} = \left(\widetilde{\sigma}^{m-j}{}_{|C_{i_j}^\nu}\right)^* \widetilde{L}_{i_0} \quad \text{and} \quad b_{i_j} = \left(\widetilde{\sigma}^{m-j}{}_{|C_{i_j}^\nu}\right)^* b_{i_0},$$

in particular $\widetilde{L}_{i_j}$ is a theta characteristic on $C_{i_j}^\nu$. Let $\widetilde{L}$ be the line bundle on $\widetilde{X}$ which is $\widetilde{L}_{i_j}$ on $C_{i_j}^\nu$ and $\widetilde{b} : \widetilde{L}^{\otimes 2} \xrightarrow{\cong} \omega_{\widetilde{X}}$ be the isomorphism which is b_{i_j} on $C_{i_j}^\nu$. Then by construction $\widetilde{\sigma}^* \widetilde{L} \cong \widetilde{L}$.

Let L be a line bundle on X which restricts to $\widetilde{L}$ on $\widetilde{X}$ and to $\mathcal{O}_E(1)$ on every exceptional component E. Moreover, extend $\widetilde{b}$ by 0 on E to get a homomorphism $b : L^{\otimes 2} \to \beta^* \omega_C$. This gives a spin curve (X, L, b) with stable model C. By Corollary 2.2.12 the automorphism σ_C lifts to (X, L, b) since $\widetilde{\sigma}^* \widetilde{L} \cong \widetilde{L}$. Therefore, $[(X, L, b)] \in \operatorname{sing} \overline{S}_g$ and $[C] \in \pi(\operatorname{sing} \overline{S}_g)$. □

Corollary 4.3.29. *Let C be a stable curve of genus $g \geq 4$ such that $\Gamma(C)$ is not tree-like. Then the fibre $\overline{S}_C$ of $\pi : \overline{S}_g \to \overline{M}_g$ over $[C]$ contains a singular point of $\overline{S}_g$, i.e. $[C] \in \pi(\operatorname{sing} \overline{S}_g)$. If in addition $\operatorname{Aut} C$ is generated by elliptic tail automorphisms of order 2, then $\overline{S}_C$ contains a smooth point of $\overline{S}_g$.*

Proof. Let C be a stable curve of genus $g \geq 4$ with $\Gamma(C)$ not tree-like. Then $\Delta = \emptyset \subset \operatorname{sing} C$ is an even subset and there exists a spin curve (X, L, b) with support X, where $\beta : X \to C$ is the blow up of C at $N = \operatorname{sing} C \setminus \Delta = \operatorname{sing} C$. Since $\Sigma(X)$ is the contraction of Δ considered as a subset of the set $E(\Gamma(C))$ of edges of $\Gamma(C)$, the graphs $\Sigma(X)$ and $\Gamma(C)$ coincide. In particular $\Sigma(X)$ is not tree-like and $[(X, L, b)] \in \overline{S}_g$ is singular.

In case $\operatorname{Aut} C$ is generated by elliptic tail automorphisms a point $[(X, L, b)] \in \overline{S}_C$ is smooth if and only if $\Sigma(X)$ is tree-like. Consider the following subset $\Delta \subset E(\Gamma(C))$. An edge $e(P)$ belongs to Δ if and only if there exists a cycle of edges in $\Gamma(C)$ which contains $e(P)$ and $e(P)$ is not a loop. Then Δ is an even subset and there exists a spin curve (X, L, b) whose support X is the blow-up of C at $N = \operatorname{sing} C \setminus \Delta$. The graph $\Sigma(X)$ is then obtained by contracting all edges of $\Gamma(C)$ contained in Δ, i.e. by contracting all cycles of edges in $\Gamma(C)$ which are not loops. The resulting graph is tree-like and $[(X, L, b)] \in \overline{S}_g$ is smooth. Note that in the same way any $\Delta' \supset \Delta$ gives rise to smooth points of $\overline{S}_g$. □

Remark 4.3.30. The question which automorphisms σ_C of C lift to a spin curve (X, L, b) with stable model C is difficult, even in the case of a smooth curve C. By Atiyah's result for every automorphism $\sigma_C \in \operatorname{Aut} C$ of a smooth curve C there exists at least one theta characteristic L on C such that σ_C lifts to the spin curve (C, L, b). Moreover, S. Kallel and D. Sjerve show in [KS06] that an automorphism $\sigma_C \in \operatorname{Aut} C$, where C is smooth, lifts to *every* theta characteristic L on C if and only if C is hyperelliptic and σ_C is the hyperelliptic involution. Therefore, for every automorphism σ_C which is not a hyerelliptic involution there exists at least one theta characteristic to which σ_C lifts and at least one to which it does not lift. It seems to be an interesting question, which subgroups of $\operatorname{Aut} C$ actually arise as the stabiliser of a theta characteristic on the smooth curve C. More generally, the question would be, which subgroups of the automorphism group of a stable curve C do arise as the image of $\operatorname{Aut}(X, L, b) \to \operatorname{Aut} C$, where (X, L, b) is any spin curve with stable model C.

Chapter 5

Global results

In this chapter the Picard groups of $\overline{M}_g$ and $\overline{S}_g$ are described as well as the connection between them. In particular the canonical divisors of $\overline{S}_g^+$ and $\overline{S}_g^-$ are determined. Afterwards an important lifting result for pluricanonical forms on $\overline{S}_g$ is proved.

5.1 The Picard group

5.1.1 Stable curves

This section summarises some properties of the rational Picard group $\mathrm{Pic}_{\mathbb{Q}}(\overline{M}_g) = \mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ of $\overline{M}_g$. It is often easier to work with the rational Picard group of the stack.

Definition 5.1.1. An element $\mathcal{F}$ of the *rational Picard group* $\mathrm{Pic}^{fun}_{\mathbb{Q}}(\overline{M}_g)$ *of the moduli stack of stable curves* is a collection of rational divisor classes $\mathcal{F}(f) \in \mathrm{Pic}_{\mathbb{Q}}(Z)$ for every family $f : \mathcal{C} \to Z$ of stable curves of genus g such that for every fibre square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\overline{\varrho}} & \mathcal{C}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Z & \xrightarrow{\varrho} & Z' \end{array}$$

$\mathcal{F}(f)$ is the pull back of $\mathcal{F}(f')$ via ϱ.

An important fact is the following

Proposition 5.1.2. [HM98, Proposition 3.88]

$$\mathrm{Pic}_{\mathbb{Q}}(\overline{M}_g) \cong \mathrm{Pic}^{fun}_{\mathbb{Q}}(\overline{M}_g)$$

Example 5.1.3. The *Hodge class* λ in $\mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{M}_g)$ is defined as follows. For any family $f : \mathcal{C} \to Z$ of stable curves of genus g the push forward $\Lambda = f_*(\omega_{\mathcal{C}/Z})$ of the relative dualizing sheaf is a vector bundle of rank g, the *Hodge bundle*. Denote by λ_i the ith Chern class $c_i(\Lambda)$ of the Hodge bundle. Then $\lambda_1 = c_1(\Lambda)$ is a line bundle on Z. Since the relative dualizing sheaf is compatible with fibre squares λ is also compatible with them. Therefore λ defines an element in $\mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{M}_g)$, called the Hodge class.

Example 5.1.4. The boundary classes $\delta_0, \dots, \delta_{[g/2]}$ in $\mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{M}_g)$ are defined as follows. Let $Y \subset \overline{M}_g$ be any proper subvariety. It is a general result (see for example [HM98, Section 3.D]) that in order to define an element $\mathcal{F} \in \mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{M}_g)$ it is enough to give the classes $\mathcal{F}(f)$ for families $f : \mathcal{C} \to Z$, where Z is smooth and one-dimensional and the image of the moduli map $Z \to \overline{M}_g$ is not contained in Y. Let $f : \mathcal{C} \to Z$ be a family with smooth one-dimensional base such that the image of the moduli map is not contained in the boundary $\partial\overline{M}_g$, i.e. the general fibre $\mathcal{C}_z$ is a smooth curve. For any $i = 0, \dots, [g/2]$ and any $z \in Z$ define the multiplicity $\mathrm{mult}_z(\delta_i)$ by considering the local universal deformation space $B(\mathcal{C}_z)$ of the fibre $\mathcal{C}_z$ and the inverse image $\widetilde{\Delta}_i$ of $\Delta_i \subset \overline{M}_g$ in $B(\mathcal{C}_z)$. The multiplicity $\mathrm{mult}_z(\delta_i)$ is then the multiplicity of the pull back of $\widetilde{\Delta}_i$ via the natural map $Z \to B(\mathcal{C}_b)$ (locally at b). Define $\delta_i(f) = \sum_z \mathrm{mult}_z(\delta_i) z$. Informally spoken $\delta_i(f)$ counts the fibres with nodes of type i in the family f.

Denote by $[\Delta_i] \in \mathrm{Pic}_{\mathbb{Q}}(\overline{M}_g)$ the class of the boundary divisor Δ_i. Then $[\Delta_i]$ can and will be considered as an element in $\mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{M}_g)$ by Proposition 5.1.2. Since the order of the automorphism group of a general point $[C] \in \Delta_1$ is two, $\delta_1 = \frac{1}{2}[\Delta_1]$ ([HM98, Proposition 3.92]). For $i = 0, 2, \dots, [g/2]$ $\delta_i = [\Delta_i]$.

Proposition 5.1.5. [AC87, Theorem 1] *For $g \geq 3$ $\mathrm{Pic}^{fun}_{\mathbb{Q}}(\overline{M}_g)$ is generated by the classes λ and $\delta_0, \dots, \delta_{[g/2]}$ and these classes are independent.*

5.1.2 Spin curves

Comparable results have been obtained for $\overline{S}_g$.

Definition 5.1.6. An element $\mathcal{F}$ of the *rational Picard group* $\mathrm{Pic}^{fun}_{\mathbb{Q}}(\overline{S}_g)$ *of the moduli stack of spin curves* is a collection of rational divisor classes $\mathcal{F}(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B}) \in \mathrm{Pic}_{\mathbb{Q}}(Z)$ for every family $(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B})$ of spin curves of genus g compatible with fibre squares. Restricting this definition to families of even resp. odd spin curves gives $\mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{S}^+_g)$ and $\mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{S}^-_g)$.

There is a natural map $\pi^* : \mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{M}_g) \to \mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{S}_g)$, mapping an element $\mathcal{F} \in \mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{M}_g)$ to $\pi^*\mathcal{F}$ defined on a family $(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B})$ of spin curves as $\pi^*\mathcal{F}(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B}) = \mathcal{F}(f_{\mathcal{C}})$ where $f_{\mathcal{C}} : \mathcal{C} \to Z$ is the stable model of f. This map is injective for $g > 2$ ([Cor89, p. 585]. The maps $\pi^{\pm *} : \mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{M}_g) \to \mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{S}^{\pm}_g)$ are defined similarly.

Example 5.1.7. In the same way as the classes δ_i on $\overline{M}_g$ the classes $\alpha_i^{\pm}, \beta_i^{+}$, $i = 0, \dots, [g/2]$ and β_i^{-}, $i = 0, \dots, [(g-1)/2]$ can be defined. Fix a family of spin curves $(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B})$ with smooth one-dimensional base Z such that the image of the moduli map $Z \to \overline{S}_g$ does not lie entirely in the boundary $\partial\overline{S}_g$. Consider the local universal deformation space $B'(\mathcal{X}_z, \mathcal{L}_z, \mathcal{B}_z)$ of the fibre over $z \in Z$ and denote by $\widetilde{A}_i^{\pm}$ resp. $\widetilde{B}_i^{\pm}$ the pull back of $A_i^{\pm}$ resp. $B_i^{\pm}$ to the deformation space. The multiplicity $\mathrm{mult}_z(\alpha_i^{\pm})$ resp. $\mathrm{mult}_z(\beta_i^{\pm})$ is the multiplicity of the pull back of $\widetilde{A}_i^{\pm}$ resp. $\widetilde{B}_i^{\pm}$ via the natural map $Z \to B'(\mathcal{X}_z, \mathcal{L}_z, \mathcal{B}_z)$ (locally at z) and the classes on Z are defined by $\alpha_i^{\pm}(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B}) = \sum_z \mathrm{mult}_z\, \alpha_i^{\pm} z$ and $\beta_i^{\pm}(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B}) = \sum_z \mathrm{mult}_z\, \beta_i^{\pm} z$. Denote by $\alpha_i = \alpha_i^{+} + \alpha_i^{-}$ and $\beta_i = \beta_i^{+} + \beta_i^{-}$ for $i = 0, \dots, [g/2]$, where $\beta_{[g/2]}^{-} = 0$.

Example 5.1.8. In addition to the (pull back via π of the) Hodge class $\lambda \in \mathrm{Pic}_{\mathbb{Q}}^{\mathrm{fun}}(\overline{S}_g)$ which can equivalently be given as $\det f_{\mathcal{C}!}\omega_{\mathcal{C}/Z}$ where $f_{\mathcal{C}} : \mathcal{C} \to Z$ is the stable model of the family of spin curves $(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B})$ a similar construction can be applied to the line bundle $\mathcal{L}$. Define the class $\mu \in \mathrm{Pic}_{\mathbb{Q}}^{\mathrm{fun}}(\overline{S}_g)$ on a family of spin curves $(f : \mathcal{X} \to Z, \mathcal{L}, \mathcal{B})$ as $\det f_! \mathcal{L}$, this is a well defined class as Cornalba shows in [Cor89, p. 584] (see also [Jar01]). The class μ can be written as $\mu = \mu^{+} + \mu^{-}$, where the classes $\mu^{\pm} \in \mathrm{Pic}_{\mathbb{Q}}^{\mathrm{fun}}(\overline{S}_g^{\pm})$ are the restrictions of μ to families of spin curves with even resp. odd spin structure $\mathcal{L}$. Here for any family of curves $f : \mathcal{X} \to Z$ and any line bundle $\mathcal{L}$ on $\mathcal{X}$ the line bundle $\det f_! \mathcal{L}$ is defined as $(\det R^0 f_* \mathcal{L}) \otimes (\det R^1 f_* \mathcal{L})^{-1}$, where $R^i f_*$ is the ith right derived functor of f_*, $R^i f_* \mathcal{L}$ is a vector bundle on the base Z and det is the highest exterior power, i.e. $\det R^i f_* \mathcal{L} = \bigwedge^{r_i} R^i f_* \mathcal{L}$ where $r_i = \mathrm{rank}\, R^i f_* \mathcal{L}$.

Proposition 5.1.9. [Cor89, Proposition 7.2] *For $g > 2$ $\mathrm{Pic}_{\mathbb{Q}}^{fun}(\overline{S}_g)$ is generated by μ^{+}, μ^{-}, $\alpha_i^{\pm}$, β_i^{+} for $i = 0, \dots, [g/2]$ and β_i^{-} for $i = 0, \dots, [(g-1)/2]$. These classes are independent and the following relations hold*

(i) $\pi^*\delta_0 = \alpha_0 + 2\beta_0$

(ii) $\pi^*\delta_i = 2\alpha_i + 2\beta_i$, $i = 1, \dots, [g/2]$

(iii) $\alpha_0 = 4\lambda + 8\mu$.

Remark 5.1.10. Jarvis proves in [Jar01] that (iii) also holds in $\mathrm{Pic}^{\mathrm{fun}}(\overline{S}_g)$ and that the restriction of $\lambda + 2\mu$ to S_g has order 4.

With this information the canonical divisors of $\overline{S}_g^{+}$ and $\overline{S}_g^{-}$ are readily calculated.

Proposition 5.1.11. *Let $g \geq 3$. Then*

$$K_{\overline{S}_g^{+}} = 13\lambda - 2\alpha_0^{+} - 3\beta_0^{+} - 6\alpha_1^{+} - 6\beta_1^{+} - 4\sum_{i=2}^{[\frac{g}{2}]}(\alpha_i^{+} + \beta_i^{+})$$

and

$$K_{\overline{S}_g^{-}} = 13\lambda - 2\alpha_0^{-} - 3\beta_0^{-} - 6\alpha_1^{-} - 6\beta_1^{-} - 4\sum_{i=2}^{[\frac{g}{2}]}(\alpha_i^{-} + \beta_i^{-})$$

Proof. The canonical divisor of $\overline{M}_g$ is well known to be

$$K_{\overline{M}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\sum_{i=2}^{[\frac{g}{2}]} \delta_i$$

[HM82, Theorem 2.bis]. Therefore the canonical divisor of $\overline{S}_g^{\pm}$ is the sum of the pull back of $K_{\overline{M}_g}$ via $\pi^{\pm}$ and the ramification divisor of $\pi^{\pm} : \overline{S}_g^{\pm} \to \overline{M}_g$, i.e.

$$\begin{aligned} K_{\overline{S}_g^+} &= \left[13\lambda - 2\alpha_0^+ - 4\beta_0^+ - 6\alpha_1^+ - 6\beta_1^+ - 4\sum_{i=2}^{[\frac{g}{2}]}(\alpha_i^+ + \beta_i^+)\right] + \beta_0^+ \\ &= 13\lambda - 2\alpha_0^+ - 3\beta_0^+ - 6\alpha_1^+ - 6\beta_1^+ - 4\sum_{i=2}^{[\frac{g}{2}]}(\alpha_i^+ + \beta_i^+) \end{aligned}$$

and

$$\begin{aligned} K_{\overline{S}_g^-} &= \left[13\lambda - 2\alpha_0^- - 4\beta_0^- - 6\alpha_1^- - 6\beta_1^- - 4\sum_{i=2}^{[\frac{g}{2}]}(\alpha_i^- + \beta_i^-)\right] + \beta_0^- \\ &= 13\lambda - 2\alpha_0^- - 3\beta_0^- - 6\alpha_1^- - 6\beta_1^- - 4\sum_{i=2}^{[\frac{g}{2}]}(\alpha_i^- + \beta_i^-). \end{aligned}$$

□

5.2 Pluricanonical forms

For a variety to have canonical singularities is a local property which can be expressed via lifting properties of pluricanonical forms to a desingularisation. But even a variety with non-canonical singularities can have the property, that all pluricanonical forms on (a subset of) the smooth locus lift to a desingularisation. This property is very helpful in the study of the Kodaira dimension of the variety.

5.2.1 Stable curves

Denote by $\overline{M}_g^0 \subset \overline{M}_g$ the open set of points $[C] \in \overline{M}_g$ corresponding to curves with trivial automorphism group. Recall that the boundary divisor Δ_1 of curves with an elliptic tail is contained in the complement of $\overline{M}_g^0$ and $\overline{M}_g^0$ is a proper subset of the open set of smooth points of $\overline{M}_g$. Furthermore consider a desingularisation $\widetilde{M}_g \to \overline{M}_g$.

Theorem 5.2.1. [HM82, Theorem 1] *For $g \geq 4$ every pluricanonical form on $\overline{M}_g^0$ extends to the desingularisation $\widetilde{M}_g$, i.e.*

$$\Gamma\left(\overline{M}_g^0, \mathcal{O}_{\overline{M}_g}\left(nK_{\overline{M}_g}\right)\right) = \Gamma\left(\widetilde{M}_g, \mathcal{O}_{\widetilde{M}_g}\left(nK_{\widetilde{M}_g}\right)\right)$$

for all n.

The proof uses the following generalisation of the Reid-Tai criterion.

Proposition 5.2.2. [HM82, Appendix 1 to §1] *Let $V = \mathbb{C}^m$ and $G \subset \mathrm{GL}(V)$ be a finite subgroup. Denote by $V_0 \subset V$ the open set, where G acts freely, and for any $M \in G$ by $V^M \subset V$ the linear subspace of fixed points of M. Let ω be a G-invariant pluricanonical form on V and $\widetilde{V/G}$ a desingularisation of V/G. The form ω induces a form on V_0/G which in turn can be considered as a meromorphic form on the desingularisation $\widetilde{V/G}$. If for all non-trivial $M \in G$, with Reid-Tai sum less than 1 for some primitive* $\operatorname{ord} M$*th root ζ of 1 ω is holomorphic along every divisor $E \subset \widetilde{V/G}$ mapping onto the image of V^M in V/G then ω extends holomorphically to $\widetilde{V/G}$.*

Remark 5.2.3. In the situation of the above proposition let $U \subset V/G$ be an open subset containing V_0/G with the following property: For every non-trivial $M \in G$ with Reid-Tai sum less than 1 (for some root) the intersection of U with the image of V^M in V/G is non-empty. Denote by $\widetilde{U} \subset \widetilde{V/G}$ the preimage of U under the desingularisation.

Claim: If the form ω extends holomorphically to $\widetilde{U}$ then it extends holomorphically to all of $\widetilde{V/G}$.

Suppose ω does not extend holomorphically to $\widetilde{V/G}$. By the above proposition this implies that there exist an $M \in G$, $M \neq \mathbb{I}$, with Reid-Tai sum less than 1 (for some root) and a divisor E in $\widetilde{V/G}$ mapping onto the image of V^M in V/G such that ω is not holomorphic along E, i.e. ω has a pole along E. Now consider the intersection $\widetilde{U} \cap E$. It is non-empty since the intersection of U with the image of V^M in V/G is non-empty: A point which lies in U as well as in the image of V^M has a preimage in E, since E surjects onto the image of V^M and this preimage also lies in $\widetilde{U}$, since $\widetilde{U}$ is just the preimage of U under the desingularisation. Therefore ω has a pole in $\widetilde{U}$ contradicting the assumption and proving the claim.

Moreover, if the form ω is only given in a neighbourhood of $0 \in V$ and one is only interested in extending ω to a desingularisation of an appropriate neighbourhood W of $0 \in V/G$ it is enough to show that ω lifts over an open subset $U \subset W$ containing $W \cap V_0/G$, which has non-empty intersection with V^M for every $\mathbb{I} \neq M \in G$ with Reid-Tai sum less than 1 (for some root).

Proof of Theorem 5.2.1. Let ω be a pluricanonical form on $\overline{M}_g^0$. It is enough to prove that for every point $[C] \in \overline{M}_g$ the form ω extends holomorphically to a desingularisation of an open analytic neighbourhood of $[C]$.

Let $W \subset \overline{M}_g$ be an open neighbourhood of $[C] \in \overline{M}_g$ such that W is isomorphic to a neighbourhood of $0 \in \mathbb{C}_t^{3g-3}/M(\operatorname{Aut} C) = V/G$ which is still denoted by W. Under this identification $W \cap \overline{M}_g^0 = W \cap V_0/G$, where $V_0 \subset V$ denotes the open subset where G acts freely. Then the form ω restricts to a form on $W \cap V_0/G$.

Consider the case where C does not have elliptic tails, i.e. $[C] \notin \Delta_1$. Theorem 4.2.3 shows that every non-trivial element $M \in M(\operatorname{Aut} C)$ has Reid-Tai sum at least 1 (for every root). Hence $U = W \cap V_0/G$ fulfills the assumption of Remark 5.2.3 and the restriction of ω to U extends holomorphically to a desingularisation $\widetilde{V/G} \to V/G$.

Now let $[C]$ be a point in Δ_1 such that C has two irreducible components C_1 and C_2 of genera $g-1$ and 1 respectively meeting at exactly one point P and $[C_1] \in M_{g-1}^0$, i.e. C_1 is smooth and has trivial automorphism group. Consider the moduli space $\overline{M}_{1,1}$ of stable 1-pointed curves of genus 1, i.e. $\overline{M}_{1,1} = \mathbb{P}^1 \supset \mathbb{A}^1 \ni j$ corresponds to the (isomorphism class of a) smooth elliptic curve E_j with j-invariant j and one marked point chosen as origin and $\infty \in \mathbb{P}^1$ corresponds to the (isomorphism class of a) rational curve E_∞ with a non-disconnecting node and one marked point chosen as origin. The map $\varphi = \varphi(C_1; P) : \mathbb{P}^1 = \overline{M}_{1,1} \to \overline{M}_g$ is defined by sending a point $j \in \mathbb{P}^1$ to the moduli point of the curve with irreducible components C_1 and E_j with $P \in C_1$ glued to the origin of E_j. Since $g \geq 4$ and $\operatorname{Aut} C_1$ is trivial, this gives an embedding of the j-line into Δ_1 and by construction $[C]$ is contained in the image $\operatorname{im} \varphi$.

Harris and Mumford construct an open analytic neighbourhood $S = S(C_1; P)$ of $\operatorname{im} \varphi$ in the following way. Let C_0 be the curve obtained from C_1 by making P into an ordinary cusp P_0, in particular $\nu_{C_0} : C_1 = C_0^\nu \to C_0$ is the normalisation of C_0 and $P \in C_1$ is the only preimage of the cusp P_0. The curve C_0 has a local universal deformation $\mathcal{C} \to B$ where B is smooth and $3g-3$-dimensional. As in the case of stable curves the tangent space to B at the base point is isomorphic to $\operatorname{Ext}^1(\Omega_{C_0}, \mathcal{O}_{C_0})$ and there is a short exact sequence

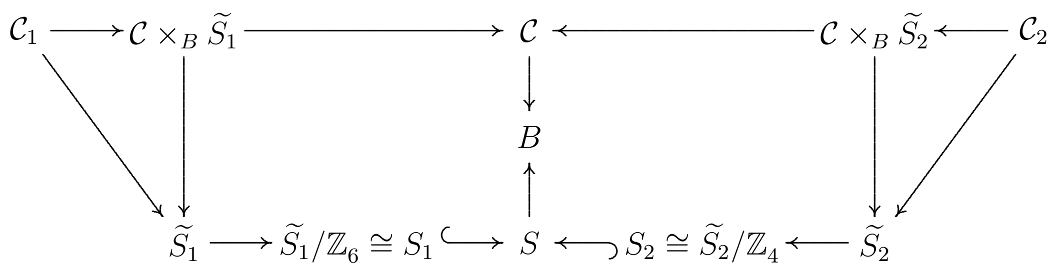
$$0 \longrightarrow H^1(C_0, \Theta_{C_0}) \longrightarrow \operatorname{Ext}^1(\Omega_{C_0}, \mathcal{O}_{C_0}) \longrightarrow H^0(C_0, \mathcal{E}xt^1(\Omega_{C_0}, \mathcal{O}_{C_0})) \longrightarrow 0$$

where the kernel gives infinitesimal deformations preserving the cusp, i.e. infinitesimal deformations coming from those of the pointed curve $(C_1; P)$ by making the deformed point into a cusp, while $H^0(C_0, \mathcal{E}xt^1(\Omega_{C_0}, \mathcal{O}_{C_0}))$ is the space of infinitesimal deformations of the ordinary cusp, which is two-dimensional. Let $s_1, \ldots, s_{3g-3}$ be coordinates of B such that $s_3, \ldots, s_{3g-3}$ are coordinates of the kernel of the above sequence. Then s_1 and s_2 correspond to deforming the cusp, i.e. all fibres over the locus $V(s_1, s_2) = \{s_1 = s_2 = 0\}$ have a cusp, while fibres over the complement $B \setminus V(s_1, s_2)$ either have one node or are smooth, in any case these fibres are stable curves of genus g and all fibres are irreducible. Let $y^2 = x^3$ be a local equation of C_0 at the cusp P_0. Choose the coordinates s_1 and s_2 such that the total space $\mathcal{C}$ of the family locally at P_0 is given by $y^2 = x^3 + s_1 x + s_2$. Harris and Mumford prove that the moduli map $f : B \setminus V(s_1, s_2) \to \overline{M}_g$ (after shrinking B if necessary) is injective and its image lies in $\overline{M}_g^0$, i.e. all fibres have trivial automorphism group.

The idea is to construct a family $\mathcal{C}' \to S$ of stable curves such that its moduli map $\widetilde{f} : S \to \overline{M}_g$ is injective and gives the sought for neighbourhood of $\operatorname{im} \varphi$ in such a way that for an appropriate closed $Z \subset S$ the restricted family $\mathcal{C}' \times_S S \setminus Z$ is isomorphic to $\mathcal{C} \times_B B \setminus V(s_1, s_2)$, in particular the restricted moduli maps $f_{|S \setminus Z}$ and $\widetilde{f}_{|B \setminus V(s_1,s_2)}$ coincide. Because of non-trivial automorphisms this idea is not realisable, nevertheless a construction nearly as good is given by Harris and Mumford:

Let $S \to B$ be the normalisation of the blow up of B in the ideal (s_1^3, s_2^2). S is covered by two charts S_1 and S_2 with coordinates $s_1, s_2, \frac{s_1^2}{s_2}, \frac{s_1^3}{s_2^2}, s_3, \ldots, s_{3g-3}$ and $s_1, s_2, \frac{s_2}{s_1}, \frac{s_2^2}{s_1^3}, s_3, \ldots, s_{3g-3}$ respectively. S_1 and S_2 can be identified with the quotients $\widetilde{S}_1/\mathbb{Z}_6$ and $\widetilde{S}_2/\mathbb{Z}_4$, where $\widetilde{S}_1$ (resp. $\widetilde{S}_2$) is smooth $(3g-3)$-dimensional with coordinates $\tilde{s}_1^{(1)}, \ldots, \tilde{s}_{3g-3}^{(1)}$ (resp. $\tilde{s}_1^{(2)}, \ldots, \tilde{s}_{3g-3}^{(2)}$) and the action of $\mathbb{Z}_6$ is $(\tilde{s}_1^{(1)}, \ldots, \tilde{s}_{3g-3}^{(1)}) \mapsto (\zeta_6^2 \tilde{s}_1^{(1)}, \zeta_6 \tilde{s}_2^{(1)}, \tilde{s}_3^{(1)}, \ldots, \tilde{s}_{3g-3}^{(1)})$ for any sixth root ζ_6 of 1 (resp. $(\tilde{s}_1^{(2)}, \ldots, \tilde{s}_{3g-3}^{(2)}) \mapsto (\zeta_4 \tilde{s}_1^{(2)}, \zeta_4^2 \tilde{s}_2^{(2)}, \tilde{s}_3^{(2)}, \ldots, \tilde{s}_{3g-3}^{(2)})$ for any fourth root ζ_4 of 1).

Harris and Mumford construct a family $\mathcal{C}_1 \to \widetilde{S}_1$ (resp. $\mathcal{C}_2 \to \widetilde{S}_2$) by pulling back $\mathcal{C}$ to $\widetilde{S}_1$ (resp. $\widetilde{S}_2$) and considering the normalisation of an appropriate blow up of the pull back. The picture is then:

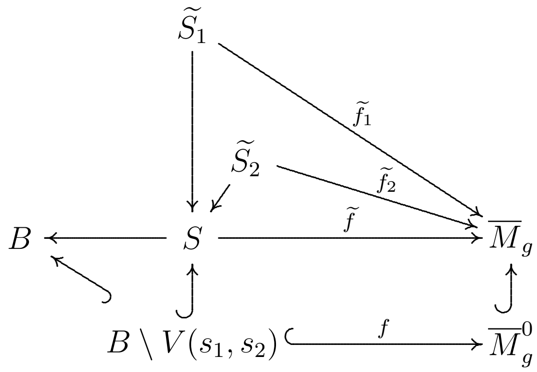


The family $\mathcal{C}_1 \to \widetilde{S}_1$ (resp. $\mathcal{C}_2 \to \widetilde{S}_2$) is a family of stable curves of genus g with the following properties:

(i) $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) is smooth
(ii) Fibres of $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) over $\{\tilde{s}_2^{(1)} = 0\}$ (resp. $\{\tilde{s}_1^{(2)} = 0\}$) are elliptic tail curves where $\tilde{s}_1^{(1)}$ (resp. $\tilde{s}_2^{(2)}$) corresponds to deforming the elliptic tail and if $\tilde{s}_1^{(1)} = 0$ (resp. $\tilde{s}_2^{(2)} = 0$) the elliptic tail has j-invariant 0 (resp. 1728). The coordinate

$\tilde{s}_2^{(1)}$ (resp. $\tilde{s}_1^{(2)}$) corresponds to smoothing the node. The remaining coordinates $\tilde{s}_3^{(1)}, \dots, \tilde{s}_{3g-3}^{(1)}$ (resp. $\tilde{s}_3^{(2)}, \dots, \tilde{s}_{3g-3}^{(2)}$) correspond to deforming the component of genus $g-1$ and the point where the elliptic tail is attached and if $\tilde{s}_3^{(1)} = \dots = \tilde{s}_{3g-3}^{(1)} = 0$ (resp. $\tilde{s}_3^{(2)} = \dots = \tilde{s}_{3g-3}^{(2)} = 0$) this component is C_1 with marked point P, i.e. the elliptic tail is attached at P. In particular the fibre over 0 has two irreducible components C_1 and an elliptic curve E_0 with j-invariant 0 (resp. E_{1728} with j-invariant 1728) attached at the node P.
(iii) The action of $\mathbb{Z}_6$ on $\widetilde{S}_1$ (resp. $\mathbb{Z}_4$ on $\widetilde{S}_2$) lifts to an action on $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) which restricts on the fibre over 0 to the action of the automorphism group of the central fibre, which is isomorphic to $\operatorname{Aut}(E_0; 0) \cong \mathbb{Z}_6$ (resp. $\operatorname{Aut}(E_{1728}; 0) \cong \mathbb{Z}_4$). All fibres over the locus $\tilde{s}_2^{(1)} = 0$ (resp. over $\tilde{s}_1^{(2)} = 0$) have non-trivial automorphism group while all fibres over the complement of this locus have trivial automorphism group.

The last point implies that the moduli map $\widetilde{f}_1 : \widetilde{S}_1 \to \overline{M}_g$ (resp. $\widetilde{f}_2 : \widetilde{S}_2 \to \overline{M}_g$) factors through the quotient $S_1 \subset S$ (resp. $S_2 \subset S$) giving the following diagram



Harris and Mumford then prove that $\widetilde{f}$ is injective and its image $\operatorname{im} \widetilde{f}$ in $\overline{M}_g$ contains $\operatorname{im} \varphi$, the locus of stable curves with two irreducible components, one being C_1 the other an arbitrary curve of genus 1. In particular $S = S(C_1; P)$ is an open neighbourhood of $[C]$. Since $B \setminus V(s_1, s_2) \subset S \cap \overline{M}_g^0$ the pluricanonical form ω on $\overline{M}_g^0$ restricts to $B \setminus V(s_1, s_2)$ and extends holomorphically to all of B, because B is smooth and $V(s_1, s_2)$ has codimension two. Let $\widetilde{S}$ be a desingularisation of S and consider the pull back of the form ω on B via the concatenation $\widetilde{S} \to S \to B$. The pull back is holomorphic and therefore ω extends locally at $[C]$ holomorphically to the desingularisation $\widetilde{S}$.

Consider now an arbitrary point $[C] \in \Delta_1$ and denote by $C_2^{(1)}, \dots, C_2^{(k)}$ all elliptic tails of C. Let $C_1^{(i)} = \overline{C \setminus C_2^{(i)}}$ for $i = 1, \dots, k$ be the closure of the complement of the ith elliptic tail and $P^{(i)}$ the node where $C_1^{(i)}$ and $C_2^{(i)}$ meet. Remember that $W \subset \overline{M}_g$ is an open neighbourhood of $[C]$ such that W is isomorphic to a neighbourhood of 0 in the quotient $\mathbb{C}_t^{3g-3}/M(\operatorname{Aut} C) = V/G$. For every $i = 1, \dots, k$

there exists a point $[C'^{(i)}] \in W$ such that $C'^{(i)}$ has two irreducible components $C_1'^{(i)}$ and $C_2'^{(i)}$ meeting at a node $P'^{(i)}$ where $(C_2'^{(i)}; P'^{(i)}) = (C_2^{(i)}; P^{(i)})$ is the ith elliptic tail and $C_1'^{(i)}$ is a nearby curve of $C_1^{(i)}$ which is smooth and has trivial automorphism group, i.e. $C'^{(i)}_1 \in M^0_{g-1}$, and $P'^{(i)} \in C'^{(i)}_1$.

Consider the maps $\varphi^{(i)} = \varphi(C_1'^{(i)}; P'^{(i)}) : \mathbb{P}^1 \to \overline{M}_g$ defined for every $j \in \mathbb{P}^1$ by attaching an elliptic tail with j-invariant j at the origin to the point $P'^{(i)} \in C_1'^{(i)}$. For all $i, i' \in \{1, \ldots, k\}$ there are exactly two possibilities: Either $(C_1'^{(i)}; P'^{(i)})$ and $(C_1'^{(i')}; P'^{(i')})$ are isomorphic and $\varphi^{(i)} = \varphi^{(i')}$ or the two one-pointed curves are not isomorphic and $\operatorname{im}\varphi^{(i)}$ and $\operatorname{im}\varphi^{(i')}$ are disjoint. Let $I \subset \{1, \ldots, k\}$ be a subset such that the $(C_1'^{(i)}; P'^{(i)})$, $i \in I$, are pairwise non-isomorphic and for every $i' \in \{1, \ldots, k\}$ the pointed curve $(C_1'^{(i')}; P'^{(i')})$ is isomorphic to $(C_1'^{(i)}; P'^{(i)})$ for an appropriate $i \in I$. Then the images $\operatorname{im}\varphi^{(i)}$, $i \in I$, are pairwise disjoint. For every $i \in I$ consider the neighbourhood $S^{(i)} = S(C_1'^{(i)}; P'^{(i)})$ of $\operatorname{im}\varphi^{(i)}$ constructed above. Since the images of the $\varphi^{(i)}$ are pairwise disjoint one may assume the $S^{(i)}$ to be pairwise disjoint (if necessary the $S^{(i)}$ can be shrinked preserving all important properties). Denote by Y the union of W and the $S^{(i)}$, $i \in I$. Let $\widetilde{Y} \to Y$ be a desingularisation and denote by $\widetilde{S}^{(i)}$ the preimage of $S^{(i)}$ under the desingularisation. As above the construction of $S^{(i)}$ implies that the restriction of the form ω to $S^{(i)} \cap \overline{M}_g^0$ extends holomorphically to a form $\omega^{(i)}$ on $\widetilde{S}^{(i)}$.

Set $W_0 = W \cap V_0/G$, where $V_0 \subset V$ denotes the open subset where the group G acts freely. In particular W_0 is smooth and its preimage under the desingularisation $\widetilde{Y} \to Y$ is isomorphic to W_0. Therefore the restriction of ω to W_0 is holomorphic and $\omega_{|W_0}$ and $\omega^{(i)}$, $i \in I$, give a holomorphic form on $W_0 \cup \bigcup_{i\in I} \widetilde{S}^{(i)}$, the preimage of $W_0 \cup \bigcup_{i\in I} S^{(i)}$ under the desingularisation $\widetilde{Y} \to Y$.

Denote by U the intersection of $W_0 \cup \bigcup_{i\in I} S^{(i)}$ with W and by $\widetilde{U}$ its preimage under the desingularisation.

Claim: U fulfills the assumption of (the local version of) Remark 5.2.3.

Clearly $W \cap V_0/G = W_0 \subset (W_0 \cup \bigcup_{i\in I} S^{(i)}) \cap W$. Let $M = M(\sigma_C) \in G = M(\operatorname{Aut} C)$ be a non-trivial matrix with Reid-Tai sum less than 1 (for some root). By Theorem 4.2.3 this implies that σ_C is an elliptic tail automorphism of order 2, 4, 3 or 6. In particular for an appropriate $i \in \{1, \ldots, k\}$ the automorphism is the identity on $C_1^{(i)}$ and $\sigma_{C_{|C_2^{(i)}}}$ is a non-trivial automorphism of the elliptic tail. Proposition 4.2.5 shows that the locus $\{t_1 = t_2 = 0\} \subset V$ is contained in the fixed point locus of M, if t_1 (resp. t_2) is the coordinate corresponding to smoothing the node $P^{(i)}$ (resp. to deforming the elliptic tail $C_2^{(i)}$). The moduli point of the curve $C'^{(i)}$ with irreducible components $C_1'^{(i)}$ and $C_2^{(i)}$ is contained in this locus and therefore an element of V^M. On the other hand this point is contained in W and in $S^{(i')}$ for an appropriate $i' \in I$, since by definition of I there exists an

$i' \in I$ such that $(C_1'^{(i)}; P'^{(i)})$ is isomorphic to $(C_1'^{(i')}; P'^{(i')})$ and if $j \in \mathbb{P}^1$ denotes the j-invariant of $C_2^{(i)}$ the point $[C'^{(i)}]$ is just the image of j under the map $\varphi^{(i')}$. Hence U fulfills the assumption.

The preimage $\widetilde{U}$ of U under the desingularisation is contained in $W_0 \cup \bigcup_{i\in I} \widetilde{S}^{(i)}$ therefore the form ω extends holomorphically to $\widetilde{U}$ and by Remark 5.2.3 it extends holomorphically to $\widetilde{W}$, the preimage of W under the desingularisation $\widetilde{Y} \to Y$. Hence ω lifts to a desingularisation locally at every point $[C] \in \overline{M}_g$ and the theorem is proved. □

5.2.2 Spin curves

Denote by $\overline{S}_g^{\text{reg}} \subset \overline{S}_g$ the smooth locus of $\overline{S}_g$ and let $\widetilde{S}_g \to \overline{S}_g$ be a desingularisation.

Theorem 5.2.4. *For $g \geq 4$ every pluricanonical form on $\overline{S}_g^{reg}$ extends to the desingularisation $\widetilde{S}_g$, i.e.*

$$\Gamma\left(\overline{S}_g^{reg}, \mathcal{O}_{\overline{S}_g}\left(nK_{\overline{S}_g}\right)\right) = \Gamma\left(\widetilde{S}_g, \mathcal{O}_{\widetilde{S}_g}\left(nK_{\widetilde{S}_g}\right)\right)$$

for all n.

Proof. Let ω be a pluricanonical form on the smooth locus $\overline{S}_g^{\text{reg}}$. As in the proof of Theorem 5.2.1 it is enough to prove that ω lifts to a desingularisation of an open neighbourhood of every point $[(X, L, b)] \in \overline{S}_g$.

By Theorem 4.3.9 the locus of non-canonical singularities is the locus of points $[(X, L, b)]$ such that X has an elliptic tail with j-invariant 0 and the theta characteristic on the elliptic tail is trivial. Let $[(X, L, b)] \in \overline{S}_g$ be a canonical singularity, i.e. the theta characteristic on every elliptic tail with j-invariant 0 is non-trivial. Let W be a neighbourhood of $[(X, L, b)]$ such that W is isomorphic to a neighbourhood of $0 \in \mathbb{C}_u^{3g-3}/K(\text{Aut}(X, L, b)) = V/G$, which will also be denoted W. Since G contains no quasireflections $\overline{S}_g^{\text{reg}} \cap W = V_0/G \cap W = W_0$, where $V_0 \subset V$ denotes the open set where G acts freely, and ω restricts to a form on W_0. All non-trivial matrices $K(\sigma, \gamma) \in G$ have Reid-Tai sum at least 1 (for every root) hence $U = W_0$ fulfils the assumption of Remark 5.2.3. If $\widetilde{W} \to W$ is a desingularisation of W and $\widetilde{U}$ the preimage of U under the desingularisation then $\widetilde{U} \cong U$ since $U = W_0$ is smooth. Hence ω is holomorphic on $\widetilde{U}$ and Remark 5.2.3 implies that ω extends holomorphically to $\widetilde{W}$.

Let $[(X, L, b)] \in \overline{S}_g$ be a spin curve such that the stable model C of X has two irreducible components C_1 and C_2 meeting at a node P, where $C_1 \in M_{g-1}^0$ and C_2 is an elliptic curve with j-invariant 0, and $L_2^\nu = \nu^* L_{|C_2} = \mathcal{O}_{C_2}$, where $\nu : X^\nu \to X$

is the normalisation. Setting $L_1^\nu = \nu^* L_{|C_1}$ consider the map $\psi = \psi(C_1, L_1^\nu; P) : \mathbb{P}^1 \to \overline{S}_g$ defined by sending a point $j \in \mathbb{P}^1$ to the moduli point of the following spin curve (X', L', b'): The stable model C' of X' has two irreducible components C_1 and C_2', where C_2' is an elliptic curve with j-invariant j, meeting in a node at $P \in C_1$ and some chosen origin in C_2'. The theta characteristic ${L_1'}^\nu$ on C_1 is just L_1^ν, while the theta characteristic ${L_2'}^\nu$ on C_2' is trivial. Remember from the proof of Theorem 5.2.1 that the map $\varphi = \varphi(C_1; P) : \mathbb{P}^1 \to \overline{M}_g$ is defined by sending $j \in \mathbb{P}^1$ to the moduli point of the above defined stable curve C', i.e. there is a commutative diagram

$$\begin{array}{ccc} & & \overline{S}_g \\ & \overset{\psi}{\nearrow} & \downarrow \pi \\ \mathbb{P}^1 & \underset{\varphi}{\searrow} & \\ & & \overline{M}_g \end{array}$$

Remember also from Theorem 5.2.1 the open neighbourhood $S(C_1; P)$ of $\operatorname{im}\varphi$.

Claim: There is an open neighbourhood $S(C_1, L_1^\nu; P)$ of $\operatorname{im}\psi$ which is isomorphic to $S(C_1; P)$ via π (after shrinking $S(C_1; P)$ if necessary).

Consider for any $j \in \mathbb{P}^1$ the point $[(X', L', b')] = \psi(j)$ in the image of ψ and its image $[C'] = \varphi(j)$ under π. Let $W = W(j)$ be a neighbourhood of $[(X', L', b')] \in \overline{S}_g$ which is isomorphic to a neighbourhood of $0 \in \mathbb{C}_u^{3g-3}/K(\operatorname{Aut}(X', L', b')) = V/G$, which is also denoted by W. Recall from Remark 4.3.8 that $K(\sigma, \gamma)$ for an automorphism $(\sigma, \gamma) \in \operatorname{Aut}(X', L', b')$ has a block form with one block for the set T of elliptic tail nodes, one for the set D of disconnecting nodes which are not elliptic tail nodes, one for the set $\overline{N}$ of exceptional nodes which are non-disconnecting, one for the set Δ of non-exceptional nodes and a last block for the set V of irreducible components. Since C' has only one node which is the elliptic tail node P, $K(\sigma, \gamma)$ has the form

$$K(\sigma,\gamma) = \begin{pmatrix} \alpha^4 & \\ & M_V \mathbb{E}_{\sigma_V} \end{pmatrix}$$

if the action of (σ, γ) on the deformation space $\mathbb{C}_\tau^{3g-3}$ is given as

$$M(\sigma,\gamma) = \begin{pmatrix} \alpha & \\ & M_V \mathbb{E}_{\sigma_V} \end{pmatrix}.$$

π on W is the restriction of

$$\begin{array}{ccc} \mathbb{C}_\tau^{3g-3}/M(\operatorname{Aut}(X', L', b')) & \longrightarrow & \mathbb{C}_t^{3g-3}/M(\operatorname{Aut} C') \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{C}_u^{3g-3}/K(\operatorname{Aut}(X', L', b')) & \longrightarrow & \mathbb{C}_{u'}^{3g-3}/K(\operatorname{Aut} C') \end{array}$$

where the isomorphism on the right hand side is defined in the proof of Proposition 4.2.7. Since P is the only node $u_i = \tau_i = t_i = u_i'$ for $i = 2, \ldots, 3g-3$

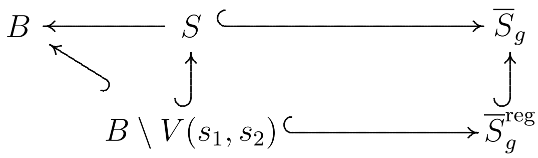
and $u_1 = \tau_1^4 = t_1^2 = u_1'$ since P is an elliptic tail node. From Remark 4.3.8 and Proposition 4.2.7 it follows that the action of the automorphism $\sigma_{C'}$ on $\mathbb{C}_t^{3g-3}$ resp. $\mathbb{C}_{u'}^{3g-3}$ is

$$M(\sigma_C) = \begin{pmatrix} \alpha^2 & \\ & M_V \mathbb{E}_{\sigma_V} \end{pmatrix} \qquad \text{resp.} \qquad K(\sigma_C) = \begin{pmatrix} \alpha^4 & \\ & M_V \mathbb{E}_{\sigma_V} \end{pmatrix}.$$

Therefore $\mathbb{C}_u^{3g-3}/K(\operatorname{Aut}(X', L', b')) \to \mathbb{C}_{u'}^{3g-3}/K(\operatorname{Aut} C')$ is an isomorphism and $\pi_{|W} : W \xrightarrow{\cong} \pi(W)$.

Now consider the union $\mathcal{W} = \bigcup_{j \in \mathbb{P}^1} W(j)$, which is an open neighbourhood of $\operatorname{im} \psi$. Then $\pi_{|\mathcal{W}} : \mathcal{W} \to \pi(\mathcal{W})$ is an isomorphism and $\pi(\mathcal{W}) \cap S(C_1; P)$ is an open neighbourhood of $\operatorname{im} \varphi$. After replacing $S(C_1; P)$ with $\pi(\mathcal{W}) \cap S(C_1; P)$ define $S(C_1, L_1^\nu; P) = \pi_{|\mathcal{W}}^{-1}(S(C_1; P))$. Then $S(C_1, L_1^\nu; P) \cong S(C_1; P)$ via π.

The neighbourhood $S = S(C_1, L_1^\nu; P)$ fits into the following diagram (see also the corresponding diagram in the proof of Theorem 5.2.1)



where $S \to B$ resp. $B \setminus V(s_1, s_2) \hookrightarrow S$ is the concatenation $S \cong S(C_1; P) \to B$ resp. $B \setminus V(s_1, s_2) \hookrightarrow S(C_2; P) \cong S$. Remember that $B \setminus V(s_1, s_2) \to \overline{M}_g^0$ is the injective moduli map of the restriction of the family $\mathcal{C} \to B$ of curves. All fibres of this family are irreducible. Because of $B \setminus V(s_1, s_2) \hookrightarrow \overline{M}_g^0$ every point in the image of $B \setminus V(s_1, s_2)$ in $\overline{S}_g$ corresponds to a spin curve whose only automorphisms are inessential and the stable model of the support is irreducible since the fibres of $\mathcal{C}$ are. This implies that the only automorphisms are $(\mathrm{id}, \pm \mathrm{id})$. Therefore $B \setminus V(s_1, s_2)$ injects into $\overline{S}_g^{\text{reg}}$.

The form ω on $\overline{S}_g^{\text{reg}}$ restricts to $B \setminus V(s_1, s_2)$ and extends holomorphically to B, since $V(s_1, s_2)$ has codimension two in the smooth B. Let $\widetilde{S} \to S$ be a desingularisation of S. Then ω extends holomorphically to $\widetilde{S}$ since the pull back via $\widetilde{S} \to B$ is holomorphic.

In the last step let $[(X, L, b)] \in \overline{S}_g$ be any non-canonical singularity and W a neighbourhood of $[(X, L, b)]$ such that W is isomorphic to a neighbourhood of $0 \in \mathbb{C}_u^{3g-3}/K(\operatorname{Aut}(X, L, b)) = V/G$. Let $C_2^{(1)}, \ldots, C_2^{(k)}$ be all elliptic tails of X with j-invariant 0 such that the theta characteristic $L_2^{(i),\nu} = \nu^* L_{|C_2^{(i)}}$ is trivial, here $\nu : X^\nu \to X$ is the normalisation. For $i = 1, \ldots, k$ denote by $C_1^{(i)} = \overline{C \setminus C_2^{(i)}}$ the closure of the complement of the ith elliptic tail in the stable model C of X and by $P^{(i)} \in C$ the corresponding elliptic tail node. Furthermore let $X_1^{(i)} = \overline{X \setminus (E^{(i)} \cup C_2^{(i)})}$ be the closure of the complement of the union of the ith elliptic tail and the incident exceptional component $E^{(i)}$ and set $L_1^{(i),\nu} = \nu^* L_{|X_1^{(i)}}$. For every $i = 1, \dots, k$ there is a point in W corresponding to a spin curve $(X'^{(i)}, L'^{(i)}, b'^{(i)})$ with the following properties: The stable model $C'^{(i)}$ of $X'^{(i)}$ has two irreducible components ${C'_1}^{(i)}$ and ${C'_2}^{(i)}$ meeting at exactly one node $P'^{(i)}$, where ${C'_1}^{(i)}$ is a smooth curve of genus $g-1$ with trivial automorphism group and ${C'_2}^{(i)} = C_2^{(i)}$ is the ith elliptic tail. Moreover, the theta characteristic $(\nu'^{(i)})^* L'^{(i)}_{|C_2^{(i)}}$ is trivial, where $\nu'^{(i)} : X'^{(i),\nu} \to X'^{(i)}$ is the normalisation. Consider the maps $\psi^{(i)} = \psi({C'_1}^{(i)}, {L'_1}^{(i),\nu}; P'^{(i)}) : \mathbb{P}^1 \to \overline{S}_g$ sending $j \in \mathbb{P}^1$ to the moduli point of a spin curve such that the stable model of its support has two irreducible components the curve ${C'_1}^{(i)}$ and an elliptic curve with j-invariant j meeting at the point $P'^{(i)} \in {C'_1}^{(i)}$ and the origin of the elliptic curve, the theta characteristic on ${C'_1}^{(i)}$ is ${L'_1}^{(i),\nu} = (\nu'^{(i)})^* L'^{(i)}_{|{C'_1}^{(i)}}$ and that on the elliptic tail is trivial. Two images $\operatorname{im}\psi^{(i)}$ and $\operatorname{im}\psi^{(i')}$ are either equal or disjoint depending on whether there exists an isomorphism of the smooth spin curves $({C'_1}^{(i)}, {L'_1}^{(i),\nu})$ and $({C'_1}^{(i')}, {L'_1}^{(i'),\nu})$ sending $P'^{(i)}$ to $P'^{(i')}$. Let $I \subset \{1, \dots, k\}$ be such that the images $\operatorname{im}\psi^{(i)}$, $i \in I$, are pairwise disjoint and for every $i' \in \{1, \dots, k\}$ there exists an $i \in I$ such that $\operatorname{im}\psi^{(i')} = \operatorname{im}\psi^{(i)}$.

Consider the neighbourhoods $S^{(i)} = S({C'_1}^{(i)}, {L'_1}^{(i),\nu}; P'^{(i)})$ of $\operatorname{im}\psi^{(i)}$, $i \in I$, constructed above. Since the images $\operatorname{im}\psi^{(i)}$, $i \in I$, are pairwise disjoint one may assume the $S^{(i)}$ to be pairwise disjoint. Denote by Y the union of W and the $S^{(i)}$, $i \in I$. Let $\widetilde{Y} \to Y$ be a desingularisation and denote by $\widetilde{S}^{(i)}$ the preimage of $S^{(i)}$ under the desingularisation. As above the construction of $S^{(i)}$ implies that the restriction of the form ω to $S^{(i)} \cap \overline{S}_g^{\text{reg}}$ extends holomorphically to a form $\omega^{(i)}$ on $\widetilde{S}^{(i)}$.

Set $W_0 = W \cap V_0/G$, where $V_0 \subset V$ denotes the open subset where the group G acts freely. In particular W_0 is smooth and its preimage under the desingularisation $\widetilde{Y} \to Y$ is isomorphic to W_0. Therefore the restriction of ω to W_0 is holomorphic and $\omega_{|W_0}$ and $\omega^{(i)}$, $i \in I$, give a holomorphic form on $W_0 \cup \bigcup_{i \in I} \widetilde{S}^{(i)}$, the preimage of $W_0 \cup \bigcup_{i \in I} S^{(i)}$ under the desingularisation $\widetilde{Y} \to Y$.

Denote by U the intersection of $W_0 \cup \bigcup_{i \in I} S^{(i)}$ with W and by $\widetilde{U}$ its preimage under the desingularisation.

Claim: U fulfills the assumption of (the local version of) Remark 5.2.3.

Clearly $W \cap V_0/G = W_0 \subset (W_0 \cup \bigcup_{i \in I} S^{(i)}) \cap W$. Let $K = K(\sigma, \gamma) \in G = K(\operatorname{Aut}(X, L, b))$ be a non-trivial matrix with Reid-Tai sum less than 1 (for some root). By Theorem 4.3.9 this implies that the automorphism σ_C of the stable model C is an elliptic tail automorphism of order 3 or 6 and the theta characteristic on the elliptic tail on which σ_C acts non-trivially is trivial. In particular for an appropriate $i \in \{1, \ldots, k\}$ the automorphism σ_C is the identity on $C_1^{(i)}$ and $\sigma_{C_{|C_2^{(i)}}}$ is an automorphism of order 3 or 6 of the elliptic tail. Remark 4.3.10 shows that the locus $\{u_1 = u_{3g-3} = 0\} \subset V$ is the fixed point locus of K, if u_1 (resp. u_{3g-3}) is the coordinate corresponding to smoothing the node $P^{(i)}$ (resp. to deforming the elliptic tail $C_2^{(i)}$). The moduli point of the spin curve $({X'}^{(i)}, {L'}^{(i)}, {b'}^{(i)})$ is contained in this locus and therefore an element of V^K. On the other hand this point is contained in W and in $S^{(i')}$ for an appropriate $i' \in I$, since by definition of I there exists an $i' \in I$ such that there exists an isomorphism of the smooth spin curves $({C'_1}^{(i)}, {L'_1}^{(i),\nu})$ and $({C'_1}^{(i')}, {L'_1}^{(i'),\nu})$ sending ${P'}^{(i)}$ to ${P'}^{(i')}$. Hence U fulfills the assumption.

The preimage $\widetilde{U}$ of U under the desingularisation is contained in $W_0 \cup \bigcup_{i \in I} \widetilde{S}^{(i)}$ therefore the form ω extends holomorphically to $\widetilde{U}$ and by Remark 5.2.3 it extends holomorphically to $\widetilde{W}$, the preimage of W under the desingularisation $\widetilde{Y} \to Y$. Hence ω lifts to a desingularisation locally at every point $[(X, L, b)] \in \overline{S}_g$ and the theorem is proved. □

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Nomenclature

$A_i^\pm$, $B_i^\pm$	boundary divisors of $\overline{S}_g$, page 33
$\alpha_i^\pm$, $\beta_i^\pm$	boundary classes in $\mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{S}_g)$, page 111
$\beta : X \to C$	stable model of quasistable curve, page 10
C	curve, always stable beginning with, page 10
$(C; Q_1, \ldots, Q_n)$	pointed nodal curve (with ordering), page 4
$(C; \{Q_1, \ldots, Q_n\})$	pointed nodal curve (without ordering), page 4
$(C_j^\nu; Q_k, P_i^\pm)$	pointed normalisation (ordered), page 7
$(C_j^\nu; \{Q_k, P_i^\pm\})$	pointed normalisation (unordered), page 7
$\mathcal{C} \to Z$, C_z	family of curves, fibre of the family over $z \in Z$, page 4
Δ_i	boundary divisors of $\overline{M}_g$, page 31
δ_i	boundary classes in $\mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{M}_g)$, page 110
$\Delta(X, L, b)$	non-exceptional nodes of X in C, page 13
$D(X, L, b)$	set of all disconnecting nodes, which are not elliptic tail nodes, page 67
$\Gamma(C; Q_1, \ldots, Q_n)$	dual graph of a pointed curve (ordered), page 5
$\Gamma(C; \{Q_1, \ldots, Q_n\})$	dual graph of a pointed curve (unordered), page 5
$K(\sigma, \gamma)$	action of $(\sigma, \gamma) \in \mathrm{Aut}(X, L, b)$ after quotienting out the quasireflections, page 68
λ	Hodge class in $\mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{M}_g)$ (also pullback to $\overline{S}_g$), page 110

M_g	moduli space of smooth curves, page 30
$\overline{M}_g$	moduli space of stable curves, page 30
$\overline{\mathcal{M}}_g$	moduli functor of stable curves, page 28
$M(\sigma, \gamma)$	action of $(\sigma, \gamma) \in \mathrm{Aut}(X, L, b)$ on $\mathbb{C}_\tau^{3g-3}$, page 44
$M(\sigma_C)$	action of $\sigma_C \in \mathrm{Aut}\, C$ on $\mathbb{C}_t^{3g-3}$, page 41
μ	class of the spin structure, page 111
$\overline{N}(X, L, b)$	set of exceptional non-disconnecting nodes, page 67
$\nu_\star : \star^\nu \to \star$	normalisation of $\star$, page 5
$N(X, L, b)$	exceptional nodes of X, page 13
$\pi : \overline{S}_g \to \overline{M}_g$	forgetful morphism, page 32
$\mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{M}_g)$	rational Picard group of the moduli stack of stable curves, page 109
$\mathrm{Pic}^{\mathrm{fun}}_{\mathbb{Q}}(\overline{S}_g)$	rational Picard group of the moduli stack of spin curves, page 110
S_g	moduli space of smooth spin curves, page 32
$\overline{S}_g$	moduli space of spin curves, page 32
$\overline{\mathcal{S}}_g$	moduli functor of spin curves, page 32
$\overline{S}_g^+$, $\overline{S}_g^-$	moduli space of even resp. odd spin curves, page 33
$\overline{S}_C$	moduli space of spin curves with stable model C, page 35
$\overline{S}_C^0$	moduli space of spin curves with stable model C with respect to inessential isomorphisms, page 34
$\overline{\mathcal{S}}_C^0$, $\overline{\mathcal{S}}_C$	moduli functors of spin curves with fixed stable model C, page 34
(σ, γ)	isomorphism of spin curves, page 14
$\Sigma(X)$	graph with vertices corresponding to connected components of non-exceptional subcurve, page 21

$T(X, L, b)$	set of elliptic tail nodes, page 67
$W_\tau(C_j)$	preimage of $W_t(C_j)$ in $\mathbb{C}^{3g-3}_\tau$, page 44
$W_\tau(P_i)$	preimage of $W_t(P_i)$ in $\mathbb{C}^{3g-3}_\tau$, page 44
$W_t(C_j) \subset \mathbb{C}^{3g-3}_t$	subspace corresponding to deforming $(C^\nu_j, \{P^\pm_i\})$, page 40
$W_t(P) \subset \mathbb{C}^{3g-3}_t$	subspace corresponding to smoothing the node P_i, page 40
X	quasistable curve of genus $g \geq 2$, page 10
(X, L, b)	spin curve, page 13
$\widetilde{X}$	non-exceptional subcurve, page 10
$(\mathcal{X} \to \mathbb{C}^{3g-3}_\tau, \mathcal{L}, \mathcal{B})$	local universal deformation of (X, L, b), page 44
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