# Singular behavior of equivariant lagrangian mean curvature flow

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**Zusammenfassung.** Inhalt dieser Dissertation ist die Untersuchung von Singularitäten des mittleren Krümmungsflußes einer equivarianten Lagrangeschen Untermannigfaltigkeit. Sei hierzu $L$ eine kompakte orientierbare Mannigfaltigkeit. Wir sagen, daß eine Ein-Parameter Familie von glatten Immersionen $F_t\colon L \to M$ den *mittleren Krümmungsfluß* erfüllt, falls gilt

$$\begin{aligned} \frac{d}{dt}F &= \overrightarrow{H}, \\ F(\cdot,0) &= F_0, \end{aligned} \tag{1}$$

hierbei ist $\overrightarrow{H}$ der mittlere Krümmungsvektor der Immersion, und $F_0\colon L \to M$ die Anfangsimmersion. Gleichung (1) ist ein quasi-lineares parabolisches System. Daher existiert ein maximales Zeitintervall $[0, T_{\text{sing}})$ in dem eine glatte Lösung von (1) existiert. Es ist aber zu erwarten, daß der Fluß Singularitäten ausbildet. Man kann zeigen, daß das genau dann der Fall ist, falls die zweite Fundamentalform explodiert.

Schwerpunkt der Arbeit ist die Analyse des singulären Verhaltens von (1) in der Klasse der equivarianten Lagrangeschen Untermannigfaltigkeiten. Sei $z_0\colon S^1 \to \mathbb{C}\backslash\{0\}$ eine geschlossene immersierte Kurve mit $z_0 = u_0 + \imath v_0$, und sei ferner $G\colon S^{n-1} \to \mathbb{R}^n$ die Standardeinbettung der Sphäre mit Radius Eins. Eine *equivariante Lagrangesche Untermannigfaltigkeit* $F_0\colon S^1 \times S^{n-1} \to \mathbb{C}^n$ ist gegeben durch

$$F_0(\phi, x) = \big(u_0(\phi)G(x), v_0(\phi)G(x)\big).$$

Da der mittlere Krümmungsfluß isotrop ist, ist das Verhalten der equivarianten Lagrangeschen Untermannigfaltigkeiten unter dem Fluß determiniert durch den Fluß der Profilkurve. Hierbei muß man zwei Fälle unterscheiden: Entweder enthält $z_0$ den Ursprung oder nicht. Die vorliegende Arbeit konzentriert sich auf den zweiten Fall. Unsere Hauptresultate lauten:

**Theorem A.** *Sei $F_0$ eine equivariante Lagrangesche Immersion von $L := S^{n-1}\times S^1$ in $\mathbb{C}^n$. Falls die Anfangskurve $z_0\colon S^1 \to \mathbb{C}\backslash\{0\}$ geschlossen und eingebettet ist, nicht den Ursprung enthält, und $\mathscr{F} > 0$ erfüllt, dann konvergiert $F_t(S^1 \times S^{n-1})$ zur Sphäre $\|p_0\|S^{n-1}$ für $t \to T_{\text{sing}}$. Desweiteren ist die singuläre Zeit gegeben durch die eingeschlossene Fläche $A_0$ der Anfangskurve $z_0$, es gilt $T_{\text{sing}} = \frac{A_0}{2\pi}$.*

**Theorem B.** *Sei $F_0$ eine equivariante Lagrangesche Immersion von $L$ in $\mathbb{C}^n$. Falls die Anfangskurve $z_0\colon S^1 \to \mathbb{C}\backslash\{0\}$ geschlossen und eingebettet ist, nicht den Ursprung enthält, $\mathscr{F} > 0$ erfüllt und alle Profilkurven $z(\cdot,t)$ konvex sind, dann ist die Singularität vom Typ-I. Nach Reskalierung und Auswahl einer Teilfolge konvergiert $F_t$ zu dem Zylinder $S^{n-1} \times \mathbb{R}$ glatt auf kompakten Teilmengen von $\mathbb{C}^n$.*

**Theorem C.** *Sei $F_0$ eine equivariante Lagrangesche Immersion von $L$ in $\mathbb{C}^n$. Falls die Anfangskurve $z_0\colon S^1 \to \mathbb{C}\backslash\{0\}$ geschlossen und eingebettet ist, den Ursprung nicht enthält, und folgende Ungleichung erfüllt*

$$\mathscr{F}_{\min}(0) \geq \frac{n-1+\sqrt{(n-1)^2+n-1}}{r_{\min}(0)},$$

*dann erfüllt $z_0$ die Voraussetzungen von Theorem B.*

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**Abstract.** In this paper we study the singular behavior of the mean curvature flow of an equivariant lagrangian submanifold. To that end let $L$ be a closed, oriented manifold. We say that a one-parameter family of immersion $F_t\colon L \to (M,g)$, $F_t = F(\cdot,t)$ satisfies the *mean curvature flow equation* if

$$\begin{aligned} \frac{d}{dt}F &= \overrightarrow{H}, \\ F(\cdot,0) &= F_0, \end{aligned} \tag{2}$$

where $\overrightarrow{H}$ is the *mean curvature vector* and $F_0\colon L \to (M,g)$ is the *initial immersion.* We recall that $H$ is the trace of the second fundamental form $A = \nabla dF$. Equation (2) is a quasi-linear parabolic system. Therefore, there exists a maximal time interval $[0, T_{\text{sing}})$ in which a smooth solution of (2) exists.

The present paper gives a detailed analysis of the singular behavior of (2) in the case of equivariant lagrangian submanifolds. Suppose that $z_0\colon S^1 \to \mathbb{C}\backslash\{0\}$ is a closed immersed curve with $z_0 = u_0 + \imath v_0$, and $G\colon S^{n-1} \to \mathbb{R}^n$ is the standard embedding of the sphere of radius one. Then the *equivariant lagrangian submanifold* $F_0\colon S^1 \times S^{n-1} \to \mathbb{C}^n$ is given by

$$F_0(\phi,x) = \big(u_0(\phi)G(x), v_0(\phi)G(x)\big).$$

One has to distinguish two different cases, namely whether $z_0$ encloses the origin or not. We will focus on the latter case, although some insight to existing results of the former case is given.

Since the mean curvature flow is isotropic, it will be determined by the flow of the corresponding profile curves. Our main results are the following theorems:

**Theorem A.** *Let $F_0$ be an equivariant lagrangian immersion of $L$ in $\mathbb{C}^n$. If the initial profile curve is closed, embedded, satisfies $\mathscr{F} > 0$, and does not contain the origin, then $F_t(S^1 \times S^{n-1})$ converges to a sphere $\|p_0\| S^{n-1}$ as time goes to $T_{\text{sing}}$. Moreover, the singular time is determined by the area $A_0$ enclosed by the initial curve. That is $T_{\text{sing}} = \frac{A_0}{2\pi}$.*

**Theorem B.** *Let $F_0$ be an equivariant lagrangian immersion of $L$ in $\mathbb{C}^n$. If the initial profile curve is closed, embedded, satisfies $\mathscr{F} > 0$, does not contain the origin, and all profile curves remain convex, then the singularity is of type-I. After rescaling and possibly choosing a subsequence $F_t$ converges to the cylinder $S^{n-1} \times \mathbb{R}$ smoothly on compact subsets of $\mathbb{C}^n$.*

**Theorem C.** *Let $F_0$ be an equivariant lagrangian immersion of $L$ in $\mathbb{C}^n$. If the initial profile curve is closed, embedded, satisfies*

$$\mathscr{F}_{\min}(0) \geq \frac{n-1+\sqrt{(n-1)^2+n-1}}{r_{\min}(0)},$$

*and does not contain the origin, then the assumptions of Theorem B are fulfilled.*

**Schlagworte.** Mittlerer Krümmungsfluß, Lagrangesche Untermannigfaltigkeit, singuläres Verhalten, mean curavture flow, lagrangian submanifold, singular behavior.

# Introduction

Let $L$ be a closed, oriented manifold. We say that a one-parameter family of immersion $F_t\colon L \to (M,g)$, $F_t = F(\cdot,t)$ satisfies the *mean curvature flow equation* if

$$\begin{aligned} \frac{d}{dt}F &= \overrightarrow{H}, \\ F(\cdot,0) &= F_0, \end{aligned} \tag{3}$$

where $\overrightarrow{H}$ is the *mean curvature vector* and $F_0\colon L \to (M,g)$ is the *initial immersion*. We recall that $H$ is the trace of the second fundamental form $A = \nabla dF$. Equation (3) is a quasi-linear parabolic system. Therefore, there exists a maximal time interval $[0, T_{\mathrm{sing}})$ in which a smooth solution of (3) exists.

The study of mean curvature flows was initiated by Brakke [11]. He was mainly interested in this flow because it is a model for the motion of grain boundaries in an annealing metal. Consequently the convenient setting is that of varifolds and geometric measure theory. An easier to read introduction to Brakke's flow has been given by Ilmanen [41].

Some time later Gage and Hamilton [27] studied the so called curve shortening flow in the plane using the classical theory of partial differential equations. Their main result is: Embedded, closed, convex curves in the plane become asymptotically spherical as they disappear, that is they shrink to a point, and after rescaling they converge smoothly to the unit circle. This result was extended by Grayson [30] who proved that embedded, closed, curves become convex. Two different proofs of the Grayson convexity theorem were given by Hamilton [33] and Huisken [37]. The curve shortening flow initiated the study of curve flows with different speed functions, we only refer to [16] for a recent exposition. The curve shortening flow has also been studied on surfaces, compare [31], [47] and many others. Let us also mention the work of Angenent who introduced Sturmian oscillation theory to the curve shortening flow, compare [5], [7], as well as some delicate singular analysis, see [6], [8].

Huisken [35] investigated the motion of convex hyper-surfaces in riemannian manifolds by the mean curvature flow. His main result is that they become asymptotically spherical. In his paper [36] he classified the singular behavior. This lead to many other papers concerning the singular behavior of the flow, see for example [9], [39], and [38]. Comprehensive estimates for graphs which evolve by their mean curvature has appeared in [18].

Let us further note that Allen and Cahn [3] conjectured mean curvature motion as the singular limit of a reaction-diffusion (phase-field) equation. This idea has been developed by de Mottoni and Schatzman [43], [44], [45], Bronsard-Kohn [12]. Chen-Giga-Goto [15] and Evans-Spruck , [20],[21],[22],[23] introduced the *level-set-flow*, in which the moving surface is the zero-set of a function, all of whose level-sets move by mean curvature. The phase-field and level-set approaches are reconciled in the paper of Evans-Soner-Souganidis [19], and unified with Brakke's work in Ilmanen's paper [40].

An immersion $F_0\colon L \to M$ in a Kähler manifold $(M^{2n}, \omega, J, g)$ is called *lagrangian* if the dimension of $L$ is $n$ and $F^*\omega = 0$. From now on we assume that this is the case. It was shown by Smoczyk [55] that the lagrangian condition is preserved under the mean curvature flow if we assume that the ambient space is Kähler-Einstein. Moreover, Wang [58], Chen and Tian [14] proved that symplectic surfaces in Kähler-Einstein manifolds remain symplectic along the mean curvature flow.

**Theorem.** *Let us assume that for any $t \in [0, T_0)$ we are given a one-parameter family of lagrangian immersions $F_t(L)$ in a Kähler-Einstein manifold $M^{2n}$ which evolves by its mean curvature. Suppose further all ambient curvatures quantities are bounded and that $\lim_{t\to T_0} \mid A \mid^2$ is bounded. Then there exists an $\epsilon > 0$ such that the mean curvature flow admits a smooth solution on the extended time interval $[0, T_0 + \epsilon)$.*

For a proof, see [56]. If the ambient space is $\mathbb{R}^{2n}$, then all ambient curvature quantities are zero. Thus, if $T_{\text{sing}}$ is finite, then the second fundamental form has to blow up. This gives: If the initial initial submanifold is compact, and the ambient space is euclidian, then the singular time is finite. This follows for example from Brakke's sphere barrier to internal varifolds. Another way to see this is to look at the evolution equation of $\mid F \mid^2$. It reads

$$\frac{d}{dt} \mid F \mid^2 = \Delta \mid F \mid^2 - 2n.$$

The parabolic maximum principle yields that the function $\mid F \mid^2 + 2nt$ is bounded from above by a constant. This gives a contradiction for $T_{\text{sing}} = \infty$. So, compact lagrangian immersions in $\mathbb{R}^{2n}$ will develop finite time singularities.

A point $p \in \mathbb{R}^{2n}$ is called *blow-up point* if there exists a point $x \in L$ such that $\lim_{t \to T_{\text{sing}}} F(x,t) = p$ and $\lim_{t \to T_{\text{sing}}} |\, A\,|^2(x) = \infty$. It holds

$$\frac{\mathtt{const}_1}{T_{\text{sing}} - t} \leq \max_{L_t} |\, A\,|^2,$$

where $\mathtt{const}_1 > 0$, this can be shown with evolution equation of $|\, A\,|^2$ and the maximum principle. We will call the singularity to be of *type-I* if there exists another constant $\mathtt{const}_2$ such that

$$\max_{L_t} |\, A\,|^2 \leq \frac{\mathtt{const}_2}{T_{\text{sing}} - t}.$$

The prime example is the sphere which shrinks self-similarly to a point. Otherwise the singularity is called of *type-II.* For an example picture an immersed convex curve with two loops. It is intuitively clear that it will develop a kink, for a proof see [8].

Let us define for $y \in \mathbb{R}^{2n}$

$$\rho(y,t) := \Big\{ \frac{1}{4\pi(T_{\text{sing}} - t)} \Big\}^{\frac{n}{2}} \exp\Big\{ - \frac{|\, y\,|^2}{4(T_{\text{sing}} - t)} \Big\}.$$

**Theorem.** *If $L_t$ is a family of closed lagrangian immersions in $\mathbb{R}^{2n}$ which evolves by its mean curvature, then we have*

$$\frac{d}{dt} \int_{L_t} \rho\big(F(x,t),t\big)\, d\mu_t = - \int_{L_t} \Big| \overrightarrow{H} + \frac{1}{2(T_{\text{sing}} - t)} F^{\perp} \Big|^2 \rho\big(F(x,t),t\big)\, d\mu_t. \quad (4)$$

For a proof see Huisken [36]. Equation (4) is called *monotonicity formula.* This formula is analogous to the monotonicity formula for minimal surfaces, compare §5.4.3 of [25], the monotonicity formula of Giga and Kohn, [29], the mean value property for harmonic functions, and for the Yang-Mills flow, [48]. A similar formula also holds for the Brakke flow, compare [41].

Now we describe the *rescaling procedure.* For simplicity we assume that the origin is a blow-up point. We define the rescaled lagrangian immersions by

$$\widetilde{F}(x,s) := \frac{1}{\sqrt{2\big(T_{\text{sing}} - t\big)}} F(x,t),$$

$$s(t) := -\frac{1}{2} \ln \big(T_{\text{sing}} - t\big).$$

Then the submanifolds $\widetilde{L}_s := \widetilde{F}(L,s)$ are defined for $s \in [-\frac{1}{2} \ln T_{\text{sing}}, \infty)$, and satisfy the equation

$$\frac{d}{ds} \widetilde{F}(x,s) = \overrightarrow{\widetilde{H}}(x,s) + \widetilde{F}(x,s).$$

Note that a rescaled lagrangian submanifold is again a lagrangian submanifold. It holds:

**Theorem.** *Let us suppose that $F_t \colon L \to \mathbb{R}^{2n}$ is a smooth one-parameter family of oriented, lagrangian immersions which moves by the mean curvature. Let us further assume that the origin is a blow-up point of type-I. Then for each sequence $s_j \to \infty$, there exists a subsequence again denoted by $s_j$ such that the rescaled lagrangian immersions $\widetilde{L}_{s_j}$ converge to a limiting immersions $\widetilde{L}_\infty$ smoothly on compact subsets. Moreover, the limit immersion satisfies the identity*

$$H_i = -\langle F, N_i \rangle . \tag{5}$$

We remark that any immersion which satisfies the above identity (5) is called *self-similar*. This terminology is due to the fact that a self-similar submanifold shrinks homothetically under the mean curvature flow. A similar result holds for the Brakke flow, see [42]. The main difficulty here is that of regularity.

Let us note some geometrically interesting properties of the mean curvature flow. As noted above it was shown by Smoczyk [55] that it preserves the lagrangian condition. Moreover in [56] it was shown: *If $L_t$ is a family of closed, oriented, lagrangian submanifolds evolving by the mean curvature flow in a Calabi-Yau manifold, then the cohomology class of the one-form $H$ is fixed.*

Wang [58] proved that no type-I singularities can occur under the lagrangian mean curvature flow if on the initial lagrangian immersion we have $\cos(\alpha) \geq 0$. Here $\alpha$ is the lagrangian angle. In particular, this condition implies $[H] = 0$ for the cohomology class of the mean curvature form $H$. Neves [46] on the other hand showed that $[H] = 0$ implies that no type-I singularity form. Finally, Li and Chen proved that type-II singularities of the lagrangian mean curvature flow in $\mathbb{C}^2$ consists of a finite union of more than one lagrangian two-plane.

Let us now describe the content of our thesis. The present paper gives a detailed analysis of the singular behavior of (3) in the case of equivariant lagrangian submanifolds. Suppose that $z_0 \colon S^1 \to \mathbb{C}\backslash\{0\}$ is a closed immersed curve with $z_0 = u_0 + \imath v_0$, and $G \colon S^{n-1} \to \mathbb{R}^n$ is the standard embedding of the sphere of radius one. Then the *equivariant lagrangian submanifold* $F_0 \colon S^1 \times S^{n-1} \to \mathbb{C}^n$ is given by

$$F_0(\phi, x) = \big(u_0(\phi)G(x), v_0(\phi)G(x)\big).$$

One has to distinguish two different cases, namely whether $z_0$ encloses the origin or not. We will focus on the latter case, although some insight to existing results of the former case is given.

Since the mean curvature flow is isotropic, it will be determined by the flow of the corresponding profile curves. Thus, to solve equation (3) in the equivariant setting described above we have to find a smooth family of curves $z\colon S^1 \times [0, T_{\text{sing}}) \to \mathbb{C}$ for which

$$\begin{aligned} \frac{d}{dt} z &= \mathscr{F} N := \Big\{ k - (n-1) \frac{\langle e_r, N \rangle}{r} \Big\} N, \\ z(\cdot, 0) &= z_0, \end{aligned} \tag{6}$$

where $k$ denotes the curvature of the curve, $N$ the inward pointing unit normal, $e_r := \frac{z}{r}$, $r := |z|$, and $n$ the dimension of $L$. The derivation of equation (6) was given by H. Anciaux [4] who also classified all self-similar solutions of this flow. These curves show that unlike in the curve shortening flow the curves do not necessarily become convex. Moreover, as shown in [32] an initial convex curve can also become non-convex and develop a type-II singularity.

Our main results are the following theorems:

**Theorem A.** *Let $F_0$ be an equivariant lagrangian immersion of $L$ in $\mathbb{C}^n$. If the initial profile curve is closed, embedded, satisfies $\mathscr{F} > 0$, and does not contain the origin, then $F_t(S^1 \times S^{n-1})$ converges to a sphere $\|p_0\| S^{n-1}$ as time goes to $T_{\text{sing}}$. Moreover, the singular time is determined by the enclosed area $A_0$ of the initial curve. That is $T_{\text{sing}} = \frac{A_0}{2\pi}$.*

Let us note that $\mathscr{F} = |\overrightarrow{H}|$. Thus, $\mathscr{F} > 0$ implies that $\overrightarrow{H}$ never vanishes.

**Theorem B.** *Let $F_0$ be an equivariant lagrangian immersion of $L$ in $\mathbb{C}^n$. If the initial profile curve is closed, embedded, satisfies $\mathscr{F} > 0$, does not contain the origin, and all profile curves remain convex, then the singularity is of type-I. After rescaling and possibly choosing a subsequence $F_t$ converge to the cylinder $S^{n-1} \times \mathbb{R}$ smoothly on compact subsets of $\mathbb{C}^n$.*

This is an example of a monotone lagrangian submanifold in $\mathbb{C}^n$ which develops a type-I singularity under the mean curvature flow.

**Theorem C.** *Let $F_0$ be an equivariant lagrangian immersion of $L$ in $\mathbb{C}^n$. If the initial profile curve is closed, embedded, satisfies*

$$\mathscr{F}_{\min}(0) \geq \frac{n - 1 + \sqrt{(n-1)^2 + n - 1}}{r_{\min}(0)},$$

*and does not contain the origin, then the assumptions of Theorem B are fulfilled.*

The present paper consists of three chapters and three appendices.

Chapter 1 takes a look at the curve shortening flow that is the case where $n = 1$. It recalls convex sets. Then we introduce tamed sets which are generalized convex sets. Finally some comments on the general planar curve flow problem are given. Chapter 2 provides the proofs of Theorem A and C.

Theorem A is proved analogously as in the curve shortening case. But we have to replace convex curves, which are characterized by $k > 0$ with tamed curves, which are characterized by $\mathscr{F} > 0$. It is shown that tamed curves enjoy most properties of convex sets, this result is established in Chapter 1. The remaning parts of the theorem is proved in Chapter 2.

Chapter 3 consits of four sections: The first section explains the blow-up procedure. The second section establishes the asymptotic behavior of several geometric quantities. The key observation is that $r_p \sim \exp(t)$. The third section proves a Bernstein-type estimate as well as a Harnack-type inequality. Finally, in the fourth section we finish the proof of Theorem B.

The structure of the proof is as follows: We use Gage's inequality to bound the length of the rescaled curves. This is the only time where we make us of the convexity assumption. Then it is shown that $\mathscr{G} \to -\langle z, N\rangle$ in $L^2(S^1)$. This is done with a monotonicity-type argument. Moreover, we show that the entropy of the curves remains bounded. Together with the Harnack-type inequality and the Arzela-Ascoli theorem the result is obtained.

In the appendices we have collected several computations which would have distracted the flow of reading.

One should keep in mind that we are considering two pairs of flow. The first pair is the *equivariant curve shortening flow* and its *reparametrized* counterpart. The second pair is the *rescaled equivariant curve shortening flow* and again its *reparametrized* counterpart. Another flow is also mentioned; the *weighted curve flow*.

# Contents

CHAPTER 1

# Reparametrization

This chapter prepares the proof of Theorem A. We begin with an expository Section 1.1 on the curve shortening flow equation. We included this paragraph because the proof of Theorem A has a similar structure at a conceptual level. The main ingredients are a reparametrization of the curves with respect to the normal angle and the notion of the support function. The section is followed by an elementary discussion of convex sets and their properties. In the third Section 1.3 we introduce what we call *tamed sets* which play a similar role as convex sets play in the curve shortening flow. Furthermore, we reparametrize the curves with respect to an new angle, and introduce a generalized support function.

## 1.1 On the curve shortening flow

The discussion of this section is of expository character and may be skipped. If the dimension is $n = 1$, then (6) is just the curve shortening flow equation, considered by Gage and Hamilton [27] and many others. In this section we recall the definition of the support function for a strictly convex plane curve along with basic properties of it. Then we recall one possible proof of Gage and Hamilton's classical result that embedded, closed, strictly convex plane curves shrink to a point in finite time. The chapter closes with some general comments on the curve flow problem.

**1.1.1.** The *curve shortening flow equation* reads; given $z_0 \colon S^1 \to \mathbb{C}$ find a smooth family $z \colon S^1 \times [0, T_{\text{sing}}) \to \mathbb{C}$ with

$$\begin{aligned} \frac{d}{dt} z &= kN, \text{ and} \\ z(\cdot, 0) &= z_0. \end{aligned} \tag{1.1}$$

Here $k$ denotes the curvature of the curve, and $N$ the inward pointing unit normal. The first key observation is that convexity of $z_0$ is preserved along the flow.

Indeed, we have

$$\frac{d}{dt}k = \Delta k + k^3;$$

the claim follows with the maximum principle. Let us recall that a strictly convex curve $z\colon S^1 \to \mathbb{C}$ admits a reparametrization $\vartheta\colon S^1 \to S^1$ as follows

$$\vartheta(p) = \int_0^p k\, d\mu.$$

This is possible because $\int_{S^1} k\, d\mu = 2\pi$, and $k > 0$. Because the flow preserves strict convexity, we may reparametrize $S^1$ as above for every $t \in [0, T_{\text{sing}})$. Whenever we do this we say that we *reparametrize the flow.* Let us now give a geometric interpretation for the parameter $\vartheta$. We note that by definition

$$\frac{\partial}{\partial\vartheta}\mu = \frac{1}{k}.$$

This gives with the Frenet formulas that

$$\frac{\partial}{\partial\vartheta}T = N \text{ and } \frac{\partial}{\partial\vartheta}N = -T.$$

In particular, we have $\frac{\partial^2}{\partial\vartheta^2}N = -N$. Therefore, we may choose the parametrization such that the *inward pointing unit normal* is given by

$$N(\vartheta) = -\begin{pmatrix}\cos\vartheta\\ \sin\vartheta\end{pmatrix}.$$

With this choice the unit circle fulfills $z(\vartheta) = -N(\vartheta)$. This is the reason why one calls $\vartheta$ the *normal angle.* Let us recall the definition of the *support function*

$$h(\vartheta) := -r\,\langle e_r, N\rangle = x(\vartheta)\cos\vartheta + y(\vartheta)\sin\vartheta.$$

The support function measures the signed distance of the supporting hyperplane to the origin. Let us denote differentiation with respect to $\vartheta$ by a prime. Observe that

$$h' = r\,\langle e_r, T\rangle\,.$$

The curve as a function of $\vartheta$ is given by

$$z(\vartheta) = -r\,\langle e_r, N\rangle\,N + r\,\langle e_r, T\rangle\,T = \left\{h + \imath h'\right\}N = \left\{h + ih'\right\}\exp(\imath\vartheta).$$

Here we have identified $\mathbb{C}$ and $\mathbb{R}^2$. Note that $z(0)$ corresponds to the point on $\gamma$ which has normal vector $\binom{-1}{0}$.

Suppose that $A, B$ are both strictly convex subsets of the plane with corresponding support functions $h_A$ and $h_B$. Then $A \subset B$ if and only if $h_A \le h_B$. Finally, let us recall that the *area* of $\gamma$ may be computed by

$$A(\gamma) = \frac{1}{2}\int_{S^1} h^2 - h'^2 \, d\vartheta,$$

and the *length* by

$$L(\gamma) = \int_{S^1} h \, d\vartheta,$$

compare [10].

**1.1.2.** Now we want to express the evolution equation of the curvature in terms of the new parameter $\vartheta$. Let us denote by $\tau$ the new time parameter then we use $\vartheta$ as the other coordinate. Thus we change variables form $(p,t)$ to $(\vartheta,\tau)$. We want to point out that $\frac{\partial}{\partial t} \neq \frac{\partial}{\partial \tau}$. We compute

$$\frac{d}{dt}\vartheta = \frac{d}{dt}\int_0^p k \, d\mu = \int_0^p \frac{\partial^2}{\partial \mu^2}k + k^3 - k^3 \, d\mu = \frac{\partial}{\partial \mu}k = k\frac{\partial}{\partial \vartheta}k.$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial \tau}k &= \frac{d}{dt}k - \frac{\partial}{\partial \vartheta}k\frac{d}{dt}\vartheta = \frac{\partial^2}{\partial \mu^2}k - k\Big\{\frac{\partial}{\partial \vartheta}k\Big\}^2 + k^3 \\ &= \frac{\partial^2}{\partial \mu^2}k - k\Big\{\frac{\partial}{\partial \vartheta}k\Big\}^2 + k^3 = k^2\frac{\partial^2}{\partial \vartheta^2} + k^3.\end{aligned}$$

In the same spirit the other evolution equations may be derived. We observe

$$\frac{\partial}{\partial \tau}z = \frac{d}{dt}z - \frac{\partial}{\partial \vartheta}z\frac{d}{dt}\vartheta = kN - k\frac{\partial}{\partial \vartheta}z\frac{\partial}{\partial \vartheta}k = kN - \frac{\partial}{\partial \vartheta}kT. \tag{1.2}$$

Note that the original flow equation (1.1) and the reparametrized equation (1.2) differ only by a tangential term. Thus, they describe the same geometric flow. The tangential contribution just makes $\vartheta$ and $\tau$ independent.

**1.1.3.** We claim that the enclosed area at singular time must vanish. If this is not the case, then there exists a small ball enclosed by all curves $z(\cdot,\tau)$ for $\tau$ closed to $T_{\text{sing}}$. That is to say that $h - h_{\text{ball}} \ge \epsilon > 0$. The maximum principle applied to the evolution equation of the function

$$f = \frac{k}{h - h_{\text{ball}}}$$

will give a contradiction. We introduce

$$m = \frac{1}{h - h_{\mathrm{ball}}}.$$

Then, there exist constants $\mathtt{const}_1, \mathtt{const}_2 > 0$ such that

$$\mathtt{const}_1 \leq m \leq \mathtt{const}_2. \tag{1.3}$$

We note that

$$m' = m^2\big\{ - \langle z, T\rangle + h'_{\mathrm{ball}}\big\},$$

and

$$m'' = 2m^3\Big\{ - \langle z, T\rangle + h'_{\mathrm{ball}}\Big\}^2 + m^2\Big\{ - \frac{1}{k} - \langle z, N\rangle + h''_{\mathrm{ball}}\Big\}.$$

Here, the prime denoted differentiation with respect to $\vartheta$. This yields

$$f' = mk' + m^2\big\{ - \langle z, T\rangle + h'_{\mathrm{ball}}\big\}k.$$

Therefore,

$$mk' = f' - m^2\big\{ - \langle z, T\rangle + h'_{\mathrm{ball}}\big\}k.$$

Moreover,

$$\begin{aligned} f'' = mk'' &+ 2m^2\big\{ - \langle z, T\rangle + h'_{\mathrm{ball}}\big\}k' \\ &+ \Big\{2m^3\big\{ - \langle z, T\rangle + h'_{\mathrm{ball}}\big\}^2 + m^2\big\{ - \langle z, N\rangle + h''_{\mathrm{ball}}\big\}\Big\}k - m^2. \end{aligned}$$

Combined with the last equality we get

$$f'' = mk'' + 2m\big\{ - \langle z, T\rangle + h'_{\mathrm{ball}}\big\}f' + m^2\big\{ - \langle z, N\rangle + h''_{\mathrm{ball}}\big\}k - m^2.$$

Now we can derive the evolution equation of $f$. We compute with the help of §A.3 that

$$\frac{d}{d\tau}f = mk^2k'' + mk^3 + m^2k^2,$$

which gives

$$\begin{aligned} \frac{d}{d\tau}f = k^2f'' &+ 2mk^2\big\{ \langle z, T\rangle - h'_{\mathrm{ball}}\big\}f' \\ &+ 2m^2k^2 + \big\{m^2\big(\langle z, N\rangle - h''_{\mathrm{ball}}\big) + m\big\}k^3. \end{aligned} \tag{1.4}$$

Let us recall that the support function of a ball centered at $p = (p_0, p_1)$, with radius $\rho$ is given by

$$h_{\text{ball}}(\vartheta) = \rho + p_0 \cos\vartheta + p_1 \sin\vartheta.$$

This implies that $h''_{\text{ball}} = \rho - h_{\text{ball}}$. Therefore

$$\begin{aligned} m^2\big(\langle z, N\rangle - h''_{\text{ball}}\big) + m &= m^2\big(\langle z, N\rangle + h_{\text{ball}} - \rho\big) + m \\ &= m^2\Big(-\frac{1}{m} - \rho\Big) + m = -\rho m^2. \end{aligned}$$

Inserting this in equation (1.4) yields

$$\frac{d}{d\tau} f = k^2 f'' + 2mk^2\big\{\langle z, T\rangle - h'_{\text{ball}}\big\} f' + 2m^2k^2 - \rho m^2 k^3.$$

It holds $\rho > 0$ because of our assumption. Invoking Inequality (1.3) we see that $f$ is bounded from above by the maximum principle. But this is a contradiction as the curvature $k$ has to blow up at $T_{\text{sing}}$ and therefore also $f$. This proves the claim.

**1.1.4.** We are left with two cases; either the limit curve is a point or a segment. But it can not be the latter because, $k$ is bounded away from zero again by the maximum principle. Thus we have shown convergence of support functions, or equivalently (in the set of compact convex bodies) convergence in the Hausdorff metric, compare Schneider [52] and also the next section. All we have to do is carry over this proof to the case $n \geq 1$.

### Notes for Section 1.1

**1.** As noted earlier, the starting point for the curve shortening flow is the paper of Gage and Hamilton [27], in which they prove that simple, strictly convex curves converge smoothly to a round point. The given proof follows ideas of Tso [57].

**2.** What we know about the curve shortening flow goes way beyond what we have sketched in this section. Gage and Hamilton further showed in their paper that the curves become asymptotically round. Grayson [31] proved that any embedded curve in the plane becomes convex before it develops a singularity. For further development we refer to the literature, see for example [16] and the reference therein.

**3.** The notion of the support function of a closed convex curve can already be found in Blaschke's classical book [10]. In fact, every weak*-closed convex subset $A \subset E'$ of the dual of a real Banach space $E$ admits a support function

$$\sigma(x) = \sup_{x' \in A} \langle x', x\rangle.$$

Moreover, it is uniquely determined by its support function by Hörmander's theorem, compare [2].

### 1.2 On convex sets

**1.2.1.** Let $(X,d)$ be a metric space. For any subset $A$ in $X$, and $\epsilon > 0$ we define the *$\epsilon$-thickening* of $A$ by

$$[A]_\epsilon := \{x \in X \,|\, d(x,A) \leq \epsilon\} .$$

Let $\mathfrak{X}$ denote the collection of compact subsets of $X$. Given $E$ and $F$ in $\mathfrak{X}$, we define their *Hausdorff distance* by

$$\delta(E,F) := \inf\{\epsilon > 0 \,|\, E \subset [F]_\epsilon, F \subset [E]_\epsilon\} .$$

One can check that $(E,F) \mapsto \delta(E,F)$ satisfies the axioms of a metric. We have the following theorem: *Assume that $(X,d)$ has the property that closed and bounded subsets are compact, then $(\mathfrak{X},\delta)$, the space of compact subsets of $X$ with Hausdorff metric $\delta$, is complete. Furthermore, if $X$ is compact, then $\mathfrak{X}$ is compact.* As a consequence: *From each bounded sequence of convex bodies one can select a subsequence converging to a convex body.* This is Blaschke famous selection theorem.

**1.2.2.** Suppose that $A$ and $B$ are subsets of the plane. Let us recall that a hyperplane is determined by a vector $N \in \mathbb{R}^2$ and a real number $\alpha \in \mathbb{R}$ as follows $H(N,\alpha) = \{y \in \mathbb{R}^2 \,|\, \langle y, N\rangle = \alpha\}$. In our case $H(N,\alpha)$ is just a straight line. We say that $H(N,\alpha)$ separates $A$ and $B$ if $A \subset H(N,\alpha)^- := \{y \in \mathbb{R}^2 \,|\, \langle y, N\rangle \leq \alpha\}$ and $B \subset H(N,\alpha)^+ := \{y \in \mathbb{R}^2 \,|\, \langle y, N\rangle \geq \alpha\}$, or vice versa. We say that $A$ and $B$ are strongly separated by $H(N,\alpha)$ if there exists $\epsilon$ such that $A$ and $B$ are separated by $\subset H(N,\alpha-\epsilon)^-$ and $H(N,\alpha+\epsilon)^+$.

The following *separation theorem* holds true: *If $A$ and $B$ are nonempty convex subsets of the plane with $A \cap B = \varnothing$, then $A$ and $B$ can be separated. If $A$ is compact and $B$ is closed, then $A$ and $B$ can be strongly separated.*

## Notes for Section 1.2

**1.** The treatment of the Hausdorff metric, and Blaschke's selection theorem is taken from [13]. Blaschke's selection theorem can be found on page 62 of [10].

**2.** Paragraph 1.2.2 relies on Section 1.3 of Schneider's book [52]. The standard reference to convex analysis is of course Rockafellar's book [49]. The main direction of this book is optimization theory.

### 1.3 Tamed curves

This section prepares the proof of Theorem A. The flow equation we are looking at is

$$\begin{aligned} \frac{d}{dt} z &= \mathscr{F} N := \Big\{ k - (n-1) \frac{\langle e_r, N \rangle}{r} \Big\} N, \\ z(\cdot, 0) &= z_0. \end{aligned} \tag{1.5}$$

The theorem was proved in Section 1.1 in the case $n = 1$. The first step is: Find a suitable notion of convex curves in the general case. This will be curves which satisfy $\mathscr{F} > 0$. We refer to such curves as being *tamed*.

**1.3.1.** A standard calculation yields the evolution equation of $\mathscr{F}$. We observe from Appendix A.4 that

$$\begin{aligned} \frac{d}{dt} \mathscr{F} &= \frac{d}{dt} k - (n-1) \frac{d}{dt} \frac{\langle e_r, N \rangle}{r} \\ &= \Delta \mathscr{F} + k^2 \mathscr{F} \\ &\quad - (n-1) \Big\{ - \frac{\langle e_r, \nabla \mathscr{F} \rangle}{r} + \Big\{ \frac{1}{r^2} - \frac{\langle e_r, N \rangle^2}{r^2} \Big\} \mathscr{F} \Big\} + (n-1) \frac{\langle e_r, N \rangle^2}{r^2} \mathscr{F}. \end{aligned}$$

Note that,

$$k^2 = \Big\{ \mathscr{F} + (n-1) \frac{\langle e_r, N \rangle}{r} \Big\}^2 = \mathscr{F}^2 + 2(n-1) \frac{\langle e_r, N \rangle}{r} \mathscr{F} + (n-1)^2 \frac{\langle e_r, N \rangle^2}{r^2}.$$

Hence,

$$\begin{aligned} \frac{d}{dt} \mathscr{F} &= \Delta \mathscr{F} + (n-1) \frac{\langle e_r, \nabla \mathscr{F} \rangle}{r} + (n-1) \Big\{ (n+1) \frac{\langle e_r, N \rangle^2}{r^2} - \frac{1}{r^2} \Big\} \mathscr{F} \\ &\qquad + 2(n-1) \frac{\langle e_r, N \rangle}{r} \mathscr{F}^2 + \mathscr{F}^3. \end{aligned} \tag{1.6}$$

This implies that $\mathscr{F} > 0$ is preserved along the flow by the maximum principle. From now on we assume that $z \colon S^1 \to \mathbb{C}$ satisfies $\mathscr{F} > 0$. Whenever this is the case we say that $z$ is *tamed*. Let us introduce the following functions:

$$\begin{aligned} \Phi &= (n-1) \Big\{ (n+1) \frac{\langle e_r, N \rangle^2}{r^2} - \frac{1}{r^2} \Big\} \\ \Psi &= 2(n-1) \frac{\langle e_r, N \rangle}{r} \end{aligned}$$

Sometimes it is useful to make use of the following evolution equation

$$\frac{d}{dt}\mathscr{F} = \frac{\partial^2}{\partial\mu^2}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\frac{\partial}{\partial\mu}\mathscr{F} + \Phi\mathscr{F} + \Psi\mathscr{F}^2 + \mathscr{F}^3. \tag{1.7}$$

That Equation (1.6) and Equation (1.7) are equivalent is shown with the help of Appendix A.4.

**1.3.2.** Moreover, we note that

$$\int_{S^1} \mathscr{F}\,d\mu = \int_{S^1} k - (n-1)\frac{\langle e_r, N\rangle}{r}\,d\mu = 2\pi\big\{\operatorname{rot}(z) - (n-1)\operatorname{wind}(z)\big\} =: 2\pi\kappa,$$

where $\operatorname{rot}(z)$ denotes the *rotation number* of the curve, and $\operatorname{wind}(z)$ its *winding number* with respect to the origin. It follows that

$$\eta(p) := \int_0^p \mathscr{F}\,d\mu \tag{1.8}$$

is a map of $\eta\colon S^1 \to S^1(\kappa)$. Here $S^1(\kappa) := \mathbb{R}/2\pi\kappa\mathbb{Z}$. Thus $z\circ\eta^{-1}\colon S^1(\kappa) \to \mathbb{C}$ is a reparametrization of our curve. Similar to the curve shortening case we perform this reparametrization for all $t \in [0, T_{\text{sing}})$. This works because $\mathscr{F} > 0$ is preserved during the flow as shown in the previous paragraph. Let us now change the parameters from $(p,t)$ to $(\eta,\tau)$. In order to make $\eta$ independent of the time parameter $\tau$ we have to add a tangential term to equation (6). We note that

$$\begin{aligned}
\frac{d}{dt}\eta(p) &= \frac{d}{dt}\int_0^p \mathscr{F}\,d\mu \\
&= \int_0^p \frac{\partial^2}{\partial\mu^2}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\frac{\partial}{\partial\mu}\mathscr{F} + (n-1)\Big\{(n+1)\frac{\langle e_r, N\rangle^2}{r^2} - \frac{1}{r^2}\Big\}\mathscr{F} \\
&\qquad\qquad + 2(n-1)\frac{\langle e_r, N\rangle}{r}\mathscr{F}^2 + \mathscr{F}^3 - \mathscr{F}^2 k\,d\mu \\
&= \int_0^p \frac{\partial^2}{\partial\mu^2}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\frac{\partial}{\partial\mu}\mathscr{F} \\
&\qquad + (n-1)\Big\{(n+1)\frac{\langle e_r, N\rangle^2}{r^2} - \frac{1}{r^2}\Big\}\mathscr{F} + (n-1)\frac{\langle e_r, N\rangle}{r}\mathscr{F}^2\,d\mu.
\end{aligned}$$

Here we refer to Appendix A.2 and equation (1.7). We note that

$$\begin{aligned}
\int_0^p \frac{\langle e_r, T\rangle}{r}\frac{\partial}{\partial\mu}\mathscr{F}\,d\mu &= \frac{\langle e_r, T\rangle}{r}\mathscr{F} - \int_0^p \Big\{\frac{1}{r^2} + \frac{\langle e_r, N\rangle}{r}k - 2\frac{\langle e_r, T\rangle^2}{r^2}\Big\}\mathscr{F}\,d\mu \\
&= \frac{\langle e_r, T\rangle}{r}\mathscr{F} - \int_0^p \Big\{\frac{1}{r^2} + \frac{\langle e_r, N\rangle}{r}\mathscr{F} + (n-1)\frac{\langle e_r, N\rangle^2}{r^2} - 2\frac{\langle e_r, T\rangle^2}{r^2}\Big\}\mathscr{F}\,d\mu \\
&= \frac{\langle e_r, T\rangle}{r}\mathscr{F} + \int_0^p \frac{\mathscr{F}}{r^2} - \frac{\langle e_r, N\rangle}{r}\mathscr{F}^2 - (n+1)\frac{\langle e_r, N\rangle^2}{r^2}\mathscr{F}\,d\mu.
\end{aligned}$$

This and the previous equation yield

$$\frac{d}{dt}\eta(p) = \Big\{\frac{\partial}{\partial\mu}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\mathscr{F}\Big\} = \mathscr{F}\Big\{\frac{\partial}{\partial\eta}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}.$$

With this equation at hand we can compute the evolution equation of $\mathscr{F}$ in new coordinates $(\eta, \tau)$. But first let us observe that

$$\frac{\partial}{\partial\eta}z = \frac{1}{\mathscr{F}}\frac{\partial}{\partial\mu}z = \frac{1}{\mathscr{F}}T.$$

This yields, combined with the previous equation, that

$$\frac{\partial}{\partial\tau}z = \frac{d}{dt}z - \frac{\partial}{\partial\eta}z\frac{d}{dt}z = \mathscr{F}N - \Big\{\frac{\partial}{\partial\eta}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}T.$$

Therefore the *reparametrized flow equation* reads

$$\frac{d}{d\tau}z = \mathscr{F}N - \Big\{\frac{d}{d\eta}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}T. \tag{1.9}$$

The tangential contribution of equation (1.9) does not alter the geometric behavior of the flow, it just makes $\tau$ and $\eta$ independent. Let us now derive the evolution equation of $\mathscr{F}$. First of all recall that

$$\frac{\partial}{\partial\mu}\mathscr{F} = \mathscr{F}\frac{\partial}{\partial\eta}\mathscr{F},$$

and

$$\frac{\partial^2}{\partial\mu^2}\mathscr{F} = \mathscr{F}^2\frac{\partial^2}{\partial\eta^2}\mathscr{F} + \mathscr{F}\Big\{\frac{\partial}{\partial\eta}\mathscr{F}\Big\}^2.$$

We have

$$\begin{aligned}
\frac{\partial}{\partial\tau}\mathscr{F} &= \frac{d}{dt}\mathscr{F} - \frac{\partial}{\partial\eta}\mathscr{F}\frac{\partial\eta}{\partial t} \\
&= \frac{\partial^2}{\partial\mu^2}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\frac{\partial}{\partial\mu}\mathscr{F} + \Phi\mathscr{F} + \Psi\mathscr{F}^2 + \mathscr{F}^3 \\
&\quad - \mathscr{F}\Big\{\frac{\partial}{\partial\eta}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}\frac{\partial}{\partial\eta}\mathscr{F} \\
&= \mathscr{F}^2\frac{\partial^2}{\partial\eta^2}\mathscr{F} + \mathscr{F}\Big\{\frac{\partial}{\partial\eta}\mathscr{F}\Big\}^2 + (n-1)\frac{\langle e_r, T\rangle}{r}\mathscr{F}\frac{\partial}{\partial\eta}\mathscr{F} \\
&\quad + \Phi\mathscr{F} + \Psi\mathscr{F}^2 + \mathscr{F}^3 - \mathscr{F}\Big\{\frac{\partial}{\partial\eta}\mathscr{F}\Big\}^2 - (n-1)\frac{\langle e_r, T\rangle}{r}\mathscr{F}\frac{\partial}{\partial\eta}\mathscr{F}.
\end{aligned}$$

Hence,

$$\frac{d}{d\tau}\mathscr{F} = \mathscr{F}^2\frac{d^2}{d\eta^2}\mathscr{F} + \Phi\mathscr{F} + \Psi\mathscr{F}^2 + \mathscr{F}^3. \tag{1.10}$$

For the readers convenience we have collected more calculations for the reparametrized flow in Appendix B.1.

**1.3.3.** A geometric characterization of $\eta$ is not so easy to explain. In some sense it is also a kind of normal angle. If we look at convex curves, then

$$N = -\begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix}$$

was the normal vector of the curve at $z(\vartheta)$, and $N$ was also normal to the supporting hyperplane going through $z(\vartheta)$. A hyperplane has the property that its curvature is equal to zero. As a first step we want to replace these hyperplanes by *supporting hypercurves* characterized by the property that $\mathscr{F} = 0$. The corresponding equivariant submanifold are the so called *lagrangian catenoids* described by Harvey and Lawson in [34].

We claim that: *Let* $\eta_0 \in [0, 2\pi]$, *and* $h \in \mathbb{R}\backslash\{0\}$. *The implicitly defined curve given by*

$$f(x, y) := \Re z^n \cos\eta_0 + \Im z^n \sin\eta_0 - h = 0,$$

*is a stationary solution of equation* (1.9)*, where* $z = x + \imath y$ *and the motion of the curve is taken in direction of the vector* $(f_y, -f_x)^T$*.* The proof of this assertion can be found in Appendix B.2. Let us list these curves for $n = 1, \dots, 6$:

$$\begin{aligned}
h_1 &= x\cos\eta + y\sin\eta \\
h_2 &= (x^2 - y^2)\cos\eta + 2xy\sin\eta \\
h_3 &= (x^3 - 3xy^2)\cos\eta + (-y^3 + 3x^2y)\sin\eta \\
h_4 &= (x^4 - 6x^2y^2 + y^4)\cos\eta + (-4xy^3 + 4x^3y)\sin\eta \\
h_5 &= (x^5 - 10x^3y^2 + 5xy^4)\cos\eta + (y^5 - 10x^2y^3 + 5x^4y)\sin\eta \\
h_6 &= (x^6 - 15x^4y^2 + 5x^2y^4 - y^6)\cos\eta + (6x^5y - 20x^3y^3 + 6xy^5)\sin\eta
\end{aligned}$$

Moreover, the normal vector is given by

$$N_{\text{hyp}} = \frac{1}{r^{n-1}}\begin{pmatrix} \Re z^{n-1}\cos\eta + \Im z^{n-1}\sin\eta \\ -\Im z^{n-1}\cos\eta + \Re z^{n-1}\sin\eta \end{pmatrix},$$

which implies that

$$h = r^n \langle e_r, N_{\text{hyp}} \rangle .$$

Let us now suppose that $z\colon S^1(\kappa) \to \mathbb{C}$ is a tamed curve, which is parametrized by $\eta(p) := \int_0^p \mathscr{F}\, d\mu$. Then to every point $z(\eta)$ there is associated a real number $h(\eta) := -r^n \langle e_r, N_{\text{curve}} \rangle$, which measures the distance of the supporting hypercurve going through $z(\eta)$ with normal angle $-N_{\text{curve}}$ at $z(\eta)$. That this heuristic picture is indeed true will be revealed in the next paragraphs §1.3.4 - §1.3.5. Moreover, we will see that tamed set enjoy most properties of convex sets.

**1.3.4.** The previous paragraph motivated to define the *generalized support function* of a tamed, reparametrized curve $z\colon S^1(\kappa) \to \mathbb{C}$ by

$$h(\eta) = -r^n \langle e_r, N \rangle .$$

This definition needs some justification. We know from §B.1.1 that

$$h' = r^n \langle e_r, T \rangle , \text{ and } h'' = n\frac{r^{n-1}}{\mathscr{F}} - h.$$

Here $'$ denotes differentiation with respect to $\eta$. These two equations imply several things. First of all we have

$$r^{2n} = h^2 + h'^2, \text{ and } \frac{\mathscr{F}}{r^{n-1}} = \frac{n}{h + h''}. \tag{1.11}$$

Therefore, we can recover $\mathscr{F}$ from the support function as follows

$$\mathscr{F} = n\frac{\left\{h^2 + h'^2\right\}^{\frac{n-1}{2n}}}{h + h''}. \tag{1.12}$$

With a little effort one can also show that

$$z^n = \Big\{h + \imath h'\Big\} \exp\{\imath\eta\}. \tag{1.13}$$

Let us derive equation (1.13). We know from the previous paragraph that the support function $h$ can also be written as

$$h(\eta) = \Re z^n \cos\eta + \Im z^n \sin\eta.$$

We recall from Appendix B.2 that

$$T_{\text{hyp}} = \frac{1}{r^{n-1}} \begin{pmatrix} -\Im z^{n-1} \cos\eta + \Re z^{n-1} \sin\eta \\ -\Re z^{n-1} \cos\eta - \Im z^{n-1} \sin\eta \end{pmatrix}.$$

This implies that

$$\begin{aligned} h'(\eta) &= r^n \langle e_r, T_{\text{curve}} \rangle = -r^{n-1} \langle z, T_{\text{hyp}} \rangle \\ &= \Big\{x\Im z^{n-1} \cos\eta - y\Re z^{n-1} \sin\eta + y\Re z^{n-1} \cos\eta + y\Im z^{n-1} \sin\eta\Big\} \\ &= \Big\{\big\{x\Im z^{n-1} + y\Re z^{n-1}\big\} \cos\eta + \big\{-y\Re z^{n-1} + y\Im z^{n-1}\big\} \sin\eta\Big\} \\ &= \Im z^n \cos\eta - \Re z^n \sin\eta. \end{aligned}$$

Therefore,

$$\begin{aligned} &\Big\{h + \imath h'\Big\} \exp\{\imath\eta\} \\ &\quad = \Big\{\Re z^n \cos\eta + \Im z^n \sin\eta + \imath\Big\{\Im z^n \cos\eta - \Re z^n \sin\eta\Big\}\Big\} \exp\{\imath\eta\} \\ &\quad = \Re z^n \cos^2\eta + \Im z^n \sin\eta\cos\eta + \imath\Im z^n \cos^2\eta - \imath\Re z^n \sin\eta\cos\eta \\ &\qquad + \imath\Re z^n \cos\eta\sin\eta + \imath\Im z^n \sin^2\eta - \Im z^n \cos\eta\sin\eta + \Re z^n \sin^2\eta \\ &\quad = \Re z^n + \imath\Im z^n = z^n, \end{aligned}$$

which shows the equation. Thus, a tamed curve is determined by its generalized support function. But more important is the following observation: *A curve* $z\colon S^1(\kappa) \to \mathbb{C}$ *is tamed if and only if* $h + h'' > 0$. Let us recall that a $2\pi\kappa$-periodic function describes a convex curve in the plane if and only if $h + h'' > 0$, convex meaning here that $k > 0$. Thus, there exists a one-to-one correspondence between tamed curves and convex curves.

**1.3.5.** Let us suppose that $z\colon S^1(\kappa) \to \mathbb{C}$ is an embedded, compact, tamed curve. The last paragraph showed that the generalized support function defined by $h := -r^n \langle e_r, N\rangle$, satisfies $h + h'' > 0$. Thus, $h$ also determines a convex curve in the plane. A little thought shows that the relation is given by the *transform* $\wedge := \{z \mapsto z^n\}\colon \mathbb{C} \to \mathbb{C}$. Indeed,

$$\hat{z} = \left\{h + \imath h'\right\} \exp\{\imath\eta\} = z^n.$$

Let us compute the curvature of $\hat{z}$ in terms of $z$

$$\hat{k} = \frac{1}{h + h''} = \frac{1}{n}\frac{\mathscr{F}}{r^{n-1}},$$

by equation (1.11). Similarly, we may compute the flow equation of $\hat{z}$. We observe with §B.1.2 and equation (1.11) that

$$\frac{d}{d\tau}h = -nr^{n-1}\mathscr{F} = -nr^{2n-2}\frac{\mathscr{F}}{r^{n-1}} = -n^2\frac{\left(h^2 + h'^2\right)^{\frac{n-1}{n}}}{h + h''} = -n^2\hat{r}^{\frac{2n-2}{n}}\hat{k}.$$

Therefore, we could also study the *weighted curve flow*. It reads in support functions

$$\frac{d}{d\tau}h = -n^2\frac{\left\{h^2 + h'^2\right\}^{\frac{2n-2}{2n}}}{h + h''},$$

or more geometrically

$$\frac{d}{d\tau}z = n^2 r^{\frac{2n-2}{n}} kN. \tag{1.14}$$

But as it turns out the asymptotic analysis of this flow is more complicated than the original version. This is due to the fact that the latter has a linear area decrease - a property the former flow does not satisfy. Nevertheless this flow gives some insight on the geometric behaviour of our flow equation (6). We will discuss some previously obtained results in the next paragraph.

**1.3.6.** Equation (1.14) and equation (6) are equivalent. This implies that a tamed curve $z\colon S^1(\kappa) \to \mathbb{C}\backslash\{0\}$ which encloses the origin and does not shrink to a point at singular time can be smoothly reparametrized. Because, $\hat{z}$ is convex as seen above. It is easy to see that $\hat{z}$ remains convex under the weighted curve flow, and continues to flow until it touches the origin. At this time $\hat{z}$ admits a supporting hyperplane through the the origin. If we take the $n$th-root to get the evolution of $z$ we see that a kink of at least $\frac{\pi}{n}$ degree has to occur, and therefore also a singularity for $n \geq 2$. But the $n$th-root admits $n$ different curves, which all appear - $z$ becomes $n$-times point symmetric, which in turn implies that one can reparametrize $z$ to a smooth immersed curve. Moreover, the curvature of the immersed curve is zero at the origin. Because the normal direction changes its direction by $\pi$ through the origin. A detailed study of the asymptotic of the weighted curve flow remains an open problem.

**1.3.7.** With the result obtained so far it is clear why tamed set and convex sets are almost equivalent. Whenever we want to use a result of *convex geometry* we apply the $\wedge$-transform to the tamed curve, obtain a convex set, for which the result may be applied and then we go back by tacking the $n$th-root of the set as a function of the complex plane to the complex plane. The assumptions of our theorems imply that this is a one-to-one correspondence. If one considers curves which contain the origin, then one has to be a little bit more careful.

With this we have: *Suppose that $A$ and $B$ are two tamed sets which do not intersect, then there exists a stationary solution separating both sets.* This follows because we can separate the convex sets $\hat{A}$ and $\hat{B}$. This implies: *Let us suppose that $z\colon S^1(\kappa) \to \mathbb{C}$ is a tamed curve which moves under the equivariant curve flow, then $z(\cdot, t_1)$ will be contained in $z(\cdot, t_2)$ for $t_1 \leq t_2$. In particular, if $z_0$ does not contain the origin, then $r_{\min} \geq \texttt{const} > 0$ for all $t \in [0, T_{\text{sing}})$.* Also whenever we speak of convergence to a point we mean convergence of support functions, which is convergence with respect to a tamed version of the Hausdorff metric. Let us recall that convergence in the space of convex, compact bodies with respect to the Hausdorff metric is equivalent to the convergence of the corresponding support functions, compare [52], [2].

**1.3.8.** Let us give an *example.* The support function of a *tamed ball* is defined by

$$h_{\text{ball}} := R + \Re p_0^n \cos \eta + \Im p_0^n \sin \eta.$$

This function has the property that

$$n \frac{r^{n-1}}{\mathscr{F}} = h''_{\text{ball}} + h_{\text{ball}} = R = \texttt{const}.$$

We remark that: *Any tamed curve whose area is not zero, contains a tamed ball. Moreover, there exists $\epsilon > 0$ such that $h - h_{\text{ball}} > \epsilon$.* This holds because the support function of a tamed ball is also the support function of a convex ball after taking the $\wedge$-transform.

**1.3.9.** To put *tamed set* into context, observe that we can look at them as generalized closed convex sets. Recall that a closed convex set is the intersection of its supporting hyperplanes. Ky Fan [24] initiated the study of so called $\Phi$-*convex sets*, where $\Phi$ is a family of function on a set $S$. A set is called $\Phi$-*convex* if it is either $S$ or the intersection of sets of the form $\{x \in S \mid f(x) \leq \alpha\}$, for $\alpha \in \mathbb{R}$ and $f \in \Phi$. The study of such sets falls in the theory of abstract convexity and optimization. For tamed set this family are the lagrangian catenoids. It is possible to deduce all needed properties without the $\wedge$-transform. The hardest part is to prove the separation property.

### Notes for Section 1.3

**1.** It is not hard to prove that convex curves stay convex under the weighted curve flow, and curves which do not contain the origin and are strictly convex will shrink to a point in finite time. But as noted above the asymptotic analysis seems to be not easy accessible.

**2.** For the theory of abstract convex sets we refer to [53] and [50].

## 1.4 Some comments on the general curve shortening equation

Here we collect some well known facts about planar curves and the general curve flow equation

$$\begin{aligned} \frac{d}{dt} z &= \mathscr{A}(z, k, \theta) N, \\ z(\cdot, 0) &= z_0. \end{aligned} \tag{1.15}$$

Where $\mathscr{A} \colon \mathbb{R}^2 \times \mathbb{R} \times S^1 \to \mathbb{R}$ is called *(normal-) speed function*, and $z_0 \colon S^1 \to \mathbb{R}^2$ is called *initial curve*. We will identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$. Let us also refer to Appendix A which is devoted to several computations and an explanation of notation can be found. Our main reference for this section has been [16].

*Planar curves*

**1.4.1.** An *immersed $C^1$-curve* is a continuous differentiable map $z\colon I \to \mathbb{R}^2$, from a closed connected subset $I$ of the sphere $S^1$ to the plane, with nonzero tangent $z' := z_p := \frac{d}{dp}z$. We will denote differentiation with respect to the parameter $p$ by a prime. If $I$ equals $S^1$, then we say that $z$ is *closed*. We call the curve *embedded* if it is one-to-one. Given a curve $z(p) = (x(p), y(p))$, its *unit tangent* is defined by $T = z'/\,|\,z'\,|$ , its *unit normal* is given by $N := Jz$, where $J$ denotes the *complex structure*, that is $J(x, y) = (-y, x)$. In coordinates we have $T = (x'^2 + y'^2)^{1/2}(x', y')$ and $N = (x'^2 + y'^2)^{1/2}(-y', x')$. With this definition the unit normal is inward pointing for counterclockwise traced curves. We will identify the map $z$ with its image $\gamma$ in $\mathbb{R}^2$.

Let $z\colon I \to \mathbb{R}^2$ be an immersed $C^1$-curve. Its metric tensor is given by

$$g(p) := \langle z'(p), z'(p)\rangle = |z'(p)|^2.$$

The one-dimensional surface measure on $z$ is

$$L(z) := \int_I \sqrt{det\, g}\, d\sigma = \int_I |z'(p)|dp.$$

We call $L(z)$ the *arc-length* of $z$. One can show that the arc-length is independent of the parametrization of $z$.

*Curvature*

The *curvature* $k$ of a curve $z\colon I \to \mathbb{R}^2$ at $p \in I$ is defined by the formula

$$k(p) := \frac{\langle z''(p), N(p)\rangle}{|\,z'(p)\,|^2}.$$

One can show the following *Frenet formulas*

$$\frac{\partial}{\partial\mu}T = kN, \text{ and } \frac{\partial}{\partial\mu}N = -kT,$$

where $d\mu = \,|\,z_p\,|\,dp$ denotes the arc-length element. In coordinate functions we have

$$k = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}.$$

If the curve is a graph of a function $f\colon \mathbb{R} \to \mathbb{R}$, then the curvature satisfies

$$k = \frac{f''}{(1 + f'^2)^{3/2}}.$$

*Tangent angle*

Let $z\colon I \to \mathbb{R}^2$ be an immersed curve, $p \in I$. The angle $\theta(p)$ between the tangent $T(p)$ at $z(p)$ and the positive $x$-axis is called the *tangent angle.* It is defined modulo $2\pi$. With this definition we have the equations $T(p) = (\cos\theta, \sin\theta)$ and $N = (-\sin\theta, \cos\theta)$. While comparing with the literature, one should carefully check whatever $\theta$ means, it is used both as tangent and as normal angle, sometimes at the same time. We always denote the normal angle by $\vartheta$. Also, it is important to check, whether $N$ is the inward pointing normal as in our case, or the outward pointing normal.

*On the curve flow problem*

**1.4.2.** A *classical solution* of (1.15) is a map $z\colon S^1 \times (0, T_{\text{sing}}) \to \mathbb{C}$ which satisfies; (i) it is continuously differentiable in $t$ and twice differentiable in $p$, (ii) for each $t$ the map $p \mapsto z(p)$ is a curve, and (iii) $z$ satisfies (1.15) and $z \to z_0$ as $t \to 0$.

We also assume that $\mathscr{A}$ is smooth in all of its argument. We say that $\mathscr{A}$ is *parabolic* if

$$\frac{\partial}{\partial q}\mathscr{A}(x, y, q, \theta) > 0.$$

We call $\mathscr{A}$ *strictly parabolic* if there are two positive real numbers $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1 \leq \frac{\partial}{\partial q}\mathscr{A} \leq \lambda_2.$$

Furthermore, we say that $\mathscr{A}$ is *symmetric* provided that

$$\mathscr{A}(x, y, \theta + \pi, -q) = -\mathscr{A}(x, y, \theta, q)$$

Let us recall that a *reparametrization* of a curve $z$ is another curve $\tilde{z} := z(\varphi(p))$ where $\varphi$ is a diffeomorphism.

**1.4.3.** Consider the flow equation

$$\begin{aligned} &\frac{d}{dt}z = \mathscr{A}(z, k, \theta)N + \mathscr{B}(z, k, \theta)T \\ &z(\cdot, 0) = z_0, \end{aligned} \tag{1.16}$$

where $\mathscr{A}$ and $\mathscr{B}$ are smooth and $2\pi$-periodic in $\theta$. *Let $z$ be a solution of* (1.16) *in $C^\infty(S^1 \times [0, T_{\text{sing}}))$. There exists $\varphi\colon S^1 \times [0, T_{\text{sing}}) \to S^1$ satisfying $\varphi' > 0$ and $\varphi(p, 0) = p$ such that $\tilde{z}(p, t) := z(\varphi(p, t), t)$ solves* (1.15). Thus, the tangential contribution $\mathscr{B}$ does not alter the geometric behavior of the flow. It is just responsible for an diffeomorphism on the parameter space. The geometry of the flow only depends on $\mathscr{A}$.

**1.4.4.** Let us suppose that $\mathscr{A}$ is smooth and parabolic, and $z_0 \in C^{2,\alpha}(S^1)$ for some $\alpha \in (0,1)$. Then there exists a solution $z \in C^{2,\alpha}(S^1 \times [0, T_{\text{sing}}))$ satisfying (1.15). Moreover, $z$ is smooth in $(0, T_{\text{sing}})$. If $T_{\text{sing}}$ is finite the curvature becomes unbounded as $t \to T_{\text{sing}}$, and if $z_0$ depends smoothly on a parameter, so does $z$.

**1.4.5.** Let $z_1, z_2 \colon S^1 \to \mathbb{C}$ be two solutions of (1.15) in $C^{0,1}(S^1 \times [0, T_{\text{sing}}))$, (i.e. $z_1, z_2$ are ), where $\mathscr{A}$ is parabolic. If $z_1(\cdot, 0) = z_2(\cdot, 0)$ in some parametrization, then $z_1(\cdot, t) = z_2(\cdot, t)$ for all $t \in [0, T_{\text{sing}})$.

**1.4.6.** Consider (1.15) where $\mathscr{A}$ is parabolic and symmetric. Then any solution $z(\cdot, t)$ in $C^{0,1}(S^1 \times [0, T_{\text{sing}}))$ is embedded if $z_0$ is embedded.

**1.4.7.** Consider (1.15) where $\mathscr{A}$ is parabolic and symmetric. Let us denote the number of intersection points of $z_1(\cdot, t)$ and $z_2(\cdot, t)$ by $Z(t)$. Suppose further that $z_1$ and $z_2$ do not coincide. Then $Z(t)$ is finite for all $t$ in $(0, T_{\text{sing}})$, and drops exactly at those instants $\tilde{t}$ when $z_1(\cdot, \tilde{t})$ and $z_2(\cdot, \tilde{t})$ touch tangentially at some point. Moreover, all these instants form a discrete subset of $(0, T_{\text{sing}})$.

### Notes for Section 1.4

**1.** The basic geometry of curves can be found for example in [51].

**2.** As noted above all results of this section and their proof can be found in [16]. Let us give the precise reference. The result of §1.4.3 is Proposition 1.1. The local existence theorem §1.4.4 is Proposition 1.2. The uniqueness result §1.4.5 is Proposition 1.4. The embeddedness theorem §1.4.6 is Proposition 1.5. The Sturmian oscillation-type theorem §1.4.7 is Proposition 1.7.

**3.** The long history of the Sturmian oscillation theorem is surveyed in [28]. Application to the general curve shortening flow have been given in Angenent papers [5] and [7]

Chapter 2

# Proof of Theorems A and C

We consider a smooth family of curves $z\colon S^1 \times [0, T_{\text{sing}}) \to \mathbb{C}$ for which

$$\begin{aligned} \frac{d}{dt} z &= \mathscr{F} N := \Big\{ k - (n-1)\frac{\langle e_r, N\rangle}{r} \Big\} N, \\ z(\cdot, 0) &= z_0, \end{aligned} \tag{2.1}$$

where $k$ denotes the curvature of the curve, $N$ the inward pointing unit normal, $e_r := \frac{z}{r}$, $r := |z|$, and $n$ the dimension of $L$.

This chapter contains two sections. In the first section we will finish the proof of Theorem A. In the second section we will prove Theorem C.

## 2.1 Proof of Theorem A

This section completes the proof of Theorem A which claims: *Let $F_0$ be an equivariant lagrangian immersion of $L$ in $\mathbb{C}^n$. If the initial profile curve is closed, embedded, satisfies $\mathscr{F} > 0$, and does not contain the origin, then $F_t$ converges to a sphere $\|p_0\| S^{n-1}$ as time goes to $T_{\text{sing}}$. Moreover, the singular time is determined by the enclosed area of the initial curve. That is $T_{\text{sing}} = \frac{A_0}{2\pi}$.*

**2.1.1.** Here we assume that the initial profile curve $z_0\colon S^1 \to \mathbb{C}$ is simple, closed, tamed, and does not contain the origin. We only have to show that the profile curves converge to a point $p_0 \in \mathbb{C}\backslash\{0\}$. The result follows from the equivariant structure of the submanifold.

Because, $z_0$ is tamed, simple, and does not contain the origin there exists a lagrangian catenoid separating $z_0$ and the origin. Therefore, $z(\cdot, t)$ is bounded away from the origin by $r_{\min}(0)$ by the maximum principle. In fact, $z(\cdot, t)$ is contained in $z_0$ for all $t$. Moreover, $\kappa = 1$.

**2.1.2.** Let us denote the *area* of a curve by $A(z_0)$. Recall that

$$\frac{d}{dt}A = -\int_{S^1} \mathscr{F}\, d\mu = 2\pi.$$

Therefore the area decays according to

$$A(t) = A(0) - 2\pi t.$$

This gives an upper bound for the singular time. Let us suppose that the area of $z(\cdot, T_{\text{sing}})$ is not zero. Then there exist $\epsilon > 0$ and a tamed ball with support function $h_{\text{ball}}$ inside of all curves $z(\cdot, t)$ for $t$ close to $T_{\text{sing}}$. That is, $h - h_{\text{ball}} > \epsilon$. Let us compute the evolution equation of

$$f := \frac{\mathscr{F}}{-r^n \langle e_r, N\rangle - h_{\text{ball}}}.$$

To get a feeling how to do this the computation will be detailed. The first step is to compute the laplacian of $f$. To this end we define

$$m := \frac{1}{-r^n \langle e_r, N\rangle - h_{\text{ball}}}.$$

Then

$$\frac{d}{d\eta}m = \Big\{h'_{\text{ball}} - r^n \langle e_r, T\rangle \Big\}m^2,$$

and

$$\frac{d^2}{d\eta^2}m = 2\Big\{h'_{\text{ball}} - r^n \langle e_r, T\rangle \Big\}^2 m^3 + \Big\{h''_{\text{ball}} - r^n \langle e_r, N\rangle - n\frac{r^{n-1}}{\mathscr{F}}\Big\}m^2.$$

This gives

$$\frac{d}{d\eta}f = m\frac{d}{d\eta}\mathscr{F} + \Big\{h'_{\text{ball}} - r^n \langle e_r, T\rangle \Big\}m^2\mathscr{F},$$

and

$$\frac{d^2}{d\eta^2}f = m\frac{d^2}{d\eta^2}\mathscr{F} + 2\Big\{h'_{\text{ball}} - r^n \langle e_r, T\rangle \Big\}m\frac{d}{d\eta}f \\ - nr^{n-1}m^2 + \Big\{h''_{\text{ball}} - r^n \langle e_r, N\rangle \Big\}m^2\mathscr{F}.$$

Now it is time to compute the time derivative of $f$. It holds

$$\frac{d}{d\tau}m = m^2 n r^{n-1}\mathscr{F}.$$

This and equation (1.7) give

$$\frac{d}{d\tau}f = m\mathscr{F}^2\frac{d^2}{d\eta^2}\mathscr{F} + (n-1)\Big\{(n+1)\frac{\langle e_r, N\rangle^2}{r^2} - \frac{1}{r^2}\Big\}m\mathscr{F}$$
$$+\Big\{2(n-1)\frac{\langle e_r, N\rangle}{r}m + m^2nr^{n-1}\Big\}\mathscr{F}^2 + m\mathscr{F}^3.$$

In the final step we replace $m\frac{d^2}{d\eta^2}\mathscr{F}$ by $\frac{d^2}{d\eta^2}f$. This yields

$$\frac{d}{d\tau}f = \mathscr{F}^2\frac{d^2}{d\eta^2}f - 2\Big\{h'_{\text{ball}} - r^n\langle e_r, T\rangle\Big\}m\mathscr{F}^2\frac{d}{d\eta}f$$
$$+\Big\{(n^2-1)\frac{\langle e_r, N\rangle^2}{r^2} - \frac{n-1}{r^2}\Big\}m\mathscr{F}$$
$$+2\Big\{(n-1)\frac{\langle e_r, N\rangle}{r} + mnr^{n-1}\Big\}m\mathscr{F}^2$$
$$+\Big\{m - \Big\{h''_{\text{ball}} - r^n\langle e_r, N\rangle\Big\}m^2\Big\}\mathscr{F}^3.$$

Let us analyze the leading order term. We know that $f \to \infty$ as $t$ approaches $T_{\text{sing}}$. This is only possible if $\{\ldots\}$ is positive, or tends to zero. By our assumption we know that $h - h_{\text{ball}} \geq \epsilon > 0$, which implies that $m$ is bounded from below. Moreover, $m$ is also bounded from above, because $z_0$ is compact. Thus,

$$0 < \texttt{const}_1 \leq m \leq \texttt{const}_2.$$

Let us take a look back to §1.3.8. It implies that $h''_{\text{ball}} = -h_{\text{ball}} + R$. Therefore,

$$m - \Big\{h''_{\text{ball}} - r^n\langle e_r, N\rangle\Big\}m^2 = m - \Big\{h - h_{\text{ball}} + R\Big\}m^2 = -Rm^2.$$

This together with the lower bounds for $m$ show that the leading is negative, and does not tend to zero. Therefore $f$ is bounded by the maximum principle. This is a contradiction.

**2.1.3.** By the Blaschke selection theorem, see for example [52], there exists a subsequence $z(\cdot, t_n)$ which converges to a tamed limit curve with respect to the tamed Hausdorff metric, compare §1.3.7. As the area of this curve is zero we are left with two possibilities, either the limit curve is a tamed segment, or it is a point as claimed. To exclude the former let us recall the evolution equation of $\mathscr{F}$, compare equation (1.6). It yields the following lower estimate

$$\frac{d}{dt}\mathscr{F}_{\min} \geq \Big\{(n^2-1)\frac{\langle e_r, N\rangle^2}{r^2} - \frac{n-1}{r^2}\Big\}\mathscr{F}_{\min} + 2(n-1)\frac{\langle e_r, N\rangle}{r}\mathscr{F}^2_{\min} + \mathscr{F}^3_{\min}$$
$$\geq -\texttt{const}\mathscr{F}_{\min}.$$

This shows that $\mathscr{F}_{\min}$ can only decrease exponentially fast to zero. Because $z_0$ is compact, the singular time is finite. Therefore there exists $\epsilon > 0$ with

$$\mathscr{F} \geq \epsilon > 0.$$

If the limit curve would be a segment, then there would exist a sequence of points with $\mathscr{F} \to 0$, this gives a contradiction. Another way to see this is to recall that the support function of a point is smooth, whereas the first derivative of the support function of a tamed segment jumps. The lower bounds on $\mathscr{F}$ and $r$ show that $h'_{\text{limit}}$ is continuous. This proves the theorem. q.e.d.

### Notes for Section 2.1

**1.** As notes earlier, our proof is inspired by the one given in [59] which dates back to [57].

## 2.2 Proof of Theorem C

In this section we give a proof of Theorem C, which states: *Let $F_0$ be an equivariant lagrangian immersion of $L$ in $\mathbb{C}^n$. If the initial profile curve is closed, embedded, satisfies*

$$\mathscr{F}_{\min}(0) \geq \frac{n-1+\sqrt{(n-1)^2+n-1}}{r_{\min}(0)},$$

*and does not contain the origin, then the assumptions of Theorem B are fulfilled.* Thus, we need to show that $\mathscr{F} > 0$, which is obvious, and that $k > 0$.

**2.2.1.** Let us suppose that

$$r_{\min}(t)\mathscr{F}_{\min}(t) \geq (n-1), \tag{2.2}$$

then the following holds

$$k = \mathscr{F} + (n-1)\frac{\langle e_r, N\rangle}{r} \geq \mathscr{F} - \frac{n-1}{r} \geq 0.$$

Thus, $\gamma$ is a convex curve. We already know that $r(t) \geq r_{\min}(0)$; remember that $\gamma(t)$ is contained in $\gamma(0)$ by the maximum principle. To prove equation (2.2) it thus suffices to show that $\mathscr{F}(t) \geq \mathscr{F}_{\min}(0)$. Note that

$$\begin{aligned}\frac{d}{dt}\mathscr{F} &= \Delta\mathscr{F} + (n-1)\frac{\langle e_r, \nabla\mathscr{F}\rangle}{r} + (n-1)\Big\{(n+1)\frac{\langle e_r, N\rangle^2}{r^2} - \frac{1}{r^2}\Big\}\mathscr{F} \\ &\quad + 2(n-1)\frac{\langle e_r, N\rangle}{r}\mathscr{F}^2 + \mathscr{F}^3 \\ &\geq \Big\{(n-1)\Big\{(n+1)\frac{\langle e_r, N\rangle^2}{r^2} - \frac{1}{r^2}\Big\} + 2(n-1)\frac{\langle e_r, N\rangle}{r}\mathscr{F} + \mathscr{F}^2\Big\}\mathscr{F}\end{aligned}$$

Recall that $r_{\min}(0) \leq r(t)$. Thus,

$$\frac{d}{dt}\mathscr{F} \geq \Big\{ -\frac{n-1}{r_{\min}(0)^2} - \frac{2(n-1)}{r_{\min}(0)}\mathscr{F} + \mathscr{F}^2 \Big\} \mathscr{F}$$

The the largest zero of the term in the brackets $\{\ldots\}$ as a function of $\mathscr{F}$ is given by

$$\frac{n-1}{r_{\min}(0)} + \sqrt{\frac{(n-1)^2}{r_{\min}(0)^2} + \frac{n-1}{r_{\min}(0)^2}} = \frac{n-1+\sqrt{(n-1)^2+n-1}}{r_{\min}(0)}$$

This proves the theorem. q.e.d.

CHAPTER 3

# Proof of Theorem B

This chapter gives a proof of Theorem B, which states that: *Let $F_0$ be an equivariant lagrangian immersion of $L$ in $\mathbb{C}^n$. If the initial profile curve is closed, embedded, satisfies $\mathscr{F} > 0$, does not contain the origin, and suppose further that all profile curves remain convex, then the singularity is of type-I. After rescaling und possibly choosing a subsequence $F_t$ converges to the cylinder $S^{n-1} \times \mathbb{R}$ smoothly on compact subsets of $\mathbb{C}^n$.*

The first section explains the blow-up procedure. The problem is reduced to the following curve flow equation

$$\begin{aligned} \frac{d}{dt}z &= \Big\{k - (n-1)\frac{\langle e_p, N\rangle}{r_p} + \langle z, N\rangle\Big\}N, \\ z(\cdot, 0) &= \sqrt{\frac{\pi}{A_0}}\big\{z_0 - p_0\big\}. \end{aligned} \tag{3.1}$$

The second section establishes the asymptotic behavior of several geometric quantities. The key observation is that $r_p \sim \exp(t)$. The third section proves a Bernstein-type estimate as well as a Harnack-type inequality. Finally, in the fourth section we finish the proof of Theorem B.

### 3.1 On the rescaled flow equation

Here we introduce the notion of a blow-up point, the different types of singularities due to Huisken, and derive the rescaled flow equation.

**3.1.1.** We say that a point $p \in \mathbb{C}^n$ is a *blow-up point* if there exists $x \in L$ with $F(x,t) \to p$ and $|A|(x,t) \to \infty$ as $t$ approaches $T_{\text{sing}}$. If the singular time is finite, then the evolution equation of the second fundamental form yields an lower bound

$$\frac{\mathtt{const}_1}{T_{\text{sing}} - t} \le \sup_{L_t} |A|^2.$$

Here $\texttt{const}_1 > 0$ is a positive constant. If the second fundamental form admits the upper bound

$$\sup_{L_t} |A|^2 \leq \frac{\texttt{const}_1}{T_{\text{sing}} - t},$$

then we say that the singularity is of *type-I*, otherwise it is of *type-II*. We remark that this terminology dates back to Huisken [36]. Let us also refer to the introduction of this paper.

**3.1.2.** By Theorem A we know that $L_t$ converges to $\|p_0\|S^{n-1}$ as $t$ approaches the singular time. Let us suppose that $\overline{p}_0 \in \mathbb{C}^n$ is the north-pole of $\|p_0\|S^{n-1}$. If we consider the mean curvature flow of $F - \overline{p}_0$, then the origin becomes a blow-up point. We rescale the flow as follows

$$\begin{aligned}
\widetilde{F}(x,s) &= \sqrt{\frac{\pi}{A_0 - 2\pi t(s)}} \Big\{ F(x, t(s)) - \overline{p}_0 \Big\}, \\
s(t) &= -\frac{1}{2} \ln \Big\{ \frac{A_0 - 2\pi t}{\pi} \Big\}.
\end{aligned}$$

Then $\widetilde{L}_s := \widetilde{F}(L, s)$ is defined for $s \in [-\frac{1}{2}\ln\frac{A_0}{\pi}, \infty)$, and is again a lagrangian submanifold of $\mathbb{C}^n$. The rescaled flow satisfies:

$$\begin{aligned}
\frac{d}{ds}\widetilde{F}(x,s) &= \overrightarrow{\widetilde{H}}(x,s) + \widetilde{F}(x,s) \\
\widetilde{F}(\cdot, -\tfrac{1}{2}\ln\tfrac{A_0}{\pi}) &= \sqrt{\frac{\pi}{A_0}} \Big\{ F_0 - \overline{p}_0 \Big\}
\end{aligned} \tag{3.2}$$

Let us introduce

$$\psi(t) := \sqrt{\frac{\pi}{A_0 - 2\pi\kappa t}} = \sqrt{\frac{\pi}{A(t)}}.$$

Equation (3.2) is determined by the flow of the profile curve. Indeed we have

$$\widetilde{F}(\phi, x, s) = \Big( u_0(\phi, s)\big(\psi(t(s))G(x)\big), v_0(\phi, s)\big(\psi(t(s))G(x)\big) \Big) - \psi(t(s))\overline{p}_0. \tag{3.3}$$

Thus it suffices to study the following curve flow problem

$$\frac{d}{ds}\widetilde{z} = \Big\{ \widetilde{k} - (n-1) \frac{\Big\langle \widetilde{z} + p_0\sqrt{\frac{\pi}{A_0}}\exp(t), \widetilde{N} \Big\rangle}{|\widetilde{z} + p_0\sqrt{\frac{\pi}{A_0}}\exp(t)|^2} + \Big\langle \widetilde{z}, \widetilde{N} \Big\rangle \Big\} \widetilde{N},$$

$$\widetilde{z}(\cdot, -\tfrac{1}{2}\ln\tfrac{A_0}{\pi}) = \sqrt{\frac{\pi}{A_0}} \big\{ z_0 - p_0 \big\}.$$

Let us return to equation (3.3). It follows that

$$\widetilde{F}(\phi,x,s)=\psi(t(s))F(\phi,x,s)-\psi(t(s))\overline{p}_0=\exp\{s\}F(\phi,x,s)-\exp\{s\}\overline{p}_0.$$

Let us introduce

$$\widetilde{\mathscr{G}}=\widetilde{k}-(n-1)\frac{\left\langle \widetilde{z}+p_0\sqrt{\frac{\pi}{A_0}}\exp(t),\widetilde{N}\right\rangle}{|\widetilde{z}+p_0\sqrt{\frac{\pi}{A_0}}\exp(t)|^2}.$$

We close this paragraph with the following observation

$$\widetilde{z}(t)=\psi(t)z(t)-p_0\sqrt{\frac{\pi}{A_0}}\exp\{t\}. \tag{3.4}$$

The last equation implies that $A(\widetilde{z}(t))=\pi$.

**3.1.3.** The previous paragraph reduced the blow-up analysis to a curve flow problem again

$$\begin{aligned}&\frac{d}{dt}z=\Big\{k-(n-1)\frac{\langle e_p,N\rangle}{r_p}+\langle z,N\rangle\Big\}N,\\&z(\cdot,0)=\sqrt{\frac{\pi}{A_0}}\{z_0-p_0\}.\end{aligned} \tag{3.5}$$

Where $e_p:=\frac{z+p}{r_p}$, $r_p:=|z+p|$, and $p=\sqrt{\frac{\pi}{A_0}}\exp(t)p_0$. We have seen in the previous paragraph that this flow keeps the area of $z$ constant to $\pi$. Let us introduce

$$\mathscr{G}:=k-(n-1)\frac{\langle e_p,N\rangle}{r_p}.$$

It further followed that $\mathscr{G}$ plays the same role as $\mathscr{F}$ in §1.3. So $\mathscr{F}>0$ for the original curve if and only if $\mathscr{G}>0$ for the rescaled curve. We say that a curve is *tamed* if it satisfies $\mathscr{G}>0$.

**3.1.4.** Let us now compute the evolution equation of $\mathscr{G}$. We refer to §C.1.2. We start with curvature. Observe that

$$\nabla\mathscr{H}=\nabla\mathscr{G}+\nabla\langle z,N\rangle=\nabla\mathscr{G}-\mathscr{G}r\nabla r-(n-1)\frac{\langle e_p,N\rangle}{r_p}r\nabla r,$$

and

$$
\begin{aligned}
\Delta\mathscr{H} = {}& \Delta\mathscr{G} - r\left\langle \nabla\mathscr{G}, \nabla r\right\rangle \\
& + \Big\{ -1 - 2(n-1)\frac{\langle z, N\rangle \langle e_p, N\rangle}{r_p} + (n-1)\frac{r}{r_p}\left\langle \nabla r, \nabla r_p\right\rangle \Big\}\mathscr{G} \\
& - \langle z, N\rangle \mathscr{G}^2 - (n-1)\frac{\langle e_p, N\rangle}{r_p} - (n-1)^2\frac{\langle z, N\rangle \langle e_p, N\rangle^2}{r_p^2} \\
& + (n^2-1)\frac{\langle e_p, N\rangle}{r_p^2} r\left\langle \nabla r, \nabla r_p\right\rangle .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{d}{dt}k = {}& \Delta\mathscr{G} - r\left\langle \nabla\mathscr{G}, \nabla r\right\rangle \\
& + \Big\{ -1 + (n-1)\frac{r}{r_p}\left\langle \nabla r, \nabla r_p\right\rangle + (n-1)^2\frac{\langle e_p, N\rangle^2}{r_p^2} \Big\}\mathscr{G} \\
& + 2(n-1)\frac{\langle e_p, N\rangle}{r_p}\mathscr{G}^2 \\
& + \mathscr{G}^3 \\
& - (n-1)\frac{\langle e_p, N\rangle}{r_p} + (n^2-1)\frac{\langle e_p, N\rangle}{r_p^2} r\left\langle \nabla r, \nabla r_p\right\rangle .
\end{aligned}
$$

Therefore, the evolution of $\mathscr{G}$ reads

$$
\begin{aligned}
\frac{d}{dt}\mathscr{G} = {}& \Delta\mathscr{G} - r\left\langle \nabla\mathscr{G}, \nabla r\right\rangle \\
& + \Big\{ -1 + (n-1)\frac{r}{r_p}\left\langle \nabla r, \nabla r_p\right\rangle + (n-1)^2\frac{\langle e_p, N\rangle^2}{r_p^2} \Big\}\mathscr{G} \\
& + 2(n-1)\frac{\langle e_p, N\rangle}{r_p}\mathscr{G}^2 \\
& + \mathscr{G}^3 \\
& - (n-1)\frac{\langle e_p, N\rangle}{r_p} + (n^2-1)\frac{\langle e_p, N\rangle}{r_p^2} r\left\langle \nabla r, \nabla r_p\right\rangle \\
& + (n-1)\frac{\langle e_p, \nabla\mathscr{G}\rangle}{r_p} + \Big\{ -(n-1)\frac{1}{r_p^2} + (n-1)\frac{\langle e_p, N\rangle^2}{r_p^2} - (n-1)\frac{r\left\langle e_p, \nabla r\right\rangle}{r_p} \Big\}\mathscr{G} \\
& - (n-1)\frac{\langle e_p, N\rangle}{r_p} + (n-1)\frac{\langle z, N\rangle \langle e_p, N\rangle^2}{r_p^2} + (n-1)\frac{\langle e_p, N\rangle \langle e_p, p\rangle}{r_p^2} \\
& - (n-1)^2\frac{r\left\langle e_p, N\right\rangle \left\langle e_p, \nabla r\right\rangle}{r_p^2} \\
& + (n-1)\frac{\langle e_p, N\rangle \langle e_p, N\rangle}{r_p^2}\mathscr{G} + (n-1)\frac{\langle e_p, N\rangle^2 \langle z, N\rangle}{r_p^2} + (n-1)\frac{\langle e_p, N\rangle \langle e_p, p\rangle}{r_p^2}.
\end{aligned}
$$

Rearranging terms yields

$$
\begin{aligned}
\frac{d}{dt}\mathscr{G} = \Delta\mathscr{G} - r\left\langle \nabla\mathscr{G}, \nabla r\right\rangle + (n-1)\frac{\left\langle e_p, \nabla\mathscr{G}\right\rangle}{r_p} \\
&+ \Big\{ -1 + (n-1)\frac{r}{r_p}\left\langle \nabla r, \nabla r_p\right\rangle + (n^2-1)\frac{\left\langle e_p, N\right\rangle^2}{r_p^2} \\
&\qquad - (n-1)\frac{1}{r_p^2} - (n-1)\frac{r\left\langle e_p, \nabla r\right\rangle}{r_p}\Big\}\mathscr{G} \\
&+ 2(n-1)\frac{\left\langle e_p, N\right\rangle}{r_p}\mathscr{G}^2 \\
&+ \mathscr{G}^3 \\
&+ (n-1)\Big\{ -2 + (n+1)\frac{r}{r_p}\left\langle \nabla r, \nabla r_p\right\rangle + 2\frac{\left\langle z, N\right\rangle\left\langle e_p, N\right\rangle}{r_p} + 2\frac{\left\langle e_p, p\right\rangle}{r_p} \\
&\qquad\qquad - (n-1)\frac{r\left\langle e_p, \nabla r\right\rangle}{r_p}\Big\}\frac{\left\langle e_p, N\right\rangle}{r_p}.
\end{aligned}
$$

Note that $\left\langle \nabla r, \nabla r_p\right\rangle = \left\langle e_p, \nabla r\right\rangle$. Therefore,

$$
(n+1)\frac{r}{r_p}\left\langle \nabla r, \nabla r_p\right\rangle + 2\frac{\left\langle z, N\right\rangle\left\langle e_p, N\right\rangle}{r_p} + 2\frac{\left\langle e_p, p\right\rangle}{r_p} - (n-1)\frac{r\left\langle e_p, \nabla r\right\rangle}{r_p} = 2,
$$

and

$$
\begin{aligned}
\frac{d}{dt}\mathscr{G} = \Delta\mathscr{G} - r\left\langle \nabla\mathscr{G}, \nabla r\right\rangle + (n-1)\frac{\left\langle e_p, \nabla\mathscr{G}\right\rangle}{r_p} \\
&+ \Big\{(n^2-1)\frac{\left\langle e_p, N\right\rangle^2}{r_p^2} - \frac{n-1}{r_p^2} - 1\Big\}\mathscr{G} + 2(n-1)\frac{\left\langle e_p, N\right\rangle}{r_p}\mathscr{G}^2 + \mathscr{G}^3.
\end{aligned}
\tag{3.6}
$$

This shows: $\mathscr{G} > 0$ *is indeed preserved during the flow.* This is of course obvious if we recall §3.1.2. Let us set

$$
\begin{aligned}
\Phi &= (n^2-1)\frac{\left\langle e_p, N\right\rangle^2}{r_p^2} - \frac{n-1}{r_p^2}, \\
\Psi &= 2(n-1)\frac{\left\langle e_p, N\right\rangle}{r_p}.
\end{aligned}
$$

Again we note that

$$
\frac{d}{dt}\mathscr{G} = \frac{\partial^2}{\partial\mu^2}\mathscr{G} + \Big\{(n-1)\frac{\left\langle e_p, T\right\rangle}{r_p} - \left\langle z, T\right\rangle\Big\}\frac{\partial}{\partial\mu}\mathscr{G} + \big\{\Phi - 1\big\}\mathscr{G} + \Psi\mathscr{G}^2 + \mathscr{G}^3.
$$

**3.1.5.** This paragraph should be compared with §1.3.2 as all ideas are similar, and most calculations almost coincide. We begin with the obvious observation that

$$\int_{S^1} \mathscr{G}\, d\mu = 2\pi\big\{ \operatorname{rot}(z) - (n-1)\operatorname{wind}(z)\big\} = 2\pi\kappa.$$

Analogously to the cited paragraph we introduce

$$\eta(a) := \int_0^a \mathscr{G}\, d\mu.$$

Thus, $\eta\colon S^1 \to S^1(\kappa)$, and therefore $z \circ \eta^{-1}\colon S^1(\kappa) \to \mathbb{C}$ is a reparametrization of our curve. *We claim that*

$$\frac{d}{dt}\eta = \Big\{\frac{\partial}{\partial\eta}\mathscr{G} + (n-1)\frac{\langle e_p, T\rangle}{r_p} - \langle z, T\rangle\Big\}\mathscr{G}. \tag{3.7}$$

This follows because,

$$\begin{aligned}
\frac{d}{dt}\eta(a) &= \frac{d}{dt}\int_0^a \mathscr{G}\, d\mu \\
&= \int_0^a \frac{\partial^2}{\partial\mu^2}\mathscr{G} + \Big\{(n-1)\frac{\langle e_p, T\rangle}{r_p} - \langle z, T\rangle\Big\}\frac{\partial}{\partial\mu}\mathscr{G} \\
&\qquad + \Big\{ -1 + (n^2-1)\frac{\langle e_p, N\rangle^2}{r_p^2} - (n-1)\frac{1}{r_p^2}\Big\}\mathscr{G} \\
&\qquad + 2(n-1)\frac{\langle e_p, N\rangle}{r_p}\mathscr{G}^2 + \mathscr{G}^3 - k\mathscr{G}\mathscr{H}\, d\mu \\
&= \int_0^a \frac{\partial^2}{\partial\mu^2}\mathscr{G} + \Big\{(n-1)\frac{\langle e_p, T\rangle}{r_p} - \langle z, T\rangle\Big\}\frac{\partial}{\partial\mu}\mathscr{G} \\
&\qquad + \Big\{ -1 + (n^2-1)\frac{\langle e_p, N\rangle^2}{r_p^2} - \frac{n-1}{r_p^2} \\
&\qquad\qquad - (n-1)\frac{\langle e_p, N\rangle}{r_p}\langle z, N\rangle\Big\}\mathscr{G} \\
&\qquad + \Big\{(n-1)\frac{\langle e_p, N\rangle}{r_p} - \langle z, N\rangle\Big\}\mathscr{G}^2\, d\mu.
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_0^a \Big\{(n-1)\frac{\langle e_p, T\rangle}{r_p} - \langle z, T\rangle\Big\}\frac{\partial}{\partial\mu}\mathscr{G} \\
&\quad = \Big\{(n-1)\frac{\langle e_p, T\rangle}{r_p} - \langle z, T\rangle\Big\}\mathscr{G} - \int_0^a \Big\{(n-1)\frac{\langle e_p, N\rangle}{r_p} - \langle z, N\rangle\Big\}\mathscr{G}^2 \\
&\qquad + \Big\{(n-1)\frac{\langle e_p, N\rangle}{r_p}\langle z, N\rangle - (n-1)\frac{\langle e_p, T\rangle^2}{r_p^2} + n(n-1)\frac{\langle e_p, N\rangle^2}{r_p^2} - 1\Big\}\mathscr{G}\, d\mu.
\end{aligned}$$

Here we refer to §C.1.1. It follows

$$\frac{d}{dt}\eta(a) = \frac{\partial}{\partial\mu}\mathscr{G} + \Big\{(n-1)\frac{\langle e_p, T\rangle}{r_p} - \langle z, T\rangle\Big\}\mathscr{G},$$

and hence the claim. As one might guess we now want to derive the evolution equation of $\mathscr{G}$ in new coordinates $(\eta, \tau)$. It holds

$$\frac{\partial}{\partial\eta}z = \frac{1}{\mathscr{G}}\frac{\partial}{\partial\mu}z = \frac{1}{\mathscr{G}}T.$$

This yields, combined with the previous equation, that

$$\frac{\partial}{\partial\tau}z = \frac{d}{dt}z - \frac{\partial}{\partial\eta}z\frac{d}{dt}z = \mathscr{H}N - \Big\{\frac{\partial}{\partial\eta}\mathscr{G} + (n-1)\frac{\langle e_p, T\rangle}{r_p} - \langle z, T\rangle\Big\}T.$$

Therefore the reparametrized flow equation reads

$$\frac{d}{d\tau}z = \Big\{\mathscr{G} + \langle z, N\rangle\Big\}N - \Big\{\frac{d}{d\eta}\mathscr{G} + (n-1)\frac{\langle e_p, T\rangle}{r_p} - \langle z, T\rangle\Big\}T. \tag{3.8}$$

The tangential contribution of equation (3.8) does not alter the geometric behavior of the flow, it just makes $\tau$ and $\eta$ independent. Let us now derive the evolution equation of $\mathscr{G}$. First of all recall that

$$\frac{\partial}{\partial\mu}\mathscr{G} = \mathscr{G}\frac{\partial}{\partial\eta}\mathscr{G} \text{ and } \frac{\partial^2}{\partial\mu^2}\mathscr{G} = \mathscr{G}^2\frac{\partial^2}{\partial\eta^2}\mathscr{G} + \mathscr{G}\Big\{\frac{\partial}{\partial\eta}\mathscr{G}\Big\}^2.$$

We have

$$\begin{aligned}\frac{\partial}{\partial\tau}\mathscr{G} &= \frac{d}{dt}\mathscr{G} - \frac{\partial}{\partial\eta}\mathscr{G}\frac{\partial\eta}{\partial t}\\ &= \mathscr{G}^2\frac{\partial^2}{\partial\eta^2}\mathscr{G} + \Big\{-1 + (n^2-1)\frac{\langle e_p, N\rangle^2}{r_p^2} - (n-1)\frac{1}{r_p^2}\Big\}\mathscr{G}\\ &\qquad + 2(n-1)\frac{\langle e_p, N\rangle}{r_p}\mathscr{G}^2 + \mathscr{G}^3.\end{aligned}$$

For the readers convenience we have collected more calculations for the reparametrized flow in Appendix C.1.

### 3.2 Asymptotic behaviour of geometric quantities

We continue our investigation of the rescaled flow equation

$$\begin{aligned}&\frac{d}{dt}z = \Big\{k - (n-1)\frac{\langle e_p, N\rangle}{r_p} + \langle z, N\rangle\Big\}N,\\ &z(\cdot, 0) = z_0.\end{aligned} \tag{3.9}$$

We adopt the notation of the previous paragraphs and set $\mathscr{G} := k-(n-1)\frac{\langle e_p,N\rangle}{r_p}$. Throughout this paragraph we only assume that $\mathscr{G} > 0$. The program consists of: At first we show that $r_p \sim \exp(t)$, which is the key observation, then we prove that $\mathscr{G} \to 0$ at most exponentially, we show that $r$ and $L$ can only increase exponentially fast to infinity, finally we will see that $r_p\mathscr{G}$ is bounded from below. Let us recall that we assume that the initial $z_0\colon S^1 \to \mathbb{C}\backslash\{0\}$ is tamed, closed, embedded, compact and does not contain the origin. As usual we abbreviate these assumption by saying that $z_0$ satisfies $(A)$.

**3.2.1.** We precede with $r_p$. Let us suppose for the moment that we rescaled by $\breve{z} := \psi(t)z$. Then clearly $\breve{\mathscr{F}} = \frac{1}{\psi}\mathscr{F}$. This implies that an lagrangian catenoid remains an lagrangian catenoid under this rescaling. Suppose that $\breve{p} \in \mathbb{C}$ is the point on the lagrangian catenoid which minimizes distance to the origin. Because $\breve{z} = \psi z$ it follows that $\breve{p}(t) = \exp(t)p$. Thus, if we are given a tamed compact closed curve which does not contain the origin, then we may bound it away from the origin by comparing it with lagrangian catenoids. It follows by the maximum principle that $\breve{r} \sim \exp(t)$. Here $r_p \sim \exp(t)$ means that $0 < \mathtt{const}_1 \exp(t) \leq \breve{r}(t) \leq \mathtt{const}_2 \exp(t)$. Note that $\breve{z} = \tilde{z} + p_o\sqrt{\frac{\pi}{A_0}}\exp(t)$. Thus, $\tilde{r}_p = \breve{r}$. The claim follows. We remark that we dropped the tilde in our notation. For this paragraph we refer to §3.1.2

The behavior of $r_p$ has some immediate consequences. For example it shows that $\mathscr{G} \to k$ exponentially fast, because

$$\mathscr{G} = k-(n-1)\frac{\langle e_p,N\rangle}{r_p}.$$

Moreover,

$$\mathscr{G}^{(1)} = k^{(1)} + (n-1)\Big\{(n+1)\frac{\langle e_p,N\rangle}{r_p\mathscr{G}} + 1\Big\}\frac{\langle e_p,T\rangle}{r_p}.$$

As will be seen in the next paragraph, $\mathscr{G}$ tends to zero at most exponentially fast. This implies that $r_p\mathscr{G}$ is bounded from below, and hence we have $\mathscr{G}^{(1)} \to k^{(1)}$. But we do not know if $\mathscr{G}^{(2)} \to k^{(2)}$. This is of course *not* true if the curves contain the origin, as in this case the blow-up point is the origin and we do not have any control over $r_p$ in this case.

**3.2.2.** Let us proceed with $\mathscr{G}$. By equation (3.6) we know that

$$\begin{aligned}\frac{d}{dt}\mathscr{G} &= \Delta\mathscr{G} - r\left\langle\nabla\mathscr{G},\nabla r\right\rangle + (n-1)\frac{\langle e_p,\nabla\mathscr{G}\rangle}{r_p}\\ &\quad + \Big\{(n^2-1)\frac{\langle e_p,N\rangle^2}{r_p^2} - \frac{n-1}{r_p^2} - 1\Big\}\mathscr{G} + 2(n-1)\frac{\langle e_p,N\rangle}{r_p}\mathscr{G}^2 + \mathscr{G}^3.\end{aligned}$$

Let us rearrange terms. This yields

$$\frac{d}{dt}\mathscr{G} = \Delta\mathscr{G} - r\left\langle\nabla\mathscr{G},\nabla r\right\rangle + (n-1)\frac{\langle e_p,\nabla\mathscr{G}\rangle}{r_p} + \left\{\left\{2(n-1)\frac{\langle e_p,N\rangle^2}{r_p^2} - \frac{n-1}{r_p^2} - 1\right\} + \left\{(n-1)\frac{\langle e_p,N\rangle}{r_p} + \mathscr{G}\right\}^2\right\}\mathscr{G}.$$

Therefore,

$$\frac{d}{dt}\mathscr{G}_{\min} \geq -\Big\{\frac{n-1}{r_p^2} + 1\Big\}\mathscr{G}_{\min}.$$

Recall that from the previous Paragraph 3.2.1 it holds $r_p \sim \exp(t)$, and

$$f' = -\Big\{c\exp\{-2t\} + 1\Big\}f$$

admits the solution

$$f(t) = f_0\exp\big\{-\frac{c}{2} + \frac{c}{2}\exp\{-2t\} - t\big\}.$$

This shows that: *$\mathscr{G}$ can decrease at most exponentially fast to zero.* Similarly to the curve shortening case we obtain a lower bound for the maximum of $\mathscr{G}$ as follows. First of all let us recall the evolution equation of $\mathscr{F}$ - it reads

$$\Big\{\frac{d}{dt} - \Delta\Big\}\mathscr{F} = (n-1)\frac{\langle e_r,\nabla\mathscr{F}\rangle}{r} + \Big\{(n^2-1)\frac{\langle e_r,N\rangle}{r^2} - \frac{n-1}{r^2}\Big\}\mathscr{F} + 2(n-1)\frac{\langle e_r,N\rangle}{r}\mathscr{F}^2 + \mathscr{F}^3.$$

We already know that $\mathscr{F} \geq \texttt{const} > 0$, provided that $\mathscr{F} > 0$ for the initial curve. Thus there exists another constant, again denoted by $\texttt{const} > 0$, such that

$$\frac{d}{dt}\mathscr{F}_{\max} \leq \texttt{const}\mathscr{F}_{\max}^3.$$

We deduce

$$\frac{d}{dt}\frac{1}{\mathscr{F}_{\max}^2} = -2\frac{1}{\mathscr{F}_{\max}^3}\frac{d}{dt}\mathscr{F}_{\max} \geq -2\texttt{const}.$$

Therefore,

$$\mathscr{F}_{\max} \geq \frac{1}{\sqrt{2\texttt{const}(T_{\text{sing}} - t)}}.$$

Recall that $\mathscr{G} = \sqrt{2(T_{\text{sing}} - 1)}\mathscr{F}$. Hence,

$$\mathscr{G}_{\max} = \sqrt{2(T_{\text{sing}} - t)}\mathscr{F}_{\max} \geq \frac{1}{\sqrt{\texttt{const}}}.$$

Thus: *If the initial curve is tamed, then there exits a lower bound for $\mathscr{G}_{\max}$.*

**3.2.3.** Here we discuss the evolution of $r$. Recall that

$$\Big\{\frac{d}{dt} - \Delta\Big\} r = -r\,|\,\nabla r\,|^{\,2} + r - \frac{\langle e_r, N\rangle^2}{r} - (n-1)\frac{\langle e_r, N\rangle\,\langle e_p, N\rangle}{r_p}.$$

Thus,

$$\frac{d}{dt} r_{\max} \leq r_{\max} + \mathtt{const}\exp\{-t\},$$

which yields

$$r_{\max} \leq \Big\{\frac{\mathtt{const}}{2}\{1 - \exp\{-2t\}\} + r_{\max}(0)\Big\}\exp\{t\}.$$

This shows: *$r$ increases at most exponentially fast to infinity.* This is actually obvious: Note that the curves are bounded for the original flow equation. Which implies the result for the rescaled curves.

**3.2.4.** Let us now take a closer look at the *length* of the curve. As usually we assume that $\gamma$ is tamed. It holds

$$\begin{aligned}
\frac{d}{dt} L &= -\int_{S^1} k\mathscr{H}\, d\mu \\
&= -\int_{S^1} k^2\, d\mu + (n-1)\int_{S^1} k\frac{\langle e_p, N\rangle}{r_p}\, d\mu - \int_{S^1} k\,\langle z, N\rangle\, d\mu \\
&\leq (n-1)\int_{S^1} \mathscr{G}\frac{\langle e_p, N\rangle}{r_p}\, d\mu + (n-1)^2\int_{S^1} \frac{\langle e_p, N\rangle^2}{r_p^2}\, d\mu + L \\
&\leq L + (n-1)\int_{S^1} \frac{\langle e_p, N\rangle}{r_p}\, d\eta + \mathtt{const}\exp\{-2t\}L \\
&\leq L + \mathtt{const}\exp\{-t\} + \mathtt{const}\exp\{-2t\}L.
\end{aligned}$$

Recall that

$$\begin{aligned}
\frac{d}{dt} f(t) &= f(t) + A\exp\{-2t\}f(t) + B\exp\{-t\}, \\
f(0) &= C,
\end{aligned}$$

admits the solution

$$f(t) = \Big\{-\frac{B}{A} + \Big\{C + \frac{B}{A}\Big\}\exp\Big\{\frac{A}{2}\{1 - \exp\{-2t\}\}\Big\}\Big\}\exp\{t\}.$$

This shows, *If $\gamma$ is tamed curve which evolves under the rescaled flow* (3.1)*, then the length of $\gamma$ increases at most exponentially.* Again this also follows from a careful analysis of the blow-up procedure. The length of our curves under the original flow equation is bounded, in fact it tends monotonically to zero.

### 3.3 Classical estimates

Here we will establish a *Harnack-type inequality*, and give a *Bernstein-type estimate*.

**3.3.1.** Let us set $\mathscr{S} := c\mathscr{G}^2 + \mathscr{G}'^2$. We claim that for any $c > 1$ the following *Bernstein-type estimate* holds true

$$\sup_{S^1\times[0,\tau]} \left\{ c\mathscr{G}^2 + {\mathscr{G}^{(1)}}^2 \right\} \leq \mathtt{const} + c \sup_{S^1\times[0,\tau]} \mathscr{G}^2. \tag{3.10}$$

We may assume that $\mathscr{G}^{(1)} \neq 0$ at the maximum of $\mathscr{S}$. Observe that

$$\begin{aligned} \frac{d}{d\eta} r_p^i &= i \frac{\langle e_p, T\rangle}{r_p} \frac{r_p^i}{\mathscr{G}}, \\ \frac{d}{d\eta} \langle e_p, N\rangle &= -\langle e_p, T\rangle - n \frac{\langle e_p, T\rangle \langle e_p, N\rangle}{r_p} \frac{1}{\mathscr{G}}, \text{ and} \\ \frac{d}{d\eta} r_p^i \langle e_p, N\rangle^j &= (i - nj) \frac{\langle e_p, T\rangle \langle e_p, N\rangle^j}{r_p} \frac{r_p^i}{\mathscr{G}} - j \langle e_p, T\rangle \langle e_p, N\rangle^{j-1} r_p^i. \end{aligned}$$

Here we refer to Appendix C.1.4. Let us recall that we introduced

$$\Phi = (n^2 - 1)\frac{\langle e_p, N\rangle^2}{r_p^2} - \frac{n-1}{r_p^2}, \text{ and } \Psi = 2(n-1)\frac{\langle e_p, N\rangle}{r_p}.$$

This implies

$$\begin{aligned} \Phi^{(1)} = &\, 2(n-1)\left\{ 1 - (n+1)^2 \langle e_p, N\rangle^2 \right\} \frac{\langle e_p, T\rangle}{r_p^3} \frac{1}{\mathscr{G}} \\ &- 2(n^2-1)\frac{\langle e_p, T\rangle \langle e_p, N\rangle}{r_p^2}, \end{aligned}$$

and

$$\Psi^{(1)} = -2(n-1)\left\{ (n+1)\frac{\langle e_p, N\rangle}{r_p} \frac{1}{\mathscr{G}} + 1 \right\} \frac{\langle e_p, T\rangle}{r_p}.$$

A routine calculation gives the evolution equation of $\mathscr{S}$. We recall that

$$\frac{d}{d\tau}\mathscr{G} = \mathscr{G}^2 \frac{d^2}{d\eta^2}\mathscr{G} + \left\{ \Phi - 1 \right\} \mathscr{G} + \Psi \mathscr{G}^2 + \mathscr{G}^3.$$

Suppose that $f$ is an arbitrary function. It holds

$$D_\eta f^2 = 2fD_\eta f, \text{ and } D_{\eta\eta} f^2 = 2fD_{\eta\eta} f + 2\Big\{D_\eta f\Big\}^2.$$

This yields

$$\frac{d}{d\tau}\mathscr{G}^2 = \mathscr{G}^2 D_{\eta\eta}\mathscr{G}^2 - 2\mathscr{G}^2{\mathscr{G}^{(1)}}^2 + 2\mathscr{G}^4 + 2\Psi\mathscr{G}^3 + 2\Phi\mathscr{G}^2 - 2\mathscr{G}^2.$$

Now,

$$\begin{aligned}\frac{d}{d\tau}\mathscr{G}^{(1)} = \mathscr{G}^2 D_{\eta\eta}\mathscr{G}^{(1)} &+ 2\mathscr{G}\mathscr{G}^{(1)} D_{\eta\eta}\mathscr{G} \\ &+ 3\mathscr{G}^2\mathscr{G}^{(1)} + \Psi^{(1)}\mathscr{G}^2 + 2\Psi\mathscr{G}\mathscr{G}^{(1)} + \Phi^{(1)}\mathscr{G} + \Phi\mathscr{G}^{(1)} - \mathscr{G}^{(1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{d\tau}{\mathscr{G}^{(1)}}^2 = \mathscr{G}^2 D_{\eta\eta}{\mathscr{G}^{(1)}}^2 &- \frac{1}{2}\frac{\mathscr{G}^2}{{\mathscr{G}^{(1)}}^2}\Big\{D_\eta{\mathscr{G}^{(1)}}^2\Big\}^2 + 2\mathscr{G}\mathscr{G}^{(1)} D_\eta{\mathscr{G}^{(1)}}^2 \\ &+ 6\mathscr{G}^2{\mathscr{G}^{(1)}}^2 + 2\Psi^{(1)}\mathscr{G}^2\mathscr{G}^{(1)} + 4\Psi\mathscr{G}{\mathscr{G}^{(1)}}^2 \\ &+ 2\Phi^{(1)}\mathscr{G}\mathscr{G}^{(1)} + 2\Phi{\mathscr{G}^{(1)}}^2 - 2{\mathscr{G}^{(1)}}^2.\end{aligned}$$

Then

$$\begin{aligned}\frac{d}{d\tau}\mathscr{S} = \mathscr{G}^2 D_{\eta\eta}\mathscr{S} &- \frac{1}{2}\frac{\mathscr{G}^2}{{\mathscr{G}^{(1)}}^2}\big\{D_\eta\mathscr{S}\big\}^2 + 2c\frac{\mathscr{G}^3}{{\mathscr{G}^{(1)}}^2}D_\eta\mathscr{S} + 2\mathscr{G}\mathscr{G}^{(1)}D_\eta\mathscr{S} \\ &+ 2(1-c)c\mathscr{G}^4 + 6(1-c)\mathscr{G}^2{\mathscr{G}^{(1)}}^2 \\ &+ 2c\Psi\mathscr{G}^3 + 2c\Phi\mathscr{G}^2 \\ &+ 2\Psi^{(1)}\mathscr{G}^2\mathscr{G}^{(1)} + 4\Psi\mathscr{G}{\mathscr{G}^{(1)}}^2 + 2\Phi^{(1)}\mathscr{G}\mathscr{G}^{(1)} + 2\Phi{\mathscr{G}^{(1)}}^2 - 2\mathscr{S}.\end{aligned}$$

We claim that $\mathscr{S}$ must be bounded. To see this we introduce the notation $o[r_p^i]$, which means that a function is bounded from above by $\texttt{const}\, r_p^i$. For example $\Phi^{(1)} \le o[r_p^{-3}]\frac{1}{\mathscr{G}} + o[r_p^{-2}]$. Therefore,

$$\begin{aligned}\frac{d}{d\tau}\mathscr{S}_{\max} \le 2(1-c)c\mathscr{G}^4 &+ 6(1-c)\mathscr{G}^2{\mathscr{G}^{(1)}}^2 + o[r_p^{-1}]\mathscr{G}^3 + o[r_p^{-2}]\mathscr{G}^2 \\ &+ \Big\{o[r_p^{-2}]\frac{1}{\mathscr{G}} + o[r_p^{-1}]\Big\}\mathscr{G}^2\mathscr{G}^{(1)} + o[r_p^{-1}]\mathscr{G}{\mathscr{G}^{(1)}}^2 \\ &+ \Big\{o[r_p^{-3}]\frac{1}{\mathscr{G}} + o[r_p^{-2}]\Big\}\mathscr{G}\mathscr{G}^{(1)} + o[r_p^{-2}]{\mathscr{G}^{(1)}}^2 - 2c\mathscr{G}^2 - 2{\mathscr{G}^{(1)}}^2.\end{aligned}$$

Rearranging terms gives

$$\begin{aligned}\frac{d}{d\tau}\mathscr{S}_{\max} \le 2(1-c)c\mathscr{G}^4 &+ 6(1-c)\mathscr{G}^2{\mathscr{G}^{(1)}}^2 + o[r_p^{-1}]\mathscr{G}^3 + o[r_p^{-2}]\mathscr{G}^2 \\ &+ o[r_p^{-2}]\mathscr{G}\mathscr{G}^{(1)} + o[r_p^{-1}]\mathscr{G}^2\mathscr{G}^{(1)} + o[r_p^{-1}]\mathscr{G}{\mathscr{G}^{(1)}}^2 \\ &+ o[r_p^{-3}]\mathscr{G}^{(1)} + o[r_p^{-2}]{\mathscr{G}^{(1)}}^2 - 2c\mathscr{G}^2 - 2{\mathscr{G}^{(1)}}^2.\end{aligned}$$

Suppose to the contrary that $\mathscr{S}$ is unbounded. If $\mathscr{G}$ is bounded, then $\mathscr{G}'$ has to blow up. Therefore,

$$\frac{d}{d\tau}\mathscr{S}_{\max} \le \texttt{const} + 6(1-c)\mathscr{G}^2{\mathscr{G}^{(1)}}^2 - 2{\mathscr{G}^{(1)}}^2 + o[r_p^{-2}]\mathscr{G}^{(1)} + o[r_p^{-1}]{\mathscr{G}^{(1)}}^2.$$

This gives a contradiction for $c \ge 1$. Therefore, $\mathscr{G}$ has to blow up. If also $\mathscr{G}'$ becomes unbounded, then we arrive again at a contradiction if we choose $c > 1$. We are left with the case that $\mathscr{G}'$ is bounded. But then

$$\begin{aligned}\frac{d}{d\tau}\mathscr{S}_{\max} \le \texttt{const} &+ 2(1-c)c\mathscr{G}^4 + 6(1-c)\mathscr{G}^2{\mathscr{G}^{(1)}}^2 + o[r_p^{-1}]\mathscr{G}^3 + o[r_p^{-1}]\mathscr{G}^2 \\ &+ o[r_p^{-2}]\mathscr{G} + o[r_p^{-2}]\mathscr{G}^2 + o[r_p^{-1}]\mathscr{G} - 2c\mathscr{G}^2,\end{aligned}$$

which again yields a contradiction if $c > 1$. Thus, $\mathscr{S}$ is bounded if $\mathscr{G}^{(1)} \ne 0$. This proves the claim.

**3.3.2.** Let us denote $\mathscr{G}_{\max}(\tau) := \max_{\eta\in S^1}\mathscr{G}(\eta,\tau)$. We choose a time $\tau$ for which $\mathscr{G}_{\max}(\tau) \ge \mathscr{G}_{\max}(\tau')$ for all $\tau' \in [0,\tau]$. Let us denote the angle at which the maximum is attained by $\eta_0$. Note that $\eta_0$ is not necessarily unique. The mean value theorem and equation (3.10) give

$$\begin{aligned}\mathscr{G}_{\max}(\tau) - \mathscr{G}(\eta_0,\tau) &\le |\,\eta_0-\eta\,| \sup_{\eta\in S^1} |\,\mathscr{G}^{(1)}(\eta,\tau)\,| \\ &\le |\,\eta_0-\eta\,| \Big\{\texttt{const} + c\mathscr{G}_{\max}(\tau)\Big\},\end{aligned}$$

where $\texttt{const} > 0$, and $c > 1$ denote new constants. This yields

$$\big\{1 - c\,|\,\eta_0-\eta\,|\,\big\}\mathscr{G}_{\max}(\tau) \le |\,\eta_0-\eta\,|\ \texttt{const} + \mathscr{G}(\eta_0,\tau).$$

We obtain the following *Harnack-type estimate* for $|\,\eta_0-\eta\,| \le \frac{1}{3}$, where we have chosen $c = \frac{3}{2}$:

$$\mathscr{G}_{\max}(\tau) \le \texttt{const} + 2\mathscr{G}(\eta_0,\tau), \tag{3.11}$$

where again $\texttt{const} > 0$ denotes yet another constant.

**3.3.3.** Let us assume that $G^{(k)}$ is bounded from above for $k = 0,\dots,n-1$, and $\mathscr{G}$ is bounded from below by a positive constant. We claim that $\Psi^{(n)}$ and $\Phi^{(n)}$ tend to zero exponentially fast. We only show the result for $\Psi^{(n)}$, the remaining part is similar. Recall that

$$\frac{\partial}{\partial\eta}\Psi = -2(n-1)\Big\{(n+1)\frac{\langle e_p, N\rangle}{r_p\mathscr{G}} + 1\Big\}\frac{\langle e_p, T\rangle}{r_p}.$$

With this we see that $\Psi^{(n)}$ only depends on terms up to order $\mathscr{G}^{(n-1)}$. Note further that $\frac{1}{\mathscr{G}}$ is also bounded. Therefore,

$$\frac{\partial}{\partial\eta}\Psi^{(n)} = \big\{\texttt{bounded terms}\big\}\frac{\langle e_p, T\rangle}{r_p}.$$

The claim follows.

**3.3.4.** Here we briefly discuss; if $\mathscr{G}$ is bounded from below by a positive constant, and we also have bounds for $\mathscr{G}^{(k)}$ for all $k = 0, \dots, n$ then we also have bounds on $\mathscr{G}^{(n+1)}$. The idea is similar to Paragraph 3.3.1. We define

$$\mathscr{U} := c\mathscr{G}^{(n)^2} + \mathscr{G}^{(n+1)^2}.$$

First of all recall that

$$\frac{d}{d\tau}\mathscr{G} = \mathscr{G}^2 \frac{d^2}{d\eta^2}\mathscr{G} + \Big\{\Phi - 1\Big\}\mathscr{G} + \Psi\mathscr{G}^2 + \mathscr{G}^3.$$

This gives

$$\begin{aligned}\frac{d}{d\tau}\mathscr{G}^{(k)} = \mathscr{G}^2 D^2\mathscr{G}^{(k)} &+ \binom{k}{1}\Big\{\mathscr{G}^2\Big\}^{(1)} D\mathscr{G}^{(k)} \\ &+ \binom{k}{2}\Big\{\mathscr{G}^2\Big\}^{(2)}\mathscr{G}^{(k)} + 2\mathscr{G}\mathscr{G}^{(2)}\mathscr{G}^{(k)} \\ &+ \Phi\mathscr{G}^{(k)} + 2\Psi\mathscr{G}\mathscr{G}^{(k)} + 3\mathscr{G}^2\mathscr{G}^{(k)} - \mathscr{G}^{(k)} \\ &+ \texttt{lower order terms}.\end{aligned}$$

Note that some terms do not appear for $k = 0, 1$. Moreover, note that $\Phi^{(k)}$ and $\Phi^{(k)}$ are bounded by the considerations of the previous Paragraph 3.3.3. Therefore,

$$\begin{aligned}\frac{d}{d\tau}\Big\{\mathscr{G}^{(k)}\Big\}^2 = \mathscr{G}^2 D^2\Big\{\mathscr{G}^{(k)}\Big\}^2 &- \frac{\mathscr{G}^2}{2\Big\{\mathscr{G}^{(k)}\Big\}^2}\Big\{D\Big\{\mathscr{G}^{(k)}\Big\}^2\Big\}^2 \\ &+ \binom{k}{1}\Big\{\mathscr{G}^2\Big\}^{(1)} D\Big\{\mathscr{G}^{(k)}\Big\}^2 \\ &+ 2\binom{k}{2}\Big\{\mathscr{G}^2\Big\}^{(2)}\Big\{\mathscr{G}^{(k)}\Big\}^2 + 4\mathscr{G}\mathscr{G}^{(2)}\Big\{\mathscr{G}^{(k)}\Big\}^2 \\ &+ 2\Phi\Big\{\mathscr{G}^{(k)}\Big\}^2 + 4\Psi\mathscr{G}\Big\{\mathscr{G}^{(k)}\Big\}^2 + 6\mathscr{G}^2\Big\{\mathscr{G}^{(k)}\Big\}^2 - 2\Big\{\mathscr{G}^{(k)}\Big\}^2 \\ &+ \Big\{\texttt{lower order terms}\Big\}\mathscr{G}^{(k)}.\end{aligned}$$

Observe,

$$-\frac{\mathscr{G}^2}{2\Big\{\mathscr{G}^{(k)}\Big\}^2}\Big\{D\Big\{\mathscr{G}^{(k)}\Big\}^2\Big\}^2 = -2\mathscr{G}^2\Big\{D\mathscr{G}^{(k)}\Big\}^2.$$

This implies

$$
\begin{aligned}
\frac{d}{d\tau}\mathscr{U} = \mathscr{G}^2 D^2\mathscr{U} &- \frac{\mathscr{G}^2}{2\Big\{\mathscr{G}^{(k+1)}\Big\}^2}\Big\{D\Big\{\mathscr{G}^{(k+1)}\Big\}^2\Big\}^2 + (n+1)\Big\{\mathscr{G}^2\Big\}^{(1)} D\mathscr{U} \\
&- 2c\mathscr{G}^2\Big\{\mathscr{G}^{(n+1)}\Big\}^2 \\
&\quad - c\Big\{\mathscr{G}^2\Big\}^{(1)} D\Big\{\mathscr{G}^{(n)}\Big\}^2 \\
&\quad + 2\binom{n+1}{2}\Big\{\mathscr{G}^2\Big\}^{(2)}\Big\{\mathscr{G}^{(n+1)}\Big\}^2 + 4\mathscr{G}\mathscr{G}^{(2)}\Big\{\mathscr{G}^{(n+1)}\Big\}^2 \\
&\quad + 2\Phi\Big\{\mathscr{G}^{(n+1)}\Big\}^2 + 4\Psi\mathscr{G}\Big\{\mathscr{G}^{(n+1)}\Big\}^2 + 6\mathscr{G}^2\Big\{\mathscr{G}^{(n+1)}\Big\}^2 - 2\Big\{\mathscr{G}^{(n+1)}\Big\}^2 \\
&\quad + 2c\binom{n}{2}\Big\{\mathscr{G}^2\Big\}^{(2)}\Big\{\mathscr{G}^{(n)}\Big\}^2 + 4c\mathscr{G}\mathscr{G}^{(2)}\Big\{\mathscr{G}^{(n)}\Big\}^2 \\
&\quad + 2c\Phi\Big\{\mathscr{G}^{(n)}\Big\}^2 + 4c\Psi\mathscr{G}\Big\{\mathscr{G}^{(n)}\Big\}^2 + 6c\mathscr{G}^2\Big\{\mathscr{G}^{(n)}\Big\}^2 - 2c\Big\{\mathscr{G}^{(n)}\Big\}^2 \\
&\quad + \Big\{\texttt{lower order terms}\Big\}\mathscr{G}^{(n)} + \Big\{\texttt{lower order terms}\Big\}\mathscr{G}^{(n+1)}.
\end{aligned}
$$

The rest of the argument is clear. We may choose the constant $c$ big enough to show that

$$
\sup_{S^1\times[0,\tau]}\Big\{c\Big\{\mathscr{G}^{(n)}\Big\}^2 + \Big\{\mathscr{G}^{(n+1)}\Big\}^2\Big\} \le \texttt{const} + c\sup_{S^1\times[0,\tau]}\Big\{\mathscr{G}^{(n)}\Big\}^2.
$$

The claim follows.

## Notes for Section 3.3

**1.** The basic idea of the above proof to the Bernstein-type estimate is classic. We refer to [16] and [59] for an approach in curve shortening flows. Our source of inspiration for the proof of the Harnack inequality have also been the above cited books.

**2.** Harnack-type estimates and Bernstein-type estimates for the classical curve shortening flow can also be found in Angenent [6]. Basically this proof compares the given solution with a specific one a so called shrinking spiral. Which is nothing but a traveling wave solution of the classical curve flow equation. If we were able to obtain such a solution in our case we would expect the proof also to work here.

**3.** For the readers convenience we compare the argument with the normalized curve shortening flow equation, that is $n = 1$, with constant area $\pi$. The evolution equation of the curvature reads

$$
\frac{d}{d\tau}k = k^2 D^2 k + k^3 - k,
$$

and therefore

$$\frac{d}{d\tau}k^2 = k^2 D^2 k^2 - 2k^2 {k^{(1)}}^2 + 2k^4 - 2k^2.$$

Differentiation of the first equation yields

$$\frac{d}{d\tau}k^{(1)} = k^2 D^2 k^{(1)} + 2kk^{(1)} D^2 k + 3k^2 k^{(1)} - k^{(1)}.$$

Hence,

$$\frac{d}{d\tau}{k^{(1)}}^2 = k^2 D^2 {k^{(1)}}^2 - \frac{1}{2}\frac{k^2}{{k^{(1)}}^2}\Big\{D{k^{(1)}}^2\Big\}^2 + 2kk^{(1)} D{k^{(1)}}^2 + 6k^2 {k^{(1)}}^2 - 2{k^{(1)}}^2.$$

We define

$$\mathscr{S} = ck^2 + {k^{(1)}}^2.$$

Then

$$\begin{aligned} D\mathscr{S} &= 2ckk^{(1)} + D{k^{(1)}}^2, \text{ and} \\ \big\{D\mathscr{S}\big\}^2 &= 4c^2k^2{k^{(1)}}^2 + \big\{Dk^{(1)}\big\}^2 + 4ckk^{(1)} D{k^{(1)}}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} -\frac{1}{2}\frac{k^2}{{k^{(1)}}^2}\{Dk^{(1)}\}^2 &= -\frac{1}{2}\frac{k^2}{{k^{(1)}}^2}\{D\mathscr{S}\}^2 + 2c^2k^4 + 2c\frac{k^3}{k^{(1)}}D{k^{(1)}}^2 \\ &= -\frac{1}{2}\frac{k^2}{{k^{(1)}}^2}\{D\mathscr{S}\}^2 + 2c\frac{k^3}{{k^{(1)}}^2}D\mathscr{S} - 2c^2k^4, \text{ and ,} \\ 2kk^{(1)}D{k^{(1)}}^2 &= 2kk^{(1)}D\mathscr{S} - 4ck^2{k^{(1)}}^2. \end{aligned}$$

This gives

$$\begin{aligned} \frac{d}{d\tau}\mathscr{S} &= k^2D^2\mathscr{S} - \frac{1}{2}\frac{k^2}{{k^{(1)}}^2}\{D{k^{(1)}}^2\}^2 + 2kk^{(1)}D{k^{(1)}}^2 \\ &\qquad - 2ck^2{k^{(1)}}^2 + 2ck^4 - 2ck^2 + 6k^2{k^{(1)}}^2 - 2{k^{(1)}}^2 \\ &= k^2D^2\mathscr{S} - \frac{1}{2}\frac{k^2}{{k^{(1)}}^2}\{D\mathscr{S}\}^2 + 2c\frac{k^3}{{k^{(1)}}^2}D\mathscr{S} + 2kk^{(1)}D\mathscr{S} \\ &\qquad + 2(1-c)\Big\{3{k^{(1)}}^2 + ck^2\Big\}k^2 - 2\mathscr{S}. \end{aligned}$$

Here $D$ denotes differentiation with respect to the normal angle. In the this case, we may choose $c = 1$. This shows: *If $k^{(1)}$ is not zero at the maximum of $\mathscr{S}$, then $\mathscr{S}$ is bounded by* $\max \mathscr{S}(0)$. *Thus,* $\mathscr{S}(\tau) \leq \max \mathscr{S}(0) + \max_\tau k^2$. Let us note that Angenent obtain a Bernstein-type estimate for the curve shortening flow by a complete different approach, compare [6].

### 3.4 Proof of Theorem B

This section finished the proof the Theorem B. We use Gage's inequality to bound the length of the rescaled curves. This is the only time where we make us of the convexity assumption. Then we show that $\mathscr{G} \to -\langle z, N\rangle$ in $L^2(S^1)$. This is done with a monotonicity type argument. Moreover, we show that the entropy of the curves remains bounded. Together with the Harnack-type inequality and the Arzela-Ascoli theorem we obtain the result.

**3.4.1.** Let us bound the *length* of $z$ for the rescaled curve equation. It is the only time that we need the convexity. First of all note that the length can tend to infinity at most exponentially fast. We recall *Gage's inequality* [26], which holds for any strictly convex simple $C^2$-curve in the plane it reads

$$\pi \frac{L}{A} \leq \int k^2 \, d\mu.$$

We bound the time derivative of $L$ as follows

$$\begin{aligned}
\frac{d}{dt} L &= -\int_{S^1} \mathscr{G} k + \langle z, N\rangle \, k \, d\mu \\
&= L - \int_{S^1} k^2 \, d\mu + (n-1)\int_{S^1} \mathscr{G} \frac{\langle e_p, N\rangle}{r_p} \, d\mu + (n-1)^2 \int_{S^1} \frac{\langle e_p, N\rangle^2}{r_p^2} \, d\mu \\
&\leq (n-1)\int_{S^1} \frac{\langle e_p, N\rangle}{r_p} \, d\eta + (n-1)^2 \int_{S^1} \frac{\langle e_p, N\rangle^2}{r_p^2} \, d\mu \\
&\leq \mathtt{const}_1 \exp(-t) + \mathtt{const}_2 \exp(-2t) L \leq \mathtt{const} \exp(-t).
\end{aligned}$$

This shows that $L$ is bounded from above, and therefore also $r$. Let us remark that the result of the theorem would follow if we had a Gage-type inequality for almost convex curves, meaning that $k > -\exp(-t)$. Alternatively it would suffice to show that $z$ is bounded under the flow, which also implies that $L$ is bounded.

**3.4.2.** Let us define the *Gauß-kernel* and the *energy* by

$$\rho := \exp\Big\{ -\frac{r^2}{2} \Big\} \text{ and } \mathcal{M} := \int_{S^1} \rho \, d\mu.$$

It follows that $\mathcal{M}$ is bounded from above by $L$. We introduce

$$\mathscr{D} := -\Big\{ \frac{d}{d\eta}\mathscr{G} + (n-1)\frac{\langle e_p, T\rangle}{r_p} - \langle z, T\rangle \Big\}.$$

We note with Appendix C.1.6 that

$$\frac{d}{d\tau} r^2 = -2\langle z, T\rangle \frac{d}{d\eta}\mathscr{G} + 2\langle z, N\rangle \mathscr{G} + 2r^2 - 2(n-1)\frac{\langle z, T\rangle \langle e_p, T\rangle}{r_p},$$

and

$$\begin{aligned}\frac{d}{d\tau}\rho &= \Big\{ \langle z,T\rangle \frac{d}{d\eta}\mathscr{G} - \langle z,N\rangle \mathscr{G} - r^2 + (n-1)\frac{\langle z,T\rangle \langle e_p,T\rangle}{r_p}\Big\}\rho \\ &= \Big\{ -\langle z,T\rangle \mathscr{D} - \langle z,N\rangle \mathscr{G} - \langle z,N\rangle^2 \Big\}\rho.\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{d}{d\tau}d\mu &= \Big\{ -\mathscr{G}k - \langle z,N\rangle k + \frac{\partial}{\partial\mu}\mathscr{D}\Big\}d\mu \\ &= \Big\{ -\mathscr{G}^2 - \langle z,N\rangle \mathscr{G} - (n-1)\frac{\langle e_p,N\rangle}{r_p}\mathscr{G} \\ &\qquad - (n-1)\frac{\langle z,N\rangle \langle e_p,N\rangle}{r_p} + \frac{\partial}{\partial\mu}\mathscr{D}\Big\}d\mu.\end{aligned}$$

We want to point out that

$$\frac{\partial}{\partial\mu}\rho = -\langle z,T\rangle \rho.$$

This yields

$$\begin{aligned}\mathcal{M}_\tau = -\int_{S^1} \Big\{ & \langle z,T\rangle \mathscr{D} + \langle z,N\rangle \mathscr{G} + \langle z,N\rangle^2 \\ & + \mathscr{G}^2 + \langle z,N\rangle \mathscr{G} + (n-1)\frac{\langle e_p,N\rangle}{r_p}\mathscr{G} \\ & + (n-1)\frac{\langle z,N\rangle \langle e_p,N\rangle}{r_p} - \frac{\partial}{\partial\mu}\mathscr{D}\Big\}d\mu \\ = -\int_{S^1} \Big\{ & \langle z,N\rangle \mathscr{G} + \langle z,N\rangle^2 \\ & + \mathscr{G}^2 + \langle z,N\rangle \mathscr{G} + (n-1)\frac{\langle e_p,N\rangle}{r_p}\mathscr{G} \\ & + (n-1)\frac{\langle z,N\rangle \langle e_p,N\rangle}{r_p}\Big\}d\mu.\end{aligned}$$

This gives

$$\mathcal{M}_\tau = -\int_{S^1} \Big\{\big(\mathscr{G} + \langle z,N\rangle\big)^2 + (n-1)\frac{\langle e_p,N\rangle}{r_p}\big(\mathscr{G} + \langle z,N\rangle\big)\Big\}\rho\, d\mu.$$

Observe,

$$\begin{aligned}\int_{S^1} \frac{\langle e_p,N\rangle}{r_p}\big(\mathscr{G} + \langle z,N\rangle\big)\rho\, d\mu &= \int_{S^1} \frac{\langle e_p,N\rangle}{r_p}\rho\, d\eta + \int_{S^1} \frac{\langle e_p,N\rangle}{r_p} \langle z,N\rangle \rho\, d\mu \\ &\leq \texttt{const}\exp\{-\tau\}.\end{aligned}$$

Note that $\mathcal{M}$ is bounded from above by $L$, which is bounded by §3.4.1. Thus,

$$-\int_0^\infty \int_{S^1} \Big\{ \big(\mathscr{G} + \langle z, N\rangle\,\big)^2 + (n-1)\frac{\langle e_p, N\rangle}{r_p}\big(\mathscr{G} + \langle z, N\rangle\,\big)\Big\}\rho\, d\mu d\tau < \infty.$$

We compute

$$\frac{d}{d\tau}\mathcal{M} = -\int_{S^1}\Big\{\Big\{\mathscr{G} + \langle z, N\rangle\,\Big\}^2 + (n-1)\frac{\langle e_p, N\rangle}{r_p}\Big\{\mathscr{G} + \langle z, N\rangle\,\Big\}\Big\}\rho\, d\mu.$$

We observe that

$$\begin{aligned}\int_{S^1}\frac{\langle e_p, N\rangle}{r_p}\Big\{\mathscr{G} + \langle z, N\rangle\,\Big\}\rho\, d\mu &= \int_{S^1}\frac{\langle e_p, N\rangle}{r_p}\rho\, d\eta + \int_{S^1}\frac{\langle e_p, N\rangle}{r_p}\,\langle z, N\rangle\,\rho\, d\mu\\ &\leq \texttt{const}_1 \exp(-t),\end{aligned}$$

because $z$ is bounded. Therefore,

$$\int_0^{\tau_0}\int_{S^1}\Big\{\mathscr{G} + \langle z, N\rangle\,\Big\}^2 \rho\, d\mu\, d\tau \leq \texttt{const}_2.$$

Therefore we can select a sequence $\tau_n \to \infty$ such that

$$\int_{S^1}\Big\{\mathscr{G} + \langle z, N\rangle\,\Big\}^2 d\mu \to 0.$$

Note that $\rho \geq \epsilon > 0$, because $z$ is bounded.

**3.4.3.** We define the *entropy* by

$$\mathscr{E}(\gamma) := \int_{S^1}\ln\mathscr{G}\, d\eta.$$

It follows from §3.4.2 that

$$\mathscr{E}(\gamma) = \int_{S^1}\mathscr{G}\ln\mathscr{G}\, d\mu \leq \int_{S^1}\mathscr{G}^2 d\mu \leq \texttt{const}.$$

**3.4.4.** We claim that $\mathscr{G}$ must be bounded. If this is not the case we can select a subsequence $\tau_j \to \infty$ such that $\mathscr{G}_{\max}(\tau_j) \geq \mathscr{G}_{\max}(\tau')$ for all $\tau' \in [0, \tau_j]$. Therefore,

$$\begin{aligned}\texttt{const} &\geq \int_{S^1}\ln\mathscr{G}(\eta,\tau_j)\, d\eta\\ &\geq \int_{|\,\eta-\eta_0\,|\leq\frac{1}{3}}\ln\mathscr{G}(\eta,\tau_j)\, d\eta + \int_{\mathscr{G}<1}\mathscr{G}\ln\mathscr{G}(\eta,\tau_j)\, d\mu\\ &\geq \int_{|\,\eta-\eta_0\,|\leq\frac{1}{3}}\ln\Big\{\frac{1}{2}\{\mathscr{G}_{\max}(\tau_j) - \texttt{const}\}\Big\}\, d\eta - \exp\{-1\}L\\ &\geq \frac{2}{3}\Big\{\ln\Big\{\frac{1}{2}\{\mathscr{G}_{\max}(\tau_j) - \texttt{const}\}\Big\}\Big\} - \exp\{-1\}L,\end{aligned}$$

because $\mathscr{G} \ln \mathscr{G} \geq -\exp\{-1\}$, and the Harnack-type estimate (3.11). But this gives a contradiction for large $\mathscr{G}_{\max}$. Therefore $\mathscr{G}$ must be bounded. An careful analysis of the evolution of $\mathscr{S}$ and an inductive argument gives bounds for $\mathscr{G}^{(n)}$ as well, compare §3.3.1.

**3.4.5.** By §3.4.2 we know that $\mathscr{G} \to \langle z, N \rangle$ in $L^2$. The bounds on $\mathscr{G}^{(n)}$ and the Arzela-Ascoli theorem imply that we actually have smooth convergence. Moreover, it follows that $\mathscr{G} \to k$ smoothly, 1compare §3.3.3. This gives that the limit curve has to satisfy

$$k = -\langle z, N \rangle .$$

The only embedded convex solution of this equation is the standard circle. This proves the theorem. q.e.d.

### Notes for Section 3.4

**1.** That we only embedded convex curve which satisfies

$$k = -\langle z, N \rangle$$

is the unit circle was proven by Abresch and Langer [1]

Appendix A

# Evolving Curves

In this appendix we derive several equations which are frequently used. Paragraph A.1 contains some basics of planar curves. In §A.2 we consider curves which satisfy the following equation

$$\frac{d}{dt}z = \mathscr{A}(z,k,\theta)N + \mathscr{B}(z,k,\theta)T, \tag{A.1}$$

where $\mathscr{A}, \mathscr{B}\colon \mathbb{R}^2 \times \mathbb{R} \times S^1 \to \mathbb{R}$, and $T$, $N$ denote the tangent respective the normal vector. We refer to equation (A.1) as the *general curve shortening problem.*

## A.1 Preliminaries

**A.1.1.** We start with

$$\frac{\partial}{\partial\mu}r = \frac{1}{\mu}\frac{d}{dp}\big(x^2+y^2\big)^{.5} = \frac{1}{\mu}\big(x^2+y^2\big)^{-.5}\langle z, z'\rangle = \langle e_r, T\rangle .$$

Therefore,

$$\frac{\partial}{\partial\mu}r^i = i\frac{\langle e_r, T\rangle}{r}r^i.$$

We have

$$\frac{\partial}{\partial\mu}\langle e_r, T\rangle = k\langle e_r, N\rangle + \frac{1}{r} - \frac{\langle e_r, T\rangle^2}{r} = k\langle e_r, N\rangle + \frac{\langle e_r, N\rangle^2}{r},$$
$$\frac{\partial}{\partial\mu}\langle e_r, N\rangle = -k\langle e_r, T\rangle - \frac{\langle e_r, T\rangle\langle e_r, N\rangle}{r}.$$

From which we deduce

$$\frac{\partial}{\partial\mu}\langle e_r, T\rangle r^i = k\langle e_r, N\rangle r^i + \frac{\langle e_r, N\rangle^2}{r}r^i + i\frac{\langle e_r, T\rangle^2}{r}r^i$$
$$\frac{\partial}{\partial\mu}\langle e_r, N\rangle r^i = -k\langle e_r, T\rangle r^i + (i-1)\frac{\langle e_r, T\rangle\langle e_r, N\rangle}{r}r^i.$$

As special cases

$$\frac{\partial}{\partial \mu}\langle z,T\rangle = k\langle z,N\rangle + 1 \text{ and } \frac{\partial}{\partial \mu}\langle z,N\rangle = -k\langle z,T\rangle .$$

Finally we compute

$$\begin{aligned}&\frac{\partial}{\partial \mu}\langle e_r,T\rangle^j \langle e_r,N\rangle^l r^i\\ &\quad = \Big\{\Big\{j\langle e_r,N\rangle^2 - l\langle e_r,T\rangle^2\Big\}k\\ &\qquad + \Big\{j\langle e_r,N\rangle^2 + (i-l)\langle e_r,T\rangle^2\Big\}\frac{\langle e_r,N\rangle}{r}\Big\} r^i\langle e_r,T\rangle^{j-1}\langle e_r,N\rangle^{l-1} .\end{aligned}$$

## A.2 Time derivatives for the general curve flow

In this paragraph we compute several time derivatives for the *general curve flow equation.* That is, we consider for given $\mathscr{A},\mathscr{B}\colon \mathbb{R}^2\times\mathbb{R}\times S^1\to\mathbb{R}$ the equation

$$\frac{d}{dt}z = \mathscr{A}(z,k,\theta)N + \mathscr{B}(z,k,\theta)T. \tag{A.2}$$

Let us refer to $\mathscr{A}$ respective $\mathscr{B}$ as the *normal* respective *tangential speed function.* Given this equation we may compute several evolution equations for geometric quantities. If we write $\frac{d}{dt}$, then $p$ is assumed to be fixed, analogously we assume for $\frac{d}{dp}$ that $t$ is fixed.

**A.2.1.** Let us recall that $\frac{\partial}{\partial\mu} = \frac{1}{|z'|}\frac{d}{dp}$. This implies for the *Frenet formulas*

$$\frac{\partial}{\partial \mu}T = kN, \text{ and } \frac{\partial}{\partial \mu}N = -kT. \tag{A.3}$$

We claim that the time derivative of $\mu$ equals

$$\frac{d}{dt}\mu = (-\mathscr{A}k + \frac{\partial}{\partial\mu}\mathscr{B})\mu. \tag{A.4}$$

To see this, recall that $\mu = |z'|$. Therefore

$$\begin{aligned}\frac{d}{dt}|z'|^2 &= \frac{d}{dt}\left\langle \frac{dz}{dp},\frac{dz}{dp}\right\rangle = 2\left\langle \frac{dz}{dp},\frac{d}{dp}\frac{d}{dt}z\right\rangle = 2\left\langle \frac{dz}{dp},\frac{d}{dp}(\mathscr{A}N+\mathscr{B}T)\right\rangle\\ &= 2\left\langle \frac{dz}{dp},\mathscr{A}\frac{d}{dp}N + (\frac{d}{dp}\mathscr{B})T\right\rangle = -2(\mathscr{A}k + \frac{\partial}{\partial\mu}\mathscr{B})\mu^2,\end{aligned}$$

and the claim follows. In the next step we compute a rule for interchanging $\frac{d}{dt}$ and $\frac{\partial}{\partial\mu}$. The operators do not commute, as $\mu$ is not independent of $t$. We have

$$\frac{d}{dt}\frac{\partial}{\partial\mu} = \frac{\partial}{\partial\mu}\frac{d}{dt} + (\mathscr{A}k - \frac{\partial}{\partial\mu}\mathscr{B})\frac{\partial}{\partial\mu}. \tag{A.5}$$

For a proof compute

$$\begin{aligned}\frac{d}{dt}\frac{\partial}{\partial\mu} &= \frac{d}{dt}\left(\mu^{-1}\frac{d}{dp}\right) = \frac{d}{dt}\left(\mu^{-1}\right) + \mu^{-1}\frac{d}{dt}\frac{d}{dp} \\ &= -\mu^{-2}(-\mathscr{A}k + \frac{\partial}{\partial\mu}\mathscr{B})\mu\frac{d}{dp} + \mu^{-1}\frac{d}{dp}\frac{d}{dt} = (\mathscr{A}k - \frac{\partial}{\partial\mu}\mathscr{B})\frac{\partial}{\partial\mu} + \frac{\partial}{\partial\mu}\frac{d}{dt}.\end{aligned}$$

**A.2.2.** Here we derive the time derivative of $T$, $N$, and the normal angle $\theta$. We claim that time derivation of $T$ gives

$$\frac{d}{dt}T = \Big\{\frac{\partial}{\partial\mu}\mathscr{A} + k\mathscr{B}\Big\}N. \tag{A.6}$$

The proof is straight forward. In fact,

$$\begin{aligned}\frac{d}{dt}T &= \frac{d}{dt}\frac{\partial}{\partial\mu}z \underset{(A.5)}{=} \frac{\partial}{\partial\mu}\frac{d}{dt}z + \Big\{\mathscr{A}k - \frac{\partial}{\partial\mu}\mathscr{B}\Big\}\frac{\partial}{\partial\mu}z \\ &= \frac{\partial}{\partial\mu}\Big\{\mathscr{A}N + \mathscr{B}T\Big\} + \Big\{\mathscr{A}k - \frac{\partial}{\partial\mu}\mathscr{B}\Big\}T \\ &= \Big\{\frac{\partial}{\partial\mu}\mathscr{A}\Big\}N + \mathscr{A}\frac{\partial}{\partial\mu}N + \Big\{\frac{\partial}{\partial\mu}\mathscr{B}\Big\}T + \mathscr{B}\frac{\partial}{\partial\mu}T + \Big\{\mathscr{A}k - \frac{\partial}{\partial\mu}\mathscr{B}\Big\}T \\ &\underset{(A.3)}{=} \Big\{\frac{\partial}{\partial\mu}\mathscr{A}\Big\}N - k\mathscr{A}T + k\mathscr{B}N + \mathscr{A}kT = \Big\{\frac{\partial}{\partial\mu}\mathscr{A} + k\mathscr{B}\Big\}N.\end{aligned}$$

Similarly,

$$\frac{d}{dt}N = -\Big\{\frac{\partial}{\partial\mu}\mathscr{A} + k\mathscr{B}\Big\}T. \tag{A.7}$$

To see this consider

$$0 = \frac{d}{dt}\langle N, T\rangle = \left\langle \frac{d}{dt}N, T\right\rangle + \left\langle N, \frac{d}{dt}T\right\rangle.$$

Now,

$$\begin{aligned}\left\langle \frac{d}{dt}N, T\right\rangle &= -\left\langle N, \frac{d}{dt}T\right\rangle \\ &\underset{(A.6)}{=} -\left\langle N, \Big\{\frac{\partial}{\partial\mu}\mathscr{A} + k\mathscr{B}\Big\}N\right\rangle = -\left\langle T, \Big\{\frac{\partial}{\partial\mu}\mathscr{A} + k\mathscr{B}\Big\}T\right\rangle,\end{aligned}$$

and the claim follows. For the tangent angle we have

$$\frac{d}{dt}\theta = \frac{\partial}{\partial\mu}\mathscr{A} + \mathscr{B}k. \tag{A.8}$$

For a proof recall that the unit tangent equals $T = (\cos\theta, \sin\theta)$. Thus,

$$\frac{d}{dt}T = \frac{d}{dt}N.$$

This gives, together with (A.6), the assertion.

**A.2.3.** It holds

$$\frac{d}{dt}r^i = i\frac{\langle e_r, T\rangle}{r}r^i\mathscr{B} + i\frac{\langle e_r, N\rangle}{r}r^i\mathscr{A}.$$

We compute

$$\frac{d}{dt}r = \frac{d}{dt}\langle z, z\rangle^{\frac{1}{2}} = \frac{\langle e_r, T\rangle}{r}\mathscr{B} + \frac{\langle e_r, N\rangle}{r}\mathscr{A}.$$

The result follows from the chain rule. With this at hand we are able to derive

$$\begin{aligned}
\frac{d}{dt}\langle e_r, T\rangle &= \frac{d}{dt}\frac{\langle z, T\rangle}{r} \\
&= -\frac{\langle e_r, T\rangle^2}{r}\mathscr{B} - \frac{\langle e_r, T\rangle\langle e_r, N\rangle}{r}\mathscr{A} + \Big\{\frac{\partial}{\partial\mu}\mathscr{A} + k\mathscr{B}\Big\}\langle e_r, N\rangle + \frac{\mathscr{B}}{r} \\
&= \Big\{\frac{\partial}{\partial\mu}\mathscr{A} - \frac{\langle e_r, T\rangle}{r}\mathscr{A} + \frac{\langle e_r, N\rangle}{r}\mathscr{B} + k\mathscr{B}\Big\}\langle e_r, N\rangle.
\end{aligned}$$

Analogously we derive

$$\frac{d}{dt}\langle e_r, N\rangle = \Big\{-\frac{\partial}{\partial\mu}\mathscr{A} + \frac{\langle e_r, T\rangle}{r}\mathscr{A} - k\mathscr{B} - \frac{\langle e_r, N\rangle}{r}\mathscr{B}\Big\}\langle e_r, T\rangle.$$

We add to our list

$$\begin{aligned}
\frac{d}{dt}\langle e_r, T\rangle r^i &= \langle e_r, N\rangle r^i\frac{\partial}{\partial\mu}\mathscr{A} + \langle e_r, N\rangle r^i k\mathscr{B} \\
&\quad + \Big\{i\frac{\langle e_r, T\rangle^2}{r} + \frac{\langle e_r, N\rangle^2}{r}\Big\}r^i\mathscr{B} + (i-1)\frac{\langle e_r, T\rangle\langle e_r, N\rangle}{r}r^i\mathscr{A}, \\
\frac{d}{dt}\langle e_r, N\rangle r^i &= -\langle e_r, T\rangle r^i\frac{\partial}{\partial\mu}\mathscr{A} - \langle e_r, T\rangle r^i k\mathscr{B} \\
&\quad + \Big\{\frac{\langle e_r, T\rangle^2}{r} + i\frac{\langle e_r, N\rangle^2}{r}\Big\}r^i\mathscr{A} + (i-1)\frac{\langle e_r, T\rangle\langle e_r, N\rangle}{r}r^i\mathscr{B}.
\end{aligned}$$

Finally we have

$$
\begin{aligned}
&\frac{d}{dt} \langle e_r, T\rangle^j \langle e_r, N\rangle^l r^i \\
&\quad = \Bigg\{ \Big\{ j \langle e_r, N\rangle^2 - l \langle e_r, T\rangle^2 \Big\} \frac{\partial}{\partial \mu} \mathscr{A} \\
&\qquad + \Big\{ (i-j) \langle e_r, N\rangle^2 + l \langle e_r, T\rangle^2 \Big\} \frac{\langle e_r, T\rangle}{r} \mathscr{A} \\
&\qquad + \Big\{ j \langle e_r, N\rangle^2 \frac{1}{r} \mathscr{B} + (i-l) \langle e_r, T\rangle^2 \Big\} \frac{\langle e_r, N\rangle}{r} \mathscr{B} \\
&\qquad + \Big\{ j \langle e_r, N\rangle^2 - l \langle e_r, T\rangle^2 \Big\} k\mathscr{B} \Bigg\} \langle e_r, T\rangle^{j-1} \langle e_r, N\rangle^{l-1} r^i.
\end{aligned}
$$

**A.2.4.** Let us now look at the curvature. We have

$$k = \frac{\partial}{\partial \mu}\theta \tag{A.9}$$

Because,

$$k \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = kN = \frac{\partial}{\partial \mu} T = \frac{\partial}{\partial \mu} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \frac{\partial}{\partial \mu}\theta,$$

where we have used the Frenet formulas. We claim that

$$\frac{d}{dt} k = \frac{d^2}{d\mu^2} \mathscr{A} + \left( \frac{\partial}{\partial \mu} k \right) \mathscr{B} + k^2 \mathscr{A}. \tag{A.10}$$

To see this compute

$$
\begin{aligned}
\frac{d}{dt} k &= \frac{d}{dt} \frac{\partial}{\partial \mu} \theta = \frac{\partial}{\partial \mu} \frac{d}{dt} \theta + (\mathscr{A} k \frac{\partial}{\partial \mu} \mathscr{B}) \frac{\partial}{\partial \mu} \theta \\
&= \frac{\partial}{\partial \mu} \left( \frac{\partial}{\partial \mu} \mathscr{A} + k\mathscr{B} \right) + \left( \mathscr{A} k - \frac{\partial}{\partial \mu} \mathscr{B} \right) k \\
&= \frac{\partial^2}{\partial \mu^2} \mathscr{A} + (\frac{\partial}{\partial \mu} k) \mathscr{B} + k \frac{\partial}{\partial \mu} \mathscr{B} + k^2 \mathscr{A} - k \frac{\partial}{\partial \mu} \mathscr{B}.
\end{aligned}
$$

**A.2.5.** We close this section while computing the time derivative of the *length* $L$ and the *area* $A$. It holds

$$\frac{d}{dt} L(t) = - \int_\gamma \mathscr{A} k \, d\mu. \tag{A.11}$$

Since the length is given by

$$L(t) = \int_{S^1} d\mu = \int_{S^1} |z'(p)|\, dp.$$

This yields

$$\frac{d}{dt}L(t) = \int_{S^1} \frac{d}{dt}|z'(p)|\, dp = -\int_\gamma \left(-k\mathscr{A} + \frac{\partial}{\partial\mu}\mathscr{B}\right) d\mu = -\int_\gamma k\mathscr{A}\, d\mu$$

and hence the claim. For an embedded curve we have

$$\frac{d}{dt}A(t) = -\int_\gamma \mathscr{A}\, d\mu. \tag{A.12}$$

We compute

$$\begin{aligned}
\frac{d}{dt}A(t) &= -\frac{d}{dt}\frac{1}{2}\int_\gamma \langle z, N\rangle d\mu \\
&= -\frac{1}{2}\int_\gamma \left\langle \frac{d}{dt}z, N\right\rangle d\mu - \frac{1}{2}\int_\gamma \left\langle z, \frac{d}{dt}N\right\rangle d\mu - \frac{1}{2}\int_\gamma \langle z, N\rangle \frac{d}{dt}d\mu \\
&= -\frac{1}{2}\int_\gamma \mathscr{A}\, d\mu - \frac{1}{2}\int_\gamma \langle z, \frac{d}{dt}N\rangle d\mu - \frac{1}{2}\int_\gamma \langle z, N\rangle \frac{d}{dt}d\mu.
\end{aligned}$$

The second integrand yields

$$-\frac{1}{2}\int_\gamma \left\langle z, \frac{d}{dt}N\right\rangle d\mu = \frac{1}{2}\int_\gamma \langle z, T\rangle \frac{\partial}{\partial\mu}\mathscr{A}\, d\mu = -\frac{1}{2}\int_\gamma \big(1 + k\langle z, N\rangle\big)\mathscr{A}\, d\mu.$$

The third finally gives

$$-\frac{1}{2}\int_\gamma \langle z, N\rangle \frac{d}{dt}d\mu = \frac{1}{2}\int_\gamma \langle z, N\rangle \mathscr{A} k d\mu.$$

Which shows the equation.

### Notes for Section A.2

**1.** The equations derived in this paragraph are all either well known facts or direct consequences of such. We have relied mostly on [27] which considers the curve shortening case and on [16], where the general curve shortening problem is discussed.

**2.** The evolution equation of arc-length element, tangent angle, normal angle, curvature, length, and area of the general curve shortening problem can be found in §1.3 of [16]. Lemma is taken from [27], Lemma 3.1.5.

### A.3 The convex case

Throughout this appendix it is assumed that the evolving curves are convex. Thus, it is possible to parametrize the curves with respect to the normal angle. The resulting time derivatives are given.

**A.3.1.** We consider for given $\mathscr{A}, \mathscr{B}\colon \mathbb{R}^2 \times \mathbb{R} \times S^1 \to \mathbb{R}$ the equation

$$\begin{aligned} \frac{d}{dt} z &= \mathscr{A}(z, k, \theta) N + \mathscr{B}(z, k, \theta) T, \\ z(\cdot, 0) &= z_0. \end{aligned} \tag{A.13}$$

In this appendix we make the *big assumption* that $z_0$ is a convex curve, and that $z(\cdot, t)$ is also convex for all $t \in [0, \zeta)$. This has to be checked for each flow individually. Recall that we have introduced $r := |\, z \,|$, $e_r := \frac{z}{r}$, and $N$ is the inward pointing unit normal.

**A.3.2.** As $z$ is convex at all times it is convenient to reparametrize (A.13) with respect to the normal angle. We denote the new variables by $(\vartheta, \tau)$. It holds $\frac{\partial}{\partial \mu} \theta = k$, compare §A.2.4. But we want to parametrize the curve with respect to the normal angle $(\vartheta, \tau)$. Because, $\vartheta = \theta + \frac{\pi}{2}$, we also have $\frac{\partial}{\partial \mu} \vartheta = k$. This yields

$$\begin{aligned} \frac{\partial}{\partial \vartheta} z &= \frac{1}{k} T, \\ \frac{\partial}{\partial \vartheta} r^i &= i \frac{\langle e_r, T \rangle}{r} r^i \frac{1}{k}, \\ \frac{\partial}{\partial \vartheta} \langle e_r, T \rangle &= \langle e_r, N \rangle + \frac{\langle e_r, N \rangle^2}{r} \frac{1}{k}, \\ \frac{\partial}{\partial \vartheta} \langle e_r, N \rangle &= - \langle e_r, T \rangle - \frac{\langle e_r, T \rangle \langle e_r, N \rangle}{r} \frac{1}{k}, \end{aligned}$$

as special cases

$$\begin{aligned} \frac{\partial}{\partial \vartheta} \langle z, T \rangle &= \frac{1}{k} + \langle z, N \rangle, \\ \frac{\partial}{\partial \vartheta} \langle z, N \rangle &= - \langle z, T \rangle, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial \vartheta} \langle e_r, T \rangle r^i &= \langle e_r, N \rangle r^i + i \frac{\langle e_r, T^2 \rangle}{r} \frac{r^i}{k} + \frac{\langle e_r, N \rangle^2}{r} \frac{r^i}{k}, \\ \frac{\partial}{\partial \vartheta} \langle e_r, N \rangle r^i &= - \langle e_r, T \rangle r^i + (i-1) \frac{\langle e_r, T \rangle \langle e_r, N \rangle}{r} \frac{r^i}{k}. \end{aligned}$$

**A.3.3.** Let us recall that for an arbitrary function $f$ it holds

$$
\begin{aligned}
\frac{\partial}{\partial\vartheta}f &= \frac{1}{k}\frac{\partial}{\partial\mu}f,\\
\frac{d}{d\tau}f &= \frac{d}{dt}f - \frac{\partial}{\partial\vartheta}f\frac{d}{dt}\vartheta = \frac{d}{dt}f - k\frac{\partial}{\partial\vartheta}f\Big\{\frac{\partial}{\partial\vartheta}\mathscr{A} + \mathscr{B}\Big\}\\
&= \frac{d}{dt}f - \frac{\partial}{\partial\mu}f\Big\{\frac{\partial}{\partial\vartheta}\mathscr{A} + \mathscr{B}\Big\}.
\end{aligned}
$$

Together with the computations of §A.2 one obtains:

$$
\begin{aligned}
\frac{d}{d\tau}z &= \mathscr{A}N + \mathscr{B} - \Big\{\frac{\partial}{\partial\vartheta}\mathscr{A} + \mathscr{B}\Big\}T = \mathscr{A}N - \Big\{\frac{\partial}{\partial\vartheta}\mathscr{A}\Big\}T\\
\frac{d}{d\tau}r^i &= i\frac{\langle e_r, T\rangle}{r}r^i\mathscr{B} + i\frac{\langle e_r, N\rangle}{r}\beta r^i\mathscr{A} - \Big\{\frac{\partial}{\partial\vartheta}\mathscr{A} + \mathscr{B}\Big\}i\frac{\langle e_r, T\rangle}{r}r^i\\
&= i\frac{\langle e_r, N\rangle}{r}r^i\mathscr{A} - i\frac{\langle e_r, T\rangle}{r}r^i\Big\{\frac{\partial}{\partial\vartheta}\mathscr{A}\Big\}\\
\frac{d}{d\tau}T &= \Big\{\frac{\partial}{\partial\mu}\mathscr{A} + k\mathscr{B}\Big\}N - k\Big\{\frac{\partial}{\partial\vartheta}\mathscr{A} + \mathscr{B}\Big\}N = 0\\
\frac{d}{d\tau}N &= 0
\end{aligned}
$$

This implies:

$$
\begin{aligned}
\frac{d}{d\tau}\langle z, T\rangle &= -\frac{\partial}{\partial\vartheta}\mathscr{A}\\
\frac{d}{d\tau}\langle z, N\rangle &= \mathscr{A}.
\end{aligned}
$$

Finally, let us compute the evolution of the curvature. We have

$$
\begin{aligned}
\frac{d}{d\tau}k &= \frac{\partial^2}{\partial\mu^2}\mathscr{A} + \frac{\partial}{\partial\mu}k\mathscr{B} + k^2\mathscr{A} - \frac{\partial}{\partial\mu}k\Big\{\frac{\partial}{\partial\vartheta}\mathscr{A} + \mathscr{B}\Big\}\\
&= \frac{\partial^2}{\partial\mu^2}\mathscr{A} + k^2\mathscr{A} - \frac{\partial}{\partial\mu}k\Big\{\frac{\partial}{\partial\vartheta}\mathscr{A}\Big\} = k^2\frac{\partial^2}{\partial\vartheta^2}\mathscr{A} + k^2\mathscr{A}.
\end{aligned}
$$

### A.4 No coordinates

**A.4.1.** This paragraph is taken from Ecker's book [17], Appendix A. To that end, let $\Omega \subset R^n$ be an open set. We will consider smooth embeddings $F\colon \Omega \to \mathbb{R}^{n+1}$, where $M := F(\Omega)$ is contained in some open set $U \subset \mathbb{R}^{n+1}$. The *tangent* *space* $T_xM$ at $x = F(p)$ is spanned by the vectors $F_i := \frac{\partial}{\partial p_i}F(p)$, $i \in \{1, \ldots n\}$. The *metric* on $M$ is given by

$$g_{ij} = \langle F_i, F_j \rangle ,$$

where $i, j \in \{1, \ldots n\}$, the *inverse metric* by

$$g^{ij} = \big(g_{ij}\big)^{-1},$$

and the *area element* of $M$ by

$$\sqrt{g} = \sqrt{\det g_{ij}}.$$

The *tangential gradient* of a function $h\colon M \to \mathbb{R}$ is defined by

$$\nabla h = g^{ij}\partial_j h F_i.$$

For a smooth tangent vector field $X = X^i F_i$ we define the *covariant derivative* tensor by

$$\nabla_i X^j = \frac{\partial}{\partial p_i}X^j + \Gamma^j_{ik}X^k = g^{jl}\Big\{\frac{\partial}{\partial p_i}X_l - \Gamma^k_{il}X_k\Big\},$$

where the *Christoffel symbols* are given by

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\Big\{g_{jl,i} + g_{il,j} - g_{ij,l}\Big\}.$$

For a smooth tangential vector field $X\colon M \to \mathbb{R}^{n+1}$, the *tangential divergence* is defined by

$$\operatorname{div} X = \nabla_i X^i = g^{ij}\nabla_i X_j,$$

and the *Laplace-Beltrami* operator of $h\colon M \to \mathbb{R}$ on $M$ is defined by

$$\Delta h = \operatorname{div}\nabla h = g^{ij}\big(\partial_j\partial_i h - \Gamma^k_{ij}\partial_k h\big).$$

**A.4.2.** A planar curve $z\colon S^1 \to \mathbb{C}$ is a one dimensional hypersurface in $\mathbb{R}^2$. Thus, its metric reads $g_{11} = |\,z'\,|^2$, the inverse of the metric is $g^{11} = |\,z'\,|^{-2}$, the area-element equals $g = |\,z'\,| = \mu$, and $\Gamma^1_{11} = \frac{\langle z', z''\rangle}{|\,z'\,|^2}$. Moreover, $F_1 = z'$. Suppose we are given a function $h\colon \gamma \to \mathbb{R}$, then the previous paragraph yields

$$\nabla h = \frac{1}{|\,z'\,|}h'T, \text{ and } \Delta h = \frac{1}{|\,z'\,|^2}\Big\{h'' - \frac{\langle z', z''\rangle}{|\,z'\,|^2}h'\Big\}.$$

This yields $\nabla h = \Big\{\frac{\partial}{\partial \mu}h\Big\}T$, and $\Delta h = \frac{\partial^2}{\partial^2\mu}h$. Hence

$$\nabla r^i = i\frac{\langle e_r, T\rangle}{r}r^i T.$$

The Laplace-Beltrami of $r$ will be derived below.

**A.4.3.** Here we collect some basic rules for covariant derivation for planar curves. Let us note that

$$\begin{aligned} 1 &= |\nabla r|^2 + \langle e_r, N\rangle^2, \\ \nabla \langle e_r, N\rangle &= -\Big\{\frac{\langle e_r, N\rangle}{r} + k\Big\}\nabla r, \\ \nabla \langle z, N\rangle &= -rk\nabla r = -\frac{1}{2}k\nabla r^2. \end{aligned}$$

This yields the laplacian of $r$

$$\begin{aligned} \Delta r &= \Big\{\frac{\langle e_r, N\rangle}{r} + k\Big\}\langle e_r, N\rangle, \\ \Delta r^i &= ir^{i-1}\langle e_r, N\rangle k - i(i-1)r^{i-2}\langle e_r, N\rangle + i(i-1)r^{i-2} + i\langle e_r, N\rangle^2 r^{i-2}. \end{aligned}$$

Let us also recall *Gauß' equation*

$$\Delta z = kN.$$

**A.4.4.** Again we look at the equation

$$\frac{d}{dt}z = \mathscr{A}(z, k, \theta)N, \tag{A.14}$$

where $\mathscr{A}\colon \mathbb{R}^2 \times \mathbb{R} \times S^1 \to \mathbb{R}$ is the *normal speed function* of the flow. Let us recall the following abbreviations $r := |z|$, $e_r := \frac{z}{|z|}$. Let us set $F(p,t) := z(p,t)$. We make use of the following abbreviations $F_i := \frac{\partial}{\partial x_i}F$. Let us recall that the *metric* and the *second fundamental form* are given by

$$g_{ij} := g_{\alpha\beta}F_i^\alpha F_j^\beta, \text{ and } h_{ij} := g_{\alpha\beta}F_{ij}^\alpha N^\beta.$$

Of course in the our case $i, j = 1$, and $\alpha$ and $\beta$ run from 1 to 2. As a start we compute the evolution equation of the metric

$$\frac{d}{dt}g_{ij} = \frac{d}{dt}\Big\{g_{\alpha\beta}F_i^\alpha F_j^\beta\Big\} = 2g_{\alpha\beta}F_i^\alpha \frac{\partial}{\partial x_j}\frac{d}{dt}F^\beta = 2\mathscr{A}g_{\alpha\beta}F_i^\alpha N_j^\beta = -2\mathscr{A}h_{ij}.$$

Moreover,

$$\begin{aligned}
\frac{d}{dt}N &= -\nabla\mathscr{A},\\
\frac{d}{dt}k &= \Delta\mathscr{A} + \mathscr{A}k^2,\\
\frac{d}{dt}d\mu &= -k\mathscr{A}\,d\mu,\\
\frac{d}{dt}\langle z, N\rangle &= \mathscr{A} - \langle z, \nabla\mathscr{A}\rangle,\\
\frac{d}{dt}\langle e_r, N\rangle &= -\langle e_r, \nabla\mathscr{A}\rangle + \left\{\frac{1}{r} - \frac{\langle e_r, N\rangle^2}{r}\right\}\mathscr{A},\\
\frac{d}{dt}r &= \langle e_r, N\rangle\,\mathscr{A}.
\end{aligned}$$

## Notes for Section A.4

The computations follow directly from the hypersurface case, as a curve in $\mathbb{R}^2$ can also be seen as a hypersurface. We refer to [54].

Appendix B

# Calculations for the original flow equation

Here we derive several useful equations for the *equivariant curve flow*: Find $z\colon S^1 \times [0, T_{\text{sing}}) \to \mathbb{C}$ for which

$$\begin{aligned} \frac{d}{dt} z &= \mathscr{F} N := \Big\{ k - (n-1) \frac{\langle e_r, N \rangle}{r} \Big\} N, \\ z(\cdot, 0) &= z_0, \end{aligned} \tag{B.1}$$

where $k$ denotes the curvature of the curve, $N$ the inward pointing unit normal, $e_r := \frac{z}{r}$, $r := |z|$, and $n$ the dimension of $L$.

### B.1 Calculation

In this section we assume that $z_0\colon S^1(\kappa) \to \mathbb{C}$ is tamed, i.e. $\mathscr{F} > 0$, $2\pi\kappa$-periodic function which evolves by equation (1.9):

$$\begin{aligned} \frac{d}{d\tau} z &= \mathscr{F} N - \Big\{ \frac{d}{d\eta} \mathscr{F} + (n-1) \frac{\langle e_r, T \rangle}{r} \Big\} T, \\ z(0, \cdot) &= z_0. \end{aligned}$$

This flow is just a reparametrized version of the Flow (B.1). It has the property that $\tau$ and $\eta$, defined by $\eta := \int_0^a \mathscr{F}\, d\mu$ are independed, as shown in §1.3.2. Our scope is to provide several calculation associated to this flow.

**B.1.1.** Let us recall from §1.3.2 that

$$\frac{\partial}{\partial \mu} \eta = \frac{1}{\mathscr{F}}.$$

With this equation at hand we easily derive the next equations from §A.1

$$\begin{aligned} \frac{\partial}{\partial \eta} z &= \frac{\partial}{\partial \mu} z \frac{\partial \eta}{\partial \mu} = \frac{1}{\mathscr{F}} T, \text{ and} \\ \frac{\partial}{\partial \eta} r^i &= i \frac{1}{\mathscr{F}} \frac{\langle e_r, T \rangle}{r} r^i. \end{aligned}$$

Moreover,

$$\frac{\partial}{\partial\eta}T = \Big\{1 + \frac{n-1}{\mathscr{F}}\frac{\langle e_r, N\rangle}{r}\Big\}N, \text{ and } \frac{\partial}{\partial\eta}N = -\Big\{1 + \frac{n-1}{\mathscr{F}}\frac{\langle e_r, N\rangle}{r}\Big\}T.$$

This yields

$$\begin{aligned}\frac{\partial}{\partial\eta}\langle e_r, T\rangle &= \frac{1}{\mathscr{F}}\frac{1}{r} - \frac{1}{\mathscr{F}}\frac{\langle e_r, T\rangle^2}{r} + \Big\{1 + \frac{n-1}{\mathscr{F}}\frac{\langle e_r, N\rangle}{r}\Big\}\langle e_r, N\rangle \\ &= \langle e_r, N\rangle + \frac{n}{\mathscr{F}}\frac{\langle e_r, N\rangle^2}{r},\end{aligned}$$

and

$$\frac{\partial}{\partial\eta}\langle e_r, N\rangle = -\langle e_r, T\rangle - n\frac{1}{\mathscr{F}}\frac{\langle e_r, T\rangle\langle e_r, N\rangle}{r}.$$

We will also need

$$\begin{aligned}\frac{\partial}{\partial\eta}\langle e_r, T\rangle r^i &= \langle e_r, N\rangle r^i + n\frac{1}{\mathscr{F}}\langle e_r, N\rangle^2 r^{i-1} + i\frac{1}{\mathscr{F}}\langle e_r, T\rangle^2 r^{i-1}, \\ \frac{\partial}{\partial\eta}\langle e_r, N\rangle r^i &= -\langle e_r, T\rangle r^i + (i-n)\frac{1}{\mathscr{F}}\langle e_r, T\rangle\langle e_r, N\rangle r^{i-1},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\eta}\langle e_r, T\rangle\langle e_r, N\rangle r^i &= \big\{\langle e_r, N\rangle^2 - \langle e_r, T\rangle^2\big\}r^i \\ &\quad + \frac{1}{\mathscr{F}}\big\{n\langle e_r, N\rangle^2 + (i-n)\langle e_r, T\rangle^2\big\}\langle e_r, N\rangle r^{i-1}.\end{aligned}$$

**B.1.2.** We rely on Section A.2 for the next computations. Let us recall that $\frac{\partial}{\partial\mu} = \mathscr{F}\frac{\partial}{\partial\eta}$. Note further that $\eta$ and $\tau$ are independent parameters. We have

$$\begin{aligned}\frac{d}{d\tau}r^i &= i\frac{\langle e_r, N\rangle}{r}r^i\mathscr{F} - i\Big\{\frac{d}{d\eta}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}\frac{\langle e_r, T\rangle}{r}r^i \\ &= -i\frac{\langle e_r, T\rangle}{r}r^i\frac{d}{d\eta}\mathscr{F} + i\frac{\langle e_r, N\rangle}{r}r^i\mathscr{F} - i(n-1)\frac{\langle e_r, T\rangle^2}{r^2}r^i.\end{aligned}$$

It holds

$$\begin{aligned}\frac{d}{d\tau}T &= \Big\{\mathscr{F}\frac{d}{d\eta}\mathscr{F} - k\frac{d}{d\eta}\mathscr{F} - (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}N \\ &= -\Big\{(n-1)\frac{\langle e_r, N\rangle}{r}\frac{d}{d\eta}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}N.\end{aligned}$$

and analogously

$$\frac{d}{d\tau}N = \Big\{(n-1)\frac{\langle e_r, N\rangle}{r}\frac{d}{d\eta}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}T.$$

Moreover,

$$\begin{aligned}
\frac{d}{d\tau}\langle e_r, T\rangle &= \langle e_r, N\rangle \mathscr{F}\frac{d}{d\eta}\mathscr{F} - \frac{\langle e_r, T\rangle\langle e_r, N\rangle}{r}\mathscr{F} \\
&\qquad - \Big\{\frac{d}{d\eta}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}\langle e_r, N\rangle k \\
&\qquad - \Big\{\frac{d}{d\eta}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}\frac{\langle e_r, N\rangle^2}{r} \\
&= \langle e_r, N\rangle \mathscr{F}\frac{d}{d\eta}\mathscr{F} - \frac{\langle e_r, T\rangle\langle e_r, N\rangle}{r}\mathscr{F} \\
&\qquad - \Big\{\frac{d}{d\eta}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}\langle e_r, N\rangle \mathscr{F} \\
&\qquad - (n-1)\Big\{\frac{d}{d\eta}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}\frac{\langle e_r, N\rangle^2}{r} \\
&\qquad - \Big\{\frac{d}{d\eta}\mathscr{F} + (n-1)\frac{\langle e_r, T\rangle}{r}\Big\}\frac{\langle e_r, N\rangle^2}{r} \\
&= -n\frac{\langle e_r, N\rangle^2}{r}\frac{d}{d\eta}\mathscr{F} - n\frac{\langle e_r, T\rangle\langle e_r, N\rangle}{r}\mathscr{F} \\
&\qquad - n(n-1)\frac{\langle e_r, T\rangle\langle e_r, N\rangle^2}{r^2}.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
\frac{d}{d\tau}\langle e_r, N\rangle &= n\frac{\langle e_r, T\rangle\langle e_r, N\rangle}{r}\frac{d}{d\eta}\mathscr{F} + n\frac{\langle e_r, T\rangle^2}{r}\mathscr{F} \\
&\qquad + n(n-1)\frac{\langle e_r, T\rangle^2\langle e_r, N\rangle}{r^2}.
\end{aligned}$$

This gives

$$\begin{aligned}
\frac{d}{d\tau}\langle e_r, T\rangle r^i &= -\Big\{n\langle e_r, N\rangle^2 + i\langle e_r, T\rangle^2\Big\}r^{i-1}\frac{d}{d\eta}\mathscr{F} \\
&\qquad + (i-n)\langle e_r, T\rangle\langle e_r, N\rangle r^{i-1}\mathscr{F} \\
&\qquad - (n-1)\Big\{n\langle e_r, N\rangle^2 + i\langle e_r, T\rangle\Big\}\langle e_r, T\rangle r^{i-2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{d\tau}\langle e_r, N\rangle r^i &= (n-i)\frac{\langle e_r, T\rangle\langle e_r, N\rangle}{r^{1-i}}\frac{d}{d\eta}\mathscr{F} + \frac{n\langle e_r, T\rangle^2 + i\langle e_r, N\rangle^2}{r^{1-i}}\mathscr{F} \\
&\qquad + (n-i)(n-1)\langle e_r, T\rangle^2\langle e_r, N\rangle r^{i-2}.
\end{aligned}$$

## B.2 Stationary solutions

Here we identify those curves which are stationary under the flow equation (1.9). Thus which satisfy $\mathscr{F} = 0$. We claimed in 1.3.3 that: *Let* $\eta_0 \in [0, 2\pi]$, *and* $h \in \mathbb{R}\backslash\{0\}$. *The implicitly defined curve given by*

$$f(x,y) := \Re z^n \cos\eta_0 + \Im z^n \sin\eta_0 - h = 0,$$

*is a stationary solution of equation* (1.9), *where* $z = x + \imath y$ *and the motion of the curve is taken in direction of the vector* $(f_y, -f_x)^T$. This claim will be justified in this section.

**B.2.1.** Before we prove the assertion let us recall some basic equations which will be frequently used. We have

$$\big(\Re z^n\big)^2 + \big(\Im z^n\big)^2 = r^{2n}$$

and

$$\begin{aligned} z^n &= [x\Re z^{n-1} - y\Im z^{n-1}] + \imath[y\Re z^{n-1} + x\Im z^{n-1}], \\ \imath z^n &= -[y\Re z^{n-1} + x\Im z^{n-1}] + \imath[x\Re z^{n-1} - y\Im z^{n-1}]. \end{aligned}$$

Which imply

$$\Re z^n = \Im \imath z^n \text{ and } \Im z^n = -\Re \imath z^n.$$

The normal vector at a point $(x, y)$ of an implicitly defined curve is given by

$$N = \frac{1}{\sqrt{f_x^2 + f_y^2}} \begin{pmatrix} f_x \\ f_y \end{pmatrix}.$$

Let us note that

$$f_x = n\big(\Re z^{n-1}\cos\eta + \Im z^{n-1}\sin\eta\big)$$

and

$$\begin{aligned} f_y &= n\big(\Re \imath z^{n-1}\cos\eta + \Im \imath z^{n-1}\sin\eta\big) \\ &= n\big(-\Im z^{n-1}\cos\eta + \Re z^{n-1}\sin\eta\big). \end{aligned}$$

Observe that

$$\begin{aligned} xf_x + yf_y &= n\Big(\big(x\Re z^{n-1} - y\Im z^{n-1}\big)\cos\eta + \big(y\Re z^{n-1} + x\Im z^{n-1}\big)\sin\eta\Big) \\ &= n\big(\Re z^n\cos\eta + \Im z^n \sin\eta\big) \\ &= nh, \end{aligned}$$

and

$$\begin{aligned} f_x^2 + f_y^2 &= n^2\Big[\big(\Re z^{n-1}\big)^2 + \big(\Im z^{n-1}\big)^2\Big] \\ &= n^2\,|\,z\,|^{\,2n-2}. \end{aligned}$$

This yields

$$(n-1)\frac{\langle z, N\rangle}{|\,z\,|^{\,2}} = \frac{(n-1)}{|\,z\,|^{\,2}\sqrt{f_x^2+f_y^2}}\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} f_x \\ f_y \end{pmatrix} \right\rangle = \frac{(n-1)h}{|\,z\,|^{\,n+1}}.$$

Let us compute the curvature of the curve. It is given by

$$k = -\frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{(f_x^2+f_y^2)^{1.5}}.$$

The minus sign is due to the motion of the curve. We observe:

$$\begin{aligned} f_{xx} &= n(n-1)\big(\Re z^{n-2}\cos\eta + \Im z^{n-2}\sin\eta\big) \\ f_{yy} &= -n(n-1)\big(\Re z^{n-2}\cos\eta + \Im z^{n-2}\sin\eta\big) \\ f_{xy} &= n(n-1)\big(-\Im z^{n-2}\cos\eta + \Re z^{n-2}\sin\eta\big) \end{aligned}$$

Let us abbreviate

$$\Re := \Re z^{n-2} \text{ and } \Im := \Im z^{n-2}.$$

Moreover, we set $\mathrm{c} := \cos\eta$ and $\mathrm{s} := \sin\eta$. We have

$$\begin{aligned} & f_y^2 f_{xx} - f_x f_y f_{xy} \\ &= n^3(n-1)\big[\big((-y\Re - x\Im)\,\mathrm{c} + (x\Re - y\Im)\,\mathrm{s}\big)^2\big(\Re\,\mathrm{c} + \Im\,\mathrm{s}\big)\big] \\ &\quad - \big((x\Re - y\Im)\,\mathrm{c} + (y\Re + x\Im)\,\mathrm{s}\big)\big((-y\Re - x\Im)\,\mathrm{c} + (x\Re - y\Im)\,\mathrm{s}\big)\big(-\Im\,\mathrm{c} + \Re\,\mathrm{s}\big)\big] \\ \\ &= n^3(n-1)\big[\,\mathrm{c}^3\,\big((y\Re + x\Im)^2\Re + (x\Re - y\Im)(-y\Re - x\Im)\Im\big) \\ &\quad + \mathrm{c}^2\,\mathrm{s}\,\big((y\Re + x\Im)^2\Im - 2(y\Re + x\Im)(x\Re - y\Im)\Re + (x\Re - y\Im)(y\Re + x\Im)\Re \\ &\quad + (x\Re - y\Im)^2\Im - (y\Re + x\Im)^2\Im\big) \\ &\quad + \mathrm{c}\,\mathrm{s}^2\,\big((x\Re - y\Im)^2\Re - 2(y\Re + x\Im)(x\Re - y\Im)\Im + (y\Re + x\Im)(x\Re - y\Im)\Im \\ &\quad - (x\Re - y\Im)^2\Re + (y\Re + x\Im)^2\Re\big) \\ &\quad + \mathrm{s}^3\,\big((x\Re - y\Im)^2\Im - (y\Re + x\Im)(x\Re - y\Im)\Re\big)\big] \\ \\ &= n^3(n-1)\big[\,\mathrm{c}^3\,\big((y\Re + x\Im)^2\Re - (x\Re - y\Im)(y\Re + x\Im)\Im\big) \\ &\quad + \mathrm{c}^2\,\mathrm{s}\,\big((x\Re - y\Im)^2\Im - (y\Re + x\Im)(x\Re - y\Im)\Re\big) \\ &\quad + \mathrm{c}\,\mathrm{s}^2\,\big((y\Re + x\Im)^2\Re - (y\Re + x\Im)(x\Re - y\Im)\Im\big) \\ &\quad + \mathrm{s}^3\,\big((x\Re - y\Im)^2\Im - (y\Re + x\Im)(x\Re - y\Im)\Re\big)\big] \end{aligned}$$

$$= n^3(n-1)\big[\,\mathrm{c}\,\big((y\Re + x\Im)^2\Re - (x\Re - y\Im)(y\Re + x\Im)\Im\big)$$
$$+\,\mathrm{s}\,\big((x\Re - y\Im)^2\Im - (y\Re + x\Im)(x\Re - y\Im)\Re\big)\big]$$

$$= n^3(n-1)\big[\,\mathrm{c}(y\Re + x\Im)\big((y\Re + x\Im)\Re - (x\Re - y\Im)\Im\big)$$
$$+\,\mathrm{s}(x\Re - y\Im)\big((x\Re - y\Im)\Im - (y\Re + x\Im)\Re\big)\big]$$

$$= n^3(n-1)(\Re^2 + \Im^2)\big[\,\mathrm{c}(y\Re + x\Im) + \mathrm{s}(-x\Re + y\Im)\big]y.$$

Analogously we obtain

$$f_x^2 f_{yy} - f_x f_y f_{xy}$$
$$= -n^3(n-1)n^3(n-1)(\Re^2 + \Im^2)\big[\,\mathrm{c}(x\Re - y\Im) + \mathrm{s}(y\Re + x\Im)\big]x.$$

This yields

$$f_x^2 f_{yy} - 2f_x f_y f_{xy} + f_y^2 f_{xx}$$
$$= -n^3(n-1)(\Re^2 + \Im^2)\big(((x^2 - y^2)\Re - 2xy\Im)\,\mathrm{c} + (2xy\Re + (x^2 - y^2)\Im)\,\mathrm{s}\,\big).$$

Let us note that

$$(x^2 - y^2)\Re z^{n-2} - 2xy\Im z^{n-2} = \Re z^n$$

and

$$2xy\Re z^{n-2} + (x^2 - y^2)\Im z^{n-2} = \Im z^n.$$

Altogether

$$k = -\frac{n^3(n-1)((\Re z^{n-2})^2 + (\Im z^{n-2})^2)(\Re z^n \cos\eta + \Im z^n \sin\eta)}{(n^2\,|\,z\,|^{\,2n-2})^{1.5}}$$
$$= -\frac{(n-1)\,|\,z\,|^{\,2n-4}h}{|\,z\,|^{\,3n-3}}$$
$$= -\frac{(n-1)h}{|\,z\,|^{\,n+1}}.$$

This proves the claim.

*Remark.* Let us point out that

$$N = \frac{1}{|\,z\,|^{\,n-1}}\begin{pmatrix} \Re z^{n-1}\cos\eta + \Im z^{n-1}\sin\eta \\ -\Im z^{n-1}\cos\eta + \Re z^{n-1}\sin\eta \end{pmatrix} \text{ and}$$
$$T = \frac{1}{|\,z\,|^{\,n-1}}\begin{pmatrix} -\Im z^{n-1}\cos\eta + \Re z^{n-1}\sin\eta \\ -\Re z^{n-1}\cos\eta - \Im z^{n-1}\sin\eta \end{pmatrix}.$$

The notation is a bit sloppy as $N_h$ depends on $z$ and $\eta$.

Appendix C

# Calculation for the rescaled flow equation

Here we derive several equations for the rescaled flow equation

$$\begin{aligned} \frac{d}{dt}z &= \Big\{k-(n-1)\frac{\langle e_p, N\rangle}{r_p}+\langle z, N\rangle\Big\}N, \\ z(\cdot,0) &= \sqrt{\frac{\pi}{A_0}}\big\{z_0-p_0\big\}. \end{aligned} \tag{C.1}$$

Where $e_p := \frac{z+p}{r_p}$, $r_p := |z+p|$, and $p=\sqrt{\frac{\pi}{A_0}}\exp(t)p_0$. We also introduce

$$\mathscr{H} = \Big\{k-(n-1)\frac{\langle e_p, N\rangle}{r_p}+\langle z, N\rangle\Big\}.$$

## C.1 The rescaled equation

**C.1.1.** Let us start with the claim $1=\langle e_p, T\rangle^2+\langle e_p, N\rangle^2$. Indeed,

$$\begin{aligned} r_p^2 &= \langle z+p, z+p\rangle \\ &= z^2+p^2+2\langle z,N\rangle\langle p,N\rangle+2\langle z,T\rangle\langle p,T\rangle \\ &= \langle z+p,T\rangle^2+\langle z+p,N\rangle^2. \end{aligned}$$

Alternatively,

$$r_p^2 = \langle z+p,N\rangle\langle z,N\rangle+\langle z,p\rangle+p^2+\langle z+p,T\rangle\langle z,T\rangle.$$

Furthermore, let us recall $\mathscr{G} := k-(n-1)\frac{\langle e_p,N\rangle}{r_p}$, and $\mathscr{H} := \mathscr{G}+\langle z,N\rangle$. We have

$$\begin{aligned} \frac{\partial}{\partial\mu}r_p^i &= i\frac{\langle e_p,T\rangle}{r_p}r_p^i, \\ \frac{\partial}{\partial\mu}\langle e_p,T\rangle &= k\langle e_p,N\rangle-\frac{\langle e_p,T\rangle^2}{r_p}+\frac{1}{r_p} = \langle e_p,N\rangle\mathscr{G}+n\frac{\langle e_p,N\rangle^2}{r_p}, \\ \frac{\partial}{\partial\mu}\langle e_p,N\rangle &= -\langle e_p,T\rangle k-\frac{\langle e_p,T\rangle\langle e_p,N\rangle}{r_p} = -\langle e_p,T\rangle\mathscr{G}-n\frac{\langle e_p,T\rangle\langle e_p,N\rangle}{r_p}. \end{aligned}$$

Moreover,

$$\frac{\partial}{\partial\mu}\langle e_p, T\rangle r_p^i = \langle e_p, N\rangle r_p^i \mathscr{G} + \Big\{ i\frac{\langle e_p, T\rangle^2}{r_p} + n\frac{\langle e_p, N\rangle^2}{r_p}\Big\} r_p^i,$$
$$\frac{\partial}{\partial\mu}\langle e_p, N\rangle r_p^i = -\langle e_p, T\rangle r_p^i \mathscr{G} + (i-n)\frac{\langle e_p, T\rangle \langle e_p, N\rangle}{r_p} r_p^i.$$

This list should be compared with §A.1.1

**C.1.2.** Here we derive several time derivatives. We refer to Appendix A.4. Recall that $\mathscr{H} = \mathscr{G} + \langle z, N\rangle$. First of all we need

$$\frac{d}{dt} r_p = \langle e_p, N\rangle \mathscr{H} + \langle e_p, p\rangle,$$
$$\frac{d}{dt}\langle e_p, N\rangle = -\langle e_p, \nabla\mathscr{H}\rangle + \Big\{\frac{1}{r_p} - \frac{\langle e_p, N\rangle^2}{r_p}\Big\}\mathscr{H} - \frac{\langle e_p, N\rangle\langle e_p, p\rangle}{r_p} + \frac{\langle p, N\rangle}{r_p},$$
$$\frac{d}{dt} k = \Delta\mathscr{H} + k^2\mathscr{H}.$$

Let us now replace $\mathscr{H}$ by $\mathscr{G}$. Then

$$\frac{d}{dt} r_p = \langle e_p, N\rangle \mathscr{G} + \langle e_p, N\rangle\langle z, N\rangle + \langle e_p, p\rangle,$$
$$\begin{aligned}\frac{d}{dt}\langle e_p, N\rangle &= -\langle e_p, \nabla\mathscr{G}\rangle + \Big\{\frac{1}{r_p} - \frac{\langle e_p, N\rangle^2}{r_p}\Big\}\mathscr{H} - \frac{\langle e_p, N\rangle\langle e_p, p\rangle}{r_p} + \frac{\langle p, N\rangle}{r_p} \\ &\quad + r\langle e_p, \nabla r\rangle \mathscr{G} + (n-1)\frac{r\langle e_p, N\rangle\langle e_p, \nabla r\rangle}{r_p} \\ &= -\langle e_p, \nabla\mathscr{G}\rangle + \Big\{\frac{1}{r_p} - \frac{\langle e_p, N\rangle^2}{r_p} + r\langle e_p, \nabla r\rangle\Big\}\mathscr{G} + n\frac{r\langle e_p, N\rangle\langle e_p, \nabla r\rangle}{r_p}.\end{aligned}$$

Here we made use of the fact that

$$\begin{aligned}&\langle z, N\rangle\langle e_p, N\rangle + \langle e_p, p\rangle + \langle z, T\rangle\langle e_p, T\rangle \\ &\quad = \langle z, N\rangle\langle e_p, N\rangle + \langle e_p, N\rangle\langle p, N\rangle + \langle e_p, T\rangle\langle p, T\rangle + \langle z, T\rangle\langle e_p, T\rangle = r_p.\end{aligned}$$

**C.1.3.** Here we basically derive the same equations as in previous paragraph. But this time we make use of $\frac{\partial}{\partial\mu}$. We start with

$$\frac{d}{dt} T = \frac{\partial}{\partial\mu}\mathscr{H} N, \text{ and } \frac{d}{dt} N = -\frac{\partial}{\partial\mu}\mathscr{H} T.$$

This gives

$$\frac{d}{dt} T = \Big\{\frac{\partial}{\partial\mu}\mathscr{G} - \langle z, T\rangle \mathscr{G} - (n-1)\frac{\langle e_p, N\rangle\langle z, T\rangle}{r_p}\Big\} N, \text{ and}$$
$$\frac{d}{dt} N = \Big\{-\frac{\partial}{\partial\mu}\mathscr{G} + \langle z, T\rangle \mathscr{G} + (n-1)\frac{\langle e_p, N\rangle\langle z, T\rangle}{r_p}\Big\} T.$$

$$\frac{d}{dt}r_p = \langle e_p, N\rangle \mathscr{G} + \langle e_p, N\rangle \langle z, N\rangle + \langle e_p, p\rangle,$$

$$\frac{d}{dt}\langle e_p, N\rangle = -\langle e_p, T\rangle \frac{\partial}{\partial\mu}\mathscr{G} + \Big\{\frac{\langle e_p, T\rangle^2}{r_p} + \langle e_p, T\rangle \langle z, T\rangle\Big\}\mathscr{G} + n\frac{\langle e_p, T\rangle \langle e_p, N\rangle \langle z, T\rangle}{r_p}.$$

We also compute

$$\begin{aligned}\frac{d}{dt}\langle e_p, T\rangle &= \langle e_p, N\rangle \frac{\partial}{\partial\mu}\mathscr{G} - \Big\{\langle e_p, N\rangle \langle z, T\rangle + \frac{\langle e_p, T\rangle \langle e_p, N\rangle}{r_p}\Big\}\mathscr{G}\\ &\quad - (n-1)\frac{\langle e_p, N\rangle^2 \langle z, T\rangle}{r_p} + \frac{\langle p, T\rangle}{r_p}\\ &\quad - \frac{\langle e_p, T\rangle \langle e_p, N\rangle \langle z, N\rangle}{r_p} - \frac{\langle e_p, T\rangle \langle e_p, p\rangle}{r_p}\\ &= \langle e_p, N\rangle \frac{\partial}{\partial\mu}\mathscr{G} - \Big\{\langle z, T\rangle \langle e_p, N\rangle + \frac{\langle e_p, T\rangle \langle e_p, N\rangle}{r_p}\Big\}\mathscr{G}\\ &\quad - n\frac{\langle e_p, N\rangle^2 \langle z, T\rangle}{r_p}.\end{aligned}$$

Another way to see this is to note that

$$D\langle e_p, T\rangle = D\sqrt{1-\langle e_p, N\rangle^2} = -\frac{\langle e_p, N\rangle}{\langle e_p, T\rangle}D\langle e_p, N\rangle.$$

Here $D$ denotes any differential operator.

**C.1.4.** Here we provide several calculation for the rescaled flow equation. That is we assume that $z_0\colon S^1(\kappa) \to \mathbb{C}$ is tamed, i.e. $\mathscr{G} > 0$ and evolves by

$$\frac{d}{d\tau}z = \Big\{\mathscr{G} + \langle z, N\rangle\Big\}N - \Big\{\frac{d}{d\eta}\mathscr{G} + (n-1)\frac{\langle e_p, T\rangle}{r_p} - \langle z, T\rangle\Big\}T,$$

$$z(0,\cdot) = z_0.$$

This flow has the property that $\tau$ and $\eta$, defined by $\eta := \int_0^a \mathscr{G}\, d\mu$ are independed, as shown in §3.1.5. To begin with let us note that $\frac{\partial}{\partial\mu} = \mathscr{G}\frac{\partial}{\partial\eta}$. The previous Paragraph B.1.1 gives

$$\frac{\partial}{\partial\eta}r_p^i = i\frac{\langle e_p, T\rangle}{r_p}\frac{r_p^i}{\mathscr{G}},$$

$$\frac{\partial}{\partial\eta}\langle e_p, T\rangle = \langle e_p, N\rangle + n\frac{\langle e_p, N\rangle^2}{r_p}\frac{1}{\mathscr{G}},$$

$$\frac{\partial}{\partial\eta}\langle e_p, N\rangle = -\langle e_p, T\rangle - n\frac{\langle e_p, T\rangle \langle e_p, N\rangle}{r_p}\frac{1}{\mathscr{G}},$$

and

$$\frac{\partial}{\partial\eta}\langle e_p,T\rangle r_p^i=\langle e_p,N\rangle r_p^i+\left\{i\frac{\langle e_p,T\rangle^2}{r_p}+n\frac{\langle e_p,N\rangle^2}{r_p}\right\}\frac{r_p^i}{\mathscr{G}},$$
$$\frac{\partial}{\partial\eta}\langle e_p,N\rangle r_p^i=-\langle e_p,T\rangle r_p^i+(i-n)\frac{\langle e_p,T\rangle\langle e_p,N\rangle}{r_p}\frac{r_p^i}{\mathscr{G}}.$$

Moreover, by Paragraph A.1.1 we have

$$\frac{\partial}{\partial\eta}r^i=i\frac{\langle e_r,T\rangle}{r}\frac{r^i}{\mathscr{G}},$$
$$\frac{\partial}{\partial\eta}\langle e_r,T\rangle=\langle e_r,N\rangle+(n-1)\frac{\langle e_p,N\rangle}{r_p}\frac{\langle e_r,N\rangle}{\mathscr{G}}+\frac{\langle e_r,N\rangle^2}{r}\frac{1}{\mathscr{G}},$$
$$\frac{\partial}{\partial\eta}\langle e_r,N\rangle=-\langle e_r,T\rangle-(n-1)\frac{\langle e_p,N\rangle}{r_p}\frac{\langle e_r,T\rangle}{\mathscr{G}}-\frac{\langle e_r,T\rangle\langle e_r,N\rangle}{r}\frac{1}{\mathscr{G}}.$$

From which we deduce

$$\begin{aligned}&\frac{\partial}{\partial\eta}\langle e_r,T\rangle r^i\\&\quad=\langle e_r,N\rangle r^i+(n-1)\frac{\langle e_p,N\rangle}{r_p}\frac{\langle e_r,N\rangle r^i}{\mathscr{G}}+\frac{\langle e_r,N\rangle^2}{r}\frac{r^i}{\mathscr{G}}+i\frac{\langle e_r,T\rangle^2}{r}\frac{r^i}{\mathscr{G}},\\&\frac{\partial}{\partial\eta}\langle e_r,N\rangle r^i\\&\quad=-\langle e_r,T\rangle r^i-(n-1)\frac{\langle e_p,N\rangle}{r_p}\langle e_r,T\rangle\frac{r^i}{\mathscr{G}}+(i-1)\frac{\langle e_r,T\rangle\langle e_r,N\rangle}{r}\frac{r^i}{\mathscr{G}}.\end{aligned}$$

Finally we compute

$$\begin{aligned}&\frac{\partial}{\partial\eta}\langle e_r,T\rangle^j\langle e_r,N\rangle^l r^i\\&\quad=\left\{\left\{j\langle e_r,N\rangle^2-l\langle e_r,T\rangle^2\right\}+(n-1)\left\{j\langle e_r,N\rangle^2-l\langle e_r,T\rangle^2\right\}\frac{\langle e_p,N\rangle}{r_p}\frac{1}{\mathscr{G}}\right.\\&\qquad\left.+\left\{j\langle e_r,N\rangle^2+(i-l)\langle e_r,T\rangle^2\right\}\frac{\langle e_r,N\rangle}{r}\frac{1}{\mathscr{G}}\right\}r^i\langle e_r,T\rangle^{j-1}\langle e_r,N\rangle^{l-1}.\end{aligned}$$

**C.1.5.** We continue the observation of the last paragraph. We rely on §C.1.3. In principle there are at least two ways to derive the time derivatives in new coordinates $(\eta,\tau)$. We make use of the following facts

$$\frac{\partial}{\partial\tau}f=\frac{d}{dt}f-\frac{\partial}{\partial\eta}f\frac{\partial\eta}{\partial t},$$

and

$$\frac{\partial}{\partial t}\eta = \Big\{\frac{\partial}{\partial \eta}\mathscr{G} + (n-1)\frac{\langle e_p, T\rangle}{r_p} - \langle z, T\rangle \Big\}\mathscr{G}.$$

Note that $\frac{\partial}{\partial \mu} = \mathscr{G}\frac{\partial}{\partial \eta}$. This gives

$$\begin{aligned}\frac{d}{d\tau}r_p = &-\langle e_p, T\rangle \frac{d}{d\eta}\mathscr{G} + \langle e_p, N\rangle \mathscr{G} \\ &\qquad + \langle e_p, N\rangle \langle z, N\rangle + \langle e_p, p\rangle + \langle e_p, T\rangle \langle z, T\rangle - (n-1)\frac{\langle e_p, T\rangle^2}{r_p} \\ = &-\langle e_p, T\rangle \frac{d}{d\eta}\mathscr{G} + \langle e_p, N\rangle \mathscr{G} + r_p - (n-1)\frac{\langle e_p, T\rangle^2}{r_p}.\end{aligned}$$

Here we made use of the fact that

$$\begin{aligned}&\langle z, N\rangle \langle e_p, N\rangle + \langle e_p, p\rangle + \langle z, T\rangle \langle e_p, T\rangle \\ &\quad = \langle z, N\rangle \langle e_p, N\rangle + \langle e_p, N\rangle \langle p, N\rangle + \langle e_p, T\rangle \langle p, T\rangle + \langle z, T\rangle \langle e_p, T\rangle = r_p.\end{aligned}$$

We easily obtain

$$\frac{d}{d\tau}r_p^i = -i\frac{\langle e_p, T\rangle}{r_p}r_p^i\frac{d}{d\eta}\mathscr{G} + i\frac{\langle e_p, N\rangle}{r_p}r_p^i\mathscr{G} + ir_p^i - i(n-1)\frac{\langle e_p, T\rangle^2}{r_p^2}r_p^i.$$

Moreover,

$$\begin{aligned}\frac{d}{d\tau}e_p = &\Big\{\frac{1}{r_p}\mathscr{G} + \frac{\langle z, N\rangle}{r_p}\Big\}N - \Big\{\frac{1}{r_p}\frac{d}{d\eta}\mathscr{G} + (n-1)\frac{\langle e_p, T\rangle}{r_p^2} - \frac{\langle z, T\rangle}{r_p}\Big\}T + \frac{p}{r_p} \\ &\qquad + \frac{\langle e_p, T\rangle}{r_p}e_p\frac{d}{d\eta}\mathscr{G} - \frac{\langle e_p, N\rangle}{r_p}e_p\mathscr{G} - e_p + (n-1)\frac{\langle e_p, T\rangle^2}{r_p^2}e_p.\end{aligned}$$

It holds by §A.2.2 that

$$\frac{d}{d\tau}N = (n-1)\Big\{\frac{\langle e_p, N\rangle}{r_p}\frac{d}{d\eta}\mathscr{G} + \frac{\langle e_p, T\rangle}{r_p}\mathscr{G} + (n-1)\frac{\langle e_p, T\rangle \langle e_p, N\rangle}{r_p^2}\Big\}T.$$

This gives

$$\frac{d}{d\tau}\langle e_p, N\rangle = n\frac{\langle e_p, T\rangle \langle e_p, N\rangle}{r_p}\frac{d}{d\eta}\mathscr{G} + n\frac{\langle e_p, T\rangle^2}{r_p}\mathscr{G} + n(n-1)\frac{\langle e_p, T\rangle^2 \langle e_p, N\rangle}{r_p^2}.$$

**C.1.6.** Here we provide further time derivatives for the reparametrized rescaled flow equation

$$\frac{d}{d\tau}z = \Big\{\mathscr{G} + \langle z, N\rangle \Big\}N - \Big\{\frac{d}{d\eta}\mathscr{G} + (n-1)\frac{\langle e_p, T\rangle}{r_p} - \langle z, T\rangle \Big\}T,$$
$$z(0,\cdot) = z_0.$$

It holds

$$\frac{d}{d\tau}r^i = -i\frac{\langle e_r, T\rangle}{r}r^i\frac{d}{d\eta}\mathscr{G} + i\frac{\langle e_r, N\rangle}{r}r^i\mathscr{G} + ir^i - i(n-1)\frac{\langle e_r, T\rangle}{r}\frac{\langle e_p, T\rangle}{r_p}r^i,$$

and

$$\frac{d}{d\tau}\mu = \Big\{ -\mathscr{G}^2 - (n-1)\frac{\langle e_p, N\rangle}{r_p}\mathscr{G} - \langle z, N\rangle \mathscr{G} - (n-1)\frac{\langle z, N\rangle \langle e_p, N\rangle}{r_p}$$
$$-\mathscr{G}\frac{d^2}{d\eta^2}\mathscr{G} - (n-1)\mathscr{G}\frac{d}{d\eta}\frac{\langle e_p, T\rangle}{r_p} + \mathscr{G}\frac{d}{d\eta}\langle z, T\rangle \Big\}d\mu.$$

Let us recall that

$$\frac{\partial}{\partial\eta}\frac{\langle e_p, T\rangle}{r_p} = \frac{\langle e_p, N\rangle}{r_p} + \Big\{n\frac{\langle e_p, N\rangle^2}{r_p^2} - \frac{\langle e_p, T\rangle^2}{r_p^2}\Big\}\frac{1}{\mathscr{G}},$$

and

$$\frac{\partial}{\partial\eta}\langle z, T\rangle = \langle e_r, N\rangle r + (n-1)\frac{\langle e_p, N\rangle}{r_p}\frac{\langle z, N\rangle}{\mathscr{G}} + \frac{1}{\mathscr{G}}.$$

Finally,

$$\frac{d}{d\tau}\mu = \Big\{ -\mathscr{G}\frac{d^2}{d\eta^2}\mathscr{G} - \mathscr{G}^2 - 2(n-1)\frac{\langle e_p, N\rangle}{r_p}\mathscr{G}$$
$$+ (n-1)\Big\{\frac{\langle e_p, T\rangle^2}{r_p^2} - n\frac{\langle e_p, N\rangle^2}{r_p^2}\Big\} + 1\Big\}\mu.$$

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