# Aspects of Solitons in Noncommutative Field Theories - The Modified Ward Model -

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*To My Beloved Family*

## Abstract

Over the last nearly 20 years noncommutative geometry has been a lively area of research in theoretical physics as well as mathematics and has provided some very interesting results especially for high energy physics. Among these is the existence of new classes of soliton solutions in field theories whose properties remain subject of intensive study.

In this thesis several aspects of solutions to the equations of motions to noncommutative field theories are investigated in detail. The main focus of the analysis will be on the integrable chiral or modified unitary sigma model with $U(n)$-valued fields as introduced by Ward and its noncommutative extension where the above mentioned new solutions arise. Of particular interest in this context are to us the question of stability of static solitons and the applicability of the so-called adiabatic approach to as a means to approximate time-dependent solutions by geodesic motion in the moduli space of static solutions.

After some introductory remarks on noncommutativity and its role in physics we begin in the second chapter by reviewing the basic ingredients of noncommutative geometry as needed for our analysis. Having set the stage, we then proceed to present the Ward model together with its noncommutative extension and give a unified exposition of its known static solutions. This model, as the prime example of an almost Lorentz-invariant field theory in 1+2 dimensions, has several virtues which make its analysis worthwhile. First of all it is integrable thus allowing for powerful, well developed, techniques to generate soliton solutions. At the same time these feature interaction among them. Furthermore, the commutative counterpart of the Ward model has been investigated in great detail such that many results are available for comparison.

Next, the question of stability for the presented static solutions is considered. This stability is governed by the quadratic form of the fluctuations, which, upon concentrating on the case of diagonal $U(1)$ solutions, is explicitly computed. We show that the considered solutions are stable within a certain subsector of possible configurations, namely the grassmannian ones, and become unstable upon embedding them into the full unitary sigma model. Finally, we remark on some possible generalization of these results.

This subject is followed, after a brief review of time-dependent Ward model solitons, by the application of the adiabatic approach, as proposed by Manton, to the static solutions. Our analysis shows that this approximation yields the same results for a wider class of noncommutative field theories as the terms in the action constituting the difference between these have no effect on the moduli space dynamics. Since exact time-dependent solutions to the Ward model are known, we are then able to compare the results of the moduli space approximation employed with the exact time behavior in the $U(1)$ case, showing a bad matching of these two. We exemplify these discrepancies with some simple multi-soliton examples and give possible reasons for them.

Even though we believe that the work presented in this thesis sheds some light on the issue of noncommutative solitons, there remains a host of unsolved questions whose investigation seems rewarding. We try to summarize the, in our opinion, most urgent ones at the end of the chapters on the stability analysis and on the adiabatic approach.

Keywords: Noncommutative Solitons, Stability Analysis, Adiabatic Approximation

## Zusammenfassung

In den letzten ungefähr 20 Jahren hat sich die nichtkommutative Geometrie zu einem aktiven Forschungsgebiet sowohl der Mathematik als auch der Physik entwickelt und insbesondere in der Hochenergiephysik zu interessanten neuen Resultaten geführt. Hierzu gehört insbesondere die Existenz neuer Klassen von feldtheoretischen Lösungen, deren Eigenschaften Gegenstand aktueller Forschung sind.

Die vorliegende Arbeit befasst sich mit verschiedenen Aspekten dieser Lösungen. Hierbei stehen das von Ward eingeführte integrable chirale oder modifizierte Sigma Modell mit $U(n)$-wertigen Feldern, sowie dessen nichtkommutative Erweiterung, im Zentrum unseres Interesses. Insbesondere widmen wir uns den Fragen nach der Stabilität der statischen Lösungen und der Anwendbarkeit des so genannten adiabatischen Zuganges auf diese. Letzterer approximiert zeitabhängige Lösungen durch eine geodätische Bewegung im Modulraum der statischen Lösungen.

Nach einigen einführenden Worten zur Nichtkommutativität und ihrer Rolle in der Physik, präsentieren wir im zweiten Kapitel die notwendigen Grundlagen der nichtkommutativen Geometrie, um anschließend das Wardmodell als nahezu lorentzinvariantes Modell in 1+2 Dimensionen einzuführen. Die statischen Lösungen der zugehörigen Bewegungsgleichung werden an dieser Stelle eingehend diskutiert. Das Wardmodell besitzt einige bemerkenswerte Eigenschaften, die eine gründliche Untersuchung interessant erscheinen lassen. Zu diesen gehört dessen Integrabilität, die uns effektive Techniken zur Konstruktion von solitonischen Lösungen zur Verfügung stellt, durch die im vorliegenden Fall insbesondere auch wechselwirkende Solitonen generiert werden. Darüber hinaus erlauben die zahlreichen im kommutativen Fall erzielten Resultate einen direkten Vergleich mit den Ergebnissen einer Untersuchung im nichtkommutativen Fall.

Im Anschluss diskutieren wir die Stabilität der präsentierten statischen Lösungen. Diese wird durch die quadratische Form für die Störungen bestimmt, die sich bei Betrachtung von diagonalen $U(1)$ Lösungen explizit angeben lässt. Wir zeigen, dass die untersuchten Lösungen in einem bestimmten Unterraum aller möglichen Konfigurationen stabil sind. Dieser Unterraum ist durch die Grassmannschen Konfigurationen gegeben und deren Einbettung in das vollständige unitäre Sigma Modell führt zu einer Instabilität der entsprechenden Lösungen. Schließlich zeigen wir einige mögliche Erweiterungen dieser Resultate auf.

Einer kurzen Darstellung der bekannten zeitabhängigen Lösungen folgt dann die Anwendung der von Manton eingeführten adiabatischen Näherung auf das vorliegende Modell. Diese Untersuchung lässt sich in gleicher Weise auf eine weitaus größere Klasse von nichtkommutativen Feldtheorien übertragen. Da zeitabhängige Lösungen des Wardmodells bekannt sind, können wir die Ergebnisse der durchgeführten Modulraumapproximation mit den exakten Resultaten vergleichen und finden eine schlechte Übereinstimmung im $U(1)$ Fall. Dieses wird an einigen Mehrsolitonenlösungen veranschaulicht, und wir geben mögliche Erklärungen für diese Diskrepanz.

Obwohl die vorliegende Arbeit einige Aspekte nichtkommutativer Solitonen klären kann, bleiben viele interessante Fragen unbeantwortet. Eine Zusammenstellung der, in unseren Augen, wichtigsten Punkte findet sich am Ende der Kapitel zur Stabilitätsuntersuchung und zur adiabatischen Näherung.


Schlagworte: Nichtkommutative Solitonen, Stabilitätsuntersuchung, Adiabatische Näherung

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# Chapter 1

# Motivation and Introduction

The subject matter of this thesis is a well developed and ubiquitous topic in theoretical physics, namely classical field theory, which has been endowed with new and interesting facets due to the incorporation of noncommutativity during the last two decades. This new aspect has brought about many truly novel results and led to a number of fruitful developments in this area which encompass a wide spectrum of applications in mathematics as well as physics.

We will address a particular example of a noncommutative field theory, the so-called Ward model, and give a concise review of its known features before investigating some new results concerned with the stability of its solutions and the approximation of time-dependent solitons in the framework of the adiabatic approach.

## Emergence of noncommutativity

It has been postulated as early as 1947 [106, 107, 129] that the structure of space itself should be modified at very small distances by the emergence of a noncommutativity of the spatial coordinates. Supposedly, the notion of spatial noncommutativity is even older and has been communicated by Heisenberg in a letter to Peierls, suggesting that problems of infinite self-energies may be ameliorated this way. According to an account of Jackiw [65] Peierls in turn conveyed this idea to Pauli, Pauli to Oppenheimer and the latter eventually to Snyder who wrote the aforementioned paper.

This idea of noncommutativity is implied, very similar to the canonical quantization of phase space, by the transition of the coordinates $x^i$ on spacetime to hermitian operators $\hat{x}^i$, satisfying a Heisenberg algebra

$$\left[\hat{x}^i, \hat{x}^j\right] \;=\; \mathrm{i}\,\theta^{ij}\,,$$

for a constant real-valued antisymmetric matrix $\theta = (\theta^{ij})$ of dimension length squared. Noncommutativity thus effectively induces a smearing out of spacetime and the notion of a point becomes pointless.[1]

Nevertheless this line of thought has not been pursued very enthusiastically by physicists in the following years and it was not until the early 80's with the pioneering work of Alain Connes [13, 14, 16] that the idea of noncommutative spaces received a notable

[1] This pun is an allusion to the work of von Neumann, who tried to define these "quantum" spaces in a rigorous way and dubbed his studies "pointless geometry" (see [112]).

increase in attention by physicists, again. Since then, huge efforts have been invested into the study of field theories on noncommutative spaces. It is impossible to list all relevant references in this vast context and we just give some examples.

One of the early instances of noncommutative field theories is Yang-Mills theory considered on a noncommutative torus [15], which can be derived from string theory [18]. In fact, string theory provides further evidence for noncommutativity in spacetime. Seiberg and Witten, extending earlier ideas in this direction, were able to prove [105] that type IIB superstring theory in a constant $B$-field background can be formulated in terms of a gauge field theory on a noncommutative spacetime upon taking the so-called Seiberg-Witten zero slope limit. Instigated by this analysis, many subsequent works investigated the relation between string theory and noncommutativity. It turns out that noncommutative field theories often inherit features from the underlying string theory thus allowing to transfer problems arising in string theory to questions in noncommutative field theories. In this way noncommutative field theory has, for example, been employed to study tachyon condensation [2, 21, 125] and other issues related to D-branes (for a review see e.g. [53, 49]). Besides the above mentioned directions, ideas of noncommutative field theories have entered many other areas of theoretical physics, of which we only want to mention the attempt to construct a noncommutative theory of gravity (see e.g. [83, 12, 5, 11, 6, 113]). More exhaustive reviews of noncommutativity and its applications in physics can, for example, be found in [17, 69, 116, 84, 45, 93, 26, 53, 112].

The introduction of the ideas of noncommutative geometry into field theory thus offers a prospective way to enlarge our current understanding of nature and is believed to provide a possible route towards the ultimate goal of unifying quantum field theory with gravity.

## Solitons in noncommutative field theory and the modified Ward model

The particular aspect of noncommutative field theory that this thesis is devoted to is the study of noncommutative solitons and their properties.

In ordinary field theory, solitonic solutions, or solitons for short, are usually defined [103] as stable finite-energy solutions to the equations of motion of some minkowskian action functional. Here, the term stable refers to the somewhat vague but crucial property that solitons emerge mainly unaltered from a possible collision and may thus be interpreted as extended particles. A thorough study of soliton solutions and their properties is motivated by their ubiquitous appearance in field theories which renders them interesting objects for mathematicians and physicists alike (for a review see e.g. [103, 86, 114]). Apart from their attractiveness as exact solutions to field theories by themselves, solitons also play a crucial role in physics as nontrivial classical solutions around which the path integral may be expanded.

A Moyal deformation of the underlying space leads to a rewarding generalization of these ideas to field theories on noncommutative spaces. Such field theories offer not only smooth deformations of well known soliton solutions but entirely new types of noncommutative solitons (see, e.g. [42, 53]). This property is most prominent when the commutative limit yields a free theory, because the soliton configurations are then forced to become singular in this limit. To mention just one example, we refer to the $U(1)$ sigma model in two dimensions. Upon replacing the ordinary product by a star-product we find a rich class of soliton solutions. Adding a temporal dimension, these not only represent static solutions of a Wess-Zumino-Witten-extended sigma model (see [123, 126] for this extensions and [73] for a discussion of solutions) but also of the Yang-Mills-Higgs BPS equations on noncommutative $\mathbb{R}^{1,2}$. In fact, the full BPS sector of the Yang-Mills-Higgs system is, in a particular gauge, given by the solutions to the sigma-model equations of motion. To make contact to our previous discussion, we note that multi-solitons in noncommutative euclidean two-dimensional sigma models are of interest as static D0-branes inside D2-branes with a constant $B$-field background [72].

The described noncommutative WZW-extended model, also known as Ward model, will be the main topic of this thesis. Even though many aspects of this model are already well understood [72, 73, 74, 9, 127, 56, 75, 76], the issues covered in this thesis certainly deserve some careful examination.

## Outline

The outline of this thesis is as follows:

To set up notation, we give a brief account on the basics of noncommutative geometry in the second chapter, where we present both the star-product as well as the operator formulation of field theories on the Moyal-deformed complex plane. This exposition includes the introduction of the Moyal-Weyl map as mediator between these two pictures. Furthermore, we fix conventions for the coherent states and show how translations and rotations are realized on them as unitary operators.

Chapter three includes some general remarks on low dimensional sigma models and shows how Derrick's theorem restricts the existence of nontrivial solutions to these. We then provide a definition of the notion of integrability and introduce the Ward model as an example of an integrable field theory in 1+2 dimensions. Its basic properties are discussed before shifting the focus to the noncommutative realm, where we describe the modified Ward model, both in the star-product as well as in the operator picture. We conclude this account by showing how this model can be related to scalar field theories via the implementation of the sigma-model constraint as a Lagrange multiplier.

The static solutions to the noncommutative Ward model are discussed in chapter four. After introducing the subset of grassmannian solutions and their BPS bound, we provide a complete description of abelian solitons. In addition to presenting nonabelian solitons, as well, we also embed the discussion into the framework of uniton theory and give examples of non-BPS solutions.

A detailed analysis on the stability of the static solutions discussed is then carried out in the fifth chapter. It is based on the work [24] together with A. Domrin and

O. Lechtenfeld. After presenting the Hessian, i.e. the quadratic form of the fluctuations, we identify its invariant subspaces, thus allowing for a decomposition of the fluctuations into different sectors. We demonstrate that solitons are stable within their respective Grassmannian but become unstable upon embedding them into the full unitary sigma model. For the case of diagonal $U(1)$ BPS solutions, the spectrum of the Hessian is explicitly computed, analytically as well as numerically, and the moduli of the solitons are identified among its zero modes. Furthermore, we address the fluctuation problem in the nonabelian case for the example of the noncommutative $U(2)$ model. Summarizing the achieved results at the end of the chapter, a list of related unsolved questions is provided.

For later comparison with the findings in the final chapter we review results on time-dependent Ward solitons in chapter six. After presenting the corresponding linear system we give some simple examples of soliton lumps moving with equal and constant velocity in the Moyal plane. Employing the dressing ansatz it is possible to generate more complicated multi-solitons with distinct velocities, some of which exhibit scattering. Unfortunately, there are no abelian solutions among these. To exemplify this, we display the breather-like behavior of the arising $U(1)$ two-soliton.

In chapter seven we devote ourselves to the application of Manton's adiabatic approach to the noncommutative Ward model. The ensuing results have been published together with M. Klawunn and O. Lechtenfeld in [67]. Section 7.1 gives a general overview of the underlying idea. We proceed by analyzing the moduli space of abelian sigma model $r$-solitons, identifying it as a smooth Kähler manifold, namely the $r$-th symmetrized power of the complex plane. Its Kähler metric is computed and we investigate the limits of large and small mutual distances. This enables us to consider moduli-space trajectories for the scattering of two multi-soliton lumps yielding a universal 90° scattering angle for head-on collision, in contrast to the exact solution. We put forward possible reasons for this mismatch and close the chapter with some comments on the encountered difficulties for the adiabatic approach in the nonabelian setting.

## Conventions

Throughout the subsequent chapters we employ the following conventions:

Greek letters $\mu, \nu, \lambda, \ldots \in \{0,1,2\}$ are used to denote the coordinates $(x^\mu) := (t,x,y)$ in the (1+2)-dimensional Minkowski space $\mathbb{R}^{1,2}$, where the metric is taken as $\eta = (\eta_{\mu\nu}) = \mathrm{diag}(-1,+1,+1)$ and the totally antisymmetric Levi-Civita tensor has the property $\varepsilon^{012} = 1$. As usual, we then use Latin letters $i,j,\ldots \in \{1,2\}$ for the restriction to the spatial coordinates. But note that these are also used for other purposes, e.g. to enumerate the moduli in chapter 7.

For the introduction of the modified nonlinear sigma model, an extension of the space-time to 1+3 dimensions is needed, where the coordinates are denoted by $(x^a) := (t,x,y,\rho)$ with $a,b,c,\ldots \in \{0,1,2,3\}$ and the totally antisymmetric tensor is defined with $\varepsilon^{0123} = 1$.

Since we take traces in different contexts, it is convenient to set up our notation here: We employ tr for the trace over a group space while $\mathrm{Tr}_{\mathcal{H}}$ denotes the trace over the Hilbert space $\mathcal{H}$ and the abbreviation $\mathrm{Tr} := \mathrm{tr}\,\mathrm{Tr}_{\mathcal{H}} = \mathrm{Tr}_{\mathcal{H}}\,\mathrm{tr}$ is used for the combination of both these operations.

Furthermore, in dealing with noncommutative field theory one usually encounters some notational ambiguities since the term “noncommutative” is also used to indicate fields taking values in some nonabelian group space. To make definitions distinct we refer to the latter situation as “nonabelian” case and reserve the notation “noncommutative” solely for noncommutative space structures.

# Chapter 2

# Basic Notions of Noncommutativity

Before analyzing field theories on noncommutative spaces we review the essential basics of noncommutative geometry in this chapter. The exposition will be restricted to material needed for the subsequent analysis. More detailed discussions of this wide area in relation to field theory can, for example, be found in the reviews [116, 26, 53, 112].

## 2.1 Deformations of the ordinary product

To understand how noncommutativity enters into field theory from a mathematical point of view, we recall [93] the remarkable results instigated by the seminal works of Gelfand, Grothendieck and von Neumann, which roughly state that the geometrical properties of a given space $\mathcal{X}$ can entirely be captured by the properties of the algebra $\mathcal{A} = (C^\infty(\mathcal{X}), \cdot)$ of smooth functions[1] on $\mathcal{X}$ together with pointwise addition and multiplication.

Noncommutativity can then be introduced into field theories on the space $\mathcal{X}$ by replacing the ordinary algebra $\mathcal{A}$ by a deformed algebra $\mathcal{A}_\star$, where the product of two functions is replaced by a noncommutative product usually called the star-product:

$$f \cdot g \quad \longrightarrow \quad f \star g, \quad \text{for } f, g \in C^\infty(\mathcal{X})\,.$$

Even though this product generically violates commutativity it still retains associativity.

The fundamental relation for the algebra $\mathcal{A}_\star$ is then, at least locally, given by the star-commutator between coordinates $x^\mu$ on $\mathcal{X}$:

$$[x^\mu, x^\nu]_\star \;:=\; x^\mu \star x^\nu \,-\, x^\nu \star x^\mu \;=\; \mathrm{i}\, C^{\mu\nu}\,, \tag{2.1}$$

where $C := (C^{\mu\nu})$ is a real antisymmetric rank two tensor field of dimension $(\text{length})^2$ parametrizing the noncommutativity. This change of algebra has drastic consequences of which some will be presented in the following analysis.

There exists an extensive amount of literature on general deformations in the above sense and its physical applications. We refer the reader to the texts [17, 69, 84, 45, 93] and references therein. Instead of dealing with the general case, we will now concentrate on more specific and quite simple examples of noncommutative deformations in the following.

For this, we employ an expansion of $C$ in terms of the coordinates $x^\mu$ as

$$C^{\mu\nu} \;=\; \theta^{\mu\nu} \,+\, f^{\mu\nu}{}_{\rho}\, x^\rho \,+\, R^{\mu\nu}{}_{\rho\tau}\, x^\rho\, x^\tau \,+\, \dots\,. \tag{2.2}$$

[1] Sticking to smooth functions will sometimes prove to be too restrictive or too general. We will therefore follow standard physicists practice and be somewhat sloppy with our notation always implying that we allow for all functions yielding well defined expressions in the context considered.

## 2.2 Constant deformations and representations of noncommutativity

Now and in the following we consider the star-commutator to be of the special form

$$[x^\mu, x^\nu]_\star \;:=\; \mathrm{i}\,\theta^{\mu\nu}\,, \tag{2.3}$$

for some constant skew-symmetric real noncommutativity tensor $\theta := (\theta^{\mu\nu})$. Note that this algebraic structure is nothing but a Heisenberg algebra which is well known from quantum mechanics, where it appears as algebra for the positions and momenta.

This star-product implies what is usually [7, 8, 26] referred to as deformation $A_\theta$ of the algebra $\mathcal{A} = (C^\infty(\mathcal{X}), \cdot)$, where the algebra $\mathcal{A}_\theta$ differs from $\mathcal{A}$ in the multiplication law only. In the limit $\theta \to 0$ we recover the commutative algebra $\mathcal{A}$ again.

### 2.2.1 The Groenewold-Moyal product

Additionally choosing $\mathcal{X} = \mathbb{R}^{2d}$, $d \in \mathbb{N}$ yields the easiest examples of star-deformed algebras and its corresponding noncommutative spaces, namely $\theta$-deformed real space $\mathbb{R}^{2d}_\theta$ and, as the particular case of $d$=1, the so-called Moyal plane.

*Example* 2.1 *($\mathbb{R}^{2d}_\theta$)*
The $\theta$-deformed real space $\mathbb{R}^{2d}$ which is denoted $\mathbb{R}^{2d}_\theta$ is defined by the deformation of the algebra of smooth functions on $\mathbb{R}^{2d}$ via the Groenewold-Moyal product, sometimes also called Moyal product, which is given for $f, g \in C^\infty(\mathbb{R}^{2d})$ as [46, 89]

$$f \star g \;:=\; f\;\mathrm{e}^{\frac{\mathrm{i}}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}}\;g\,. \tag{2.4}$$

Thus the above star-product is defined via the Poisson bi-vector field $\theta^{\mu\nu}\partial\mu \otimes \partial\nu$ induced by the noncommutativity tensor $\theta$. Note that this noncommutativity parameter can always be brought to a canonical form $\theta_\mathrm{c}$ by some orthogonal transformation $O$ as

$$\theta \;=\; O^T \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & \theta_i \\ -\,\theta_i & 0 \end{pmatrix} O \;=:\; O^T\,\theta_\mathrm{c}\,O\,, \quad \theta_i \in \mathbb{R}_{\geq 0}\,.$$

For notational simplicity we will always assume that $\theta$ is of the particular form $\theta_\mathrm{c}$.

*Example* 2.2 *(The noncommutative plane)*
The noncommutative plane, also known as the Moyal plane, derives from the above example as special case of two dimensions. We thus have coordinates $x, y$ with nontrivial star-commutator

$$[x, y]_\star \;:=\; \mathrm{i}\,\theta\,, \tag{2.5}$$

for a constant real parameter $\theta$ which, without loss of generality, we assume to be positive. Even though the noncommutative plane will be of primary importance in most of this thesis we will, for now, stick to the more general setting of $\mathbb{R}^{2d}_\theta$ and specialize later, where necessary.

To analyze the properties of the star-product (2.4) we expand it for some functions $f$ and $g$ into

$$\begin{aligned} f \star g &= f\, e^{\frac{i}{2}\overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu}\, g \\ &= f \cdot g + \tfrac{i}{2}\theta^{\mu\nu}\, \partial_\mu f\, \partial_\nu g + \ldots \\ &= f \cdot g + \text{total derivatives} \end{aligned}$$

and deduce that the integral of a product of two square integrable functions does not change under replacement of the ordinary product with the star-product introduced above, i.e.

$$\int d^{2d}x\; f \star g \;=\; \int d^{2d}x\; f \cdot g\,. \tag{2.6}$$

Furthermore, it will prove convenient to introduce complex coordinates $z^i$ on $\mathbb{R}^{2d}$ by

$$z^i \;:=\; x^{2i-1} + i\, x^{2i}\,, \quad \text{for } i \in \{1,\ldots,d\}\,, \tag{2.7}$$

and to rescale them to dimensionless coordinates via

$$z^i \quad\longrightarrow\quad a^i \;:=\; \frac{z^i}{\sqrt{2\,\theta_i}}\,, \tag{2.8}$$

for $\theta_i \neq 0$, such that the basic star-commutation relation now reads

$$\left[a^i, \bar{a}^j\right]_\star \;=\; \delta^{ij}\,. \tag{2.9}$$

Formula (2.9) thus exhibits the intimate connection between the considered noncommutativity and the quantum mechanics of harmonic oscillators that we already alluded to in the beginning of this section. This relation will now be exploited further by expressing the $\theta$-deformed real space by operators on some Hilbert space.

### 2.2.2 Operator formalism

As mentioned above, another way to approach noncommutativity in the case of a constant deformation parameter $\theta$ is the formulation on operator space. Consider again the commutation relation (2.3). This Heisenberg algebra may also be realized by promoting the coordinates to linear operators acting on on some auxiliary Hilbert space $\mathcal{H}$ and satisfying (2.3). This different representation will be introduced in this subsection and is usually much more convenient for carrying out computations explicitly.

Replacing the (rescaled) coordinates (2.7) and (2.8) by the operators $\hat{z}^i, \hat{z}^{j\,\dagger}$ and $\hat{a}^i, \hat{a}^{j\,\dagger}$ satisfying

$$\left[\hat{z}^i, \hat{z}^{j\,\dagger}\right] \;=\; 2\,\theta_i\,\delta^{ij}\,\mathbb{1} \quad \text{and} \quad \left[\hat{a}^i, \hat{a}^{j\,\dagger}\right] \;=\; \delta^{ij}\,\mathbb{1}\,, \tag{2.10}$$

we see that the rescaled coordinate operators are just creation and annihilation operators of a $d$-dimensional harmonic oscillator Fock space which we denote by $\mathcal{H}_{\mathrm{HO}}^{\otimes d} =: \mathcal{H}$. A basis for $\mathcal{H}_{\mathrm{HO}}^{\otimes d}$ is, as usual, given by applying the creation operators $\hat{a}^{i\,\dagger}$ to the ground state or vacuum $|0\rangle$

$$|k_1, \ldots, k_d\rangle \;=\; \prod_{i=1}^{d} \frac{\big(\hat{a}^{i\,\dagger}\big)^{k_i}}{\sqrt{k_i!}}\;|0\rangle\,, \quad \text{for } k_i \in \mathbb{N}_0\,, \tag{2.11}$$

where $\hat{a}^i\,|0\rangle \;=\; 0$ for all $i \in \{1, \ldots, d\}$. A remark on notation seems in order here. We will in this section explicitly indicate operators by a "ˆ" but leave this out in the following sections for notational simplicity as long as we are convinced that there will be no confusions.

*The Moyal-Weyl map*

The reason why the representation of the noncommutativity algebra by operators on a Hilbert space provides a useful tool for computation is that the two mentioned descriptions of $\theta$-deformed real space are equivalent. There actually exists an isomorphism between an appropriate algebra of functions on noncommutative real space and an algebra of linear operators on the Hilbert space introduced above.

Consider [112] the commutative algebra of functions on $\mathbb{R}^{2d}$ belonging to the Schwarz class, i.e. decreasing rapidly with all their derivatives at infinity. Elements of this algebra are mapped to corresponding linear operators by the Moyal-Weyl map [124, 89]. In complex coordinates (2.7) it is given for a function $f(\bar{z}, z)$, where $z := (z^i)$ is a collective notation for the coordinates, by

$$\begin{aligned} f(\bar{z}, z) \;\mapsto\; F(\hat{a}^\dagger, \hat{a}) \;&:=\; \int \tfrac{\mathrm{d}^d\bar{p}\,\mathrm{d}^d p}{(2\,\pi)^d}\;\tilde{f}(\bar{p}, p)\;\mathrm{e}^{-\,\mathrm{i}\,(\bar{p}\cdot\hat{z}+p\cdot\hat{\bar{z}}^\dagger)} \\ &=\; \text{Weyl-ordered}\,\big(f(\hat{a}, \hat{a}^\dagger)\big) \end{aligned} \tag{2.12}$$

with the Fourier transform $\tilde{f}(\bar{p}, p)$ of $f$ given by

$$\tilde{f}(\bar{p}, p) \;:=\; \int \tfrac{\mathrm{d}^d\bar{z}\,\mathrm{d}^d z}{(2\,\pi)^d}\;f(\bar{z}, z)\;\mathrm{e}^{\mathrm{i}\,(\bar{p}\cdot z+p\cdot\bar{z})}\,. \tag{2.13}$$

The term "Weyl-ordered" in the definition above denotes the linear operation that maps any monomial in operators to its symmetrized expression, such that, for example, $\hat{a}^\dagger\,\hat{a}^2$ gets mapped to Weyl-ordered $\big(\hat{a}^\dagger\,\hat{a}^2\big) \;=\; \frac{1}{3}(\hat{a}^\dagger\,\hat{a}^2 + \hat{a}\,\hat{a}^\dagger\,\hat{a} + \hat{a}^2\,\hat{a}^\dagger)$. In this context $\hat{F} := F(\hat{a}^\dagger, \hat{a})$ is called the symbol or Weyl-symbol of $f$. The Moyal-Weyl map thus promotes functions of coordinates multiplied by the star-product to operators on the Hilbert space $\mathcal{H}$ with the usual operator composition as product. As a side remark let us mention that apart from the Weyl-ordering introduced above there exist also other possible ordering prescriptions to assign unique operator expressions to classical functions. These are induced by allowing for a weight function $w(\bar{p}, p)$ in the integrand of (2.12) (see e.g. [110] for a review on this topic). Note also that the normalization in (2.12) is chosen such that $1 \mapsto \hat{\mathbb{1}}$.

To recover the configuration space function $f$ from an operator $\hat{F}$ we need to employ the inverse mapping to (2.12) which is given by

$$\begin{aligned}\hat{F} \;\mapsto\; f(\bar{z},z) \;&:=\; (\mathrm{i}\,4\,\pi)^d\,\mathrm{Pf}(\theta)\int\tfrac{\mathrm{d}^d\bar{p}\,\mathrm{d}^d p}{(2\,\pi)^{2\,d}}\,\mathrm{Tr}_{\mathcal{H}}\Big(\hat{F}\,\mathrm{e}^{-\,\mathrm{i}\,(\bar{p}\cdot z+p\cdot\bar{z})}\,\mathrm{e}^{\mathrm{i}\,(\bar{p}\cdot\hat{z}+p\cdot\hat{z}^\dagger)}\Big)\\ &=\; F_\star(\bar{z},z)\,,\end{aligned} \tag{2.14}$$

where $F_\star$ derives from $F$ by replacing the composition by the star-product and $\mathrm{Pf}(\theta)$ denotes the Pfaffian of the noncommutativity tensor $\theta$ which for positive determinant is given by $\sqrt{\det\theta}$ and otherwise defined as, for example, in [70]. Furthermore, the symbol $\mathrm{Tr}_{\mathcal{H}}$ is used here to denote the trace over the Hilbert space.

We thus have a bijection of the form

$$(f(\bar{z},z),\star) \quad\longleftrightarrow\quad \Big(\hat{F}(\hat{a}^\dagger,\hat{a}),\cdot\Big) \tag{2.15}$$

allowing us to investigate problems on $\theta$-deformed real space via its equivalent formulation in the operator formalism introduced here.

*Derivative and integral*

In order to properly analyze field theories in the operator formalism it is still necessary to introduce analogues of derivatives and integration on the level of operators. The Moyal-Weyl map (2.12) enables us to state the following transformations under the transition between star-product and operator formalism:

$$\partial_{\bar{z}_i} f \quad\longrightarrow\quad \tfrac{1}{\sqrt{2\,\theta_i}}\big[\hat{a}^i,\hat{F}\big] \quad\text{and}\quad \partial_{z_i} f \quad\longrightarrow\quad -\tfrac{1}{\sqrt{2\,\theta_i}}\big[\hat{a}^{i\,\dagger},\hat{F}\big]\,, \tag{2.16}$$

as well as

$$\int\frac{\mathrm{d}^d\bar{z}\,\mathrm{d}^d z}{(2\,\mathrm{i})^d}\,f \quad\longrightarrow\quad (2\,\pi)^d\,\mathrm{Pf}(\theta)\,\mathrm{Tr}_{\mathcal{H}}(\hat{F})\,. \tag{2.17}$$

Furthermore, we may introduce the hermitian Laplace operator $-\Delta$ which is defined via

$$\Delta\hat{F} \;:=\; \big[a,\big[a^\dagger,\hat{F}\big]\big] \;=\; \big[a^\dagger,\big[a,\hat{F}\big]\big]\,. \tag{2.18}$$

Its kernel is spanned by functions of $a$ or $a^\dagger$ only.

Summarizing, we get the following dictionary for the transition between star-product and operator formulation on the $\theta$-deformed real space mediated by the Moyal-Weyl map:

| | star-product | | | operator | | |
|---|---|---|---|---|---|---|
| coordinates | $\bar{z}^i$, | $z^j$ | $\leftrightarrow$ | $\sqrt{2\,\theta_i}\,\hat{a}^{i\,\dagger}$, | $\sqrt{2\,\theta_j}\,\hat{a}^j$ | operators |
| derivatives | $\partial_{\bar{z}_i}$, | $\partial_{z_j}$ | $\leftrightarrow$ | $\frac{1}{\sqrt{2\,\theta_i}}\,\mathrm{ad}(\hat{a}^i)$, | $-\frac{1}{\sqrt{2\,\theta_j}}\,\mathrm{ad}(\hat{a}^{j\,\dagger})$ | commutators |
| integral | $\int\frac{\mathrm{d}^d\bar{z}\,\mathrm{d}^d z}{(2\,\mathrm{i})^d}$ | | $\leftrightarrow$ | $(2\,\pi)^d\,\mathrm{Pf}(\theta)\,\mathrm{Tr}_{\mathcal{H}}$ | | trace. |

Note that instead of $\frac{\mathrm{d}^d\bar{z}\,\mathrm{d}^d z}{(2\,\mathrm{i})^d}$ we will also make use of the more convenient notation $\mathrm{d}^{2d}z$ to denote the same thing.

*Coherent states and some unitary transformations*

In addition to the aforementioned harmonic oscillator states, there exists another special class of states in $\mathcal{H}_{\mathrm{HO}}$ which merit our attention. These, so-called coherent states play an important role in our discussion of the noncommutative sigma model on the operator level and thus deserve a thorough inspection. To this end, we collect some of the basic facts about them as far as needed for the analysis in this thesis. A more detailed discussion can, for example, be found in [99]. In addition, some special unitary operators are presented which induce translations and rotations in the complex plane underlying the Hilbert space spanned by the coherent states.

In the single harmonic oscillator Fock space $\mathcal{H}_{\mathrm{HO}}$ the coherent states $|\alpha\rangle$ are defined as eigenstates of the annihilation operator $\hat{a}$ with eigenvalue $\alpha \in \mathbb{C}$ and can be written as linear combination of basis states

$$|\alpha\rangle \;:=\; \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}}\,|k\rangle \;=\; \mathrm{e}^{\alpha\,\hat{a}^\dagger}\,|0\rangle \;=\; \mathrm{e}^{\frac{1}{2}\,\alpha\bar{\alpha}}\,\mathrm{e}^{\alpha\,\hat{a}^\dagger-\bar{\alpha}\,\hat{a}}\,|0\rangle\,. \tag{2.19}$$

Note that this definition is chosen such that $|\alpha\rangle$ is not normalized to unity but rather

$$\langle\alpha|\alpha'\rangle \;=\; \mathrm{e}^{\bar{\alpha}\alpha'}\,, \tag{2.20}$$

i.e., $\langle\alpha|\alpha\rangle = \exp(\alpha\bar{\alpha})$. From the formula (2.19) the action of the creation and annihilation operators on $|\alpha\rangle$ can be deduced yielding in summary

$$\hat{a}\,|\alpha\rangle \;=\; \alpha\,|\alpha\rangle \quad \text{and} \quad \hat{a}^\dagger\,|\alpha\rangle \;=\; \mathrm{e}^{\alpha\,\hat{a}^\dagger}\,|1\rangle \;=\; \frac{\partial}{\partial\alpha}\,|\alpha\rangle\,. \tag{2.21}$$

Furthermore, the coherent states (2.19) form an overcomplete basis of $\mathcal{H}_{\mathrm{HO}}$ with separation of unity

$$\hat{\mathbb{1}} \;=\; \int \frac{\mathrm{d}\alpha\mathrm{d}\bar{\alpha}}{2\pi\mathrm{i}}\,\mathrm{e}^{-\alpha\bar{\alpha}}\,|\alpha\rangle\langle\alpha|\,. \tag{2.22}$$

Of special interest in the context of coherent states and its role in noncommutative field theory will be the transformation of coherent states under the unitary transformations mediating translations and rotations in the complex plane.

For the first of these, define the shift operator $\mathcal{D}(\beta)$ for some $\beta \in \mathbb{C}$ by

$$\hat{\mathcal{D}}(\beta) \;:=\; \mathrm{e}^{\beta\,\hat{a}^\dagger-\bar{\beta}\,\hat{a}} \qquad \Longrightarrow \qquad \hat{\mathcal{D}}(\beta)\,\hat{a}^\dagger\,\hat{\mathcal{D}}^\dagger(\beta) \;=\; \hat{a}^\dagger - \bar{\beta}\,, \tag{2.23}$$

from which we find that $\hat{\mathcal{D}}(\beta)$ indeed translates coherent states as

$$\hat{\mathcal{D}}(\beta)\,|\alpha\rangle \;=\; \mathrm{e}^{-\bar{\beta}(\beta+\frac{1}{2}\,\alpha)}\,|\alpha+\beta\rangle\,. \tag{2.24}$$

Rotations of coherent states by some real angle $\vartheta$, on the other hand, are induced by

$$\hat{\mathcal{R}}(\vartheta) \;:=\; \mathrm{e}^{\mathrm{i}\,\hat{a}^\dagger\,\vartheta\,\hat{a}} \qquad \Longrightarrow \qquad \hat{\mathcal{R}}(\vartheta)\,\hat{a}^\dagger\,\hat{\mathcal{R}}^\dagger(\vartheta) \;=\; \mathrm{e}^{\mathrm{i}\,\vartheta}\,\hat{a}^\dagger, \tag{2.25}$$

such that

$$\hat{\mathcal{R}}(\vartheta)\,|\alpha\rangle ~=~ |\mathrm{e}^{\mathrm{i}\,\vartheta}\,\alpha\rangle\,. \tag{2.26}$$

Another unitary transformation needed later is the squeezing or Bogoliubov transformation, given for some complex parameter $\beta$ by

$$\hat{\mathcal{S}}(\beta) ~:=~ \mathrm{e}^{\frac{1}{2}(\bar\beta\,\hat a^2-\beta\,a^{\dagger\,2})}\,, \tag{2.27}$$

which implies

$$\hat{\mathcal{S}}(\beta)\;\hat a^\dagger\;\hat{\mathcal{S}}^\dagger(\beta) ~=~ \hat a^\dagger\;\cosh(|\beta|)\,\frac{\beta}{|\beta|}+\hat a\;\sinh(|\beta|)\,\frac{\beta}{|\beta|}\,. \tag{2.28}$$

# Chapter 3

# Ward Model and Noncommutative Extension

A predominant aim in field theory is to find nontrivial solutions to the equations of motion for some given model. Unfortunately, the equations arising in realistic models are usually highly nonlinear and solving them in full generality is often an infeasible task. It is therefore natural to look for possible simplifications which might allow for finding at least subclasses of solutions. In many cases, this is indeed possible and there exist some particularly rewarding situations where the considered solutions also have an interpretation in terms of topological quantities. We will elucidate this idea in the following sections where we will consider primarily solitonic solutions, or solitons for short, which, for our purposes, are defined as finite-energy solutions to a minkowskian field theory usually arising from some topological argument.[1]

Still the complete classification of these nonperturbative solutions may well be an arduous, if not impossible, task. Therefore, one might consider other routes for simplifying the posed problem. Along these lines huge efforts have been invested into the study of lower dimensional models descended from or similar to interesting higher dimensional models in the hope that these may be more easily tractable. When one seeks to understand realistic field theories by means of these lower dimensional models one would usually require these to preserve as many of the features the higher dimensional model possesses as possible. Since in this thesis we are primarily interested in classical field theories the key feature we want to keep is the existence of nontrivial solutions to the equations of motion.[2]

Combining the two mentioned tracks, we are thus set out to analyze solitonic solutions to a particular low dimensional field theoretical model, the so-called Ward chiral or nonlinear sigma model, and its extension to the noncommutative realm. After briefly reviewing the commutative setting we proceed to the noncommutative case clarifying also the relation of the noncommutative Ward model to noncommutative scalar field theories.

Before going into detail let us present some references for background reading. The analysis of nonperturbative nontrivial field configurations in general is an old subject in field theory and there exists a fair amount of literature on it. For a good overview on the topic of solitons and instantons in the commutative setting and in particular for the

[1] Strictly speaking this condition rather defines what is known as "solitary waves", instead of "solitons", since the latter phrase often refers to a more stringent definition as, for example, in [103]. Yet, we will not make this somewhat subtle discrimination here and thus adhere to common practice in physics literature.

[2] Even though there exist, of course, other reasonable features as e.g. conformal invariance of the action, asymptotic freedom or confinement of fermions upon introducing them [133].

nonlinear sigma model presented here, we suggest the books [103, 133, 86]. More recently, stimulated by the advances in the area of noncommutative field theory, nonperturbative solutions received a lot of attention in the noncommutative setting, as well. For general references on the topic of noncommutative solitons and instantons we refer the reader to [42, 47, 94, 53, 49, 75, 76] and references therein.

## 3.1 Low dimensional sigma models

The models of primary interest to us are the (non-)linear sigma models, scalar field theories on $(1+d)$-dimensional flat spacetime with field taking values in a riemannian target manifold and defined by a lagrangian action of the form

$$S[\Phi] \;:=\; \int \mathrm{d}^{1+d}x \left(\tfrac{1}{2}\, G_{ab}\, \partial_\mu \Phi^a\, \partial^\mu \Phi^b \;-\; V(\Phi)\right) , \tag{3.1}$$

where $G$ denotes the metric on the target space and $V$ is some potential. The notation "(non-)linear" refers to the form of the metric and a model is usually called nonlinear if the components of $G$ depend on the field $\Phi$ itself.

One of the most prominent examples of a nonlinear sigma model is the well understood $O(n)$ sigma model as first introduced by Gell-Mann and Levy [40] where the fields are given by $n$-component real unit vectors and the lagrangian density simply reads $\mathcal{L} = \partial_\mu \Phi^a \partial^\mu \Phi^a$. A direct generalization of the Lagrangian is achieved by allowing a group target manifold, e.g. for $U(n)$-valued fields, yielding the unitary sigma or chiral model with $\mathcal{L} = \mathrm{Tr}\, \partial_\mu \Phi^\dagger\, \partial^\mu \Phi$. Obviously these models enjoy a global $O(n)$ or $U(n)$ symmetry, respectively, to which their names refer. Of particular importance for later applications will be the restriction of the chiral model to hermitian fields $\Phi^\dagger = \Phi$ which allows us to put $\Phi = \mathbb{1} - 2P$ with a hermitian projector $P$ of some rank $r$. The projector $P$ itself parametrizes subspaces of the full configuration space with dimension given by its rank. The set of all projectors unitarily equivalent to $P$, and thus of same rank, is called the Grassmannian, and it is given by the coset[3]

$$\mathrm{Gr}(P) \;=\; \frac{U(n)}{U(\mathrm{im}\, P) \times U(\ker P)} , \tag{3.2}$$

where $U(\mathcal{X}) := \{\mathcal{U} \in GL(\mathcal{X})\,|\,\mathcal{U} = \mathcal{U}^\dagger\}$ denotes the set of all unitary linear transformations on $\mathcal{X}$. Therefore, each given $P$, and thus each hermitian $\Phi$, belongs to a certain Grassmannian, and the space of all hermitian $\Phi$ decomposes into a disjoint union of Grassmannians. The restriction to hermitian $\Phi$ reduces the unitary to a grassmannian sigma model, whose configuration space is parametrized by projectors $P$. Finally, if we consider only projectors of rank 1 we are left with the so-called $\mathbb{C}P^{n-1}$ models [30, 41, 31, 133] which are subject of extensive studies of their own. There is much more to be said about these models and their properties but we will not do so in this thesis and rather concentrate on general features and the specific setup which will be introduced in the next section.

[3]In the literature one usually encounters the notation $\mathrm{Gr}(k, n)$ for the set of all $k$-complex-dimensional subspaces of $\mathbb{C}^n$.

### 3.1.1 Derrick's Theorem

The models introduced above seem easy enough to give explicit solutions, at least in low dimensions, and thus serve as toy examples for features of more realistic higher-dimensional models. Nevertheless, these models do not feature nontrivial solutions in arbitrary dimensions. The restrictions on the number of allowed dimensions for nontrivial solutions to exist are due a famous theorem by Derrick [55, 22] which states that under certain assumptions there cannot exist any other static finite-energy solutions than the vacuum itself. Let us follow the lines of [86] as well as Derrick's original paper and go into a little more detail. To this end consider again an action of the form (3.1), where, for simplicity, we assume $\Phi$ to be scalar. Static solutions $\Phi$ need to extremize the energy functional

$$E[\Phi] \; := \; \int \mathrm{d}^d x \left( \tfrac{1}{2}\, \partial_i \Phi\, \partial^i \Phi \; + \; V(\Phi) \right) \; =: \; E_2[\Phi] + E_0[\Phi]\,, \tag{3.3}$$

where $i, j \in \{1, 2, \ldots, d\}$, implying that it should in particular be stationary under spatial rescaling $\Phi(x) \to \Phi(\lambda x)$. The energy transforms under this rescaling as

$$E[\Phi] \quad \longrightarrow \quad \lambda^{2-d} E_2[\Phi] + \lambda^{-d} E_0[\Phi] \; =: \; E_\lambda\,, \tag{3.4}$$

thus accounting for the choice of terminology above. Stationarity of the energy leads us to the condition

$$0 \; \stackrel{!}{=} \; \left. \frac{\mathrm{d}E_\lambda}{\mathrm{d}\lambda} \right|_{\lambda=1} \; = \; (2-d)\, E_2 - d\, E_0 \qquad \Longrightarrow \qquad E_0 \; = \; \frac{2-d}{d}\, E_2\,, \tag{3.5}$$

Hence we see that the behavior of the energy depends crucially on the number of spatial dimensions. Generically the kinetic energy $E_2$ and the potential energy $E_0$ are both positive and therefore (3.5) can only hold for $d = 1$ whereas for $d \geq 2$ it is violated. The case $d = 2$ poses an exception insofar as there it is still possible to consider the situation of vanishing potential which might still allow for solutions. Another way to evade this theorem in scalar field theories is to include additional terms with higher order derivatives or higher powers of the derivatives in the Lagrangian and thus also change the corresponding energy in a way which allows for solutions even in the case of higher dimensions.[4]

One important implication due to the introduction of a length-scale $\sqrt{\theta}$ via the star-product (2.4) is that in this noncommutative setting the scaling argument used to prove the above theorem is generically no longer valid [26]. Therefore static finite-energy solutions can and do exist even in higher dimensions (see [43] for the example of $d = 2$ with nonzero potential). For $V \equiv 0$, however, the energy reduces to the commutative expression (3.3) due to the property (2.6) and Derrick's theorem does still apply.

It is also worthwhile mentioning that besides the negative non-existence result discussed above we can also put this theorem to some constructive use by realizing that the

[4] Apart from this, the introduction of gauge fields circumvents the obstructions of this theorem by a different scaling behavior.

compulsory invariance of the energy under spatial rescaling implies a relation between the different parts of the energy usually referred to as virial theorem [103, 86]. In $d = 1$, for example, the condition (3.5) yields $E_2 = E_0$ and the energy of a solution is thus evenly split between kinetic and potential contribution.

### 3.1.2 Integrability

Eventually, one would also like to analyze time-dependent solutions to the respective model at hand. An important question arising in this context of classifying solutions to a given model is whether it is possible to completely solve the equations of motions or, in other words, whether the model is integrable. In field theory, the term integrability usually refers to the existence of an infinite number of conservation laws,[5] which, for a wide class of models, is intrinsically tied to the existence of a so-called Lax-pair formulation [71, 131], where the equations of motions arise as compatibility condition of a set of equations for an auxiliary field. To exemplify this statement, let us briefly consider [133] as a simple (1+1)-dimensional case the Korteweg-de Vries equation on the real line. The equation of motion is given by

$$\partial_t q + \alpha\, q\, \partial_x q + \partial_x^3 q \;=\; 0 \quad \text{with } \alpha \in \mathbb{C}\,.$$

Introduce the linear operators [23, 95]

$$L \;:=\; -\left(\partial_x^2 + \frac{\alpha}{6}\, q\right)\,, \qquad\qquad A \;:=\; -\left(4\,\partial_x^3 + \alpha\, q\, \partial_x + \frac{\alpha}{2}\,\frac{\partial q}{\partial x} + f\right)\,,$$

where $f$ is an arbitrary function depending on $t$ and an additional variable $k$. Calculating the compatibility condition to the set of equations

$$L\,\Psi \;=\; k^2\,\Psi\,, \qquad\qquad \frac{\partial \Psi}{\partial t} \;=\; A\,\Psi\,,$$

for some auxiliary field $\Psi(t, x, k)$ and $\alpha{\neq}0$ gives

$$\partial_t L + \big[L, A\big] \;=\; 0\,,$$

which upon inserting the definitions of $L$ and $A$, the so-called Lax pair, reproduces the original equation of motion for the field $q$. We have thus rewritten the equation of motion as a compatibility condition to a set of linear equation, the linear system to this model. In view of the connection between the existence of a Lax pair and integrability mentioned above, we will adopt common practice and call a model integrable whenever there exist such a Lax-pair formulation. Besides the Korteweg-de Vries equation, there are a lot of important integrable systems in this sense. For example, the nonlinear sigma model in 2 dimensions, i.e. $d = 1$, is known to be integrable [133]. Due to the existence of powerful

[5]For a mathematically more rigorous definition of integrability one needs to employ a little more subtlety (see e.g. [87]) from which we will refrain in this framework and be content with the following Lax-pair condition.

techniques for generating and studying solutions entailed by integrability, it is very useful to retain this feature as we will see during the subsequent analysis.

Now that we presented the necessary preliminaries let us be more specific and introduce the model that will be at the center of our attention in the following chapters. The main aim in this context will be to define a model which features nontrivial static and also interacting time-dependent solutions to the equation of motion and at the same time exhibits integrability. Since we are eventually interested in noncommutative generalizations of the model we focus on two spatial dimensions in order to be then able to Moyal-deform the spatial coordinates.

## 3.2 Modified nonlinear sigma model and noncommutative extension

In the previous section we reviewed well known results on general low dimensional sigma models emphasizing the role of the existence of interesting static solutions and integrability of the models. Upon further pursuing this quest of finding integrable models featuring nontrivial field configurations one is naturally led to the question whether there exist such models in (1+$d$)-dimensional minkowskian spacetime which are also Lorentz-invariant, i.e. invariant under the action of the group $SO(1,d)$. This question can be answered to the positive for $d=1$, where models of this type have been found (see e.g. [131] and references therein for a review). In contrast to this, there are no known models in 1+2 dimensions comprising simultaneously the features of integrability, Lorentz-invariance and the existence of interacting soliton solutions [79, 27]. Even though there do exists examples of models which are integrable in more than one space dimension, these usually feature only non-moving or trivially scattering soliton solutions and thus no interesting time evolution with respect to interaction of solitons (see e.g. [111, 121] for a brief exposition). [6] On the other hand models exhibiting soliton scattering in 1+2 dimensions have been introduced and investigated (see e.g. [100, 101, 64, 102]) but these were found not to be integrable.

However, there exists a model which comes, in a sense, close to achieving the above set goal by providing a system which is integrable but explicitly breaks Lorentz-invariance through an additional term in the Lagrangian compared to (3.1). Even though Lorentz-invariance is broken, this model is nevertheless very interesting since it provides an integrable theory with moving and scattering soliton solutions in 1+2 dimensions. It is therefore that this model has experienced a thorough analysis since its introduction by Ward [119] nearly 20 years ago and is still subject of current research [60, 27, 61].

[6] The Kadomtsev-Petviashvili equation poses an exception since interacting solutions have been found in this model (see [1] and references therein).

### 3.2.1 Ward's chiral model

To arrive at a modified sigma model in 1+2 dimensions we may start with the $U(n)$ chiral model in two spatial dimensions and extend it to an additional time dimension. Even though there exist diverse ways of doing this, only a very specific modification leads to an integrable modified sigma model, the so-called Ward model. The modification to the standard sigma-model action needed for this Ward model is originally due to Wess, Zumino and Witten [123, 126] who introduced the appearing WZW-term for a rather different purpose (see also [133]). As mentioned above its equations of motion have been introduced and thoroughly investigated by Ward in the context of the $SU(2)$ chiral model [119, 79, 120, 59]. For this reason the model is also known as Ward's chiral model.

One way of deriving the field equations for this modified $U(n)$ sigma model is by a dimensional reduction of the selfdual Yang-Mills theory from 2+2 to 1+2 dimensions [73, 86, 68]. The latter is implied by the Bogomolnyi equations

$$\partial_\mu H + \left[A_\mu, H\right] \;=\; \tfrac{1}{2}\,\varepsilon_\mu{}^{\nu\lambda}\left(\partial_{[\nu}A_{\lambda]} + \left[A_\nu, A_\lambda\right]\right)\,, \tag{3.6}$$

where the Yang-Mills gauge potential $A_\mu$ and the Higgs field $H$ take values in the Lie-algebra $\mathfrak{u}(n)$ of $U(n)$. Here, $\mu, \nu, \lambda \in \{0,1,2\}$ are used to denote the coordinates $\{t, x, y\}$ in $\mathbb{R}^{1,2}$ with metric $(\eta_{\mu\nu}) = \mathrm{diag}(-1,+1,+1)$ and the Levi-Civita tensor is defined with $\varepsilon^{012} = 1$. Furthermore, the square brackets, as usual, denote antisymmetrization with respect to the indices, i.e. $A_{[\mu}B_{\nu]} := A_\mu B_\nu - A_\nu B_\mu$. Choosing the light-cone gauge $A_x = -H$, $A_y = A_t$ and introducing a group valued prepotential $\Phi \in U(n)$ with ansatz

$$A_x \;=\; -H \;=\; \tfrac{1}{2}\,\Phi^\dagger\,\partial_x\Phi \quad \text{and} \quad A_t \;=\; A_y \;=\; \tfrac{1}{2}\,\Phi^\dagger\,(\partial_t + \partial_y)\Phi \tag{3.7}$$

leads to the following Yang-type [130] equation for the prepotential $\Phi$:

$$(\eta^{\mu\nu} \;+\; X_\lambda\,\varepsilon^{\lambda\mu\nu})\,\partial_\mu(\Phi^\dagger\,\partial_\nu\Phi) \;=\; \partial_x(\Phi^\dagger\,\partial_x\Phi) \;+\; \partial_{y-t}(\Phi^\dagger\,\partial_{y+t}\Phi) \;=\; 0 \tag{3.8}$$

with fixed one-form $X = X_\mu\,\mathrm{d}x^\mu = \mathrm{d}x$ responsible for the breaking of Lorentz-invariance. Equation (3.8) is just the advertised equation of motion for the Ward model and will therefore play a paramount role in the following discussions.

There exists another way to arrive at the modified $U(n)$ sigma model in 1+2 dimensions which allows us to derive the equation of motion from an action principle. For this one considers a Nair-Shiff sigma model-type action [91, 82] and dimensionally reduces it from four to three dimensions (see also [73]). The ensuing model is again the non-Lorentz invariant but possibly integrable modified sigma model with $U(n)$ valued field and action

$$\begin{aligned} S[\Phi] \;&:=\; -\tfrac{1}{2}\int \mathrm{tr}\,J\wedge *J \;+\; \tfrac{1}{3}\int_0^1\!\!\int \mathrm{tr}\,X\wedge\widetilde{J}\wedge\widetilde{J}\wedge\widetilde{J} \\ &=:\; S_2 \;+\; S_3\,, \end{aligned} \tag{3.9}$$

where the antihermitian flat gauge connection $J$ is defined by

$$J \;:=\; \Phi^\dagger\,\mathrm{d}\Phi \qquad\Longrightarrow\qquad F \;:=\; \mathrm{d}J + J\wedge J \;=\; 0\,, \tag{3.10}$$

and the Hodge-dual is, as usual, given as

$$*J \;:=\; \tfrac{1}{2}\,\varepsilon_{\mu\nu\lambda}\,J^{\mu}\,\mathrm{d}x^{\nu}\wedge\mathrm{d}x^{\lambda}\,. \tag{3.11}$$

Finally, the symbol "tr" is used to denote the trace over the $U(n)$ group space. The two parts of this action may be identified as the standard sigma model term $S_2$ plus an additional Wess-Zumino-Witten-type (WZW) term $S_3$ consisting of an integral over $\mathbb{R}^{1,2}\times[0,1]$, with the extension $\widetilde{J}$ interpolating along the interval $[0,1]$ between $\widetilde{J}=0$ and $\widetilde{J}=J$. Furthermore, there appears the generically Lorentz-breaking constant one-form $X$, which will be kept arbitrary again for the moment and will eventually render the model integrable for special choices of $X$.

Expanding the gauge connection in terms of the fields and employing coordinates $(x^{\mu}):=(t,x,y)$, $\mu\in\{0,1,2\}$, $(x^{a}):=(t,x,y,\rho)$, $a\in\{0,1,2,3\}$, we get the explicit form

$$\begin{aligned} S[\Phi] \;=&\; -\tfrac{1}{2}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\;\mathrm{tr}\,\eta^{\mu\nu}\,\partial_{\mu}\Phi^{\dagger}\,\partial_{\nu}\Phi \\ &+\tfrac{1}{3}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\int_{0}^{1}\mathrm{d}\rho\;X_{a}\,\varepsilon^{abcd}\;\mathrm{tr}\left(\tilde{\Phi}^{\dagger}\,\partial_{b}\tilde{\Phi}\,\tilde{\Phi}^{\dagger}\,\partial_{c}\tilde{\Phi}\,\tilde{\Phi}^{\dagger}\,\partial_{d}\tilde{\Phi}\right)\,, \end{aligned} \tag{3.12}$$

where the totally antisymmetric tensor in 1+3 dimensions is defined with $\varepsilon^{0123}:=1$.

If we now vary the action with respect to $\Phi$ we need to take into account the unitarity of the fields which is to be conserved under perturbations. A convenient way of doing this is to perturb

$$\Phi \quad\longrightarrow\quad \Phi\,\mathrm{e}^{\mathrm{i}\,\chi} \;=\; \Phi\left(\mathbb{1}+\mathrm{i}\,\chi-\tfrac{1}{2}\,\chi^{2}+\mathcal{O}(\chi^{3})\right)\,, \tag{3.13}$$

for some hermitian field $\chi$. The variation of the action (3.12) is then straightforwardly computed to

$$\begin{aligned} \delta S[\Phi,\chi] \;=&\; -\,\mathrm{i}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\;\mathrm{tr}\left\{(\eta^{\mu\nu}+X_{\lambda}\,\varepsilon^{\lambda\mu\nu})\,\partial_{\mu}(\Phi^{\dagger}\,\partial_{\nu}\Phi)\right\}\chi \\ &+\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\;\mathrm{tr}\,\chi\left\{\tfrac{1}{2}\,\partial^{\mu}\partial_{\mu}+\Phi^{\dagger}\,\partial^{\mu}\Phi\,\partial_{\mu}-X_{\lambda}\,\varepsilon^{\lambda\mu\nu}\,\Phi^{\dagger}\,\partial_{\mu}\Phi\,\partial_{\nu}\right\}\chi \\ &+\tfrac{1}{2}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\;\mathrm{tr}\,\chi\left\{(\eta^{\mu\nu}+X_{\lambda}\,\varepsilon^{\lambda\mu\nu})\,\partial_{\mu}(\Phi^{\dagger}\,\partial_{\nu}\Phi)\right\}\chi+\mathcal{O}(\chi^{3})\,. \end{aligned} \tag{3.14}$$

To make contact to the commonly considered perturbations of the form $\Phi\to\Phi+\phi$ for some small $\phi$, we infer from (3.13)

$$\phi \;=\; \mathrm{i}\,\Phi\,\chi-\tfrac{1}{2}\,\Phi\,\chi^{2}+\mathcal{O}(\chi^{3}) \qquad\Longrightarrow\qquad \chi \;=\; -\,\mathrm{i}\,\Phi^{\dagger}\,\phi+\tfrac{\mathrm{i}}{2}\,\Phi^{\dagger}\,\phi\,\Phi^{\dagger}\phi+\mathcal{O}(\phi^{3})\,, \tag{3.15}$$

and thus find for the variation of the action in terms of this perturbation

$$\begin{aligned} \delta S[\Phi,\phi] \;=&\; -\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\;\mathrm{tr}\left\{(\eta^{\mu\nu}+X_{\lambda}\,\varepsilon^{\lambda\mu\nu})\,\partial_{\mu}(\Phi^{\dagger}\,\partial_{\nu}\Phi)\,\Phi^{\dagger}\right\}\phi \\ &+\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\;\mathrm{tr}\,\phi^{\dagger}\left\{\tfrac{1}{2}\,\partial^{\mu}\partial_{\mu}-\tfrac{1}{2}\,(\partial^{\mu}\partial_{\mu}\Phi)\,\Phi^{\dagger}-X_{\lambda}\,\varepsilon^{\lambda\mu\nu}\,\partial_{\mu}\Phi\,(\partial_{\nu}\Phi^{\dagger}+\Phi^{\dagger}\,\partial_{\nu})\right\}\phi \\ &+\mathcal{O}(\phi^{3})\,. \end{aligned} \tag{3.16}$$

The unitarity condition provides in this case a constraint on the perturbation $\phi$ and its hermitian conjugate:

$$(\Phi+\phi)^\dagger(\Phi+\phi) \stackrel{!}{=} \mathbb{1} \qquad \Longrightarrow \qquad \phi^\dagger \;=\; -\Phi^\dagger\,\phi\,\Phi^\dagger \;+\; \Phi^\dagger\,\phi\,\Phi^\dagger\,\phi\,\Phi^\dagger \;+\; \mathcal{O}(\phi^3)\,, \tag{3.17}$$

which allows us to express $\phi^\dagger$ in the variation by $\phi$ up to arbitrary order and thus gives a contribution to the second order variation of $S$ from the term of first order in $\phi^\dagger$. Mathematically speaking, to compute the variation to second order in the perturbation it is not sufficient to consider perturbations in the tangent space $\phi \in T_\Phi U(n)$ but rather one needs to include the leading correction stemming from the exponential map onto $U(n)$ [24] as can be more explicitly seen from the first way of perturbing.

From either one of these variations the equation of motion for Ward's chiral model is read off as

$$(\eta^{\mu\nu} \;+\; X_\lambda\,\varepsilon^{\lambda\mu\nu})\,\partial_\mu(\Phi^\dagger\,\partial_\nu\Phi) \;=\; \partial_x(\Phi^\dagger\,\partial_x\Phi) \;+\; \partial_{y-t}(\Phi^\dagger\,\partial_{y+t}\Phi) \;=\; 0\,, \tag{3.18}$$

which is just the same as in (3.8). Furthermore, we may subject the field to a general coordinate transformation $\delta\Phi = \xi^\mu(x)\,\partial_\mu\Phi$ leading to a change of the action by

$$\delta S \;=\; \int \mathrm{tr}\left(\mathrm{d}t\mathrm{d}x\mathrm{d}y\;\partial^{(\mu}\xi^{\nu)}\,T_{\mu\nu} \;+\; X(\xi)\,J\wedge J\wedge J\right)\,, \tag{3.19}$$

where $\xi = \xi^\mu\,\partial_\mu$. Writing $J = J_\mu \mathrm{d}x^\mu$, yields the standard energy-momentum tensor

$$T_{\mu\nu} \;=\; \mathrm{tr}\left(J_\mu\,J_\nu - \tfrac{1}{2}\eta_{\mu\nu}\,\eta^{\rho\sigma}\,J_\rho\,J_\sigma\right)\,. \tag{3.20}$$

For a choice of $X$ with $X(\partial_t)$, the action is invariant under time translations and thus the energy functional

$$E[\Phi] \;=\; \tfrac{1}{2}\int \mathrm{d}x\mathrm{d}y\;\mathrm{tr}\left(\partial_t\Phi^\dagger\partial_t\Phi + \partial_x\Phi^\dagger\partial_x\Phi + \partial_y\Phi^\dagger\partial_y\Phi\right)\,. \tag{3.21}$$

is conserved [119]. It is also known that the model defined by (3.9) is integrable in the sense of possessing a Lax-pair formulation if and only if $X$ meets the above requirement $X(\partial_t)$ [34, 118, 57] (see also chapter 6). To retain this integrability we will from now on only consider the case with

$$X \;:=\; \mathrm{d}x \qquad \Longleftrightarrow \qquad (X_a) \;=\; (0,1,0,0)\,, \tag{3.22}$$

which yields the original Ward chiral model and leaves the $y$- and $t$-translations unbroken.[7] Furthermore, this choice of $X$ breaks the Lorentz group $SO(1,2)$ down to the linear group $GL(1,\mathbb{R})$ generated by boosts in $y$-direction.

We conclude this account of the commutative Ward model by noticing that for static solutions living on some Grassmannian, i.e. $\Phi^\dagger = \Phi$ we may rewrite the energy functional (3.21) to

$$E[P] \;=\; 2\int \mathrm{d}x\mathrm{d}y\;\mathrm{tr}\,\partial_i P\,\partial^i P \;=\; 8\int \mathrm{d}^2 z\;\mathrm{tr}\,\partial_z P\,\partial_{\bar z} P\,, \tag{3.23}$$

[7] Under $x$-translations, $\delta S \sim \int J\wedge J\wedge J = 24\pi^2\,n$ with $n \in \pi_3(U(n)) = \mathbb{Z}$, where $\pi_3$ denotes the third homotopy group [92].

where the field has been parametrized by a projector $P = P^2 = P^\dagger$ via $\Phi = \mathbb{1} - 2\,P$ (see also section 3.1). Exploiting the cyclicity of the trace, we derive the decomposition

$$\begin{aligned} E &= 4 \int \mathrm{d}^2 z\ \mathrm{tr}\,\{\partial_z P\,P\,\partial_{\bar z}P + P\,\partial_z P\,\partial_{\bar z}P\,P + \partial_{\bar z}\,P\,\partial_z P + P\,\partial_{\bar z}\,\partial_z P\,P\} \\ &= 4 \int \mathrm{d}^2 z\ \mathrm{tr}\,\left\{F\,F^\dagger + G\,G^\dagger\right\}\,, \end{aligned} \tag{3.24}$$

with the convenient abbreviations

$$F \;:=\; (\partial_{\bar z}P)\,PG \;:=\; (\partial_z P)\,P\,. \tag{3.25}$$

Upon closer inspection we find using $P\,(\partial P)\,P = 0$ that

$$E \;=\; 8 \int \mathrm{d}^2 z\ \mathrm{tr}\,P\,\{\partial_{\bar z}P\,\partial_z P - \partial_z P\,\partial_{\bar z}P\} + 16 \int \mathrm{d}^2 z\ \mathrm{tr}\,F\,F^\dagger\,, \tag{3.26}$$

and by similar reasoning

$$E \;=\; -8 \int \mathrm{d}^2 z\ \mathrm{tr}\,P\,\{\partial_{\bar z}P\,\partial_z P - \partial_z P\,\partial_{\bar z}P\} + 16 \int \mathrm{d}^2 z\ \mathrm{tr}\,G\,G^\dagger\,, \tag{3.27}$$

from which it follows that the energy of static hermitian solutions is bounded from below by a BPS bound as

$$Q \;:=\; \int \mathrm{d}^2 z\ \mathrm{tr}\,P\,\{\partial_{\bar z}P\,\partial_z P - \partial_z P\,\partial_{\bar z}P\} \quad \text{with} \quad E \geq |Q|\,. \tag{3.28}$$

The appearing topological charge $Q$ admits [10] a geometrical interpretation[8] and the BPS or Bogomolnyi bound (3.28) is saturated for configurations satisfying either

$$\begin{aligned} F &= (\partial_{\bar z}P)\,P \;=\; (\mathbb{1} - P)\,\partial_{\bar z}P \;=\; 0 \quad \text{(BPS-equation)} \\ \text{or}\quad G &= (\partial_z P)\,P \;=\; (\mathbb{1} - P)\,\partial_z P \;=\; 0 \quad \text{(anti-BPS-equation)}\,. \end{aligned} \tag{3.29}$$

These solutions define what we will henceforth call solitons and anti-solitons, respectively, and it is them that our discussion will focus on in the following since they provide interesting configurations with topological origin. In order to see that solutions to (3.29) are indeed also solutions to the Ward model equation of motion (3.18), we rewrite the latter for static hermitian configurations in terms of the projector as

$$\partial_{\bar z}F^\dagger - \partial_z F \;=\; 0 \;=\; \partial_z G^\dagger - \partial_{\bar z}G\,. \tag{3.30}$$

Finally, note also that in this commutative setting solitons will only arise for non-abelian group spaces like e.g. $U(n)$, $SU(n)$ or $O(n)$ all with $n > 1$ since otherwise $Q = 0$.

A wide variety of soliton solutions to the model just presented or rather to its $SU(n)$ descendant are known and their properties seem quite well understood [118, 119, 79, 115, 120, 64, 111, 121, 57, 58, 3, 59, 60, 27, 61]. We will in later sections refer to specific results in order to compare them to the analysis in the noncommutative setting but for now refrain from giving a more lengthy exposition of these and instead continue by extending the model to the noncommutative case. The interested reader may consult the literature mentioned above for detailed treatments of further issues in this commutative setting.

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[8] In fact, $Q$ is an element of $\pi_2(\mathrm{Gr}\,P) = \mathbb{Z}$.

### 3.2.2 Noncommutative extension of the Ward model

Let us now proceed to the noncommutative extension of the (1+2)-dimensional Ward model which results from Moyal-deforming the Wess-Zumino-Witten-modified integrable sigma model action discussed in the last subsection. The base space for our model is thus exchanged from $\mathbb{R}^{1,2}$ into $\mathbb{R} \times \mathbb{R}^2_\theta$ with the metric kept fixed. Properties and solutions of this model have been subject to intensive study during the last couple of years [72, 73, 74, 9, 127, 56, 75, 76] leading to a good understanding of this model as one of the rare showcases for noncommutative integrable models in 1+2 dimensions. In this subsection, we start by presenting the model itself, its equations of motions and some elementary properties before extending the scope to solutions and their properties in the next sections. We will also postpone an account of the topological properties arising in the noncommutative setting like the BPS bound or the topological charge until then. A recent discussion of some of this material is also given in the diploma thesis [68]. For the subsequent applications we present the results in the star-product as well as in the operator formulation where necessary for later considerations.

*Noncommutative Ward model in star-product formulation*

To shift the model introduced in section 3.2.1 to the noncommutative realm in the star-product formalism as introduced via the Groenewold-Moyal product in section 2.2, we consider $U_\star(n)$ valued fields $\Phi$, where the ordinary group-, i.e. matrix-, multiplication is modified by changing the multiplication of matrix elements to the star-product of the Moyal plane. The field thus satisfies

$$\Phi \star \Phi^\dagger \;:=\; \left( \sum_{k=1}^{n} \Phi_{ik} \star \Phi^\dagger_{kj} \right)_{i,j=1\dots n} \;=\; \mathbb{1}_n \;=\; \Phi^\dagger \star \Phi\,, \tag{3.31}$$

where $\mathbb{1}_n$ denotes here the $n{\times}n$ unit matrix. In analogy to the commutative case we then define the noncommutative nonlinear sigma model action by

$$\begin{aligned} S[\Phi] \;:=\; &-\tfrac{1}{2}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\;\mathrm{tr}\,\eta^{\mu\nu}\,\partial_\mu\Phi^\dagger \star \partial_\nu\Phi \\ &+\tfrac{1}{3}\int \mathrm{d}t\mathrm{d}x\mathrm{d}y\int_0^1 \mathrm{d}\rho\; X_a\,\varepsilon^{abcd}\;\mathrm{tr}\left(\tilde\Phi^\dagger \star \partial_b\tilde\Phi \star \tilde\Phi^\dagger \star \partial_c\tilde\Phi \star \tilde\Phi^\dagger \star \partial_d\tilde\Phi\right)\,, \end{aligned} \tag{3.32}$$

where the Lorentz-breaking constant one-form $X$ is again chosen as in (3.22), rendering the model, similar to the commutative case, integrable at the cost of losing Lorentz invariance. In addition the Lorentz group is independently broken down to the rotations in the $x{-}y$-plane by the introduction of the star product such that in total there is nothing left of the Lorentz invariance of this model. Nevertheless, it is still invariant under $y$- and $t$-translations and thus the conservation of the energy functional

$$\begin{aligned} E[\Phi] \;&=\; \tfrac{1}{2}\int \mathrm{d}x\mathrm{d}y\;\mathrm{tr}\left(\partial_t\Phi^\dagger \star \partial_t\Phi + \partial_x\Phi^\dagger \star \partial_x\Phi + \partial_y\Phi^\dagger \star \partial_y\Phi\right) \\ &=\; \tfrac{1}{2}\int \mathrm{d}^2z\;\mathrm{tr}\left(\partial_t\Phi^\dagger \star \partial_t\Phi + 4\,\partial_z\Phi^\dagger\,\partial_{\bar z}\Phi\right) \end{aligned} \tag{3.33}$$

also pertains to the noncommutative case [67]. Furthermore, the existence of a Lax-pair formulation [73] as in the commutative setting with product replaced by the star-product explicitly shows that the integrability of this model is not impeded by the introduction of noncommutativity (see also section 6.1.1).

Finally, the equation of motion under the usual assumption of vanishing boundary terms is given by

$$(\eta^{\mu\nu} + X_\rho\,\varepsilon^{\rho\mu\nu})\,\partial_\mu(\Phi^\dagger \star \partial_\nu\Phi) \;=\; \partial_x(\Phi^\dagger \star \partial_x\Phi) + \partial_{y-t}(\Phi^\dagger \star \partial_{y+t}\Phi) \;=\; 0\,. \tag{3.34}$$

*Noncommutative Ward model in operator formalism*

Now switching to the operator formalism as introduced in section 2.2.2, the $U_\star(n)$ valued fields $\Phi$ descend to unitary $n{\times}n$ matrices with operator-valued entries[9], i.e. $\Phi \in U(\mathcal{H}^{\oplus n}) = U(\mathbb{C}^n \otimes \mathcal{H})$, where $\mathcal{H} = \mathcal{H}_{\rm HO}$ is the 1-dimensional harmonic oscillator Fock space, and are thus subject to the constraint

$$\Phi\,\Phi^\dagger \;=\; \Phi^\dagger\,\Phi \;=\; \mathbb{1}_n \otimes \mathbb{1}_{\mathcal{H}} \;=:\; \mathbb{1}\,. \tag{3.35}$$

In this setting the corresponding expressions for the noncommutative action (3.32) and energy (3.33) read:

$$\begin{aligned} S[\Phi] \;=\;& \pi \int \mathrm{d}t\ \mathrm{Tr}\left(\theta\,\partial_t\Phi^\dagger\partial_t\Phi + \left[a,\Phi^\dagger\right]\left[a^\dagger,\Phi\right] + \left[a^\dagger,\Phi^\dagger\right]\left[a,\Phi\right]\right) \\ &- \mathrm{i}\,\pi\sqrt{2\,\theta} \int \mathrm{d}t \int_0^1 \mathrm{d}\rho\ \mathrm{Tr}\left(\tilde\Phi^{-1}\,\partial_t\tilde\Phi\ \left[\tilde\Phi^{-1}\,\partial_\rho\tilde\Phi\,,\tilde\Phi^{-1}\left[a{+}a^\dagger,\tilde\Phi\right]\right]\right) \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} E[\Phi] \;&=\; \pi\ \mathrm{Tr}\Big\{\theta\,\partial_t\Phi^\dagger\,\partial_t\Phi + \left[a,\Phi\right]^\dagger\left[a,\Phi\right] + \left[a^\dagger,\Phi\right]^\dagger\left[a^\dagger,\Phi\right]\Big\}\,, \\ &=\; \pi\ \mathrm{Tr}\Big\{\theta\,\partial_t\Phi^\dagger\,\partial_t\Phi + \Delta\Phi^\dagger\,\Phi + \Phi^\dagger\,\Delta\Phi\Big\}\,, \\ &=\; \pi\,\theta\left|\partial_t\Phi\right|^2 + 2\,\pi\left|\left[a,\Phi\right]\right|^2\,, \end{aligned} \tag{3.37}$$

where $|A|^2 := \mathrm{Tr}(A^\dagger A)$ denotes the squared Hilbert-Schmidt norm of $U(n)$-valued operators on $\mathcal{H}$ and Tr denotes the double trace over the $U(n)$ group space as well as over the Hilbert space $\mathcal{H}$. Furthermore, the equation of motion (3.34) is now given by

$$\theta\,\partial_t(\Phi^\dagger\partial_t\Phi) + \Phi^\dagger\,\Delta\Phi - \Delta\Phi^\dagger\,\Phi + \mathrm{i}\,\sqrt{\tfrac{\theta}{2}}\left(\left[a{+}a^\dagger,\Phi^\dagger\right]\partial_t\Phi - \partial_t\Phi^\dagger\left[a{+}a^\dagger,\Phi\right]\right) \;=\; 0\,. \tag{3.38}$$

A remark on the admissible classes of operators seems in order here. Since we only take into account configurations of finite energy, we have to demand that the two terms in the last line of (3.37) are finite and hence, that $\partial_t\Phi$ as well as $\left[a,\Phi\right]$ are Hilbert-Schmidt operators, i.e. their Hilbert-Schmidt norm is finite. Furthermore, we only consider

[9]Note again that we refrain from indicating operators by a different symbol since the operator picture will from now on be our preferred point of view.

solutions for which $\Delta\Phi$, $\partial_t^2\Phi$ are trace-class and $\Phi$ is a bounded operator[10] in order for the other expressions in (3.37) to be well defined. As we will shortly see, imposing more stringent requirements on the allowed operators may facilitate the analysis of solutions to the field equations (3.37) and we will do so if deemed appropriate but otherwise stick to this minimalistic choice.

*Symmetries of the action*

For the following discussion of solutions to the equations of motion (3.34), (3.38) it is advisable to analyze symmetries of the action since these yield configurations of equal action and hence additional solutions capturing moduli of the configurations under consideration. We will therefore now briefly summarize obvious symmetries of the action (3.36) in operator formalism postponing a detailed discussion of the moduli space to later sections.

First realize that the annihilation and creation operators above should be read as $\mathbb{1}_n \otimes a$ and $\mathbb{1}_n \otimes a^\dagger$, respectively, such that the model enjoys a global $U(1) \times SU(n) \times SU(n)$ invariance under

$$\Phi \quad\longrightarrow\quad (\mathcal{V} \otimes \mathbb{1}_{\mathcal{H}})\, \Phi\, (\mathcal{W} \otimes \mathbb{1}_{\mathcal{H}}) \quad \text{with} \quad \mathcal{V}\mathcal{V}^\dagger = \mathbb{1}_n = \mathcal{W}\mathcal{W}^\dagger\,, \tag{3.39}$$

which generates a moduli space to any nontrivial solution of (3.38).

Another symmetry, at least of the energy (3.37), is induced by the $ISO(2)$ euclidean group transformations of the noncommutative plane, as generated by the adjoint action of $a$, $a^\dagger$ and $N$. Specifically, a global translation of the noncommutative plane,

$$(z\,,\, \bar z) \quad\longrightarrow\quad (z+\zeta\,,\, \bar z + \bar\zeta) \;=\; \sqrt{2\theta}\,(a+\alpha\,,\, a^\dagger + \bar\alpha)\;, \tag{3.40}$$

induces on operator-valued scalar functions $f$ the unitary operation

$$\begin{aligned} f \quad\longrightarrow\quad \mathrm{e}^{-\zeta\partial_z - \bar\zeta\partial_{\bar z}}\, f \;&=\; \mathrm{e}^{\alpha\,\mathrm{ad}(a^\dagger) - \bar\alpha\,\mathrm{ad}(a)}\, f \;=\; \mathrm{e}^{\alpha a^\dagger - \bar\alpha a}\, f\, \mathrm{e}^{-\alpha a^\dagger + \bar\alpha a} \\ &=:\; \mathcal{D}(\alpha)\, f\, \mathcal{D}(\alpha)^\dagger\,, \end{aligned} \tag{3.41}$$

where the shift operator $\mathcal{D}$ is the same as defined in (2.23). On the other hand a global rotation of the noncommutative plane,

$$(z\,,\, \bar z) \quad\longrightarrow\quad (\mathrm{e}^{\mathrm{i}\,\vartheta} z\,,\, \mathrm{e}^{-\mathrm{i}\,\vartheta}\bar z) \;=\; \sqrt{2\theta}\,(\mathrm{e}^{\mathrm{i}\,\vartheta} a\,,\, \mathrm{e}^{-\mathrm{i}\,\vartheta} a^\dagger)\;, \tag{3.42}$$

induces the unitary transformation

$$f \quad\longrightarrow\quad \mathrm{e}^{\mathrm{i}\vartheta\,\mathrm{ad}(a^\dagger a)}\, f \;=\; \mathrm{e}^{\mathrm{i}\vartheta\, a^\dagger a}\, f\, \mathrm{e}^{-\mathrm{i}\vartheta\, a^\dagger a} \;=:\; \mathcal{R}(\vartheta)\, f\, \mathcal{R}(\vartheta)^\dagger \tag{3.43}$$

for $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$, because $[N, f] = \bar z\partial_{\bar z} f - \partial_z f z$.

Since the adjoint actions of both $\mathcal{D}$ and $\mathcal{R}$ commute with that of $\Delta$, the energy functional (3.37) is invariant under them. Furthermore, since the adjoint action of $\mathcal{D}$ also commutes with $\mathrm{ad}(a)$ and $\mathrm{ad}(a^\dagger)$, all translates of a solution to (3.38) also qualify as solutions, with equal energy and action. In addition the same holds true for rotated static solutions since they experience no change in the action due to the rotation. However, other unitary transformations, and in particular rotations of time-dependent solutions, will in general change the value of $S$.

[10]As a short reminder: trace-class means of finite trace, whereas for a bounded operator $A$ we have $|A\,|\Psi\rangle|^2 \leq c\,||\Psi\rangle|^2$ with some fixed $c \in \mathbb{R}$ for all $|\Psi\rangle \in \mathcal{H}$.

## 3.3 Relation to noncommutative scalar field theory

Before now delving into the explicit construction of solutions to the noncommutative Ward model we will in this section show that the discussed noncommutative extension of this model is intimately connected to noncommutative scalar field theory allowing us to transfer results obtained in the setting just described to these kind of models which have been a lively area of research since the beginning of this century. This connection, which also holds in the commutative case, is just the manifestation of the fact that to embed a possibly curved group manifold into some euclidean space, we have to implement the constraint to live on this manifold in a scalar field theory via a Lagrange multiplier.

Let us follow the lines of [67] to make this explicit and begin with a rather generic action $S_\theta$ of a real scalar field $\Phi$ living on the Moyal plane with complex coordinates $(z, \bar{z})$ and depending on time $t$,

$$S_\theta[\Phi] \;=\; \int \mathrm{d}t\, \mathrm{d}^2 z \, \left( \frac{1}{2}\, \partial_t \Phi \star \partial_t \Phi - \partial_z \Phi \star \partial_{\bar{z}} \Phi - V_\star(\Phi) \right) , \tag{3.44}$$

where the subscript on the potential signifies star-product multiplication again. The associated equation of motion reads

$$\Box \Phi - V'_\star(\Phi) \;=\; 0 \,. \tag{3.45}$$

Various types of this scalar model have been investigated in recent literature, e.g. in [42, 44, 135, 81, 108, 88, 28, 80, 43, 48, 4, 109, 29, 98], yielding mainly approximate results for infinite noncommutativity parameter $\theta$ under some assumptions on the general form of the potential. To make contact between these papers and the model analyzed here, we further specify $V$ to be nonnegative with minimum 0 at $\Phi_0$ and assume $V'$ to be polynomial, i.e.

$$V(\Phi) \;\geq\; 0 \,, \quad V(\Phi_0) \;=\; 0 \quad \text{and} \quad V'(\Phi) \;=\; v \prod_i (\Phi - \Phi_i) \,. \tag{3.46}$$

Static classical configurations $\Phi_{\mathrm{cl}}$ extremize the corresponding energy functional

$$E_\theta[\Phi] \;=\; \int \mathrm{d}^2 z \; (\partial_z \Phi \star \partial_{\bar{z}} \Phi + V_\star(\Phi)) \,, \tag{3.47}$$

which for large values of $\theta$ is dominated by the potential term, because $z = \mathcal{O}(\sqrt{\theta})$. Expanding around $\theta = \infty$, one obtains an expansion for the solution [43]

$$\Phi_{\mathrm{cl}} \;=\; \Phi^{(0)} + \frac{1}{\theta}\, \Phi^{(1)} + \ldots \quad \text{with} \quad \Phi^{(0)} \;=\; \sum_i \phi_i \, P_i \,, \quad \phi_i \in \mathbb{C} \,, \tag{3.48}$$

where $\{P_i\}$ is an arbitrary orthogonal resolution of the star-algebra identity,

$$P_i \star P_j \;=\; \delta_{ij}\, P_i \quad \text{and} \quad \sum_i P_i \;=\; \mathbb{1} \,. \tag{3.49}$$

Therefore $\Phi^{(0)}$ solves (3.45) for infinite value of $\theta$. Note that for $\Phi^{(0)}$ to be a solution it is necessary that the collection $\{P_i\}$ appearing in (3.48) is complete, i.e. satisfies the second equation of (3.49). If it were not, $V'_\star(\Phi^{(0)}) \neq 0$ in general.

For later use we also introduce the rank $k_i$ of the projector $P_i$ via[11]

$$\int \frac{\mathrm{d}^2 z}{2\pi\theta} P_i \;=\; k_i\,, \quad \text{for } k_i \in \mathbb{N}_0\,. \tag{3.50}$$

Since $\Phi^{(1)}$ and all further terms in the expansion are determined by $\Phi^{(0)}$ [43], any classical solution is fixed by an assignment of projectors $P_i$ to the extrema $\Phi_i$ of the potential. This appearance of projectors parametrizing possible solutions is a recurring theme and will be most prominent in the discussion of the abelian noncommutative grassmannian sigma model as we shall see in section 4.1. Now, restricting ourselves to stable solutions, we always associate the zero projector, which is admissible, to the local maxima of $V$. The expansion of the classical energy reads

$$E_\theta[\Phi_{\mathrm{cl}}] \;=\; \theta\, E_0 + E_1 + \frac{1}{\theta}\, E_2 + \ldots \quad \text{with} \quad E_0 \;=\; 2\,\pi \sum_i k_i\, V(\phi_i) \tag{3.51}$$

and

$$E_1 \;=\; \int \mathrm{d}^2 z\, \partial_z \Phi^{(0)} \star \partial_{\bar z} \Phi^{(0)} \;=\; \sum_{i,j} \phi_i\, \phi_j \int \mathrm{d}^2 z\, \partial_z P_i \star \partial_{\bar z} P_j\,. \tag{3.52}$$

Any complete collection $\{P_i\}$ thus extremizes $E_\theta$ at leading order in $\theta$. Beyond this, however, $E_1$ lifts this infinite degeneracy: its extremization selects a finite-dimensional class of identity resolutions.

To exemplify what was just described we follow the paper [43], where an asymmetric double-well potential with local minima $V(0) = 0$ and $V(\lambda) > 0$ for some $\lambda \in \mathbb{R}$ was chosen. The authors assigned

$$P \quad \leftrightarrow \quad \Phi \;=\; \lambda \quad \text{and} \quad \mathbb{1} - P \quad \leftrightarrow \quad \Phi \;=\; 0\,, \tag{3.53}$$

which led to the asymptotic solution

$$\Phi^{(0)} \;=\; \lambda\, P \quad \text{with} \quad E_0 \;=\; 2\,\pi\, k\, V(\lambda) \quad \text{and} \quad E_1 \;=\; \lambda^2 \int \mathrm{d}^2 z\, |\partial_z P|^2\,. \tag{3.54}$$

Instead of this asymmetric potential we now consider the symmetric double-well potential

$$V_\star(\Phi) \;=\; \beta\,(\Phi \star \Phi - 1)^2 \quad \text{with } \beta \in \mathbb{R}\,, \tag{3.55}$$

where the square is also implied to be taken with the star-product. It follows that $V'_\star(\Phi) = 4\,\beta\,(\Phi + 1) \star \Phi \star (\Phi - 1)$ and we associate

$$P \quad \leftrightarrow \quad \Phi \;=\; -1 \quad \text{and} \quad \mathbb{1} - P \quad \leftrightarrow \quad \Phi \;=\; +1\,, \tag{3.56}$$

[11] This definition is justified by the transition to the operator formulation, where $\int \mathrm{d}^2 z \longrightarrow 2\,\pi\,\theta\, \mathrm{Tr}_{\mathcal{H}}$

which implies a solution

$$\Phi^{(0)} \;=\; \mathbb{1} - 2P\,, \quad \text{as well as} \quad E_0 \;=\; 0 \quad \text{and} \quad E_1 \;=\; 4\int \mathrm{d}^2 z\, |\partial_z P|^2\,, \tag{3.57}$$

as above. It is easy to see that all higher corrections, i.e. $\Phi^{(1)}$, $E_2$ etc., come with negative powers of $\beta$. Therefore, we have the exact result

$$\Phi_{\mathrm{cl}} \quad\longrightarrow\quad \Phi^{(0)} \quad \text{and} \quad E_\theta[\Phi_{\mathrm{cl}}] \quad\longrightarrow\quad E_1 \tag{3.58}$$

in the limit of infinite stiffness, $\beta \to \infty$, and there is no effective potential on the moduli space. This limit nails the value of $\Phi$ to $-1$ (in $\mathrm{im}\,P$) or to $+1$ (in $\ker P$).

Note that these classical configurations are idempotent, i.e. $\Phi^{(0)} \star \Phi^{(0)} \;=\; \mathbb{1}$. This idempotency has another perspective since idempotent fields also appear in the unitary sigma model discussed in section 3.1, where they define grassmannian submanifolds of the defining group via $P = \frac{1}{2}(1 - \Phi)$. The simplest example in this respect is the $U(1)$ model with complex unimodular field $\Phi$. The theory, originally considered in the commutative case, where it is free, becomes interacting when being Moyal-deformed. Its two grassmannian submanifolds correspond precisely to the two idempotent values above, namely $\Phi = \pm 1$. Hence, if we extend our double-well model to a Mexican-hat model for complex $\Phi$, the stiff limit will yield the constraint $\Phi^\dagger \star \Phi = 1$, defining the $U_\star(1)$ sigma model, and our trial configurations $\Phi^{(0)}$ for $P$ of rank $k$ parametrize precisely the Grassmannian $\mathrm{Gr}(k, \mathcal{H})$, the coset of all unitary transformations on $\mathcal{H}$ modulo the product space of those transformations in $\mathrm{im}\,P$ or $\ker P$ (see also (4.6)). The only modification owed to the extension described is a factor of two in the energy functional which, for $k < \infty$, can be manipulated to

$$E_1 \;=\; 8\int \mathrm{d}^2 z\, |\partial_z P|^2 \;=\; 8\,\pi\, k \,+\, 16\int \mathrm{d}^2 z\, |(\mathbb{1} - P)\star \partial_{\bar z} P|^2 \;\geq\; 8\,\pi\, k\,, \tag{3.59}$$

revealing the Bogomolnyi bound already encountered in 3.2.1.

These considerations explicitly show the close connection between scalar field theory on the one side and the nonlinear sigma model on the other hand which derives from the implementation of the sigma-model constraint as a Lagrange multiplier. In order to comprise both into our discussion as far as possible we are thus led to the idea of considering a whole family of noncommutative sigma models with action

$$S_\gamma[\Phi] \;=\; S_2[\Phi] + \gamma\, S_3[\Phi]\,, \tag{3.60}$$

where the constituents $S_2$ and $S_3$ are given by the noncommutative version of the expressions in (3.9) and the fields are taken to be $U_\star(1)$- or more generally $U_\star(n)$-valued. By varying $\gamma$, we thus get a family of actions connecting the ordinary sigma model at $\gamma = 0$ to the Ward model at $\gamma = 1$, both based on $U_\star(n)$. In the static situation $S_3$ vanishes and the ordinary sigma model is equivalent to the Ward model. This connection will give us important information in the context of the adiabatic approach considered in chapter 7.

Finally we would like to mention that apart from the analysis of noncommutative scalar field theories huge efforts have simultaneously been made to also understand other noncommutative theories. As an example, the noncommutative $\mathbb{C}P^n$ model has recently come under scrutiny [77, 37, 38, 32, 90, 96, 97] with one result being that the equivalence between the $U(2)$ chiral model and the $\mathbb{C}P^1$ model, known to exist for the commutative setting, is lost in the noncommutative case [97], thus showing that the investigation of noncommutative extensions to known models may offer new features of otherwise well understood issues.

# Chapter 4

# Static Solutions to the Noncommutative Sigma Model

After having introduced the noncommutative Ward model in the preceding chapter we now address the solutions to this model concentrating, for the time being, solely on static solutions. Time-dependent solutions will be discussed later on in chapter 6. Most of the material presented here appeared in [72, 73, 74, 9, 127, 56, 75, 24, 67, 76], and we try to give a unified account of these results. For pedagogical reasons, we will begin by presenting special classes of solutions in the beginning and comment on more general ones in the last part of this chapter.

The euclidean action of the static noncommutative Ward model coincides with the energy functional of the (1+2)-dimensional time-dependent model (3.37) evaluated on static configurations [73]. We thus have in the operator formalism

$$E[\Phi] \;=\; 2\pi\,\big|\big[a,\Phi\big]\big|^2 \;=\; \pi\,\mathrm{Tr}\left(\Delta\Phi^\dagger\,\Phi+\Phi^\dagger\Delta\Phi\right)\,, \tag{4.1}$$

with Laplacian given as in (2.18).

Varying $E[\Phi]$ under the above constraint, one deduces the static equation of motion corresponding to (3.38),

$$0 \;=\; \big[a\,,\Phi^\dagger[a^\dagger,\Phi]\big]+\big[a^\dagger,\Phi^\dagger[a\,,\Phi]\big] \;=\; \Phi^\dagger\,\Delta\Phi-\Delta\Phi^\dagger\,\Phi\,. \tag{4.2}$$

There is a wealth of finite-energy configurations $\Phi$ which fulfill the equation of motion (4.2). A subclass of those is distinguished by possessing a smooth commutative limit, in which $\Phi$ merges with a commutative solution. The latter have been classified by [115, 128]. We call these "nonabelian" because they cannot appear in the $U(1)$ case, where the commutative model is a free field theory and does not admit finite-energy solutions. Yet, there can and do exist non-trivial abelian finite-energy solutions at finite $\theta$ whose $\theta\to 0$ limit is necessarily singular and produces a discontinuous configuration. We term such solutions "abelian" even in case they are embedded in a nonabelian group. They provide the genuinely new features arising in the noncommutative setting. Both abelian and nonabelian solutions are interesting in their own right as abelian solutions present new classical configurations and the latter allow a direct comparison with the behavior of configurations in the commutative case.

As an introductory example of a static solution let us consider the simplest possible case, namely solutions of zero energy. They are given by $\Phi=\mathcal{V}\otimes\mathbb{1}_{\mathcal{H}},\ \mathcal{V}\in U(n)$ as can be seen from the following line of thoughts:

Energy zero implies $\big[a,\Phi\big]=0$ and thus $\Phi$ must only depend on the annihilation operator $a$ but not on $a^\dagger$. We may then (at least formally) expand the field in terms of $a$ with $n{\times}n$-matrix valued constants to see that due to the non-boundedness of the annihilation operator $\Phi$ can actually only depend on $a^0$, i.e. be proportional to unity, if it is to be bounded. Hence we indeed find

$$E[\Phi] \;=\; 0 \qquad \Longrightarrow \qquad \Phi \;=\; \mathcal{V}\otimes \mathbb{1}_{\mathcal{H}}\,. \tag{4.3}$$

Having found a very simple solution we will now take a more systematic approach towards the analysis of all possible solutions and encounter many additional ones on the way.

## 4.1 Grassmannian configurations and BPS bound

It is a daunting task to classify all solutions to the full equation of motion (4.2), and we begin by focusing on the subset of hermitian ones. Any, not necessarily classical, hermitian configuration obeys

$$\Phi^\dagger \;=\; \Phi \qquad \Longrightarrow \qquad \Phi^2 \;=\; \mathbb{1}_n\otimes\mathbb{1}_{\mathcal{H}} \;=:\; \mathbb{1} \tag{4.4}$$

and is conveniently parametrized by a hermitian projector $P=P^\dagger$ via

$$\Phi \;=:\; \mathbb{1}-2P \;=\; P^\perp - P \;=\; \mathrm{e}^{\mathrm{i}\pi P} \quad \text{with} \quad P^2 \;=\; P\,, \tag{4.5}$$

where $P^\perp=\mathbb{1}-P$ denotes the complementary projector. The set of all projectors unitarily equivalent to $P$ is called the Grassmannian, and it is given by the coset

$$\mathrm{Gr}(P) \;=\; \frac{U(\mathcal{H}^{\oplus n})}{U(\mathrm{im}\,P)\times U(\ker P)}\,, \tag{4.6}$$

similar to the definition (3.2) in the finite dimensional case. Thus, each given $P$, and thus each hermitian $\Phi$, belongs to a certain Grassmannian, and the space of all hermitian $\Phi$ decomposes into a disjoint union of Grassmannians. The restriction to hermitian $\Phi$ reduces the unitary to a grassmannian sigma model, whose configuration space is parametrized by projectors $P$.

Equation (4.1) for the energy of a solution is given in terms of the projectors $P$ as

$$\begin{aligned}\frac{1}{8\,\pi}\,E[\Phi] \;&=\; \mathrm{Tr}\,\big(\big[a^\dagger,P\big]\,\big[P,a\big]\big)\\ &=\; \mathrm{Tr}\,\big(P\,(a\,a^\dagger+a^\dagger a)\,P - P\,a\,P\,a^\dagger P - P\,a^\dagger P\,a\,P\big)\\ &=\; \mathrm{Tr}\,\big(P\,a\,(\mathbb{1}-P)\,a^\dagger P + P\,a^\dagger\,(\mathbb{1}-P)\,a\,P\big)\\ &=\; \big|P\,\big[a,P\big]\big|^2 + \big|\big[a,P\big]\,P\,\big|^2\,,\end{aligned} \tag{4.7}$$

which, similar to the commutative case, may be compared with a topological charge defined[1] as [88, 39]

$$\begin{aligned} Q[\Phi] &= \frac{1}{4}\,\theta\,\mathrm{Tr}\,(\Phi\,\partial_z\Phi\,\partial_{\bar z}\Phi - \Phi\,\partial_{\bar z}\Phi\,\partial_z\Phi) \\ &= \mathrm{Tr}\left(P\,[a^\dagger,P]\,[a,P] - P\,[a,P]\,[a^\dagger,P]\right) \\ &= \mathrm{Tr}\left(P - P\,a\,P\,a^\dagger P + P\,a^\dagger P\,a\,P\right) \\ &= \mathrm{Tr}\left(P\,a\,(\mathbb{1}-P)\,a^\dagger P - P\,a^\dagger\,(\mathbb{1}-P)\,a\,P\right) \\ &= \left|P\,[a,P]\right|^2 - \left|[a,P]\,P\right|^2 . \end{aligned} \tag{4.8}$$

Note that each hermitian projector $P$ gives rise to the complementary projector $P^\perp := \mathbb{1} - P$, with the properties[2]

$$E[\mathbb{1}-P] \;=\; E[P] \qquad \text{and} \qquad Q[\mathbb{1}-P] \;=\; -Q[P]\,. \tag{4.9}$$

Now, comparing (4.7) and (4.8) we get the relations

$$\frac{1}{8\,\pi}\,E[P] \;=\; Q[P] + 2\left|[a,P]\,P\right|^2 \;=\; -Q[P] + 2\left|P\,[a,P]\right|^2 , \tag{4.10}$$

which yield the BPS bound (see also section 3.2.1)

$$E[P] \;\geq\; 8\,\pi\;|Q[P]| \tag{4.11}$$

for any hermitian configuration. Supposed that either $\mathrm{im}\,P$ or $\ker P = \mathrm{im}\,\mathbb{1} - P$ is finite dimensional the topological charge $Q$ is integer valued and admits a simple interpretation given by

$$Q[P] \;=\; \begin{cases} \mathrm{Tr}\,P = \mathrm{rank}\,P = \dim(\mathrm{im}\,P) & , \quad \dim(\mathrm{im}\,P) < \infty \\ -\mathrm{Tr}(\mathbb{1}-P) = -\mathrm{corank}\,P = -\,\dim(\ker P)\,, & \quad \dim(\ker P) < \infty \end{cases} , \tag{4.12}$$

as can be easily checked with (4.8) and cyclicity of the trace in this finite dimensional case.

Nevertheless, the interpretation of the nature of the topological charge in general is not as clear as in the commutative case where the topological charge $Q$ is an element of $\pi_2(\mathrm{Gr}(P)) = \mathbb{Z}$, with $S^2$ being the compactified plane. Hence, it is an invariant of the Grassmannian if one excludes from the definition of the Grassmannian (3.2) any singular unitary transformations with nontrivial winding at infinity. It is less obvious how to properly extend this consideration to the infinite-dimensional cases encountered here [52]. In the case that $P$ has a finite rank or corank the argument still carries through but subtleties may arise in other cases. We therefore take a pragmatic viewpoint and demand

[1] Note that $Q$ as given here is well defined for all hermitian solutions of finite energy.

[2] By a slight abuse of notation, we denote the energy $E$ and topological charge $Q$ as functionals of $P$ again.

$Q[P]$ to be constant throughout the Grassmannian. This may downsize the above coset $\mathrm{Gr}(P)$ by restricting the set of admissible unitary transformations $\mathcal{U}$.

Let us make this more explicit by computing $Q[\mathcal{U}P\mathcal{U}^\dagger]$ for some unitary transformation $\mathcal{U} \in U(\mathcal{H}^{\oplus n})$ on the Hilbert space. To this end, we define

$$\omega \;=\; \mathcal{U}^\dagger\,[a,\mathcal{U}] \quad \text{and} \quad \omega^\dagger \;=\; \mathcal{U}^\dagger\,[a^\dagger,\mathcal{U}] \quad \text{with} \quad \big[a+\omega, a^\dagger+\omega^\dagger\big] \;=\; \mathbb{1}\,, \tag{4.13}$$

as elements of the Lie algebra $\mathfrak{u}(\mathcal{H}^{\oplus n})$ of $U(\mathcal{H}^{\oplus n})$.[3] A short calculation yields

$$\begin{aligned} Q[\mathcal{U}P\mathcal{U}^\dagger] \;&=\; \mathrm{Tr}\,\big(P\,[a^\dagger+\omega^\dagger,P]\,[a+\omega,P] - P\,[a+\omega,P]\,[a^\dagger+\omega^\dagger,P]\big) \\ &=\; Q[P] \;+\; \mathrm{Tr}\,P\Big(\big\{[a^\dagger,P]\,,\,\omega\big\} - \big\{[a,P]\,,\,\omega^\dagger\big\} \\ &\qquad\qquad - \omega^\dagger(\mathbb{1}-P)\omega + \omega(\mathbb{1}-P)\omega^\dagger\Big)P\,, \end{aligned} \tag{4.14}$$

with the anticommutator $\{a,b\} := a\,b+b\,a$. This equation constrains $\omega$ in terms of $P$. In case of $\mathrm{Tr}P<\infty$, this indeed reduces to

$$Q[\mathcal{U}P\mathcal{U}^\dagger] \;=\; Q[P] \;+\; \mathrm{Tr}\,P\,\big([\omega,a^\dagger] + [a,\omega^\dagger] + [\omega,\omega^\dagger]\big) \;=\; Q[P]\,. \tag{4.15}$$

Alternatively, we may calculate the infinitesimal variation of $Q[P]$ under $P \to P{+}\delta P$. Remembering that $P$ and $\delta P$ are bounded and that their commutators with $a$ or $a^\dagger$ are Hilbert-Schmidt, we get

$$\begin{aligned} \delta Q[P] \;&=\; \mathrm{Tr}\,\big([a^\dagger,[P,\delta P]\,[a,P]] - [a,[P,\delta P]\,[a^\dagger,P]]\big) \\ &\quad + 3\,\mathrm{Tr}\,\delta P\,\big([a^\dagger,P]\,[a,P] - [a,P]\,[a^\dagger,P]\big)\,, \end{aligned} \tag{4.16}$$

with the first trace being a "boundary term". Variations inside the Grassmannian are given by

$$\delta P \;=\; \big[\Lambda_{\rm o},P\big]\,, \quad \text{with} \quad \Lambda_{\rm o}^\dagger \;=\; -\Lambda_{\rm o} \quad \text{and} \quad P\Lambda_{\rm o}P \;=\; 0 \;=\; (\mathbb{1}-P)\Lambda_{\rm o}(\mathbb{1}-P) \tag{4.17}$$

(see section 5.1.3), and thus

$$\begin{aligned} \delta Q[P] \;&=\; \mathrm{Tr}\,\big([a,\Lambda_{\rm o}[a^\dagger,P]] - [a^\dagger,\Lambda_{\rm o}[a,P]]\big) \\ &\quad + 3\,\mathrm{Tr}\,\big[P\,,\,\Lambda_{\rm o}[a,P]\,[a^\dagger,P] - \Lambda_{\rm o}[a^\dagger,P]\,[a,P]\big]\,. \end{aligned} \tag{4.18}$$

Hence, if $\Lambda_{\rm o}$ is bounded, the second term vanishes and $\delta Q[P]$ reduces to the boundary term.[4]

Formally, another invariant of grassmannian configurations is the rank of the projector,

$$r \;:=\; \mathrm{rank}(P) \;=\; \dim(\mathrm{im}\,P)\,, \tag{4.19}$$

[3] In fact, the finite-energy condition (see remark below (3.38)) enforces $[a\,,\mathcal{U}P\mathcal{U}^\dagger]$ to be Hilbert-Schmidt which implies that $[\omega\,,P]$ is Hilbert-Schmidt, as well. Unfortunately, this does not suffice to guarantee the constancy of $Q$ in $\mathrm{Gr}(P)$. On the other hand, $\omega$ itself need not even be bounded, as in the example of (3.43) where $\omega=(\mathrm{e}^{\mathrm{i}\vartheta}{-}1)a$.

[4] Yet again, this condition is too strong a demand as the examples of (3.41) and (3.43) indicate.

which may differ from $Q$ when being infinite. For simplicity, let us consider a special class of projectors consisting of a finite rank piece and a part proportional to the unit operator in the Hilbert space. Such a projector can be decomposed as

$$P \;=\; \mathcal{U}\,\big(\widehat{P} + P'\big)\,\mathcal{U}^\dagger \quad \text{with} \quad \widehat{P} \;=\; \bar{P}\otimes\mathbb{1}_{\mathcal{H}} \quad \text{and} \quad \widehat{P}\,P' \;=\; 0 \;=\; P'\,\widehat{P}\,, \tag{4.20}$$

where $\mathcal{U}$ is an admissible unitary transformation, $\bar{P}$ denotes a constant projector, and the rank $r'$ of $P'$ is finite. In this case the difference of $r$ and $Q$ is determined by the invariant $U(n)$ trace [96, 97]

$$R[P] \;=\; \operatorname{tr}\bar{P} \;\in\{0,1,\ldots,n\}\;, \tag{4.21}$$

and these projectors are characterized by the pair $(R,Q)$. Formally, the total rank then becomes $r=R{\cdot}\infty+Q$. Since $a$ and $a^\dagger$ commute with $\widehat{P}$, the topological charge depends only on the finite-rank piece,

$$Q[P] \;=\; Q[\widehat{P}+P'] \;=\; Q[P'] \;=\; \operatorname{Tr}P' \;=\; r'\,. \tag{4.22}$$

For $\bar{P}=0$ one obtains the abelian solutions because they fit into $U(\mathcal{H})$. For $R{>}0$ we have nonabelian solutions because more than one copy of $\mathcal{H}$ is needed to accommodate them.

Having presented general properties of grassmannian configurations we introduce a convenient formulation for the configurations under consideration realizing that any projector in $\mathbb{C}^n\otimes\mathcal{H}=\mathcal{H}^{\oplus n}$ can be parametrized as

$$P \;=\; |T\rangle\,\langle T|T\rangle^{-1}\langle T|\,, \tag{4.23}$$

where

$$|T\rangle \;=\; \Big(|T^1\rangle\,|T^2\rangle\,\ldots\,|T^r\rangle\Big) \;=\; \begin{pmatrix} |T^1_1\rangle & |T^2_1\rangle & \ldots & |T^r_1\rangle \\ |T^1_2\rangle & |T^2_2\rangle & \ldots & |T^r_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ |T^1_n\rangle & |T^2_n\rangle & \ldots & |T^r_n\rangle \end{pmatrix} \;=\; \begin{pmatrix} |T_1\rangle \\ |T_2\rangle \\ \vdots \\ |T_n\rangle \end{pmatrix} \tag{4.24}$$

denotes an $n{\times}r$ array of kets in $\mathcal{H}$, with $r$ possibly being infinite. Thus,

$$\langle T|T\rangle \;=\; \Big(\langle T^\ell|T^m\rangle\Big) \;=\; \Big(\sum_{i=1}^n\langle T^\ell_i|T^m_i\rangle\Big) \tag{4.25}$$

stands for an invertible $r{\times}r$ matrix, and $r$ is the rank of $P$. The not necessarily orthonormal column vectors $|T^\ell\rangle$ span the image $\operatorname{im}P$ of the projector in $\mathcal{H}^{\oplus n}$. There is some ambiguity in the definition (4.23) of $|T\rangle$ since

$$|T\rangle \quad\to\quad |T\rangle\,\Gamma\,, \quad \text{for } \Gamma\in GL(r) \tag{4.26}$$

amounts to a change of basis in $\operatorname{im}P$ and does not change the projector $P$. This freedom may be used to normalize $\langle T|T\rangle=\mathbb{1}_r$, which is still compatible with $\Gamma\in U(r)$. On the other hand, a unitary transformation

$$|T\rangle \quad\to\quad \mathcal{U}\,|T\rangle\,, \quad \text{for } \mathcal{U}\in U(\mathcal{H}^{\oplus n}) \tag{4.27}$$

yields a unitarily equivalent projector $\mathcal{U}P\mathcal{U}^\dagger$.

Altogether, we have the bijection

$$\tilde{P} \;=\; \mathcal{U}\,P\,\mathcal{U}^\dagger \qquad\Longleftrightarrow\qquad |\tilde{T}\rangle \;=\; \mathcal{U}\,S\,|T\rangle \;=\; \mathcal{U}\,|T\rangle\,\Gamma \quad\text{with}\quad S\in GL(\operatorname{im}P)\,. \tag{4.28}$$

Here, the trivial action of $\mathcal{U}\in U(\operatorname{im}P)\times U(\ker P)$ on $P$ can be subsumed in the $S$ action on $|T\rangle$.

We may be more explicit for a diagonal projector which can be cast into the form

$$P_{\rm d} \;=\; \operatorname{diag}\Bigl(\underbrace{\mathbb{1}_{\mathcal{H}},\ldots,\mathbb{1}_{\mathcal{H}}}_{R\text{ times}},\,P_Q\,,\underbrace{\mathbf{0}_{\mathcal{H}},\ldots,\mathbf{0}_{\mathcal{H}}}_{n-R-1\text{ times}}\Bigr) \quad\text{with}\quad P_Q \;=\; \sum_{m=0}^{Q-1}|m\rangle\langle m|\,. \tag{4.29}$$

Formally, $P_{\rm d}$ has rank $r$ in $\mathcal{H}^{\oplus n}$. A corresponding $r{\times}n$ array of kets via (4.23) would be

$$|T_{\rm d}\rangle \;=\; \begin{pmatrix} |\mathcal{H}\rangle & \emptyset & \ldots & \emptyset & 0_Q \\ \emptyset & |\mathcal{H}\rangle & & \emptyset & 0_Q \\ \vdots & & \ddots & & \vdots \\ \emptyset & \emptyset & & |\mathcal{H}\rangle & 0_Q \\ \emptyset & \emptyset & \ldots & \emptyset & |T_Q\rangle \\ \vdots & \vdots & & \vdots & \\ \emptyset & \emptyset & \ldots & \emptyset & 0_Q \end{pmatrix}\,, \tag{4.30}$$

where we used

$$\begin{aligned} |\mathcal{H}\rangle \;&:=\; \bigl(|0\rangle\;|1\rangle\;|2\rangle\;|3\rangle\;\ldots\bigr)\,, \qquad & \emptyset \;&:=\; \bigl(\,0\quad 0\quad 0\quad 0\quad\ldots\bigr)\,, \\ |T_Q\rangle \;&:=\; \bigl(|0\rangle\;|1\rangle\;\ldots\;|Q{-}1\rangle\bigr)\,, \qquad & 0_Q \;&:=\; \bigl(\,\underbrace{0\quad 0\quad\ldots\quad 0}_{Q\text{ times}}\,\bigr)\,. \end{aligned} \tag{4.31}$$

In the abelian case, $n=1$ and $R=0$, this reduces to $|T_{\rm d}\rangle=|T_Q\rangle$, and any projector is unitarily equivalent to $P_Q$. Due to (4.9), analogous results are valid in the complementary case $P\to\mathbb{1}{-}P$.

The above paragraph yields a useful way of parametrizing solutions which will be used extensively throughout the following sections.

## 4.2 BPS solutions

After having presented general properties of grassmannian configurations let us now delve a little deeper and focus on special solutions within a Grassmannian. For $\Phi=\mathbb{1}{-}2P$, the equation of motion (4.2) reduces to

$$\begin{aligned} 0 \;&=\; \bigl[\Delta P,P\bigr] \\ &=\; \bigl[a^\dagger,(\mathbb{1}{-}P)aP\bigr]+\bigl[a,Pa^\dagger(\mathbb{1}{-}P)\bigr] \\ &=\; \bigl[a,(\mathbb{1}{-}P)\,a^\dagger P\bigr]+\bigl[a^\dagger,P\,a\,(\mathbb{1}{-}P)\bigr]\,, \end{aligned} \tag{4.32}$$

which may be compared with the commutative result (3.30). Still, it is not clear how to characterize its full solution space. However, (4.32) is identically satisfied by projectors subject to [43, 48]

$$\text{either the BPS equation} \qquad 0 \;=\; \big[a,P\big]\,P \;=\; (\mathbb{1}-P)\,a\,P \tag{4.33}$$

$$\text{or the anti-BPS equation} \qquad 0 \;=\; \big[a^\dagger,P\big]\,P \;=\; (\mathbb{1}-P)\,a^\dagger P\,, \tag{4.34}$$

which are only of "first-order". Solutions to (4.33) are called solitons while those to (4.34) are named anti-solitons in analogy to the commutative case discussed in section 3.2.1. Hermitian conjugation shows that the latter are obtained from the former by exchanging $P \leftrightarrow \mathbb{1}-P$, and so, without loss of generality, we can ignore the anti-BPS solutions for most of this thesis. Equation (4.33) means that

$$a \quad \text{maps} \quad \operatorname{im} P \hookrightarrow \operatorname{im} P \tag{4.35}$$

and, hence, characterizes subspaces of $\mathcal{H}^{\oplus n}$ which are stable under the action of $a$. The term "BPS equation" derives from the observation that (4.33) inserted in (4.10) implies the saturation of the BPS bound (4.11), which simplifies to

$$E[P] \;=\; 8\,\pi\,Q[P] \;=\; 8\,\pi\,\mathrm{Tr}\big(P\,a\,(\mathbb{1}-P)\,a^\dagger P\big) \;=\; 8\,\pi\,\mathrm{Tr}\big(P\,a\,a^\dagger P - a\,P\,a^\dagger\big)\,. \tag{4.36}$$

For projectors of finite dimensional image or kernel we find $E = 8\pi\,Q$ with $Q$ given as in (4.12) and for BPS solutions of the particular form (4.20) we indeed recover that $E[P] = 8\,\pi\,\mathrm{Tr}P'$. Clearly, these BPS solutions constitute the absolute minima of the energy functional within each Grassmannian.

When the parametrization (4.23) is used, the BPS condition (4.33) simplifies to

$$a\,|T_i^{\ell}\rangle \;=\; |T_i^{\ell'}\rangle\,\gamma_{\ell'}{}^{\ell} \quad \text{for some } r{\times}r \text{ matrix } \gamma = \big(\gamma_{\ell'}{}^{\ell}\big)\,, \tag{4.37}$$

or abbreviated

$$a|T\rangle \;=\; |T\rangle\,\gamma\,, \tag{4.38}$$

which represents the action of $a$ in the basis chosen for $\operatorname{im} P$. For instance, the ket $|T_Q\rangle$ in (4.30) indeed obeys

$$a\,|T_Q\rangle \;=\; |T_Q\rangle\,\gamma_Q \quad \text{with} \quad \gamma_Q \;=\; \begin{pmatrix} 0 & \sqrt{1} & 0 & \dots & 0 \\ 0 & 0 & \sqrt{2} & & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \dots & 0 & \sqrt{Q-1} \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}\,, \tag{4.39}$$

and the diagonal projector $P_{\mathrm{d}}$ in (4.29) is BPS, but it is by far not the only one.

Any basis change in $\operatorname{im} P$ induces a similarity transformation $\gamma \mapsto \Gamma\,\gamma\,\Gamma^{-1}$, which leaves $P$ unaltered and thus has no effect on the value of the energy. Therefore, it suffices to consider $\gamma$ to be of Jordan normal form. More generally, a unitary transformation (4.28) is compatible with the BPS condition (4.38) if and only if

$$\mathcal{U}^\dagger a\,\mathcal{U}\,|T\rangle \;=\; |T\rangle\,\gamma_{\mathcal{U}} \quad \text{for some } r{\times}r \text{ matrix } \gamma_{\mathcal{U}}\,, \tag{4.40}$$

which in the notation of (4.13) implies that

$$\omega\,|T\rangle \;=\; |T\rangle\,\gamma_\omega \quad \text{with} \quad \gamma_\omega = \gamma_{\mathcal{U}} - \gamma\,. \tag{4.41}$$

The trivially compatible transformations are those in $U(\mathrm{im}\,P)\times U(\ker P)$, which can be subsumed in $S\in GL(\mathrm{im}\,P)$ and lead to $\gamma_{\mathcal{U}}=\Gamma^{-1}\,\gamma\,\Gamma$. Another obvious choice are rigid symmetries of the energy functional, as given in (3.41), (3.43), and (3.39) for $\mathcal{W}=\mathcal{V}^\dagger$. The challenging task then is to identify the nontrivial BPS-compatible unitary transformations, since these relate different $a$-stable subspaces of fixed dimension in $\mathcal{H}^{\oplus n}$ and thus generate the multi-soliton moduli space. For $|T_Q\rangle$ in (4.30) we shall accomplish this infinitesimally in section 5.1.3.

## 4.3 Classification of solitons

### 4.3.1 Abelian solitons

Let us now take a closer look at the BPS solutions of the noncommutative $U(1)$ sigma model. All finite-energy configurations are based on $R=0$ and have rank $r=Q<\infty$, thus $E=8\pi r$. Using the parametrization (4.23) the task is to solve the "eigenvalue equation"

$$a\,|T\rangle \;=\; |T\rangle\,\gamma\,, \quad \text{for} \quad \gamma \;=\; \bigoplus_{s=1}^{q}\begin{pmatrix} \alpha_s & 1 & 0 & \dots & 0\\ 0 & \alpha_s & 1 & & 0\\ \vdots & & \ddots & \ddots & \\ 0 & 0 & & \alpha_s & 1\\ 0 & 0 & \dots & 0 & \alpha_s \end{pmatrix} \quad \text{with} \quad \alpha_s\in\mathbb{C}\,, \tag{4.42}$$

where the Jordan cells have sizes $r_s$ for $s=1,\dots,q$ with $\sum_{s=1}^{q} r_s = r$. For a given rank $r$, the above matrices $\gamma$ parametrize the $r$-soliton moduli space. The general solution, unique up to cell-wise normalization and basis changes $|T^{(s)}\rangle \mapsto |T^{(s)}\rangle\Gamma^{(s)}$, reads

$$\begin{aligned} |T\rangle \;&=\; \Big(|T^{(1)}\rangle \;\dots\; |T^{(q)}\rangle\Big) \\ \text{with} \quad |T^{(s)}\rangle \;&=\; \Big(|\alpha_s\rangle \;\; a^\dagger|\alpha_s\rangle \;\dots\; \tfrac{1}{(r_s-1)!}(a^\dagger)^{r_s-1}|\alpha_s\rangle\Big) \end{aligned} \tag{4.43}$$

and is based on the coherent states (2.19). In the star-product picture, the corresponding $\Phi$ represents $r$ lumps centered at positions $\alpha_s$ with degeneracies $r_s$ in the $x-y$ plane. Note that a Jordan block of size $r_s$ yields a sub-basis $\big\{\,|\alpha_s\rangle, a^\dagger|\alpha_s\rangle,\dots,(a^\dagger)^{r_s-1}|\alpha_s\rangle\big\}$, whose span is obviously invariant under the action of $a$.

Lifting a degeneracy by "point-splitting", the related Jordan cell dissolves into different eigenvalues. Hence, the generic situation has $r_s=1\ \forall s$, and so

$$\gamma \;=\; \mathrm{diag}(\alpha_1,\alpha_2,\dots,\alpha_r) \qquad \Longleftrightarrow \qquad |T\rangle \;=\; \Big(|\alpha_1\rangle\;|\alpha_2\rangle\;\dots\;|\alpha_r\rangle\Big)\,, \tag{4.44}$$

revealing again the key role of coherent states (2.19). We note that our solution depends on $r$ complex moduli. The ensuing BPS projector

$$P_\alpha \;=\; \sum_{k,l=1}^{r} |\alpha_k\rangle\,\big(\langle\alpha_.|\alpha_.\rangle\big)^{-1}_{kl}\,\langle\alpha_l| \tag{4.45}$$

generates a superposition of $r$ Gaussian lumps in the Moyal plane. The fusion of these lumps by means of taking the limit of coinciding positions and taking appropriate coordinates smoothly produces lumps of higher weight or degeneracy again [43].

Besides the (inessential) choice of normalizations, only a residual permutation freedom remains in the solution $|T\rangle = |\alpha\rangle := \big(|\alpha_1\rangle, |\alpha_2\rangle, \ldots, |\alpha_r\rangle\big)$. This corresponds to a relabeling of the lumps and emphasizes their bosonic character [50, 51].

Each solution can be translated via a unitary transformation mediated by $\mathcal{D}(\beta)$ (2.23), which essentially shifts $\alpha_\ell \to \alpha_\ell+\beta\ \forall\ell$ and rotated by $\mathcal{R}(\vartheta)$ (2.25), which moves $\alpha_\ell \to \mathrm{e}^{\mathrm{i}\vartheta}\alpha_\ell\ \forall\ell$. The individual values of $\alpha_\ell$ (the soliton locations) may also be moved around by appropriately chosen unitary transformations, so that any $r$-soliton configuration can be reached from the diagonal one, which describes $r$ solitons on top of each other at the coordinate origin [48]:

$$|T\rangle \ = \ \mathcal{U}\,|T_r\rangle\,\Gamma \qquad \Longleftrightarrow \qquad P \ = \ \mathcal{U}\,P_r\,\mathcal{U}^\dagger \quad \text{with} \quad P_r \ = \ \sum_{m=0}^{r-1} |m\rangle\langle m|\,. \tag{4.46}$$

We illustrate the latter point with the example of $r=2$. Generically,

$$\begin{aligned} \big(|\alpha_1\rangle\ |\alpha_2\rangle\big) \ &= \ \mathcal{U}\,S\,\big(|0\rangle\ |1\rangle\big) \ = \ \mathcal{U}\,\big(|0\rangle\ |1\rangle\big)\,\Gamma \\ \text{with} \quad \mathcal{U}\,S \ &= \ |\alpha_1\rangle\langle 0| + |\alpha_2\rangle\langle 1| + \ldots\,, \end{aligned} \tag{4.47}$$

where the omitted terms annihilate $|0\rangle$ and $|1\rangle$. Factorizing $\mathcal{U}S$ yields

$$\begin{aligned} \mathcal{U} \ = \ &\frac{1}{\sqrt{1{-}|\sigma|^2}}\Big(\frac{|\alpha_1\rangle}{\sqrt{\langle\alpha_1|\alpha_1\rangle}}\ \frac{|\alpha_2\rangle}{\sqrt{\langle\alpha_2|\alpha_2\rangle}}\Big)\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\gamma} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\gamma'}\end{pmatrix} \\ &\begin{pmatrix}-\sin\beta' & \cos\beta'\\ \sin\beta & -\cos\beta\end{pmatrix}\begin{pmatrix}\langle 0|\\ \langle 1|\end{pmatrix} \ + \ \ldots \end{aligned} \tag{4.48}$$

$$\text{and} \quad \Gamma \ = \ \begin{pmatrix}\cos\beta & \cos\beta'\\ \sin\beta & \sin\beta'\end{pmatrix}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\gamma} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\gamma'}\end{pmatrix} \tag{4.49}$$

$$\text{with} \quad \sigma \ = \ \frac{\langle\alpha_1|\alpha_2\rangle}{\sqrt{\langle\alpha_1|\alpha_1\rangle}\sqrt{\langle\alpha_2|\alpha_2\rangle}} \ = \ \mathrm{e}^{\bar\alpha_1\alpha_2-\frac12|\alpha_1|^2-\frac12|\alpha_2|^2} \ = \ \mathrm{e}^{-\mathrm{i}(\gamma-\gamma')}\cos(\beta{-}\beta')\,,$$

so that $|\sigma|^2 = \mathrm{e}^{-|\alpha_1-\alpha_2|^2}$, and $\beta{+}\beta'$ and $\gamma{+}\gamma'$ remain undetermined. The rank-2 projector becomes

$$\begin{aligned} P \ &= \ \frac{1}{1{-}|\sigma|^2}\Big(\frac{|\alpha_1\rangle\langle\alpha_1|}{\langle\alpha_1|\alpha_1\rangle} + \frac{|\alpha_2\rangle\langle\alpha_2|}{\langle\alpha_2|\alpha_2\rangle} - \frac{\sigma|\alpha_1\rangle\langle\alpha_2| + \bar\sigma|\alpha_2\rangle\langle\alpha_1|}{\sqrt{\langle\alpha_1|\alpha_1\rangle}\sqrt{\langle\alpha_2|\alpha_2\rangle}}\Big) \\ &= \ \mathcal{U}\,\Big(|0\rangle\langle 0| + |1\rangle\langle 1|\Big)\,\mathcal{U}^\dagger\,. \end{aligned} \tag{4.50}$$

### 4.3.2 Nonabelian solitons

We proceed to infinite-rank projectors. For simplicity, let us discuss the case of $U(2)$ solitons – the results will easily generalize to $U(n)$. Clearly, we can embed two finite-rank

BPS solutions (with $R=0$) into $U(\mathcal{H}\oplus\mathcal{H})$ by letting each act on a different copy of $\mathcal{H}$. Such configurations are noncommutative deformations of the trivial projector $\bar{P}=\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$ and thus represent a combination of abelian solitons. Therefore, we turn to projectors with $R=1$ and so, formally, $r=\infty+Q$. Given (4.20) such solutions may be considered as noncommutative deformations of the $U(2)$ projector $\bar{P}=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. For this reason, one expects the generic solution $|T\rangle$ to (4.38) to be combined from a full set of states for the first copy of $\mathcal{H}$ and a finite set of coherent states $|\alpha_\ell\rangle$ in the second copy of $\mathcal{H}$,

$$|T\rangle \;=\; \begin{pmatrix}|\mathcal{H}\rangle & 0 & 0 & \dots & 0\\ \emptyset & |\alpha_1\rangle & |\alpha_2\rangle & \dots & |\alpha_Q\rangle\end{pmatrix}, \tag{4.51}$$

which yields the projector

$$P \;=\; \mathbb{1}_{\mathcal{H}} \,\oplus\, \sum_{k,\ell=1}^{Q} |\alpha_k\rangle \Big(\langle\alpha_.|\alpha_.\rangle\Big)^{-1}_{k\ell} \langle\alpha_\ell|\,, \tag{4.52}$$

with $E=8\pi\,Q$. This projector is of the special form (4.20), with $\bar{P}=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $\mathcal{U}=\mathbb{1}$. By a unitary transformation on the second copy of $\mathcal{H}$ such a configuration can be mapped to the diagonal form

$$|T_{\rm d}\rangle \;=\; \begin{pmatrix}|\mathcal{H}\rangle & 0 & 0 & \dots & 0\\ \emptyset & |0\rangle & |1\rangle & \dots & |Q{-}1\rangle\end{pmatrix} \qquad\Longrightarrow\qquad P_{\rm d} \;=\; \mathbb{1}_{\mathcal{H}} \,\oplus\, P_Q\,. \tag{4.53}$$

It is convenient to reorder the basis of $\operatorname{im} P_{\rm d}$ such that

$$|T_{\rm d}\rangle \;=\; \begin{pmatrix}0 & 0 & \dots & 0 & |\mathcal{H}\rangle\\ |0\rangle & |1\rangle & \dots & |Q{-}1\rangle & \emptyset\end{pmatrix} \;=\; \begin{pmatrix}S_Q\\ P_Q\end{pmatrix}|\mathcal{H}\rangle \;=:\; \hat{T}_{\rm d}\,|\mathcal{H}\rangle\,, \tag{4.54}$$

where

$$P_Q \;=\; \sum_{m=0}^{Q-1}|m\rangle\langle m|\,, \quad\text{and}\quad S_Q \;=\; (a\,\tfrac{1}{\sqrt{N}})^Q \;=\; \sum_{m=Q}^{\infty}|m{-}Q\rangle\langle m| \tag{4.55}$$

denotes the $Q$-th power of the shift operator. The form of (4.54) suggests to pass from states $|T\rangle=\begin{pmatrix}|T_1\rangle\\ |T_2\rangle\end{pmatrix}$ with $R=1$ to operators $\hat{T}=\begin{pmatrix}\hat{T}_1\\ \hat{T}_2\end{pmatrix}$ on $\mathcal{H}$[5]:

$$|T\rangle \;=\; \hat{T}\,|\mathcal{H}\rangle \qquad\Longrightarrow\qquad P \;=\; \hat{T}\,(\hat{T}^\dagger\hat{T})^{-1}\hat{T}^\dagger\,. \tag{4.56}$$

In fact, it is always possible to introduce $\hat{T}$ as

$$\hat{T}_i \;=\; \sum_{\ell=1}^{r}|T_i^\ell\rangle\langle\ell{-}1| \qquad\Longleftrightarrow\qquad |T_i^\ell\rangle \;=\; \hat{T}_i\,|\ell{-}1\rangle \quad\text{for } i=1,2\,,\ \ell=1,\dots,r\,. \tag{4.57}$$

[5]The "ˆ" is used here again to avoid confusion.

We may even put $\hat T^\dagger \hat T = \mathbb{1}_{\mathcal H}$ by using the freedom $\hat T \to \hat T\,\hat\Gamma$ with an operator $\hat\Gamma = (\hat T^\dagger \hat T)^{-1/2}$. Our example of $|T_{\rm d}\rangle$ in (4.54) is already normalized since

$$S_1\,S_1^\dagger \;=\; \mathbb{1}_{\mathcal H} \quad \text{but} \quad S_1^\dagger S_1 \;=\; \mathbb{1}_{\mathcal H} - |0\rangle\langle 0| \qquad \Longrightarrow \qquad S_Q^\dagger S_Q + P_Q \;=\; \mathbb{1}_{\mathcal H}\,. \tag{4.58}$$

It is instructive to turn on a BPS-compatible unitary transformation in (4.20). In our example (4.54), we apply [78]

$$\mathcal{U}(\mu) \;=\; \begin{pmatrix} S_Q\sqrt{\frac{N_Q}{N_Q+\mu\bar\mu}}\,S_Q^\dagger & S_Q\frac{\bar\mu}{\sqrt{N_Q+\mu\bar\mu}} \\ \frac{\mu}{\sqrt{N_Q+\mu\bar\mu}}\,S_Q^\dagger & \frac{\mu\,P_Q-\sqrt{N_Q}}{\sqrt{N_Q+\mu\bar\mu}} \end{pmatrix} \tag{4.59}$$
$$\text{with} \quad N_Q \;=\; a^{\dagger Q} a^Q \;=\; N(N{-}1)\cdots(N{-}Q{+}1)$$

to $\hat T_{\rm d} = (S_Q, P_Q)^t$ of (4.54). With the help of

$$S_Q P_Q \;=\; 0 \;=\; N_Q P_Q \quad \text{and} \quad S_Q\sqrt{N_Q} \;=\; a^Q \tag{4.60}$$

we arrive at

$$\hat T(\mu) \;=\; \mathcal{U}(\mu)\,\hat T_{\rm d} \;=\; \begin{pmatrix} a^Q \\ \mu \end{pmatrix} \frac{1}{\sqrt{N_Q+\mu\bar\mu}} \;=\; \begin{pmatrix} a^Q \\ \mu \end{pmatrix}\hat\Gamma \;=:\; \check T(\mu)\,\hat\Gamma\,. \tag{4.61}$$

This transformation can be regarded as a regularization of $\hat T_{\rm d}$ since

$$\lim_{\mu\to 0}\hat T(\mu) \;=\; \left(\begin{smallmatrix} 1 & 0 \\ 0 & {\rm e}^{{\rm i}\delta}\end{smallmatrix}\right)\hat T_{\rm d} \quad \text{with} \quad \delta = \lim_{\mu\to 0}\arg\mu \tag{4.62}$$

and thus

$$\lim_{\mu\to 0} P(\mu) \;=\; P_{\rm d}\,, \tag{4.63}$$

but note the singular normalization in the limit! For completeness, we also display the transformed projector,

$$P(\mu) \;=\; \mathcal{U}(\mu)\begin{pmatrix} \mathbb{1}_{\mathcal H} & \mathbf{0}_{\mathcal H} \\ \mathbf{0}_{\mathcal H} & P_Q \end{pmatrix}\mathcal{U}^\dagger(\mu) \;=\; \begin{pmatrix} a^Q\frac{1}{N_Q+\mu\bar\mu}a^{\dagger Q} & a^Q\frac{\bar\mu}{N_Q+\mu\bar\mu} \\ \frac{\mu}{N_Q+\mu\bar\mu}a^{\dagger Q} & \frac{\mu\bar\mu}{N_Q+\mu\bar\mu} \end{pmatrix}. \tag{4.64}$$

It remains to be seen that such projectors are indeed BPS? Writing $|T\rangle = \hat T|\mathcal H\rangle$, the BPS condition

$$a\,|T_i^\ell\rangle \;=\; |T_i^{\ell'}\rangle\,\gamma_{\ell'}{}^{\ell} \qquad \text{implies} \qquad a\,\hat T_i \;=\; \hat T_i\,\hat\gamma \quad \text{with } i=1,2 \tag{4.65}$$

for some operator $\hat\gamma$ in $\mathcal H$. We do not know the general solution for arbitrary $\hat\gamma$. However, an important class of solutions arises for the choice $\hat\gamma = a$ where the BPS condition reduces to the "holomorphicity condition"[6]

$$\big[a, \hat T_i\big] \;=\; 0\,, \quad \text{for } i=1,2\,. \tag{4.66}$$

[6] This may even be the general case: If there exists an invertible operator $\hat\Gamma$ solving $a\hat\Gamma = \hat\Gamma\hat\gamma$, then the general solution to (4.65) reads $\hat T_i = \check T_i\hat\Gamma$ with $[a, \check T_i] = 0$, and $\hat\Gamma$ can be scaled to unity.

These equations are satisfied by any set of functions $\{\hat{T}_1, \hat{T}_2\}$ of $a$ alone, i.e. not depending on $a^\dagger$. Indeed, for the example of $\hat{T}_\mathrm{d}$ in (4.54), we concretely have

$$\hat{\gamma} \;=\; a\,P_Q \;+\; (\mathbb{1}_\mathcal{H}{-}P_Q)\,a\,\sqrt{\tfrac{N-Q}{N}}\,, \tag{4.67}$$

while $\check{T}(\mu)$ in (4.61) is obviously holomorphic and thus BPS. Because $\hat{T}(\mu)$ emerges from $\hat{T}_\mathrm{d}$ via a BPS-compatible unitary transformation it shares the topological charge $Q$ and the energy $E = 8\pi Q$ with the latter.

The generalization to arbitrary values of $n$ and $R < n$ is straightforward: $\hat{T}$ becomes an $n{\times}R$ array of operators $(\hat{T}_i^L)$, on which left multiplication by $a$ amounts to right multiplication by an $R{\times}R$ array of operators $\hat{\gamma}$. For the special choice $\hat{\gamma}_{L'}{}^L = \delta_{L'}{}^L a$, any collection of holomorphic functions of $a$ serves as a solution for $\hat{T}_i^L$. Quite generally, one can show that for polynomial functions $\hat{T}_i^L(a)$ the resulting projector has a finite topological charge $Q$ given by the degree of the highest polynomial and is thus of finite energy [77, 73, 32]. In this formulation it becomes evident that nonabelian solutions have a smooth commutative limit, where $\sqrt{2\theta}a \to z$ and $\theta \to 0$. Indeed, they are seen as deformations of the well known solitons in the $\mathrm{Gr}(n,R) = \frac{U(n)}{U(R)\times U(n-R)}$ grassmannian sigma model.[7] Hence, the moduli space of the nonabelian solitons coincides with that of their commutative cousins. By rescaling $\hat{T} \to \hat{T}\,\Gamma$ with a $\mathbb{C}$-valued $R{\times}R$ matrix $\Gamma$ we can eliminate $R$ complex parameters from $nR$ independent polynomials. For a charge-$Q$ solution, there remain $nRQ + (n{-}1)R$ complex moduli, of which $(n{-}1)R$ parametrize the vacuum and $nRQ$ describe the position and shape of the multi-soliton [77]. For the case of $\mathbb{C}P^1$ this yields a complex $2Q$-dimensional soliton moduli space represented by $\binom{\check{T}_1}{\check{T}_2} = \binom{a^Q+\ldots+\nu}{\lambda a^Q+\ldots+\mu}$. Our sample calculation above suggests that taking $\check{T}_2 \to \mu$ and then performing the limit $\mu \to 0$ one recovers the complex $Q$-dimensional moduli space of the abelian solitons given by $|T\rangle = \big(|\alpha_1\rangle\,\ldots\,|\alpha_{Q-1}\rangle\big)$ as a boundary.

## 4.4 Uniton theory and non-BPS solutions

In the preceding discussion we derived general properties of BPS solutions and presented some generic examples. These may be embedded in a more general framework, the so-called uniton theory, instigated in the commutative case by the work of Uhlenbeck et al. [115, 120], which allows for the classification of solutions to two-dimensional euclidean sigma models (for a review see, e.g. [133]). The main result of this analysis is the assertion that under certain assumptions all solutions to the equations of motion can be constructed from basic ones by the addition of such unitons. In fact, any classical solution $\Phi$ for the commutative $U(n)$ sigma model can be constructed iteratively, with at most $n{-}1$ so-called unitons as building blocks [115, 128]. In this context the encountered BPS configurations or solitons are precisely the one-uniton solutions. Following the analysis of [25] we will now present a generalization of this scheme to the noncommutative setting.

[7] For the above-discussed example one has $\mathrm{Gr}(2,1) = \mathbb{C}P^1$.

Let us start with a general noncommutative configuration $\Phi \in U(\mathcal{H}^{\oplus n})$ and assume we are given a hermitian projector $P$ satisfying

$$P\,\Phi_-\,(\mathbb{1}-P) \;=\; 0 \quad \text{and} \quad (\mathbb{1}-P)\,\Phi_-\,P + 2\,(\mathbb{1}-P)\big[a,P\big] \;=\; 0\,, \tag{4.68}$$

where we employed the following notation similar to the introduction of $J$ in (3.10):

$$\chi_- \;:=\; \chi^\dagger\,\big[a,\chi\big] \tag{4.69}$$

with $\chi$ being an operator on $\mathcal{H}^{\oplus n}$ not to be confused with the perturbation considered in (3.13). Defining the configuration $\Psi$ as

$$\Psi \;:=\; \Phi\,(\mathbb{1}-2\,P) \tag{4.70}$$

we can then prove the following

**Proposition 4.1 (Uniton construction)**

*i.* $\Psi_- \;=\; \Phi_- + 2\,\big[a,P\big]$

*ii. The energy* (4.1) *of* $\Psi$ *is given by* $E[\Psi] \;=\; E[\Phi] + 8\pi\,Q[P]$.

*iii. If* $\Phi$ *is a solution to the equations of motion* (4.2) *then* $\Psi$ *also solves them.*

*Proof.* The first proposition is easily proved by direct calculation:

$$\begin{aligned}
\Psi_- \;&=\; (\mathbb{1}-2\,P)\,\Phi^\dagger\,\big[a,\phi\,(\mathbb{1}-2\,P)\big] \\
&=\; -2\,(\mathbb{1}-2\,P)\,\big[a,P\big] + (\mathbb{1}-2\,P)\,\Phi_-\,(\mathbb{1}-2\,P) \\
&=\; 2\,\big(P\,\big[a,P\big] + \big[a,P\big]\,P\big) + P\,\Phi_-\,P + (\mathbb{1}-P)\,\Phi_-\,P + (\mathbb{1}-P)\,\Phi_-\,(\mathbb{1}-P)\,.
\end{aligned}$$

Here we made use of the fact that every field $\chi$ can be decomposed as

$$\chi \;=\; P\,\chi\,P + (\mathbb{1}-P)\,\chi\,P + P\,\chi\,(\mathbb{1}-P) + (\mathbb{1}-P)\,\chi\,(\mathbb{1}-P)\,, \tag{4.71}$$

and the projector property $P^2 = P$. Since these four parts are mutually orthogonal with respect to the squared Hilbert-Schmidt norm, the energy of $\Psi$ also decomposes into a sum allowing us to rewrite it as

$$\begin{aligned}
\tfrac{1}{2\pi}\,E[\Psi] \;&=\; \big|\big[a,\Psi\big]\big|^2 \;=\; |\Psi_-|^2 \\
&=\; |P\,\Psi_-\,P|^2 + |(\mathbb{1}-P)\,\Psi_-\,P|^2 + |P\,\Psi_-\,(\mathbb{1}-P)|^2 + |(\mathbb{1}-P)\,\Psi_-\,(\mathbb{1}-P)|^2 \\
&=\; |\Phi_-|^2 + \Big(\big|P\,\big[a,P\big]\big|^2 - \big|\big[a,P\big]\,P\big|^2\Big) \\
&=\; \tfrac{1}{2\pi}\,E[\Phi] + 4\,Q[P]\,.
\end{aligned}$$

Finally, the equation of motion may be rewritten to

$$\big[a^\dagger,\Phi_-\big] + \big[a,\Phi_-^\dagger\big] \;=\; 0\,, \tag{4.72}$$

from which one derives upon insertion that if (4.72) holds for $\Phi$ then $\Psi$ also is a solution. ∎

We have thus seen that it is possible to construct a new solution from a given one by the addition of a projector $P$ subject to certain conditions. A projector satisfying (4.68) is called a $\Phi$-uniton and in this way $\Psi$ is obtained from $\Phi$ by adding the $\Phi$-uniton $P$. Note that the addition of a uniton is invertible in the sense that if we constructed $\Psi$ in the above way then $\mathbb{1}-P$ is a $\Psi$-uniton and adding $\mathbb{1}-P$ to $\Psi$ yields $-\Phi$, as can be easily checked.

Let us turn to a more special case in which the full beauty of the uniton approach emerges. For this we now consider a restricted class of configurations, namely those for which $\dim(\operatorname{im}\big[a,\Phi\big]) < \infty$.[8] The solutions given by (4.20) definitely belong to this class and furthermore, we know of no solutions which are not included in it. In the case at hand we can lower the energy of a solution by the following construction.

**Proposition 4.2 (Addition of negative uniton)**
*For a solution $\Phi$ with $\dim(\operatorname{im}\big[a,\Phi\big]) < \infty$ define $P$ as the projector onto $\operatorname{im}\Phi_- = \operatorname{im}\big(\Phi^+\big[a,\Phi\big]\big)$. Then $\mathbb{1}-P$ is a $\Phi$-uniton and we have for the energy of the ensuing solution*

$$E[\Phi(\mathbb{1}-2\,(\mathbb{1}-P))] \;=\; E[\Phi] - 8\pi\,\dim\operatorname{im}\Phi_-\,. \tag{4.73}$$

*Proof.* First note that $\dim(\operatorname{im}\big[a,\Phi\big]) < \infty$ implies the same property for $\operatorname{im}\Phi_-$ and hence $\infty > \dim(\operatorname{im}\Phi_-) = \dim(\ker(\mathbb{1}-P)) = -\,Q[\mathbb{1}-P]$ establishing the above energy formula if $\mathbb{1}-P$ is a $\Phi$-uniton. To prove that this is indeed the case we rewrite the condition (4.68) in the form

$$(\mathbb{1}-P)\,\Phi_-\,P \;=\; 0 \quad\wedge\quad (\mathbb{1}-P)\,\Big\{\Phi_-^\dagger + 2\,a^\dagger\Big\}\,P \;=\; 0\,,$$

which shows that $\mathbb{1}-P$ is a $\Phi$-uniton if and only if $\Phi_-$ as well as $\Phi_-^\dagger + 2\,a^\dagger$ map $\operatorname{im}P$ into itself. Obviously this is the case for $\Phi_-$ by definition. Furthermore, we find from the equation of motion (4.2)

$$0 \;=\; \big[a,\Phi_-\big] + \big[a^\dagger,\Phi_-^\dagger\big] \;=\; \big[\Phi_-,\Phi_-^\dagger + 2\,a^\dagger\big]\,.$$

Hence, considering the image of $\chi\,P := (\Phi_-^\dagger + 2\,a^\dagger)\,P$ yields

$$\chi\,P\,\mathcal{H}^{\oplus n} \;=\; \chi\,P\,\Phi_-\,\mathcal{H}^{\oplus n} \;=\; \chi\,\Phi_-\,\mathcal{H}^{\oplus n} \;=\; \Phi_-\,\chi\,\mathcal{H}^{\oplus n} \subset \operatorname{im}\Phi_- \;=\; \operatorname{im}P\,,$$

which concludes the proof. ■

This results entails some important conclusions.

**Corollary 4.3 (Uniton factorization)**
*For every solution $\Phi$ as above there exists a sequence of solutions $\{\Phi_i\}_{i=0,\dots,m}$ with $m \in \mathbb{N}_0$ such that $\Phi_m = \Phi$, $\Phi_0$ is a zero energy solution as given in (4.3) and each $\Phi_{i+1}$ is obtained from $\Phi_i$ by the addition of a $\Phi_i$-uniton.*

[8] Actually, it would have been sufficient to demand $\dim\operatorname{im}\Phi_- < \infty$ but in view of our former definition (see section 3.2.2) we deemed the given condition more appropriate and are not aware of any configuration not included in either both or none of these two.

*Proof.* By the above assertion we can lower the energy of the solution $\Phi$ and repeat this procedure until we arrive at a solution of zero energy (note here that the energy is lowered in integer steps). Since this process is invertible we repeat the procedure in the reverse way thus establishing the result. ∎

**Corollary 4.4 (Quantization of energy)**
*The energy of all solutions to the equation of motion belonging to the above defined class is an integer multiple of* $8\pi$.

*Proof.* This is a direct consequence of the uniton factorization. Let $P_i$ be the corresponding $\Phi_i$-unitons, assumed to have finite dimensional image. The energy is then given by

$$E[\Phi] \;=\; 8\pi \sum_{i=0}^{m-1} \dim(\operatorname{im} P_i) \;=\; 8\pi \sum_{i=0}^{m-1} \operatorname{rank} P_i \,. \tag{4.74}$$

∎

Solutions of the next-to-lowest possible energy values $8\pi$ and $16\pi$ play a special role in this context as can be seen from the following

**Corollary 4.5 (Solutions of small energy)**
*A solution of the class defined above with energy* $8\pi k$ *for* $k = 1, 2$ *is, up to unitary transformations in the group space as given by* (3.39), *a rank* $k$ *BPS solution.*

*Proof.* This claim is obvious for a solution of energy $8\pi$ which derives from the addition of a rank 1 uniton to the basic solution (4.3). A solution of energy $16\pi$ may now be either constructed from a solution of zero energy or a solution of energy $8\pi$. The first case is easily handled by the same argument whereas for the latter we show that the addition of a rank 1 uniton to a BPS solution of energy $8\pi$ always yields a BPS solution. The general case can be reduced to this setting.[9] Now, let $\Phi_1 = 1 - 2\,P_1$ be such a BPS solution and let P be a $\Phi_1$-uniton of rank 1 with image $\operatorname{im} P := \mathcal{I}$. The uniton-condition (4.68) may be rephrased as

$$(\mathbb{1}{-}P)\,\Phi_{1\,-}^{\dagger}\,P \;=\; 0 \quad\wedge\quad (\mathbb{1}{-}P)\,\{\Phi_{1\,-} + 2\,a\}\,P \;=\; 0\,,$$

making it obvious that $\mathcal{I}$ has to be an invariant subspace of $\Phi_{1\,-}^{\dagger}$ as well as $\Phi_{1\,-} + 2\,a$. The image $\mathcal{I}_1 := \operatorname{im} P_1$ is a 1-dimensional subspace of the Hilbert space and we find that the image of $\Phi_{1\,-}^{\dagger} \;=\; -\,2\,(\mathbb{1}{-}P_1)\,a^{\dagger}\,P_1$ is a 1-dimensional subspace of $\mathcal{I}_1^{\perp}$. Furthermore, $\ker\,\Phi_{1\,-}^{\dagger} = \mathcal{I}_1^{\perp}$ and we thus see that invariance of $\mathcal{I}$ under $\Phi_{1\,-}^{\dagger}$ implies $\mathcal{I} \subset \mathcal{I}_1^{\perp}$. Hence the new solution $\Phi = (\mathbb{1} - 2\,P_1)(\mathbb{1} - 2\,P) = \mathbb{1} - 2\,(P_1 + P)$ is grassmannian with projector $P_1 + P$ onto $\mathcal{I}_1 \oplus \mathcal{I}$. On $\mathcal{I}$ the operator $\Phi_{1\,-} + 2\,a$ can be written as $2\,(\mathbb{1}{-}P_1)\,a$ and the invariance of $\mathcal{I}$ under it is thus equivalent to $a\,\mathcal{I} \subset \mathcal{I}_1 \oplus \mathcal{I}$. Finally, using that $\Phi_1$ is a BPS solutions, i.e. $a\,\mathcal{I}_1 \subset \mathcal{I}_1$ we conclude that $a\,(\mathcal{I}_1 \oplus \mathcal{I}) \subset \mathcal{I}_1 \oplus \mathcal{I}$ and therefore $\Phi$ is also a BPS solution, as claimed. ∎

[9] To see this take a general solution $\Phi_1 = (\mathcal{V} \otimes \mathbb{1}_{\mathcal{H}})\,\tilde{\Phi}_1(\mathcal{W} \otimes \mathbb{1}_{\mathcal{H}})$ of energy $8\pi$ with $\tilde{\Phi}_1$ BPS and a rank 1 $\Phi_1$-uniton $P$. The new solution $\Phi = \Phi_1\,(\mathbb{1} - 2\,P) = (\mathcal{V} \otimes \mathbb{1}_{\mathcal{H}})\,\tilde{\Phi}_1\,(\mathbb{1} - 2\,\tilde{P})\,(\mathcal{W} \otimes \mathbb{1}_{\mathcal{H}})$, where $\tilde{P} := (\mathcal{W}^{\dagger} \otimes \mathbb{1}_{\mathcal{H}})\,P\,(\mathcal{W} \otimes \mathbb{1}_{\mathcal{H}})$ is a $\tilde{\Phi}_1$-uniton and the situation is indeed traced back to the BPS case.

Note that it is not possible to extend the result of the previous corollary to solutions of energy $8\pi\, r$ with $r > 3$, which may be illustrated by the following $U(1)$ example of a solution of energy $3 \cdot 8\pi$.

*Example* 4.1 *(block-diagonal solutions of energy* $3 \cdot 8\pi$*)*
Consider the simple BPS solution $\Phi_1 := \mathbb{1} - 2\,P_0$ with $P_0 := |0\rangle\langle 0|$ and define a rank 2 projector $P$ by

$$P \;:=\; |1\rangle\langle 1| + \tfrac{1}{1+|c|^2}\,(|0\rangle + c\,|2\rangle)\,(\langle 0| + \bar{c}\,\langle 2|)\ , \quad \text{for } c \in \mathbb{C}\,.$$

It is then easy to check that $P$ is a $\Phi_1$-uniton and the ensuing new solution $\Phi$ is given by

$$\Phi \;=\; \Phi_1\,(\mathbb{1} - 2\,P) \;=\; \begin{pmatrix} \frac{1-|c|^2}{1+|c|^2} & 0 & \frac{2\,\bar{c}}{1+|c|^2} & & \\ 0 & -1 & 0 & & \\ \frac{-2\,c}{1+|c|^2} & 0 & \frac{1-|c|^2}{1+|c|^2} & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}\,. \tag{4.75}$$

There exist some special choices for the parameter $c$. First of all, if $c = \infty$ we find that (4.75) reduces to the known diagonal rank 3 BPS solution (4.46) parametrized by the projector $\sum_{k=0}^{2} |k\rangle\langle k|$. Furthermore, for $c = 0$ we find a grassmannian diagonal non-BPS solution with projector $|1\rangle\langle 1|$. All other values of $c$ imply non-grassmannian solutions of equal energy and hence lead to a family, more explicitly a sphere $\mathbb{C}P^1$, of solutions interpolating between the rank 3 BPS solutions and a diagonal grassmannian solution of the same energy thus providing moduli of the solution at hand.

Configurations of the form above, namely such that they act as unity on all basis states $|k\rangle$, with $k \geq r$ for some $r \in \mathbb{N}$ will be called block-diagonal of rank $r$. Domrin is able to prove [25] that all block-diagonal solutions of rank 3 are either diagonal or given by (4.75).

Before continuing to analyze the block-diagonal solutions in more generality let us, for the sake of clarity, first take a closer look at those configurations which are diagonal in the oscillator basis (2.11) as well as in $\mathbb{C}^n$, namely

$$\Phi \;=\; \bigoplus_{i=1}^{n} \text{diag}\big(\{\mathrm{e}^{\mathrm{i}\alpha_i^\ell}\}_{\ell=0}^{\infty}\big)\,, \tag{4.76}$$

with $\alpha_i^\ell \in \mathbb{R}$. From the action of $\Delta$ on a basis operator,

$$\begin{aligned} \Delta\,|m\rangle\langle n| \;&=\; (m{+}n{+}1)\,|m\rangle\langle n| - \sqrt{mn}\,|m{-}1\rangle\langle n{-}1| \\ &\quad - \sqrt{(m{+}1)(n{+}1)}\,|m{+}1\rangle\langle n{+}1|\,, \end{aligned} \tag{4.77}$$

we infer that $\Delta$ maps the $k$-th off-diagonal into itself, so in particular it retains the diagonal:

$$\Delta\,|m\rangle\langle m| \;=\; (2m{+}1)\,|m\rangle\langle m| - m\,|m{-}1\rangle\langle m{-}1| - (m{+}1)\,|m{+}1\rangle\langle m{+}1|\,. \tag{4.78}$$

A short computation now reveals that the configuration (4.76) obeys the equation of motion (4.2) if and only if the phases satisfy $e^{i\alpha_i^\ell} = e^{i(\alpha_i^0 + n^\ell \pi)} = \pm e^{i\alpha_i^0}$ where $n^\ell \in \{0,1\}$ and the sign thus depends on $\ell$. Its energy is then formally given by

$$E[\Phi] \;=\; 2\pi \sum_{i=1}^{n} \sum_{\ell\in\mathbb{N}_0} \Big\{ (2\ell+1) - (-1)^{n^\ell + n^{\ell+1}} (\ell+1) - (-1)^{n^\ell + n^{\ell-1}} \ell \Big\} ,$$

which is finite if the number of positive signs or the number of negative signs - in other words if the number of transitions from positive to negative signs and vice versa - in each block labeled by $i$ is finite. Turning to grassmannian configurations, we find that *every* diagonal hermitian configuration $\Phi = \mathrm{diag}\{\pm 1\}$ is a solution[10] and hence any diagonal projector $P$ solves (4.32). Let us first consider the case of $U(1)$. Given the natural ordering of the basis $\{|m\rangle\}$ of $\mathcal{H}$, any diagonal projector $P$ of finite rank $r = Q$ can be written as

$$P \;=\; \sum_{s=1}^{q} \sum_{k=0}^{r_s-1} |m_s{+}k\rangle\langle m_s{+}k| \;=\; \sum_{s=1}^{q} S_{m_s}^\dagger \, P_{r_s} \, S_{m_s} \tag{4.79}$$

with $m_{s+1} > m_s{+}r_s \;\forall s$ and $\sum_{s=1}^q r_s = r$. Here, $S_{m_s}$ again denotes the $m_s$-th power of the shift operator as in (4.55). Via

$$\begin{aligned} \Delta P \;=\; \sum_{s=1}^{q} \Big\{ & m_s \, |m_s\rangle\langle m_s| \;-\; m_s \, |m_s{-}1\rangle\langle m_s{-}1| \\ & -\; (m_s{+}r_s)\, |m_s{+}r_s\rangle\langle m_s{+}r_s| \;+\; (m_s{+}r_s)\, |m_s{+}r_s{-}1\rangle\langle m_s{+}r_s{-}1| \Big\} \end{aligned} \tag{4.80}$$

its energy is easily calculated as

$$\tfrac{1}{8\pi}\, E[P] \;=\; Q \;+\; 2 \sum_{s=1}^{q} m_s \;, \tag{4.81}$$

which is obviously minimized for the BPS case $q = 1$ and $m_1 = 0$. The energy is additive as long as the two projectors to be combined are not getting "too close". This picture generalizes to the nonabelian case by formally allowing $r_s$ and $m_s$ to become infinite. In particular, the energy does not change when one embeds $P'$ (or several copies of it) into $U(\mathcal{H}^{\oplus n})$ and adds to it a constant projector as in (4.20). Hence, with the proper redefinition of the $m_s$, (4.81) holds for nonabelian diagonal solutions, as well.

This analysis ties in quite nicely with what we have already learned about block-diagonal solutions since the diagonal grassmannian solutions encountered above may be put into a more combinatorial context upon noting again that the freedom we have to construct these is given by the distribution of plus or minus signs in front of the diagonal entries. We therefore find that the set of all block-diagonal solutions of rank $r$ contains precisely $\sum_{i=1}^{r} \binom{i}{r} = 2^r$ diagonal grassmannian solutions whose energy varies from 0 to $\frac{r(r+1)}{2} \cdot 8\pi$ in integer steps without gaps. The number $N(r,k)$ of diagonal grassmannian

[10] albeit conceivably not of finite energy and therefore possibly not in the class of configurations that we wish to consider.

solutions which are block-diagonal of rank $r$ and feature energy $k \cdot 8\pi$ therefore generically grows very large with $r$. More precisely, it may be computed as the number of distinct partitions into positive integers of the number $k$, which can be explicitly written in terms of generalized hypergeometric functions [122].

It is tempting to try to generalize the geometric understanding implied by the example 4.1 to higher rank solutions. The investigation of these cases is still under way and there only exists a conjecture on the structure of the corresponding block-diagonal solutions.

**Conjecture 4.6 (Block-diagonal solutions )**
*The set of all block-diagonal solutions of rank $r$ and energy $k \cdot 8\pi$ is a connected compact complex manifold of complex dimension $N(r,k)-1$ which interpolates between the $N(r,k)$ diagonal solutions of the same rank and energy.*

Without being able to give a proof to this conjecture, we present some examples similar to (4.75) in favor of it. The construction of these is based on a diagonal grassmannian solution parametrized by a projector whose image is spanned by a sequence of basis states $\{|m\rangle, \dots, |m{+}r{-}1\rangle$ for $m \in \mathbb{N}_0$ and rank $r \in \mathbb{N}_{>2}$. The energy of this solution is given by $(m{+}r) \cdot 8\pi$. Another diagonal grassmannian solution of equal energy is then given by a projector whose image is spanned by $\{|m{+}1\rangle, \dots, |m{+}r{-}2\rangle$, i.e. constructed by removing the first and the last basic vector in the above set. It is again possible to interpolate between these two solutions as the following argument shows:
Consider the diagonal BPS solution $\Phi_{m+1} := \mathbb{1} - 2\,P_{m+1}$ with $\mathrm{im}\,P_{m+1} = \mathrm{span}\{|0\rangle, \dots, |m\rangle\}$ as in (4.46) and an additional projector $P(c)$, whose image is spanned by $\{|0\rangle, \dots, |m{-}1\rangle$, $|m{+}1\rangle, \dots, |m{+}r{-}2\rangle$, $|m\rangle + c\,|m{+}r{-}1\rangle\}$, where $c \in \mathbb{C}$.[11] Using

$$\Phi_{m+1} \;=\; -\sum_{k=0}^{m} |k\rangle\langle k| + \sum_{k=m+1}^{\infty} |k\rangle\langle k| \qquad \Longrightarrow \qquad \Phi_{0\,-} \;=\; \sqrt{m+1}\,|m\rangle\langle m{+}1|$$

and

$$\begin{aligned} P \;=\; & \sum_{k=0}^{m+r-2} |k\rangle\langle k| + \tfrac{|c|^2}{1+|c|^2}\,\big(|m{+}r{-}1\rangle\langle m{+}r{-}1| - |m\rangle\langle m|\big) \\ & + \tfrac{\bar c}{1+|c|^2}\,|m\rangle\langle m{+}r{-}1| + \tfrac{c}{1+|c|^2}\,|m{+}r{-}1\rangle\langle m|\,, \end{aligned}$$

it is straightforward to deduce that $P$ is a $\Phi_0$-uniton and that the ensuing new solution is given by

$$\Phi \;=\; \Phi_0(1-2P) \;=\; \begin{pmatrix} \mathbb{1}_m & & & & \\ & \frac{1-|c|^2}{1+|c|^2} & & \frac{2\,\bar c}{1+|c|^2} & \\ & & -\,\mathbb{1}_{r-2} & & \\ & -\,\frac{2\,c}{1+|c|^2} & & \frac{1-|c|^2}{1+|c|^2} & \\ & & & & \mathbb{1} \end{pmatrix}\,, \tag{4.82}$$

[11]We could just as well have chosen $c\,|m\rangle + |m{+}r{-}1\rangle$ for the last vector the important fact being that we choose any nonzero vector of the space spanned by $|m\rangle$ and $|m{+}r{-}1\rangle$.

which includes both the solutions of equal energy that we started with as special cases for $c=\infty$ and $c=0$, respectively. Note also that this procedure may be iterated until one eventually obtains a diagonal BPS solution based on a projector of rank 1 or 2. In this way we get a chain of solutions each parametrized by a 2-dimensional sphere. Still, this procedure does not exhaust all block-diagonal solutions in the general case as further examples found in [25] show.

For the record, we finally note that in the case of a BPS solution the inversion $P \to \mathbb{1}-P$ generates additional solutions, which for the structure in (4.20) and BPS-compatible unitaries $\mathcal{U}$ are represented as

$$P \;=\; \mathcal{U}\,\big(\widehat{P}-P'\big)\,\mathcal{U}^\dagger \tag{4.83}$$

with

$$\widehat{P} \;=\; (\mathbb{1}_n-\bar{P})\otimes\mathbb{1}_{\mathcal{H}} \quad\text{and}\quad (\mathbb{1}-\widehat{P})P' \;=\; 0 \;=\; P'(\mathbb{1}-\widehat{P})\,.$$

When $\mathbb{1}-\widehat{P}+P'$ is BPS, $P$ becomes anti-BPS with topological charge $Q[P]=-\mathrm{Tr}P'$, producing an anti-soliton with energy $E[P]=8\pi\mathrm{Tr}P'$. It is possible to combine solitons and anti-solitons to a non-BPS solution via

$$P \;=\; P_{\mathrm{sol}}+P_{\overline{\mathrm{sol}}} \quad\text{provided}\quad P_{\mathrm{sol}}P_{\overline{\mathrm{sol}}} \;=\; 0 \;=\; P_{\overline{\mathrm{sol}}}P_{\mathrm{sol}}\;, \tag{4.84}$$

so that their topological charges and energies simply add to

$$Q[P] \;=\; Q[P_{\mathrm{sol}}]+Q[P_{\overline{\mathrm{sol}}}] \quad\text{and}\quad E[P] \;=\; E[P_{\mathrm{sol}}]+E[P_{\overline{\mathrm{sol}}}]\;. \tag{4.85}$$

For the diagonal case, this is included in the solutions discussed above. Examples for $U(1)$ and $U(2)$, which appeared in [38], are (for $m>r$)

$$\begin{aligned} P \;&=\; \mathbb{1}_{\mathcal{H}}-P_m+P_r \qquad &&\text{with}\quad Q=r{-}m \quad\text{and}\quad \tfrac{1}{8\pi}E=r{+}m\;,\\ P \;&=\; P_{r_1}\oplus(\mathbb{1}_{\mathcal{H}}{-}P_{r_2}) \qquad &&\text{with}\quad Q=r_1{-}r_2 \quad\text{and}\quad \tfrac{1}{8\pi}E=r_1{+}r_2\,, \end{aligned} \tag{4.86}$$

respectively. Besides the global translations and rotations, other unitary transformations are conceivably compatible with the equation of motion $\big[\Delta P,P\big]=0$, generating moduli spaces of non-BPS grassmannian solutions. Outside the grassmannian manifolds, many more classical configurations are to be found as we have already seen above.

# Chapter 5

# Stability Analysis within the Noncommutative Ward Model

The aim of this chapter is to analyze perturbations of the previously discussed multi-solitons within the configuration space of the two-dimensional, i.e. static, sigma model, either within their Grassmannian or, more widely, within the whole group manifold. A linear stability analysis in the commutative as well as in the noncommutative setting then admits a two-fold interpretation. First, it is relevant for the semi-classical evaluation of the euclidean path integral, revealing potential *quantum* instabilities of the two-dimensional model whose current knowledge in the context of the commutative $U(n)$ model is summarized in [133]. Second, it yields the (infinitesimal) time evolution of fluctuations around the static multi-soliton in the time-extended three-dimensional theory, indicating *classical* instabilities if they are present. More concretely, any static perturbation of a classical configuration can be taken as (part of the) Cauchy data for a classical time evolution, and any negative eigenvalue of the quadratic fluctuation operator will give rise to an exponential runaway behavior, at least within the linear response regime. Furthermore, fluctuation zero modes are expected to belong to moduli perturbations of the classical configuration under consideration.

Concentrating again on soliton solutions in the Moyal deformation of the euclidean two-dimensional sigma model, we note that each BPS configuration belongs to some Grassmannian, whose rank $r$ can be finite or infinite. The latter case represents a smooth deformation of the known commutative multi-solitons, while the former situation realizes a noncommutative novelty, namely abelian multi-solitons which even occurred for the $U(1)$ model as we have seen in section 4.3. Although these solutions are available in explicit form, very little is known about their stability.[1] The following discussion based mainly on the work [24] sheds some light on this issue, in particular for the interesting abelian case.

## 5.1 General remarks on the fluctuation analysis

In order to investigate the stability of the classical configurations constructed in the previous chapter, we must study the energy functional $E$ in the neighborhood of the solution under consideration. Since the latter is a minimum or a saddle point of $E$, all the linear stability information is provided by the Hessian, i.e. the second variation of $E$

[1] A few recent works [108, 2, 47, 48, 36, 29] address perturbations of noncommutative solitons but not for sigma models.

evaluated at the solution. The Hessian is viewed as a linear map on the solution's tangent space of fluctuations, and its spectrum encodes the invariant information: zero modes belong to field directions of marginal stability and extend to moduli if they remain zero to higher orders, while negative eigenvalues signal instabilities. A perturbation in such a field direction provides (part of) the initial conditions of a runaway solution in a time extension of the model. We cannot be very specific about the stability of a general classical configuration. Therefore, we shall restrict our attention to the stability of BPS solutions as introduced above, at which the Hessian simplifies sufficiently to obtain concrete results. Since any BPS configuration is part of a moduli space which is embedded into some Grassmannian which itself lies inside the full configuration space of the noncommutative $U(n)$ sigma model, the total fluctuation space contains the subspace of grassmannian fluctuations which in turn includes the subspace of BPS perturbations, the latter being zero modes associated with moduli. In order to simplify the problem of diagonalizing the Hessian we shall search for decompositions of the fluctuation space into subspaces which are invariant under the action of the Hessian. As we consider configurations of finite energy only, admissible fluctuations $\phi$ must render $\delta^2 E$ finite and keep the "background" $\Phi$ unitary. Furthermore, they need to be subject to the same conditions as $\Phi$ itself: $\phi$ is bounded, $[a,\phi]$ and $[a^\dagger,\phi]$ are Hilbert-Schmidt, and $\Delta\phi$ is traceclass (see also the remark below (3.38)). Finally, in keeping with our restricted notion of Grassmannians, we do not admit hermitian fluctuations[2] which alter the topological charge.

### 5.1.1 The Hessian

The Taylor expansion of the energy functional (3.37) around some finite-energy configuration $\Phi$ can be deduced from the corresponding commutative expression (3.16) and reads

$$\begin{aligned} E[\Phi{+}\phi] &= E[\Phi] + \int \mathrm{d}^2 z\, \frac{\delta E}{\delta\Phi(z)}\,\phi(z) + \frac{1}{2}\int \mathrm{d}^2 z \int \mathrm{d}^2 z'\, \phi(z)\,\frac{\delta^2 E}{\delta\Phi(z)\,\delta\Phi(z')}\,\phi(z') + \ldots \\ &=: E[\Phi] + E^{(1)}[\Phi,\phi] + E^{(2)}[\Phi,\phi] + \ldots\,, \end{aligned} \tag{5.1}$$

where the $U(n)$ traces are included in $\int \mathrm{d}^2 z$. As already mentioned in section 3.2.1, the perturbation $\phi$ is to be constrained as to keep the background $\Phi$ unitary. Since we compute to second order in $\phi$, it does not suffice to take $\phi \in T_\Phi U(\mathcal{H}^{\oplus n})$. Rather, we must include the leading correction stemming from the exponential map onto $U(\mathcal{H}^{\oplus n})$, which implies the constraint (3.17) that is used to eliminate $\phi^\dagger$ from the variations. It is important to again realize that in this way the term linear in $\phi^\dagger$ generates a contribution to $E^{(2)}[\Phi,\phi]$. Performing the expansion for the concrete expression (4.1) and using (3.16) we arrive at

[2] Perturbations inside the Grassmannian must be hermitian (see section 5.1.2 below).

the explicit expressions

$$\begin{aligned} E^{(1)}[\Phi,\phi] &= \pi\,\mathrm{Tr}\big\{\big[a^\dagger,\Phi^\dagger\phi\,\Phi^\dagger\big]\big[a,\Phi\big]+\big[a^\dagger,\Phi\big]\big[a,\Phi^\dagger\phi\,\Phi^\dagger\big] \\ &\qquad -\big[a^\dagger,\Phi^\dagger\big]\big[a,\phi\big]-\big[a^\dagger,\phi\big]\big[a,\Phi^\dagger\big]\big\} \\ &= 2\pi\,\mathrm{Tr}\big\{(\Delta\Phi^\dagger\,\Phi-\Phi^\dagger\,\Delta\Phi)\,\Phi^\dagger\,\phi\big\} \\ &= 0\,, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} E^{(2)}[\Phi,\phi] &= \pi\,\mathrm{Tr}\big\{\big[a^\dagger,\Phi^\dagger\phi\,\Phi^\dagger\big]\big[a,\phi\big]+\big[a^\dagger,\phi\big]\big[a,\Phi^\dagger\phi\,\Phi^\dagger\big] \\ &\qquad -\big[a^\dagger,\Phi^\dagger\phi\,\Phi^\dagger\phi\,\Phi^\dagger\big]\big[a,\Phi\big]-\big[a^\dagger,\Phi\big]\big[a,\Phi^\dagger\phi\,\Phi^\dagger\phi\,\Phi^\dagger\big]\big\} \\ &= 2\pi\,\mathrm{Tr}\big\{\Phi^\dagger\phi\,\Phi^\dagger\phi\,\Phi^\dagger\,\Delta\Phi-\Phi^\dagger\phi\,\Phi^\dagger\,\Delta\phi\big\} \\ &= 2\pi\,\mathrm{Tr}\big\{\phi^\dagger\,\Delta\phi-\phi^\dagger\,(\Phi\Delta\Phi^\dagger)\,\phi\big\}+\mathcal{O}(\phi^3) \\ &=: 2\pi\,\mathrm{Tr}\big\{\phi^\dagger\,H\,\phi\big\}+\mathcal{O}(\phi^3)\,, \end{aligned} \tag{5.3}$$

defining the Hessian

$$H \;:=\; \Delta-(\Phi\Delta\Phi^\dagger) \tag{5.4}$$

as a self-adjoint operator. Hence, our task is essentially reduced to working out the spectrum of the Hessian. Since $\Delta$ is clearly a positive semidefinite operator, an instability can only occur in directions for which $\langle\Phi\Delta\Phi^\dagger\rangle$ is sufficiently large.

For later reference, we present the action of $H$ in the oscillator basis,

$$H\sum_{m,\ell}\phi_{m,\ell}\,|m\rangle\langle\ell| \;=\; \sum_{m,\ell}(H\phi)_{m,\ell}\,|m\rangle\langle\ell|$$

with the components

$$\begin{aligned} (H\phi)_{m,\ell} &= (m{+}\ell{+}1)\,\phi_{m,\ell}-\sqrt{(m{+}1)(\ell{+}1)}\,\phi_{m+1,\ell+1}-\sqrt{m\ell}\,\phi_{m-1,\ell-1} \\ &\quad -\sum_{j,k}\Phi_{m,j}\,\big\{(j{+}k{+}1)\,\Phi_{j,k}-\sqrt{(j{+}1)(k{+}1)}\,\Phi_{j+1,k+1}-\sqrt{jk}\,\Phi_{j-1,k-1}\big\}\,\phi_{k,\ell}\,, \end{aligned} \tag{5.5}$$

where $\Phi_{m,\ell}$ as well as $\phi_{m,\ell}$ are still $n{\times}n$ matrix-valued. At diagonal abelian backgrounds, as given in (4.79), the matrices reduce to numbers and the latter expression simplifies to

$$(H\phi)_{m,\ell} \;=\; (m{+}\ell{+}1-2b_m)\,\phi_{m,\ell}-\sqrt{(m{+}1)(\ell{+}1)}\,\phi_{m+1,\ell+1}-\sqrt{m\ell}\,\phi_{m-1,\ell-1} \tag{5.6}$$

with the abbreviation

$$b_m \;=\; \sum_{s=1}^{q}\big\{\delta_{m,m_s}m_s+\delta_{m,m_s-1}m_s+\delta_{m,m_s+r_s}(m_s{+}r_s)+\delta_{m,m_s+r_s-1}(m_s{+}r_s)\big\}\,, \tag{5.7}$$

which for fixed $\ell$ differs from $(\Delta\phi)_{m,\ell}$ in at most $4q$ entries. For diagonal $U(1)$ BPS backgrounds $\Phi=\Phi_r=\mathbb{1}-2P_r$ this further reduces to

$$b_m \;=\; r\,\big(\delta_{m,r-1}+\delta_{m,r}\big)\,. \tag{5.8}$$

### 5.1.2 Decomposition into even and odd fluctuations

Let us now specialize to grassmannian backgrounds, $\Phi = \mathbb{1} - 2P = \Phi^\dagger$, characterized by a hermitian projector $P$ and obeying $\Phi^2 = \mathbb{1}$. Any such projector induces an orthogonal decomposition

$$\mathbb{C}^n \otimes \mathcal{H} \;=\; P\,(\mathbb{C}^n \otimes \mathcal{H}) \oplus (\mathbb{1}{-}P)(\mathbb{C}^n \otimes \mathcal{H}) \;=:\; \mathrm{im}\,P \oplus \ker P\ , \tag{5.9}$$

and a fluctuation $\phi$ decomposes accordingly as

$$\phi \;=\; \underbrace{P\,\phi\,P \;+\; (\mathbb{1}{-}P)\,\phi\,(\mathbb{1}{-}P)}_{\phi_\mathrm{e}} \;+\; \underbrace{P\,\phi\,(\mathbb{1}{-}P) \;+\; (\mathbb{1}{-}P)\,\phi\,P}_{\phi_\mathrm{o}}\,, \tag{5.10}$$

where the subscripts refer to "even" and "odd", respectively. Since $\Phi$ acts as $-\mathbb{1}$ on $\mathrm{im}\,P$ but as $+\mathbb{1}$ on $\ker P$, we infer that

$$\begin{array}{rcl} \Phi\,\phi_\mathrm{e} &=& \phi_\mathrm{e}\Phi \\ \text{and}\quad \Phi\,\phi_\mathrm{o} &=& -\phi_\mathrm{o}\Phi \end{array} \quad\Longrightarrow\quad \begin{array}{rcl} \phi_\mathrm{e}^\dagger &=& -\phi_\mathrm{e} \\ \text{and}\quad \phi_\mathrm{o}^\dagger &=& \phi_\mathrm{o} \end{array} \tag{5.11}$$

to leading order from (3.17), i.e. even fluctuations are anti-hermitian while odd ones are hermitian. This implies that odd fluctuations keep $\Phi$ inside its Grassmannian, but even ones perturb away from it. It also follows that in

$$\mathrm{Tr}\big(\phi^\dagger\,\Delta\phi\big) \;=\; \mathrm{Tr}\big(\phi_\mathrm{e}^\dagger\,\Delta\phi_\mathrm{e}\big) + \mathrm{Tr}\big(\phi_\mathrm{o}^\dagger\,\Delta\phi_\mathrm{o}\big) + \mathrm{Tr}\big(\phi_\mathrm{e}^\dagger\,\Delta\phi_\mathrm{o}\big) + \mathrm{Tr}\big(\phi_\mathrm{o}^\dagger\,\Delta\phi_\mathrm{e}\big) \tag{5.12}$$

the last two terms cancel each other. Furthermore, the equation of motion (4.32), $[\Delta P, P] = 0$, implies that

$$(\Phi\Delta\Phi)_\mathrm{o} \;=\; 0 \qquad\Longrightarrow\qquad \mathrm{Tr}\big(\phi_\mathrm{e}^\dagger\;\Phi\Delta\Phi\;\phi_\mathrm{o}\big) \;=\; 0 \;=\; \mathrm{Tr}\big(\phi_\mathrm{o}^\dagger\;\Phi\Delta\Phi\;\phi_\mathrm{e}\big)\,, \tag{5.13}$$

because only an even number of odd terms in a product survives under the trace. Combining (5.12) and (5.13) we conclude that

$$E^{(2)}[\Phi,\phi_\mathrm{e}{+}\phi_\mathrm{o}] \;=\; E^{(2)}[\Phi,\phi_\mathrm{e}] + E^{(2)}[\Phi,\phi_\mathrm{o}]\ , \tag{5.14}$$

which allows us to treat these two types of fluctuations separately.

The above decomposition has another perspective. Recall that any background configuration $\Phi$, being unitary, can be diagonalized by some unitary transformation,

$$\Phi \;=\; \mathcal{U}\,\Phi_\mathrm{d}\,\mathcal{U}^\dagger \;=\; \mathcal{U}\,\mathrm{diag}\big(\{\mathrm{e}^{\mathrm{i}\lambda_i}\}\big)\,\mathcal{U}^\dagger \quad \text{with } \lambda_i \in \mathbb{C}\,. \tag{5.15}$$

When $\Phi$ is hermitian, i.e. inside some Grassmannian, the diagonal phase factors can just be $+1$ or $-1$, and $\mathcal{U}$ is determined only up to a factor $\mathcal{V} \in U(\mathrm{im}\,P) \times U(\ker P)$ which keeps the two eigenspaces $\mathrm{im}\,P$ and $\ker P$ invariant. Adding a perturbation $\phi$ lifts the high degeneracy of $\Phi$, so that the diagonalization of $\Phi + \phi$ requires an infinitesimal "rotation"

$K$ of $\mathrm{im}\,P$ and $\ker P$ inside $\mathcal{H}$ as well as a "large" rediagonalization $\mathcal{V}$ inside the two eigenspaces. Modulo higher order terms we may write

$$\begin{aligned} \Phi+\phi \;&=\; \mathcal{U}\,(1{+}K)\,\mathcal{V}\,(\Phi_\mathrm{d}+\phi_\mathrm{d})\,\mathcal{V}^\dagger(1{-}K)\,\mathcal{U}^\dagger \\ &=\; \Phi+\mathcal{U}\left\{\mathcal{V}\,\phi_\mathrm{d}\,\mathcal{V}^\dagger \;+\; \big[K,\Phi_\mathrm{d}\big]\right\}\mathcal{U}^\dagger\,, \\ \text{with}\quad \big[\mathcal{V},\Phi_\mathrm{d}\big] \;&=\; 0 \quad\text{and}\quad K \;=\; -K^\dagger \quad \text{infinitesimal}\,, \end{aligned} \tag{5.16}$$

where $\phi_\mathrm{d}$ is a purely diagonal and anti-hermitian fluctuation. Since $\mathcal{V}$ depends on $\phi$ it should not be absorbed into $\mathcal{U}$. It rather generates all non-diagonal fluctuations inside $\mathrm{im}\,P$ and $\ker P$, allowing us to rewrite

$$\mathcal{V}\,\phi_\mathrm{d}\,\mathcal{V}^\dagger \;=\; \phi'_\mathrm{d} \;+\; [\Lambda_\mathrm{e}\,,\,\phi_\mathrm{d}] \quad\text{with}\quad \Lambda_\mathrm{e}^\dagger \;=\; -\Lambda_\mathrm{e} \tag{5.17}$$

being a generator of $U(\mathrm{im}\,P)\times U(\ker P)$ and a modified diagonal perturbation $\phi'_\mathrm{d}$. Redenoting also $K=\epsilon\Lambda_\mathrm{o}$ with a real and infinitesimal $\epsilon$ and a generator $\Lambda_\mathrm{o}$ of the Grassmannian, the general fluctuation is parametrized as

$$\phi \;=\; \mathcal{U}\left\{\phi'_\mathrm{d} \;+\; \big[\Lambda_\mathrm{e},\phi_\mathrm{d}\big] \;+\; \big[\Lambda_\mathrm{o},\epsilon\Phi_\mathrm{d}\big]\right\}\mathcal{U}^\dagger \tag{5.18}$$

and decomposed after diagonalizing the background via $\mathcal{U}$ into a "radial" part $\phi'_\mathrm{d}$ and an "angular" part $\phi_\mathrm{a}=[\Lambda,\text{any}]$ with $\Lambda$ generating $U(\mathcal{H}^{\oplus n})$ [42]. For a grassmannian background all terms have definite hermiticity properties, and we can identify

$$\phi_\mathrm{e} \;=\; \mathcal{U}\left\{\phi'_\mathrm{d}+\big[\Lambda_\mathrm{e},\phi_\mathrm{d}\big]\right\}\mathcal{U}^\dagger \quad\text{and}\quad \phi_\mathrm{o} \;=\; \mathcal{U}\left\{\big[\Lambda_\mathrm{o},\epsilon\Phi_\mathrm{d}\big]\right\}\mathcal{U}^\dagger = \epsilon\,\big[\mathcal{U}\Lambda_\mathrm{o}\mathcal{U}^\dagger,\Phi\big]\,. \tag{5.19}$$

We have seen in (5.14) above that the even and odd fluctuations can be disentangled in $E^{(2)}$. It is not clear, however, whether the diagonal perturbations can in turn be separated from the even angular ones in the fluctuation analysis. We will see, though, in section 5.2.2 that this is indeed possible for the special case of diagonal $U(1)$ backgrounds.

### 5.1.3 Odd or grassmannian perturbations

As far as stability of BPS configurations is concerned, the odd perturbations are easily dealt with by a general argument. Since a shift by $\phi_\mathrm{o}$ keeps $\Phi$ inside its Grassmannian, wherein $\Phi$ already minimizes the energy, such a perturbation cannot lower the energy any further and we can be sure that negative modes are absent here. Therefore, solitons in the noncommutative grassmannian sigma model are stable, up to possible zero modes. An obvious zero mode is generated by the translational and rotational symmetry (see section 3.2.2). A glance at (3.41) and (3.43) shows that the corresponding infinitesimal operators $\Lambda$ are given by

$$\Lambda_\mathrm{trans} \;=\; \alpha\,a^\dagger-\bar\alpha\,a \quad\text{and}\quad \Lambda_\mathrm{rot} \;=\; \mathrm{i}\vartheta\,a^\dagger a\;, \tag{5.20}$$

which indeed leads to the annihilation of $[\Lambda_\mathrm{trans},\Phi]$ and $[\Lambda_\mathrm{rot},\Phi]$ by $H$ as we shall see in (5.94) and (5.95).

From the discussion at the end of section 4.3 we know that for $r>1$ there are additional zero modes inside the Grassmannian, because the multi-soliton moduli spaces are higher-dimensional. Parametrizing these BPS-compatible perturbation as follows,

$$\Phi + \phi_{\text{o}} \;=\; \mathcal{U}_{\text{B}}\,\Phi\,\mathcal{U}_{\text{B}}^{\dagger} \;=\; \Phi + \epsilon\,\big[\Lambda_{\text{B}},\Phi\big] + \mathcal{O}(\epsilon^2)\,, \tag{5.21}$$

for $\mathcal{U}_{\text{B}} = \mathrm{e}^{\epsilon\Lambda_{\text{B}}}$ with $\Lambda_{\text{B}}^{\dagger} = -\Lambda_{\text{B}}$, the corresponding Lie-algebra element (4.13) becomes

$$\omega \;=\; \mathcal{U}_{\text{B}}^{\dagger}\,\big[a,\mathcal{U}_{\text{B}}\big] \;=\; (\mathrm{e}^{-\epsilon\,\mathrm{ad}\Lambda_{\text{B}}} - \mathbb{1})\,a \;=\; \epsilon\,\big[a,\Lambda_{\text{B}}\big] + \mathcal{O}(\epsilon^2)\,, \tag{5.22}$$

and the condition (4.41) of BPS compatibility to leading order in $\epsilon$ reads

$$\big[a,\Lambda_{\text{B}}\big]\,|T\rangle \;=\; |T\rangle\,\gamma_{\omega}\,, \quad \text{for some } r{\times}r \text{ matrix } \gamma_{\omega}\,. \tag{5.23}$$

Let us try to find $\Lambda_{\text{B}}$ in the abelian case by perturbing around the diagonal BPS configuration

$$|T_r\rangle \;=\; \big(|0\rangle\ |1\rangle\ \dots\ |r{-}1\rangle\big) \qquad\Longleftrightarrow\qquad \Phi_r \;=\; \mathbb{1}_{\mathcal{H}} \;-\; 2\sum_{m=0}^{r-1}|m\rangle\langle m|\,. \tag{5.24}$$

Expanding the generator

$$\Lambda_{\text{B}} \;=\; \sum_{m,\ell}\Lambda_{m,\ell}\,|m\rangle\langle\ell| \quad \text{with} \quad \bar{\Lambda}_{m,\ell} \;=\; -\Lambda_{\ell,m}\,, \tag{5.25}$$

it is easy to see that the even part of $\Lambda_{\text{B}}$ automatically fulfills (5.23), and so restrictions to $\Lambda_{m,\ell}$ arise only for the odd components. In fact, only the terms with $m \ge r$ and $\ell < r$ in the above sum may violate (5.23), which leads to the conditions

$$\sqrt{m{+}1}\,\Lambda_{m+1,\ell} - \sqrt{\ell}\,\Lambda_{m,\ell-1} \;=\; 0\,, \quad \text{for all } m \ge r \text{ and } \ell < r\,. \tag{5.26}$$

Since $\ell = 0$ yields $\Lambda_{m+1,0} = 0$ as a boundary condition, this hierarchy of equations puts most components to zero, except for $m = r,\dots,r{+}\ell$ at any fixed $\ell < r$. These remaining $r(r{+}1)/2$ components are subject to $r(r{-}1)/2$ equations from (5.26), whose solution

$$\Lambda_{r+j,\ell+j} \;=\; \sqrt{\tfrac{(\ell+j)(\ell+j-1)\cdots(\ell+1)}{(r+j)(r+j-1)\cdots(r+1)}}\,\Lambda_{r,\ell}\,, \quad \text{for } j = 1,\dots,r{-}1{-}\ell \text{ at } \ell \le r{-}2 \tag{5.27}$$

fixes $r(r{-}1)/2$ components in terms of the $r{-}1$ components appearing on the right hand side, which therefore are free complex parameters. The $r$-th free parameter $\Lambda_{r,r-1}$ does not enter and is associated with the rigid translation mode.[3] Finally, the ensuing BPS perturbation (5.21) is found to be

$$\phi_{\text{o}} \;=\; \epsilon\,[\Lambda_{\text{B}},\Phi_r] \;=\; \epsilon\sum_{m=r}^{\infty}\sum_{\ell=0}^{r-1}\Big(\Lambda_{m,\ell}\,|m\rangle\langle\ell| \;+\; \bar{\Lambda}_{m,\ell}\,|\ell\rangle\langle m|\Big) \tag{5.28}$$

with $\Lambda_{m,\ell}$ taken from (5.27). To higher orders in $\epsilon$, the BPS-compatibility condition is not automatically satisfied by our solution (5.27) but this can be repaired by adding suitable even components to $\Lambda_{\text{B}}$. Our result ties in nicely with the observation of $r$ complex moduli $\alpha_k$ in (4.45) whose shifts produce precisely $r$ complex zero modes. For the simplest non-trivial case of $r = 2$, one can also extract these modes by differentiating (4.49) with respect to $\alpha_1$ or $\alpha_2$.

[3] The rigid rotation mode is absent because $\Phi_r$ is spherically symmetric.

### 5.1.4 Even or non-grassmannian perturbations

The stability analysis of BPS configurations inside the full noncommutative $U(n)$ sigma model requires the investigation of the even fluctuations $\phi_{\mathrm{e}}$, as well. Here, we have only partial results to offer.[4] Yet, there is the following general argument which produces an unstable even fluctuation mode $\phi_{\mathrm{neg}}$ (but not an eigenmode) for any noncommutative multi-soliton $\Phi = \mathbb{1} - 2P$ with topological charge $Q > 0$. For this, consider some other multi-soliton $\tilde{\Phi} = \mathbb{1} - 2\tilde{P}$ which is contained in $\Phi$ in the sense that

$$\mathrm{im}\,(\tilde{P}) \subset \mathrm{im}\,(P) \qquad \Longleftrightarrow \qquad \tilde{P}\,P \;=\; P\,\tilde{P} \;=\; \tilde{P}\,. \tag{5.29}$$

We then simply say that $\tilde{P} \subset P$. It follows that their difference $\Pi$ is the orthogonal complement of $\tilde{P}$ in $\mathrm{im}\,(P)$,

$$\Pi \;=\; P - \tilde{P} \;\subset\; P \qquad \Longrightarrow \qquad \Pi^2 \;=\; \Pi \quad \text{and} \quad \Pi\,\tilde{P} \;=\; 0 \;=\; \tilde{P}\,\Pi\,. \tag{5.30}$$

In particular, we may choose $\tilde{P} = 0$. For any such pair $(P, \tilde{P})$ there exists a continuous path

$$\Phi(s) \;=\; \mathrm{e}^{\mathrm{i}s\Pi}(\mathbb{1}{-}2P) \;=\; \mathbb{1} - 2P + (1{-}\mathrm{e}^{\mathrm{i}s})\Pi \;=\; \mathbb{1} - 2\tilde{P} - (1{+}\mathrm{e}^{\mathrm{i}s})\Pi \tag{5.31}$$

connecting $\Phi(0) = \Phi = \mathbb{1} - 2P$ with $\Phi(\pi) = \tilde{\Phi} = \mathbb{1} - 2\tilde{P}$. Note that $\Phi(s)$ interpolates between two different Grassmannians, touching them only at $s = 0$ and $s = \pi$. Since we assumed that $\Phi$ and $\tilde{\Phi}$ are BPS we know that

$$E[\Phi] \;=\; 8\pi\,Q \quad \text{and} \quad E[\tilde{\Phi}] \;=\; 8\pi\,\tilde{Q} \tag{5.32}$$

with $Q$ and $\tilde{Q}$ being the topological charges of $P$ and $\tilde{P}$, respectively.[5] Inserting (5.31) into the expression (4.1) for the energy and abbreviating $1 - \mathrm{e}^{\mathrm{i}s} =: \rho$, we compute

$$\begin{aligned} E[\Phi(s)] \;&=\; 8\pi\,Q - 2\pi(\rho{+}\bar{\rho})\,\mathrm{Tr}\big([a,P][\Pi,a^\dagger] + [a^\dagger,P][\Pi,a]\big) \\ &\qquad\qquad + 2\pi\,\rho\bar{\rho}\,\mathrm{Tr}\big([a,\Pi][\Pi,a^\dagger]\big) \\ &=\; 8\pi\,Q - 4\pi(1{-}\cos s)\,\mathrm{Tr}\big([a,P][\Pi,a^\dagger] + [a^\dagger,P][\Pi,a] - [a,\Pi][\Pi,a^\dagger]\big)\,, \end{aligned} \tag{5.33}$$

where the two traces could be combined due to the relation

$$\rho + \bar{\rho} \;=\; \rho\bar{\rho} \;=\; 2(1{-}\cos s)\,. \tag{5.34}$$

Luckily, we do not need to evaluate the traces above. Knowing that $E[\Phi(\pi)] = 8\pi\tilde{Q}$ we infer that the last trace in (5.33) must be equal to $Q - \tilde{Q}$ and hence

$$\begin{aligned} E[\Phi(s)] \;&=\; 8\pi\,Q \;-\; 4\pi(1{-}\cos s)(Q - \tilde{Q}) \;=\; 8\pi\big(Q\,\cos^2\tfrac{s}{2} + \tilde{Q}\,\sin^2\tfrac{s}{2}\big) \\ &=\; 8\pi\,Q \;-\; 2\pi(Q - \tilde{Q})\,s^2 \;+\; \mathcal{O}(s^4)\;. \end{aligned} \tag{5.35}$$

[4]Even for the commutative sigma model this is an open problem [133].

[5]Note that $\tilde{Q}$ need not be smaller than $Q$ when $r(P)$ is infinite.

Evidently, for $Q > 0$ we can always lower the energy of a given BPS configuration by applying an even perturbation $\phi_{\text{neg}} = -\mathrm{i}\epsilon\Pi$ towards a BPS solution with smaller topological charge $\tilde{Q} < Q$. The higher the charge of $P$ the more such modes are present. Yet, they are not independent of one another but rather span a cone extending from the background, as we shall see in examples below. In fact, there is no reason to expect any of these unstable fluctuations to represent an eigenmode of the Hessian, and in general they do not. Nevertheless, their occurrence again demonstrates that there must be (at least) one negative eigenvalue of $H$, and for diagonal $U(1)$ BPS backgrounds we shall prove in section 5.2.3 that there is exactly one. The only stable BPS solutions are therefore the "vacua" defined by $P' = 0$ in (4.20) and based on a constant $U(n)$ projector $\bar{P}$. This leaves no stable solutions in the abelian case besides $\Phi = \mathbb{1}$, even though Derrick's theorem does not forbid them in this setting.

In addition to the unstable mode, there exist also a number of non-grassmannian zero modes around each BPS configuration, which generate nearby non-BPS solutions to the equation of motion. This will become explicit in the examples presented below.

## 5.2 Perturbations of $U(1)$ backgrounds

After these general remarks we will in this section concentrate on the stability analysis of abelian background solutions where, for the sake of simplicity, these are further specialized to diagonal $U(1)$ backgrounds $\Phi$ as given, up to an irrelevant phase, by (4.79). The solutions considered are thus not necessarily BPS and furthermore, parametrized by a diagonal projector with finite dimensional image.

### 5.2.1 Invariant subspaces

Using (5.5) it is easy to see that the Hessian $H$ maps any off-diagonal into itself. Let us parametrize the $k$-th upper diagonal $\mathcal{D}_k$ as

$$\phi_{(+k)} \;=\; \sum_{m=0}^{\infty} \mu_{(k)m}\, |m\rangle\langle m{+}k| \quad \text{with } \mu_{(k)m} \in \mathbb{C}, \text{ for } k = 0,1,2,\ldots\,. \tag{5.36}$$

The hermiticity properties (5.11) of the perturbations demand that we combine $\mathcal{D}_k$ and $\mathcal{D}_k^\dagger$ into a subspace $\mathcal{E}_k$ of the $k$-th upper plus lower diagonals by defining

$$\mathcal{E}_k \;:=\; \big\{\phi_{(k)} \;\big|\; \phi_{(k)} \;=\; \phi_{(+k)} - \Phi\,\phi_{(+k)}^\dagger \Phi\big\}\;, \quad \text{implying} \quad \phi_{(k)}^\dagger \;=\; -\Phi\,\phi_{(k)}\,\Phi\,. \tag{5.37}$$

The direct sum of all $\mathcal{E}_k$ is the full admissible tangent space to $U(\mathcal{H})$, and we may thus decompose

$$\phi \;=\; \sum_{k=0}^{\infty} \phi_{(k)} \;=\; \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\Big\{\mu_{(k)m}\,|m\rangle\langle m{+}k| \;\mp\; \bar{\mu}_{(k)m}\,|m{+}k\rangle\langle m|\Big\}\,, \tag{5.38}$$

with the sign depending on whether the component is even or odd. Since $H$ maps $\mathcal{E}_k$ into itself, the bilinear form for the quadratic energy correction defined in (5.3) is block-diagonal on the set of $\mathcal{E}_k$,

$$2\pi\,\mathrm{Tr}\big\{\phi^\dagger_{(k)}\,H\,\phi_{(\ell)}\big\} \;\sim\; \delta_{k\ell}\ , \tag{5.39}$$

which implies the factorization

$$E^{(2)}\,[\Phi,\textstyle\sum_k\phi_{(k)}] \;=\; \textstyle\sum_k E^{(2)}\,[\Phi,\phi_{(k)}]\ . \tag{5.40}$$

In other words, $\mathcal{E}_k$ forms an $H$-invariant subspace for each value of $k$. In particular, $\mathcal{E}_0$ is the space of admissible skew-hermitian diagonal matrices, and the purely imaginary diagonal fluctuations $\phi_{(0)} \equiv \phi_{\mathrm{d}} \in \mathcal{E}_0$ can be considered on their own.

### 5.2.2 Results for diagonal $U(1)$ backgrounds

Apparently, for diagonal $U(1)$ backgrounds it suffices to study the second variation form $E^{(2)}[\Phi,.]$ on each subspace $\mathcal{E}_k$ ($k \geq 0$) separately. Let us give more explicit formulae for the restriction of $E^{(2)}[\Phi,.]$ to these subspaces. To this end, we denote the non-zero entries of $\Phi$ by $\delta_j := \Phi_{jj}$ ($j = 0,1,2,\ldots$). We have $\delta_j = \pm 1$ for each $j$, and the set

$$J \;:=\; \{j \geq 0 \mid \delta_{j+1} \neq \delta_j\} \tag{5.41}$$

is finite. Then, a straightforward calculation shows that the restriction of $E^{(2)}[\Phi,.]$ to

$$\mathcal{E}_0 \;=\; \Big\{\phi_{(0)} \;\Big|\; \phi_{(0)} = \mathrm{i}\sum_{m=0}^{\infty}\phi_m\,|m\rangle\langle m| \text{ with } \phi_m \in \mathbb{R}\Big\} \tag{5.42}$$

is given by

$$\tfrac{1}{2\pi}E^{(2)}[\Phi,\phi_{(0)}] \;=\; \sum_{m=0}^{\infty}(m{+}1)\,(\phi_{m+1}-\phi_m)^2 - 2\sum_{j\in J}(j{+}1)\,(\phi_{j+1}^2+\phi_j^2)\,. \tag{5.43}$$

The formula for $E^{(2)}[\Phi,.]$ on $\mathcal{E}_{k>0}$ is more complicated. Considering a fixed $k$-th upper diagonal $\mathcal{D}_k$ as parametrized in (5.36), we first evaluate (suppressing the subscript $(k)$)

$$\begin{aligned}\mathrm{Tr}\{\phi^\dagger_{(+k)}\,\Delta\,\phi_{(+k)}\} \;&=\; \sum_{m=0}^{\infty}|\sqrt{m{+}1}\,\mu_{m+1}-\sqrt{m{+}k{+}1}\,\mu_m|^2\\ &=\; \tfrac12\,k\,|\mu_0|^2+\tfrac12\sum_{m=0}^{\infty}R_m(\mu)\end{aligned} \tag{5.44}$$

with

$$\begin{aligned}R_m(\mu) \;:=\;& |\sqrt{m{+}1}\,\mu_{m+1}-\sqrt{m{+}k{+}1}\,\mu_m|^2+|\sqrt{m{+}k{+}1}\,\mu_{m+1}-\sqrt{m{+}1}\,\mu_m|^2\\ =\;& \big(\sqrt{m{+}k{+}1}-\sqrt{m{+}1}\big)^2\big(|\mu_{m+1}|^2+|\mu_m|^2\big)\\ &+2\sqrt{m{+}k{+}1}\sqrt{m{+}1}\,|\mu_{m+1}-\mu_m|^2\,.\end{aligned} \tag{5.45}$$

Armed with this expressions, we compute the second variation on $\mathcal{E}_k$. To state the outcome, it is useful to introduce the index sets

$$J - k \ := \ \{j{-}k \in \mathbb{N}_0 \,|\, j \in J\} \quad \text{and} \quad A \ := \ (J \cup (J-k)) \setminus (J \cap (J-k)) \ , \tag{5.46}$$

i.e. $A$ is the symmetric difference of $J$ and $J{-}k$. For $\phi_{(k)} \in \mathcal{E}_k$ a direct calculation results in

$$\begin{aligned} \tfrac{1}{2\pi}E^{(2)}[\Phi,\phi_{(k)}] \ &= \ k\,|\mu_0|^2 \ + \sum_{j\in\mathbb{N}_0\setminus A} R_j(\mu) \ + \ \sum_{j\in A}(2j{+}2{+}k)\big(|\mu_j|^2+|\mu_{j+1}|^2\big) \\ &\quad - 2\sum_{j\in J}(j{+}1)\big(|\mu_j|^2+|\mu_{j+1}|^2\big) \ - \ 2\sum_{j\in J-k}(j{+}k{+}1)|\mu_j|^2 \ - \ 2\sum_{j\in J-k+1}(j{+}k)|\mu_j|^2 \, . \end{aligned} \tag{5.47}$$

This expression simplifies when $k$ is greater than the largest element of $J{+}1$. Then the sets $J{-}k$ and $J{-}k{+}1$ are empty, so $A = J$, and (5.47) takes the form

$$\tfrac{1}{2\pi}E^{(2)}[\Phi,\phi_{(k)}] \ = \ k\,|\mu_0|^2 \ + \sum_{j\in\mathbb{N}_0\setminus J} R_j(\mu) \ + \ k\sum_{j\in J}\big(|\mu_j|^2+|\mu_{j+1}|^2\big) \, . \tag{5.48}$$

One sees that for each $j \in J$ the corresponding coefficient $\mu_j$ decouples from all successive coefficients $\mu_{m>j}$. Since the elements of $J$ signify the boundaries between even and odd fluctuations in the string of $\mu_m$, this observation confirms the decomposition (5.14) of $E^{(2)}$ into an even and odd part also after the restriction to $\mathcal{E}_k$.[6] Furthermore, we note that $R_j(\mu) > 0$ unless $\mu_j = \mu_{j+1} = 0$, which makes it obvious from the expression (5.48) that the quadratic form $E^{(2)}$ is strictly positive on each $\mathcal{E}_k$ with $k > \max_{j\in J}(j{+}1)$. For the remaining $\mathcal{E}_k$ we have to work a little harder.

### 5.2.3 Results for diagonal $U(1)$ BPS backgrounds

The idea is to pursue the reduction of $E^{(2)}$ to a sum of squares as in the previous section whenever possible. For diagonal BPS solutions (4.46)

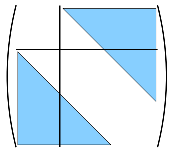
$$\Phi_r \ = \ \mathbb{1} \ - \ 2\sum_{m=0}^{r-1}|m\rangle\langle m| \ = \ \sum_{m=0}^{\infty}\delta_m\,|m\rangle\langle m| \quad \text{with } \delta_m \ := \ \begin{cases} -1 & \text{for } m < r \\ +1 & \text{for } m \geq r \end{cases} \ , \tag{5.49}$$

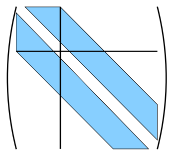
this strategy turns out to be successful at all $k \geq 1$ but breaks down at $k = 0$. We note that now $J = \{r{-}1\}$, which implies a possible distinction of cases: "very off-diagonal" perturbations have $k > r$, "slightly off-diagonal" perturbations occur for $1 \leq k \leq r$, and diagonal perturbations mean $k = 0$, to be discussed last. Let us try to visualize this in matrix form. The solution $\Phi_r$ is given as

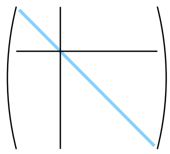
$$\Phi_r \ = \ \left(\begin{array}{ccc|ccc} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \end{array}\right) \begin{array}{c} 0 \\ \vdots \\ r{-}1 \\ r \\ \vdots \end{array} \ , \tag{5.50}$$

[6] For $|J| > 1$ the even and/or odd part of $E^{(2)}$ is split further. In total, $E^{(2)}$ decomposes into at least $|J|{+}1$ blocks.

whereas the fluctuations decompose according to

very off-diagonal perturbations $\phi =$  $r$ ,

slightly off-diagonal perturbations $\phi =$  $r$ , (5.51)

diagonal perturbations $\phi =$  $r$ .

It will also be instructive to express the Hessian on the space $\mathcal{E}_k$ in matrix form,

$$E^{(2)}[\Phi_r, \phi_{(k)}] \;=\; 2\pi \sum_{m,\ell=0}^{\infty} \bar{\mu}_m \, H^{(k)}_{m\ell} \, \mu_\ell \;, \tag{5.52}$$

defining an infinite-dimensional matrix $H^{(k)} = \big(H^{(k)}_{m\ell}\big)$.

*Very off-diagonal perturbations*

Since $J$ consists just of one element, at $k > r$ the matrix $H^{(k)}$ splits into two parts only, of which the odd one has finite size $r$. More explicitly,

$$H^{(k)} \;=\; H^{(k)}_{\mathrm{Gr}(P)} \oplus H^{(k)}_{\ker P} \tag{5.53}$$

with

$$\tfrac{1}{2}H^{(k)}_{\mathrm{Gr}(P)} = \begin{pmatrix} k+1 & -\sqrt{1(k{+}1)} & & & & \\ -\sqrt{1(k{+}1)} & k+3 & -\sqrt{2(k{+}2)} & & & \\ & -\sqrt{2(k{+}2)} & k+5 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 2r+k-3 & -\sqrt{(r{-}1)(r{+}k{-}1)} \\ & & & & -\sqrt{(r{-}1)(r{+}k{-}1)} & 2r+k-1-\boldsymbol{r} \end{pmatrix} \tag{5.54}$$

$$\tfrac{1}{2}H^{(k)}_{\ker P} = \begin{pmatrix} 2r+k+1-\boldsymbol{r} & -\sqrt{(r{+}1)(r{+}k{+}1)} & & \\ -\sqrt{(r{+}1)(r{+}k{+}1)} & 2r+k+3 & -\sqrt{(r{+}2)(r{+}k{+}2)} & \\ & -\sqrt{(r{+}2)(r{+}k{+}2)} & 2r+k+5 & \ddots \\ & & \ddots & \ddots \end{pmatrix} , \tag{5.55}$$

where the boldface contributions $\boldsymbol{r}$ disturb the systematics and originate from the $\Phi\Delta\Phi^\dagger$ term in the Hessian. The previous paragraph asserts strict positivity of $H$ for $k>r$. Due to the finiteness of $H^{(k)}_{\mathrm{Gr}(P)}$, the Hessian on $\mathcal{E}_{k>r}$ features precisely $r$ positive eigenvalues[7] in its spectrum. We shall argue below that the infinite part $H^{(k)}_{\ker P}$ contributes a purely continuous spectrum $\mathbb{R}_+$ for any $k$. Thus, the above eigenvalues are not isolated but embedded in the continuum.

It is instructive to look at the edge of the continuum. The non-normalizable zero mode of $H^{(k)}_{\mathrm{Gr}(P)}$ is explicitly given by

$$\begin{aligned}\mu^{\text{zero}}_{r+m} &= \mu_r\sqrt{\frac{(r+k+1)(r+k+2)\cdots(r+k+m)}{(r+1)(r+2)\cdots(r+m)}}\\ &= \mu_r\sqrt{\frac{(r+m+1)(r+m+2)\cdots(r+m+k)}{(r+1)(r+2)\cdots(r+k)}}\,,\end{aligned}\tag{5.56}$$

which grows like $m^{k/2}$ when $m=0,1,2,\ldots$ gets large. Being unbounded for $k>0$ this infinite vector does not yield an admissible perturbation of $\Phi_r$, however, since $\delta^2E$ is infinite.

*Slightly off-diagonal perturbations*

For each value of $k$ in the range $1\le k\le r$ we already established in section 5.1.3 the existence of an odd complex normalizable zero mode connected with a moduli. Nevertheless, as we will show now, $E^{(2)}[\Phi_r,\phi_{(k)}]$ remains positive for $1\le k\le r$ albeit not strictly so. In order to simplify (5.47) we make use of the following property for $R_j(\mu)$ as defined in (5.45):

$$\begin{aligned}\sum_{j=m}^{l-1} R_j(\mu) &= k\,|\mu_l|^2 - k\,|\mu_m|^2 + 2\sum_{j=m}^{l-1}|\sqrt{j{+}1}\,\mu_{j+1}-\sqrt{j{+}k{+}1}\,\mu_j|^2\\ &= k\,|\mu_m|^2 - k\,|\mu_l|^2 + 2\sum_{j=m}^{l-1}|\sqrt{j{+}k{+}1}\,\mu_{j+1}-\sqrt{j{+}1}\,\mu_j|^2\,.\end{aligned}\tag{5.57}$$

Here, by definition, all integers are non-negative and all sums are taken to vanish if $m\ge l$. The two equations above are easily proved by induction over $l$, starting from the trivial case $l=m$. We now employ the algebraic identities (5.57) to rewrite (5.47) for $1\le k\le r$ and obtain

$$\begin{aligned}\tfrac{1}{2\pi}E^{(2)}[\Phi_r,\phi_{(k)}] &= k\,|\mu_r|^2 + \sum_{j=r}^{\infty}R_j(\mu) + 2\sum_{j=0}^{r-k-2}|\sqrt{j{+}1}\,\mu_{j+1}-\sqrt{j{+}k{+}1}\,\mu_j|^2\\ &\quad + 2\sum_{j=r-k}^{r-2}|\sqrt{j{+}k{+}1}\,\mu_{j+1}-\sqrt{j{+}1}\,\mu_j|^2\,,\end{aligned}\tag{5.58}$$

[7] meaning that the corresponding modes are normalizable eigenvectors, i.e. of Hilbert-Schmidt class

which is indeed positive semi-definite. We observe that the even part of $E^{(2)}$ now consists of two disjoint pieces, containing $\{\mu_0, \dots, \mu_{r-k-1}\}$ and $\{\mu_{j\geq r}\}$, which are separated by the odd perturbations $\{\mu_{r-k}, \dots, \mu_{r-1}\}$. If the right-hand side of (5.58) is equal to zero, then all components $\mu_{j\geq r}$ must vanish because each $R_j$ is a strictly positive quadratic form of $\mu_j$ and $\mu_{j+1}$. In contrast, the remaining coefficients $\mu_{j<r}$ need not vanish at the zero set of $E^{(2)}[\Phi_r, \phi_{(k)}]$. In order to find the zero modes of the Hessian, it is convenient to visualize it again in matrix form, namely

$$H^{(k)} \;=\; H^{(k)}_{\mathrm{im}\,P} \oplus H^{(k)}_{\mathrm{Gr}(P)} \oplus H^{(k)}_{\ker P} \;, \tag{5.59}$$

where the blocks have sizes $r{-}k$, $k$, and $\infty$, respectively. From (5.58) we extract

$$\tfrac12 H^{(k)}_{\mathrm{im}\,P} = \begin{pmatrix} k+1 & -\sqrt{1(k{+}1)} & & & & \\ -\sqrt{1(k{+}1)} & k+3 & -\sqrt{2(k{+}2)} & & & \\ & -\sqrt{2(k{+}2)} & k+5 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 2r-k-3 & -\sqrt{(r{-}k{-}1)(r{-}1)} \\ & & & & -\sqrt{(r{-}k{-}1)(r{-}1)} & 2r-k-1-\boldsymbol{r} \end{pmatrix} \tag{5.60}$$

$$\tfrac12 H^{(k)}_{\mathrm{Gr}(P)} = \begin{pmatrix} 2r-k+1-\boldsymbol{r} & -\sqrt{(r{-}k{+}1)(r{+}1)} & & & \\ -\sqrt{(r{-}k{+}1)(r{+}1)} & 2r-k+3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2r+k-3 & -\sqrt{(r{-}1)(r{+}k{-}1)} \\ & & & -\sqrt{(r{-}1)(r{+}k{-}1)} & 2r+k-1-\boldsymbol{r} \end{pmatrix} \tag{5.61}$$

and $H^{(k)}_{\ker P}$ being identical to the matrix in (5.55). The extreme cases are $H^{(1)}_{\mathrm{Gr}(P)} = (0)$ and $H^{(r-1)}_{\mathrm{im}\,P} = (0)$, while $H^{(r)}_{\mathrm{im}\,P}$ is empty.

We already know that $H^{(k)}_{\ker P}$ is strictly positive definite. As claimed earlier, its spectrum is $\mathbb{R}_+$ and purely continuous, with a non-normalizable zero mode given by (5.56). Being finite-dimensional, the other two blocks jointly yield again just $r$ non-negative and non-isolated proper eigenvalues for fixed $k > 0$. Interestingly, two of these eigenvalues are now zero and give us one even and one odd complex normalizable zero mode. With $\beta_k, \gamma_k \in \mathbb{C}$ the latter take the following form:[8]

[8] The even zero mode is, of course, absent for $k = r$.

$$H^{(k)}_{\mathrm{im}\,P}: \qquad (\mu_\ell)_{\mathrm{zero}} \;=\; \gamma_k \begin{pmatrix} \sqrt{1}\sqrt{2}\sqrt{3}\cdots\sqrt{r{-}k{-}2}\sqrt{r{-}k{-}1} \\ \sqrt{k{+}1}\sqrt{2}\sqrt{3}\cdots\sqrt{r{-}k{-}2}\sqrt{r{-}k{-}1} \\ \sqrt{k{+}1}\sqrt{k{+}2}\sqrt{3}\cdots\sqrt{r{-}k{-}2}\sqrt{r{-}k{-}1} \\ \vdots \\ \sqrt{k{+}1}\sqrt{k{+}2}\sqrt{k{+}3}\cdots\sqrt{r{-}2}\sqrt{r{-}k{-}1} \\ \sqrt{k{+}1}\sqrt{k{+}2}\sqrt{k{+}3}\cdots\sqrt{r{-}2}\sqrt{r{-}1} \end{pmatrix}, \tag{5.62}$$

$$H^{(k)}_{\mathrm{Gr}(P)}: \qquad (\mu_\ell)_{\mathrm{zero}} \;=\; \beta_k \begin{pmatrix} \sqrt{r{+}1}\sqrt{r{+}2}\sqrt{r{+}3}\cdots\sqrt{r{+}k{-}2}\sqrt{r{+}k{-}1} \\ \sqrt{r{-}k{+}1}\sqrt{r{+}2}\sqrt{r{+}3}\cdots\sqrt{r{+}k{-}2}\sqrt{r{+}k{-}1} \\ \sqrt{r{-}k{+}1}\sqrt{r{-}k{+}2}\sqrt{r{+}3}\cdots\sqrt{r{+}k{-}2}\sqrt{r{+}k{-}1} \\ \vdots \\ \sqrt{r{-}k{+}1}\sqrt{r{-}k{+}2}\sqrt{r{-}k{+}3}\cdots\sqrt{r{-}2}\sqrt{r{+}k{-}1} \\ \sqrt{r{-}k{+}1}\sqrt{r{-}k{+}2}\sqrt{r{-}k{+}3}\cdots\sqrt{r{-}2}\sqrt{r{-}1} \end{pmatrix}. \tag{5.63}$$

Altogether, there are $2r{-}2$ real zero modes in $H_{\mathrm{im}\,P}$ and $2r$ real zero modes in $H_{\mathrm{Gr}(P)}$. Identifying $\mu_{(k)j} = \Lambda_{j,k+j} = -\bar{\Lambda}_{k+j,j}$, the latter precisely agree with the BPS moduli found in (5.27). The former zero modes correspond to flat directions outside the Grassmannian. Some of these generate nearby non-diagonal non-BPS solutions and are therefore non-BPS-moduli. In contrast, the remaining ones are not moduli but arise from the fact that $E^{(2)}\big|_{U(\mathrm{im}\,P)}$ is indefinite near the saddle point given by $\Phi_r$ and thus separate the hill from the valley.

*Diagonal perturbations*

Next, we discuss the diagonal or radial perturbations $\phi_{(0)} \in \mathcal{E}_0$ (see (5.43)). The transformation of

$$\tfrac{1}{2\pi}E^{(2)}[\Phi_r,\phi_{(0)}] \;=\; \sum_{m=0}^{\infty}(m{+}1)\,(\phi_{m+1}-\phi_m)^2 - 2r\,(\phi_r^2+\phi_{r-1}^2) \tag{5.64}$$

to a sum of squares is much easier than that of (5.47), but the result shows one minus in the signature:

$$\tfrac{1}{2\pi}E^{(2)}[\Phi_r,\phi_{(0)}] \;=\; -r\,(\phi_{r-1}+\phi_r)^2 \;+ \sum_{j\geq 0,\, j\neq r-1} (j{+}1)\,(\phi_{j+1}-\phi_j)^2\,. \tag{5.65}$$

However, we can use this expression to conclude that the second variation form $E^{(2)}$ cannot have more than one negative mode on $\mathcal{E}_0$.

**Proposition 5.1**
*Let $V \subset \mathcal{E}_0$ be a real vector subspace such that $E^{(2)}[Phi,\phi] < 0$ for all nonzero $\phi \in V$. Then $V$ is at most one-dimensional. In other words, one cannot find linearly independent vectors $\phi,\psi \in \mathcal{E}_0$ such that $E^{(2)}[\Phi,.]$ takes negative values on each non-zero linear combination of $\phi$ and $\psi$.*

*Proof.* A necessary and sufficient condition for a pair of linearly independent vectors $\phi, \psi \in \mathcal{E}_0$ to span a negative subspace for $E^{(2)}(.) := E^{(2)}[\Phi, .]$ is that

$$E^{(2)}(\phi) \;<\; 0 \;, \quad E^{(2)}(\psi) \;<\; 0 \quad \text{and} \quad B(\phi,\psi)^2 - E^{(2)}(\phi)E^{(2)}(\psi) \;<\; 0 \;, \tag{5.66}$$

where $B(\phi,\psi) := \frac{1}{2}\{E^{(2)}(\phi{+}\psi) - E^{(2)}(\phi) - E^{(2)}(\psi)\}$ is the corresponding bilinear form. If $\phi, \psi \in \mathcal{E}_0$ satisfy these inequalities, then the same holds for their truncations (that is, vectors obtained by replacing all the coordinates $\phi_j$, $\psi_j$ with $j \geq n$ by zero) if $n$ is large enough. Thus, we may assume the existence of $n$ such that $\phi_j = \psi_j = 0$ for $j \geq n$. But the restriction of $E^{(2)}$ to the space $\mathbb{R}^n$ of all vectors $\phi = (\phi_0, \phi_1, \ldots, \phi_{n-1}, 0, 0, \ldots)$ has signature $(1, n{-}1)$ according to (5.65) and, therefore, this space contains no two-dimensional negative subspaces. ∎

For an alternative visualization, we write the second-order fluctuation form as

$$E^{(2)}[\Phi_r, \phi_{(0)}] \;=\; 2\pi \sum_{m,\ell=0}^{\infty} \phi_m \, H^{(0)}_{m\ell} \, \phi_\ell \;, \tag{5.67}$$

where the matrix entries $H^{(0)}_{m\ell}$ can be extracted from (5.6) or from (5.65) as

$$\tfrac{1}{2\pi}\, H^{(0)}_{m\ell} \;=\; (2m{+}1 - 2r\delta_{m,r-1} - 2r\delta_{m,r})\, \delta_{m,\ell} - m\, \delta_{m,\ell+1} - (m{+}1)\, \delta_{m+1,\ell} \;. \tag{5.68}$$

In matrix form this reads for $m, \ell \in \mathbb{N}_0$ as

$$\begin{aligned}
(H^{(0)}_{m\ell}) \;&=\; \begin{pmatrix} 1 & -1 & & & \\ -1 & 3 & -2 & & \\ & -2 & 5 & -3 & \\ & & -3 & 7 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \;-\; 2\,r\,(\delta_{m,r-1}\delta_{\ell,r-1} \;+\; \delta_{m,r}\delta_{\ell,r}) \\
&=\; \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 2r{-}3 & -r{+}1 & & & \\ & -r{+}1 & \mathbf{-1} & -r & & \\ & & -r & \mathbf{+1} & -r{-}1 & \\ & & & -r{-}1 & 2r{+}3 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} .
\end{aligned} \tag{5.69}$$

In distinction to the earlier cases, this infinite Jacobian (i.e. tri-diagonal) matrix does not split. We shall compute its spectrum in a moment. Prior to this, some observations can already be made.

First of all, the unique zero mode of $H^{(0)}$ is found from (5.56) for $k=0$ and given by $(\gamma_0 \in \mathbb{R})$

$$\phi_{\text{zero}} \;=\; \mathrm{i}\gamma_0\, \Phi_r \qquad \Longleftrightarrow \qquad (\phi_m)_{\text{zero}} \;=\; \gamma_0\, (\underbrace{-1, \ldots, -1}_{r \text{ times}}, +1, +1, \ldots) \;, \tag{5.70}$$

which clearly generates the global phase rotation symmetry $\Phi \to \mathrm{e}^{\mathrm{i}\gamma_0}\Phi$ already noted in (3.39) and below. In contrast to the zero modes in $\mathcal{E}_{k>0}$ depicted in (5.56), (5.62) and (5.63), $\phi_{\mathrm{zero}}$ is not Hilbert-Schmidt but bounded. Although it does not belong to a proper zero eigenvalue of the Hessian, $\phi_{\mathrm{zero}}$, being proportional to $\Phi_r$, still meets all our requirements for an admissible perturbation.

In order to identify unstable modes, we turn on all diagonal perturbations in a particular manner suggested by (5.31),

$$\Phi(\{\alpha_m\}) \;=\; \mathbb{1} - \sum_{m=0}^{r-1}(\mathrm{e}^{\mathrm{i}\alpha_m}{+}1)\,|m\rangle\langle m| + \sum_{m=r}^{\infty}(\mathrm{e}^{\mathrm{i}\alpha_m}{-}1)\,|m\rangle\langle m|\,. \tag{5.71}$$

The energy of this configuration is easily calculated to be

$$\begin{aligned}\tfrac{1}{8\pi}\,E(\{\alpha_m\}) \;=\; & \sin^2\tfrac{\alpha_0-\alpha_1}{2} + \dots + (r{-}1)\sin^2\tfrac{\alpha_{r-2}-\alpha_{r-1}}{2} \\ & + r\cos^2\tfrac{\alpha_{r-1}-\alpha_r}{2} + (r{+}1)\sin^2\tfrac{\alpha_r-\alpha_{r+1}}{2} + \dots\end{aligned} \tag{5.72}$$

with a single $\cos^2$ appearing only in the indicated place. Expanding up to second order in the real parameters $\alpha_m$ and putting $\alpha_m = 0$ for $m \geq r$, we find that

$$\begin{aligned} H^{(0)} < 0 \qquad \Longleftrightarrow \qquad & (\alpha_0{-}\alpha_1)^2 + 2(\alpha_1{-}\alpha_2)^2 + \dots + (r{-}1)(\alpha_{r-2}{-}\alpha_{r-1})^2 \\ & \qquad -r\,\alpha_{r-1}^2 \;\leq\; 0\,, \end{aligned} \tag{5.73}$$

which determines a convex cone in the $r$-dimensional restricted fluctuation space. The most strongly negative mode is given by ($\alpha \in \mathbb{R}$)

$$\phi_{\mathrm{neg}} \;=\; -\mathrm{i}\alpha P_r \qquad \Longleftrightarrow \qquad (\phi_m)_{\mathrm{neg}} \;=\; -\alpha\,(\underbrace{1,\dots,1}_{r\ \text{times}},0,0,\dots)\,. \tag{5.74}$$

Comparison with (5.35) reveals that following this mode one arrives at $\Phi = \mathbb{1}$. More generally,

$$(\phi_m) \;=\; -\alpha\,(\underbrace{0,\dots,0}_{\tilde{r}\ \text{times}},\underbrace{1,\dots,1}_{(r-\tilde{r})\ \text{times}},0,0,\dots) \tag{5.75}$$

perturbs in the direction of $\Phi = \mathbb{1} - 2P_{\tilde{r}}$. Nevertheless, none of these modes are eigenvectors of the matrix $H^{(0)}$.

It is instructive to confirm these assertions numerically. To this end, we truncate the matrix $(H^{(0)}_{m\ell})$ at some cut-off value $m_{\max} = \ell_{\max}$ and diagonalize the resulting finite-dimensional matrix (see also the corresponding Mathematica code in appendix A.1). The truncation at values $m_{\max} < r$ only allows for positive eigenvalues, whereas for values $m_{\max} \geq r$ we obtain exactly one negative eigenvalue. Its numerical value depends roughly linearly on $r$ and converges very quickly with increasing cut-off as can be seen from the following graphs.

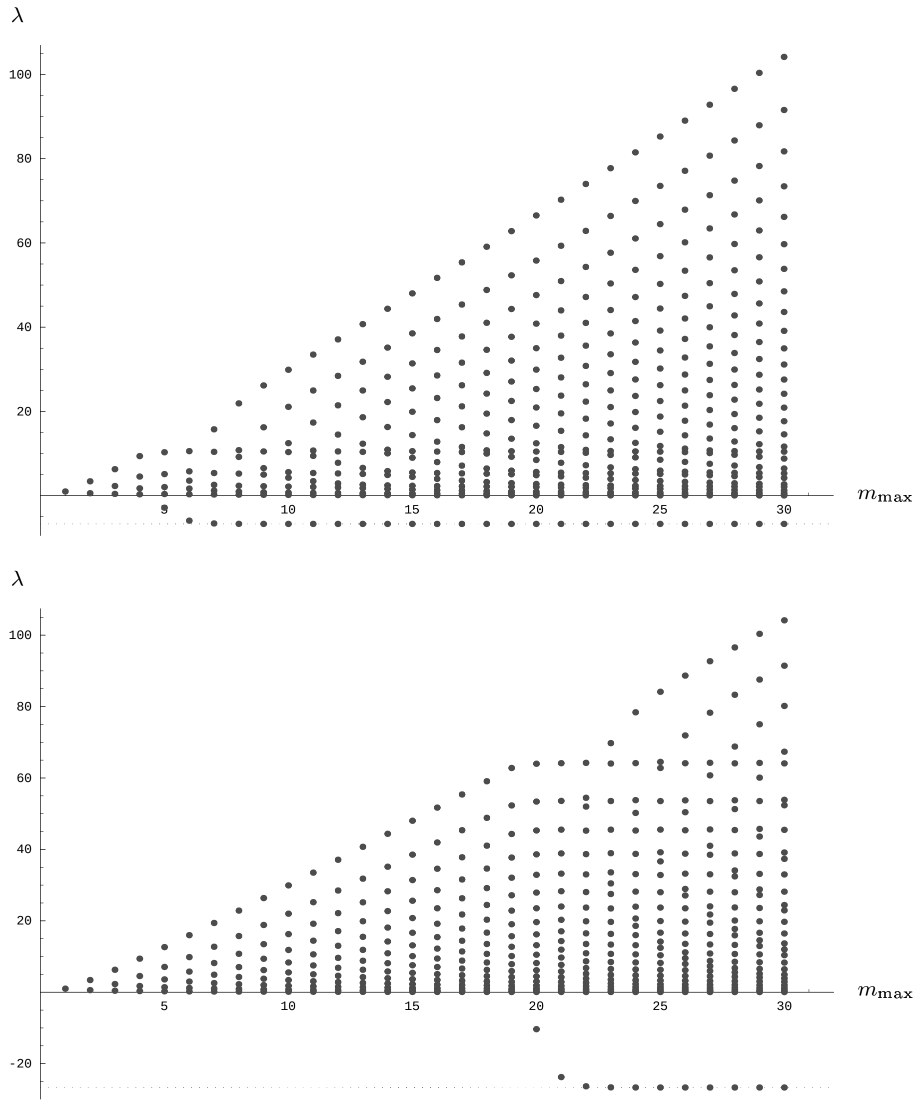



Figure 5.1: Possible eigenvalues $\lambda$ versus the cut-off parameter $m_{\max}$ for $r = 5$ and $r = 20$.

*Explicit spectrum of the Hessian*

The spectrum of the restriction $H^{(k)} = \Delta_k - (\Phi_r \Delta \Phi_r)_k$ of the Hessian $H$ to $\mathcal{E}_k$ can in this simple case and under an additional assumptions actually be explicitly calculated by passing to a basis of Laguerre polynomials. The following considerations are based on spectral theory and require all vectors to be in $l_2$, i.e. all operators to be Hilbert-Schmidt.

Let us first consider $\mathcal{E}_0$. We use the following property of the noncommutative Laplacian $-\Delta_0$: sending each basis vector $|m\rangle$ of $\mathcal{H}$ to the $m$-th Laguerre polynomial $L_m(x)$, we get a unitary map $\mathcal{U} : \mathcal{H} \to L^2(\mathbb{R}_+, \mathrm{e}^{-x}\mathrm{d}x)$ such that $\mathcal{U}\Delta_0\mathcal{U}^{-1}$ is just the operator of multiplication by $x$:

$$x\,L_m(x) \;=\; -m\,L_{m-1}(x) + (2m{+}1)L_m(x) - (m{+}1)L_{m+1}(x)\ , \tag{5.76}$$

which is to be compared with (5.69). Hence, in the basis of the Laguerre polynomials, the eigenvalue equation $H^{(0)}|\phi\rangle = \lambda|\phi\rangle$ is rewritten in terms of $f := \mathcal{U}|\phi\rangle$ as

$$x\,f(x) - 2r\big(\langle f, L_{r-1}\rangle L_{r-1}(x) + \langle f, L_r\rangle L_r(x)\big) \;=\; \lambda\,f(x)\,, \tag{5.77}$$

where

$$\langle f, g\rangle \;:=\; \int_{\mathbb{R}_+} \mathrm{d}x\,\bar{f}(x)\,g(x)\,\mathrm{e}^{-x}\,. \tag{5.78}$$

Clearly, a function $f \in L^2(\mathbb{R}_+, \mathrm{e}^{-x}\mathrm{d}x)$ satisfies (5.77) if and only if it is given by

$$f(x) \;=\; \frac{c_0 L_{r-1}(x) + c_1 L_r(x)}{x-\lambda} \tag{5.79}$$

and the constant coefficients $c_0, c_1 \in \mathbb{R}$ satisfy the linear system

$$\begin{aligned} \Big(I_{r-1,r-1}(\lambda) - \frac{1}{2r}\Big)\,c_0 + I_{r-1,r}(\lambda)\,c_1 \;&=\; 0 \\ \text{and}\quad I_{r,r-1}(\lambda)\,c_0 + \Big(I_{r,r}(\lambda) - \frac{1}{2r}\Big)\,c_1 \;&=\; 0\ , \end{aligned} \tag{5.80}$$

which is obtained simply by inserting (5.79) into (5.77). Here we use the notation

$$I_{k,l}(\lambda) \;:=\; \int_0^\infty \frac{\mathrm{e}^{-x}\,\mathrm{d}x}{x-\lambda}\,L_k(x)\,L_l(x)\,, \quad \text{for} \quad k,l \geq 0\ . \tag{5.81}$$

The integrals (5.81) are simple, though not elementary, special functions of $\lambda$. Indeed, $I_{00}(\lambda)$ is a version of the integral logarithm: $I_{00}(\lambda) = -\mathrm{e}^{-\lambda}\,\mathrm{li}(\mathrm{e}^{\lambda})$ for all $\lambda < 0$, where $\mathrm{li}(x) := \int_0^x \frac{\mathrm{d}t}{\ln t}$. On the other hand, using the recursion relations

$$(k{+}1)\,L_{k+1}(x) \;=\; (2k{+}1{-}x)\,L_k(x) \;-\; k\,L_{k-1}(x)\ , \tag{5.82}$$

one can show by induction over $k$ and $l$ that all functions $I_{k,l}(\lambda)$ are expressed in terms of $I_{00}(\lambda)$:

$$I_{kl}(\lambda) \;=\; A_{kl}(\lambda)\,I_{00}(\lambda) \;+\; B_{kl}(\lambda)\,, \tag{5.83}$$

where $A_{kl}$ and $B_{kl}$ are polynomials in $\lambda$ of degree at most $k{+}l$. Hence, the determinant

$$F_r(\lambda) \;:=\; \begin{Vmatrix} I_{r-1,r-1}(\lambda) - \frac{1}{2r} & I_{r-1,r}(\lambda) \\ I_{r,r-1}(\lambda) & I_{r,r}(\lambda) - \frac{1}{2r} \end{Vmatrix} \tag{5.84}$$

of the linear system (5.80) is a known special function of $\lambda$, whose zeros $\lambda_r$ on the negative semiaxis are precisely the negative eigenvalues of the Hessian operator $H^{(0)}[\Phi_r]$ for the diagonal BPS background of rank $r$.

One can now prove the existence of negative eigenvalues. By verifying that the real numbers $F_r(-\infty) := \lim_{\lambda\to-\infty} F_r(\lambda)$ and $F_r(0) := \lim_{\lambda\to 0-} F_r(\lambda)$ have different signs, we can conclude that $F_r(\lambda)$ possesses at least one zero on the negative semiaxis. For example, take $r = 1$. In this case we find

$$\begin{aligned} I_{01}(\lambda) \;=\; I_{10}(\lambda) &\;=\; (1-\lambda)I_{00}(\lambda) - 1 \\ \text{and} \quad I_{11}(\lambda) &\;=\; (1-\lambda)^2 I_{00}(\lambda) + \lambda - 1\,, \end{aligned} \tag{5.85}$$

from which it follows that

$$2\,F_1(\lambda) \;=\; -\lambda - \tfrac{1}{2} - \lambda^2 I_{00}(\lambda)\,. \tag{5.86}$$

Passing to the limit under the integral sign, we see that $F_1(-\infty) = \frac{1}{4}$ and $F_1(0) = -\frac{1}{4}$. This proves the existence of a negative eigenvalue of $H^{(0)}[\Phi_1]$.

Can $H^{(0)}[\Phi_r]$ also have non-negative eigenvalues? To disprove this possibility, we consider the eigenvalue equation (5.77) and show that it admits only the trivial solution $f(x) \equiv 0$ if $\lambda \geq 0$. Indeed, for non-negative $\lambda$, the solution (5.79) is not square-integrable unless

$$c_0 L_{r-1}(\lambda) + c_1 L_r(\lambda) \;=\; 0\,, \tag{5.87}$$

which implies

$$c_0 \;=\; c\,L_r(\lambda) \quad \text{and} \quad c_1 \;=\; -c\,L_{r-1}(\lambda) \tag{5.88}$$

for some constant $c$. Therefore, the solution (5.79) is completely determined as

$$f(x) \;=\; c\,\frac{L_r(\lambda)L_{r-1}(x) - L_{r-1}(\lambda)L_r(x)}{x-\lambda} \;=\; c\sum_{k=0}^{r-1} L_k(\lambda)\,L_k(x) \tag{5.89}$$

via the Christoffel-Darboux formula [54]. It follows that $f$ is orthogonal to $L_r$. Inserting $\langle f, L_r\rangle = 0$ into (5.77), we learn that the polynomial $(x-\lambda)f(x)$ is proportional to $L_{r-1}(x)$. But then the first equality in (5.89) demands that $L_{r-1}(\lambda) = 0$, constraining $\lambda$. Feeding this into the second expression for $f$ in (5.89) we see that the sum runs to $r-2$ only, implying that $f$ is orthogonal to $L_{r-1}$, as well. This finally simplifies the eigenvalue equation (5.77) to the one for $\Delta_0$, namely

$$x\,f(x) \;=\; \lambda\,f(x)\;, \quad \text{implying} \quad f(x) \;\equiv\; 0\;, \tag{5.90}$$

as required. Hence, the non-negative part of the spectrum of $H^{(0)}$ is purely continuous.

A similar analysis can be applied to the Hessian on $\mathcal{E}_{k>0}$. In this case, the appropriate polynomials are the normalized (generalized) Laguerre polynomials $L_m^k$, which form an orthonormal basis for $L^2(\mathbb{R}_+, x^k \mathrm{e}^{-x}\mathrm{d}x)$. One rediscovers the $r$ proper eigenvalues as zeroes of the characteristic polynomial for $H^{(k)}_{\mathrm{im}\,P} \oplus H^{(k)}_{\mathrm{Gr}(P)}$, accompanied by a continuous spectrum covering the positive semiaxis for $H^{(k)}_{\ker P}$, as claimed earlier.

If we do not care for resolving non-isolated eigenvalues, we may also infer the spectrum of the Hessian $H$ from the classical Weyl theorem (see [66], Theorem IV.5.35 and [104], § XIII.4, Example 3).

**Proposition 5.2**

*Let $\Phi = \Phi_r$ be the diagonal BPS solution of rank $r > 0$. Then*

$$\text{spectrum}\,(H[\Phi_r]) \;=\; \{\lambda_r\} \cup [0, +\infty)\;, \tag{5.91}$$

*where $\lambda_r$ is a negative eigenvalue of multiplicity 1 and $[0, +\infty)$ is the essential spectrum.*[9] *Furthermore, we have $-2r < \lambda_r < 0$.*

*Proof.* Weyl's theorem asserts that the essential spectrum is preserved under compact perturbations. The essential spectrum of $\Delta$ (on all subspaces $\mathcal{E}_k$) is known to be equal to $[0, +\infty)$. This follows, for example, from the explicit diagonalization of $\Delta_k$ in terms of the Laguerre polynomials, as indicated above. Since the perturbation $\Phi_r\Delta\Phi_r$ is finite-dimensional, Weyl's theorem applies to show that the essential spectrum of $H$ is also equal to $[0, +\infty)$. Although we have explicitly shown that $H^{(k)}$ is positive semi-definite for $k{>}0$, the essential spectrum $\mathbb{R}_+$ cannot exhaust the whole spectrum because the operator $H^{(0)}$ is not positive semi-definite on $\mathcal{E}_0$. Indeed, (5.65) shows that the quadratic form $E^{(2)}$ is negative on many finite vectors. Hence, $H^{(0)}$ must have negative eigenvalues of finite multiplicity. The above proposition implies that there can be only one such eigenvalue, and its multiplicity must be equal to 1. This proves (5.91) and the inequality $\lambda_r < 0$. The estimate $-2r < \lambda_r$ follows because the norm of the operator $(\Phi_r\Delta\Phi_r)$ is equal to $2r$, so $H[\Phi_r]{+}2r\mathbb{1}_{\mathcal{H}} \geq \Delta$ is non-negative definite and hence cannot have negative eigenvalues. ■

### 5.2.4 Results for non-diagonal $U(1)$ BPS backgrounds

Even though any non-diagonal BPS background $\Phi$ can be reached from a diagonal one by a unitary transformation $\mathcal{U}$ which does not change the value of the energy, the fluctuation problem will get modified under such a transformation because

$$\mathcal{U}\,a\,\mathcal{U}^\dagger \;=\; f(a, a^\dagger) \quad\text{and}\quad \mathcal{U}\,a^\dagger\mathcal{U}^\dagger \;=\; f^\dagger(a, a^\dagger) \tag{5.92}$$

will in general not have an action as simple as $a$ and $a^\dagger$ in the original oscillator basis. The exceptions are the symmetry transformations, of which only the rigid translations $\mathcal{D}(\alpha)$ (3.41) and rotations $\mathcal{R}(\vartheta)$ (3.43) keep $\Phi$ within the Grassmannian. To be specific, we recall that

$$\begin{aligned}
\mathcal{D}(\alpha)^\dagger\, a\, \mathcal{D}(\alpha) \;&=\; a + \alpha \qquad\Longrightarrow\qquad E\,[\mathcal{D}(\alpha)\,\Phi\,\mathcal{D}(\alpha)^\dagger] \;=\; E\,[\Phi]\;, \\
\mathcal{R}(\vartheta)^\dagger\, a\, \mathcal{R}(\vartheta) \;&=\; \mathrm{e}^{\mathrm{i}\vartheta} a \qquad\Longrightarrow\qquad E\,[\mathcal{R}(\vartheta)\,\Phi\,\mathcal{R}(\vartheta)^\dagger] \;=\; E\,[\Phi]\,.
\end{aligned} \tag{5.93}$$

The invariance of the spectrum of $H$ under translations (or rotations) can also be seen directly:

$$H\,\phi_n \;=\; \epsilon_n\,\phi_n \quad\text{and}\quad \Delta\,(\mathcal{D}\,f\,\mathcal{D}^\dagger) \;=\; \mathcal{D}\,(\Delta f)\,\mathcal{D}^\dagger \tag{5.94}$$

[9] We recall that the essential spectrum of a self-adjoint operator is the whole spectrum minus all isolated eigenvalues of finite multiplicity. The multiplicity of an eigenvalue is the dimension of the corresponding eigenspace.

imply

$$H'\phi'_n \;\equiv\; \big[\Delta-(\mathcal{D}\Phi\mathcal{D}^\dagger)\Delta(\mathcal{D}\Phi\mathcal{D}^\dagger)\big]\mathcal{D}\phi_n\mathcal{D}^\dagger \;=\; \mathcal{D}\left([\Delta-\Phi\Delta\Phi]\phi_n\right)\mathcal{D}^\dagger \;=\; \epsilon_n\,\phi'_n\,. \tag{5.95}$$

More general unitary transformations will not simply commute with the action of $\Delta$ and thus change the spectrum of $H$.

In the rank-one BPS situation, any solution is a translation of $\Phi_1$, and hence our previous discussion of fluctuations around diagonal $U(1)$ BPS backgrounds completely covers that case. This is no longer true for higher-rank BPS backgrounds. For a complete stability analysis of abelian $r$-solitons it is therefore necessary to investigate separately the spectrum of $H$ around each soliton configuration (4.45) based on $r$ coherent states, whose center of mass may be chosen to be the origin. Since the background holomorphically depends on $r$ parameters $\alpha_1,\dots,\alpha_r$ and passes to $\Phi_r$ when all parameters go to zero, one may hope to show that the spectrum of $H$ does not change qualitatively when $\Phi$ varies inside the rank-$r$ moduli space.

## 5.3 Perturbations of $U(2)$ backgrounds

In this section we exemplify the analysis in the nonabelian case by investigating the behavior of a simple $U(2)$ BPS solution under perturbations concluding that the fluctuation analysis reduces to the $U(1)$ case discussed in the previous section.

### 5.3.1 Results for diagonal $U(2)$ BPS backgrounds

Consider a nonabelian BPS projector of the diagonal type considered in (4.53):

$$P \;=\; \mathbb{1}_{\mathcal{H}}\oplus P_Q\,,\quad \text{where } P_Q \;=\; \sum_{k=0}^{Q-1}|k\rangle\langle k|\,. \tag{5.96}$$

Clearly this describes a BPS solution

$$\Phi \;=\; \begin{pmatrix}\Phi^{(1,1)} & \Phi^{(1,2)}\\ \Phi^{(2,1)} & \Phi^{(2,2)}\end{pmatrix} \;=\; \begin{pmatrix}-\mathbb{1}_{\mathcal{H}} & 0\\ 0 & \mathbb{1}_{\mathcal{H}}-2P_Q\end{pmatrix} \tag{5.97}$$

of energy $E=8\pi Q$. Inserting this into the expression (5.3) for the quadratic energy correction $E^{(2)}$ one obtains

$$\begin{aligned} E^{(2)}[\Phi,\phi] \;&=\; 2\pi\,\mathrm{Tr}\big\{\phi^\dagger\,\Delta\phi \;-\; \phi^\dagger\,(\Phi\Delta\Phi^\dagger)\,\phi\big\}\\ &=\; 2\pi\,\mathrm{Tr}\big\{\phi^\dagger\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\Delta\phi \;-\; \phi^\dagger\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)2Q\big(|Q{-}1\rangle\langle Q{-}1|+|Q\rangle\langle Q|\big)\,\phi\big\}\,. \end{aligned} \tag{5.98}$$

Since the action of the Hessian is obviously diagonal, we find that

$$\phi \;=\; \begin{pmatrix}\phi^{(1,1)} & \phi^{(1,2)}\\ \phi^{(2,1)} & \phi^{(2,2)}\end{pmatrix} \qquad\Longrightarrow\qquad E^{(2)}[\Phi,\phi] \;=\; \sum_{i,j=1}^{2}E^{(2)}\big[\Phi^{(i,i)},\phi^{(i,j)}\big]\,, \tag{5.99}$$

which reduces the fluctuation analysis to a collection of abelian cases. For the case at hand, the Hessian in the $(1,.)$ sectors is given by the Laplacian and thus non-negative, while in each $(2,.)$ sector it is identical to the Hessian for the abelian rank-$Q$ diagonal BPS case. Hence, the relevant results of the previous section carry over completely.

This mechanism extends to any diagonal $U(2)$ background (not necessarily BPS), mapping the $U(2)$ fluctuation spectrum to a collection of fluctuation spectra around corresponding diagonal $U(1)$ configurations. The fluctuation problem around non-diagonal backgrounds, in contrast, is not easily reduced to an abelian one, except when the background is related to a diagonal one by a rigid symmetry, as defined in (3.39), (3.41) and (3.43). Only a small part of the moduli, however, is generated by rigid symmetries, as our example of $\mathcal{U}(\mu)$ in (4.59) demonstrates even for $Q = 1$. Finally, it is straightforward to extend these considerations to the general $U(n)$ case.

## 5.4 Summary on fluctuation analysis

Concluding the fluctuation analysis carried out in this chapter, we summarize what has been achieved:
Starting from grassmannian configurations we found that thanks to the BPS property (anti-)solitons within the grassmannian sigma model are always stable. As in the commutative case, their embedding into a unitary sigma model renders them unstable, however, as there always exists one negative eigenvalue of the Hessian which triggers a decay to the vacuum configuration.

Apart from these general statements we obtained explicit and complete results for abelian and nonabelian $r$-soliton configurations which are diagonal in the oscillator basis or related to such by global symmetry. For this case, we proved that the spectrum of the Hessian consists of the essential spectrum $[0, \infty)$ and an eigenvalue $\lambda_r$ of multiplicity one with $-2r < \lambda_r < 0$. This assertion was confirmed numerically, and the value of $\lambda_r$ was given as a zero of a particular function composed of monomials in $\lambda$ and the special function $\mathrm{e}^{-\lambda}\,\mathrm{li}(\mathrm{e}^{\lambda})$.

Furthermore, the complete set of zero modes of the Hessian was identified. Each abelian diagonal $r$-soliton background is characterized by a diagonal projector $P$ of rank $r$, whose image and kernel trigger a decomposition of the space of fluctuations into three invariant subspaces, namely $u(\mathrm{im}\, P)$, $u(\ker P)$ and $\mathrm{d}\,\mathrm{Gr}(P)$. In addition, every side diagonal together with its transpose is separately invariant under the action of the Hessian. This leads to a particular distribution of the admissible zero modes of the Hessian, displayed here for the example of $r = 4$:

$\phi$ =

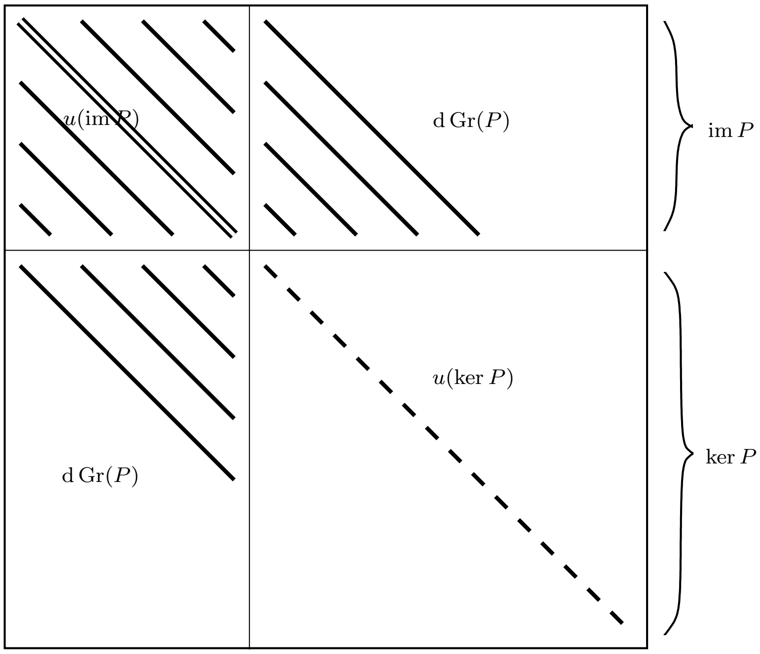



Figure 5.2: Admissible zero modes for abelian diagonal rank 4 soliton.

The double line denotes the single negative eigenvector, each solid diagonal segment represents a real normalizable zero mode, the dashed line depicts an admissible non-normalizable zero mode, and empty areas do not contain admissible zero modes. In addition, each side diagonal features a non-admissible zero mode at the edge of the continuous part $[0, \infty)$ of the spectrum.

For visualization we plot the complete spectrum of the Hessian at $r = 4$ (cut off at size $m_{\text{max}} = 30$) for each invariant subspace $\mathcal{E}_k^{\text{Gr}(P)}$ (boxes), $\mathcal{E}_k^{\text{im}\,P}$ (stars), $\mathcal{E}_k^{\ker P}$ (crosses) and $\mathcal{E}_0$ (circles), up to $k = 6$:

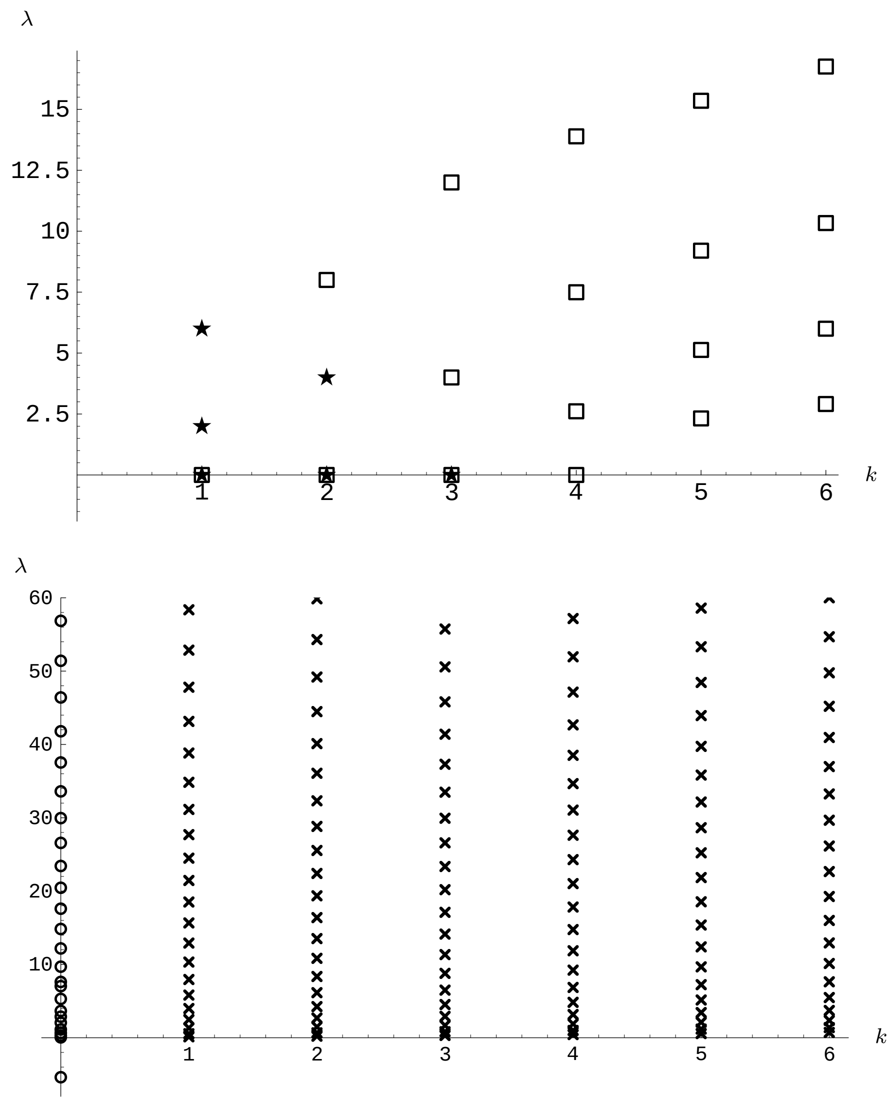



Figure 5.3: Eigenvalues $\lambda$ of the cut-off Hessian (size 30) for $Q = 4$ in subspaces $\mathcal{E}_k$.

Since most soliton moduli are not associated with global symmetries, our explicit results for diagonal backgrounds do not obviously extend to generic (non-diagonal) backgrounds. We have not been able to compute the fluctuation spectrum across the entire soliton moduli space. The only exception is the abelian single-soliton solution which always is a translation of the diagonal configuration and therefore fully covered by our analysis. Already for the case of two $U(1)$ solitons, the unitary transformation which changes their distance in the noncommutative plane is only partially known.

This leads us to a number of unsolved problems. The most pressing one seems to be the extension of our fluctuation analysis to non-diagonal backgrounds. Next, we saw in section 4.4 that following the even zero modes one can find new non-BPS solutions whose further study certainly deserves interest. Another interesting aspect is the existence of infinite-rank abelian projectors, i.e. via an infinite array of coherent states, associated with BPS solutions of infinite energy. Finally, it is worthwhile to investigate the commutative limit of the Hessian and its spectrum.

# CHAPTER 6

# TIME-DEPENDENT WARD MODEL SOLITONS

In this chapter we review results on time-dependent solutions to the Ward model. These will be needed for the comparison with the adiabatic approach presented in the following chapter. The presented material is based on [73, 74, 127, 76] and, to a great extent, also applies to the commutative case. We will therefore refrain from denoting derivatives as commutators and rather stick to the formulation in the star-product formalism, where the symbols are the same as in the commutative setting.[1]

After reviewing the Lax-pair formulation of the Ward model we proceed by presenting boosted static solitons which are given by lumps moving with a constant velocity and hence feature no scattering. To obtain more complex solutions, the dressing approach as an effective means to generate time-dependent multi-solitons is then introduced. The examples to be given are mainly concerned with the abelian case in consideration of the comparison with the adiabatic approach. The discussion is finally followed by an exposition of the scattering properties for these multi-solitons.

## 6.1 LINEAR SYSTEM AND SIMPLE SOLUTIONS

### 6.1.1 Lax-pair formulation of the Ward model

To prove our earlier claim that the equation of motion (3.18) of the Ward model does indeed arise as compatibility condition for some linear system, we introduce an auxiliary $n\times n$ matrix $\psi$ depending on the coordinates $t, x, y$ and a so-called spectral parameter $\zeta \in \mathbb{C} \cup \{\infty\}$ subject to the reality condition

$$\psi(t,x,y,\zeta)\,\psi^{\dagger}(t,x,y,\bar{\zeta}) \;=\; \mathbb{1}\,. \tag{6.1}$$

Furthermore, we may impose some asymptotic conditions [63] on the field $\psi$ of which we will need

$$\lim_{\zeta\to 0}\;\psi(t,x,y,\zeta) \;=\; \Phi^{\dagger} + \mathcal{O}(\zeta)\,, \tag{6.2}$$

where $\Phi$ is a $U(n)$-valued field. This asymptotic condition is used to connect $\psi$ with the Ward model solutions.

Consider now the linear systems of equations

$$\begin{pmatrix} A(t,x,y) \\ B(t,x,y) \end{pmatrix} \;=\; -\,\psi(t,x,y,\zeta)\,L(\zeta)\,\psi^{\dagger}(t,x,y,\bar{\zeta})\,, \tag{6.3}$$

[1]For simplicitly we also leave out the star-symbol for the multiplication.

where

$$L(\zeta) \;:=\; \begin{pmatrix} \zeta\,\partial_x - \partial_{y+t} \\ -\,\zeta\,\partial_{y-t} - \partial_x \end{pmatrix} \;=\; \begin{pmatrix} (\zeta-\mathrm{i})\,\partial_z + (\zeta+\mathrm{i})\,\partial_{\bar z} - \partial_t \\ -\,\mathrm{i}\,(\zeta-\mathrm{i})\,\partial_z + \mathrm{i}\,(\zeta+\mathrm{i})\,\partial_{\bar z} + \zeta\,\partial_t \end{pmatrix}\,, \tag{6.4}$$

and the matrices $A$, $B$ on the left hand side are of the same type as $\psi$ but independent of the spectral parameter $\zeta$ [73, 74]. In fact, this linear system can be obtained from the Lax pair of the self-dual Yang-Mills equation on $\mathbb{R}^{2,2}$ [62, 63] by gauge-fixing and imposing the condition $\partial_3\psi = 0$. The compatibility condition for the linear system (6.3) reads

$$\partial_x B + \partial_{y-t} A \;=\; 0 \quad \text{and} \quad \partial_x A - \partial_{y+t} B - \big[A, B\big] \;=\; 0\,, \tag{6.5}$$

of which, making an ansatz of the form[2]

$$A \;=\; \Phi^\dagger\,\partial_{y+t}\Phi \quad \text{and} \quad B \;=\; \Phi^\dagger\,\partial_x\Phi \tag{6.6}$$

the second equation is automatically satisfied whereas the first one is transformed to the equation of motion of the Ward model (3.18). This shows that it is indeed induced by a Lax-pair formulation and thus, according to the given definition, this model is integrable, as claimed. Reading this statement a different way we see that from each solution $\psi$ of the linear system (6.3) with some $A$ and $B$ we can infer a solution $\Phi$ to the Ward model equation of motion via (6.2) and (6.6). However, it is not possible to simply prescribe some $A$ and $B$ since they depend on the solution $\Phi$ themselves.

### 6.1.2 Dressing approach

The integrability of the Ward model entails some powerful techniques of constructing solutions to the linear system given above and thus to the equation of motion to the Ward model. In this context the dressing approach will be of great importance to us since it provides genuine multi-soliton solutions with relative velocity. It is a recursive procedure to generate new solutions from a given seed solution to (6.3) by an ansatz of the form

$$\bar\psi(t,x,y,\zeta) \;:=\; \chi(t,x,y,\zeta)\,\psi(t,x,y,\zeta)\,, \tag{6.7}$$

where $\psi$ is a solution to the linear system (6.3) for some left hand side $(A, B)$. The dressing factor $\chi$ is, for our purposes, restricted to the simple one pole form

$$\chi \;:=\; \mathbb{1} + \frac{\mu-\bar\mu}{\zeta-\mu}\,P\,, \tag{6.8}$$

for $\mu \in \mathbb{C}\backslash\mathbb{R}$ and some hermitian projector $P$ [73]. Compare also [131, 132, 33] for a more general exposition of this approach.

[2]Note that this ansatz is compatible with the asymptotic condition (6.2) and that it implies algebra-valued $A$ and $B$.

Inserting this ansatz into the linear system (6.3) yields together with the reality condition (6.1)

$$\begin{pmatrix} \tilde{A}(t,x,y) \\ \tilde{B}(t,x,y) \end{pmatrix} = -\,\chi(t,x,y,\zeta)\left\{\begin{pmatrix} A(t,x,y) \\ B(t,x,y) \end{pmatrix} + L(\zeta)\right\}\chi^\dagger(t,x,y,\bar{\zeta})\,, \tag{6.9}$$

where $\tilde{A}$ and $\tilde{B}$ are of the same type as $A$ and $B$. Note that the left hand sides of (6.3) and the equation above are independent of $\zeta$ and that therefore the $\zeta$-dependence on the right hand sides of these equations has to drop out.

### 6.1.3 Simple time-dependent solutions and co-moving coordinates

To exemplify the discussion of the preceding section, we give an easy example for the construction of a time-dependent solution. Consider a seed solution $\psi_0 := \mathbb{1}$, which solves the linear system (6.3) with $A_0 = B_0 = 0$ and the ansatz (6.8) for the dressing factor. The new solution $\psi = \chi\,\psi_0$ is then constructed by expanding the right-hand side of (6.9) into a Laurent-series around $\zeta = \bar{\mu}$ [3] and demanding the vanishing of poles to all orders. This yields the BPS-type conditions

$$(\mathbb{1}-P)(\bar{\mu}\,\partial_x - \partial_{y+t})P = 0 \quad \text{and} \quad (\mathbb{1}-P)(\bar{\mu}\,\partial_{y-t} + \partial_x)P = 0\,, \tag{6.10}$$

for the projector $P$. The ensuing Ward model solution is then found via the asymptotic condition (6.2) as

$$\Phi = \mathbb{1} - \frac{\bar{\mu}-\mu}{\bar{\mu}}\,P\,, \tag{6.11}$$

which is similar in form to a hermitian static solution except for the deviation from 2 of the factor in front of $P$. This renders the solution non-grassmannian in general. An analysis of (6.11) [73, 76] reveals that each finite-rank projector $P$ satisfying (6.10) in this way yields a solution whose energy density is localized with peak moving with constant velocity $\vec{v}$ given by

$$\vec{v} = (v_x, v_y) = -\left(\frac{\mu+\bar{\mu}}{\mu\bar{\mu}+1}\,,\,\frac{\mu\bar{\mu}-1}{\mu\bar{\mu}+1}\right)\,, \tag{6.12}$$

with inverse given by

$$\mu = -\frac{v_x + \mathrm{i}\sqrt{1-\vec{v}}}{1-v_y^2} \in \mathbb{C}\backslash\mathbb{R}\,. \tag{6.13}$$

This solution is thus interpreted as moving lump in the Moyal plane whose total energy $E[\Phi]$ reads

$$E[\Phi] = \frac{\sqrt{1-\vec{v}^2}}{1-v_y^2}\,8\pi\,\mathrm{tr}P = \left(1 - \tfrac{1}{2}\,v_x^2 + \tfrac{1}{2}\,v_y^2 + \ldots\right)8\pi\,\mathrm{tr}P\,, \tag{6.14}$$

[3] This is only one possibility, but in this case a very useful one.

making obvious the Lorentz symmetry breaking.

Since the solutions feature a constant velocity, it is possible to shift to their rest frame via a linear coordinate transformation $(x, y, t) \to (w, \bar{w}, s)$ given by [76]

$$\begin{aligned} w &= \nu\left[\tfrac{1}{2}\, t\,(\bar{\mu} + \tfrac{1}{\bar{\mu}}) \;+\; \tfrac{1}{2}\, y\,(\bar{\mu} - \tfrac{1}{\bar{\mu}}) + x\right] , \\ \bar{w} &= \bar{\nu}\left[\tfrac{1}{2}\, t\,(\mu + \tfrac{1}{\mu}) \;+\; \tfrac{1}{2}\, y\,(\mu - \tfrac{1}{\mu}) + x\right] \\ \text{and} \quad s &= \ldots , \end{aligned} \tag{6.15}$$

where $\nu \in \mathbb{C}$ is to be chosen later and the shifted time-coordinate $s$ is of no relevance for the solutions considered. Using the commutation relations $[x, y]_\star = \mathrm{i}\,\theta$ we find for the new coordinates the star-commutation relation

$$[w, \bar{w}]_\star \;=\; 2\,\mathrm{i}\,\theta\,\nu\bar{\nu}\,\frac{\sqrt{1 - v_x^2 - v_y^2}}{1 - v_y^2} . \tag{6.16}$$

Note that the shift (6.15) is mediated by a unitary transformation $\mathcal{U}(\mu, t)$ effecting a boost with velocity (6.12).

Furthermore, the condition (6.10) is given in the rest frame as

$$(\mathbb{1} - P)\,\partial_{\bar{w}} P \;=\; 0 \quad \text{and} \quad (\mathbb{1} - P)\,\partial_s P \;=\; 0 , \tag{6.17}$$

which is nothing but the BPS equation (3.29) in co-moving coordinates if the solution is static in this frame, i.e. $\partial_s P = 0$, and we will therefore call these solutions solitons, as well. The generic $r$-soliton moving with velocity implied by $\mu$ is thus parametrized by the projector

$$\tilde{P}(t) \;=\; \mathcal{U}(\mu, t)\, P_\alpha\, \mathcal{U}^\dagger(\mu, t) \tag{6.18}$$

where $P_\alpha$ is the generic static BPS solution (4.45) expressed, here, in the star-product picture. The static situation is recovered for the choice $\mu = -\,\mathrm{i}$ which implies $v = 0$ and $w = \nu\,(x + \mathrm{i}\, y)$. To give an example, the simple rank 1 moving soliton computes to [73]

$$\Phi_{1\star}(t, x, y) \;=\; 1 - 2\,\frac{\bar{\mu} - \mu}{\bar{\mu}}\, e^{-\frac{1}{\theta}\left|w(t,x,y) - \sqrt{2\,\theta}\,\alpha\right|} , \tag{6.19}$$

which represents a squeezed Gaussian moving in the Moyal plane because $w$ is time dependent. Note that even though higher rank solutions may feature seperate lumps in the energy density on the Moyal plane these move with equal velocity and thus do not represent genuine multi-solitons as the latter term is reserved for multi-lump solutions with mutually different velocities. These will be the subject of discussion in the next section.

As in the static case it is worthwhile to investigate this solution in the operator formalism, as well. Employing the Moyal-Weyl map (2.12) we associate to the co-moving coordinate $w$ an annihilation operator $c$ via

$$w \quad \leftrightarrow \quad \sqrt{2\,\theta}\, c , \quad \text{where } \nu \text{ is chosen such that } \left[c, c^\dagger\right] \;=\; \mathbb{1} . \tag{6.20}$$

By definition, $c$ annihilates the co-moving vacuum $|\vec{v}\rangle$, i.e. $c\,|\vec{v}\rangle = 0$, and the BPS equation (6.17) now becomes

$$(\mathbb{1} - P)\, c\, P \;=\; 0\,, \tag{6.21}$$

which, upon parametrizing the projector again as

$$P \;=\; |T\rangle\, \langle T|T\rangle^{-1} \langle T|$$

is solved in the abelian case similar to the static situation by

$$|T\rangle \;=\; \Big( |T_1\rangle\, |T_2\rangle\, \dots\, |T_r\rangle \Big) \quad \text{with} \quad |T_i\rangle \;:=\; \mathrm{e}^{\alpha_i\, c^\dagger}\, |\vec{v}\rangle\,, \tag{6.22}$$

for $\alpha_i \in \mathbb{C}$.

A change $\vec{v} \to \vec{v}\,'$ of the velocity of this solution is mediated by an inhomogeneous $SU(1,1)$ squeezing transformation given by a successive time-dependent translation (2.23)[4] followed by a squeezing (2.27) transformation with

$$c' \;=\; \mathcal{S}(\vec{v}\,')\, \mathcal{D}(\vec{v}\,', t)\, c\, \mathcal{D}^\dagger(\vec{v}\,', t)\, \mathcal{S}(\vec{v}\,') \quad \text{as well as} \quad |\vec{v}\,'\rangle \;=\; \mathcal{S}(\vec{v}\,')\, \mathcal{D}(\vec{v}\,', t)\, |\vec{v}\rangle \tag{6.23}$$

and all co-moving vacua are generated in this way from $|0\rangle$. The simple rank 1 moving-soliton is thus given by

$$\Phi_1(t) \;=\; \mathrm{e}^{\alpha\, c^\dagger - \bar{\alpha}\, c} \left( \mathbb{1} - \frac{\bar{\mu} - \mu}{\bar{\mu}}\, |\vec{v}\rangle\langle\vec{v}| \right) \mathrm{e}^{-(\alpha\, c^\dagger - \bar{\alpha}\, c)}\,. \tag{6.24}$$

We may map this soliton again back to the Moyal plane using the inverse Moyal-Weyl map (2.14) to obtain (6.19).

## 6.2 Time-dependent multi-solitons

Let us now proceed to multi-soliton solutions with nontrivial time dependence which may also be generated by the dressing approach introduced above. Mentioning only the most important results of the given references we remark that a generic $U(1)$ $r$-soliton is constructed in a manner similar to (6.22) via rows of states

$$\Big( |T_i^{(k)}\rangle \Big)_{i=1\dots r_k}^{k=1\dots r}\,,$$

parametrized by $(\mu_1, \dots, \mu_r)$ and thus by corresponding velocities $(\vec{v}_1, \dots, \vec{v}_r)$. These states are subject to the BPS conditions

$$c_k\, |T^{(k)}\rangle \;=\; |T^{(k)}\rangle\, \Gamma^{(k)} \quad \text{with} \quad c_k \;=\; \mathcal{S}(\vec{v}_k)\, \mathcal{D}(\vec{v}_k, t)\, a\, \mathcal{D}(\vec{v}_k, t)^\dagger \mathcal{S}^\dagger(\vec{v}_k) \tag{6.25}$$

[4] the precise dependence on the time $t$ being more complicated [73]

and $r_k \times r_k$ matrices $\Gamma^{(k)}$. For a basis in which $\Gamma^{(k)}$ is diagonal the constituents of the solution are given by

$$|T_i^{(k)}\rangle \;=\; \mathrm{e}^{\alpha_i^k c_k^\dagger}\,|v_k\rangle \quad \text{with} \quad |v_k\rangle \;=\; \mathcal{S}(\vec{v}_k)\,\mathcal{D}(\vec{v}_k,t)\,|0\rangle\,. \tag{6.26}$$

Hence the $|v_k\rangle$ are the co-moving vacua with velocities $\vec{v}_k$. Solutions constructed in this way describe $r$-soliton solutions where each lump moves with a different velocity in the noncommutative plane and which do not scatter.

As an example, the exact two-soliton solution of rank two in the abelian Ward model reads [73]

$$\begin{aligned}\Phi_2(t) \;=\; \mathbb{1} \;&-\; \frac{(1-\frac{\mu_1}{\bar\mu_1})|1\rangle\langle 2|2\rangle\langle 1| + (1-\frac{\mu_2}{\bar\mu_2})|2\rangle\langle 1|1\rangle\langle 2|}{\langle 1|1\rangle\langle 2|2\rangle - \mu\langle 1|2\rangle\langle 2|1\rangle} \\ &+\; \frac{\mu(1-\frac{\mu_2}{\bar\mu_1})|1\rangle\langle 1|2\rangle\langle 2| + \mu(1-\frac{\mu_1}{\bar\mu_2})|2\rangle\langle 2|1\rangle\langle 1|}{\langle 1|1\rangle\langle 2|2\rangle - \mu\langle 1|2\rangle\langle 2|1\rangle}\end{aligned} \tag{6.27}$$

with

$$\mu \;=\; \tfrac{(\mu_1-\bar\mu_1)(\mu_2-\bar\mu_2)}{(\mu_1-\bar\mu_2)(\mu_2-\bar\mu_1)} \quad \text{and} \quad |\,k\,\rangle \;=\; \mathcal{S}(\mu_k)\,\mathcal{D}(\mu_k,t)\,|\alpha_k\rangle \quad \text{for } k=1,2\,. \tag{6.28}$$

Here, $\mu_k$ parametrize the distinct constant velocities of the two solitons like in (6.12), and $\alpha_1 \neq \alpha_2$ are their positions at $t=0$. The unitary transformations $\mathcal{S}\,\mathcal{D}$ boost the vacuum state, and so the time-dependent states $|\,k\,\rangle$ are just co-moving frame vacua for the two lumps.

In the static limit $\mu_i \to -\mathrm{i}$ the configuration (6.27) tends to the static solution (4.45) as long as $\alpha_1 \neq \alpha_2$. Furthermore, for large times, the overlap $\langle 1|2\rangle$ dies away and we are left with widely seperated solitons moving independently of one another in the noncommutative plane. The constancy of the velocities implies that the two solitons do not scatter off one another since scattering would lead to different velocities in the limits $t \to \pm\infty$.[5] This is also evinced by a no-force property of Ward solitons, borne out by their energy additivity:

$$E[\Phi_2] \;=\; E([\Phi_1(\vec{v}_1)] \;+\; E[\Phi_1(\vec{v}_2)]\,. \tag{6.29}$$

In fact, (6.27) was constructed by dressing (6.24) with a copy.

It is possible to extend these considerations to solutions exhibiting scattering by employing an ansatz (6.7) with higher order poles. This can be achieved by considering the limit of coincident velocities in the above setting. A thorough analysis of this idea [74, 127] indeed yields scattering solutions in the case of $U(n)$ with $n>1$ where the lumps are found to scatter at rational angles $\frac{\pi}{q}$ for $q \in \mathbb{N}$. Scattering solutions are of particular interest to us since the adiabatic approximation considered in the next chapter yields scattering of solitons in the $U(1)$ case.

Let us therefore exemplify this coincident velocity limit, or fusion, by considering the fusion of the abelian two-soliton (6.27). This is achieved by putting $\alpha_1 = \alpha_2 = \alpha$ and

[5] The same argument applies to the nonabelian case as well as long as the lump velocities are distinct.

sending both velocities to zero.[6] In this limit a new type of time dependence emerges. Putting in (6.27)

$$\mu_1 \;=\; -\mathrm{i}+\epsilon\,, \quad \text{as well as} \quad \mu_2 \;=\; -\mathrm{i}-\epsilon \quad \text{with } \epsilon\in\mathbb{C}\,,\ \text{infinitesimal} \tag{6.30}$$

and observing that [73, 74]

$$\mathcal{S}(-\mathrm{i}\pm\epsilon)\,\mathcal{D}(-\mathrm{i}\pm\epsilon,t)\,|\alpha\rangle \;=\; \mathrm{e}^{-|\epsilon|^2 t^2/4\theta}\,\mathrm{e}^{\pm\alpha\,\epsilon\,t/\sqrt{2\theta}}\left(1 \;\mp\; \tfrac{\bar{\epsilon}t}{\sqrt{2\theta}}\,a^\dagger \;+\; \mathcal{O}(\epsilon^2)\right)|\alpha\rangle\,, \tag{6.31}$$

we learn that any time dependence comes in the combination of $\epsilon\,t/\sqrt{2\theta}$. It is crucial to observe that the limits $\epsilon\to 0$ and $|t|\to\infty$ do not commute, and so the asymptotic behavior of (6.27) is modified under fusing. The result is

$$\widetilde{\Phi}_2(t) \;:=\; \lim_{\epsilon\to 0}\Phi_2(t) \;=\; \left(\mathbb{1}-2\tfrac{|\alpha\rangle\langle\alpha|}{\langle\alpha|\alpha\rangle}\right)\left(\mathbb{1}-2\tfrac{|\widetilde{\alpha}\rangle\langle\widetilde{\alpha}|}{\langle\widetilde{\alpha}|\widetilde{\alpha}\rangle}\right), \tag{6.32}$$

where the time dependence hides in

$$|\widetilde{\alpha}\rangle \;=\; |\alpha\rangle \;-\; \mathrm{i}\,t\sqrt{\tfrac{2}{\theta}}\,|\alpha^\perp\rangle \quad \text{with} \quad |\alpha^\perp\rangle \;=\; (a^\dagger-\bar{\alpha})\,|\alpha\rangle \tag{6.33}$$

being orthogonal to $|\alpha\rangle$. More explicitly,

$$\widetilde{\Phi}_2(t) \;=\; \mathbb{1} \;-\; \frac{2}{\theta+2t^2}\Big\{2\,t^2\left(|\alpha\rangle\langle\alpha|+|\alpha^\perp\rangle\langle\alpha^\perp|\right) \;-\; \mathrm{i}\,t\sqrt{2\theta}\left(|\alpha\rangle\langle\alpha^\perp|+|\alpha^\perp\rangle\langle\alpha|\right)\Big\}, \tag{6.34}$$

which at $t=0$ momentarily degenerates to $\widetilde{\Phi}_2=\mathbb{1}$.

Putting $\alpha=0$ for simplicity, the energy density $\mathcal{E}$ of (6.34) is readily computed to be [74]

$$\begin{aligned}\mathcal{E}[\widetilde{\Phi}_2] \;=\; \frac{4\theta}{(\theta+2t^2)^2}\Big\{&\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right) \;+\; \frac{2t^2}{\theta}\left(2\,|0\rangle\langle 0|+|1\rangle\langle 1|+|2\rangle\langle 2|\right)\\ &+\;\frac{4t^4}{\theta^2}\left(|1\rangle\langle 1|+|2\rangle\langle 2|\right)\frac{2t^2}{\theta}\left(\tfrac{1}{\sqrt{2}}|2\rangle\langle 0|+\tfrac{1}{\sqrt{2}}|0\rangle\langle 2|\right)\\ &-\;\mathrm{i}\,\frac{2^{3/2}t^3}{\theta^{3/2}}\left(|1\rangle\langle 0|-|0\rangle\langle 1|+\tfrac{1}{\sqrt{2}}|2\rangle\langle 1|-\tfrac{1}{\sqrt{2}}|1\rangle\langle 2|\right)\Big\}\end{aligned} \tag{6.35}$$

with $E[\widetilde{\Phi}_2]=2\pi\,\theta\,\mathrm{Tr}_{\mathcal{H}}\,\mathcal{E}=16\pi$, as should be since the energy is additive under dressing and unchanged under fusing. Employing the Moyal-Weyl correspondence, the energy density in the Moyal plane takes the form [74]

$$\begin{aligned}\mathcal{E}_\star[\widetilde{\Phi}_2] \;=\; \frac{16\,\mathrm{e}^{-\rho^2/\theta}}{\theta\,(1+2t^2/\theta)^2}\Big\{&\frac{\rho^2}{\theta} \;+\; \left(1-\frac{\rho^2}{\theta}+\frac{\rho^4}{\theta^2}\right)\frac{2t^2}{\theta} \;+\; \left(-\frac{\rho^2}{\theta}+\frac{\rho^4}{\theta^2}\right)\frac{4t^4}{\theta^2}\\ &-\;\left(\frac{x^2}{\theta}-\frac{y^2}{\theta}\right)\frac{2t^2}{\theta} \;-\; \frac{4y\rho^2t^3}{\theta^3}\Big\},\end{aligned} \tag{6.36}$$

[6] The more general situation of merely equal velocities is related by boosting the center of mass.

where $\rho := \sqrt{x^2 + y^2}$. Unfortunately, this energy distribution is invariant under space-time inversion and has a ring-like structure in the Moyal plane, localized at the origin like a bound state or breather solution as can be seen from the graphs below.

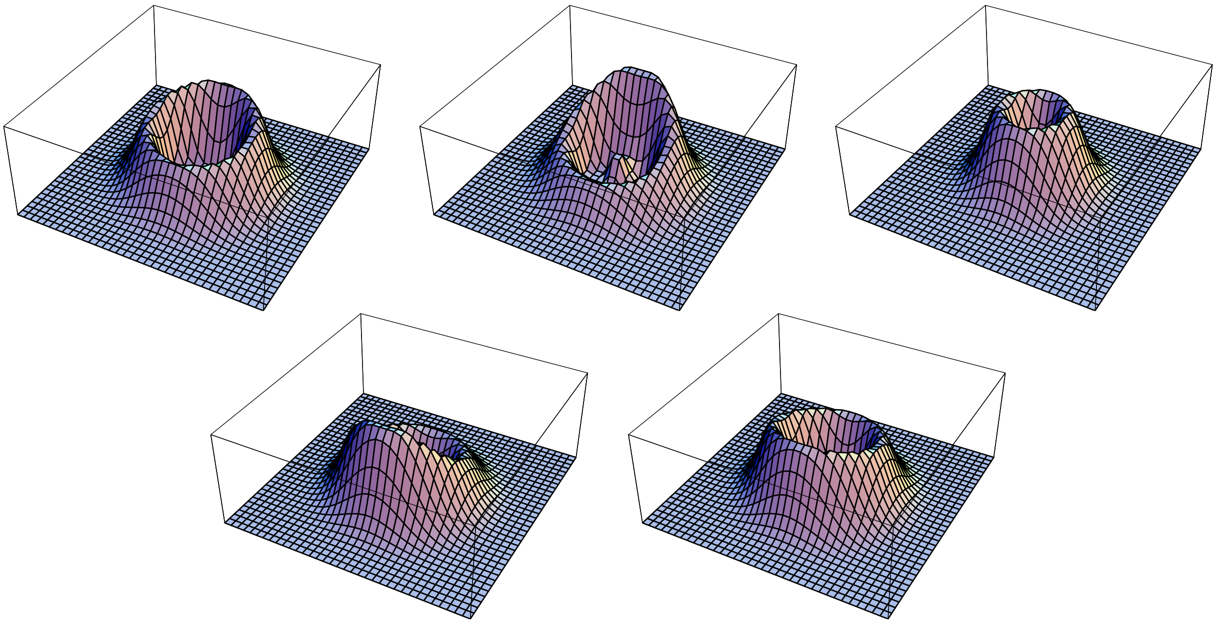


Figure 6.1: Energy density of breather solution for $\theta = 1$ at $t = -5$, $t = 1.5$, $t = 0$, $t = 1.5$ and $t = 5$ (from left to right).

Hence, we do not find any scattering solutions with $r = 2$ in the noncommutative $U(1)$ Ward model. This may, of course, be an artefact of our restricted ansatz. A more careful analysis by means of relaxing the first projector, allowing for higher-rank projectors and admitting time-dependent coefficients, however, also leads to no scattering solutions.

Recently it has been established [19, 20] for the commutative case that all Ward model multi-solitons are obtained from the one-soliton configurations (6.24) by dressing and fusing operations[7] and it therefore seems doubtful whether abelian scattering solutions exist at all.

Interestingly, the nonabelian Ward model is very different in this respect because of its larger moduli space. Fusing the $U(2)$ two-soliton solution also features ring-like configurations but also admits moduli choices which produce genuine $90°$ scattering, in the commutative [121, 58, 3, 59] as well as in the noncommutative [74] case.

[7] This result presumably extends to the noncommutative case.

# Chapter 7

# Adiabatic Approximation and Soliton Dynamics

Field theories with a nontrivial vacuum structure often feature static localized finite-energy solutions as was exemplified for the Ward model in chapter 4. These lumps can be boosted to moving solitons with constant velocity. An interesting aspect in this analysis of time-dependent solutions is the question of soliton scattering, which in most theories, however, is accessible only numerically.[1] Alternatively, a qualitative understanding of soliton scattering can be achieved for small relative velocity via the adiabatic or moduli-space dynamics invented by Manton [85, 86] whose application to the noncommutative Ward model will be discussed in this chapter.

In order to test the validity of this adiabatic method, one would need to apply it to an integrable model, where exact multi-soliton solutions are available for comparison. As we have seen before, such theories are rare in two or more spatial dimensions, which are required for interesting trajectories and our prime example of the modified nonlinear sigma model seems ideally suited for these purposes due to its integrability and the existence of genuine time-dependent multi-soliton solutions. Recently, the adiabatic approach was tested in this model for the commutative setting and the case of $SU(2)$ [60, 27, 61]. It appears rewarding to extend this analysis to field theories on noncommutative spaces investigating the new features appearing there and some effort has already been put into this endeavor for the case of the $\mathbb{C}P^1$ model [37].

As we have seen in section 3.3, the Ward model is closely related to noncommutative scalar field theories upon implementing the sigma-model constraint in unconstrained (multi-component) scalar field models by choosing an appropriate potential and performing an infinite-stiffness limit. Therefore, the soliton analysis of generic noncommutative field theories in 1+2 dimensions [53, 42, 2, 81, 43, 48] applies to noncommutative sigma models without WZW-like term, as well. Furthermore, it will turn out that the adiabatic approach does not see the WZW-term. By considering the family of extended $U(1)$ sigma models introduced in (3.60) the results presented here thus allow us to encompass both the noncommutative sigma models without WZW-like term and the Ward model with our analysis of the moduli-space dynamics.

In the next section we review the basic ideas of Manton's adiabatic approach and afterwards comment on general features as applied to the model at hand. This exposition is then followed by an analysis of the moduli-space scattering trajectories for two Ward

[1] Exceptions are integrable theories, which allow for analytic multi-soliton configurations and an exact S-matrix.

solitons which in this approach exhibit scattering angles between 0 and $\frac{\pi}{2}$. In the final section of this chapter we remark on how this compares to the known exact results which do not show this scattering behavior. It thus appears that the adiabatic approximation fails in this integrable case and we comment on possible reasons for this mismatch.

## 7.1 Manton's adiabatic approach

The general idea of the adiabatic approach [85, 86] is to approximate the exact $r$-soliton scattering configuration by a time sequence of static $r$-lump solutions. Thereby one introduces a time dependence for the latter's moduli $\alpha_i$, which is determined by extremizing the action on the moduli space $\mathcal{M}_r$. Being a functional of finitely many moduli $\alpha_i(t)$, this action describes the motion of a point particle in $\mathcal{M}_r$, equipped with a metric $g_{ij}(\alpha)$ and a magnetic field $A_i(\alpha)$. The geodesic motion in this geometric setup is believed to approximate exact time-dependent solutions in the limit of small velocities.

The program to carry out the adiabatic approach thus looks as follows [27]:

i. Construct finite-dimensional families of solutions to the static equations of motion (3.38).

ii. Allow for time dependence of the moduli $\{\alpha_i\}$ and rewrite the Lagrangian in a canonical form, i.e. as kinetic term plus possibly a magnetic and a potential term yielding the metric $g$, the magnetic one-form $A$ and the potential $V$.

iii. Analyze the geodesic motion, possibly with magnetic forcing and a potential, to approximate the slow velocity limit of non-static solutions.

## 7.2 Soliton moduli space for abelian sigma model solitons

### 7.2.1 General procedure

Let us turn our attention again to the abelian version of our family of sigma model actions (3.60) or its operator picture analogue. For the adiabatic approximation, we need to find the static multi-lump solutions $\Phi$ with $\partial_t\Phi = 0$ to the corresponding equation of motion (4.2). Since static configurations do not contribute to $S_3$ the moduli space of static multi-lumps is the same for all $\gamma$. It can be more precisely defined by our knowledge of the $r$-soliton solutions (4.43) or (4.45) encountered in chapter 4. The $r$-lump moduli space $\mathcal{M}_r$ is parametrized by all collections of $r$ harmonic-oscillator coherent states and is therefore given by the $r$-fold symmetric product of the single-soliton moduli space $\mathbb{C}$, i.e.

$$\mathcal{M}_r \;=\; \mathbb{C}^r/S_r\,, \tag{7.1}$$

where the symmetrization is induced by the fact that a permutation of the soliton locations leaves the configuration unchanged. This moduli space is a smooth Kähler manifold despite the coordinate singularities at the coincidence loci [43], which can be shown by passing to a different set of coordinates,[2] and we will deal in more detail with its Kähler potential and metric below. In fact, $\mathcal{M}_r$ is a complex submanifold of the Grassmannian $\mathrm{Gr}(r,\mathcal{H})$ introduced in (3.2) and (4.6). Moreover, it is known that $\mathbb{C}^r/S_r \cong \mathbb{C}^r$ as a complex manifold.

Manton posits that the motion of $r$ slowly scattering solitons is well described by a geodesic trajectory in $\mathcal{M}_r$, possibly with magnetic forcing. Since among the moduli are the spatial locations of the individual quasi-static lumps, (a projection of) the geodesic in $\mathcal{M}_r$ may be viewed as trajectories of the various lumps in the common ambient space. Abbreviating the $r$ complex moduli by $\alpha$, we denote the static $r$-lump solution by $\Phi(\alpha)$. To extract the time dependence in the action, we delineate the latter in a simplified form as

$$S_\gamma[\Phi] \;=\; \pi\int \mathrm{d}t\, \mathrm{Tr}_{\mathcal{H}}\,\big[\theta\,\dot\Phi^2 \;+\; C(\Phi,\Phi')\,\dot\Phi \;-\; W(\Phi,\Phi')\big] \tag{7.2}$$

with $\Phi' \;:=\; \big(\big[a^\dagger,\Phi\big],\,\big[a,\Phi\big]\big)$. The adiabatic method implies the approximation

$$\Phi(t) \;\approx\; \Phi(\alpha(t))\,, \tag{7.3}$$

thus replacing dynamics for $\Phi(t)$ with dynamics for $\alpha(t)$. We are instructed to compute

$$\begin{aligned} S_{\mathrm{mod}}[\alpha] \;&:=\; S[\Phi\,(\alpha(t))] \\ &=\; \pi\int \mathrm{d}t\,\Big[\theta\,\Big\{\mathrm{Tr}_{\mathcal{H}}\,(\partial_\alpha\Phi(\alpha))^2\Big\}\,\dot\alpha^2 \;+\; \Big\{\mathrm{Tr}_{\mathcal{H}}\,C(\Phi(\alpha),\Phi'(\alpha))\;\partial_\alpha\Phi(\alpha)\Big\}\,\dot\alpha \\ &\qquad\qquad -\;\mathrm{Tr}_{\mathcal{H}}\,W(\Phi(\alpha),\Phi'(\alpha))\;\Big] \\ &=:\; \int \mathrm{d}t\,\big[\tfrac12\,g_{\alpha\alpha}(\alpha)\,\dot\alpha^2 \;+\; A_\alpha(\alpha)\,\dot\alpha \;-\; V(\alpha)\big] \end{aligned} \tag{7.4}$$

to read off the metric $g$, magnetic field $F=\mathrm{d}A$ and potential $V$ on the moduli space. As mentioned above this moduli space has a natural Kähler structure, whose Kähler potential will turn out to be given by the determinant of the matrix of coherent-state overlaps [43]. Furthermore, we will see that there is no magnetic background field and no potential in the case considered.

### 7.2.2 Application to the noncommutative abelian Ward model

Let us implement this program for the extended deformed abelian sigma model and consider the solution (4.44). Putting $\alpha_i \to \alpha_i(t)$ introduces $t$-dependence into

$$|\alpha\rangle \;=\; \big(|\alpha_1\rangle,\ldots,|\alpha_r\rangle\big) \quad\longrightarrow\quad P_\alpha \;=\; |\alpha\rangle\tfrac{1}{\langle\alpha|\alpha\rangle}\langle\alpha| \quad\longrightarrow\quad \Phi(\alpha) \;=\; \mathbb{1}{-}2\,P_\alpha\,. \tag{7.5}$$

Inserting this into the action (3.36) we make two important observations. Firstly,

$$S_3[\Phi(\alpha)] \;\sim\; \int \mathrm{d}t\,\partial_t\,\mathrm{Tr}_{\mathcal{H}}\big[\,(a{+}a^\dagger)\,P_\alpha\,\big] \;=\; \int \mathrm{d}t\,\Big\{\partial_{\alpha_i}\,\mathrm{Tr}_{\mathcal{H}}\big[\,(a{+}a^\dagger)\,P_\alpha\,\big]\Big\}\,\dot\alpha_i \;+\; c.c.\,, \tag{7.6}$$

[2]The new coordinates are precisely given by the states in (4.43)

which reveals the magnetic potential to be exact, $A_i = \partial_{\alpha_i}\Omega$. Thus, magnetic forcing is absent,[3] and $S_{\rm mod}$ is independent of $\gamma$. Secondly, we get

$$\begin{aligned} S_2[\Phi(\alpha)] &= \int \mathrm{d}t\ \mathrm{Tr}_{\mathcal{H}}\Big[\,\pi\theta\,\big|\dot\Phi(\alpha)\big|^2 \;-\; 2\pi\,\big|[a,\Phi(\alpha)]\big|^2\Big] \\ &= \int \mathrm{d}t\ \Big[\,4\pi\theta\,\mathrm{Tr}_{\mathcal{H}}\,\dot P_\alpha^2 \;-\; E[\Phi(\alpha)]\Big]\,. \end{aligned} \tag{7.7}$$

Because $E[\Phi(\alpha)] = 8\pi\, r$ is constant inside $\mathcal{M}_r$, the second term yields an irrelevant constant potential $V$ and can be dropped.

As a result, $S_{\rm mod}$ reduces to the kinetic part of $S_2[\Phi(\alpha)]$, which simplifies to

$$\begin{aligned} S_{\rm mod}[\Phi(\alpha)] \;=\; 4\pi\theta\int \mathrm{d}t\ \mathrm{Tr}_{\mathcal{H}}\,\dot P_\alpha^2 &= 8\pi\theta\int \mathrm{d}t\ \mathrm{Tr}_{\mathcal{H}}\,(\mathbb{1}-P_\alpha)\,|\dot\alpha\rangle\,\langle\alpha|\alpha\rangle^{-1}\langle\dot\alpha| \\ &= 8\pi\theta\int \mathrm{d}t\ \mathrm{tr}_r\ \langle\alpha|\alpha\rangle^{-1}\langle\dot\alpha|\mathbb{1}-P_\alpha|\dot\alpha\rangle \\ &=: \int \mathrm{d}t\ \sum_{i,j=1}^{r} g_{\bar\imath j}\ \dot{\bar\alpha}_i\,\dot\alpha_j\,, \end{aligned} \tag{7.8}$$

where

$$|\dot\alpha\rangle \;:=\; \partial_t|\alpha\rangle \;=\; a^\dagger|\alpha\rangle\,\dot\Gamma \quad \text{with} \quad \Gamma := \mathrm{diag}(\{\alpha_i\})\,. \tag{7.9}$$

Hence, abbreviating $\partial_{\alpha_j} := \partial_j$ and $\partial_{\bar\imath} := \partial_{\bar\alpha_i}$, the metric on $\mathcal{M}_r$ is given by

$$g_{\bar\imath j} \;=\; 8\pi\theta\ \mathrm{Tr}_{\mathcal{H}}\ \partial_{\bar\imath}P_\alpha\,\partial_j P_\alpha \;=\; 8\pi\theta\ \mathrm{tr}_r\ \langle\alpha|\alpha\rangle^{-1}\,\partial_{\bar\imath}\Gamma^\dagger\,\langle\alpha|a\,(\mathbb{1}-P_\alpha)\,a^\dagger|\alpha\rangle\,\partial_j\Gamma\,. \tag{7.10}$$

Using the shorthand notation

$$M \;=\; (M_{ij}) \;:=\; \big(\langle\alpha|\alpha\rangle_{ij}\big) \;=\; \big(\langle\alpha_i|\alpha_j\rangle\big) \;=\; \big(\mathrm{e}^{\bar\alpha_i\alpha_j}\big)\,, \tag{7.11}$$

one computes

$$\begin{aligned} S_{\rm mod} &= 8\pi\theta\int \mathrm{d}t\ \mathrm{tr}_r\ \langle\alpha|\alpha\rangle^{-1}\,\dot\Gamma^\dagger\,\langle\alpha|a\,(\mathbb{1}-P_\alpha)\,a^\dagger|\alpha\rangle\,\dot\Gamma \\ &= 8\pi\theta\int \mathrm{d}t\sum_{i,j=1}^{r} M^{-1}_{ji}\,\dot{\bar\alpha}_i\,\langle\alpha_i|a\,(\mathbb{1}-P_\alpha)\,a^\dagger|\alpha_j\rangle\,\dot\alpha_j \\ &= 8\pi\theta\int \mathrm{d}t\sum_{i,j=1}^{r} M^{-1}_{ji}\,\Big\{M_{ij}(1+\bar\alpha_i\alpha_j) - \sum_{m,n=1}^{r} M_{im}\,\alpha_m\,M^{-1}_{mn}\,\bar\alpha_n\,M_{nj}\Big\}\ \dot{\bar\alpha}_i\,\dot\alpha_j \\ &= 8\pi\theta\int \mathrm{d}t\sum_{i,j=1}^{r} M^{-1}_{ji}\,\Big\{M \;+\; \Gamma^\dagger\,M\,\Gamma - M\,\Gamma\,M^{-1}\Gamma^\dagger\,M\Big\}_{ij}\ \dot{\bar\alpha}_i\,\dot\alpha_j\,, \end{aligned} \tag{7.12}$$

---

[3] The holonomy of $A = A_\alpha \mathrm{d}\alpha$ may yet be nontrivial.

which reveals the hermitian metric $(g_{\bar{\imath}j})$ on $\mathcal{M}_r$. With the help of the identities

$$\partial_j M \;=\; \Gamma^\dagger\, M\, \partial_j \Gamma \quad \text{and} \quad \partial_{\bar{\imath}} M \;=\; \partial_{\bar{\imath}} \Gamma^\dagger\, M\, \Gamma\,, \tag{7.13}$$

it is straightforward to check that this metric is indeed Kähler and derives from the Kähler potential

$$K \;=\; 8\pi\theta \,\ln\det M \;=\; 8\pi\theta \,\ln\det \big(\langle\alpha_i|\alpha_j\rangle\big) \;=\; 8\pi\theta\,\ln\det\big(\mathrm{e}^{\bar{\alpha}_i\alpha_j}\big)\,. \tag{7.14}$$

This result agrees with the geometric intuition: up to the prefactor of $8\pi\theta$, the metric $g_{\bar{\imath}j} = \partial_{\bar{\imath}}\partial_j K$ is the natural one on the Grassmannian $\mathrm{Gr}(r,\mathcal{H})$, which also has an interesting interpretation in terms of a system of classical identical particles [50, 51].

Global rotations (2.25) and translations (2.23) shift the Kähler potential by a gauge transformation,

$$K \quad\longrightarrow\quad K \;+\; 8\pi\theta \sum_{i=1}^{r} \big(\bar{\beta}\alpha_i + \beta\bar{\alpha}_i + \beta\bar{\beta}\big)\,, \quad \text{for } \beta\in\mathbb{C}\,, \tag{7.15}$$

leaving the metric unchanged. Furthermore, (7.14) is invariant under permutations of the $\alpha_i$. When passing to center-of-mass and barycentric coordinates

$$s \;=\; \tfrac{1}{r}\sum_{i=1}^{r}\alpha_i \quad\text{and}\quad w_i \;=\; \alpha_i - s\,, \quad \text{such that} \quad \sum_{i=1}^{r} w_i = 0\,. \tag{7.16}$$

we thus get the decomposition

$$K \;=\; 8\pi\theta\, r\, |s|^2 \;+\; 8\pi\theta\,\ln\det\big(\mathrm{e}^{\bar{w}_i w_j}\big)\,, \tag{7.17}$$

which shows that the metric depends only on difference coordinates $\alpha_i - \alpha_j$. One may also extract the diagonal (free) part via

$$K \;=\; 8\pi\theta \sum_{i=1}^{r} |\alpha_i|^2 \;+\; 8\pi\theta\,\ln\det\big(\mathrm{e}^{-\frac{1}{2}|\alpha_i-\alpha_j|^2 \;+\; \frac{1}{2}(\bar{\alpha}_i\alpha_j - \bar{\alpha}_j\alpha_i)}\big)\,. \tag{7.18}$$

From this expression it is easy to see a cluster decomposition property: Upon splitting the moduli into two groups, $\{\alpha_i\} = \{\alpha'_\ell, \alpha''_m\}$, and separating these to infinity,

$$\lim_{|\alpha'_\ell - \alpha''_m|\to\infty} K\big(\{\alpha_i\}\big) \;=\; K\big(\{\alpha'_\ell\}\big) \;+\; K\big(\{\alpha''_m\}\big)\,, \tag{7.19}$$

such that the potential decomposes into two disjoint parts. In particular, an isolated single lump at $\alpha_q$ asymptotically contributes with $|\alpha_q|^2$ to $K$. Therefore, the moduli-space metric becomes flat for large mutual separations, $|\alpha_i - \alpha_j| \to \infty$.

More interesting is the limit of coinciding lumps, say $\alpha_i \to \alpha$ for $i = q_1, \ldots, q_r$. Some lengthy algebra then shows that

$$K \quad\longrightarrow\quad 8\pi\theta \sum_{q_\ell > q_m} \ln|\alpha_{q_\ell} - \alpha_{q_m}|^2 \;+\; K'\,, \quad \text{where} \quad K' \;=\; \ln\det\big(\langle T|T\rangle\big) \tag{7.20}$$

with the following replacement inside $|T\rangle = \big(|\alpha_1\rangle, \ldots, |\alpha_r\rangle\big)$:

$$\big\{ |\alpha_{q_1}\rangle, |\alpha_{q_2}\rangle, \ldots, |\alpha_{q_r}\rangle \big\} \quad \longrightarrow \quad \big\{ |\alpha\rangle, a^\dagger|\alpha\rangle, \ldots, \tfrac{1}{(r-1)!}(a^\dagger)^{r-1}|\alpha\rangle \big\}\,. \tag{7.21}$$

The coordinate singularity in (7.20) can be removed by passing to new coordinates, namely elementary symmetric polynomials in $\alpha_{q_\ell}-\alpha$, which correspond precisely to the new states in (7.21) (see also the discussion at the beginning of this section). The expression $K'$ produces the same metric as $K$ but is smooth at the coincidence locus. In the most general situation, $|T\rangle$ is composed of various blocks like in (7.21), of different sizes $r$ (see also the general solution (4.43)), but the formula for the smooth Kähler potential $K'$ in (7.20) remains correct.

## 7.3 Moduli-space trajectories for two-soliton scattering

For concreteness, let us display the simplest nontrivial case, i.e. $r$=2. We are thus dealing with a rank two soliton. The moduli space $\mathcal{M}_2$ of rank-two BPS projectors is parametrized by $\{\alpha,\beta\} \sim \{\beta,\alpha\} \in \mathbb{C}^2/S_2$. Since the details have been given in [81], we can be short here.

The static two-lump configuration is derived from (4.45) as

$$\begin{aligned} \Phi_{\alpha\beta} \;&=\; \mathbb{1} - 2\,P_{\alpha\beta} \\ &=\; \mathbb{1} \;-\; 2\,\frac{|\alpha\rangle\langle\beta|\beta\rangle\langle\alpha| + |\beta\rangle\langle\alpha|\alpha\rangle\langle\beta| - |\alpha\rangle\langle\alpha|\beta\rangle\langle\beta| - |\beta\rangle\langle\beta|\alpha\rangle\langle\alpha|}{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle - \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}\,. \end{aligned} \tag{7.22}$$

Writing $\alpha = s + w$ and $\beta = s - w$ as well as $2\,w =: \rho\,\mathrm{e}^{\mathrm{i}\varphi}$, the corresponding Kähler potential reads

$$\begin{aligned} K \;&=\; 8\pi\theta\,\ln\big(\mathrm{e}^{\alpha\bar\alpha+\beta\bar\beta} - \mathrm{e}^{\alpha\bar\beta+\beta\bar\alpha}\big) \;=\; 8\pi\theta\big[2\,s\bar s \;+\; 2\,w\bar w \;+\; \ln\big(1-\mathrm{e}^{-4w\bar w}\big)\big] \\ &=\; 8\pi\theta\,\ln\big(2\mathrm{e}^{2s\bar s}\sinh\tfrac{\rho^2}{2}\big) \;=\; 8\pi\theta\big[2\,s\bar s \;+\; \tfrac12\rho^2 \;+\; \ln\big(1-\mathrm{e}^{-\rho^2}\big)\big]\,, \end{aligned} \tag{7.23}$$

with the limits

$$\begin{aligned} K \;&=\; 8\pi\theta\big[2s\bar s + \tfrac12\rho^2 - \mathrm{e}^{-\rho^2} + \mathcal{O}(\mathrm{e}^{-2\rho^2})\big] \\ \text{and}\quad K \;&=\; 8\pi\theta\big[2s\bar s + \ln\rho^2 + \tfrac{1}{24}\rho^4 + \mathcal{O}(\rho^8)\big]\,. \end{aligned} \tag{7.24}$$

It yields the metric

$$\mathrm{d}\ell^2 \;=\; 16\pi\theta\big[\mathrm{d}s\mathrm{d}\bar s \;+\; \Omega\,\mathrm{d}w\mathrm{d}\bar w\big] \;=\; 4\pi\theta\big[4\,\mathrm{d}s\mathrm{d}\bar s \;+\; \Omega(\rho^2)\,(\mathrm{d}\rho^2+\rho^2\mathrm{d}\varphi^2)\big] \tag{7.25}$$

with the conformal factor

$$\Omega(\rho^2) \;=\; \tfrac{1}{4\pi\theta}\,\partial_{\rho^2}\big(\rho^2\,\partial_{\rho^2}K\big) \;=\; \frac{1-2\rho^2\mathrm{e}^{-\rho^2}-\mathrm{e}^{-2\rho^2}}{(1-\mathrm{e}^{-\rho^2})^2} \;=\; \frac{\sinh\rho^2-\rho^2}{\cosh\rho^2-1}\,, \tag{7.26}$$

possessing the limits

$$\begin{aligned} \Omega(\rho^2) \;&=\; 1 + (1{-}2\rho^2)\mathrm{e}^{-\rho^2} + \mathcal{O}(\mathrm{e}^{-2\rho^2})\,, \\ \text{as well as}\quad \Omega(\rho^2) \;&=\; \tfrac13\rho^2 - \tfrac{1}{90}\rho^6 + \mathcal{O}(\rho^{10})\,, \end{aligned} \tag{7.27}$$

which can also be seen from the following graph.

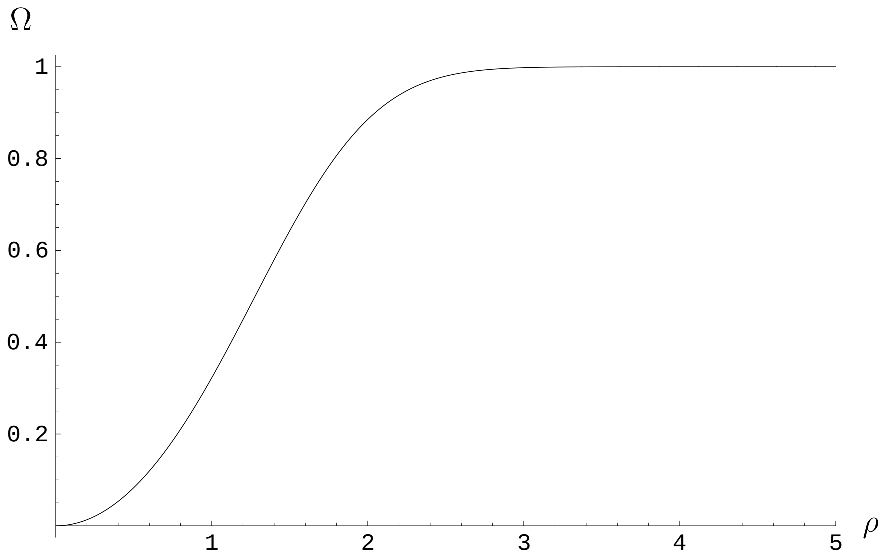



Figure 7.1: Conformal factor for moduli-space metric of rank 2 soliton.

Clearly, the metric becomes flat for $\rho \to \infty$ but develops a conical singularity with an angle of $4\pi$ at $\rho = 0$. The latter is removed by passing to the symmetric coordinate $\sigma = w^2$, in terms of which one finds

$$\begin{aligned} \mathrm{d}\ell^2 &= 16\pi\theta\Big[\mathrm{d}s\mathrm{d}\bar{s} \;+\; \tfrac{\Omega(\rho^2\to 4\sqrt{\sigma\bar{\sigma}})}{4\sqrt{\sigma\bar{\sigma}}}\,\mathrm{d}\sigma\mathrm{d}\bar{\sigma}\Big] \\ &= 16\pi\theta\Big[\mathrm{d}s\mathrm{d}\bar{s} \;+\; \big(\tfrac{1}{3} - \tfrac{8}{45}\sigma\bar{\sigma} + \mathcal{O}((\sigma\bar{\sigma})^2)\big)\,\mathrm{d}\sigma\mathrm{d}\bar{\sigma}\Big]\,. \end{aligned} \tag{7.28}$$

Due to the decoupling of the trivial center-of-mass dynamics, $\mathcal{M}_2 = \mathbb{C} \times \mathcal{M}_{\text{rel}}$, with $\mathcal{M}_{\text{rel}} \simeq \mathbb{C}$ rotationally symmetric, asymptotically flat, and of positive curvature, which may be computed from the metric to $R = \frac{1}{4\pi\theta}[\frac{5}{4} - \frac{6}{175}\rho^4 + \mathcal{O}(\rho^8)]$. Head-on scattering of two lumps corresponds to a single radial trajectory in $\mathcal{M}_{\text{rel}}$, which in the smooth coordinate $\sigma$ must pass straight through the origin. In the 'doubled coordinate' $w = \sqrt{\sigma}$, we then see two straight trajectories with 90° scattering off the singularity in the Moyal plane.

This picture persists for the scattering of two composite lumps, i.e. lumps obtained by fusing

$$\begin{aligned} \alpha_i &\longrightarrow \alpha \quad \text{for } i = 1, \ldots, r_1 \\ \text{and} \quad \alpha_{r_1+j} &\longrightarrow \beta \quad \text{for } j = 1, \ldots, r_2\,. \end{aligned} \tag{7.29}$$

The decoupling of the center-of-mass coordinate, now for $r = r_1 + r_2$, is achieved by writing

$$\alpha \;=\; s + r_2\, w \quad \text{and} \quad \beta \;=\; s - r_1\, w\,, \quad \text{such that} \quad \alpha - \beta \;=\; (r_1 + r_2)\, w \;=:\; \rho\, \mathrm{e}^{\mathrm{i}\varphi}\,, \tag{7.30}$$

and one obtains

$$\begin{aligned} K &= 8\pi\theta\big[(r_1 + r_2)s\bar{s} \;+\; \tfrac{r_1 r_2}{r_1 + r_2}\,\rho^2 \;+\; \ln\big(1 - \mathcal{P}\mathrm{e}^{-\rho^2} + \mathcal{O}(\mathrm{e}^{-2\rho^2})\big)\big] \\ &\stackrel{\rho\to 0}{\longrightarrow} 8\pi\theta\big[(r_1 + r_2)s\bar{s} \;+\; c_0 \;+\; c_1 \ln\rho^2 \;+\; c_2\,\rho^4 \;+\; \mathcal{O}(\rho^6)\big]\,, \end{aligned} \tag{7.31}$$

where $\mathcal{P}$ is a polynomial in $\rho^2$ and $c_0$, $c_1$ and $c_2$ are constants. As in (7.24), the absence of the $\rho^2$ term (and $c_2 \neq 0$) leads to a conformal factor $\Omega \sim \rho^2$ for $\rho \to 0$ and the same conical singularity for any value of $r_1$ or $r_2$. Its remedy by employing the coordinate $\sigma = w^2$ demonstrates that the 90° scattering angle is universal for head-on motion. Only for more special situations with simultaneous head-on collision of $r$ (>2) solitons one will get $\frac{\pi}{r}$ scattering.

Let us return to the simple case of $r_1 = r_2 = 1$ and drop the center-of mass coordinate. The motion in $\mathcal{M}_{\rm rel}$ is geodesic with conformal factor $\Omega(\rho^2)$ given in (7.26). It conserves angular momentum and energy,

$$l \;=\; \Omega\,\rho^2\,\dot\varphi \;=\; v_\infty\,b \quad\text{and}\quad e \;=\; \tfrac12\,\Omega\,\dot\rho^2 + \frac{l^2}{2\Omega\rho^2} \;=\; \tfrac12\,v_\infty^2\,, \tag{7.32}$$

respectively, with the asymptotic speed $v_\infty$ and the impact parameter

$$b \;=\; l/\sqrt{2e} \;=\; \rho_{\rm min}\,\sqrt{\Omega(\rho_{\rm min}^2)}\,, \tag{7.33}$$

where $\rho_{\rm min}$ denotes the soliton distance at closest approach. Hence, the trajectory is given by

$$\frac{{\rm d}\rho}{{\rm d}\varphi} \;=\; \frac{\rho^2}{b}\,\sqrt{\Omega - b^2/\rho^2}\,, \tag{7.34}$$

and we obtain the scattering angle

$$\Theta(b) \;=\; \pi \;-\; 2\int_{\rho_{\rm min}}^{\infty} \frac{b\;{\rm d}\rho}{\rho^2\,\sqrt{\Omega - b^2/\rho^2}}\,, \tag{7.35}$$

which varies between 0 (for $b \to \infty$ and $\Omega \to 1$) and $\frac{\pi}{2}$ (for $b \to 0$ and $\Omega \to \frac13 r^2$). Therefore, if we fix $b$ and vary $v_\infty$, the trajectory is unchanged. The total energy of the $r = 2$ system is

$$E_{\rm mod} \;:=\; E[\Phi(\alpha)] \;=\; 16\pi + 8\pi\theta\,e \;=\; 16\pi + 4\pi\,\theta\,v_\infty^2\,. \tag{7.36}$$

Good agreement with the full field-theory dynamics is expected only for small values of $v_\infty$. Note however, that the trajectory $\rho(\varphi)$ does not depend on $v_\infty$.

For visualization the scattering angle $\Theta(b)$ is plotted below for the cases $r_1 = 1$ and $r_2 = 1$ (triangles), 2 (diamonds), 4 (boxes). The underlying Mathematica code is given in appendix A.2

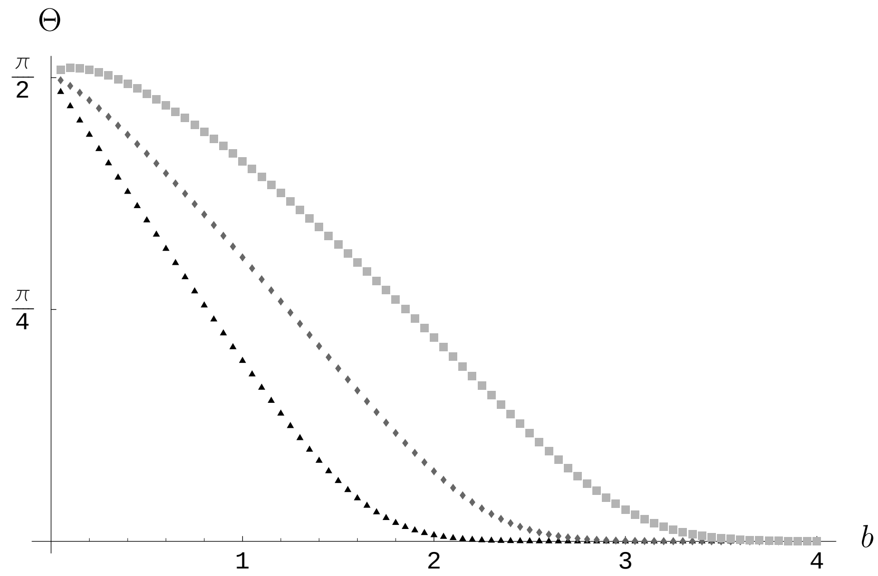



Figure 7.2: Scattering angle $\Theta$ vs. impact parameter $b$.

## 7.4 Comparison with exact solutions and conclusion

According to the general arguments about the adiabatic approximation, the moduli-space dynamics described in the previous section should apply to the whole family $S_\gamma$ of actions in (3.36). To test the quality of the approach, one would like to compare the moduli-space scattering trajectories with the time evolution of the energy-density maxima of the corresponding classical field configurations. Since widely separated lumps roam essentially independently of each other, we already know the large-time asymptotics: a (multiplicative) superposition of several one-soliton configurations of the form $\Phi(\alpha) = \mathbb{1}-2P_\alpha$, after applying individual translations and boosts. To simplify the discussion, let us consider just two lumps of rank one each, i.e. combine two copies of $\Phi_0 = \mathbb{1}-2|0\rangle\langle 0|$ and comment on their scattering behavior. For large times we must have

$$\begin{aligned}\Phi(t \to \pm\infty) \;&\simeq\; \big(\mathbb{1} - (1-\mathrm{e}^{\mathrm{i}\delta_1})\,\mathcal{U}_1\,|0\rangle\langle 0|\,\mathcal{U}_1^\dagger\big)\,\big(\mathbb{1} - (1-\mathrm{e}^{\mathrm{i}\delta_2})\,\mathcal{U}_2\,|0\rangle\langle 0|\,\mathcal{U}_2^\dagger\big) \\ &\simeq\; \mathbb{1} \;-\; (1-\mathrm{e}^{\mathrm{i}\delta_1})\,\mathcal{U}_1\,|0\rangle\langle 0|\,\mathcal{U}_1^\dagger \;-\; (1-\mathrm{e}^{\mathrm{i}\delta_2})\,\mathcal{U}_2\,|0\rangle\langle 0|\,\mathcal{U}_2^\dagger\,,\end{aligned} \tag{7.37}$$

where

$$\mathcal{U}_i \;=\; \mathcal{U}(\vec{v}_i^{\,\pm}, \vec{\rho}_i^{\,\pm}, t) \quad \text{for } i = 1,2 \text{ and } t \to \pm\infty \tag{7.38}$$

are unitary transformations implementing translations by $\vec{\rho}_i^{\,\pm}$ and boosts with velocities $\vec{v}_i^{\,\pm}$ in the Moyal plane. Note that time-dependent solitons need no longer be grassmannian, and so we must allow in (7.37) for the slightly more general prefactors with velocity-dependent phases $\delta_i$ [73, 74]. If the scattering angle differs from $\pi$, i.e. if nontrivial scattering occurs, then the late velocities $\vec{v}_i^{\,+}$ must differ from the early ones $\vec{v}_i^{\,-}$.

Outside the value $\gamma$=1, the solitons are affected by each other's presence, and no integrability protects them from shrinking and decay. Yet, in cases where their lifetime is sufficiently long the configurations (7.37) can still be approached for not too large times, and scattering data are viable. In the absence of exact time-dependent solutions, however, numerical investigations are needed for confirmation. To the author's knowledge, computer analysis has been applied only in the commutative case ($\theta$=0) for $\gamma$=0 in the $O(3)$ sigma model, where it established the universality of 90° head-on two-soliton scattering [134]. Passing to the noncommutative realm, the equation of motion to solve in the operator formulation for the $U(1)$ sigma model, i.e. $\gamma$=0, can be deduced from (3.38) as

$$\begin{aligned} 0 &= \theta\,\partial_t(\Phi^\dagger\partial_t\Phi) \;+\; \big[a\,,\,\Phi^\dagger[a^\dagger,\,\Phi]\big] \;+\; \big[a^\dagger,\,\Phi^\dagger[a\,,\,\Phi]\big] \\ &= \theta\,\partial_t(\Phi^\dagger\partial_t\Phi) \;+\; \Phi^\dagger\,\Delta\Phi \;-\; \Delta\Phi^\dagger\,\Phi\,. \end{aligned} \tag{7.39}$$

With $\Phi \in U(\mathcal{H})$ viewed as an infinite-size matrix $(\Phi_{mn})$ in the Fock space basis (2.11), it reads

$$\begin{aligned} \theta\,\partial_t(\Phi^*_{nm}\partial_t\Phi_{n\ell}) \;=\; & (m{-}\ell)\,\Phi^*_{nm}\Phi_{n\ell} + \sqrt{(n{+}1)(\ell+1)}\,\Phi^*_{nm}\Phi_{n+1\,\ell+1} \\ & + \sqrt{n\,\ell}\,\Phi^*_{nm}\Phi_{n-1\,\ell-1} - \sqrt{nm}\,\Phi^*_{n-1\,m-1}\Phi_{n\ell} \\ & - \sqrt{(n{+}1)(m{+}1)}\,\Phi^*_{n+1\,m+1}\Phi_{n\ell}\,. \end{aligned} \tag{7.40}$$

It would be interesting to analyze this coupled initial-value problem numerically.

For the Ward model ($\gamma = 1$), the situation is entirely different since exact multi-soliton solutions are available as we have already seen in the previous chapter. Nevertheless, these do not feature any scattering in the abelian case. Let us make this more explicit for the two-soliton (6.34). This solution can also be constructed directly by the dressing method, starting from the ansatz (6.32) with an unknown state $|\widetilde{\alpha}\rangle$. In this way one arrives at the conditions [74]

$$\begin{aligned} a\,|\widetilde{\alpha}\rangle + \big[a,\tfrac{|\alpha\rangle\langle\alpha|}{\langle\alpha|\alpha\rangle}\big]\,|\widetilde{\alpha}\rangle \;&=\; |\widetilde{\alpha}\rangle\,Z_1 \\ \text{and}\quad \partial_t\,|\widetilde{\alpha}\rangle + \mathrm{i}\sqrt{\tfrac{2}{\theta}}\,\big[a^\dagger,\tfrac{|\alpha\rangle\langle\alpha|}{\langle\alpha|\alpha\rangle}\big]\,|\widetilde{\alpha}\rangle \;&=\; |\widetilde{\alpha}\rangle\,Z_2\,, \end{aligned} \tag{7.41}$$

where $Z_1$ and $Z_2$ are functions of $t$ to be determined. We read off that $Z_1 = \alpha$ and fix the (inessential) normalization such that $Z_2 = 0$. It is not hard then to recover (6.33) as the general solution indeed. To compare this solution with the adiabatic approach we need to set same initial conditions. Matching of the energy $E = 16\pi$ with $E_{\text{mod}}$ in (7.36) enforces $v_\infty = 0$ which, however, does not restrict $b$ in any way. Therefore even for small velocities, the energy density of the two-soliton solution (6.34) does not follow the moduli-space dynamics of the previous section, except, of course, at very large impact parameter where the scattering disappears. As mentioned, the search for exact abelian two-soliton solutions exhibiting scattering also turned out to be fruitless.

In summary, the moduli-space motion of the considered solution therefore does not approximate the extended abelian sigma-model soliton scattering in the Moyal plane equally well for all values of the family parameter $\gamma$. Numerical analysis is needed to make the case for $\gamma < 1$. For the integrable value $\gamma = 1$, i.e. the abelian Ward model, however, we are curiously lacking the field-theory dynamics which the moduli-space kinematics is supposed to mimic.[4] It is therefore conceivable that in this case, even for arbitrary small velocities, the soliton scattering takes place far away from their moduli space, if it occurs at all! Certainly, the known no-scattering multi-solitons are not seen in the moduli space, which challenges Manton's paradigm. In part responsible for this failure seems to be the absence of magnetic forcing in the moduli space in contrast to the crucial importance of the WZW-like action term for integrability. Another obstruction could be the proven instability of each static multi-soliton configuration in the unitary model (see section 5.1.4). Further analysis of this conundrum is definitely expedient and perhaps a numerical study can give further insight.

Finally we remark that an attempt to extend these considerations to the nonabelian case has, at least up to now, been to no avail due to the encountered technical difficulties. To make this more comprehensible consider the kinetic part of the action (7.8) given for the parametrization $P = |T\rangle\langle T|T\rangle^{-1}\langle T|$ by

$$S_{\text{kin}}[P] \;=\; 4\pi\theta \int \mathrm{d}t\; \mathrm{Tr}\,\dot{P}^2 \;=\; 8\pi\theta \int \mathrm{d}t\; \mathrm{Tr}\,(\mathbb{1}-P)\,|\dot{T}\rangle\,\langle T|T\rangle^{-1}\langle\dot{T}| \tag{7.42}$$

with $M = \langle T|T\rangle$ similar to (7.11). The main difference to the abelian case being that $M$ is now an infinite dimensional matrix. In order to carry through the adiabatic approximation in this setting we need to consider more general solutions than the simple ones given in (4.51) to incorporate all the moduli. The $U(2)$ rank 1 soliton, for example, is in analogy to the commutative case (compare the literature cited at the beginning of section 3.2.1) given by

$$|T\rangle \;=\; \begin{pmatrix} \lambda \\ a+\alpha \end{pmatrix} |\mathcal{H}\rangle \quad \text{with } \lambda,\, \alpha \in \mathbb{C}\,. \tag{7.43}$$

Upon endowing the moduli, of which $\alpha$ corresponds to the center of mass position, with a time dependence this leads to

$$\mathrm{Tr}\,\dot{P}^2 \;=\; \dot{\lambda}\dot{\bar{\lambda}} \sum_{k=0}^{\infty} \frac{k}{(\lambda\bar{\lambda}+k)^2} \;+\; \dot{\alpha}\dot{\bar{\alpha}} \sum_{k=0}^{\infty} \frac{\lambda\bar{\lambda}}{(\lambda\bar{\lambda}+k)^2}\,. \tag{7.44}$$

To get a finite contribution to the kinetic energy we have to demand $\dot{\lambda}{=}0$ and thus encounter a phenomenon known as frozen degrees of freedom [85, 27], where some of the moduli have to be kept at fixed value to get results. The remaining term then yields $\mathrm{Tr}\,\dot{P}^2 = \dot{\alpha}\dot{\bar{\alpha}}$, as expected for the single soliton. Turning now to rank 2, one has to start with

$$|T\rangle \;=\; \begin{pmatrix} \lambda \\ a^2+\epsilon a+\xi \end{pmatrix} |\mathcal{H}\rangle\,, \quad \text{for } \lambda,\, \epsilon,\, \xi \in \mathbb{C}\,. \tag{7.45}$$

We do not know how to invert the matrix $M$, which in this case turns out to be an infinite dimensional pentadiagonal matrix, i.e. with entries only on the diagonal and the first two upper and lower diagonals. It is interesting to compare this to the commutative case,

[4] This also applies to the no-scattering solutions of the nonabelian models.

where the same analysis has been done recently in [27] using the earlier work of [117]. The authors arrive at elliptic integrals in this context and it seems worth trying to map our problem back to the noncommutative plane via the Moyal-Weyl map to see whether similar results can be achieved.

# Appendix A

# Numerical Code

In this appendix we collect the Mathematica code used for the computation of the presented numerical results. To make the exposition more readable, we omit some technicalities and concentrate on the main ideas.

## A.1 Eigenvalues for the Hessian

Computing the eigenvalues of the Hessian as explained in section 5.2.3 is a rather straightforward task. For a rank $r$ diagonal $U(1)$ BPS-solution and a diagonal perturbation the main task is to input the matrix representation of the corresponding Hessian (5.65), which is done by

```
del[i_, j_] := KroneckerDelta[i, j];

matrix[maxel_] :=
  Table[(2j+1) del[i,j] -  j del[i+1,j] - (j+1) del[i,j+1]
        -2r (del[j,r-1] + del[j,r]) del[i,j],
        {i, 0, maxel}, {j, 0,maxel}];
```

Here, `maxel` denotes the value of the cut-off parameter. Mathematica then allows to compute the eigenvalues, sort them and plot their dependence on the cut-off value by

```
evlist[maxel_] := Sort[Eigenvalues[N[matrix[maxel]]]];

plotlist =  Flatten[Table[{l,Part[evlist[l],k]},
                          {l,1,maxel}, {k,1,l}],1];

ListPlot[plotlist];
```

The same code may be used to obtain corresponding results for off-diagonal fluctuations by changing the input matrix.

## A.2 Scattering angle for two-soliton scattering

The following code supplements the analysis of moduli space trajectories in section 7.3. It is used to compute the scattering angle as a function of the impact parameter for a single soliton scattering off an $r$-fold degenerate soliton.

As a first step, the overlap matrix $M$ (7.11) is computed for the $r$-soliton at position $\alpha$ (`alpha`) and the one-soliton at position $\beta$ (`beta`).

```
m[i_, j_, alpha_] :=
  Which[i<j, Conjugate[alpha]^(j-i)
          Sum[alpha Conjugate[alpha]^k / (k!(j-i+k)!(i - k)!),{k,0,i}]
        i>=j, alpha^(i-j)
          Sum[alpha Conjugate[alpha]^k / (k!(i-j+k)!(j - k)!),{k,0,j}];

sigma[alpha_, beta_] := Exp[Conjugate[alpha] beta]
                        - 1/2 (Abs[alpha]^2 +  Abs[beta]^2)];

Mel[r_, alpha_, beta_][i_, j_] :=
  Which[i<=r   && j<=r,   m[i-1,j-1, alpha],
        i<=r   && j==r+1, beta^(i-1) sigma[alpha,beta] / (i-1)!,
        i==r+1 && j<=r,   Conjugate[beta]^(j-1)
                          Conjugate[sigma[alpha, beta]] / (j-1)!,
        i==r+1 && j==r+1, 1];

M[r_, alpha_, beta_] := Array[Mel[r, alpha, beta], {r+1, r+1}];
```

This allows us to compute the Kähler potential $K$ and conformal factor $\Omega$ in relative coordinates[1]

$$\alpha = \frac{1}{r+1}\left((r+1)\,s + w\right) \quad \text{and} \quad \beta = \frac{1}{r+1}\left((r+1)\,s - r\,w\right)\,, \tag{A.1}$$

by

```
alpha[r_, s_, w_] := (1/(r+1)) ((r+1)s + w);

beta[r_, s_, w_]  := (1/(r+1)) ((r+1)s - r w);

K[r_, s_, w_] := r alpha[r, s, w] Conjugate[alpha[r, s, w]]
                 + beta[r, s, w] Conjugate[beta[r, s, w]]
                 + Log[Det[M[r, alpha[r, s, w], beta[r, s, w]]]];

Omega[w_] = 1/4 (Derivative[0, 2][K][r, 0, w]
                 + 1/w Derivative[0, 1][K][r, 0, w]);
```

Finally we can compute the scattering angle $\Theta$ as a function of the impact parameter $b$ using (7.35):

```
Theta[b_] := Pi - 2 NIntegrate[1 / (w((w\b)^2 Omega[w] - 1)^(1/2)),
               {w, w/.FindRoot[w^2 Omega[w] == b^2, {w,1}], Infinity}];
```

[1] The definition employed slightly differs from the one in (7.30) but the final result does not change.

# Bibliography


[1] M.J. Ablowitz and J. Villarroel, *On the scattering of multipole lumps for the Kadomtsev-Petviashvili equation,* in Scattering and Inverse Scattering in Pure and Applied Science, Eds. P. Sabatier and E.R. Pike, Academic Press (2001), 1792-1806.

[2] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, *Unstable solitons in noncommutative gauge theory,* JHEP **0104** (2001) 001 [hep-th/0009142].

[3] C. Anand, *Ward's solitons,* Geom. Topol. **1** (1997) 9.

[4] T. Araki and K. Ito, *Scattering of noncommutative (n,1) solitons,* Phys. Lett. B **516** (2001) 123 [hep-th/0105012].

[5] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp and J. Wess, *A gravity theory on noncommutative spaces,* Class. Quant. Grav. **22** (2005) 3511 [hep-th/0504183].

[6] P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, *Noncommutative geometry and gravity,* Class. Quant. Grav. **23** (2006) 1883 [hep-th/0510059].

[7] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization. 1. Deformations of symplectic structures,* Annals Phys. **111** (1978) 61.

[8] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization. 2. Physical applications,* Annals Phys. **111** (1978) 111.

[9] S. Bieling, *Interaction of noncommutative plane waves in 2+1 dimensions,* J. Phys. A **35** (2002) 6281 [hep-th/0203269].

[10] F.E. Burstall and J.H. Rawnsley, *Stability of classical solutions of two-dimensional grassmannian models,* Commun. Math. Phys. **110** (1987) 311.

[11] X. Calmet and A. Kobakhidze, *Noncommutative general relativity* Phys. Rev. D **72** (2005) 045010 [hep-th/0506157].

[12] A.H. Chamseddine, *Noncommutative gravity,* Annales Henri Poincare **4S2**, S881 (2003) [hep-th/0301112].

[13] A. Connes, *$C^\star$ Algebras and differential geometry,* Compt. Rend. Math. A **290** (1980) 599 [hep-th/0101093].

[14] A. Connes, *Noncommutative differential geometry,* Inst. Hautes Études Sci. Publ. Math. **62** (1985) 257.

[15] A. Connes and M.A. Rieffel, *Yang-Mills for noncommutative two-tori,* Contemp. Math. **62** (1987) 237.

[16] A. Connes, *The action functional in noncommutative geometry,* Commun. Math. Phys. **117** (1988) 673.

[17] A. Connes, "Noncommutative geometry," Academic Press (1994).

[18] A. Connes, M.R. Douglas and A.S. Schwarz, *Noncommutative geometry and matrix theory: Compactification on tori,* JHEP **9802** (1998) 003 [hep-th/9711162].

[19] B. Dai and C.L. Terng, *Baecklund transformations, Ward solitons, and unitons,* math.dg/0405363.

[20] B. Dai, C.L. Terng and K. Uhlenbeck, *On the space-time monopole equation,* math.dg/0602607.

[21] J.R. David, *Tachyon condensation in the D0/D4 system,* JHEP **0010**, 004 (2000) [hep-th/0007235].

[22] G.H. Derrick, *Comments on nonlinear wave equations as models for elementary particles,* J. Math. Phys. **5** (1964) 1252.

[23] R.K. Dodd, H.C. Morris, J.C. Eilbeck and J.D. Gibbon, "Solitons and nonlinear wave equations," Academic Press (1982).

[24] A.V. Domrin, O. Lechtenfeld and S. Petersen, *Sigma-model solitons in the noncommutative plane: Construction and stability analysis,* JHEP **0503** (2005) 045 [hep-th/0412001].

[25] A.V. Domrin, *work in progress*

[26] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory,* Rev. Mod. Phys. **73** (2001) 977 [hep-th/0106048].

[27] M. Dunajski and N.S. Manton, *Reduced dynamics of Ward solitons,* Nonlinearity **18** (2005) 1677 [hep-th/0411068].

[28] B. Durhuus, T. Jonsson and R. Nest, *Noncommutative scalar solitons: Existence and nonexistence,* Phys. Lett. B **500**, 320 (2001) [hep-th/0011139].

[29] B. Durhuus, T. Jonsson and R. Nest, *The existence and stability of noncommutative scalar solitons,* Commun. Math. Phys. **233** (2003) 49 [hep-th/0107121].

[30] H. Eichenherr, *$SU(N)$ Invariant nonlinear sigma models,* Nucl. Phys. B **146**, 215 (1978) [Erratum-ibid. B **155**, 544 (1979)].

[31] H. Eichenherr and M. Forger, *On the dual symmetry of the nonlinear sigma models,* Nucl. Phys. B **155**, 381 (1979).

[32] O. Foda, I. Jack and D.R.T. Jones, *General classical solutions in the noncommutative $CP^{N-1}$ model,* Phys. Lett. B **547** (2002) 79 [hep-th/0209111].

[33] P. Forgács, Z. Horváth and L. Palla, *Solution generating technique for selfdual monopoles,* Nucl. Phys. B **229** (1983) 77.

[34] M. Forger and P. Zizzi, *Twisted chiral models with Wess-Zumino terms, and strings,* Nucl. Phys. B **287** (1987) 131.

[35] K. Fujii, *Classical solutions of higher dimensional nonlinear sigma models on spheres,* Lett. Math. Phys. **10**, 49 (1985).

[36] A. Fujii, Y. Imaizumi and N. Ohta, *Supersymmetry, spectrum and fate of D0-Dp systems with B-field,* Nucl. Phys. B **615** (2001) 61 [hep-th/0105079].

[37] K. Furuta, T. Inami, H. Nakajima and M. Yamamoto, *Low-energy dynamics of noncommutative $CP^1$ solitons in 2+1 dimensions,* Phys. Lett. B **537** (2002) 165 [hep-th/0203125].

[38] K. Furuta, T. Inami, H. Nakajima and M. Yamamoto, *Non-BPS solutions of the noncommutative $CP^1$ model in 2+1 dimensions,* JHEP **0208** (2002) 009 [hep-th/0207166].

[39] K. Furuuchi, *Topological charge of $U(1)$ instantons on noncommutative $R^4$,* Prog. Theor. Phys. Suppl. **144** (2001) 79 [hep-th/0010006].

[40] M. Gell-Mann and M. Levy, *The axial vector current in beta decay,* Nuovo Cim. **16** (1960) 705.

[41] V.L. Golo and A.M. Perelomov, *Solution of the duality equations for the two-dimensional $SU(N)$ invariant chiral model,* Phys. Lett. B **79** (1978) 112.

[42] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons,* JHEP **0005** (2000) 020 [hep-th/0003160].

[43] R. Gopakumar, M. Headrick and M. Spradlin, *On noncommutative multi-solitons,* Commun. Math. Phys. **233** (2003) 355 [hep-th/0103256].

[44] A.S. Gorsky, Y.M. Makeenko and K.G. Selivanov, *On noncommutative vacua and noncommutative solitons,* Phys. Lett. B **492**, 344 (2000) [hep-th/0007247].

[45] J.M. Gracia-Bondía, J.C. Várilly, H. Figueroa, "Elements of noncommutative geometry," Birkhäuser (2000).

[46] H.J. Groenewold, *On the principles of elementary quantum mechanics,* Physica **12** (1946) 405.

[47] D.J. Gross and N.A. Nekrasov, *Solitons in noncommutative gauge theory,* JHEP **0103** (2001) 044 [hep-th/0010090].

[48] L. Hadasz, U. Lindstrøm, M. Roček and R. von Unge, *Noncommutative multi-solitons: Moduli spaces, quantization, finite theta effects and stability,* JHEP **0106** (2001) 040 [hep-th/0104017].

[49] M. Hamanaka, *Noncommutative solitons and D-branes,* hep-th/0303256.

[50] T.H. Hansson, S.B. Isakov, J.M. Leinaas, U. Lindström, *Classical phase space and statistical mechanics of identical particles,* quant-ph/0003121.

[51] T.H. Hansson, S.B. Isakov, J.M. Leinaas, U. Lindström, *Exclusion statistics in classical mechanics,* quant-ph/0004108.

[52] J.A. Harvey, *Topology of the gauge group in noncommutative gauge theory,* hep-th/0105242.

[53] J.A. Harvey, *Komaba lectures on noncommutative solitons and D-branes,* hep-th/0102076.

[54] F.B. Hildebrand, "Introduction to numerical analysis," Dowver Publications (1987).

[55] R.H. Hobart, *On the instability of a class of unitary field models,* Proc. Phys. Soc. **82** (1963) 201.

[56] M. Ihl and S. Uhlmann, *Noncommutative extended waves and soliton-like configurations in N=2 string theory,* Int. J. Mod. Phys. A **18** (2003) 4889 [hep-th/0211263].

[57] T.A. Ioannidou and R.S. Ward, *Conserved quantities for integrable chiral equations in 2+1 dimensions,* Phys. Lett. A **208** (1995) 209.

[58] T.A. Ioannidou, *Soliton solutions and nontrivial scattering in an integrable chiral model in 2+1 dimensions,* J. Math. Phys. **37** (1996) 3422 [hep-th/9604126].

[59] T.A. Ioannidou and W.J. Zakrzewski, *Solutions of the modified chiral model in (2+1) dimensions,* J. Math. Phys. **39** (1998) 2693 [hep-th/9802122].

[60] T. Ioannidou and N.S. Manton, *The energy of scattering solitons in the Ward model,* Proc. Roy. Soc. Lond. A **461** (2005) 1965 [hep-th/0409168].

[61] T. Ioannidou and N.S. Manton, *Energy of scattering solitons in the Ward model,* Czech. J. Phys. **55** (2005) 1427.

[62] T.A. Ivanova and A.D. Popov, *Soliton equations and selfdual gauge fields,* Phys. Lett. A **170**, 293 (1992).

[63] T.A. Ivanova and O. Lechtenfeld, *Hidden symmetries of the open $N = 2$ string,* Int. J. Mod. Phys. A **16** (2001) 303 [hep-th/0007049].

[64] J.M. Izquierdo, M.S. Rashid, B. Piette and W.J. Zakrzewski, *Models with solitons in (2+1)-dimensions,* Z. Phys. C **53** (1992) 177.

[65] R. Jackiw, *Physical instances of noncommuting coordinates,* Nucl. Phys. Proc. Suppl. **108** (2002) 30, Phys. Part. Nucl. **33** (2002) S6, Lect. Notes Phys. **616** (2003) 294-304 [hep-th/0110057].

[66] T. Kato, "Perturbation theory for linear operators," Springer (1966).

[67] M. Klawunn, O. Lechtenfeld and S. Petersen, *Moduli-space dynamics of noncommutative Abelian sigma-model solitons,* JHEP **0606** (2006) 028 [hep-th/0604219].

[68] M. Klawunn, "Adiabatic approach for noncommutative multi-solitons in $2 + 1$ dimensions," Diploma thesis, University of Hannover (2006).

[69] G. Landi, *An introduction to noncommutative spaces and their geometry,* hep-th/9701078.

[70] S. Lang, "Algebra," Addison-Wesley (1993).

[71] P.D. Lax, *Integrals of nonlinear equations of evolution and solitary waves,* Commun. Pure Appl. Math. **21** (1968) 467.

[72] O. Lechtenfeld, A.D. Popov and B. Spendig, *Noncommutative solitons in open N=2 string theory,* JHEP **0106** (2001) 011 [hep-th/0103196].

[73] O. Lechtenfeld and A.D. Popov, *Noncommutative multi-solitons in 2+1 dimensions,* JHEP **0111** (2001) 040 [hep-th/0106213].

[74] O. Lechtenfeld and A.D. Popov, *Scattering of noncommutative solitons in 2+1 dimensions,* Phys. Lett. B **523** (2001) 178 [hep-th/0108118].

[75] O. Lechtenfeld, *Noncommutative instantons and solitons,* Fortsch. Phys. **52** (2004) 596 [hep-th/0401158].

[76] O. Lechtenfeld, *Noncommutative solitons,* hep-th/0605034.

[77] B.H. Lee, K.M. Lee and H.S. Yang, *The CP(n) model on noncommutative plane,* Phys. Lett. B **498** (2001) 277 [hep-th/0007140].

[78] K.M. Lee, *Chern-Simons solitons, chiral model, and (affine) Toda model on noncommutative space,* JHEP **0408** (2004) 054 [hep-th/0405244].

[79] R. Leese, *Extended wave solutions in an integrable chiral model in (2+1) dimensions,* J. Math. Phys. **30** (1989) 2072.

[80] M. Li, *Quantum corrections to noncommutative solitons,* hep-th/0011170.

[81] U. Lindström, M. Roček and R. von Unge, *Non-commutative soliton scattering,* JHEP **0012** (2000) 004 [hep-th/0008108].

[82] A. Losev, G.W. Moore, N. Nekrasov and S. Shatashvili, *Four-dimensional avatars of two-dimensional RCFT,* Nucl. Phys. Proc. Suppl. **46**, 130 (1996) [hep-th/9509151].

[83] J. Madore and J. Mourad, *A Noncommutative extension of gravity,* Int. J. Mod. Phys. D **3** (1994) 221 [gr-qc/9307030].

[84] J. Madore, "An introduction to noncommutative geometry and its physical aplications," Cambridge University Press (1999).

[85] N.S. Manton, *A remark on the scattering of BPS monopoles,* Phys. Lett. B **110** (1982) 54.

[86] N.S. Manton and P. Sutcliffe, "Topological solitons," Cambridge University Press (2004).

[87] L.J. Mason and N.M.J. Woodhouse, "Integrability, selfduality, and twistor theory," Clarendon Press (1996)

[88] Y. Matsuo, *Topological charges of noncommutative soliton,* Phys. Lett. B **499** (2001) 223 [hep-th/0009002].

[89] J.E. Moyal, *Quantum mechanics as a statistical theory,* Proc. Cambridge Phil. Soc. **45** (1949) 99.

[90] J. Murugan and R. Adams, *Comments on noncommutative sigma models,* JHEP **0212** (2002) 073 [hep-th/0211171].

[91] V.P. Nair and J. Schiff, *Kahler Chern-Simons theory and symmetries of antiselfdual gauge fields,* Nucl. Phys. B **371** (1992) 329.

[92] M. Nakahara, "Geometry, topology and physics," Institute of Physics Publishing (1996)

[93] N.A. Nekrasov, *Noncommutative instantons revisited,* Commun. Math. Phys. **241** (2003) 143 [hep-th/0010017].

[94] N.A. Nekrasov, *Trieste lectures on solitons in noncommutative gauge theories,* hep-th/0011095.

[95] A.C. Newell, "Solitons in mathematics and physics," Society of Industrial and Applied Mathematics - SIAM (1985).

[96] H. Otsu, T. Sato, H. Ikemori and S. Kitakado, *New BPS solitons in 2+1 dimensional noncommutative CP(1) model,* JHEP **0307** (2003) 054 [hep-th/0303090].

[97] H. Otsu, T. Sato, H. Ikemori and S. Kitakado, *Lost equivalence of nonlinear sigma and $CP^1$ models on noncommutative space,* JHEP **0406** (2004) 006 [hep-th/0404140].

[98] H. Otsu, T. Sato, H. Ikemori and S. Kitakado, *Solitons of sigma model on noncommutative space as solitons of electron system,* JHEP **0507**, 052 (2005) [hep-th/0503118].

[99] A.M. Perelomov, “Generalized coherent states and their applications,” Springer (1986).

[100] M. Peyrard, B. Piette and W.J. Zakrzewski, *Soliton scattering in the skyrme model in (2+1)-dimensions. 1. Soliton-soliton case,* Nonlinearity **5** (1992) 563.

[101] M. Peyrard, B. Piette and W.J. Zakrzewski, *Soliton scattering in the skyrme model in (2+1)-dimensions. 2. More general systems,* Nonlinearity **5** (1992) 585.

[102] B. Piette and W.J. Zakrzewski, *Skyrmion model in (2+1)-dimensions with bound states,* Nucl. Phys. B **393** (1993) 65.

[103] R. Rajaraman, “Solitons and instantons,” North Holland Publishing Company (1982).

[104] M. Reed and B. Simon, “Methods of modern mathematical physics, volume IV: Analysis of operators,” Academic Press (1978).

[105] N. Seiberg and E. Witten, *String theory and noncommutative geometry,* JHEP **9909** (1999) 032 [hep-th/9908142].

[106] H.S. Snyder, *Quantized space-time,* Phys. Rev. **71** (1947) 38.

[107] H.S. Snyder, *The electromagnetic field in quantized space-time,* Phys. Rev. **72**, 68 (1947).

[108] A. Solovyov, *On noncommutative solitons,* Mod. Phys. Lett. A**15** (2000) 2205 [hep-th/0008199].

[109] M. Spradlin and A. Volovich, *Noncommutative solitons on Kaehler manifolds,* JHEP **0203** (2002) 011 [hep-th/0106180].

[110] D. Sternheimer, *Deformation quantization: Twenty years after,* AIP Conf. Proc. **453** (1998) 107 [math.qa/9809056].

[111] P.M. Sutcliffe, *Nontrivial soliton scattering in an integrable chiral model in (2+1)-dimensions,* J. Math. Phys. **33** (1992) 2269.

[112] R.J. Szabo, *Quantum field theory on noncommutative spaces,* Phys. Rept. **378** (2003) 207 [hep-th/0109162].

[113] R.J. Szabo, *Symmetry, gravity and noncommutativity,* hep-th/0606233.

[114] D. Tong, *TASI lectures on solitons,* hep-th/0509216.

[115] K. Uhlenbeck, *Harmonic maps into Lie groups: classical solutions of the chiral model,* J. Diff. Geom. **30** (1989) 1–50.

[116] J.C. Varilly, *An introduction to noncommutative geometry,* physics/9709045.

[117] R.S. Ward, *Slowly moving lumps in the* $CP^1$ *model in (2+1)-dimensions,* Phys. Lett. B **158** (1985) 424.

[118] R.S. Ward, *Integrability of the chiral equations with torsion term,* Nonlinearity **1** (1988) 671–679.

[119] R.S. Ward, *Soliton solutions in an integrable chiral model in 2+1 dimensions,* J. Math. Phys. **29** (1988) 386–389.

[120] R.S. Ward, *Classical solutions of the chiral model, unitons, and holomorphic vector bundles,* Commun. Math. Phys. **128** (1990) 319.

[121] R.S. Ward, *Nontrivial scattering of localized solitons in a (2+1)-dimensional integrable system,* Phys. Lett. A **208** (1995) 203.

[122] E.C. Weisstein, *Partition Function Q,* From MathWorld – A Wolfram Web Resource. http://mathworld.wolfram.com/PartitionFunctionQ.html

[123] J. Wess and B. Zumino, *Consequences of anomalous Ward identities,* Phys. Lett. B **37** (1971) 95.

[124] H. Weyl, *Quantum mechanics and group theory,* Z. Phys. **46** (1927) 1.

[125] R. Wimmer, *D0-D4 brane tachyon condensation to a BPS state and its excitation spectrum in noncommutative super Yang-Mills theory,* JHEP **0505** (2005) 022 [hep-th/0502158].

[126] E. Witten, *Global aspects of current algebra,* Nucl. Phys. B **223** (1983) 422.

[127] M. Wolf, *Soliton antisoliton scattering configurations in a noncommutative sigma model in 2+1 dimensions,* JHEP **0206** (2002) 055 [hep-th/0204185].

[128] J.C. Wood, *Explicit construction and parameterization of harmonic two spheres in the unitary group,* Proc. London Math. Soc. **58** (1989) 608.

[129] C.N. Yang, *On quantized space-time,* Phys. Rev. **72**, 874 (1947).

[130] C.N. Yang, *Condition of selfduality for $SU(2)$ gauge fields on euclidean four-dimensional space,* Phys. Rev. Lett. **38** (1977) 1377.

[131] V.E. Zakharov and A.V. Mikhailov, *Relativistically invariant two-dimensional models in field theory integrable by the inverse problem technique. (In Russian),* Sov. Phys. JETP **47** (1978) 1017 [Zh. Eksp. Teor. Fiz. **74** (1978) 1953]. Reprinted in C. Rebbi, G. Soliani, (Ed.), "Solitons And Particles," World Scientific (1984).

[132] V.E. Zakharov and A.B. Shabat, *Integration of non-linear equations of mathematical physics by the inverse scattering method,* Funct. Anal. Appl. **13** (1979) 166.

[133] W.J. Zakrzewski, "Low dimensional sigma models," Adam Hilger (1989).

[134] W.J. Zakrzewski, *Soliton-like scattering in the O(3) sigma model in 2+1 dimensions,* Nonlinearity **4** (1991) 429–475.

[135] C.G. Zhou, *Noncommutative scalar solitons at finite $\theta$,* hep-th/0007255.

# Publications

The subsequent list provides a brief abstract of the publications [24, 67] presenting the scientific work of the author underlying this thesis.

Michael Klawunn, Olaf Lechtenfeld, Stefan Petersen

*Moduli-Space Dynamics of Noncommutative Abelian Sigma-Model Solitons*

JHEP **0606** (2006) 028 [hep-th/0604219]

In the noncommutative (Moyal) plane, we relate exact $U(1)$ sigma-model solitons to generic scalar-field solitons for an infinitely stiff potential. The static $k$-lump moduli space $\mathbb{C}^k/S_k$ features a natural Kähler metric induced from an embedding Grassmannian. The moduli-space dynamics is blind against adding a WZW-like term to the sigma-model action and thus also applies to the integrable $U(1)$ Ward model. For the latter's two-soliton motion we compare the exact field configurations with their supposed moduli-space approximations. Surprisingly, the two do not match, which questions the adiabatic method for noncommutative solitons.

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Andrei V. Domrin, Olaf Lechtenfeld, Stefan Petersen

*Sigma-Model Solitons in the Noncommutative Plane: Construction and Stability Analysis*

JHEP **0503** (2005) 045 [hep-th/0412001]

Noncommutative multi-solitons are investigated in Euclidean two-dimensional $U(n)$ and Grassmannian sigma models, using the auxiliary Fock-space formalism. Their construction and moduli spaces are reviewed in some detail, unifying abelian and nonabelian configurations. The analysis of linear perturbations around these backgrounds reveals an unstable mode for the $U(n)$ models but shows stability for the Grassmannian case. For multi-solitons which are diagonal in the Fock-space basis we explicitly evaluate the spectrum of the Hessian and identify all zero modes. It is very suggestive but remains to be proven that our results qualitatively extend to the entire multi-soliton moduli space.

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# Curriculum Vitae

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