hp-version of the boundary element method for electromagnetic problems – error analysis, adaptivity, preconditioners

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Dipl.-Math. Florian Leydecker

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Referent:	Prof. Dr. E. P. Stephan, Hannover University
Korreferent:	Prof. Dr. N. Heuer, Uxbridge University, Brunel, UK
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Abstract

This thesis deals with the *hp*-version for the coupling of finite elements and boundary elements in $\mathbb{R}^3$. We present preconditioners as well as reliable and efficient a posteriori error estimates for the *hp*-version.

In the first part we consider the hypersingular integral equation of the normal derivative of the double layer potential on surfaces and perform the Galerkin *hp*-version of the boundary element method (BEM) on triangles. This method is known to converge rapidly for smooth as well as for singular solutions. On the other hand the arising linear system is highly ill-conditioned. Hence, for an efficient solution procedure appropriate preconditioners are necessary to reduce the number of CG-iterations. We present an iterative substructuring method which uses the functions concentrated on the wire basket and the bubble functions in the interior of the elements separately. We prove that the condition number of the preconditioned stiffness matrix has a bound which is independent of the mesh size h and which grows only polylogarithmically in p, the maximum polynomial degree.

An essential tool for the construction of such preconditioners is the use of suitable polynomial extension operators from the boundary of a triangle into the interior. We discuss different extensions in fractional Sobolev spaces and prove their continuity.

In the second part we present an *hp*-version of the symmetric finite element/boundary element coupling method solving the eddy current problem for the time-harmonic Maxwell's equations. We use $\mathbf{H}(\mathbf{curl},\Omega)$-conforming vector-valued polynomials to approximate the electric field in the conductor Ω and surface curls of continuous piecewise polynomials on the boundary Γ of Ω to approximate the twisted tangential trace of the magnetic field on Γ. We present both a priori and a posteriori error estimates. For the a posterior estimate we prove efficiency and reliability on quasi-uniform meshes. As a second example of Maxwell's equations we discuss the time-harmonic scattering problem.

A further topic is the construction of an $\mathbf{H}(\mathbf{curl},\Omega)$-stable decomposition of the space of Nédélec elements $\mathcal{ND}_p(\mathcal{T}_h)$. Considering the trace of this space and certain extension operators we get an $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$-stable decomposition of the space of Raviart-Thomas elements $\mathcal{RT}_p(\mathcal{T}_h)$. These results can be used to construct certain preconditioners and reliable and efficient error estimates.

Furthermore, we present numerical results that underline our theoretical results. Therefore, we have to discuss the construction of suitable polynomial spaces and their transformations.

Key words. extension operators, iterative substructuring, preconditioners, FEM/BEM-coupling, Maxwell's equations, a posteriori error estimates

Zusammenfassung

Diese Arbeit behandelt die *hp*-Version der Kopplung von finiten Elementen und Randelementen in $\mathbb{R}^3$. Wir präsentieren sowohl Vorkonditionierer als auch zuverlässige und effiziente a posteriori Fehlerschätzer für die *hp*-Version.

Im ersten Teil betrachten wir die hypersinguläre Integralgleichung als Normalenableitung des Doppelschichtpotentials auf Oberflächen und analysieren die Galerkin *hp*-Version der Randelementmethode (BEM) auf Dreiecken. Diese Methode ist bekannt dafür, für glatte als auch für singuläre Lösungen sehr schnell zu konvergieren. Andererseits ist das zugehörige lineare Gleichungssystem sehr schlecht konditioniert. Folglich benötigt man für ein effizientes Lösungsverfahren geeignete Vorkonditionierer, um die Anzahl der Iterationen beim CG-Verfahren zu reduzieren. Wir präsentieren eine iterative Substruktur-Methode, bei der die Wirebasket-Funktionen, d.h. die auf dem Rand der Elemente konzentrierten Funktionen, und die inneren Funktionen auf den Dreiecken getrennt betrachtet werden. Wir zeigen, dass die Konditionszahl der so vorkonditionierten Steifigkeitsmatrix bezüglich der Gitterweite h beschränkt bleibt, während sie lediglich polylogarithmisch in p, dem maximalen Polynomgrad, anwächst.

Als wichtiges Hilfsmittel bei der Konstruktion eines solchen Vorkonditionierers erweisen sich polynomiale Fortsetzungsoperatoren vom Rand eines Dreiecks in sein Inneres. Wir diskutieren verschiedene Fortsetzungen in gebrochenen Sobolev-Räumen und beweisen ihre Stetigkeit.

Im zweiten Teil präsentieren wir eine *hp*-Version der symmetrischen Kopplung von finiten Elementen und Randelementen zur Lösung des Wirbelstromproblems der zeitharmonischen Maxwell-Gleichungen. Wir verwenden $\mathbf{H}(\mathbf{curl}, \Omega)$-konforme vektorwertige Polynome zur Approximation des elektrischen Feldes im Leiter Ω und Flächenrotationen von stetigen, stückweisen Polynomen auf dem Rand Γ von Ω zur Approximation der gedrehten Tangentialspur des magnetischen Feldes auf Γ. Wir beweisen sowohl a priori als auch a posteriori Fehlerabschätzungen. Für den a posteriori Fehlerschätzer zeigen wir Effizienz und Zuverlässigkeit auf quasi-uniformen Gittern. Als weiteres Beispiel der Maxwell-Gleichungen diskutieren wir auch das zeitharmonische Streuproblem.

Ein weiterer Punkt dieser Arbeit ist die Konstruktion einer $\mathbf{H}(\mathbf{curl}, \Omega)$-stabilen Zerlegung des Raumes der Nédélec -Elemente $\mathcal{ND}_p(\mathcal{T}_h)$. Unter Benutzung von Spurbildung und eines Fortsetzungsoperators erhalten wir eine $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$-stabile Zerlegung des Raumes der Raviart-Thomas Elemente $\mathcal{RT}_p(\mathcal{T}_h)$. Diese Ergebnisse können zur Konstruktion von Vorkonditionierern sowie effizienten und zuverlässigen Fehlerabschätzungen genutzt werden.

Weiterhin präsentieren wir numerische Ergebnisse, die unsere theoretischen Resultate unterstreichen. Dazu haben wir die Konstruktion der passenden Polynomräume und ihrer Transformationen eingehend untersucht.

Schlagwörter. Fortsetzungsoperatoren, Iterative Substruktur-Methoden, Vorkonditionierer, FEM/BEM-Kopplung, Maxwell-Gleichungen, a posteriori Fehlerabschätzungen.

Contents

Introduction

This thesis deals with the hp-version for the coupling of finite elements and boundary elements in $\mathbb{R}^3$. We present preconditioners as well as reliable and efficient a posteriori error estimates for the hp-version.

The thesis is divided into two parts. In the first part (Chapters 1 and 2) an additive Schwarz based preconditioner is presented for the hp-version of the boundary element method (BEM), applied to a first kind integral equation on surfaces Γ. High order Galerkin methods as the p- and the hp-versions are known to converge rapidly for smooth as well as for singular solutions. On the other hand, the arising linear systems are highly ill-conditioned and their iterative solutions require efficient preconditioners. For piecewise polynomial spaces on meshes, consisting of quadrilateral or hexahedral elements, overlapping and iterative substructuring methods define such optimal or quasi-optimal preconditioners, see Pavarino, Widlund, Heuer, Stephan, Guo, Cao [87, 88, 89, 55, 57, 61, 53, 33]. On triangular or tetrahedral meshes for problems in three dimensions, however, the complete analysis of such domain decomposition based preconditioners is still an open problem. This concerns the finite element method (FEM) with tetrahedral meshes as well as the boundary element method (BEM) with triangular meshes. We present here the analysis of an iterative substructuring method for the p-version of the BEM with the hypersingular operator in $\mathbb{R}^3$, thus acting on surfaces, considering triangular meshes. The integral equation under consideration is the hypersingular integral equation

$$Dv(x) := -\frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma v(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}ds_y = f(x),\ x \in \Gamma. \tag{0.1}$$

On Γ we consider a quasi-uniform mesh of triangles Γ_i, $i = 1, \ldots, n$, and take the space $S_h^p(\Gamma)$ of continuous functions whose restrictions on Γ_i are polynomials of degree $\leq p$. We perform the p-version boundary element method for equation (0.1):
Find $u_p^* \in S_h^p(\Gamma)$ *such that*

$$\langle Du_p^*, v_p\rangle_{L^2(\Gamma)} = \langle g, v_p\rangle_{L^2(\Gamma)} \qquad \textit{for all } v_p \in S_h^p(\Gamma). \tag{0.2}$$

For the stability and the convergence of the scheme, see Stephan & Suri [100]. In the p-version Galerkin scheme (0.2), the arising linear systems are highly ill-conditioned. Using standard tensor product shape functions on rectangles based on antiderivatives of Legendre polynomials, the condition number of the Galerkin matrix A_N behaves like

$\mathrm{cond}(A_N) = \mathcal{O}(p^6)$, see Heuer [56]. Therefore, the iterative solutions require efficient preconditioners.

In this work, for (0.1) the p-version of the Galerkin method is studied on a quasi-uniform triangular mesh using special low energy basis functions, introduced by Pavarino & Widlund [89], together with suitable polynomial extensions of vertex functions and edge functions into triangles. We present an iterative substructuring method which is based on a splitting of the trial space into wire basket functions and interior functions (bubbles). The resulting additive Schwarz preconditioner has a block-diagonal structure and the condition number of this Schwarz operator behaves like $\mathcal{O}((1 + \log p)^4)$.

In the second part (Chapters 3–6) we consider an hp-version of the FEM/BEM-coupling for the eddy current problem. The latter models a time-harmonic interface problem in electromagnetics where a conductor and a monochromatic exciting current are given and displacement currents are neglected. The task is to compute the resulting magnetic and electric fields in the conductor Ω as well as in the exterior domain. The use of boundary elements for exterior problems in electromagnetics goes back to the early works of Bendali [17], Nédélec [81, 83] and MacCamy & Stephan [71, 70, 72]. We also refer to the work of Buffa, Costabel, Hiptmair & Schwab *et al.* [29, 31, 32, 68]. For the coupling of FEM and BEM in electromagnetics, see Bossavit [22], Costabel & Stephan [40], Nédélec *et al.* [7, 9, 8] and Hiptmair [66, 67]. Here, we consider the field-based symmetric coupling formulation which was introduced by Hiptmair [66]. The unknowns are $\mathbf{u}$ corresponding to the electrical field in the bounded conductor Ω and $\boldsymbol{\lambda}$ corresponding to the twisted tangential trace of the magnetic field on the boundary Γ of the conductor. The natural Sobolev space for $\mathbf{u}$ is $\mathbf{H}(\mathbf{curl}, \Omega)$, which is the space of $\mathbf{L}^2$-fields in Ω with rotation in $\mathbf{L}^2(\Omega)$. The space for $\boldsymbol{\lambda}$ is $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$ which is a trace space of $\mathbf{H}(\mathbf{curl}, \Omega)$ with vanishing surface divergence. The Galerkin discretization uses the space $\mathcal{X}_{h,p}$ of $\mathbf{H}(\mathbf{curl})$-conforming vector-valued polynomials (for $\mathbf{u}$) on a regular mesh $\mathcal{T}_h$ of tetrahedrons and the space $\mathcal{Y}_{h,p}$ of surface curls of continuous, piecewise polynomials (for $\boldsymbol{\lambda}$) on a regular mesh $\mathcal{K}_h$ on Γ (which is induced by $\mathcal{T}_h$). We derive a priori error estimates for the hp-version of the FEM/BEM-coupling which use suitable projection-based interpolation operators as introduced in Chapter 4. We also give corresponding reliable and efficient residual a posteriori error estimates.

Preliminary work was done in two PhD-theses (Bică [21] and Teltscher [103]) and is here reused, completed and generalized. For Chapters 1 and 2 the main reference is the thesis of Bică [21] where an iterative substructuring method for the p-version of the finite element method on tetrahedrons is presented. He uses assumptions on the continuity of polynomial preserving extension operators from the boundary of a triangle into the interior. But he could not prove his extension theorem and introduces in his estimates a value $N(p)$ which he assumes to be constant. In Chapter 1 we prove continuity of the extension with a factor $(1 + \log p)^{1/2}$.

For the electromagnetic problems basic work was done in the thesis of Teltscher [103] who presented a residual and a p-hierarchical error estimator for the coupling of finite elements and boundary elements of electromagnetic problems, see also Teltscher *et al.* [104, 105, 106] and Maischak & Stephan [99]. The work of Teltscher is based on several articles of Hiptmair [66, 67] and Beck, Wohlmuth *et al.* [15, 16]. While Teltscher considered only the h-version with lowest polynomial degree we extend his results to the hp-version.

In the following some details are listed.

In Chapter 1 we present different polynomial preserving extension operators from the boundary of a triangle T into the interior. Our main result is Theorem 1.2.1. Here, we prove the existence of an extension U such that holds

$$\|U\|_{\tilde{H}^{1/2}(T,\Gamma)} \leq C(1+\log p)^{1/2}\|f\|_{L^2(\partial T)}$$

where f is a polynomial of degree p that vanishes on $\Gamma \subset \partial T$ which consists of one or two edges of T.

For the proof of this result we have to consider different extension operators which extend a polynomial from one side of the triangle into the interior where the polynomial possesses a root at one or two vertices. The operator under consideration is the operator

$$E(f)(x,y) := \frac{x}{y}\int_x^{x+y} \frac{f(t)}{t}\,dt$$

which extends a polynomial f of degree p defined on one side I of the unit triangle with $f(0)=0$ to a polynomial of degree p into the interior of the triangle T. This operator was introduced by Bică [21] using ideas from Muñoz-Sola [79]. Bică could only postulate the continuity of this extension from $L^2(I)$ to $H^{1/2}(T)$. We prove the continuity of this extension, i.e. there holds

$$\|E(f)\|_{H^{1/2}(T)} \leq C\,(\log p)^{1/2}\|f\|_{L^2(I)},$$

see Theorem 1.2.2.

Using this result we can prove different extensions from the boundary into the triangle using the $\tilde{H}^{1/2}$-norm, see Theorem 1.2.3.

The proof of Theorem 1.2.2 is done in Section 1.3. Therefore, we show continuity of the extension from $\tilde{H}^{1/2}(I,0)$ to $H^1(T)$ (Theorem 1.3.4) and continuity with a factor $(1+\log p)$ from $H^{-1/2}(I)$ to $L^2(T)$ (Theorem 1.3.7). The result then follows with interpolation between the spaces.

Having proven the continuity of the extension operator we can construct in Chapter 2 a preconditioner for the hp-version for the hypersingular operator on quasi-uniform triangular meshes. It uses an iterative substructuring method using the so-called wire basket space (consisting of nodal and side functions) and the space of bubble functions concentrated in the interior of the triangles. Therefore, we decompose the polynomial space $S_h^p(\Gamma)$ into functions which belong to the wire basket of the mesh, i.e. all basis functions which are associated to the nodes and the edges of the mesh, and functions which belong to the interior of the elements, i.e. functions that are zero on all edges:

$$S_h^p(\Gamma) = V_W + \sum_{i=1}^{n} V_{\Gamma_i}.$$

As vertex basis functions we use so-called low energy functions on the edges, see Pavarino & Widlund [89], which are extended into the triangle via our polynomial lifting operators, introduced in Chapter 1. As edge functions we take affine images of antiderivatives of Legendre polynomials $\mathcal{L}_n(x)$ together with their polynomial lifting, whereas as bubble functions we take linear combinations of antiderivatives of Legendre polynomials.

For the bubble spaces $V_{\Gamma_1}, \ldots, V_{\Gamma_n}$ we set

$$b_j(v, w) := \langle Dv, w \rangle \quad \forall\, v,\, w \in V_{\Gamma_j},\ j = 1, \ldots, n.$$

On the other hand, for the wire basket functions, we can take both the energy bilinear form $\langle D\cdot, \cdot \rangle$ or the L^2-bilinear form

$$\hat{a}_W(v, w) := (1 + \log p)^3 \sum_{i=1}^{n} \inf_{c_i \in \mathbb{R}} (v - c_i, w - c_i)^2_{L^2(\partial\Gamma_i)}.$$

In Theorem 2.1.1 we show that the condition number of the preconditioned system grows only polylogarithmically. The proof of this theorem is given in Section 2.3. The numerical results in the example in §2.4 show for both wire basket preconditioners the same behavior. Of course, the L^2-bilinear form leads to a sparse matrix whereas the energy bilinear form gives a dense block for the Galerkin matrix due to the non-locality of the integral operator D. Chapter 2 ends with Section 2.5 where we prove a stability estimate for discrete harmonic extensions from the faces of a tetrahedron into its interior. Furthermore, we give detailed proofs of some results of Bică [21] which we need here.

The following chapters deal with the *hp*-version for the coupling of finite elements and boundary elements for electromagnetic problems. In Chapter 3 we present the definition of the used Sobolev spaces for Maxwell's equations. These are $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}(\mathrm{div}, \Omega)$ for the bounded domain Ω. On the boundary Γ we define the tangential trace operator $\gamma_D \mathbf{u} := \mathbf{n} \times (\mathbf{u} \times \mathbf{n})$ and the twisted tangential trace $\gamma_t^\times := \mathbf{u} \times \mathbf{n}$. Then, we get the trace spaces

$$\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \gamma_t^\times(\mathbf{H}(\mathbf{curl}, \Omega)), \qquad \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) = \gamma_D(\mathbf{H}(\mathbf{curl}, \Omega)).$$

In Section 3.2 we define the used boundary integral operators for Maxwell's equations and we collect their mapping properties on the spaces $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

These results are then extended to the spaces $\mathbf{H}^s_\perp(\mathrm{curl}_\Gamma, \Gamma)$ and $\mathbf{H}^s_\parallel(\mathrm{div}_\Gamma, \Gamma)$ in Section 3.3. In Section 3.2.1 we introduce the Stratton-Chu representation formula which is an essential tool for the construction of the coupling of finite elements and boundary elements.

The approximation in the relevant spaces $\mathbf{H}(\mathbf{curl}, \Omega)$, $\mathbf{H}(\mathrm{div}, \Omega)$, $\mathbf{H}^{-1/2}_\parallel(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{H}^{-1/2}_\perp(\mathrm{curl}_\Gamma, \Gamma)$ is described in Chapter 4. For this, we define a regular mesh of tetrahedrons or hexahedrons of mesh size h on the domain Ω, and this induces a mesh $\mathcal{K}_h$ of triangles or of quadrilaterals on the boundary $\Gamma = \partial\Omega$. For the approximation in the space $\mathbf{H}(\mathbf{curl}, \Omega)$ one uses usually the so-called Nédélec space $\mathcal{ND}_p(\mathcal{T}_h)$, see Nédélec [82]. These functions fulfill the conformity condition for $\mathbf{H}(\mathbf{curl}, \Omega)$, i.e. the tangential trace between two elements has to be continuous. In order to achieve this condition Nédélec [82] introduces degrees of freedom which are based on integral moments that can be used for the definition of the basis functions. In Section 4.1 we describe the basis functions on the reference cube and on the reference tetrahedron. For the basis functions on the reference cube we introduce in §4.1.1 a general scheme for the calculation of these basis functions for higher polynomial degrees p using different test and ansatz functions. This leads to linear systems with condition numbers depending on the polynomial degrees and also on the used basis functions. We present different approaches and compare them in numerical experiments. In §4.1.3 we describe how to transform the Nédélec functions on the reference element to a local element of size h, considering an $\mathbf{H}(\mathbf{curl}, \Omega)$-conforming transformation. Using this transformation we derive an inverse inequality for the space of Nédélec functions, see Lemma 4.1.3. Finally, using the above moments again we define an interpolation operator. In Section 4.2 we present the finite element method for the eddy current problem in a bounded domain and confirm the results for the p-version numerically.

In Section 4.3 we briefly describe the main properties of the Raviart-Thomas space $\mathcal{RT}_p(\mathcal{T}_h)$ for the approximation in $\mathbf{H}(\mathrm{div}, \Omega)$. More important for our coupling formulation is the space $\mathbf{H}^{-1/2}_\parallel(\mathrm{div}_\Gamma, \Gamma)$. Therefore, we use the Raviart-Thomas space $\mathcal{RT}_p(\mathcal{K}_h) := \gamma_t^\times(\mathcal{ND}_p(\mathcal{T}_h))$, see Section 4.4. We describe the calculation of the basis functions on the reference square and derive transformation formulas in §4.4.3 and also an inverse inequality. For the discretization in $\mathbf{H}^{-1/2}_\perp(\mathrm{curl}_\Gamma, \Gamma)$ we introduce in Section 4.5 the space $\mathcal{TND}_p(\mathcal{T}_h) := \gamma_D(\mathcal{ND}_p(\mathcal{T}_h))$ as the tangential trace space of the Nédélec space. We derive the transformation formulas which are essential for our calculations. In Section 4.6 we consider the de Rham diagram which gives us the connection between the different finite element spaces. Furthermore, we consider in Section 4.7 a continuous extension operator as right inverse of the operator $\gamma_t^\times : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}^{-1/2}_\parallel(\mathrm{div}_\Gamma, \Gamma)$. The existence of such an operator was proven in Alonso & Valli [5]. But here we present another construction which was communicated by Ralf Hiptmair. We describe the details. In Section 4.8 we consider the hp-interpolation in the $\mathbf{H}(\mathbf{curl}, \Omega)$-conforming space. Furthermore, we discuss a special interpolation operator for the hp-version, which is based on Demkowicz *et al.* [44, 45, 46]. Therefore, our polynomial extension from Chapters 1 and 2 are helpful.

Finally, in Section 4.9 we give a $\mathbf{H}(\mathbf{curl},\Omega)$-stable 2-Level decomposition of the Nédélec space $\mathcal{ND}_p(\mathcal{T}_h)$ also for higher polynomial degrees. This can be used to construct an additive Schwarz preconditioner for the $\mathbf{H}(\mathbf{curl},\Omega)$-bilinear form

$$a(\mathbf{u},\mathbf{v}) := (\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_\Omega + (\mathbf{u},\mathbf{v})_\Omega.$$

Using these results we can construct an $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$-stable decomposition of $\mathcal{RT}_2(\mathcal{K}_h)$, see Section 4.10. In order to prove this we have to use the result on the extension of $\mathcal{RT}_2(\mathcal{K}_h)$ to $\mathcal{ND}_2(\mathcal{T}_h)$ from Section 4.7. Unfortunately, this extension is not local for single basis functions. Thus, we can only prove a decomposition into two spaces. Using this decomposition we can construct a two-block additive Schwarz preconditioner for the bilinear form

$$b(\boldsymbol{\lambda},\boldsymbol{\zeta}) := \langle V(\operatorname{div}_\Gamma\boldsymbol{\lambda}),\operatorname{div}_\Gamma\boldsymbol{\zeta}\rangle_\Gamma + \langle\mathcal{V}\boldsymbol{\lambda},\boldsymbol{\zeta}\rangle_\Gamma.$$

We present a numerical experiment underlining the theoretical result. For an adaptive *hp*-version using quadrilaterals and quadrangles one has to use hanging nodes. The construction is described in Section 4.11, also for higher polynomial degrees.

In Chapter 5 we present the eddy current problem. For a further discussion of the mathematical background of this problem we refer to the work of Ammari, Buffa & Nédélec [6]. We derive a coupling formulation of finite elements and boundary elements which goes back to the work of Hiptmair [66]. Starting from the Stratton-Chu representation theorem we derive a coupling formulation for the variable $\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega)$, representing the electric field in the domain Ω, and for the Neumann trace $\gamma_N\mathbf{u}\in\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma 0,\Gamma)$, which corresponds to the magnetic field. For the discretization we use finite elements in the interior of the domain to approximate the electric field and surface curls of hat functions to approximate the twisted tangential trace of the magnetic field. We derive a residual error estimator for the *hp*-version. Singular, weakly singular and hypersingular boundary integral operators appearing in the variational coupling formulation show up in the terms of the error estimators as well. Our formulation holds for non-smooth boundaries. If we fix the polynomial degree we regain the *h*-version as considered in Teltscher *et al.* [106]. For the proof we use the *hp*-interpolation operators $\tilde{\Pi}_p^1$ introduced in §4.8. We prove the reliability in Theorem 5.3.1 following the ideas of Beck *et al.* [14] and Teltscher [103, 106]. In §5.3.1 we present a three-fold algorithm which can be used to achieve suitably refined *hp*-meshes. In Section 5.4 we show the efficiency for the *h*-version using some ideas of Beck *et al.* [15] for the finite element indicators and of Carstensen [34] for the coupling of finite elements and boundary elements using the Poincaré-Steklov operator.

We present numerical experiments using hanging nodes for the polynomial degree $p = 1$. Furthermore, we perform experiments for the *p*-version. These experiments underline our theoretical results. As far as we know there has been no implementation of the p-version of the coupling method up to now. Furthermore, we present a numerical example of a 2-level hierarchical error estimator introduced by Teltscher [103]. While Teltscher only considered the so-called bubble indicators we use all indicators. The implementation of hanging nodes and edges for higher polynomial degrees still has to be done, therefore we can't present experiments for the adaptive hp-version, but the numerical experiments for both h- and p-version show the power of these algorithms.

In Chapter 6 we consider as a further application of Maxwell's equations the time-harmonic scattering problem. Here, an incident wave is scattered at a dielectric body. We derive a coupling formulation for the electric field $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and the twisted tangential trace of the magnetic field on the boundary $\boldsymbol{\lambda} := \gamma_N \mathbf{u} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. The formulation is quite similar to Hiptmair [67] but we use different integral operators. For the discretization of the variables we use Nédélec and Raviart-Thomas functions. We also consider the electric field integral equation (EFIE) which is a part in the coupling formulation. Furthermore, we investigate the calculation of the Galerkin elements. Therefore, we have to use the transformations considered in Chapter 4. These results can also be applied to the calculation of the Galerkin elements for the eddy current problem. The chapter ends with some numerical experiments.

Throughout this work, vector-valued functions or spaces are written in bold letters, scalar functions in normal typed letters. C denotes a generic positive constant, usually independent of the characteristic mesh size h, that can also change its value throughout equations. The symbol $\lesssim$ signifies "$\leq$ up to a multiplicative constant". Such constants are always assumed to be independent of the mesh size h (if present in the context). The symbol $\simeq$ means "$\lesssim$ and $\gtrsim$".

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1 An extension theorem for polynomials on triangles

An important tool for the analysis of p- and hp-approximation methods is the construction of suitable polynomial preserving extension operators from the boundary of the elements into the interior. In this chapter we consider polynomial liftings from the boundary of a triangle $\tilde{T}$ into its interior. We will present different operators and prove the stability of their extension. For the construction of a preconditioner for the hypersingular operator in Chapter 2 it is essential to have an extension operator that extends a polynomial which is vanishing on a part of the boundary.

Several work has been done before. At first we mention the extension constructed by Babuška & Suri [13] where a stable polynomial extension operator from $H^{1/2}(\partial\tilde{T})$ to $H^1(\tilde{T})$ is developed, see also Babuška *et al.* [12]. Ainsworth & Demkowicz [3] construct a polynomial preserving extension operator $\mathcal{E}$ such that $\|\mathcal{E}F\|_{H^1(\tilde{T})} \leq C\|F\|_{H^{1/2}(\partial\tilde{T})}$ where F is a polynomial and $C > 0$ a positive constant, independent of the polynomial degree. Their operator is also shown to be uniformly stable from $L^2(\partial\tilde{T})$ to $H^{1/2}(\tilde{T})$.

Polynomial liftings on a tetrahedron were developed by Muñoz-Sola [79] following ideas mentioned by Maday [73]. The main result of Muñoz-Sola is the existence of continuous extension operator $\mathcal{R} : H^{1/2}(\partial K) \to H^1(K)$ for a tetrahedron K. Using the extension of Muñoz-Sola Bică constructed in his thesis [21] a suitable extension operator, but could not proof its stability. Here, we consider his operator and close the gaps in the proof.

The main theorem in this chapter is Theorem 1.2.1. It states that there exists such an extension from $L^2(\partial\tilde{T})$ to $\tilde{H}^{1/2}(\tilde{T},\Gamma)$, where Γ denotes a part of the boundary where the extended function is zero. The proof is done in several steps. Therefore we consider different extension operators which preserve zeros on edges. For the operator $E(f)(x,y) := \frac{x}{y}\int_x^{x+y} \frac{f(t)}{t}\,dt$ we prove the continuity of the mapping from $L^2(\partial\tilde{T})$ to $H^{1/2}(\tilde{T})$ in Theorem 1.2.2. The proof is done in section 1.3. Afterwards, using Theorem 1.2.3 and Theorem 1.2.1 the main theorem follows.

1.1 Definitions

On an open surface segment Γ we introduce the spaces $H^{1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$ where the latter space is most often denoted by $H^{1/2}_{00}(\Gamma)$ in the finite element literature. Let $\tilde{\Gamma}$ be a closed surface (in our case a polyhedral surface) with $\Gamma \subset \tilde{\Gamma}$. We define

$$H^{1/2}(\tilde{\Gamma}) := \{\phi|_{\tilde{\Gamma}};\ \phi \in H^1(\mathbb{R}^3)\}, \quad H^{1/2}(\Gamma) := \{\phi|_{\Gamma};\ \phi \in H^{1/2}(\tilde{\Gamma})\},$$

and

$$\tilde{H}^{1/2}(\Gamma) := \{\phi \in H^{1/2}(\Gamma);\ \tilde{\phi} \in H^{1/2}(\tilde{\Gamma})\},$$

where $\tilde{\phi}$ denotes the extension of ϕ by 0 from Γ onto $\tilde{\Gamma}$.
For $\Omega \subset \mathbb{R}^n$ and $0 < s < 1$ a norm in $H^s(\Omega)$ is given by (see Lions & Magenes [69])

$$\|\cdot\|^2_{H^s(\Omega)} = \|\cdot\|^2_{L^2(\Omega)} + |\cdot|^2_{H^s(\Omega)}$$

with semi-norm

$$|v|^2_{H^s(\Omega)} := \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^2}{|x-y|^{2s+n}}\, dx\, dy.$$

In order to calculate the $H^{1/2}$-norm over two adjacent elements Γ_i and Γ_j (e.g. triangles) we consider the following equivalent norm, compare Grisvard [50],

$$\|u\|^2_{H^{1/2}(\Gamma_i \cup \Gamma_j)} = \|u\|^2_{H^{1/2}(\Gamma_i)} + \|u\|^2_{H^{1/2}(\Gamma_j)} + \int_{\Gamma_i} \int_{\Gamma_j} \frac{(u(x) - u(y))^2}{|x-y|^3}\, dy\, dx.$$

For a Lipschitz domain $\Omega \subset \mathbb{R}^2$ and $0 < s < 1$ the space $\tilde{H}^s(\Omega, \Gamma)$, $\Gamma \subset \partial\Omega$, can be defined using the norm

$$\|u\|^2_{\tilde{H}^s(\Omega,\Gamma)} := \|u\|^2_{H^s(\Omega)} + \int_\Omega \frac{u(x)^2}{(\mathrm{dist}(x,\Gamma))^{2s}}\, dx,$$

(see e.g. Lions & Magenes [69, Theorem 11.7]). The second parameter Γ is omitted if $\Gamma = \partial\Omega$.

We also note that the spaces $H^{1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$ can be equivalently defined as intermediate spaces between $L^2(\Gamma)$ and $H^1(\Gamma)$ or $H^1_0(\Gamma)$ ($H^1_0(\Gamma)$ is the completion of $C^\infty_0(\Gamma)$ within $H^1(\Gamma)$).

For $s > 0$ the spaces $H^{-s}(\Omega)$ (resp. $\tilde{H}^{-s}(\Omega)$) are the dual spaces of $\tilde{H}^s(\Omega)$ (resp. $H^s(\Omega)$) with respect to the L^2-inner product.

Furthermore, we consider special subspaces of $H^{1/2}(T)$ related to one or two edges of the triangle $\tilde{T}$. These are also needed in Chapter 2. Let λ_i be the barycentric function related to the edge I_i of $\tilde{T}$. Thus, $\tilde{H}^{1/2}(\tilde{T}, I_i)$ consists of these functions $u \in H^{1/2}(\tilde{T})$ which vanish on the edge I_i and satisfy $\lambda_i^{-1/2} \cdot u \in L^2(\tilde{T})$, with the norm

$$\|u\|^2_{\tilde{H}^{1/2}(\tilde{T},I_i)} = |u|^2_{H^{1/2}(\tilde{T})} + \|\lambda_i^{-1/2} u\|^2_{L^2(\tilde{T})}, \tag{1.1}$$

and for $i \neq j$ it is $\tilde{H}^{1/2}(\tilde{T}, I_i, I_j) = \tilde{H}^{1/2}(\tilde{T}, I_i) \cap \tilde{H}^{1/2}(\tilde{T}, I_j)$ with the norm

$$\|u\|^2_{\tilde{H}^{1/2}(\tilde{T},I_i,I_j)} := |u|^2_{H^{1/2}(\tilde{T})} + \|\lambda_i^{-1/2}u\|^2_{L^2(\tilde{T})} + \|\lambda_j^{-1/2}u\|^2_{L^2(\tilde{T})}. \tag{1.2}$$

On the interval I we furthermore define the space $\tilde{H}^{1/2}(I, 0)$ using the norm

$$\|u\|^2_{\tilde{H}^{1/2}(I,0)} := \|u\|^2_{H^{1/2}(I)} + \|x^{-1/2}u\|^2_{L^2(I)}.$$

1.2 An extension theorem

First of all, we state here the main extension theorem. The proof is given below in the proof for Theorem 1.2.4.

Theorem 1.2.1 *Let f be a continuous function on the triangle $\tilde{T}$ such that f is a piecewise polynomial of degree p on each side of the triangle. Furthermore, we assume that f vanishes on Γ which consists of one or two sides of $\tilde{T}$. Thus, there exits an extension U, which is a polynomial of degree at most p, with $U = f$ on $\partial\tilde{T}$ and*

$$\|U\|_{\tilde{H}^{1/2}(\tilde{T},\Gamma)} \leq C\,(1 + \log p)^{1/2}\|f\|_{L^2(\partial\tilde{T})} \tag{1.3}$$

where the constant $C > 0$ is independent of f and p.

As the $\tilde{H}^{1/2}(\tilde{T})$-norm transforms as the $L^2(\partial\tilde{T})$-norm we can use the unit triangle $T := \{(x, y) : 0 \leq x, y;\ x + y \leq 1\}$ without loss of generality. The edges are denoted by I_i, $i = 1, 2, 3$, see Figure 1.1. Furthermore, we abbreviate $I := I_1 := [0, 1]$. Finally, we define the space of polynomials of degree p on the triangle T and on the interval I as

$$P^p(T) := \operatorname{span}\{x^i y^j,\ 0 \leq i + j \leq p,\ (x, y) \in T\},\ \ P^p(I) := \operatorname{span}\{x^i,\ 0 \leq i \leq p\}.$$

Furthermore, we define $P^p(I_1, 0) := \{f \in P^p(I),\ f(0) = 0\}$.

Next, we present different extension operators that extend a polynomial from one side of the triangle into the triangle. There is a "classical" extension operator $F : H^{1/2}(I) \to H^1(T)$ which is defined by

$$F(f)(x, y) := \frac{1}{y}\int_x^{x+y} f(t)\,dt,$$

cf. Babuška *et al.* [13, 12]. The value of the extension in the point (x, y) is the average of f on the interval $[x, x + y]$, compare Figure 1.1. A disadvantage of this operator is that it is not possible to control the behavior of the extension along the other edges, e.g. a root of f in 0 does not extend to a zero trace of $F(f)$ on I_3. But for the construction of suitable basis functions we need to get functions which vanish on one or two edges. We need this operator only for the analysis of the other operators.

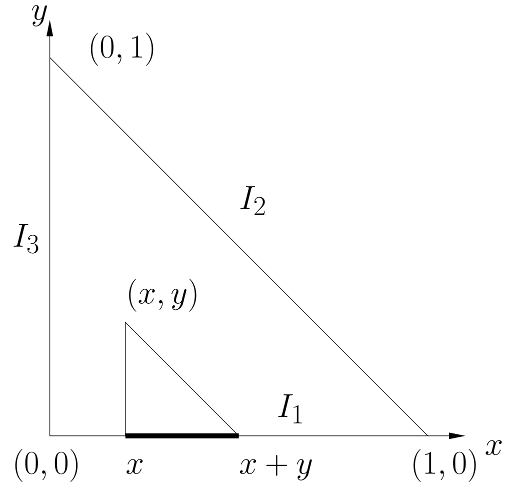

Figure 1.1: The reference triangle T.

The following operator is used by Bică [21] and its three-dimensional counterpart, for tetrahedrons by Muñoz-Sola [79]. The extension operators depend on the zeros of the polynomials in the corners and the extension vanishes on these edges which belong to the vertex where the polynomial is vanishing.

Therefore, we define the operator E by

$$E(f)(x,y) := \frac{x}{y}\int_x^{x+y} \frac{f(t)}{t}\,dt \qquad \text{if } f(0)=0.$$

More generally, for $f \in P^p(I)$ we define extension operators from I_1 by

$$\begin{aligned}
E_1^1(f)(x,y) :=& E(f)(x,y) = \frac{x}{y}\int_x^{x+y} \frac{f(t)}{t}\,dt \qquad \text{if } f(0)=0,\\
E_2^1(f)(x,y) :=& \frac{1-x-y}{y}\int_x^{x+y} \frac{f(t)}{1-t}\,dt \qquad \text{if } f(1)=0,\\
E^1(f)(x,y) :=& \frac{x(1-x-y)}{y}\int_x^{x+y} \frac{f(t)}{t(1-t)}\,dt \qquad \text{if } f(0)=f(1)=0.
\end{aligned}$$

We note that there holds

$$E^1(f)(x,y) = (1-x-y)E_1^1(f)(x,y) + xE_2^1(f)(x,y).$$

Moreover, $E_2^1(f)=0$ on I_2 and $E^1(f)=0$ on $I_2 \cup I_3$.

Extension operators E_3^3 (for $f \in P^p(I_3)$ with $f(1)=0$), E_1^3 (if $f(0)=0$) and E^3 (if $f(0)=f(1)=0$) from I_3 onto T are defined analogously.

For a polynomial $f \in P^p(I_2)$ we define

$$\begin{aligned}
E_2^2(f)(x,y) &:= \frac{y}{1-x-y}\int_x^{1-y} \frac{f(t,1-t)}{(1-t)}\,dt \qquad \text{if } f(1,0)=0,\\
E_3^2(f)(x,y) &:= \frac{x}{1-x-y}\int_x^{1-y} \frac{f(t,1-t)}{t}\,dt \qquad \text{if } f(0,1)=0,\\
E^2 f(x,y) &:= \frac{xy}{1-x-y}\int_x^{1-y} \frac{f(t,1-t)}{t(1-t)}\,dt \qquad \text{if } f(1,0)=f(0,1)=0.
\end{aligned}$$

There holds

$$E^2 f(x,y) = xE_2^2(f) + yE_3^2(f)$$

and $E_2^2(f)=0$ on I_1, $E_3^2(f)=0$ on I_3, $E^2(f)=0$ on $I_1 \cup I_3$.

It is easy to see that all the extensions are polynomials of degree p on T. Furthermore, all the operators which deal with polynomials that vanish in only one vertex are linear transformations of the operator $E = E_1^1$. Therefore we only have to study this operator.

In Theorem 1.3.4 we show that

$$E : \tilde{H}^{1/2}(I,0) \longrightarrow H^1(T) \tag{1.4}$$

is a continuous mapping and there holds

$$\|E(f)\|_{H^1(T)} \le C\|f\|_{\tilde{H}^{1/2}(I,0)}. \tag{1.5}$$

In Theorem 1.3.7 we show that

$$E : (P^p(I,0),\, H^{-1/2}(I)) \longrightarrow (P^p(T),\, L^2(T)) \tag{1.6}$$

is continuous with the estimate

$$\|E(f)\|_{L^2(T)} \le C(1+\log p)\,\|f\|_{H^{-1/2}(I)}. \tag{1.7}$$

Using interpolation between the spaces $L^2(T)$ and $H^1(T)$ and the spaces $H^{-1/2}(I)$ and $\tilde{H}^{1/2}(I,0)$ we get the following theorem.

Theorem 1.2.2 *The extension operator $E : (P^p(I,0), L^2(I)) \longrightarrow (P^p(T), H^{1/2}(T))$ is a continuous mapping and there holds*

$$\|E(f)\|_{H^{1/2}(T)} \le C(1+\log p)^{1/2}\,\|f\|_{L^2(I)}$$

for all polynomials $f \in P^p(I,0)$.

We are now in the position to prove the following extension theorem. The proof is similar to Lemma 4.10 in the dissertation of Bică [21].

Theorem 1.2.3 *For $f \in P^p(I,0)$ there holds*

$$\|E(f)\|_{\tilde{H}^{1/2}(T,I_3)} \le C(1+\log p)^{1/2}\|f\|_{L^2(I)}. \tag{1.8}$$

For $f \in P^p(I)$ and $f(1)=0$ there holds

$$\|E_2^1(f)\|_{\tilde{H}^{1/2}(T,I_2)} \le C(1+\log p)^{1/2}\|f\|_{L^2(I)}. \tag{1.9}$$

For $f \in P^p(I)$ and $f(0)=f(1)=0$ there holds

$$\|E^1(f)\|_{\tilde{H}^{1/2}(T,I_2,I_3)} \le C(1+\log p)^{1/2}\|f\|_{L^2(I)}. \tag{1.10}$$

Proof. Due to Theorem 1.2.2 there holds

$$\|E(f)\|_{H^{1/2}(T)} \le C(1+\log p)^{1/2}\,\|f\|_{L^2(I)}.$$

Thus, we only have to bound the weighted L^2-norms. Therefore we use Lemma 1.3.2 to get

$$\begin{aligned}
\|x^{-1/2}E(f)\|^2_{L^2(T)} &= \int_0^1\int_0^{1-x}\frac{x}{y^2}\left(\int_x^{x+y}\frac{f(t)}{t}\,dt\right)^2 dy\,dx \\
&\le 4\int_0^1 x\left(\int_x^1 \frac{f(t)^2}{t^2}\,dt\right)dx \\
&= 4\int_0^1 \frac{f(t)^2}{t^2}\int_0^t x\,dx\,dt \\
&= 4\int_0^1 \frac{f(t)^2}{t^2}\frac{t^2}{2}\,dt \\
&= 2\|f\|^2_{L^2(I)}
\end{aligned}$$

which finishes the proof of (1.8). The proof of (1.9) can be done in the same way as for (1.8).
Therefore, we only have to examine the operator E^1. First of all, we know that there holds

$$E^1(f)(x,y) = (1-x-y)E(f)(x,y) + xE_2^1(f)(x,y).$$

Here, we only consider the first term. The second one can be estimated the same way. In order to estimate the $H^{1/2}$-semi-norm we first see that there holds for $(x,y)\in T$ and $(x',y')\in T$

$$\begin{aligned}
&|(1-x-y)E(f)(x,y) - (1-x'-y')E(f)(x',y')|^2 \\
&\quad= |(1-x-y)E(f)(x,y) - (1-x-y)E(f)(x',y') \\
&\qquad\qquad + (1-x-y)E(f)(x',y') - (1-x'-y')E(f)(x',y')|^2 \\
&\quad\le 2(1-x-y)^2|E(f)(x,y) - E(x',y')|^2 + 2(x'-x+y'-y)^2|E(f)(x',y')|^2 \\
&\quad\le |E(f)(x,y) - E(f)(x',y')|^2 + 2(x'-x+y'-y)^2|E(f)(x',y')|^2.
\end{aligned}$$

Thus, we get using the definition of the $H^{1/2}(T)$-norm and Theorem 1.2.2

$$\begin{aligned}
&|(1-x-y)E(f)|^2_{H^{1/2}(T)} \\
&\le C\left(|E(f)|^2_{H^{1/2}} + \int_0^1\int_0^{1-y}\int_0^1\int_0^{1-y'} \frac{(x'-x+y'-y)^2\big(E(f)(x',y')\big)^2}{\big((x'-x)^2+(y'-y)^2\big)^{3/2}}\,dx'\,dy'\,dx\,dy\right) \\
&\le C\left(|E(f)|^2_{H^{1/2}} + \|E(f)\|^2_{L^2}\right) \\
&= C\,\|E(f)\|^2_{H^{1/2}(T)} \le C\,(1+\log p)\,\|f\|^2_{L^2(I)}.
\end{aligned}$$

Finally, we estimate the weighted L^2-norm on the edge I_3

$$\begin{aligned}
\|x^{-1/2}(1-x-y)E(f)\|^2_{L^2(T)} &= \int_0^1\int_0^{1-y} \frac{x(1-x-y)^2}{y^2}\left(\int_x^{x+y}\frac{f(t)}{t}\,dt\right)^2 dx\,dy \\
&\le \int_0^1\int_0^{1-y}\frac{x}{y^2}\left(\int_x^{x+y}\frac{f(t)}{t}\,dt\right)^2 dx\,dy \le C\|f\|^2_{L^2(I)}.
\end{aligned}$$

The last step is the same as above. □

We are now in the position to prove the main extension theorem (Theorem 1.2.1) which we only consider on the reference triangle T. As a further result we get the existence of an extension from $L^2(\partial T)$ to $H^{1/2}(T)$ for an arbitrary polynomial f.

Theorem 1.2.4 *Let f be a continuous function on the reference triangle T such that $f_i := f|_{I_i} \in P^p(I_i)$, $i = 1, 2, 3$. Thus, there exists $U \in P^p(T)$ such that $U = f$ on ∂T and*

$$\|U\|_{H^{1/2}(T)} \le C\,(1+\log p)^{1/2}\,\|f\|_{L^2(\partial T)}. \tag{1.11}$$

If $f_2 = 0$ there holds

$$\|U\|_{\tilde H^{1/2}(T,I_2)} \le C\,(1+\log p)^{1/2}\|f\|_{L^2(\partial T)}. \tag{1.12}$$

If $f_2 = f_3 = 0$ then there holds

$$\|U\|_{\tilde H^{1/2}(T,I_2,I_3)} \le C\,(1+\log p)^{1/2}\|f\|_{L^2(\partial T)}. \tag{1.13}$$

Proof. The estimate (1.13) is shown in Theorem 1.2.3.

Next, we show (1.11). Therefore we extend f_1 by using the extension operator F. Let $U_1 := F(f_1)$ and let g_3 be its trace on I_3. Due to the continuity of f there holds $(f_3 - g_3)(0,0) = 0$. Using the extension operator E^3 we can extend $f_3 - g_3$ from I_3 to $U_3 \in P^p(T)$ with $U_3 = 0$ on I_1 and

$$\|U_3\|_{\tilde H^{1/2}(T,I_1)} \le C\,(1+\log p)^{1/2}\|f_3-g_3\|_{L^2(I_3)} \le C(1+\log p)^{1/2}\|f\|^2_{L^2(\partial T)}.$$

The last estimate follows using Lemma 1.3.2

$$\|g_3\|^2_{L^2(I_3)} = \|F(f_1)\|^2_{L^2(I_3)} = \int_0^1 \frac{1}{y^2}\left(\int_0^y f_1(t)\,dt\right)^2 dy \le 4\int_0^1 f_1(t)^2\,dt = 4\|f_1\|^2_{L^2(I_1)}.$$

Now let g_2 be the trace of U_1+U_3 on I_2. Thus, there holds $(f_2-g_2)(1,0) = (f_2-g_2)(0,1) = 0$. Using the extension operator E^2 we can extend $f_2 - g_2$ from I_2 to $U_2 \in P^p(T)$ with $U_2 = 0$ on I_1 and I_3 and

$$\|U_2\|_{\tilde{H}^{1/2}(T,I_1,I_3)} \leq C\,(1+\log p)^{1/2}\,\|f_2 - g_2\|_{L^2(I_2)} \leq C\,(1+\log p)^{1/2}\,\|f\|^2_{L^2(\partial T)}.$$

In the end, we have to estimate the $L^2(I_2)$-norm for

$$g_2 = (U_1 + U_3)|_{I_2} = (F(f_1) + E^3(f_3 - g_3))|_{I_2} = F(f_1)|_{I_2} + E^3\big(f_2 - F(f_1)|_{I_3}\big)|_{I_2}.$$

As the extension operators behave like E we use (1.17) to estimate

$$\|g_2\|^2_{L^2(I_2)} \leq \|f_1\|^2_{L^2(I_1)} + \|f_3\|^2_{L^2(I_3)}.$$

Finally, we set $U := U_1 + U_2 + U_3$.

In order to prove (1.12) let $U_1 := E^1_2(f_1)$ and let g_3 be its trace on I_3. Due to the continuity of f there holds $(g_3 - f_3)(1,0) = (g_3 - f_3)(0,1) = 0$. Using E^3 we extend $g_3 - f_3$ to a polynomial $U_3 \in P^p(T)$ with $U_3 = 0$ on I_1 and I_2 and

$$\|U_3\|_{\tilde{H}^{1/2}(T,I_1,I_2)} \leq C\,(1+\log p)^{1/2}\|g_3 - f_3\|_{L^2(I_3)} \leq C\,(1+\log p)^{1/2}\|f\|_{L^2(\partial T)}.$$

The last estimate can be shown as above. Finally, we set $U := U_1 - U_3$. □

1.3 Proof of Theorem 1.2.2

As described on page 19, the proof of Theorem 1.2.2 is done in several steps using the Theorems 1.3.4 and Theorem 1.3.7 which are proven below. First of all, we state some technical results.

Lemma 1.3.1 (Hardy's inequality) *For $p > 1$ and $r \neq 0$ there holds*

$$\int_0^\infty y^{-r}(F(y))^p\,dy \leq \left(\frac{p}{|r-1|}\right)^p \int_0^\infty y^{-r}(yf(y))^p\,dy,$$

where $F(y) = \int_y^\infty f(t)\,dt$ for $r < 1$, and $F(y) = \int_0^y f(t)\,dt$ for $r > 1$.

Proof. See Hardy *et al.* [54, Theorem 330]. □

Applying this we can easily estimate

Lemma 1.3.2 *Let $0 \leq x \leq 1$ and $f \in L^2(x,1)$. Thus, there holds*

$$\int_0^{1-x} \frac{1}{y^2}\left(\int_x^{x+y} f(t)\,dt\right)^2 dy \leq 4\int_x^1 f^2(t)\,dt. \tag{1.14}$$

Proof. We use Hardy's inequality with $r = 2$, $p = 2$ and

$$F(y) := \int_x^{x+y} f(t)\,dt = \int_0^y f(x+t)\,dt.$$

F is extended to 0 if $y > 1 - x$. Thus, we can estimate

$$\begin{aligned}
\int_0^{1-x} \frac{1}{y^2}\left(\int_x^{x+y} f(t)\,dt\right)^2 dy &= \int_0^{1-x} \frac{1}{y^2}\left(\int_0^{y} f(x+t)\,dt\right)^2 dy \\
\leq \int_0^{\infty} \frac{1}{y^2}\left(\int_0^{y} f(x+t)\,dt\right)^2 &\leq 4\int_0^{1-x} f^2(x+y)\,dy \\
&= 4\int_x^1 f^2(y)\,dy.
\end{aligned}$$

□

Next, we give estimates to the extension operator F.

Lemma 1.3.3 *There exists $C > 0$ such that for all $f \in H^{1/2}(I)$ there holds*

$$\|F(f)\|_{H^1(T)} \leq C\|f\|_{H^{1/2}(I)}, \tag{1.15}$$

$$\|F(f)\|_{L^2(T)} \leq C\|x^{1/2} f\|_{L^2(I)}. \tag{1.16}$$

Proof. (1.15) can be found in Babuška *et al.* [12, Lemma 7.1].
For the proof of (1.16) (compare Bică [21]) we use Lemma 1.3.2 to get

$$\begin{aligned}
\|F(f)\|^2_{L^2(T)} &= \int_0^1 \int_0^{1-x} \frac{1}{y^2}\left(\int_x^{x+y} f(t)\,dt\right)^2 dy\,dx \\
&\leq 4\int_0^1 \int_x^1 f(t)^2\,dt\,dx \\
&= 4\int_0^1 f(t)^2 \int_0^t dx\,dt \\
4&\int_0^1 t\,f(t)^2\,dt = 4\|t^{1/2} f\|^2_{L^2(I)}.
\end{aligned}$$

□

Theorem 1.3.4 *There exists a constant $C > 0$ such that for all $f \in P^p(I,0)$ there holds*

$$\|E(f)\|_{H^1(T)} \leq C\,\|f\|_{\tilde{H}^{1/2}(I,0)}.$$

Proof. The proof uses ideas of Muñoz-Sola [79, Lemma 6] where a similar result for the three-dimensional case is proven.
Using (1.16) we can estimate

$$\|E(f)\|^2_{L^2(T)} \leq \|F(|f|)\|^2_{L^2(T)} \leq C\|x^{1/2}\,f\|^2_{L^2(I)} \leq C\|f\|^2_{L^2(I)}. \tag{1.17}$$

In order to estimate the $H^1(T)$-seminorm we calculate the first order derivatives of E and F.

$$\frac{\partial E(f)}{\partial x} = \frac{1}{y}\int_x^{x+y} \frac{f(t)}{t}\,dt + \frac{x}{y}\left(\frac{f(x+y)}{x+y} - \frac{f(x)}{x}\right), \tag{1.18}$$

$$\frac{\partial E(f)}{\partial y} = -\frac{x}{y^2}\int_x^{x+y} \frac{f(t)}{t}\,dt + \frac{x}{y}\frac{f(x+y)}{x+y}, \tag{1.19}$$

$$\frac{\partial F(f)}{\partial x} = \frac{1}{y}f(x+y) - \frac{1}{y}f(x), \tag{1.20}$$

$$\frac{\partial F(f)}{\partial y} = -\frac{1}{y^2}\int_x^{x+y} f(t)\,dt + \frac{1}{y}f(x+y). \tag{1.21}$$

From (1.15) we now get the estimate

$$\|F(f)\|_{H^1(T)} \le C\|f\|_{H^{1/2}(I)}.$$

Thus, we bound the differences of the operators E and F in the $H^1(T)$-semi-norm. There holds

$$\frac{\partial E(f)}{\partial x} - \frac{\partial F(f)}{\partial x} = \frac{1}{y}\int_x^{x+y} \frac{f(t)}{t}\,dt + \frac{f(x+y)}{y}\left(\frac{x}{x+y} - 1\right). \tag{1.22}$$

We then define

$$R_1 := \frac{1}{y}\int_x^{x+y} \frac{f(t)}{t}\,dt, R_2 := \frac{f(x+y)}{y}\left(\frac{x}{x+y} - 1\right).$$

It is clear that

$$|R_1| = \left|\frac{1}{y}\int_x^{x+y} \frac{f(t)}{t}\,dt\right| \le \frac{1}{y}\int_x^{x+y} \frac{|f(t)|}{t}\,dt = F\left(\frac{|f(t)|}{t}\right).$$

Using (1.16) there holds

$$\|R_1\|_{L^2(T)} \le \left\|F\left(\frac{|f(t)|}{t}\right)\right\|_{L^2(T)} \le C\left\|x^{1/2}\frac{f(x)}{x}\right\|_{L^2(I)} = C\|x^{-1/2}f(x)\|_{L^2(I)}. \tag{1.23}$$

Next, we estimate R_2. Due to $0 \le 1 - \frac{x}{x+y} \le 1$ we get

$$\begin{aligned} |R_2| &= \left|\frac{f(x+y)}{y}\left(\frac{x}{x+y} - 1\right)\right| \\ &= \frac{1}{y}\left|f(x+y)\frac{y}{x+y}\right| \\ &\le \frac{1}{y}|f(x+y) - f(x)|\frac{y}{x+y} + \frac{|f(x)|}{x+y} \\ &\le \frac{1}{y}|f(x+y) - f(x)| + \frac{|f(x)|}{x+y}. \end{aligned}$$

The first term can be estimated by

$$\begin{aligned}\left\|\frac{1}{y}\big(f(x+y)-f(x)\big)\right\|^2_{L^2(T)} &= \int_0^1\int_0^{1-x}\frac{\big(f(x+y)-f(x)\big)^2}{y^2}\,dy\,dx\\ &= \int_0^1\int_x^1\frac{\big(f(z)-f(x)\big)^2}{(z-x)^2}\,dz\,dx\\ &\le \int_0^1\int_0^1\frac{\big(f(z)-f(x)\big)^2}{(z-x)^2}\,dz\,dx\\ &= |f|^2_{H^{1/2}(I)}.\end{aligned}$$

For the second term there holds

$$\int_0^1\int_0^{1-x}\frac{f(x)^2}{(x+y)^2}\,dy\,dx \le \int_0^1\frac{f(x)^2}{x}\,dx = \|x^{-1/2}f(x)\|^2_{L^2(I)}.$$

Finally, there holds

$$\|R_2\|^2_{L^2(T)} \le C\left(|f|^2_{H^{1/2}(I)} + \|x^{-1/2}f(x)\|^2_{L^2(I)}\right). \tag{1.24}$$

Now we examine the derivative with respect to y.

$$\frac{\partial E(f)}{\partial y} - \frac{\partial F(f)}{\partial y} = \frac{1}{y^2}\int_x^{x+y} f(t)\left(1-\frac{x}{t}\right)dt + \frac{f(x+y)}{y}\left(\frac{x}{x+y}-1\right) = R_3 + R_2,$$

where we define

$$R_3 := \frac{1}{y^2}\int_x^{x+y} f(t)\left(1-\frac{x}{t}\right)dt.$$

The term R_2 has already been estimated. Thus, we only have to examine R_3. Due to $0 \le 1-\frac{x}{t} \le \frac{y}{t}$ for $t\in[x,x+y]$ we can estimate

$$|R_3| \le \frac{1}{y}\int_x^{x+y}\frac{|f(t)|}{t}\,dt = F\left(\frac{|f(t)|}{t}\right)$$

which can be estimated like R_1 and it follows

$$\|R_3\|_{L^2(T)} \le C\|x^{-1/2}f(x)\|_{L^2(I)}. \tag{1.25}$$

Using (1.15) together with (1.17), (1.23), (1.24) and (1.25) the theorem follows. □

For the proof of Theorem 1.3.7 we need the following two technical lemmas.

Lemma 1.3.5 *Let f be uniformly continuous on $[0,1]$. Thus, for fixed $y\in[0,1]$ the function $\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}$ is continuous with respect to $x\in[0,1-y]$.*

Proof. f is uniformly continuous. Therefore, for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all x_1, $x_2 \in [0, 1-y]$ with $|x_1 - x_2| < \delta$ there holds $|f(x_1 + x) - f(x_2 + x)| < \varepsilon$ for all $x \in [0, y]$. Thus, we can estimate

$$\begin{aligned}\big|\|f\|_{H^{-1/2+\varepsilon}(x_1,x_1+y)} - \|f\|_{H^{-1/2+\varepsilon}(x_2,x_2+y)}\big| &= \big|\|f(x_1+\cdot)\|_{H^{-1/2+\varepsilon}(0,y)} - \|f(x_2+\cdot)\|_{H^{-1/2+\varepsilon}(0,y)}\big| \\ &\le \|f(x_1+\cdot) - f(x_2+\cdot)\|_{H^{-1/2+\varepsilon}(0,y)} \\ &\le \|f(x_1+\cdot) - f(x_2+\cdot)\|_{L^2(0,y)} \\ &\le \varepsilon\sqrt{y}.\end{aligned}$$

This gives the desired result. □

Lemma 1.3.6 *For $f \in P^p(I)$ and $\varepsilon > 0$ there holds*

$$\big\|y^{\varepsilon-1}\,\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}\big\|_{L^2(T)} \le C\,\frac{1}{\sqrt{2\varepsilon}}\|f\|_{H^{-1/2+\varepsilon}(I)}$$

Proof. For $y \in [0,1]$ the function $\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}$ is continuous with respect to $x \in [0, 1-y]$, see Lemma 1.3.5. Therefore it is Riemann-integrable in x. Thus, we calculate the integral $\int_0^{1-y} \|f\|^2_{H^{-1/2+\varepsilon}(x,x+y)}\,dx$ as the limit of Riemann-sums. Therefore we define on the interval $[0, 1-y]$ for $N_h \in \mathbb{N}$ the points

$$x_i := i\frac{1-y}{N_h}, \quad i = 0, \ldots, N_h, \quad h := \frac{1-y}{N_h}.$$

Thus, there holds

$$\int_0^{1-y} \|f\|^2_{H^{-1/2+\varepsilon}(x,x+y)}\,dx = \lim_{h\to 0}\sum_{i=0}^{N_h-1} h\|f\|^2_{H^{-1/2+\varepsilon}(x_i,x_i+y)}$$

Every interval $(x_i, x_i + y) = (ih, ih + y)$, $i = 0, N_h - 1$, overlaps at most with $\mathcal{O}(\frac{y}{h})$ intervals. Therefore we can use a coloring argument to estimate

$$\lim_{h\to 0}\sum_{i=0}^{N_h-1} h\|f\|^2_{H^{-1/2+\varepsilon}(x_i,x_i+y)} \le C\lim_{h\to 0} h\,\frac{y}{h}\|f\|^2_{H^{-1/2+\varepsilon}(0,1)} = C\,y\|f\|^2_{H^{-1/2+\varepsilon}(0,1)}.$$

Here, we have used that $\sum_{i=0}^{N_h-1}\|f\|^2_{H^{-1/2+\varepsilon}(x_i,x_i+y)} \le C\|f\|^2_{H^{-1/2+\varepsilon}(0,1)}$, cf. v.Petersdorff [90].

Finally, there holds

$$\begin{aligned}&\big\|y^{\varepsilon-1}\,\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}\big\|^2_{L^2(T)} \\ &= \int_0^1 y^{2\varepsilon-2}\int_0^{1-y}\|f\|^2_{H^{-1/2+\varepsilon}(x,x+y)}\,dx\,dy \\ &\le C\int_0^1 y^{2\varepsilon-2}\,y\|f\|^2_{H^{-1/2+\varepsilon}(0,1)}\,dy \\ &= C\,\|f\|^2_{H^{-1/2+\varepsilon}(0,1)}\int_0^1 y^{2\varepsilon-1}\,dy \\ &= C\,\frac{1}{2\varepsilon}\|f\|^2_{H^{-1/2+\varepsilon}(I)}.\end{aligned}$$

□

Theorem 1.3.7 *There exists a constant $C > 0$, independent of p, such that there holds*

$$\|E(f)\|_{L^2(T)} \leq C(1 + \log p)\,\|f\|_{H^{-1/2}(I)}. \tag{1.26}$$

Proof. Let $\varepsilon \in (0, 1/4)$ and $(x, y) \in T$, $x > 0$. Thus, there holds due to the duality of the spaces $H^{-1/2+\varepsilon}(x, x+y)$ and $\tilde{H}^{1/2-\varepsilon}(x, x+y)$

$$\int_x^{x+y} \frac{f(t)}{t}\,dt \leq \|t^{-1}\|_{\tilde{H}^{1/2-\varepsilon}(x,x+y)} \|f\|_{H^{-1/2+\varepsilon}(x,x+y)}. \tag{1.27}$$

The $\tilde{H}^{1/2-\varepsilon}(x, x+y)$-norm can be bounded by

$$\begin{aligned}\|t^{-1}\|^2_{\tilde{H}^{1/2-\varepsilon}(x,x+y)} &= |t^{-1}|^2_{H^{1/2-\varepsilon}(x,x+y)} + \int_x^{x+y} \frac{t^{-2}}{\operatorname{dist}(t; x, x+y)^{2(1/2-\varepsilon)}}\,dt \\ &\lesssim |t^{-1}|^2_{H^{1/2-\varepsilon}(x,x+y)} + \int_x^{x+y} \frac{t^{-2}}{|t-x|^{1-2\varepsilon}}\,dt + \int_x^{x+y} \frac{t^{-2}}{|t-x-y|^{1-2\varepsilon}}\,dt.\end{aligned} \tag{1.28}$$

In order to estimate the $H^{1/2-\varepsilon}(x, x+y)$-semi-norm we calculate

$$\|t^{-1}\|^2_{L^2(x,x+y)} = \frac{y}{x(x+y)}$$

and

$$|t^{-1}|^2_{H^1(x,x+y)} = \frac{x^2y + xy^2 + \frac{1}{3}y^3}{x^3(x+y)^3} \leq \frac{y}{x^3(x+y)}.$$

Thus, there holds due to an interpolation inequality, see Bergh & Löfström [18],

$$\begin{aligned}|t^{-1}|_{H^{1/2-\varepsilon}(x,x+y)} &\lesssim \|t^{-1}\|^{1/2+\varepsilon}_{L^2(x,x+y)} \|t^{-1}\|^{1/2-\varepsilon}_{H^1(x,x+y)} \\ &\leq \left(\frac{y^{1/2+\varepsilon}}{x^{1/2+\varepsilon}(x+y)^{1/2+\varepsilon}} \frac{y^{1/2-\varepsilon}}{x^{3/2-3\varepsilon}(x+y)^{1/2-\varepsilon}}\right)^{1/2} \\ &= \frac{y^{1/2}}{\sqrt{x^{2-2\varepsilon}(x+y)}} = \frac{y^{1/2}x^{\varepsilon}}{x(x+y)^{1/2}} \\ &\leq \frac{y^{1/2}x^{\varepsilon}}{x^{3/2}}.\end{aligned} \tag{1.29}$$

The first integral of (1.28) can be estimated by

$$\begin{aligned}\int_x^{x+y} \frac{t^{-2}}{|t-x|^{1-2\varepsilon}}\,dt &\leq \frac{1}{x^2}\int_x^{x+y} \frac{1}{(t-x)^{1-2\varepsilon}}\,dt = \frac{1}{x^2}\left[\frac{(t-x)^{2\varepsilon}}{2\varepsilon}\right]_{t=x}^{x+y} \\ &= \frac{1}{x^2}\frac{y^{2\varepsilon}}{2\varepsilon}.\end{aligned} \tag{1.30}$$

For the second integral we calculate

$$\begin{aligned}\int_x^{x+y} \frac{t^{-2}}{|t-x-y|^{1-2\varepsilon}}\,dt &\leq \frac{1}{x^2}\int_x^{x+y} \frac{1}{(x+y-t)^{1-2\varepsilon}}\,dt = \frac{1}{x^2}\left[-\frac{(x+y-t)^{2\varepsilon}}{2\varepsilon}\right]_{t=x}^{x+y} \\ &= \frac{1}{x^2}\frac{y^{2\varepsilon}}{2\varepsilon}.\end{aligned} \tag{1.31}$$

Therefore, we get from (1.27) together with (1.28), (1.29), (1.30) and (1.31)

$$\begin{aligned}\int_x^{x+y} \frac{f(t)}{t}\,dt &\leq \|t^{-1}\|_{\tilde{H}^{1/2-\varepsilon}(x,x+y)}\,\|f\|_{H^{-1/2+\varepsilon}(x,x+y)} \\ &\lesssim \left(\frac{y^{1/2}x^{\varepsilon}}{x^{3/2}} + \frac{1}{\sqrt{\varepsilon}}\frac{y^{\varepsilon}}{x}\right)\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}.\end{aligned}$$

Thus, there holds for $f \in P^p(I,0)$

$$\begin{aligned}\|E(f)\|_{L^2(T)} &\lesssim \left\|\left(x^{\varepsilon-1/2}y^{-1/2} + \frac{1}{\sqrt{\varepsilon}}y^{\varepsilon-1}\right)\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}\right\|_{L^2(T)} \\ &\leq \left\|x^{\varepsilon-1/2}y^{-1/2}\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}\right\|_{L^2(T)} + \frac{1}{\sqrt{\varepsilon}}\left\|y^{\varepsilon-1}\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}\right\|_{L^2(T)}. \qquad (1.32)\end{aligned}$$

The factor $\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}$ in the first part can be estimated as follows using scalings of the norms and transformation forward and backward to the interval $(x, x+1)$, compare Heuer [58],

$$\begin{aligned}\|f\|^2_{H^{-1/2+\varepsilon}(x,x+y)} &\leq Cy^{1-2(-1/2+\varepsilon)}\|\tilde{f}\|^2_{H^{-1/2+\varepsilon}(x,x+1)} \\ &= Cy^{2\varepsilon}(y^{2-4\varepsilon})\|\tilde{f}\|^2_{H^{-1/2+\varepsilon}(x,x+1)} \\ &\leq Cy^{2\varepsilon}(y^{2-4\varepsilon})\|\tilde{f}\|^2_{H^{-1/2+2\varepsilon}(x,x+1)} \\ &\leq Cy^{2\varepsilon}(y^{2-4\varepsilon})\frac{1}{y^{1-2(-1/2+2\varepsilon)}}\|f\|^2_{H^{-1/2+2\varepsilon}(x,x+y)} \\ &= Cy^{2\varepsilon}\|f\|^2_{H^{-1/2+2\varepsilon}(x,x+y)} \\ &\leq Cy^{2\varepsilon}\|f\|^2_{H^{-1/2+2\varepsilon}(I)}\end{aligned}$$

and there holds

$$\begin{aligned}\left\|x^{-1/2+\varepsilon}y^{-1/2}\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}\right\|_{L^2(T)} &\leq C\|x^{-1/2+\varepsilon}y^{-1/2+\varepsilon}\|_{L^2(T)}\cdot\|f\|_{H^{-1/2+2\varepsilon}(I)} \\ &\leq C\|x^{-1/2+\varepsilon}\|_{L^2(I)}\cdot\|y^{-1/2+\varepsilon}\|_{L^2(I)}\cdot\|f\|_{H^{-1/2+2\varepsilon}(I)} \\ &\leq C\frac{1}{2\varepsilon}\|f\|_{H^{-1/2+2\varepsilon}(I)}\end{aligned}$$

The second term of (1.32) can be estimated using Lemma 1.3.6. Finally, we get

$$\begin{aligned}\|E(f)\|_{L^2(T)} &\leq C\left(\frac{1}{\varepsilon}\|f\|_{H^{-1/2+2\varepsilon}(I)} + \frac{1}{\varepsilon}\|f\|_{H^{-1/2+\varepsilon}(I)}\right) \\ &\leq C\frac{1}{\varepsilon}\left(p^{4\varepsilon}+p^{2\varepsilon}\right)\|f\|_{H^{-1/2}(I)} \\ &\leq C(1+\log p)\,\|f\|_{H^{-1/2}(I)}.\end{aligned}$$

Where we have used the inverse inequality for $f \in P^p(I)$, see e.g. Heuer [58, Lemma 4], and have chosen in the last step $\epsilon := (\log p)^{-1}$. This gives the result of the Theorem. □

2 An iterative substructuring method for the hp-version of the BEM on quasi-uniform triangular meshes

In this chapter we give the analysis for an iterative substructuring method for the hp-version of the boundary element method (BEM) with the hypersingular operator in $\mathbb{R}^3$, thus acting on surfaces, considering quasi-uniform triangular meshes.

Our substructuring method uses the so-called wire basket space (consisting of nodal and side basis functions) with L^2-bilinear form and, for each triangle, the space of bubble functions on that element with energy bilinear form (defined by the integral operator). Main technical details therefore deal with traces and extensions for polynomials acting between L^2 on sides of triangles and $\tilde{H}^{1/2}$ (the energy space of the hypersingular operator) on triangles. Such traces and extensions, for tetrahedral meshes and the FEM, have been analyzed by Muñoz-Sola in [79]. Essential tool is an appropriate extension operator. The counterpart of this operator in $\mathbb{R}^2$ for triangles, in combination with the discrete harmonic extension, has been used by Bică in his PhD-thesis [21] to study iterative substructuring methods for the p-version of the FEM in $\mathbb{R}^3$ with tetrahedral meshes. In Chapter 1 we have analyzed this operator and have proved the stability of the extension, filling the gaps in the proofs of Bică. Based on this extension operator in $\mathbb{R}^2$ for triangles, we analyze the p-version of the BEM on surfaces with triangular meshes. The technical tools are quite similar as in [21] for the FEM, where in [21], however, several details and proofs were left open.

Our model problem is the hypersingular integral equation

$$a(u,v) := \langle Du, v\rangle_\Gamma = \langle f, v\rangle_\Gamma \quad \text{for all} \quad v \in \tilde{H}^{1/2}(\Gamma) \tag{2.1}$$

on a plane polygonal surface segment $\Gamma \subset \mathbb{R}^3$ where $f \in H^{-1/2}(\Gamma)$ is a given function. Here, D is the hypersingular integral operator

$$Du(x) = -\frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,dS_y, \quad x \in \Gamma,$$

which is a continuous and positive definite mapping from $\tilde{H}^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$, cf. [98]. Hence, there holds the equivalence of norms

$$\langle Dv, v\rangle_\Gamma \simeq \|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \quad \text{for all} \quad v \in \tilde{H}^{1/2}(\Gamma). \tag{2.2}$$

The Galerkin scheme for (2.1) reads as follows. Given a finite dimensional subspace $\Psi \subset \tilde{H}^{1/2}(\Gamma)$ with $\dim \Psi = N$, find $u_N \in \Psi$ such that

$$\langle Du_N, v\rangle_\Gamma = \langle f, v\rangle_\Gamma \quad \text{for all} \quad v \in \Psi. \tag{2.3}$$

The space Ψ under consideration consists of continuous piecewise polynomials of degree p on regular quasi-uniform meshes formed by triangles. Our iterative substructuring method defines a preconditioner for the stiffness matrix A of system (2.3). Equivalently, the method results in a preconditioned stiffness matrix which can be considered as the additive Schwarz operator P corresponding to the underlying subspace decomposition with given bilinear forms. The main result of this chapter states that the condition number of the preconditioned matrix P is bounded by $\mathcal{O}((1 + \log p)^4)$ with a constant that is independent of the mesh parameter h.

The outline of this chapter is as follows. In §2.1 we define basis functions for the p-version and a decomposition of the ansatz space, and state the main result (Theorem 2.1.1). For the definition of the basis functions we need special extension operators which are presented in Chapter 1. In §2.2 we prove several technical lemmas. The proof of the main result is given in §2.3. Finally, in §2.4 we present some numerical experiments which underline the asymptotic behavior of the preconditioner. For the convenience of the reader we collect some technical results from other authors in Section 2.5. In particular, we indicate proofs of some of Bică's results which are used in here.

2.1 Basis functions and preconditioners

First of all, we consider the construction of basis functions for the p-version. To this end we will use extension operators as described below.

Extensions can be defined locally on patches of elements. For the extension of basis functions associated with edges (so-called edge basis functions) the situation is as indicated in Figure 2.1(a). A polynomial f defined on the edge I vanishes at the endpoints of I and needs to be extended to a piecewise polynomial U on $K := T_1 \cup T_2$ such that it can be extended continuously by zero onto an enlarged patch $\tilde{K}$ which contains K.

For functions associated with nodes (nodal functions) the situation is analogous. Namely, for a given patch as in Figure 2.1(b), values on the skeleton of the edges of the patch are given including 1 in the center node and 0 on the boundary. As for edge functions these values are extended locally onto the triangles, see the construction on the reference triangle below.

For our analysis we explicitly consider the situation on the reference triangle $T := \{(x, y);\ x \geq 0,\ y \geq 0,\ x + y \leq 1\}$. The vertices and edges of T are denoted by V_i and I_i, $i = 1, 2, 3$, respectively, see Figure 2.2. The edges I_1 and I_3 will be identified with the

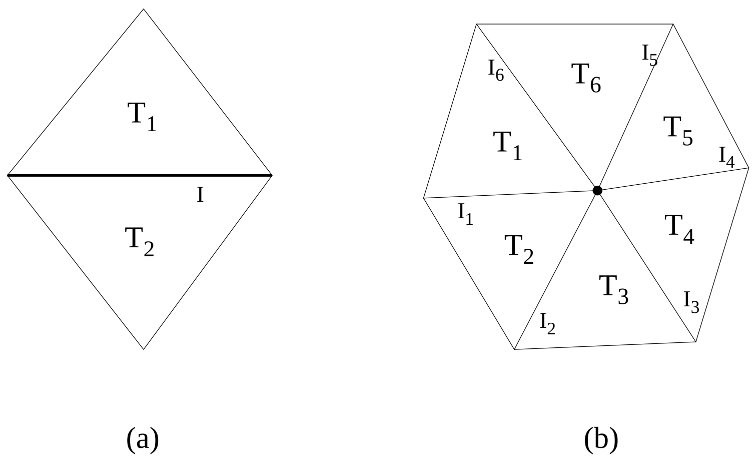

Figure 2.1: Constructing edge and nodal basis functions by extension.

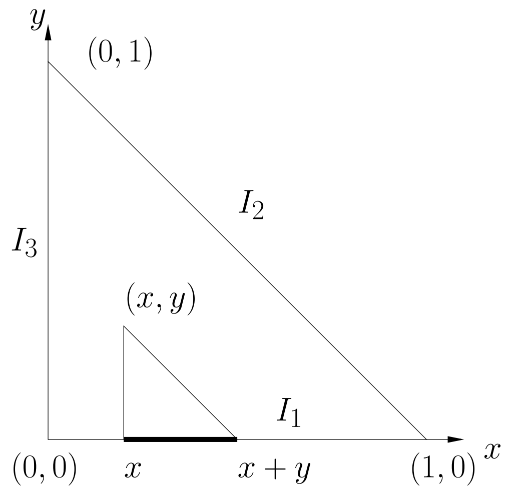

Figure 2.2: The reference triangle T.

Interval $I := (0, 1)$, and $I = I_1$ will be used without further notice. We also need the polynomial spaces

$$P^p(I) := \text{span}\{x^i,\, 0 \leq i \leq p\}, \qquad P^p(T) := \text{span}\{x^i y^j,\, 0 \leq i + j \leq p\}.$$

For the construction of our basis functions we need the extension operator which are defined in Chapter 1. For the construction of vertex basis functions we then consider special low energy functions, cf. Pavarino & Widlund [89]. Let ϕ_0 be the polynomial of degree p that minimizes the $L^2(0,1)$-norm and satisfies $\phi_0(0) = 1$ and $\phi_0(1) = 0$. The corresponding polynomial satisfying $\phi_0(0) = 0$ and $\phi_0(1) = 1$ is denoted by $\phi_0^-(x) = \phi_0(1-x)$.

These polynomials are L^2-orthogonal to $P_0^p(0,1)$ (the polynomials with homogeneous boundary values), and there holds

$$\|\phi_0\|^2_{L^2(0,1)} = 1/(p^2+p) \text{ and } (\phi_0, \phi_0^-)_{L^2(0,1)} = \frac{(-1)^{p+1}}{2(p+1)}\|\phi_0\|^2_{L^2(0,1)}, \tag{2.4}$$

see [89]. The expansion of such polynomials as a linear combination of Legendre polynomials is also given in [89]. For illustration see Figure 2.3 where ϕ_0 for $p = 10$ is given.

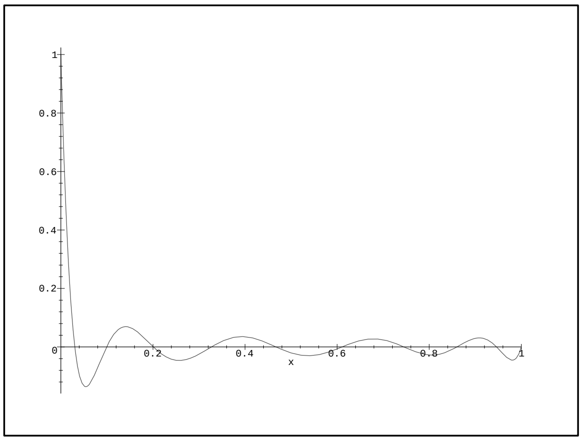

Figure 2.3: Low energy function ϕ_0 for $p = 10$.

A vertex basis function $\tilde{\phi}_{V_1}$, e.g. for vertex V_1, is defined as follows. Set $\tilde{\phi}_{V_1} = \phi_0$ on I_1 and I_3, and $\tilde{\phi}_{V_1} = 0$ on I_2. Extend $\tilde{\phi}_{V_1}$ from I_1 onto T by using the extension operator E_2^1, $\psi_1 := E_2^1\tilde{\phi}_{V_1} = E_2^1\phi_0$. Let g_3 be the trace of ψ_1 on I_3 and define $\psi_3 := E^3(g_3 - \tilde{\phi}_{V_1})$, the extension of $g_3 - \tilde{\phi}_{V_1}$ from I_3 onto T with $\psi_3 = 0$ on I_1 and I_2. Eventually we set $\tilde{\phi}_{V_1} := \psi_1 - \psi_3$. The other vertex functions are defined analogously.

As basis for the edges we use affine images of antiderivatives of Legendre polynomials that vanish in the corners. The antiderivatives of the Legendre polynomials are defined on the interval $[-1, 1]$ by

$$\mathcal{L}_0(x) := \frac{1-x}{2}, \quad \mathcal{L}_1(x) := \frac{1+x}{2}, \quad \mathcal{L}_n(x) := \frac{L_n(x) - L_{n-2}(x)}{2n-1} = \int_{-1}^{x} L_{n-1}(y)\,dy,$$

where L_n denotes the Legendre polynomial of degree n. These basis functions are extended onto the triangle using the extension operators E^i, $i = 1, 2, 3$. There are $p-1$ basis functions on each edge.

As interior (or bubble) functions we use tensor products of antiderivatives of Legendre polynomials. On the reference triangle T these are the functions

$$\phi_{k,l}(x,y) = \frac{\mathcal{L}_{k+1}(2x-1)}{1-x}\frac{\mathcal{L}_l(2y-1)}{1-y}(1-x-y), \quad 1 \le k,\ 2 \le l,\ k+l \le p.$$

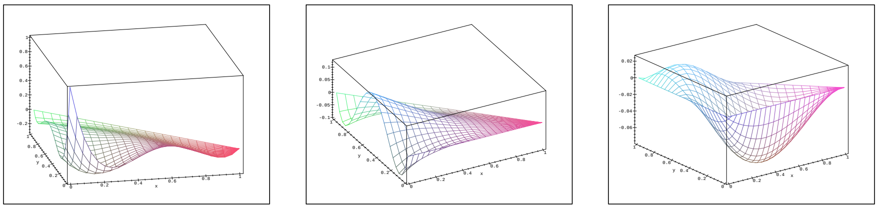

Figure 2.4: Basis functions, $p = 4$.

There are $(p-1)(p-2)/2$ interior functions per triangle.

For a sample set of nodal, edge and interior basis functions see Figure 2.4.

For a given triangle $\tilde{T}$, affine transformations of the basis functions defined above are used to span the polynomial space $P^p(\tilde{T})$. Given $u \in P^p(\tilde{T})$ this function has the unique representation $u = \sum_{i=1}^{3} \tilde{u}_{V_i} + \sum_{i=1}^{3} \tilde{u}_{I_i} + \tilde{u}_{\tilde{T}}$ where $\tilde{u}_{V_i}$ $\tilde{u}_{I_i}$ and $\tilde{u}_{\tilde{T}}$ are the vertex, edge and interior components, respectively. An interpolation operator $\tilde{I}^W$ onto the space of wire basket functions is defined by

$$\tilde{I}^W u := \sum_{i=1}^{3} \tilde{u}_{V_i} + \sum_{i=1}^{3} \tilde{u}_{I_i}. \tag{2.5}$$

Since the space of wire basket functions does not contain constants on $\tilde{T}$ we redefine the vertex and edge functions as follows. Let $\mathcal{F}$ denote the part of the expansion of the constant function 1 which belongs to the interior functions, i.e. $\mathcal{F} := 1 - \tilde{I}^W 1$. Then, we define a new interpolation operator by

$$I^W u := \tilde{I}^W u + \mathcal{F}\overline{u}_W, \tag{2.6}$$

where $\overline{u}_W := \frac{\int_{\partial\tilde{T}} u}{\int_{\partial\tilde{T}} 1}$. This operator maps a constant function onto itself. The new vertex and edge components of u for the changed basis functions are denoted by u_{V_i} and u_{E_i}, $i = 1, \ldots, 3$. They are images under I^W of the preliminary components $\tilde{u}_{V_i}$ and $\tilde{u}_{E_i}$. The interior basis functions are unchanged.

Now, in order to define the boundary element space Ψ we introduce a quasi-uniform mesh $\overline{\Gamma} = \cup_{i=1}^{n} \overline{\Gamma}_i$ consisting of triangles Γ_i and define

$$\Psi := S_h^p := \{u \in \mathcal{C}^0(\Gamma); u|_{\Gamma_i} \in P^p(\Gamma_i)\} \subset \tilde{H}^{1/2}(\Gamma).$$

Here, h denotes the maximum diameter of the elements of the mesh. In a standard way we utilise the local basis functions defined above to generate a basis for S_h^p. In particular we use the notation for components in (2.5) and the wire basket interpolation operator in (2.6) for the global setting. Additionally, W denotes the wire basket of the mesh, i.e. the union of nodes and edges.

Next, we introduce a preconditioner in the additive Schwarz framework. For simplicity we consider the situation that $\Gamma \subset \mathbb{R}^3$ is a surface piece. In fact, the case of a closed surface Γ is implicitly covered by our theory without any complication. The analysis for open surfaces is more involved since in this case the energy space of hypersingular operators must incorporate homogeneous boundary conditions.

The additive Schwarz preconditioner is based upon a subspace decomposition

$$S_h^p(\Gamma) = H_0 + H_1 + \cdots + H_n.$$

For our method we choose $H_0 := \Psi_W(\Gamma)$ being the space of wire basket functions and H_j consisting of the interior functions on Γ_j, $j = 1, \ldots, n$. Accordingly any $u \in S_n^p$ has a unique representation

$$u = u_W + \sum_{i=1}^{n} u_{\Gamma_i}, \tag{2.7}$$

where $u_W \in \Psi_W(\Gamma)$ and u_{Γ_i} are the interior functions with support in Γ_i.

Thus, the **additive Schwarz method** reads: Solve

$$Pu_N := (P_0 + P_1 + \cdots + P_n)u_N = f_N$$

where $P_j : S_h^p(\Gamma) \to H_j$, $j = 0, \ldots, n$, are projection operators defined by

$$b_j(P_j v, \varphi) = \langle Dv, \varphi \rangle_\Gamma \qquad \forall\, \varphi \in H_j.$$

For the interior spaces $H_1, \ldots, H_n$, b_0 is the energy bilinear form

$$b_j(v, w) := a(v, w) = \langle Dv, w \rangle_\Gamma, \qquad v, w \in H_j,\ j = 1, \ldots, n, \tag{2.8}$$

and for the wire basket space Ψ_W we consider two bilinear forms b_0. For our first method we choose

$$b_0(u, u) := \hat{a}_W(u, u) := (1 + \log p)^3 \sum_{i=1}^{n} \inf_{c_i \in \mathbb{R}} \|u - c_i\|^2_{L^2(W_i)}, \tag{2.9}$$

where W_i denotes the boundary of Γ_i. The corresponding additive Schwarz operator P will be denoted by P_W.

For the second method we use the energy bilinear form,

$$b_0(v, w) := a(v, w) = \langle Dv, w \rangle_\Gamma, \qquad v, w \in \Psi_W(\Gamma). \tag{2.10}$$

In this case we denote the additive Schwarz operator by $P = P_D$.

The main result of this Chapter is the following theorem.

Theorem 2.1.1 *Let b_0 denote one of the bilinear forms $\hat{a}_W$ or a. Then, for any $u = u_W + \sum_{i=1}^n u_{\Gamma_i} \in S_h^p(\Gamma)$, there holds*

$$\begin{aligned} C_0\,(1+\log p)^{-4}\Big(b_0(u_W,u_W)+\sum_{i=1}^n a(u_{\Gamma_i},u_{\Gamma_i})\Big) \\ \le a(u,u) \qquad (2.11) \\ \le C_1\Big(b_0(u_W,u_W)+\sum_{i=1}^n a(u_{\Gamma_i},u_{\Gamma_i})\Big). \end{aligned}$$

Here, the constants C_0, $C_1 > 0$ are independent of h, p and u. Therefore, the minimum and maximum eigenvalues of the additive Schwarz operator P ($P = P_W$ if $b_0 = \hat{a}_W$ or $P = P_D$ if $b_0 = a$) are bounded like

$$\lambda_{min}(P) \ge C_0\,(1+\log p)^{-4}, \quad \lambda_{max}(P) \le C_1,$$

and the condition number satisfies with a constant $C > 0$, independent of h and p,

$$\kappa(P) = \frac{\lambda_{max}(P)}{\lambda_{min}(P)} \le C\,(1+\log p)^4.$$

The bounds on the eigenvalues of P are immediate implications of the inequalities (2.11), see, e.g., Zhang [108]. The inequalities are proved by Theorems 2.3.1, 2.3.2, 2.3.3 in Section 2.3.

2.2 Technical tools

In this section we collect some technical lemmas which are needed to prove our main result (Theorem 2.1.1).

Lemma 2.2.1 *Let I be one side of the reference triangle T. Then, for any polynomial v of degree p on T there holds*

$$\|v\|^2_{L^2(I)} \le C(1+\log p)\|v\|^2_{H^{1/2}(T)}.$$

Let $\overline{v}_{W_T} := \frac{1}{|\partial T|}\int_{\partial T} v\,ds$. Then, there holds

$$\|v-\overline{v}_{W_T}\|^2_{L^2(\partial T)} \le C(1+\log p)|v|^2_{H^{1/2}(T)}. \qquad (2.12)$$

Proof. Let Q denote the reference square $(0,1)\times(0,1)$ and $I = (0,1)$. For a polynomial u of degree p there holds

$$\begin{aligned} \|u\|^2_{L^2(I)} &= \int_0^1 u(x,y=0)^2\,dx \le \int_0^1 \|u(x,\cdot)\|^2_{L^\infty(0,1)}\,dx \\ &\le C(1+\log p)\int_0^1 \|u(x,\cdot)\|^2_{H^{1/2}(0,1)}\,dx. \end{aligned} \qquad (2.13)$$

The last estimate is due to Theorem 6.2 in Babuška *et al.* [12].
For the special case of a square we use an equivalent definition for the $H^{1/2}$-semi-norm (see Lemma 5.3, Chap. 2, in Nečas [80])

$$|u|^2_{H^{1/2}(Q)} \cong \int_0^1\int_0^1 \frac{\|u(s_1,\cdot)-u(t_1,\cdot)\|^2_{L^2(0,1)}}{(s_1-t_1)^2}\,ds_1\,dt_1 + \int_0^1\int_0^1 \frac{\|u(\cdot,s_2)-u(\cdot,t_2)\|^2_{L^2(0,1)}}{(s_2-t_2)^2}\,ds_2\,dt_2.$$

Therefore, we can estimate

$$\begin{aligned}\int_0^1 |u(x,\cdot)|^2_{H^{1/2}(0,1)}\,dx &= \int_0^1\int_0^1\int_0^1 \frac{(u(x,y_1)-u(x,y_2))^2}{(y_1-y_2)^2}\,dy_1\,dy_2\,dx \\ &= \int_0^1\int_0^1 \frac{\|u(\cdot,y_1)-u(\cdot,y_2)\|^2_{L^2(0,1)}}{(y_1-y_2)^2}\,dy_1\,dy_2 \\ &\leq C|u|^2_{H^{1/2}(Q)}. \end{aligned} \tag{2.14}$$

Furthermore, it is

$$\int_0^1 \|u(x,\cdot)\|^2_{L^2(0,1)}\,dx = \|u\|^2_{L^2(Q)}.$$

Combining this relation with (2.13) and (2.14) we obtain

$$\|u\|^2_{L^2(I)} \leq C(1+\log p)\|u\|^2_{H^{1/2}(Q)}.$$

Now for the reference triangle T we extend the function v from T onto the reference square Q by reflecting it at I_2. The reflected function on the reflected triangle $\tilde{T}$ is denoted by $\tilde{v}$. By symmetry the coupling term between v and $\tilde{v}$ in the $H^{1/2}$-norm vanishes. Therefore we deduce that there holds

$$\begin{aligned}\|v\|^2_{L^2(I)} &\leq C(1+\log p)\|v\|^2_{H^{1/2}(Q)} \\ &\leq C(1+\log p)\left(|v|^2_{H^{1/2}(T)} + |\tilde{v}|^2_{H^{1/2}(\tilde{T})} + \|v\|^2_{L^2(T)} + \|\tilde{v}\|^2_{L^2(\tilde{T})}\right) \\ &= 2C(1+\log p)\|v\|^2_{H^{1/2}(T)}.\end{aligned}$$

In order to prove (2.12) we use the minimizing property of $\overline{v}_{W_T}$ and a quotient space argument as follows:

$$\begin{aligned}\|v-\overline{v}_{W_T}\|^2_{L^2(\partial T)} &\leq \|v-c\|^2_{L^2(\partial T)} = \sum_{i=1}^3 \|v-c\|^2_{L^2(I_i)} \\ &\leq C(1+\log p)\|v-c\|^2_{H^{1/2}(T)} \qquad \forall\, c\in\mathbb{R}.\end{aligned}$$

Therefore,

$$\|v-\overline{v}_{W_T}\|^2_{L^2(\partial T)} \leq C(1+\log p)\inf_{c\in\mathbb{R}}\|v-c\|^2_{H^{1/2}(T)} \leq C(1+\log p)|v|^2_{H^{1/2}(T)}.$$

□

Lemma 2.2.2 *Let $\tilde{T}$ be a triangle of diameter h and $u \in H^{1/2}(\tilde{T})$. Then, the mean value $\overline{u}_{W_{\tilde{T}}} = \frac{1}{|\partial\tilde{T}|}\int_{\partial\tilde{T}} u\,ds$ of u on the boundary of $\tilde{T}$ can be bounded by*

$$\overline{u}_{W_{\tilde{T}}}^2 \leq C\,h^{-1}\|u\|^2_{L^2(\partial\tilde{T})}.$$

On the reference triangle T there holds for $u \in P^p(T)$

$$\overline{u}_{W_T}^2 \leq C(1+\log p)\|u\|^2_{H^{1/2}(T)}.$$

Proof. Using the Cauchy-Schwarz inequality we get

$$\overline{u}_{W_{\tilde{T}}}^2 \leq \frac{1}{|\partial\tilde{T}|^2}\int_{\partial\tilde{T}} 1\,ds \int_{\partial\tilde{T}} u^2\,ds \cong C\,h^{-1}\|u\|^2_{L^2(\partial\tilde{T})},$$

which is the first assertion of the lemma. Analogously on the reference triangle T we have

$$\overline{u}_{W_T}^2 \leq C\,\|u\|^2_{L^2(\partial T)}$$

and using Lemma 2.2.1 we obtain the second assertion. □

Lemma 2.2.3 *For a polynomial f of degree p which vanishes on the boundary of T there holds*

$$\|f\|_{\tilde{H}^{1/2}(T)} \leq C(1+\log p)\|f\|_{H^{1/2}(T)}. \tag{2.15}$$

Proof. See Lemma 6 in [58]. □

Lemma 2.2.4 *Let $u \in P^p(T)$ with representation $u = u_W + u_T$. Thus, there holds*

$$|u_W|^2_{H^{1/2}(T)} \leq C(1+\log p)^2\|u\|^2_{H^{1/2}(T)}.$$

Proof. Using the definition of the interpolation operator I^W in (2.6) we get

$$\begin{aligned} |u_W|^2_{H^{1/2}(T)} &= \left|\sum_{i=1}^{3}\tilde{u}_{V_i} + \sum_{i=1}^{3}\tilde{u}_{I_i} + \mathcal{F}\overline{u}_{W_T}\right|^2_{H^{1/2}(T)} \\ &\leq C\left(\sum_{i=1}^{3}|\tilde{u}_{V_i}|^2_{H^{1/2}(T)} + \sum_{i=1}^{3}|\tilde{u}_{I_i}|^2_{H^{1/2}(T)} + \overline{u}_{W_T}^2|\mathcal{F}|^2_{H^{1/2}(T)}\right). \end{aligned}$$

In the beginning, let us consider the vertex function for the vertex V_1.
It is constructed using the extensions $\psi_1 = E_2^1\phi_0$ and $\psi_3 = E^3(\psi_1|_{I_3} - \phi_0)$ with ϕ_0 defined in §2.1. The vertex function associated with V_1 is $\tilde{u}_{V_1} = c_1(\psi_1 - \psi_3)$ (here, $c_1 = \tilde{u}_{V_1}(V_1)$). Thus, we can estimate using Theorem 1.2.3 as follows.

$$|\tfrac{1}{c_1}\tilde{u}_{V_1}|_{H^{1/2}(T)} \leq |\psi_1|_{H^{1/2}(T)} + |\psi_3|_{H^{1/2}(T)} \leq (1+\log p)^{1/2}\left(\|\phi_0\|_{L^2(I_1)} + \|\psi_1|_{I_3} - \phi_0\|_{L^2(I_3)}\right) \tag{2.16}$$

By definition of ψ_1 we have

$$\|\psi_1|_{I_3}\|^2_{L^2(I_3)} = \int_0^1 \left(\frac{1-y}{y}\right)^2 \left(\int_0^y \frac{\phi_0(t)}{1-t}\,dt\right)^2 dy \le \int_0^1 \frac{1}{y^2}\left(\int_0^y |\phi_0(t)|\,dt\right)^2 dy.$$

Using Lemma 1.3.2 this yields

$$\|\psi_1|_{I_3}\|^2_{L^2(I_3)} \le C\|\phi_0\|^2_{L^2(I_3)}, \tag{2.17}$$

and with (2.16)

$$|\tilde{u}_{V_1}|_{H^{1/2}(T)} \le C(1+\log p)^{1/2}\|c_1\phi_0\|_{L^2(I_1)} = C(1+\log p)^{1/2}\|\tilde{u}_{V_1}\|_{L^2(I_1)}. \tag{2.18}$$

In order to bound the edge component of u we use (1.10) and obtain

$$|\tilde{u}_{I_1}|^2_{H^{1/2}(T)} \le \|\tilde{u}_{I_1}\|^2_{\tilde{H}^{1/2}(T,I_2\cup I_3)} \le C(1+\log p)\|\tilde{u}_{I_1}\|^2_{L^2(I_1)}. \tag{2.19}$$

Therefore we have the intermediate result

$$|\tilde{u}_W|^2_{H^{1/2}(T)} \le C(1+\log p)\left(\sum_{i=1}^3 \|\tilde{u}_{V_i}\|^2_{L^2(\partial T)} + \sum_{i=1}^3 \|\tilde{u}_{I_i}\|^2_{L^2(I_i)}\right). \tag{2.20}$$

Recalling that ϕ_0 is orthogonal on $(0,1)$ to any edge function and using the relation $(\phi_0,\phi_0^-)_{L^2(0,1)} = \frac{(-1)^{p+1}}{2(p+1)}\|\phi_0\|^2_{L^2(0,1)}$ (see [89]) one easily deduces that

$$\|\tilde{u}_W\|_{L^2(I_1)} = \|\tilde{u}_{V_1}+\tilde{u}_{V_2}+\tilde{u}_{I_1}\|_{L^2(I_1)} \quad \text{and} \quad \left(\|\tilde{u}_{V_1}\|^2_{L^2(I_1)} + \|\tilde{u}_{V_2}\|^2_{L^2(I_1)} + \|\tilde{u}_{I_1}\|^2_{L^2(I_1)}\right)^{1/2} \tag{2.21}$$

are equivalent norms. Analogous equivalences hold on the edges I_2 and I_3.

It follows from (2.20) that

$$|\tilde{u}_W|^2_{H^{1/2}(T)} \le C(1+\log p)\|\tilde{u}_W\|^2_{L^2(\partial T)} = C(1+\log p)\|u\|^2_{L^2(\partial T)}, \tag{2.22}$$

and using Lemma 2.2.1 we find

$$|\tilde{u}_W|^2_{H^{1/2}(T)} \le C(1+\log p)^2\|u\|^2_{H^{1/2}(T)}. \tag{2.23}$$

In order to bound $|\mathcal{F}|_{H^{1/2}(T)}$ we use (2.22):

$$\begin{aligned} |\mathcal{F}|^2_{H^{1/2}(T)} = |1-\tilde{I}^W 1|^2_{H^{1/2}(T)} &\le 2|1|^2_{H^{1/2}(T)} + 2|\tilde{I}^W 1|^2_{H^{1/2}(T)} \\ &\le C(1+\log p)\|1\|^2_{L^2(\partial T)}. \end{aligned} \tag{2.24}$$

The assertion of the lemma follows by combining (2.23) and (2.24) with Lemma 2.2.2. □

Lemma 2.2.5 *For a wire basket function u_W there holds*

$$\|u_W\|_{L^2(T)} \le C\|u_W\|_{L^2(\partial T)}.$$

Proof. By the definition of u_W we get

$$\|u_W\|^2_{L^2(T)} \le C \left(\sum_{i=1}^{3} \|\tilde{u}_{V_i}\|^2_{L^2(T)} + \sum_{i=1}^{3} \|\tilde{u}_{I_i}\|^2_{L^2(T)} + \|\mathcal{F}\|^2_{L^2(T)} \overline{u}^2_{W_T} \right).$$

By Lemma 1.3.2 there holds

$$\begin{aligned} \|\tilde{u}_{I_1}\|^2_{L^2(T)} &= \int_0^1 \int_0^{1-x} \left(\frac{x(1-x-y)}{y} \right)^2 \left(\int_x^{x+y} \frac{\tilde{u}_{I_1}(t,0)}{t(1-t)}\, dt \right)^2 dy\, dx \\ &\le 4 \int_0^1 \int_x^1 (\tilde{u}_{I_1}(t,0))^2\, dt\, dx \\ &\le C \|\tilde{u}_{I_1}\|^2_{L^2(I_1)}. \end{aligned}$$

Similarly, with $\tilde{u}_{V_1} = c_1(\psi_1 - \psi_3) = c_1(E_2^1 \phi_0 - E^3(\psi_1|_{I_3} - \phi_0))$ we get

$$\|\tilde{u}_{V_1}\|^2_{L^2(T)} \le C\, c_1^2 \left(\|\psi_1\|^2_{L^2(T)} + \|\psi_3\|^2_{L^2(T)} \right) \le C \|\tilde{u}_{V_1}\|^2_{L^2(I_1 \cup I_3)}.$$

Analogous estimates hold for the other vertex and edge functions.

Proceeding as in the proof of Lemma 2.2.4 (see after (2.20)) we therefore get

$$\|\tilde{u}_W\|^2_{L^2(T)} \le C \|\tilde{u}_W\|^2_{L^2(\partial T)} = C \|u\|^2_{L^2(\partial T)}. \tag{2.25}$$

Analogously, we find

$$\|\tilde{I}^W 1\|^2_{L^2(T)} \le C\, \|1\|^2_{L^2(\partial T)}$$

and therefore

$$\|\mathcal{F}\|^2_{L^2(T)} = \|1 - \tilde{I}^W 1\|^2_{L^2(T)} \le C \|1\|^2_{L^2(T)} + C \|\tilde{I}^W 1\|^2_{L^2(T)} \le C. \tag{2.26}$$

From Lemma 2.2.2 we know that

$$\overline{u}^2_{W_T} \le C \|u\|^2_{L^2(\partial T)} = C \|u_W\|^2_{L^2(\partial T)}.$$

Combining this with (2.25) and (2.26) finishes the proof. □

2.3 Proof of the main result

In this section we prove our main result, Theorem 2.1.1. In Theorem 2.3.1 and Theorem 2.3.2 we consider the bilinear forms corresponding to the additive Schwarz operator P_W and Theorem 2.3.3 deals with P_D.

Theorem 2.3.1 *There exists a positive constant C, independent of h and p, such that for any $u = u_W + \sum_{i=1}^{n} u_{\Gamma_i} \in S_h^p$ there holds*

$$(1 + \log p)^3 \sum_{i=1}^{n} \|u - \overline{u}_{W_i}\|^2_{L^2(W_i)} + \sum_{i=1}^{n} \|u_{\Gamma_i}\|^2_{\tilde{H}^{1/2}(\Gamma_i)} \le C\, (1 + \log p)^4 |u|^2_{H^{1/2}(\Gamma)}. \tag{2.27}$$

Proof. Let Γ_i be an arbitrary triangle with boundary W_i and diameter h. Thus, using a transformation to the reference triangle T and Lemma 2.2.1 we obtain

$$\begin{aligned} \|u-\overline{u}_{W_i}\|^2_{L^2(W_i)} &\le C\,h\|v-\overline{v}_{W_i}\|^2_{L^2(\partial T)} \le Ch(1+\log p)|v|^2_{H^{1/2}(T)} \\ &\le C(1+\log p)|u|^2_{H^{1/2}(\Gamma_i)}, \end{aligned} \tag{2.28}$$

see, e.g., [58]. Here, v denotes the linearly transformed function u. Therefore,

$$(1+\log p)^3 \sum_{i=1}^{n} \|u-\overline{u}_{W_i}\|^2_{L^2(W_i)} \le C(1+\log p)^4 \sum_{i=1}^{n} |u|^2_{H^{1/2}(\Gamma_i)} \le C(1+\log p)^4 |u|^2_{H^{1/2}(\Gamma)}.$$

It remains to bound the norms of the interior components of u. On the reference triangle T we have $u_T = (u-u_W)|_T$. By Lemma 2.2.4 there holds

$$|u_W|^2_{H^{1/2}(T)} \le C(1+\log p)^2 \|u\|^2_{H^{1/2}(T)},$$

and Lemmas 2.2.5 and 2.2.1 yield

$$\|u_W\|^2_{L^2(T)} \le C\|u_W\|^2_{L^2(\partial T)} \le C(1+\log p)\|u\|^2_{H^{1/2}(T)}.$$

Therefore,

$$\|u_W\|^2_{H^{1/2}(T)} = \|u_W\|^2_{L^2(T)} + |u_W|^2_{H^{1/2}(T)} \le C(1+\log p)^2 \|u\|^2_{H^{1/2}(T)}.$$

Using Lemma 2.2.3 and the triangle inequality we obtain

$$\begin{aligned} \|u-u_W\|^2_{\tilde{H}^{1/2}(T)} &\le C(1+\log p)^2 \left(\|u\|^2_{H^{1/2}(T)} + \|u_W\|^2_{H^{1/2}(T)} \right) \\ &\le C(1+\log p)^4 \|u\|^2_{H^{1/2}(T)}. \end{aligned}$$

Since the wire basket functions contain the constants we thus have for any $c \in \mathbb{R}$

$$\begin{aligned} \|u-I^W u\|^2_{\tilde{H}^{1/2}(T)} &= \|u+c-I^W(u+c)\|^2_{\tilde{H}^{1/2}(T)} \\ &\le C(1+\log p)^4 \|u+c\|^2_{H^{1/2}(T)}. \end{aligned} \tag{2.29}$$

By a quotient space argument we conclude that

$$\|u-I^W u\|^2_{\tilde{H}^{1/2}(T)} \le C(1+\log p)^4 |u|^2_{H^{1/2}(T)}.$$

Since the norm in $\tilde{H}^{1/2}(T)$ scales like the semi-norm in $H^{1/2}(T)$ this proves

$$\|u_{\Gamma_i}\|^2_{\tilde{H}^{1/2}(\Gamma_i)} \le C(1+\log p)^4 |u|^2_{H^{1/2}(\Gamma_i)}, \tag{2.30}$$

and summing over all elements this finishes the proof of this theorem. □

Theorem 2.3.2 *There exists a positive constant C, independent of h and p, such that for any $u = u_W + \sum_{i=1}^{n} u_{\Gamma_i} \in S_h^p$ there holds*

$$\|u\|^2_{\tilde{H}^{1/2}(\Gamma)} \le C \left((1+\log p)^3 \sum_{i=1}^{n} \|u-\overline{u}_{W_i}\|^2_{L^2(W_i)} + \sum_{i=1}^{n} \|u_{\Gamma_i}\|^2_{\tilde{H}^{1/2}(\Gamma_i)} \right).$$

Proof. We extend our basis functions in a discrete harmonic way into the interior of a tetrahedron, see Bică [21] for details. We consider a reference tetrahedron Ω_{ref} with T as one of its sides and maintain the notation for the basis functions on Ω_{ref}.

Let $u = u_W + \sum_{i=1}^n u_{\Gamma_i} \in S_h^p$ be given. We remark that there holds (see von Petersdorff [90])

$$\|u\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C \left(\|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{i=1}^n \|u - u_W\|^2_{\tilde{H}^{1/2}(\Gamma_i)} \right) \tag{2.31}$$

with $(u - u_W)|_{\Gamma_i} = u_{\Gamma_i}$. Therefore we only have to prove

$$\|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C\,(1 + \log p)^3 \sum_{i=1}^n \|u - \overline{u}_{W_i}\|^2_{L^2(W_i)}.$$

Consider a three dimensional domain Ω such that $\Gamma \subset \partial\Omega$. We decompose Ω into tetrahedra Ω_i such that the trace of this mesh is compatible with the mesh on Γ. For an arbitrary extension U_W of u_W with $U_W = 0$ on $\partial\Omega \setminus \Gamma$ there holds

$$\|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C\,|U_W|^2_{H^1(\Omega)} = C \sum_i |U_{W_i}|^2_{H^1(\Omega_i)}. \tag{2.32}$$

Now we consider the reference tetrahedron Ω_{ref} and the reference triangle $T \subset \partial\Omega_{ref}$. We extend the wire basket component u_W defined on ∂T onto Ω_{ref} by using Theorem 1.2.1 and the discrete harmonic extension. Similarly as in the proof Theorem 1.2.1 there holds

$$\|u_W\|_{H^{1/2}(T)} \leq C\,(1 + \log p)^{1/2} \|u_W\|_{L^2(\partial T)}.$$

In the same way we extend u_W onto the other sides of Ω_{ref}. Thus, we get a continuous function on $\partial\Omega_{ref}$. For the discrete harmonic extension from the faces of the tetrahedron into the interior there holds

$$\|U_W\|_{H^1(\Omega_{ref})} \leq C\,\|u_W\|_{H^{1/2}(\partial\Omega_{ref})}. \tag{2.33}$$

This follows from the minimizing property of the discrete harmonic extension and the extension theorem of Muñoz-Sola [79, Theorem 1]. Using Lemma 2.5.4 (see the Section 2.5) it follows that

$$|U_W|^2_{H^1(\Omega_{ref})} \leq C(1 + \log p)^3 \|u_W\|^2_{L^2(\partial T)}.$$

All the extension operators used reproduce constant functions and therefore we get for any $c \in \mathbb{R}$

$$|U_W|^2_{H^1(\Omega_{ref})} = |U_W + c|^2_{H^1(\Omega_{ref})} \leq C\,(1 + \log p)^3 \|u_W + c\|^2_{L^2(\partial T)}.$$

Transforming this result to an arbitrary element we get

$$|U_{W_i}|^2_{H^1(\Omega_i)} \leq C\,(1 + \log p)^3 \|u_{W_i} + c\|^2_{L^2(\partial\Gamma_i)}.$$

Together with (2.32) this yields

$$\begin{aligned} \|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} &\leq C\,(1+\log p)^3 \inf_{c\in\mathbb{R}} \sum_i \|u-c\|^2_{L^2(W_i)} \\ &= C\,(1+\log p)^3 \sum_i \|u-\overline{u}_{W_i}\|^2_{L^2(W_i)}, \end{aligned} \tag{2.34}$$

which was left to be proved. □

Theorem 2.3.3 *There exist positive constants C_0, C_1, independent of h and p, such that for any $u = u_W + \sum_{i=1}^n u_{\Gamma_i} \in S_h^p$ there holds*

$$C_0\,\|u\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq \|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{i=1}^n \|u_{\Gamma_i}\|^2_{\tilde{H}^{1/2}(\Gamma_i)} \leq C_1(1+\log p)^4 \|u\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

Proof. The first inequality has already been proved by (2.31).

It remains to prove the second inequality. The bound

$$\sum_{i=1}^n \|u_{\Gamma_i}\|^2_{\tilde{H}^{1/2}(\Gamma_i)} \leq C\,(1+\log p)^4 \sum_{i=1}^n |u|^2_{H^{1/2}(\Gamma_i)} \leq C(1+\log p)^4 \|u\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

is an immediate consequence of Theorem 2.3.1.

From inequality (2.34) we know that there holds

$$\|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C(1+\log p)^3 \sum_{i=1}^n \inf_{c_i\in\mathbb{R}} \|u_W - c_i\|^2_{L^2(\partial\Gamma_i)}$$

and by (2.28) we get

$$\inf_{c\in\mathbb{R}} \|u-c\|^2_{L^2(\partial\Gamma_i)} \leq C(1+\log p)|u|^2_{H^{1/2}(\Gamma_i)}.$$

Since $u = u_W$ on $\partial\Gamma_i$ the latter two estimates imply

$$\|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C\,(1+\log p)^4 \sum_{i=1}^n |u|^2_{H^{1/2}(\Gamma_i)} \leq C\,(1+\log p)^4 \|u\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

This finishes the proof of the theorem. □

Proof of Theorem 2.1.1. The assertions are direct consequences of the previous theorems by noting that there holds

$$a(u,u) = \langle Du, u\rangle_\Gamma \cong \|u\|^2_{\tilde{H}^{1/2}(\Gamma)} \cong \|u\|^2_{\tilde{H}^{1/2}(\Gamma_i)},$$

for any $u \in \tilde{H}^{1/2}(\Gamma)$ with support on $\bar{\Gamma}_i \subset \Gamma$, see (2.2). □

2.4 Numerical results

In this section we present some numerical experiments to confirm our theoretical results about the behavior of the condition number of the preconditioned boundary element matrix.

First of all, we comment on the implementation of the preconditioner. When ordering the basis functions of the boundary element space appropriately the preconditioning matrix has a block diagonal form

$$S := \begin{pmatrix} S_W & 0 & 0 & 0 \\ 0 & S_{\Gamma_1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & S_{\Gamma_n} \end{pmatrix}.$$

Here, S_W is the discretisation of the bilinear form b_0 involving the wire basket functions and S_{Γ_i} discretises the energy bilinear form involving interior functions defined on Γ_i, cf. (2.8), (2.9), (2.10). For the calculation of the bilinear form $\hat{a}_W$ we remark that the mean value $\overline{u}_{W_i} = \frac{\int_{W_i} u\,ds}{\int_{W_i} 1\,ds}$ of u on W_i minimizes $\|u - c_i\|_{L^2(W_i)}$ with respect to c_i and therefore,

$$\begin{aligned} \min_{c_i \in \mathbb{R}} \|u - c_i\|^2_{L^2(W_i)} = \|u - \overline{u}_{W_i}\|^2_{L^2(W_i)} &= \int_{W_i} \left(u(t) - \frac{1}{|W_i|} \int_{W_i} u\,ds \right)^2 dt \\ &= \int_{W_i} u^2\,dt - \frac{1}{|W_i|} \left(\int_{W_i} 1 \cdot u\,ds \right)^2, \end{aligned}$$

see also [89]. The first term of the right-hand side above is calculated by using the mass matrix $M^{(i)}$ for the wire basket functions on W_i. Furthermore, there holds $|W_i| = \int_{W_i} 1 \cdot 1\,ds = \vec{z}^{(i)T} M^{(i)} \vec{z}^{(i)}$. Here, $\vec{z}^{(i)}$ contains the coefficients for the constant function 1 on Γ_i. With this notation we can also write

$$\int_{W_i} 1 \cdot u\,ds = \vec{z}^{(i)T} M^{(i)} \vec{u} = \left(M^{(i)} \vec{z}^{(i)} \right)^T \vec{u},$$

where $\vec{u}$ contains the coefficients of u for the basis in use, and

$$\left(\int_{W_i} 1 \cdot u\,ds \right)^2 = \left(\left(M^{(i)} \vec{z}^{(i)} \right)^T \vec{u} \right)^2 = \vec{u}^T \left(M^{(i)} \vec{z}^{(i)} \right) \left(M^{(i)} \vec{z}^{(i)} \right)^T \vec{u}.$$

Thus, locally on one element we obtain

$$S_{W_i} = (1 + \log p)^3 \left(M^{(i)} - \frac{(M^{(i)} \vec{z}^{(i)}) \cdot (M^{(i)} \vec{z}^{(i)})^T}{\vec{z}^{(i)T} M^{(i)} \vec{z}^{(i)}} \right).$$

In order to calculate the preconditioning block S_W we sum over all the elements.

For our model problem we choose the domain $\Gamma = (-1/2, 1/2)^2 \times \{0\}$ and use uniform triangular meshes. We do not specify any right-hand side function f in (2.1) since we only report on the spectral behavior of the stiffness matrix. For smooth right-hand side functions f the hp-version with quasi-uniform meshes converges like $\mathcal{O}(h^{1/2}p^{-1})$ in the energy norm, see [19, 20].

In Figure 2.5 we plot the condition numbers and the maximum and minimum eigenvalue of the Galerkin matrix with preconditioner based upon the energy bilinear form. In the plot we also give the curve of $(1+\log p)^2$ and $(1+\log p)^3$, and the numerical results seem to be slightly better than $(1+\log p)^3$ in the given range of p. This result is better as in the main theorem. This may occur due to the fact that the extension result in Chapter 1 is not optimal.

In Figure 2.6 we consider the wire basket preconditioner using the L^2-bilinear form, i.e. the additive Schwarz operator P_W based upon the bilinear form $\hat{a}_W$ defined by (2.9). In the plot we also give the curve of $(1+\log p)^4$ and $(1+\log p)^5$, and the numerical results seem to be slightly better than $(1+\log p)^5$ in the given range of p. Exact numbers are given in Table 2.1 alongside with the condition numbers of the un-preconditioned stiffness matrix. We also give the iteration numbers of the conjugate gradient method needed for fixed precision. As expected, the iteration numbers increase only moderately when one of the preconditioners is used and grow substantially without preconditioner.

p	DOF	cond(w/o)	niter	cond(pre, $\tilde{H}^{1/2}$)	niter	cond(pre, L^2)	niter
1	1	0.1000E+01	0	0.1000E+01	0	0.10000E+01	1
2	9	0.7818E+01	7	0.1000E+01	2	0.56189E+01	7
3	25	0.7345E+02	20	0.5452E+01	18	0.44387E+02	18
4	49	0.7388E+03	50	0.7966E+01	27	0.74572E+02	34
5	81	0.1027E+05	105	0.1033E+02	32	0.11207E+03	44
6	121	0.1815E+06	219	0.1202E+02	34	0.15042E+03	56
7	169	0.4289E+07	338	0.1311E+02	38		
8	225	0.1156E+09	450	0.1462E+02	40		
9	289	0.2010E+10	578	0.1632E+02	43		

Table 2.1: Condition numbers and iteration numbers for the p-version without and with preconditioning (using the L^2- and the energy bilinear form).

In Figure 2.7 we present condition numbers of the preconditioned matrix P_W for the h-version and different polynomial degrees. The results confirm the asymptotic independence of the condition numbers on h.

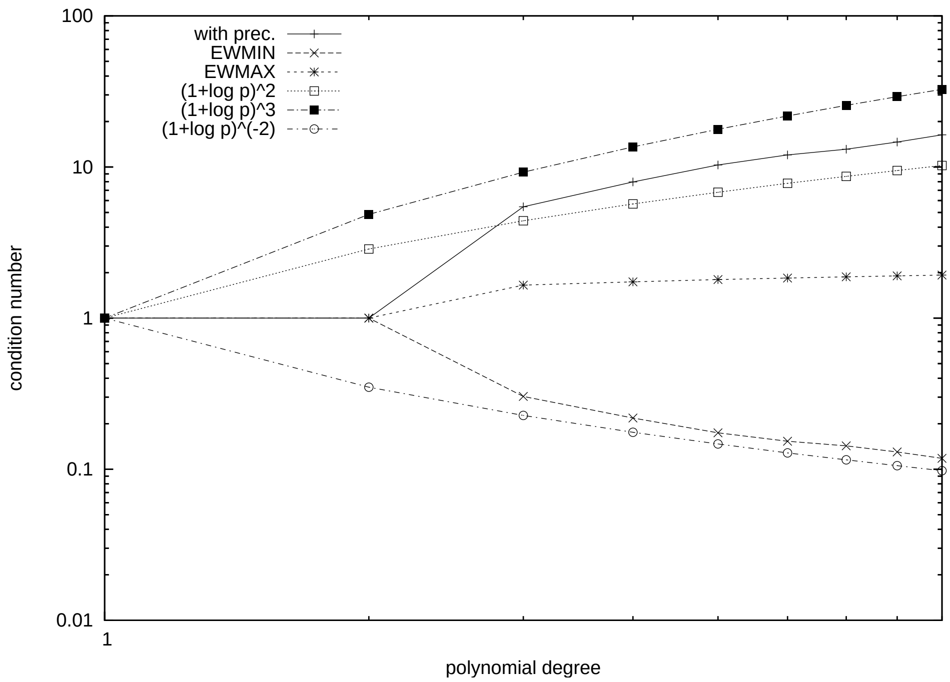

Figure 2.5: Condition number and maximum and minimum eigenvalue of the preconditioned Galerkin matrix, p-version, using the energy bilinear form.

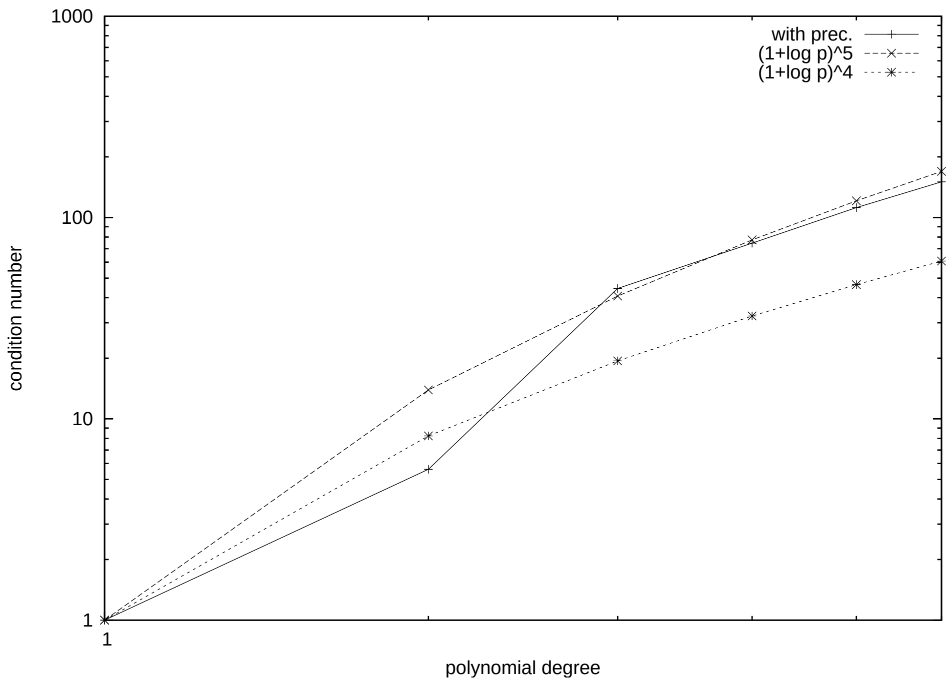

Figure 2.6: Condition number of the preconditioned Galerkin matrix, p-version, using the L^2-bilinear form.

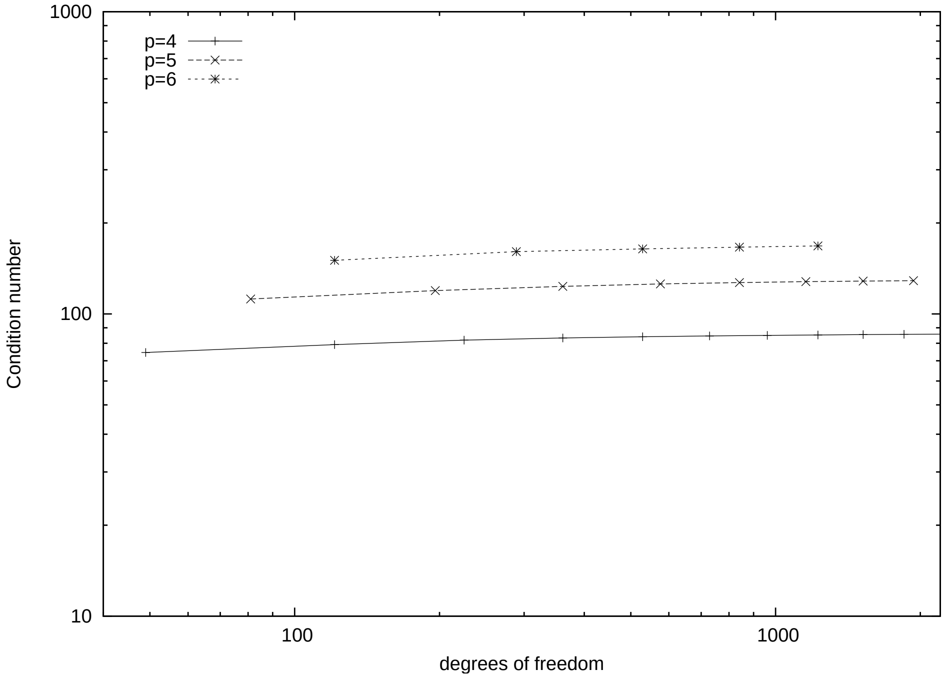

Figure 2.7: Condition number of the preconditioned Galerkin matrix, L^2-bilinear form, h-version.

Remarks on the algorithm

The algorithm here consists of three parts: first assembling the Galerkin matrix, second the setup of the preconditioner and finally applying iteratively the preconditioner and performing matrix-vector multiplications. In the standard Galerkin method used here we have $N \approx np^2/2$ unknowns, i.e., we need $\mathcal{O}(N^2) = \mathcal{O}(n^2p^4)$ operations for assembling the Galerkin matrix. The setup of the preconditioner consists in computing the inverses of the local diagonal blocks for wire basket and bubble spaces, i.e., $\mathcal{O}((np)^3) + n\mathcal{O}\big((p^2)^3\big)$ operations. One matrix-vector multiplication needs $\mathcal{O}\big((np^2)^2\big)$ operations and one application of the preconditioner $\mathcal{O}\big((np)^2 + n(p^2)^2\big)$. Altogether with the number of preconditioned CG-iterations $k_{it} = \mathcal{O}(\sqrt{\kappa}(\log n + \log p))$ with $\kappa = (1 + \log p)^4$ we obtain $\mathcal{O}\big(n^2p^4 + n^3p^3 + np^6 + k_{it}(n^2p^4 + n^2p^2 + np^4)\big)$ operations for solving the problem.
A possible improvement would be the additional use of fast boundary element methods like panel clustering (see for example Sauter & Schwab [94]), i.e., the solution procedure would hopefully only consist (in each iteration step) of one matrix-vector multiplication with linear complexity and of application of our preconditioner which consists in solving the linear system for the wire basket space with dimension np and solving the n linear systems of dimension p^2 for the bubble spaces. Denoting the numbers of CG-iterations for solving the wire basket and bubble blocks by $k_{it,w}$ and $k_{it,b}$, respectively, we have altogether $\mathcal{O}\big(k_{it}(np^2 + k_{it,w}np + nk_{it,b}p^2)\big)$ operations. The condition numbers for the wire basket and bubble spaces are unknown and the existence of appropriate optimal

preconditioners is topic of further research. The investigation of the action of such inexact solvers will also lead to an improvement of the standard implementation of our wire basket preconditioner considered here.

2.5 Additional technical results

2.5.1 Results of Bică

For the convenience of the reader we repeat some of the results and proofs of Bică [21] who deals with wire basket preconditioners for the p-version of the finite element method on tetrahedral meshes. In fact, at several places he uses an unknown factor $N(p)$ stemming from an unproved extension theorem. Here, we use the extension theorem from Chapter 1 to fill this gap.

We denote by Ω_{ref} the reference tetrahedron

$$\Omega_{\mathrm{ref}} := \{(x,y,z);\ 0 \le x,y,z \le 1,\ x+y+z \le 1\}$$

and define for integer p

$$P^p(\Omega_{\mathrm{ref}}) := \mathrm{span}\{x^i y^j z^k;\ 0 \le i+j+k \le p\}.$$

Before dealing with results from [21] we collect three technical lemmas needed below.

Lemma 2.5.1 *[93] Let $u \in P^p(0,1)$. Then,*

$$\max_{[0,1]} \left| \frac{d}{dx} u(x) \right| \le 2p^2 \max_{[0,1]} |u(x)|.$$

Lemma 2.5.2 *[12, Theorem 6.2] Let $u \in P^p(0,1)$. Then,*

$$\|u\|^2_{L^\infty(0,1)} \le C(1+\log p)\|u\|^2_{H^{1/2}(0,1)}.$$

Lemma 2.5.3 *[89, Lemma 5.3] Let I be any line segment in the closure of the reference tetrahedron $\overline{\Omega}_{\mathrm{ref}}$ and let $u \in P^p(\Omega_{\mathrm{ref}})$. Then,*

$$\|u\|^2_{L^2(I)} \le C(1+\log p)\|u\|^2_{H^1(\Omega_{\mathrm{ref}})}.$$

If $\overline{u}_W$ is the average of u over the wire basket W, then,

$$\|u - \overline{u}_W\|^2_{L^2(I)} \le C(1+\log p)|u|^2_{H^1(\Omega_{\mathrm{ref}}.)}$$

Now, let us turn to the theory of finite elements. The wire basket decomposition used in [21] is the three-dimensional analogue of our decomposition obtained by discrete harmonic extensions of basis functions onto the reference tetrahedron. The corresponding wire basket interpolation operators in three dimensions will be again denoted by I^W and $\tilde{I}^W$. Here, in this section, W denotes the wire basket of the reference tetrahedron. As in the two-dimensional setting we define $\mathcal{F} := 1 - \tilde{I}^W 1$ on the boundary of the reference element. But now $\mathcal{F}$ has four components, each associated with one of the faces and vanishing on the other faces, $\mathcal{F} = \sum_{k=1}^4 \mathcal{F}_k$. Note that by definition of the basis functions $\mathcal{F}_k$ is discrete harmonic.

The next lemma is needed for the proof of Theorem 2.3.2 and its proof is based on the two lemmas that follow.

Lemma 2.5.4 *(Compare [21, Lemma 4.16]) Setting $U_W := I^W u$ there holds*

$$|U_W|^2_{H^1(\Omega_{ref})} \le C\,(1+\log p)^3 \|u\|^2_{L^2(W)} \quad \forall\, u \in P^p(\Omega_{ref})$$

Proof. Let F_k, $k = 1, \dots, 4$, denote the faces of Ω_{ref}. Since $I^W u = \tilde{I}^W u + \sum_{k=1}^4 \overline{u}_{\partial F_k} \mathcal{F}_k$ we get

$$|I^W u|^2_{H^1(\Omega_{ref})} \le 5 \left(|\tilde{I}^W u|^2_{H^1(\Omega_{ref})} + \sum_{k=1}^4 \overline{u}^2_{\partial F_k} |\mathcal{F}_k|^2_{H^1(\Omega_{ref})} \right).$$

By Lemma 2.5.5 there holds

$$|\tilde{I}^W u|^2_{H^1(\Omega_{ref})} \le C\,(1+\log p) \|u\|^2_{L^2(W)}.$$

Using the discrete harmonicity of $\mathcal{F}$ and combining Lemma 2.2.3 with (2.24) and (2.26) we obtain

$$|\mathcal{F}_k|^2_{H^1(\Omega_{ref})} \le C\,(1+\log p)^3.$$

Together with

$$\overline{u}^2_{\partial F_k} = \left(\frac{\int_{\partial F_k} u}{\int_{\partial F_k} 1} \right)^2 \le C \|u\|^2_{L^2(W)}$$

we get

$$\begin{aligned} |I^W u|^2_{H^1(\Omega_{ref})} &\le C \left((1+\log p) \|u\|^2_{L^2(W)} + \sum_{k=1}^4 \overline{u}^2_{\partial F_k} (1+\log p)^3 \right) \\ &\le C\,(1+\log p)^3 \|u\|^2_{L^2(W)}. \end{aligned}$$

□

Lemma 2.5.5 *(Compare [21, Lemma 4.13])*

$$|\tilde{I}^W u|^2_{H^1(\Omega_{ref})} \le C\,(1+\log p) \|u\|^2_{L^2(W)} \quad \forall\, u \in P^p(\Omega_{ref})$$

Proof. Using the estimates for the vertex and edge functions in Lemma 2.5.6, and noting that $L^2(W)$-inner products between different wire basket components are negligible (see (2.21)), we get

$$\begin{aligned}
|\tilde{I}^W u|^2_{H^1(\Omega_{ref})} &\leq C \left(\sum_{i=1}^{4} |\tilde{u}_{V_i}|^2_{H^1(\Omega_{ref})} + \sum_{j=1}^{6} |\tilde{u}_{E_j}|^2_{H^1(\Omega_{ref})} \right) \\
&\leq C\,(1+\log p) \left(\sum_{i=1}^{4} \|\tilde{u}_{V_i}\|^2_{L^2(W)} + \sum_{j=1}^{6} \|\tilde{u}_{E_j}\|^2_{L^2(W)} \right) \\
&\leq C\,(1+\log p) \|\tilde{I}^W u\|^2_{L^2(W)} = C\,(1+\log p)\|u\|^2_{L^2(W)}.
\end{aligned}$$

□

Lemma 2.5.6 *(Compare [21, Lemmas 4.11, 4.12]) Let Φ_V and Φ_E be a vertex function and an edge function, respectively. There holds*

$$\|\Phi_V\|_{H^1(\Omega_{ref})} \leq C\,(1+\log p)^{1/2} \|\Phi_V\|_{L^2(W)}$$

and

$$\|\Phi_E\|_{H^1(\Omega_{ref})} \leq C(1+\log p)^{1/2} \|\Phi_E\|_{L^2(W)}.$$

Proof. This follows by using the property of discrete harmonic extensions, cf. (2.33), and Theorem 1.2.1. □

2.5.2 Discrete harmonic functions

In this paragraph we explain the idea of a discrete harmonic extension. Therefore, we consider a domain Ω and the bilinear form

$$a(u,v) := \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx.$$

Here, u, $v \in V \subset H^1(\Omega)$ where V is determined by the certain boundary conditions. Next, we consider a finite dimensional subspace $V^p \subset V$ which consists of piecewise polynomials of degree up to p. Then, the space of **discrete harmonic functions** $\tilde{V}^p \subset V^p$ is defined by

$$\tilde{V}^p := \{u \in V^p : \, a(u,v) = 0 \,\forall\, v \in V^p,\, v = 0 \text{ on } \partial\Omega\}.$$

A discrete harmonic functions $u \in \tilde{V}^p$ fulfills the following minimizing property

$$|u|_{H^1(\Omega)} \leq |v|_{H^1(\Omega)} \qquad \forall\, v \in V^p \text{ with } u = v \text{ on } \Gamma.$$

The proof of this result is included in the proof of the following lemma.

Using the extension of Muñoz-Sola [79] we can prove the following minimizing property for discrete harmonic functions.

Lemma 2.5.7 *Let Ω_{ref} denote the reference tetrahedron. For a discrete harmonic function u on Ω_{ref} there holds*

$$\|u\|_{H^1(\Omega_{ref})} \leq C\|u\|_{H^{1/2}(\partial\Omega_{ref})}.$$

Proof. Here, we abbreviate $\Omega := \Omega_{ref}$. Let u be discrete harmonic on Ω, i.e. there holds $a(u,v) = 0 \;\forall\, v \in V^p$ with $v = 0$ on $\Gamma := \partial\Omega$. Using the extension of Muñoz-Sola [79] we know that there exists an extension $w \in V^p$ of $u|_\Gamma$ with

$$\|w\|_{H^1(\Omega)} \leq C\,|u|_{H^{1/2}(\Gamma)}. \tag{2.35}$$

Thus, there exists a $v \in V_0^p := \{v \in V^p :\; v = 0 \text{ on } \partial\Omega\}$ with $w = u + v$ and it holds

$$a(w,w) = a(u+v,u+v) = \underbrace{a(u,u)}_{\geq 0} + 2\underbrace{a(u,v)}_{=0} + \underbrace{a(v,v)}_{\geq 0}.$$

Thus, the minimizing of the H^1-semi norm by discrete harmonic functions follows.

$$|u|_{H^1(\Omega)} \leq |w|_{H^1(\Omega)}. \tag{2.36}$$

As $u - w = 0$ on Γ we can use the inequality of Poincaré-Friedrich to get

$$\|u\|_{L^2(\Omega)} \leq \underbrace{\|u-w\|_{L^2(\Omega)}}_{\leq C|u-w|_{H^1(\Omega)}} + \|w\|_{L^2(\Omega)} \leq C\left(|u|_{H^1(\Omega)} + \|w\|_{H^1(\Omega)}\right). \tag{2.37}$$

Combining this estimate with (2.36) and (2.35) we get

$$\|u\|^2_{H^1(\Omega)} \leq C\|w\|^2_{H^1(\Omega)} \leq C\|u\|^2_{H^{1/2}(\partial\Omega)}. \tag{2.38}$$

This finishes the proof. □

3 Spaces, operators, theorems for the Maxwell's equations

3.1 Spaces and trace operators

In the following, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz continuous boundary $\Gamma := \partial\Omega$. We then refer to Ω as a Lipschitz domain. Every polyhedral domain is a Lipschitz domain. Furthermore, the unbounded complement is denoted by $\Omega_E := \mathbb{R}^3 \setminus \overline{\Omega}$. The outer unit normal vector $\mathbf{n}$ on Γ is pointing from Ω into Ω_E. For Lipschitz domains this exists only almost everywhere.

In this section we introduce the spaces which are necessary for the investigation of the Maxwell's equations. In three dimensions these are the spaces $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}(\mathrm{div}, \Omega)$. Furthermore, we have to consider the trace spaces on Γ of $\mathbf{H}(\mathbf{curl}, \Omega)$ using the tangential trace γ_D and the twisted tangential trace $\gamma_t^\times$. These are the spaces $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. On smooth boundaries the theory is well established, see Paquet [86], Alonso & Valli [4], Cessenat [35, Section 2.1] and Nédélec [84, Section 5.4.1]. Their results have been extended to polyhedra by Buffa [24] and Buffa & Ciarlet [27, 28]. For the case of Lipschitz domains, see Buffa *et al.* [30].

On Ω we consider the spaces $\mathbf{L}^2(\Omega) := (L^2(\Omega))^3$ and the space of tangential vector fields

$$\mathbf{L}_t^2(\Gamma) := \{\mathbf{u} \in \mathbf{L}^2(\Gamma) : \mathbf{u} \cdot \mathbf{n} = 0 \text{ a.e. on } \Gamma\}$$

with the complex dualities

$$\begin{aligned}
(\mathbf{u}, \mathbf{v}) &:= \int_\Omega \mathbf{u}(\mathbf{x}) \cdot \overline{\mathbf{v}(\mathbf{x})}\, d\mathbf{x}, \qquad && \mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega), \\
\langle \boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle_\Gamma &:= \int_\Gamma \boldsymbol{\lambda}(\mathbf{x}) \cdot \overline{\boldsymbol{\zeta}(\mathbf{x})}\, d\mathbf{x}, \qquad && \boldsymbol{\lambda}, \boldsymbol{\zeta} \in \mathbf{L}_t^2(\Gamma).
\end{aligned}$$

Besides the usual Sobolev spaces $H^s(\Omega)$ for scalar functions and $\mathbf{H}^s(\Omega) := (H^s(\Omega))^3$ for vector fields of order $s \in \mathbb{R}$ (cf. Grisvard [51]), we use the spaces

$$\begin{aligned}
\mathbf{H}(\mathbf{curl}, \Omega) &:= \left\{\mathbf{u} \in \mathbf{L}^2(\Omega) : \mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2(\Omega)\right\}, \\
\mathbf{H}_0(\mathbf{curl}, \Omega) &:= \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega) : \mathbf{u} \times \mathbf{n} = 0 \text{ on } \Gamma\},
\end{aligned}$$

$$
\begin{aligned}
\mathbf{H}(\mathbf{curl}\,\mathbf{curl},\Omega) &:= \{\mathbf{u} \in \mathbf{H}(\mathbf{curl},\Omega) :\ \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2(\Omega)\},\\
\mathbf{H}(\mathrm{div},\Omega) &:= \{\mathbf{u} \in \mathbf{L}^2(\Omega) :\ \mathrm{div}\,\mathbf{u} \in L^2(\Omega)\},\\
\mathbf{H}_0(\mathrm{div},\Omega) &:= \{\mathbf{u} \in \mathbf{H}(\mathrm{div},\Omega) :\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma\},\\
\mathbf{H}_0(\mathrm{div}\,0,\Omega) &:= \{\mathbf{u} \in \mathbf{H}_0(\mathrm{div},\Omega) :\ \mathrm{div}\,\mathbf{u} = 0\}.
\end{aligned}
$$

Furthermore, we define for $s \in \mathbb{R}$

$$\mathbf{H}^s(\mathbf{curl},\Omega) := \{\mathbf{u} \in \mathbf{H}^s(\Omega) :\ \mathbf{curl}\,\mathbf{u} \in \mathbf{H}^s(\Omega)\}.$$

The norms in $\mathbf{H}(\mathbf{curl},\Omega)$, $\mathbf{H}(\mathrm{div},\Omega)$ and $\mathbf{H}^s(\mathbf{curl},\Omega)$ are given by

$$
\begin{aligned}
\|\mathbf{u}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} &:= \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\,\mathbf{curl}\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)},\\
\|\mathbf{u}\|^2_{\mathbf{H}(\mathrm{div},\Omega)} &:= \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\,\mathrm{div}\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)},\\
\|\mathbf{u}\|^2_{\mathbf{H}^s(\mathbf{curl},\Omega)} &:= \|\mathbf{u}\|^2_{\mathbf{H}^s(\Omega)} + \|\,\mathbf{curl}\,\mathbf{u}\|^2_{\mathbf{H}^s(\Omega)}.
\end{aligned}
$$

Next, we define the trace operators. The trace of a scalar function $\phi \in H^1(\Omega)$ on Γ is denoted by $\gamma\phi$. For a vectorial function $\mathbf{u} \in \mathcal{C}(\overline{\Omega})^3$ we define the **Dirichlet trace** (**tangential trace**) on Γ by

$$\gamma_D\mathbf{u}(\mathbf{x}) := \mathbf{n}(\mathbf{x}) \times (\mathbf{u}(\mathbf{x}) \times \mathbf{n}(\mathbf{x})) = \mathbf{u}(\mathbf{x}) - \big(\mathbf{n}\cdot\mathbf{u}(\mathbf{x})\big)\mathbf{n}(\mathbf{x}).$$

The **twisted tangential trace** is then defined by

$$\gamma_t^\times\mathbf{u}(\mathbf{x}) := \mathbf{u}(\mathbf{x}) \times \mathbf{n}(\mathbf{n}).$$

Thus, there holds

$$\gamma_t^\times(\gamma_D\mathbf{u}) = \gamma_D(\gamma_t^\times\mathbf{u}) = \gamma_t^\times\mathbf{u} \tag{3.1}$$

because of

$$
\begin{aligned}
\gamma_D(\gamma_t^\times\mathbf{u}) &= \mathbf{n} \times \big((\mathbf{u}\times\mathbf{n})\times\mathbf{n}\big) = \mathbf{n} \times \big((\mathbf{u}\cdot\mathbf{n})\mathbf{n} - (\mathbf{n}\cdot\mathbf{n})\mathbf{u}\big)\\
&= 0 - \mathbf{n}\times\mathbf{u} = \gamma_t^\times\mathbf{u}.
\end{aligned}
$$

We also define the **rotating operator** R by

$$R\boldsymbol{\lambda} := \mathbf{n} \times \boldsymbol{\lambda}. \tag{3.2}$$

There holds

$$R^{-1} = -R, \tag{3.3}$$

see Buffa, Costabel, Sheen [30]. Finally, we define the **normal trace operator** γ_n for $\mathbf{u} \in \mathcal{C}^1(\overline{\Omega})^3$ by

$$\gamma_n\mathbf{u}(\mathbf{x}) := \mathbf{u}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}).$$

Furthermore, we define the jumps across the boundary Γ by $[\gamma\cdot]_\Gamma := \gamma^+\cdot - \gamma^-\cdot$ where γ^+ denotes the trace from the outer domain Ω_E and γ^- denotes the traces from Ω.

Let $\phi \in H^2(\Omega)$ be a scalar function. We then define the **surface gradient** of ϕ on Γ by

$$\mathbf{grad}_\Gamma\, \phi := \gamma_D(\mathbf{grad}\, \phi)$$

and the **vectorial surface rotation** on Γ by

$$\mathbf{curl}_\Gamma\, \phi := \gamma_t^\times(\mathbf{grad}\, \phi) = \mathbf{grad}_\Gamma\, \phi \times \mathbf{n}.$$

The **scalar surface rotation** on Γ of a vectorial function $\mathbf{u} \in \mathbf{H}^2(\Omega)$ with $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ is given by

$$\mathrm{curl}_\Gamma\, \mathbf{u} := \mathbf{curl}\, \mathbf{u} \cdot \mathbf{n}$$

and the **surface divergence** by

$$\mathrm{div}_\Gamma\, \mathbf{u} := \mathrm{div}(\gamma_D \mathbf{u}) = -\,\mathrm{curl}_\Gamma(\mathbf{u} \times \mathbf{n}) = -\,\mathbf{curl}(\mathbf{u} \times \mathbf{n}) \cdot \mathbf{n}.$$

The above definitions are valid on all regular points of Γ but can be extended to Lipschitz domains, see e.g. Buffa & Ciarlet [27, 28].

On smooth domains there hold the following dualities

$$\langle \mathbf{grad}_\Gamma\, \phi, \mathbf{u} \rangle_\Gamma = -\langle \phi, \mathrm{div}_\Gamma\, \mathbf{u} \rangle_\Gamma,$$
$$\langle \mathbf{curl}_\Gamma\, \phi, \mathbf{u} \rangle_\Gamma = \langle \phi, \mathrm{curl}_\Gamma\, \mathbf{u} \rangle_\Gamma.$$

Next, we define spaces of tangential traces on non-smooth domains. We refer to Buffa & Ciarlet [27, 28], see also Hiptmair & Schwab [68].

We consider a polyhedral domain Ω. The boundary Γ is assumed to be separated into n faces Γ_i with $\Gamma = \bigcup_{i=1}^n \Gamma_i$. For two faces Γ_i and Γ_j with a common edge e_{ij} we define $\mathbf{t}_{ij}$ as the unit tangential vector and $\mathbf{t}_{i(j)} := \mathbf{t}_{ij} \times \mathbf{n}_i$ where $\mathbf{n}_i$ denotes the unit normal vector on e_{ij} w.r.t. Γ_i. Furthermore, let $\mathcal{I}_j$ denote the set of those indices i such that Γ_i shares an edge with Γ_j. Then, we define

$$\mathbf{H}_*^{1/2}(\Gamma) := \left\{\mathbf{u} \in \mathbf{L}_t^2(\Gamma) :\ \mathbf{u}_{|\Gamma_j} \cdot \mathbf{t}_{j(i)},\, \mathbf{u}_{|\Gamma_j} \cdot \mathbf{t}_{ij} \in H^{1/2}(\Gamma_j)\, \forall\, i \in \mathcal{I}_j,\, \forall\, j = 1, \ldots, n\right\}.$$

and

$$\mathbf{H}_\|^{1/2}(\Gamma) := \left\{\mathbf{u} \in \mathbf{H}_*^{1/2}(\Gamma) : \mathcal{N}_{i,j}^{\|}(\mathbf{u}) < \infty \quad \forall i \in \mathcal{I}_j\ \forall j = 1, \ldots, n\right\}, \tag{3.4}$$
$$\mathbf{H}_\perp^{1/2}(\Gamma) := \left\{\mathbf{u} \in \mathbf{H}_*^{1/2}(\Gamma) : \mathcal{N}_{i,j}^{\perp}(\mathbf{u}) < \infty \quad \forall i \in \mathcal{I}_j\ \forall j = 1, \ldots, n\right\}, \tag{3.5}$$

with the functionals

$$\mathcal{N}_{i,j}^{\|}(\mathbf{u}) := \int_{\Gamma_i} \int_{\Gamma_j} \frac{|(\mathbf{u} \cdot \mathbf{t}_{ij})(\mathbf{x}) - (\mathbf{u} \cdot \mathbf{t}_{ij})(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^3} dS(\mathbf{x}) dS(\mathbf{y}),$$
$$\mathcal{N}_{i,j}^{\perp}(\mathbf{u}) := \int_{\Gamma_i} \int_{\Gamma_j} \frac{|(\mathbf{u} \cdot \mathbf{t}_{i(j)})(\mathbf{x}) - (\mathbf{u} \cdot \mathbf{t}_{j(i)})(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^3} dS(\mathbf{x}) dS(\mathbf{y}).$$

Loosely spoken, $\mathbf{H}_{\|}^{1/2}(\Gamma)$ contains the tangential surface vector fields that are in $\mathbf{H}^{1/2}(\Gamma_i)$ for each smooth surface piece Γ_i of Γ and fulfill a suitable "weak tangential continuity" across the edges of the Γ_i. For $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ a corresponding "weak normal continuity" is fulfilled.

The spaces $\mathbf{H}_{\perp}^{-1/2}(\Gamma)$ and $\mathbf{H}_{\|}^{-1/2}(\Gamma)$ are then defined as the dual spaces of $\mathbf{H}_{\perp}^{1/2}(\Gamma)$ and $\mathbf{H}_{\|}^{1/2}$, resp., with $\mathbf{L}_t^2(\Gamma)$ as pivot space, see [27].

The above defined surface differential operators can now be extended to other Sobolev spaces. The following Lemma can be found in the articles of Buffa & Ciarlet [27, Sect. 3.1] and [28, Sect. 4.2]

Lemma 3.1.1 *Assuming that Γ is Lipschitz regular we can extend the surface differential operators $\mathbf{grad}_\Gamma$ and $\mathbf{curl}_\Gamma$ to linear and continuous mappings*

$$\begin{aligned} \mathbf{grad}_\Gamma &: H^{1/2}(\Gamma) \to \mathbf{H}_{\perp}^{-1/2}(\Gamma), \\ \mathbf{curl}_\Gamma &: H^{1/2}(\Gamma) \to \mathbf{H}_{\|}^{-1/2}(\Gamma) \end{aligned}$$

and their adjoints

$$\begin{aligned} \mathrm{div}_\Gamma &: \mathbf{H}_{\perp}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \\ \mathrm{curl}_\Gamma &: \mathbf{H}_{\|}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \end{aligned}$$

are linear, continuous and surjective. There holds

$$\begin{aligned} \mathrm{Ker}\big(\mathrm{curl}_\Gamma(\mathbf{H}_{\perp}^{-1/2}(\Gamma))\big) &= \mathrm{Im}\big(\mathbf{grad}_\Gamma(H^{1/2})\big), \\ \mathrm{Ker}\big(\mathrm{div}_\Gamma(\mathbf{H}_{\|}^{-1/2}(\Gamma))\big) &= \mathrm{Im}\big(\mathbf{curl}_\Gamma(H^{1/2})\big). \end{aligned}$$

Furthermore, there hold the duality pairings

$$\begin{aligned} \langle \mathbf{grad}_\Gamma\, \phi, \mathbf{u}\rangle_\Gamma &= -\langle \phi, \mathrm{div}_\Gamma\, \mathbf{u}\rangle_\Gamma && \forall\, \phi \in H^{1/2}(\Gamma),\ \mathbf{u} \in \mathbf{H}_{\perp}^{1/2}(\Gamma), \\ \langle \mathbf{curl}_\Gamma\, \phi, \mathbf{u}\rangle_\Gamma &= \langle \phi, \mathrm{curl}_\Gamma\, \mathbf{u}\rangle_\Gamma && \forall\, \phi \in H^{1/2}(\Gamma),\ \mathbf{u} \in \mathbf{H}_{\|}^{1/2}(\Gamma). \end{aligned}$$

We are now in the position to define the following trace spaces.

$$\begin{aligned} \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) &:= \left\{ \mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\Gamma) :\ \mathrm{curl}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma) \right\}, \\ \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) &:= \left\{ \mathbf{u} \in \mathbf{H}_{\|}^{-1/2}(\Gamma) :\ \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma) \right\}, \\ \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma\, 0, \Gamma) &:= \{ \mathbf{u} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) :\ \mathrm{div}_\Gamma\, \mathbf{u} = 0 \}. \end{aligned}$$

Lemma 3.1.2 *The spaces $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ are dual to each other with respect to $\mathbf{L}_t^2(\Gamma)$ as pivot space.*

Proof. See Buffa & Ciarlet [28, Section 4]. □

We now get the following mapping properties of the trace operators.

Lemma 3.1.3 *The trace operators* γ_D *and* $\gamma_t^\times$ *can be extended to linear, continuous and surjective mappings*

$$\begin{aligned}
\gamma_D &: \mathbf{H}^1(\Omega) \to \mathbf{H}_{\|}^{1/2}(\Gamma),\\
\gamma_D &: \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma),\\
\gamma_t^\times &: \mathbf{H}^1(\Omega) \to \mathbf{H}_{\perp}^{1/2}(\Gamma),\\
\gamma_t^\times &: \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma).
\end{aligned}$$

Furthermore, the trace mappings $\gamma_D : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *and* $\gamma_t^\times : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *possess both a continuous right inverse.*

Proof. The proof for smooth domains can be found in Nédélec [84] and for Lipschitz domains in the articles of Buffa & Ciarlet [27, Proposition 2.7, 2.8, Theorem 3.9, 3.10] and [28, Theorem 5.4]. □

The following result can be found in Buffa & Ciarlet [27, Section 3.2] and is helpful in the computations.

Lemma 3.1.4 *For* $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ *there holds*

$$\mathrm{div}_\Gamma(\mathbf{u} \times \mathbf{n}) = \mathbf{n} \cdot \mathbf{curl}\,\mathbf{u}. \tag{3.6}$$

There holds the following **Green formula**, see Buffa & Ciarlet [27, Section 3.2].

Lemma 3.1.5 *For* $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ *and* $\mathbf{v} \in \mathbf{H}^1(\Omega)$ *there holds*

$$\int_\Omega (\mathbf{curl}\,\mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{curl}\,\mathbf{u})\, d\mathbf{x} = \langle \gamma_t^\times \mathbf{u},\, \gamma_D \mathbf{v} \rangle_{\|,1/2,\Gamma}. \tag{3.7}$$

Here, $\langle \cdot, \cdot \rangle_{\|,1/2,\Gamma}$ *denotes the* $\mathbf{H}_{\|}^{-1/2}(\Gamma)$-$\mathbf{H}_{\|}^{1/2}(\Gamma)$*-duality with* $\mathbf{L}_t^2(\Gamma)$ *as pivot space.*

For $\mathbf{u} \in \mathbf{H}(\mathbf{curl}\,\mathbf{curl}, \Omega)$ the **Neumann trace** $\gamma_N \mathbf{u} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is defined by (see Hiptmair [66])

$$\langle \gamma_N \mathbf{u}, \gamma_D \mathbf{v} \rangle_\Gamma = \pm(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_\Omega \mp (\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}, \mathbf{v})_\Omega \quad \forall\, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega). \tag{3.8}$$

Here, the upper signs are applied to the interior domain $\Omega = \Omega$. The lower signs are used for the exterior domain $\Omega = \Omega_E$. As for smooth fields there also holds $\gamma_N \mathbf{u} = \gamma_t^\times(\mathbf{curl}\,\mathbf{u})$.

Lemma 3.1.6 *The trace operator*

$$\gamma_N : \mathbf{H}(\mathbf{curl}\,\mathbf{curl}, \Omega) \to \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

is linear and continuous and there holds for $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ *with* $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = 0$

$$\|\gamma_N \mathbf{u}\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \leq C\|\,\mathbf{curl}\,\mathbf{u}\|_{\mathbf{L}^2(\Omega)}.$$

Proof. The proof can be found in Hiptmair [66, Section 3]. □

Furthermore, we define for $\mathbf{u} \in \mathbf{H}(\mathrm{div}, \Omega)$ the **weak normal trace** $\gamma_n \mathbf{u}$ by

$$\langle \gamma_n \mathbf{u} \phi \rangle_{1/2,\Gamma} = (\mathrm{div}\,\mathbf{u}, \phi)_\Omega + (\mathbf{u}, \mathbf{grad}\,\phi)_\Omega \qquad \forall\, \phi \in H^1(\Omega). \tag{3.9}$$

Here, $\langle \cdot, \cdot \rangle_{1/2,\Gamma}$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Lemma 3.1.7 $\gamma_n : \mathbf{H}(\mathrm{div}, \Omega) \to H^{-1/2}(\Gamma)$ *is continuous and surjective.*

Proof. The continuity can be found in Girault & Raviart [49, Theorem 2.5] and the surjectivity is proven in Nédélec [84, Theorem 5.4.1]. □

For $\mathbf{u} \in \mathcal{C}^1(\overline{\Omega})$ there holds $\gamma_n \mathbf{u} = \mathbf{u} \cdot \mathbf{n}$.

3.2 Boundary integral operators

Here, we define the boundary integral operators which are used for the coupling formulations. For $\kappa \geq 0$ we define by

$$\Phi(\mathbf{x}, \mathbf{y}) := \frac{1}{4\pi} \frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|}$$

($\mathbf{x} \neq \mathbf{y}$), the fundamental solution of the Helmholtz equation, for $\kappa = 0$ we get the fundamental solution of the Laplace equation. There holds $\Delta\Phi(\mathbf{x}, \mathbf{y}) = -\kappa^2 \Phi(\mathbf{x}, \mathbf{y})$ and $\mathbf{grad}_{\mathbf{x}}\, \Phi(\mathbf{x}, \mathbf{y}) = -\,\mathbf{grad}_{\mathbf{y}}\, \Phi(\mathbf{x}, \mathbf{y})$. We then define the **scalar single layer potential** for $u \in L^2(\Gamma)$ by

$$S(u)(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x}, \mathbf{y}) u(\mathbf{x})\, dS(\mathbf{y}), \quad \mathbf{x} \notin \Gamma.$$

It can be extended to a continuous mapping $S : H^{-1/2}(\Gamma) \to H^1_{\mathrm{loc}}(\mathbb{R}^3)$ and satisfies the jump relations

$$[\gamma S(u)]_\Gamma = 0, \quad [\gamma\, \mathbf{grad}\, S(u)]_\Gamma = -u\mathbf{n}$$

with the normal $\mathbf{n}$ on Γ pointing into the exterior domain, where $[\gamma u]_\Gamma := \gamma^+ u - \gamma^- u$ denotes the jump of the trace γ of a function u over the boundary Γ and γ^+ and γ^- denote the exterior and interior traces. The second relation can be written as

$$[\gamma_n \,\mathbf{grad}\, S(u)]_\Gamma = -u, \quad [\mathbf{grad}_\Gamma\, S(u)]_\Gamma = 0. \tag{3.10}$$

This leads to the definition of the boundary integral operator

$$V(u)(\mathbf{x}) := \gamma S(u)(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \tag{3.11}$$

which is continuous from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ and defines a positive definite bilinear form on $H^{-1/2}(\Gamma)$ (cf. Costabel [39]). Analogously, we define the **vectorial single layer potential** for $\boldsymbol{\lambda} \in \mathbf{L}^2(\Gamma)$ by

$$\mathbf{V}(\boldsymbol{\lambda})(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}(\mathbf{y})\, dS(\mathbf{y}), \quad \mathbf{x} \notin \Gamma,$$

which can be extended to a continuous mapping from $\mathbf{H}_\parallel^{1/2}(\Gamma)$ to $\mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ (see Hiptmair [66, Section 5] or Buffa *et al.* [32, Theorem 3.8]). We will make use of the following result by MacCamy & Stephan [72] (see also [66]):

Lemma 3.2.1 *For* $\boldsymbol{\lambda} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *there holds*

$$\mathrm{div}\, \mathbf{V}(\boldsymbol{\lambda}) = V(\mathrm{div}_\Gamma\, \boldsymbol{\lambda}) \ \textit{in}\ \mathbf{L}^2(\mathbb{R}^3).$$

We define the **vectorial double layer potential** for $\boldsymbol{\lambda} \in \mathbf{H}_\perp^{-1/2}(\Gamma)$ by

$$\mathbf{K}(\boldsymbol{\lambda}) := \mathbf{curl}\, \mathbf{V}(\mathbf{n} \times \boldsymbol{\lambda})$$

and further

$$\mathbf{W}(\boldsymbol{\lambda}) := \mathbf{curl}\, \mathbf{K}(\boldsymbol{\lambda}) = \kappa^2 \mathbf{V}(\mathbf{n} \times \boldsymbol{\lambda}) + \mathbf{grad}\, \mathbf{V}(\mathrm{div}_\Gamma(\mathbf{n} \times \boldsymbol{\lambda})). \tag{3.12}$$

The last equation follows from the identity $\mathbf{curl}\,\mathbf{curl} \equiv \mathbf{grad}\,\mathrm{div} - \boldsymbol{\Delta}$, the fact that $\Delta\Phi = -\kappa^2\Phi$ and Lemma 3.2.1. Using the continuity of $\mathbf{V}$ and the fact that the mapping $\boldsymbol{\lambda} \to \mathbf{n} \times \boldsymbol{\lambda}$ is an isometry between $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ (this is a consequence of Lemma 3.1.3), one sees that $\mathbf{K}$ is a continuous mapping from $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ to $\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}\,\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) \cap \mathbf{H}(\mathrm{div}\, 0, \mathbb{R}^3 \setminus \Gamma)$, see Buffa *et al.* [32, section 3.3] and Hiptmair [66, section 5]).

The vectorial single and double layer potentials satisfy the following jump relations, compare [32, 66]: For $\boldsymbol{\lambda} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ there hold

$$[\gamma_D \mathbf{V}(\boldsymbol{\lambda})]_\Gamma = 0, \qquad [\gamma_N \mathbf{V}(\boldsymbol{\lambda})]_\Gamma = -\boldsymbol{\lambda}, \tag{3.13}$$

and for $\boldsymbol{\lambda} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ there hold

$$[\gamma_D \mathbf{K}(\boldsymbol{\lambda})]_\Gamma = \boldsymbol{\lambda}, \qquad [\gamma_N \mathbf{K}(\boldsymbol{\lambda})]_\Gamma = 0. \tag{3.14}$$

We now define the following vectorial boundary integral operators as exterior traces of the layer potentials on Γ

$$\begin{aligned}
\mathcal{V}(\boldsymbol{\lambda}) &:= \gamma_t \mathbf{V}(\boldsymbol{\lambda}) && = \gamma_D^+ \int_\Gamma \Phi(x,y)\boldsymbol{\lambda}(y)\, ds(y),\\
\mathcal{K}(\boldsymbol{\lambda}) &:= \gamma_t^+ \mathbf{K}(\boldsymbol{\lambda}) && = \gamma_D^+ \,\mathbf{curl}_x \int_\Gamma \Phi(x,y)(\mathbf{n}\times\boldsymbol{\lambda})(y)\, ds(y),\\
\tilde{\mathcal{K}}(\boldsymbol{\lambda}) &:= \gamma_N^+ \mathbf{V}(\boldsymbol{\lambda}) = (\gamma_t^\times)^+ \mathbf{K}(\boldsymbol{\lambda}\times\mathbf{n}) && = \gamma_N^+ \int_\Gamma \Phi(x,y)\boldsymbol{\lambda}(y)\, ds(y),\\
\mathcal{W}(\boldsymbol{\lambda}) &:= \gamma_N \mathbf{K}(\boldsymbol{\lambda}) = (\gamma_t^\times)^+ \mathbf{W}(\boldsymbol{\lambda}) && = \gamma_N^+ \,\mathbf{curl}_x \int_\Gamma \Phi(x,y)(\mathbf{n}\times\boldsymbol{\lambda})(y)\, ds(y)
\end{aligned}$$

for $\mathbf{x} \in \Gamma$.

From the regularity properties of the potentials and the trace operators we get the following lemma, see also Hiptmair [66] for the case $\kappa = 0$ and Hiptmair [67] for $\kappa > 0$.

Lemma 3.2.2 *The operators*

$$\begin{aligned}
\mathcal{V} &: \mathbf{H}_\parallel^{-1/2}(\Gamma) \to \mathbf{H}_\parallel^{1/2}(\Gamma),\\
\mathcal{K} &: \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma),\\
\tilde{\mathcal{K}} &: \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma),\\
\mathcal{W} &: \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)
\end{aligned}$$

are continuous.

For the case of the Laplace kernel with $\kappa = 0$ there holds

Lemma 3.2.3 *The boundary integral operators satisfy the following properties:*

1. *The bilinear form induced on $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$ by $\mathcal{V}_0$ is symmetric and elliptic, i.e. there exists a constant $c > 0$, such that*

$$\langle \mathcal{V}_0 \mathbf{u}, \mathbf{u}\rangle_\Gamma \geq c\|\mathbf{u}\|^2_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)} \quad \forall\, \mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma).$$

2. *The boundary integral operator $\tilde{\mathcal{K}}_0$ is adjoint to $\mathcal{K}_0 - \mathcal{I}$, i.e.*

$$\langle \tilde{\mathcal{K}}_0 \mathbf{u}, \mathbf{v}\rangle_\Gamma = \langle \mathbf{u}, (\mathcal{K}_0 - \mathcal{I})\mathbf{v}\rangle_\Gamma \quad \forall \mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma), \mathbf{v} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma).$$

3. *There holds with the pairing* $\langle\cdot,\cdot\rangle_{-1/2,\Gamma}$ *between* $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ *and* $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$

$$\langle \mathcal{W}_0\mathbf{u},\mathbf{v}\rangle_\Gamma = -\langle V_0(\mathrm{curl}_\Gamma\,\mathbf{u}),\mathrm{curl}_\Gamma\,\mathbf{v}\rangle_{-1/2,\Gamma} \quad \forall \mathbf{u},\mathbf{v}\in\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma).$$

4. *The bilinear form induced on* $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ *by* $\mathcal{W}_0$ *is symmetric and negative semidefinite, in particular there exists a constant* $C>0$ *such that*

$$-\langle \mathcal{W}_0\mathbf{u},\mathbf{u}\rangle_\Gamma \geq C\|\mathrm{curl}_\Gamma\,\mathbf{u}\|^2_{H^{-1/2}(\Gamma)} \quad \forall \mathbf{u}\in\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma).$$

Proof. See [66] for all proofs. The fourth statement is a direct consequence of 3. □

We now define integral operators for $\boldsymbol{\lambda}\in\mathbf{L}_t^2(\Gamma)$ and $\mathbf{x}\in\Gamma$ by

$$\boldsymbol{\mathcal{L}}\boldsymbol{\lambda}(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}(\mathbf{y})\,dS(\mathbf{y}),$$

$$\mathcal{M}\boldsymbol{\lambda}(\mathbf{x}) := \int_\Gamma \mathbf{curl}_\mathbf{x}(\Phi(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}(\mathbf{y}))\,dS(\mathbf{y}) = \int_\Gamma \mathbf{grad}_\mathbf{x}\,\Phi(\mathbf{x},\mathbf{y})\times\boldsymbol{\lambda}(\mathbf{y})\,dS(\mathbf{y}).$$

The above integral can be defined as Cauchy-principal value. Using the jump conditions one can prove the following representation of the boundary integral operators, see e.g. Mitrea *et al.* [76, Section 3] and Colton & Kress [38, Section 6.3].

$$\begin{aligned}
\mathcal{V}\boldsymbol{\lambda} &= -\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\mathcal{L}}\boldsymbol{\lambda}),\\
\mathcal{K}\boldsymbol{\lambda} &= \mathcal{M}(\mathbf{n}\times\boldsymbol{\lambda})+\frac{1}{2}\boldsymbol{\lambda},\\
\tilde{\mathcal{K}}\boldsymbol{\lambda} &= -\mathbf{n}\times\mathcal{M}\boldsymbol{\lambda}-\frac{1}{2}\boldsymbol{\lambda},\\
\mathcal{W}\boldsymbol{\lambda} &= -\kappa^2\mathbf{n}\times\boldsymbol{\mathcal{L}}(\mathbf{n}\times\boldsymbol{\lambda})-\mathbf{n}\times\mathbf{grad}\,V(\mathrm{div}_\Gamma(\mathbf{n}\times\boldsymbol{\lambda}))\\
&= -\kappa^2\mathbf{n}\times\boldsymbol{\mathcal{L}}(\mathbf{n}\times\boldsymbol{\lambda})-\mathbf{curl}_\Gamma\,V(\mathrm{curl}_\Gamma\boldsymbol{\lambda}).
\end{aligned} \tag{3.15}$$

The last equation holds due to $\mathbf{n}\times\mathbf{grad}\,\phi = -\,\mathbf{curl}_\Gamma\,\phi$ and $\mathrm{div}_\Gamma(\mathbf{n}\times\boldsymbol{\lambda}) = -\,\mathrm{curl}_\Gamma\,\boldsymbol{\lambda}$.

Using these relations we can prove the useful equation:

Lemma 3.2.4 *For* $\mathbf{u}$, $\mathbf{v}\in\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ *there holds*

$$\langle \mathcal{W}\mathbf{u},\mathbf{v}\rangle = \kappa^2\langle \mathcal{V}(\mathbf{u}\times\mathbf{n}),\mathbf{v}\times\mathbf{n}\rangle - \langle V\,\mathrm{curl}_\Gamma(\mathbf{u}),\mathrm{curl}_\Gamma(\mathbf{v})\rangle. \tag{3.16}$$

Proof. If $\mathbf{u}$, $\mathbf{v}\in\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$, we get

$$\langle \mathcal{W}\mathbf{u},\mathbf{v}\rangle = -\kappa^2\langle \mathbf{n}\times\boldsymbol{\mathcal{L}}(\mathbf{n}\times\mathbf{u}),\mathbf{v}\rangle - \langle \mathbf{curl}_\Gamma\,V(\mathrm{curl}_\Gamma\,\mathbf{u}),\mathbf{v}\rangle. \tag{3.17}$$

For the second term there holds due to Lemma 3.1.1

$$\langle \mathbf{curl}_\Gamma\,V(\mathrm{curl}_\Gamma\,\mathbf{u}),\mathbf{v}\rangle = \langle V(\mathrm{curl}_\Gamma\,\mathbf{u}),\mathrm{curl}_\Gamma\,\mathbf{v}\rangle.$$

The first term in 3.17 can be modified the following way. First of all, we remark that there holds for $\mathbf{v}$ with $\mathbf{v} \cdot \mathbf{n} = 0$ the equation $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u} \times \mathbf{n}, \mathbf{v} \times \mathbf{n} \rangle$ because of

$$(\mathbf{u} \times \mathbf{n}) \cdot (\mathbf{v} \times \mathbf{n}) = (\mathbf{u} \cdot \mathbf{v})(\mathbf{n} \cdot \mathbf{n}) - (\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}) = \mathbf{u} \cdot \mathbf{v}.$$

Thus, we get

$$\begin{aligned} \langle \mathbf{n} \times \boldsymbol{\mathcal{L}}(\mathbf{n} \times \mathbf{u}), \mathbf{v} \rangle &= -\langle \big(\boldsymbol{\mathcal{L}}(\mathbf{n} \times \mathbf{u}) \times \mathbf{n}\big) \times \mathbf{n} \rangle, \mathbf{v} \times \mathbf{n} \rangle \\ &= -\langle \gamma_D \boldsymbol{\mathcal{L}}(\mathbf{u} \times \mathbf{n}), \mathbf{v} \times \mathbf{n} \rangle \\ &= -\langle \mathcal{V}(\mathbf{u} \times \mathbf{n}), \mathbf{v} \times \mathbf{n} \rangle \end{aligned}$$

which finishes the proof. □

The following Lemma (cf. Teltscher [103]) is necessary when deriving the residual error estimator in Chapter 5. It holds for the case of the Laplace kernel with $\kappa = 0$.

Lemma 3.2.5 *For* $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega_E)$, $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$ *there holds*

1. $\mathrm{div}_\Gamma \tilde{\mathcal{K}}_0 \boldsymbol{\lambda} = 0$ *in* $H^{-1/2}(\Gamma)$,
2. $\mathrm{div}_\Gamma \mathcal{W}_0 \gamma_D \mathbf{u} = 0$ *in* $H^{-1/2}(\Gamma)$.

Proof. For simplicity, we omit the index 0 at the integral operators.
For $\eta \in H^{1/2}(\Gamma)$ the definitions of div_Γ, $\tilde{\mathcal{K}}$ and γ_N together with the Green formula in Lemma 3.1.5 yield

$$\begin{aligned} \langle \mathrm{div}_\Gamma \tilde{\mathcal{K}} \boldsymbol{\lambda}, \eta \rangle_\Gamma &= -\langle \tilde{\mathcal{K}} \boldsymbol{\lambda}, \mathbf{grad}_\Gamma \eta \rangle_\Gamma = -\langle \gamma_N^+ \mathbf{V} \boldsymbol{\lambda}, \mathbf{grad}_\Gamma \eta \rangle_\Gamma \\ &= \int_{\Omega_E} (\mathbf{curl}\, \mathbf{V} \boldsymbol{\lambda} \cdot \mathbf{curl}\, \mathbf{grad}\, \eta - \mathbf{curl}\, \mathbf{curl}\, \mathbf{V} \boldsymbol{\lambda} \cdot \mathbf{grad}\, \psi) \, d\mathbf{x}. \end{aligned}$$

The first term vanishes because of $\mathbf{curl}\, \mathbf{grad}\, \eta = 0$ and for the second one we get $\mathbf{curl}\, \mathbf{curl}\, \mathbf{V} \boldsymbol{\lambda} = (\mathbf{grad}\, \mathrm{div} - \boldsymbol{\Delta}) \mathbf{V} \boldsymbol{\lambda} = \mathbf{grad}\, \mathrm{div}\, \mathbf{V} \boldsymbol{\lambda}$, since $\boldsymbol{\Delta}\, \mathbf{V} = 0$ in $\mathbb{R}^3 \setminus \Gamma$. From Lemma 3.2.1 we get $\mathbf{grad}\, \mathrm{div}\, \mathbf{V} \boldsymbol{\lambda} = \mathbf{grad}\, V(\mathrm{div}_\Gamma \boldsymbol{\lambda}) = 0$, because $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$. Thus, we have $\mathbf{curl}\, \mathbf{curl}\, \mathbf{V} \boldsymbol{\lambda} = 0$, and altogether $\langle \mathrm{div}_\Gamma \tilde{\mathcal{K}} \boldsymbol{\lambda}, \eta \rangle_\Gamma = 0$, which proves the first assertion. The proof of the second proposition uses the same ideas: For $\mu \in H^{1/2}(\Gamma)$ there holds

$$\begin{aligned} &\langle \mathrm{div}_\Gamma \mathcal{W} \gamma_D \mathbf{u}, \mu \rangle_\Gamma = -\langle \mathcal{W} \gamma_D \mathbf{u}, \mathbf{grad}_\Gamma \mu \rangle_\Gamma = -\langle \gamma_N^+ \mathbf{curl}\, \mathbf{V}(\mathbf{n} \times \mathbf{u}), \mathbf{grad}_\Gamma \mu \rangle_\Gamma \\ &= \int_{\Omega_E} (\mathbf{curl}\, \mathbf{curl}\, \mathbf{V}(\mathbf{n} \times \mathbf{u}) \cdot \mathbf{curl}\, \mathbf{grad}\, \mu - \mathbf{curl}\, \mathbf{curl}\, \mathbf{curl}\, \mathbf{V}(\mathbf{n} \times \mathbf{u}) \cdot \mathbf{grad}\, \mu) \, d\mathbf{x}. \end{aligned}$$

Again, we have $\mathbf{curl}\, \mathbf{grad}\, \mu = 0$, and there also holds $\mathbf{curl}\, \mathbf{curl}\, \mathbf{curl}\, \mathbf{V} = -\, \mathbf{curl}\, \boldsymbol{\Delta}\, \mathbf{V} = 0$ such that we get $\langle \mathrm{div}_\Gamma \mathcal{W} \gamma_D \mathbf{u}, \mu \rangle_\Gamma = 0$. □

3.2.1 The Stratton-Chu representation formula

In this Section we introduce an integral representation formula for the solutions of the Maxwell's equations. This is the main ingredient to derive the coupling formulations in the next Chapters. The formula is based on the results of Stratton & Chu [101]. We cite here Colton & Kress [38] for smooth boundaries but the results also hold for Lipschitz boundaries, see e.g. Buffa, Costabel & Schwab [29, Theorem 3].

Here, we consider the Maxwell's equations

$$\mathbf{curl}\,\mathbf{E} - i\kappa\mathbf{H} = 0, \tag{3.18}$$

$$\mathbf{curl}\,\mathbf{H} + i\kappa\mathbf{E} = 0 \tag{3.19}$$

where $\mathbf{E}$ and $\mathbf{H}$ denote the electric and the magnetic field, resp. Thus, there holds for $\mathbf{E}$

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = \kappa^2\mathbf{E}.$$

Let $\Phi(\mathbf{x},\mathbf{y}) := \frac{1}{4\pi}\frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$, $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \neq \mathbf{y}$ be the fundamental solution of the Helmholtz equation. We get the following representation theorem, see Colton & Kress [38].

Theorem 3.2.6 (Stratton-Chu formula) *Let Ω be a bounded domain with smooth boundary and let $\mathbf{n}$ denote the unit normal vector to the boundary $\Gamma = \partial\Omega$ directed into the exterior of Ω. Let $\mathbf{E}$, $\mathbf{H} \in \mathcal{C}^1 \cap \mathcal{C}(\overline{\Omega})$ be a solution to the Maxwell's equations* (3.18) *and* (3.19) *in Ω. Thus, there hold the Stratton-Chu formulas*

$$\begin{aligned}\mathbf{E}(\mathbf{x}) = &-\mathbf{curl}\int_\Gamma \big(\mathbf{n}(\mathbf{y}) \times \mathbf{E}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\,dS(\mathbf{y})\\ &+\frac{1}{i\kappa}\,\mathbf{curl}\,\mathbf{curl}\int_\Gamma \big(\mathbf{n}(\mathbf{y}) \times \mathbf{H}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\,dS(\mathbf{y}), \qquad \mathbf{x} \in \Omega,\end{aligned}$$

and

$$\begin{aligned}\mathbf{H}(\mathbf{x}) = &-\mathbf{curl}\int_\Gamma \big(\mathbf{n}(\mathbf{y}) \times \mathbf{H}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\,dS(\mathbf{y})\\ &-\frac{1}{i\kappa}\,\mathbf{curl}\,\mathbf{curl}\int_\Gamma \big(\mathbf{n}(\mathbf{y}) \times \mathbf{E}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\,dS(\mathbf{y}), \qquad \mathbf{x} \in \Omega.\end{aligned}$$

For the unbounded domain there holds

Theorem 3.2.7 (Stratton-Chu formula) *Let $\Omega_E := \mathbb{R}^3 \setminus \overline{\Omega}$, where Ω is a smooth domain and let $\mathbf{n}$ denote the unit normal vector to the boundary $\partial\Omega$ directed into the exterior of Ω_E. Let $\mathbf{E}$, $\mathbf{H} \in \mathcal{C}^1(\Omega_E) \cap \mathcal{C}(\Omega_E)$ be a solution to the Maxwell's equations* (3.18) *and* (3.19) *in Ω_E. Furthermore, we assume that $\mathbf{E}$ and $\mathbf{H}$ satisfy the Silver-Müller radiation conditions*

$$\lim_{|\mathbf{x}|\to\infty}(\mathbf{H} \times \mathbf{x} - |\mathbf{x}|\mathbf{E}) = 0 \tag{3.20}$$

or

$$\lim_{|\mathbf{x}|\to\infty} (\mathbf{E} \times \mathbf{x} - |\mathbf{x}|\mathbf{H}) = 0 \tag{3.21}$$

uniformly in all directions $\frac{\mathbf{x}}{|\mathbf{x}|}$. *Then, there holds*

$$\begin{aligned}\mathbf{E}(\mathbf{x}) = &\,\mathbf{curl} \int_\Gamma \big(\mathbf{n}(\mathbf{y}) \times \mathbf{E}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}) \\ &- \frac{1}{i\kappa}\,\mathbf{curl}\,\mathbf{curl} \int_\Gamma \big(\mathbf{n}(\mathbf{y}) \times \mathbf{H}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}), \qquad \mathbf{x} \in \Omega_E,\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\mathbf{H}(\mathbf{x}) = &\,\mathbf{curl} \int_\Gamma \big(\mathbf{n}(\mathbf{y}) \times \mathbf{H}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}) \\ &+ \frac{1}{i\kappa}\,\mathbf{curl}\,\mathbf{curl} \int_\Gamma \big(\mathbf{n}(\mathbf{y}) \times \mathbf{E}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}), \qquad \mathbf{x} \in \Omega_E.\end{aligned} \tag{3.23}$$

Furthermore, there holds, see [38, (6.10)],

$$\begin{aligned}&\frac{1}{i\kappa}\,\mathbf{curl}\,\mathbf{curl} \int_\Gamma \big(\mathbf{n}(\mathbf{y}) \times \mathbf{H}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}) \\ &= -i\kappa \int_\Gamma \big(\mathbf{n}(\mathbf{y}) \times \mathbf{H}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}) + \mathbf{grad} \int_\Gamma \big(\mathbf{n}(\mathbf{y}) \cdot \mathbf{E}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}).\end{aligned} \tag{3.24}$$

Thus, using $\mathbf{H} = \frac{1}{i\kappa}\,\mathbf{curl}\,\mathbf{E}$, the relation (3.22) can be rewritten as

$$\begin{aligned}\mathbf{E}(\mathbf{x}) = &\,\mathbf{curl} \int_\Gamma \big(\mathbf{n}(\mathbf{y}) \times \mathbf{E}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}) \\ &+ \int_\Gamma \big(\mathbf{n}(\mathbf{y}) \times \mathbf{curl}\,\mathbf{E}(\mathbf{y})\big)\, \Phi(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}) \\ &- \mathbf{grad} \int_\Gamma \big(\mathbf{n}(\mathbf{y}) \cdot \mathbf{E}(\mathbf{y})\big)\Phi(\mathbf{x},\mathbf{y})\, dS(\mathbf{y}), \quad \mathbf{x} \in \Omega_E.\end{aligned} \tag{3.25}$$

In Chapters 5 and 6 we use this formula for the derivation of the the coupling formulations.

3.3 Trace spaces of order s

In this section we extend the definitions and results of the previous sections to the spaces of order s on polyhedral and Lipschitz domains. This section provides a generalization of the definition of $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ to trace spaces of $\mathbf{H}^s(\mathbf{curl}, \Omega) := \{\mathbf{u} \in \mathbf{H}^s(\Omega) : \mathbf{curl}\,\mathbf{u} \in \mathbf{H}^s(\Omega)\}$.

Here, we refer to Buffa *et al* [30, 31, 26, 25] as main references.

We consider a polyhedral domain Ω with boundary faces Γ_j, $j = 1, \ldots, N_\Gamma$. First of all, we define the spaces

$$\begin{aligned} \mathbf{H}_{\perp}^{s}(\Gamma) &:= \gamma_t^{\times}(\mathbf{H}^{s+1/2}(\Omega)), \\ \mathbf{H}_{\|}^{s}(\Gamma) &:= \gamma_D(\mathbf{H}^{s+1/2}(\Omega)), \end{aligned}$$

for $0 < s < 1$.

Remark 3.3.1 *For $s = \frac{1}{2}$ we get the same definitions as in* (3.4) *and* (3.5)*, cf. Buffa [25].*

The spaces $\mathbf{H}_{\perp}^{-s}(\Gamma)$ and $\mathbf{H}_{\|}^{-s}(\Gamma)$, $0 < s < 1$ are then defined as the dual spaces of $\mathbf{H}_{\perp}^{s}(\Gamma)$ and $\mathbf{H}_{\|}^{s}(\Gamma)$, resp., with $\mathbf{L}_t^2(\Gamma)$ as pivot space.

For any $s > \frac{1}{2}$ we define

$$\mathbf{H}_{-}^{s}(\Gamma) := \{\mathbf{u} \in \mathbf{L}_t^2(\Gamma) \,:\, \mathbf{u}_{|\Gamma_j} \in \mathbf{H}_t^s(\Gamma_j),\, j = 1, \ldots, N_\Gamma\}.$$

We further define

$$\mathbf{H}_{\|}^{s}(\mathrm{div}_\Gamma, \Gamma) := \begin{cases} \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma), & s = -\frac{1}{2}, \\ \{\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{s}(\Gamma),\, \mathrm{div}_\Gamma\,\boldsymbol{\lambda} \in H^s(\Gamma)\}, & -\frac{1}{2} < s < \frac{1}{2}, \\ \{\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{s}(\Gamma),\, \mathrm{div}_\Gamma\,\boldsymbol{\lambda} \in H_{-}^{s}(\Gamma)\}, & s > \frac{1}{2}, \end{cases}$$

$$\mathbf{H}_{\perp}^{s}(\mathrm{curl}_\Gamma, \Gamma) := \begin{cases} \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma), & s = -\frac{1}{2}, \\ \{\boldsymbol{\lambda} \in \mathbf{H}_{\perp}^{s}(\Gamma),\, \mathrm{curl}_\Gamma\,\boldsymbol{\lambda} \in H^s(\Gamma)\}, & -\frac{1}{2} < s < \frac{1}{2}, \\ \{\boldsymbol{\lambda} \in \mathbf{H}_{\perp}^{s}(\Gamma),\, \mathrm{curl}_\Gamma\,\boldsymbol{\lambda} \in H_{-}^{s}(\Gamma)\}, & s > \frac{1}{2}. \end{cases}$$

The following results can be found in Buffa & Christiansen [26], see also Buffa & Hiptmair [31].

Lemma 3.3.2 *The trace mappings γ_D and $\gamma_t^{\times}$ can be extended to continuous mappings*

$$\begin{aligned} \gamma_D &: \mathbf{H}^s(\mathbf{curl}, \Omega) \to \mathbf{H}_{\perp}^{s-1/2}(\mathrm{curl}_\Gamma, \Gamma), \\ \gamma_t^{\times} &: \mathbf{H}^s(\mathbf{curl}, \Omega) \to \mathbf{H}_{\|}^{s-1/2}(\mathrm{div}_\Gamma, \Gamma) \end{aligned}$$

for all $0 \le s < 1$.

3.3.1 Mapping properties of the integral operators

Next, we examine the mapping behavior of the integral operators due to Maxwell's equations. At first we only consider smooth surfaces and use the Fourier transformation and some results of Costabel & Stephan [40]. These results can also be extended to the case of polyhedral domains Ω.

Here, we only consider the integral operators with Laplace kernel, i.e. $\Phi(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}$.

Theorem 3.3.3 *Let Γ be a smooth surface. Then,*

$$\mathcal{W}_0 : \mathbf{H}^s_\perp(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}^s_\parallel(\mathrm{div}_\Gamma, \Gamma), \quad -\frac{1}{2} \le s < \frac{1}{2},$$

is a continuous mapping, i.e. there exists a constant $C > 0$, such that

$$\|\mathcal{W}_0\boldsymbol{\lambda}\|_{\mathbf{H}^s_\parallel(\mathrm{div}_\Gamma,\Gamma)} \le C\|\boldsymbol{\lambda}\|_{\mathbf{H}^s_\perp(\mathrm{curl}_\Gamma,\Gamma)}$$

for all $\boldsymbol{\lambda} \in \mathbf{H}^s_\perp(\mathrm{curl}_\Gamma, \Gamma)$.

Proof. In the beginning, we remark that there holds

$$\mathcal{W}_0\boldsymbol{\lambda} = -\mathbf{n} \times \mathbf{grad}\, V(\mathbf{n} \cdot \mathbf{curl}\,\boldsymbol{\lambda}) = -\mathbf{n} \times \mathbf{grad}\, V(\mathrm{div}_\Gamma(\mathbf{n} \times \boldsymbol{\lambda})). \tag{3.26}$$

Since Γ is smooth we consider a conformal mapping of the neighborhood of a point $\mathbf{x} \in \Gamma$ onto $\mathbb{R}^2$. Thus, we examine the behavior of $\mathcal{W}_0$ on $\mathbb{R}^2$ as subset of $\mathbb{R}^3$.
We consider the Fourier transformation of a function

$$\hat{\mathbf{u}}(\boldsymbol{\zeta}) = \int_{\mathbb{R}^2} e^{i\mathbf{x}\cdot\boldsymbol{\zeta}}\mathbf{u}(\mathbf{x})\, d\mathbf{x}, \quad \boldsymbol{\zeta} = (\zeta_1, \zeta_2) \in \mathbb{R}^2.$$

First of all, we construct the Fourier transformation of (3.26). Therefore, we remark that there holds

$$\widehat{\frac{\partial \mathbf{u}}{\partial x_j}}(\boldsymbol{\zeta}) = \int_{\mathbb{R}^2} e^{i\mathbf{x}\cdot\boldsymbol{\zeta}}\frac{\partial \mathbf{u}}{\partial x_j}(\mathbf{x})\, d\mathbf{x} = i\zeta_j\hat{\mathbf{u}}(\boldsymbol{\zeta}), \qquad j = 1, 2. \tag{3.27}$$

For the Fourier transformation of a convolution

$$(g * f)(x) := \int_{\mathbb{R}^2} g(x-y)f(y)\, dy$$

there holds

$$\widehat{(g * f)}(\zeta) = \hat{g}(\zeta)\,\hat{f}(\zeta).$$

Hence, the Fourier transformation of the single layer potential is

$$\widehat{V\psi}(\boldsymbol{\zeta}) = -\frac{1}{|\boldsymbol{\zeta}|}\hat{\psi}(\boldsymbol{\zeta}),$$

with $|\boldsymbol{\zeta}| = \sqrt{\zeta_1^2 + \zeta_2^2}$, see Costabel & Stephan [40]. Using [40, Lemma 4.5] we derive the Fourier transformations of the differential operators in (3.26).

$$\widehat{\operatorname{div}_\Gamma(\mathbf{n} \times \boldsymbol{\lambda})}(\boldsymbol{\zeta}) = i(\zeta_1, \zeta_2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{\lambda}_1(\boldsymbol{\zeta}) \\ \hat{\lambda}_2(\boldsymbol{\zeta}) \end{pmatrix} = -i\zeta_1 \hat{\lambda}_2(\boldsymbol{\zeta}) + i\zeta_2 \hat{\lambda}_1(\boldsymbol{\zeta})$$

$$\begin{aligned} \widehat{\mathbf{n} \times \mathbf{grad}\,\boldsymbol{\tau}} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{\frac{\partial \boldsymbol{\tau}}{\partial x_1}} \\ \widehat{\frac{\partial \boldsymbol{\tau}}{\partial x_2}} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i\zeta_1 \hat{\boldsymbol{\tau}} \\ i\zeta_2 \hat{\boldsymbol{\tau}} \end{pmatrix} \\ &= \begin{pmatrix} -i\zeta_2 \hat{\tau} \\ i\zeta_1 \hat{\tau} \end{pmatrix}. \end{aligned}$$

Finally, we get for the hypersingular operator

$$\begin{aligned} \widehat{\mathcal{W}_0 \boldsymbol{\lambda}}(\boldsymbol{\zeta}) &= \widehat{\mathbf{n} \times \mathbf{grad}\, V(\mathbf{n} \cdot \mathbf{curl}\, \boldsymbol{\lambda})}(\boldsymbol{\zeta}) \\ &= \frac{1}{|\boldsymbol{\zeta}|} \begin{pmatrix} -\zeta_1\zeta_2 \hat{\lambda}_2(\boldsymbol{\zeta}) + \zeta_2^2 \hat{\lambda}_1(\boldsymbol{\zeta}) \\ \zeta_1^2 \hat{\lambda}_2(\boldsymbol{\zeta}) - \zeta_1\zeta_2 \hat{\lambda}_1(\boldsymbol{\zeta}) \end{pmatrix}. \end{aligned}$$

Furthermore, there holds

$$\begin{aligned} \widehat{\operatorname{div}_\Gamma \mathcal{W}_0 \boldsymbol{\lambda}}(\boldsymbol{\zeta}) &= i(\zeta_1, \zeta_2) \begin{pmatrix} -\zeta_1\zeta_2 \hat{\lambda}_2 + \zeta_2^2 \hat{\lambda}_1 \\ \zeta_1^2 \hat{\lambda}_2 - \zeta_1\zeta_2 \hat{\lambda}_1 \end{pmatrix} \frac{1}{|\boldsymbol{\zeta}|} \\ &= i(-\zeta_1^2\zeta_2 \hat{\lambda}_2 + \zeta_1\zeta_2^2 \hat{\Lambda}_1 + \zeta_1^2\zeta_2 \hat{\lambda}_2 - \zeta_1\zeta_2^2 \hat{\lambda}_1) \frac{1}{|\boldsymbol{\zeta}|} = 0. \end{aligned} \tag{3.28}$$

For $\mathbf{u} \in \mathbf{H}^s(\mathbb{R}^2)$ the norm can be defined by

$$\|\mathbf{u}\|^2_{\mathbf{H}^s(\mathbb{R}^2)} := \int_{\mathbb{R}^2} |\hat{\mathbf{u}}(\boldsymbol{\zeta})|^2 (1 + |\boldsymbol{\zeta}|^2)^s \, d\boldsymbol{\zeta}. \tag{3.29}$$

In order to calculate the $\mathbf{H}^s_\|(\operatorname{div}_\Gamma, \Gamma)$-norm of $\mathbf{z}(\boldsymbol{\zeta}) := \mathcal{W}_0 \boldsymbol{\lambda}$, we thus have to consider solely (because of (3.28))

$$\|\mathcal{W}_0 \boldsymbol{\lambda}\|^2_{\mathbf{H}^s(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \hat{\mathbf{z}}^\mathsf{T}(\boldsymbol{\zeta}) \overline{\hat{\mathbf{z}}}(\boldsymbol{\zeta}) (1 + |\boldsymbol{\zeta}|^2)^s \, d\boldsymbol{\zeta}.$$

There holds

$$\begin{aligned} |\widehat{\mathcal{W}_0 \boldsymbol{\lambda}}(\boldsymbol{\zeta})|^2 &= \frac{1}{|\boldsymbol{\zeta}|^2} \Big\{ (-\zeta_1\zeta_2 \hat{\lambda}_2 + \zeta_2^2 \hat{\lambda}_1)(-\zeta_1\zeta_2 \overline{\hat{\lambda}_2} + \zeta_2^2 \overline{\hat{\lambda}_1}) \\ &\qquad + (\zeta_1^2 \hat{\lambda}_2 - \zeta_1\zeta_2 \hat{\lambda}_1)(\zeta_1^2 \overline{\hat{\lambda}_2} - \zeta_1\zeta_2 \overline{\hat{\lambda}_1}) \Big\} \\ &= \frac{1}{|\boldsymbol{\zeta}|^2} \Big\{ \zeta_1^2\zeta_2^2 |\hat{\lambda}_2|^2 - \zeta_1\zeta_2^3 \overline{\hat{\lambda}_1} \hat{\lambda}_2 - \zeta_1\zeta_2^3 \hat{\lambda}_1 \overline{\hat{\lambda}_2} + \zeta_2^4 |\hat{\lambda}_1|^2 + \zeta_1^4 |\hat{\lambda}_2|^2 \\ &\qquad - \zeta_1^3\zeta_2 \overline{\hat{\lambda}_1} \hat{\lambda}_2 - \zeta_1^3\zeta_2 \hat{\lambda}_1 \overline{\hat{\lambda}_2} + \zeta_1^2\zeta_2^2 |\hat{\lambda}_1|^2 \Big\} \\ &= \zeta_1^2 |\hat{\lambda}_2|^2 - \zeta_1\zeta_2 \hat{\lambda}_1 \overline{\hat{\lambda}_2} - \zeta_1\zeta_2 \overline{\hat{\lambda}_1} \hat{\lambda}_2 + \zeta_2^2 |\hat{\lambda}_1|^2. \end{aligned} \tag{3.30}$$

The calculation of the $\mathbf{H}^s_\perp(\mathrm{curl}_\Gamma,\Gamma)$-norm of $\boldsymbol{\lambda}$ is done via

$$\widehat{\mathrm{curl}_\Gamma \boldsymbol{\lambda}} = i(-\zeta_2,\zeta_1)\begin{pmatrix}\hat{\lambda}_1\\ \hat{\lambda}_2\end{pmatrix} = i(-\zeta_2\hat{\lambda}_1+\zeta_1\hat{\lambda}_2) \tag{3.31}$$

Here, we used again [40, Theorem 4.5], and we get

$$\begin{aligned} |\widehat{\mathrm{curl}_\Gamma \boldsymbol{\lambda}}|^2 &= (-\zeta_2\hat{\lambda}_1+\zeta_1\hat{\lambda}_2)(-\zeta_2\overline{\hat{\lambda}_1}+\zeta_1\overline{\hat{\lambda}_2})\\ &= \zeta_2^2|\hat{\lambda}_1|^2-\zeta_1\zeta_2\hat{\lambda}_1\overline{\hat{\lambda}_2}-\zeta_1\zeta_2\overline{\hat{\lambda}_1}\hat{\lambda}_2+\zeta_1^2|\hat{\lambda}_2|^2. \end{aligned} \tag{3.32}$$

A comparison of (3.30) and (3.32) shows that there holds

$$|\widehat{\mathcal{W}_0\boldsymbol{\lambda}}| = |\widehat{\mathrm{curl}_\Gamma \boldsymbol{\lambda}}|.$$

Therefore, we get

$$\|\mathcal{W}_0\lambda\|_{\mathbf{H}^s_\parallel(\mathrm{div}_\Gamma,\Gamma)} \le \|\boldsymbol{\lambda}\|_{\mathbf{H}^s_\perp(\mathrm{curl}_\Gamma,\Gamma)}.$$

This finishes the proof. □

Theorem 3.3.4 *Let Γ be a smooth surface. Then,*

$$\mathcal{V}_0:\mathbf{H}^s_\parallel(\mathrm{div}_\Gamma,\Gamma)\to\mathbf{H}^s_\perp(\mathrm{curl}_\Gamma,\Gamma),\quad -\frac{1}{2}\le s<\frac{1}{2},$$

is a continuous mapping.

Proof. The proof is follows the same lines as the proof of Theorem 3.3.3. We consider the Fourier transformation of $\mathcal{V}_0\boldsymbol{\lambda} = -\mathbf{n}\times(\mathbf{n}\times\mathbf{V}\boldsymbol{\lambda})$. Using the results of Costabel & Stephan [40] we get for the Fourier transformation

$$\widehat{\mathbf{V}\boldsymbol{\psi}}(\boldsymbol{\zeta}) = -\frac{1}{|\boldsymbol{\zeta}|}\widehat{\boldsymbol{\psi}}(\boldsymbol{\zeta}),$$

$$\widehat{\mathbf{n}\times\boldsymbol{\lambda}} = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\widehat{\lambda}_1\\ \widehat{\lambda}_2\end{pmatrix}.$$

It follows that

$$\mathbf{n}\times\widehat{(\mathbf{n}\times\boldsymbol{\lambda})} = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\widehat{\lambda}_1\\ \widehat{\lambda}_2\end{pmatrix} = -\begin{pmatrix}\widehat{\lambda}_1\\ \widehat{\lambda}_2\end{pmatrix}$$

and

$$\widehat{\mathcal{V}_0\boldsymbol{\lambda}}(\boldsymbol{\zeta}) = \widehat{\gamma_D V\boldsymbol{\lambda}}(\boldsymbol{\zeta}) = \widehat{\mathbf{V}\boldsymbol{\lambda}}(\boldsymbol{\zeta}) = -\frac{1}{|\boldsymbol{\zeta}|}\widehat{\boldsymbol{\lambda}}(\boldsymbol{\zeta}).$$

Due to

$$\widehat{\operatorname{div}_\Gamma \boldsymbol{\lambda}}(\boldsymbol{\zeta}) = i(\zeta_1, \zeta_2)\begin{pmatrix}\widehat{\lambda}_1(\boldsymbol{\zeta})\\ \widehat{\lambda}_2(\boldsymbol{\zeta})\end{pmatrix} = i\left(\zeta_1\widehat{\lambda}_1(\boldsymbol{\zeta}) + \zeta_2\widehat{\lambda}_2(\boldsymbol{\zeta})\right),$$

$$\widehat{\operatorname{curl}_\Gamma \boldsymbol{\lambda}}(\boldsymbol{\zeta}) = i(-\zeta_2, \zeta_1)\begin{pmatrix}\widehat{\lambda}_1(\boldsymbol{\zeta})\\ \widehat{\lambda}_2(\boldsymbol{\zeta})\end{pmatrix} = i\left(-\zeta_2\widehat{\lambda}_1(\boldsymbol{\zeta}) + \zeta_1\widehat{\lambda}_2(\boldsymbol{\zeta})\right)$$

we finally get

$$\widehat{\operatorname{curl}_\Gamma \mathcal{V}_0\boldsymbol{\lambda}}(\boldsymbol{\zeta}) = -\frac{1}{|\boldsymbol{\zeta}|}i\left(-\zeta_2\widehat{\lambda}_1(\boldsymbol{\zeta}) + \zeta_2\widehat{\lambda}_2(\boldsymbol{\zeta})\right). \tag{3.33}$$

Using these results we will show the estimate

$$\|\mathcal{V}_0\boldsymbol{\lambda}\|_{\mathbf{H}^s_\perp(\operatorname{curl}_\Gamma,\Gamma)} \le C\|\boldsymbol{\lambda}\|_{\mathbf{H}^s_\parallel(\operatorname{div}_\Gamma,\Gamma)}.$$

First of all, we compare the $\mathbf{H}^s$-norms, using the definition of the norms in (3.29). The estimate

$$|\widehat{\mathcal{V}_0\boldsymbol{\lambda}}(\boldsymbol{\zeta})|^2 \le |\widehat{\boldsymbol{\lambda}}(\boldsymbol{\zeta})|^2$$

$$\iff \frac{1}{|\boldsymbol{\zeta}|^2}\left(|\widehat{\lambda}_1|^2 + |\widehat{\lambda}_2|^2\right) \le |\widehat{\lambda}_1|^2 + |\widehat{\lambda}_2|^2$$

holds for $|\boldsymbol{\zeta}| \ge 1$. Therefore, we only have to estimate the Fourier transformation of the surface curl of $\mathcal{V}_0\boldsymbol{\lambda}$ by the Fourier transformation of $\boldsymbol{\lambda}$. Hence, we have to show that

$$|\widehat{\operatorname{curl}_\Gamma \mathcal{V}_0\boldsymbol{\lambda}}|^2 \le |\widehat{\boldsymbol{\lambda}}|^2.$$

This holds due to

$$\frac{1}{|\boldsymbol{\zeta}|^2}\left|-\zeta_2\widehat{\lambda}_1 + \zeta_1\widehat{\lambda}_2\right|^2 \le |\widehat{\lambda}_1|^2 + |\widehat{\lambda}_2|^2$$

$$\iff \zeta_2^2|\widehat{\lambda}_1|^2 - \zeta_1\zeta_2(\widehat{\lambda}_1\overline{\widehat{\lambda}_2} + \overline{\widehat{\lambda}_1}\widehat{\lambda}_2) + \zeta_1^2|\widehat{\lambda}_2|^2 \le (\zeta_1^2 + \zeta_2^2)\left(|\widehat{\lambda}_1|^2 + |\widehat{\lambda}_2|^2\right)$$

$$\iff 0 \le \left|\zeta_1\widehat{\lambda}_1 + \zeta_2\widehat{\lambda}_2\right|^2 = |\widehat{\operatorname{div}_\Gamma \boldsymbol{\lambda}}|^2.$$

This gives the desired result. □

Remark 3.3.5 *It is also well known that*

$$\mathcal{V}_0 : \mathbf{H}^s_\parallel(\Gamma) \to \mathbf{H}^{s+1}_\parallel(\Gamma)$$

is continuous, see e.g. Hiptmair & Schwab [68].

As above one can prove the following result.

Theorem 3.3.6 *Let Γ be a smooth surface. Then,*

$$\mathcal{K}_0 : \mathbf{H}^s_\perp(\operatorname{curl}_\Gamma,\Gamma) \to \mathbf{H}^s_\perp(\operatorname{curl}_\Gamma,\Gamma),$$

$$\tilde{\mathcal{K}}_0 : \mathbf{H}^s_\parallel(\operatorname{div}_\Gamma,\Gamma) \to \mathbf{H}^s_\parallel(\operatorname{div}_\Gamma,\Gamma)$$

are continuous mappings for $-\frac{1}{2} \le s < \frac{1}{2}$.

4 Basis functions and interpolation operators

On the domain Ω we define a triangulation $\mathcal{T}_h$ with hexahedral or tetrahedral elements T_i. We assume that $\mathcal{T}_h$ is quasi-uniform with mesh size $h > 0$ and shape-regular in the sense of Ciarlet [36]. This mesh induces a mesh $\mathcal{K}_h$ of triangles or quadrilaterals on the boundary. On these meshes we define our polynomial spaces. For the approximation in $\mathbf{H}(\mathbf{curl}, \Omega)$ we use the so-called Nédélec space $\mathcal{ND}_p(\mathcal{T}_h)$. Furthermore, we have to consider the trace spaces on the boundary. These are $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. For the first one we use the space of Raviart-Thomas functions $\mathcal{RT}_p(\mathcal{K}_h)$ and for the second one we introduce $\mathcal{TND}$-functions $\mathcal{TND}_p(\mathcal{K}_h)$, i.e. the Dirichlet trace of the Nédélec space. We have to consider higher polynomial degrees For the p-version. Here, the polynomial degree is assigned with p. We first consider the calculation of the basis functions on the reference elements. Using suitable transformations we then get the polynomial spaces. The investigation of the transformations is very important for the calculation of the Galerkin elements for the FEM/BEM coupling, as described in §6.2. Furthermore, we prove inverse inequalities for the spaces $\mathcal{ND}_p(\mathcal{T}_h)$ and $\mathcal{RT}_p(\mathcal{K}_h)$. In §4.7 we prove a continuous extension of the space $\mathcal{RT}_p(\mathcal{K}_h)$ to the space $\mathcal{ND}_p(\mathcal{T}_h)$ as an inverse of the trace operator $\gamma_t^\times$. In Section 4.8 we investigate a quasi-optimal hp-interpolation operator for the space $\mathbf{H}(\mathbf{curl}, \Omega)$.

Finally, we present a $\mathbf{H}(\mathbf{curl}, \Omega)$-stable decomposition of $\mathcal{ND}_p(\mathcal{T}_h)$ and a $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$-stable decomposition of $\mathcal{RT}_2(\mathcal{K}_h)$. Using these results we can construct certain additive Schwarz preconditioners and hierarchical error estimates.

4.1 Nédélec basis functions for higher polynomial degrees

In the beginning, we consider the approximation in the space $\mathbf{H}(\mathbf{curl}, \Omega)$. The constraint for the $\mathbf{H}(\mathbf{curl}, \Omega)$-conformity is that the tangential component on adjacent elements has to be continuous, see Nédélec [82].

In order to fulfill this constraint Nédélec [82] defines on the element T local spaces

$\mathcal{ND}_p(T)$ of degree $p \in \mathbb{N}$ via integral moments for the degrees of freedom. The functions in $\mathcal{ND}_p(T)$ are also referred to as **edge functions** because the lowest order basis functions are closely related to the edges of the elements.

4.1.1 Definition on the reference cube

We consider the reference cube $\widehat{T} = [-1,1]^3$. Furthermore, $\mathbb{Q}_{k,l,m}(\widehat{T})$ denotes the space of polynomials with maximum degrees k in x-, l in y- and m in z-direction and the local space is defined by

$$\mathcal{ND}_p(\widehat{T}) := \mathbb{Q}_{p-1,p,p}(\widehat{T}) \times \mathbb{Q}_{p,p-1,p}(\widehat{T}) \times \mathbb{Q}_{p,p,p-1}(\widehat{T}).$$

The dimension is $\dim \mathcal{ND}_p(\widehat{T}) = 3p(p+1)^2$. In order to calculate the basis functions we use the following degrees of freedom defined by integral moments. Here, e denotes an edge of $\widehat{T}$ with unit tangential vector $\mathbf{t}$ and F denotes a face with unit normal vector $\mathbf{n}$ of $\widehat{T}$.

$$m_j(\mathbf{u}) := \begin{cases} \int_e \mathbf{u}\cdot\mathbf{t}\, q\, ds & \text{for all } q \in \mathbb{Q}_{p-1}, \\ \int_F (\mathbf{u}\times\mathbf{n})\cdot\mathbf{q}\, dS & \text{for all } \mathbf{q} \in \mathbb{Q}_{p-2,p-1}\times\mathbb{Q}_{p-1,p-2}, \\ \int_{\widehat{T}} \mathbf{u}\cdot\mathbf{q}\, d\mathbf{x} & \text{for all } \mathbf{q} \in \mathbb{Q}_{p-1,p-2,p-2}\times\mathbb{Q}_{p-2,p-1,p-2}\times\mathbb{Q}_{p-2,p-2,p-1} \end{cases}, \tag{4.1}$$

$j = 1,\dots,3p(p+1)^2$.

Remark 4.1.1 *These moments are not well-defined for all functions* $\mathbf{u} \in \mathbf{H}(\mathbf{curl},\Omega)$. *The main difficulty is the regularity of the operator* $\int_e \mathbf{t}\cdot\mathbf{u}\, ds$. *Monk [78, Lemma 5.38] shows that one has to demand that* $\mathbf{u} \in \mathbf{H}^{1/2+\delta}(T)$ *and* $\mathbf{curl}\,\mathbf{u} \in \mathbf{L}^p(T)$ *for some constants* $\delta > 0$ *and* $p > 2$. *Amrouche et al. [10] give a weaker result. They demand that* $\mathbf{u} \in \mathbf{L}^p(T)$, $\mathbf{curl}\,\mathbf{u} \in \mathbf{L}^p(T)$ *and* $\mathbf{u}\times\mathbf{n} \in (L^p(\partial T))^2$ *for some* $p > 2$.

We now demand that the basis functions $\mathbf{b}_i$ of $\mathcal{ND}_p(\widehat{T})$ have to satisfy the conditions

$$m_j(\mathbf{b}_i) = \delta_{ij}, \qquad i,j = 1,\dots,3p(p+1)^2.$$

This leads to a linear system depending on the choice of test and trial functions. One possibility is to use monomials as basis for $\mathcal{ND}_p(\widehat{T})$. For computations they are ordered by

$$\boldsymbol{\psi}_i(x,y,z) := \begin{cases} x^r y^s z^t\, \mathbf{e}_1, & r \le p-1,\ s \le p,\ t \le p \quad \text{if } i = 1,\dots,p(p+1)^2 \\ x^r y^s z^t\, \mathbf{e}_2, & r \le p,\ s \le p-1,\ t \le p \quad \text{if } i = p(p+1)^2+1,\dots,2p(p+1)^2 \\ x^r y^s z^t\, \mathbf{e}_3, & r \le p,\ s \le p,\ t \le p-1 \quad \text{if } i = 2p(p+1)^2+1,\dots,3p(p+1)^2 \end{cases}.$$

Here, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ denote the unit Cartesian vectors. Then, there holds $\mathcal{ND}_p(\widehat{T}) = \operatorname{span}\{\boldsymbol{\psi}_i,\ i = 1,\dots,3p(p+1)^2\}$. In order to fulfill the continuity of the tangential trace of the basis functions one has to consider the moments (4.1). Hence, we get p basis functions associated to an edge e, $2p(p+1)$ basis functions associated to a face F and $3p(p-1)^2$ basis functions associated to the interior.

The basis functions $\mathbf{b}_i$ then have a representation

$$\mathbf{b}_i = \sum_{l=1}^{3p(p+1)^2} a_{il} \boldsymbol{\psi}_l$$

with the coefficients a_{il} as the solution of the linear system

$$m_j(\mathbf{b}_i) = \sum_{l=1}^{3p(p+1)^2} a_{il} m_j(\boldsymbol{\psi}_l) = \delta_{ij}, \quad i, j = 1, \ldots, 3p(p+1)^2.$$

In order to calculate the moments m_j in (4.1) one could use monomials as test functions. It is also possible to use different polynomial basis functions of the polynomial spaces.

For the lowest order $p = 1$ we get the following basis functions associated to the edges of the reference element, see Figure 4.1.

$$\begin{aligned}
\mathbf{b}^{(e_0)} &= \frac{1}{8}(1-y)(1-z)\mathbf{e}_1, & \mathbf{b}^{(e_1)} &= \frac{1}{8}(1+y)(1-z)\mathbf{e}_1,\\
\mathbf{b}^{(e_2)} &= \frac{1}{8}(1-y)(1+z)\mathbf{e}_1, & \mathbf{b}^{(e_3)} &= \frac{1}{8}(1+y)(1+z)\mathbf{e}_1,\\
\mathbf{b}^{(e_4)} &= \frac{1}{8}(1-x)(1-z)\mathbf{e}_2, & \mathbf{b}^{(e_5)} &= \frac{1}{8}(1+x)(1-z)\mathbf{e}_2,\\
\mathbf{b}^{(e_6)} &= \frac{1}{8}(1-x)(1+z)\mathbf{e}_2, & \mathbf{b}^{(e_7)} &= \frac{1}{8}(1+x)(1+z)\mathbf{e}_2,\\
\mathbf{b}^{(e_8)} &= \frac{1}{8}(1-x)(1-y)\mathbf{e}_3, & \mathbf{b}^{(e_9)} &= \frac{1}{8}(1+x)(1-y)\mathbf{e}_3,\\
\mathbf{b}^{(e_{10})} &= \frac{1}{8}(1-x)(1+y)\mathbf{e}_3, & \mathbf{b}^{(e_{11})} &= \frac{1}{8}(1+x)(1+y)\mathbf{e}_3.
\end{aligned}$$

We remark that the edge functions are constant on the edge which they are associated to.

Here are some examples of the 54 basis functions for the polynomial degree $p = 2$.

- There are two edge functions associated to the edge e_0 ($y = -1$, $z = -1$).

  $$\begin{aligned}
  \mathbf{b}_1^{(e_0)} &:= \frac{1}{32}\ (3y+1)(y-1)(3z+1)(z-1)\mathbf{e}_1,\\
  \mathbf{b}_2^{(e_0)} &:= \frac{3}{32}x(3y+1)(y-1)(3z+1)(z-1)\mathbf{e}_1.
  \end{aligned}$$

  The functions are constant on the edge e_0 with the value $\frac{1}{2}$. Furthermore, the tangential component vanishes everywhere except on the two faces which are adjacent to the edge.

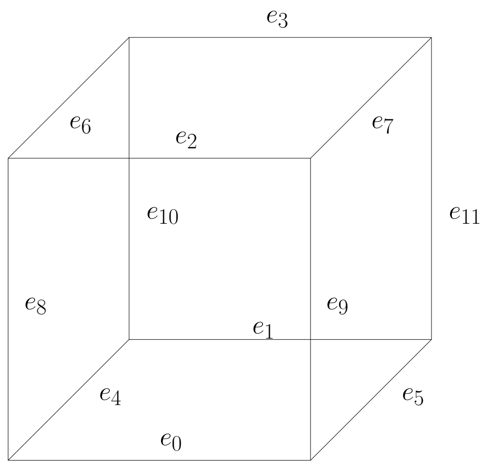

Figure 4.1: Numbering of the edges on the unit cube.

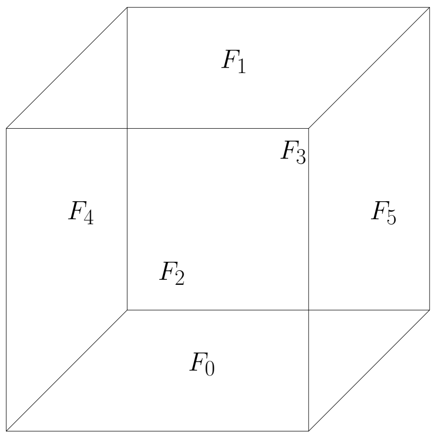

Figure 4.2: Numbering of the faces on the unit cube.

- There are four face functions associated to the face f_0 ($z = -1$), two in each direction.

$$\begin{aligned}
\mathbf{b}_1^{(F_0)} &:= \frac{3}{32}\,(1-y^2)(3z+1)(z-1)\mathbf{e}_1,\\
\mathbf{b}_2^{(F_0)} &:= \frac{9}{32}x\,(1-y^2)(3z+1)(z-1)\mathbf{e}_1,\\
\mathbf{b}_3^{(F_0)} &:= -\frac{3}{32}\,(1-x^2)(3z+1)(z-1)\mathbf{e}_2,\\
\mathbf{b}_4^{(F_0)} &:= -\frac{9}{32}y\,(1-x^2)(3z+1)(z-1)\mathbf{e}_2.
\end{aligned}$$

The tangential component of the face function is only non-zero on its associated face.

- There are six interior functions

$$\begin{aligned}
\mathbf{b}_1^{(T)} &:= \frac{9}{32}\ \ (1-y^2)(1-z^2)\mathbf{e}_1,\\
\mathbf{b}_2^{(T)} &:= \frac{27}{32}x\,(1-y^2)(1-z^2)\mathbf{e}_1,\\
\mathbf{b}_3^{(T)} &:= \frac{9}{32}\ \ (1-x^2)(1-z^2)\mathbf{e}_2,\\
\mathbf{b}_4^{(T)} &:= \frac{27}{32}y\,(1-x^2)(1-z^2)\mathbf{e}_2,\\
\mathbf{b}_5^{(T)} &:= \frac{9}{32}\ \ (1-x^2)(1-y^2)\mathbf{e}_3,\\
\mathbf{b}_6^{(T)} &:= \frac{27}{32}z\,(1-x^2)(1-y^2)\mathbf{e}_3.
\end{aligned}$$

  The interior functions are zero on four faces and have a vanishing normal component on the other two faces.

The problem in calculating these bases for higher polynomial degrees using monomials as test and trial functions is that the associated matrix $A = (a_{il})$ becomes very ill-conditioned, see Figure 4.3 and Table 4.1, and the solution of the linear system gets very unstable. This makes it solely possible to calculate the basis functions up to the polynomial degree $p = 4$.

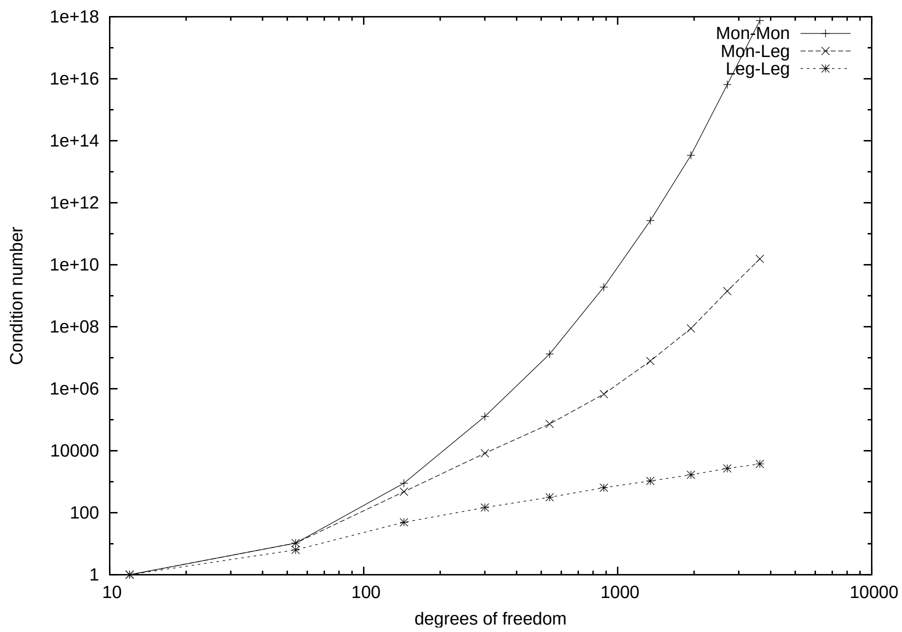

Figure 4.3: Condition numbers of the linear system related to the calculation of the Nédélec basis functions using as test and trial functions monomials and or Legendre polynomials.

p	DOF	max. EV	min. EV	Condition
1	12	4.0000	4.0000	1.0000
2	54	8.9370	0.8530	10.4767
3	144	7.1712	8.0553E-03	8.9025E+02
4	300	10.9814	8.6148E-05	1.2747E+05
5	540	10.6589	8.1041E-07	1.3153E+07
6	882	13.6520	7.2142E-09	1.8924E+09
7	1344	12.5764	4.7357E-11	2.6557E+11
8	1944	14.2408	4.1618E-13	3.4218E+13
9	2700	14.2741	2.2056E-15	6.4718E+15
10	3630	15.9238	2.1061E-17	7.5607E+17

Table 4.1: Condition numbers of the linear system with monomials as test and trial functions.

p	DOF	max. EV	min. EV	Condition
1	12	4.0000	4.0000	1.0000
2	54	8.9370	0.8530	10.4767
3	144	6.2788	1.3266E-02	4.7328E+02
4	300	7.7106	9.2799E-04	8.3089E+03
5	540	7.3115	1.0001E-04	7.3111E+04
6	882	7.6405	1.1401E-05	6.7018E+05
7	1344	7.3207	9.3746E-07	7.8091E+06
8	1944	6.5290	7.4206E-08	8.7984E+07
9	2700	7.2397	5.1370E-09	1.4093E+09
10	3630	6.5449	4.2245E-10	1.5493E+10

Table 4.2: Condition numbers of the linear system with Legendre polynomials as test and monomials as trial functions.

Another possibility is to use Legendre polynomials as test functions in (4.1). Thereafter, one gets a quite lower condition number, see Table 4.2 and we can calculate the basis functions up to the degree $p = 7$. The numerical experiments in Chapter 5 use this possibility. A third possibility would be to use Legendre polynomials as both test and trial functions and the condition number of the system is again much lower, see Table 4.3.

These properties can also be seen if we consider the maximal value of the residual $A\widetilde{A^{-1}} - I$. Here, $\widetilde{A^{-1}}$ denotes an approximate to the inverse of the matrix A which is used for solving the linear system. In Table 4.4 we compare those values which should be nearly zero. One sees that the system for monomials as test and trial functions becomes unstable at a polynomial degree of $p = 6$ while using Legendre polynomials gives good

p	DOF	max. EV	min. EV	Condition
1	12	4.0000	4.0000	1.0000
2	54	8.3392	1.3147	6.3432
3	144	5.9976	0.1225	48.94023
4	300	7.1480	4.8693E-02	1.46797E+02
5	540	6.6341	2.1061E-02	3.14998E+02
6	882	7.0189	1.0968E-02	6.39968E+02
7	1344	6.0532	5.7435E-03	1.05392E+03
8	1944	6.6939	4.0187E-03	1.66570E+03
9	2700	6.5044	2.4335E-03	2.67288E+03
10	3630	6.8200	1.8133E-03	3.75547E+03

Table 4.3: Condition numbers of the linear system with Legendre polynomials as test and trial functions.

p	DOF	Mon-Mon	Mon–Leg	Leg–Leg
1	12	2.2204E-16	2.2204E-16	2.2204E-16
2	54	6.2911E-16	6.2911E-16	3.0554E-16
3	144	4.2834E-14	1.5744E-14	1.0236E-15
4	300	4.9430E-11	2.4735E-12	6.2025E-15
5	540	1.0467E-08	7.6089E-11	1.0029E-14
6	882	1.1608E-04	3.9369E-09	3.0330E-14
7	1344	2.3527E-02	6.8362E-08	4.4390E-14
8	1944	2.9619E+03	6.3707E-06	7.3622E-14
9	2700	1.2866E+06	2.4736E-04	7.3936E-14
10	3630	5.4289E+10	1.1923E-02	1.0635E-13

Table 4.4: Maximal value of the residual $A\widetilde{A^{-1}} - I$.

results up to high polynomial degrees.

Another problem using this construction with the integral moments is that one doesn't get a hierarchical basis for the space $\mathcal{ND}_h$. Hence, for every polynomial degree one has to calculate a new set of basis functions.

Other possibilities for basis functions are introduced, e.g., by Ainsworth & Coyle [1, 2]. In order to construct such a basis one uses Legendre polynomials L_i of degree $i = 0, \dots, p$ and their anti-derivatives $\mathcal{L}_i$ defined on $[-1, 1]$ by

$$\mathcal{L}_0(s) := \frac{1}{2}(1-s), \qquad \mathcal{L}_1(s) := \frac{1}{2}(1+s)$$

and
$$\mathcal{L}_i(s) := \int_{-1}^{s} L_{i-1}(t)\,dt, \qquad i = 2,\dots,p+1.$$
On the reference cube $\widehat{T} = [-1,1]^3$ the basis functions are given by
$$\left.\begin{array}{l} L_i(x_1)\mathcal{L}_j(x_2)\mathcal{L}_k(x_3)\mathbf{e}_1 \\ \mathcal{L}_j(x_1)L_i(x_2)\mathcal{L}_k(x_3)\mathbf{e}_2 \\ \mathcal{L}_j(x_1)\mathcal{L}_k(x_2)L_i(x_3)\mathbf{e}_3 \end{array}\right\} \quad i = 0,\dots,p,\ j,k = 0,\dots,p+1,$$
where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ denote the unit Cartesian vectors. For the lowest polynomial degree $p = 1$ these basis functions are the same as given above using the degrees of freedom given by Nédélec.
These basis functions can also be separated into into edge, face and interior functions. The advantage of these basis functions is, beside the easy calculation, that one can construct easily a non-uniform mesh. In order to assure continuity of the basis functions the minimum-rule is applied, compare Demkowicz & Vardapetyan [48]. The transformation is the same as for the usual basis functions, see below.

A very flexible generalization of the spaces $\mathcal{ND}_p(\mathcal{T}_h)$ offer the *hp*-elements of Demkowicz *et al.* [43, 46, 91]. They are using local variable polynomial degrees, even variable in all directions.

For our calculations we only use the Nédélec functions introduced above. The implementation of the basis of Ainsworth or the *hp*-basis due to Demkowicz still has to be done.

4.1.2 Basis functions on a tetrahedron

For the sake of completeness we give here the basis functions and degrees of freedom for the reference tetrahedron $\widehat{T} := \{\mathbf{x} \in \mathbb{R}^3 : x_1, x_2, x_3 \geq 0,\ x_1 + x_2 + x_3 \leq 1\}$. The local Nédélec space of order p is given by, see Nédélec [82],
$$\mathcal{ND}_p(\widehat{T}) := (\mathbb{P}_{p-1}(\widehat{T}))^3 + \{\mathbf{p} \in (\mathbb{P}_p(\widehat{T}))^3 |\ \mathbf{p}(\mathbf{x}) \cdot \mathbf{x} = 0\,\forall\,\mathbf{x} \in \widehat{T}\}.$$
For the case $p = 1$ we have the representation
$$\mathcal{ND}_1(\widehat{T}) := \{\mathbf{x} \mapsto \mathbf{a} + \mathbf{b} \times \mathbf{x}, \mathbf{a},\ \mathbf{b} \in \mathbb{R}^3\}.$$
For arbitrary $p \in \mathbb{N}$ the moments are defined by

1. $\int_e \mathbf{u} \cdot \mathbf{t}\, q\, ds \qquad \forall\, q \in \mathbb{P}_{p-1},\ e$ edge of $\widehat{T}$,
2. $\int_F (\mathbf{u} \times \mathbf{n}) \cdot \mathbf{q}\, dS \qquad \forall\, \mathbf{q} \in (\mathbb{P}_{p-2})^2$, F face of $\widehat{T}$,
3. $\int_{\widehat{T}} \mathbf{u} \cdot \mathbf{q}\, d\mathbf{x} \qquad \forall\, \mathbf{q} \in (\mathbb{P}_{p-3})^3$.

Therefore, the dimension of $\mathcal{ND}_p(\widehat{T})$ is $\frac{1}{2}p(p+2)(p+3)$.

4.1.3 Transformations and an inverse inequality for Nédélec functions

In order to construct a global $\mathbf{H}(\mathbf{curl}, \Omega)$-conforming space one has to consider mappings from the reference element to a local element. Let T denote the image of the reference element $\widehat{T}$ under the affine transformation

$$F_T : \quad \mathbf{x} = B_T\hat{\mathbf{x}} + \mathbf{d}, \quad B_T \in \mathcal{L}(\widehat{T}, \mathbb{R}^3),\, \mathbf{d} \in \mathbb{R}^3, \tag{4.2}$$

and $\{\hat{\mathbf{b}}_j,\, j = 1, \ldots, n_p\}$ be a basis of $\mathcal{ND}_p(\widehat{T})$, then a local basis on T is given by

$$\mathbf{b}_j(\mathbf{x}) = (B_T^{\mathsf{T}})^{-1}\, \hat{\mathbf{b}}_j(\hat{\mathbf{x}}), \quad j = 1, \ldots, n_p. \tag{4.3}$$

This transformation is $\mathbf{H}(\mathbf{curl})$-conforming, see Nédélec [82] or Monk [78, (5.33)].
Thus, we can define the global finite element space $\mathcal{ND}_p(\mathcal{T}_h)$, if we connect those local basis functions that belong to an edge or a face to a global basis function. The space $\mathcal{ND}_p(\mathcal{T}_h)$ is invariant under the affine transformation (4.2) if we transform the basis functions by (4.3).

Next, we examine the behavior of the norms under the transformation. Therefore we consider the reference tetrahedron or the reference hexahedron $\widehat{T}$ and an arbitrary element T of size h. For $\hat{\mathbf{u}} \in \mathcal{ND}_p(\widehat{T})$ and $\mathbf{u} \in \mathcal{ND}(T)$ we get the transformation

$$\mathbf{u} \circ F_T = (B_T^{\mathsf{T}})^{-1}\hat{\mathbf{u}} \tag{4.4}$$

and the **curl** is transformed by, see e.g. Monk [78, (5.33)],

$$\mathbf{curl}\,\mathbf{u} = \frac{1}{\det B_T} B_T \widehat{\mathbf{curl}}\; \hat{\mathbf{u}}. \tag{4.5}$$

Next, we consider the transformation of the Sobolev norms, see Alonso & Valli [5, Lemma 5.5] or Monk [78, Lemma 5.43].

Lemma 4.1.2 *For $s \geq 0$ and a regular mesh $\mathcal{T}_h$ there holds for all functions $\mathbf{u}$ that are transformed using* (4.4)

$$\|\hat{\mathbf{u}}\|_{\mathbf{L}^2(\widehat{T})} \simeq h_T^{-1/2} \|\mathbf{u}\|_{\mathbf{L}^2(T)} \tag{4.6}$$

and for the semi-norms

$$|\hat{\mathbf{u}}|_{\mathbf{H}^s(\widehat{T})} \simeq h_T^{s-1/2} |\mathbf{u}|_{\mathbf{H}^s(T)}, \tag{4.7}$$

$$|\widehat{\mathbf{curl}}\; \hat{\mathbf{u}}|_{\mathbf{H}^s(\widehat{T})} \simeq h_T^{s+1/2} |\,\mathbf{curl}\,\mathbf{u}|_{\mathbf{H}^s(T)}, \tag{4.8}$$

where the constant is independent of $\mathbf{u}$ and the mesh size h_T.

Proof. First of all, we consider the $\mathbf{L}^2$-norm. Using the transformation (4.4) we get

$$\begin{aligned}\|\hat{\mathbf{u}}\|^2_{\mathbf{L}^2(\widehat{T})} &= \int_{\widehat{T}} |\hat{\mathbf{u}}(\hat{\mathbf{x}})|^2 \, d\hat{\mathbf{x}} \\ &= |\det B_T|^{-1} \int_T |B_T^{\mathsf{T}} \mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} \\ &\le |\det B_T|^{-1} \, \|B_T^{\mathsf{T}}\|^2 \int_T |\mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} \\ &\le C \, h_T^{-1} \|\mathbf{u}\|^2_{\mathbf{L}^2(T)}.\end{aligned}$$

Here, $\|\cdot\|$ denotes the spectral norm of a matrix and we have used the estimates

$$\|B_T\| \simeq h_T, \qquad |\det B_T| \simeq h_T^3,$$

see, e.g., Ciarlet [36].
The other integer norms can be estimated in a similar way, see e.g. Monk [78].

In order to estimate the non-integer semi-norms $|\cdot|_{\mathbf{H}^s(\widehat{T})}$ $(0 < s < 1)$ we consider, cf. Chapter 1,

$$\begin{aligned}|\hat{\mathbf{u}}|^2_{\mathbf{H}^s(\widehat{T})} &= \int_{\widehat{T}} \int_{\widehat{T}} \frac{|\hat{\mathbf{u}}(\hat{\mathbf{x}}) - \hat{\mathbf{u}}(\hat{\mathbf{y}})|^2}{|\hat{\mathbf{x}} - \hat{\mathbf{y}}|^{3+2s}} \, d\hat{\mathbf{x}} \, d\hat{\mathbf{y}} \\ &= |\det B_T|^{-2} \int_T \int_T \frac{|B_T^{\mathsf{T}} \big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\big)|^2}{|B_T^{-1}(\mathbf{x} - \mathbf{y})|^{3+2s}} \, d\mathbf{x} \, d\mathbf{y}.\end{aligned}$$

In order to estimate the denominator we write

$$|\mathbf{x} - \mathbf{y}| = |B_T \, B_T^{-1}(\mathbf{x} - \mathbf{y})| \le \|B_T\| \, |B_T^{-1}(\mathbf{x} - \mathbf{y})|$$

and we get

$$|B_T^{-1}(\mathbf{x} - \mathbf{y})| \ge \|B_T\|^{-1} \, |\mathbf{x} - \mathbf{y}|.$$

It follows that

$$\begin{aligned}|\hat{\mathbf{u}}|^2_{\mathbf{H}^s(\widehat{T})} &\le |\det B_T|^{-2} \|B_T\|^{3+2s} \int_T \int_T \frac{|B_T^{\mathsf{T}} \big(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\big)|^2}{|\mathbf{x} - \mathbf{y}|^{3+2s}} \, d\mathbf{x} \, d\mathbf{y} \\ &\le |\det B_T|^{-2} \|B_T\|^{3+2s} \|B_T^{\mathsf{T}}\|^2 |\mathbf{u}|^2_{\mathbf{H}^s(T)} \\ &\le C \, h_T^{-1+2s} |\mathbf{u}|^2_{\mathbf{H}^s(T)}.\end{aligned}$$

This is (4.7). The result for the **curl** can be obtained the same way using the transformation (4.5).
The results for $s \ge 1$ can be achieved similarly. □

Now, we can prove an inverse inequality for $\mathbf{H}(\mathbf{curl}, \Omega)$-conforming elements.

Lemma 4.1.3 *Let $\mathcal{T}_h$ be a regular and quasi-uniform triangulation of Ω with mesh size h. Then, there holds for $\mathbf{u} \in \mathcal{ND}_p(\mathcal{T}_h)$ and $s \geq r \geq 0$*

$$\|\mathbf{u}\|_{\mathbf{H}^s(\mathbf{curl},\Omega)} \leq Ch^{r-s}\|\mathbf{u}\|_{\mathbf{H}^r(\mathbf{curl},\Omega)} \tag{4.9}$$

with a constant $C > 0$ depending only on the polynomial degree p and the reference element $\widehat{T}$.

Proof. Using (4.4) and (4.5) we can estimate on an arbitrary element $T \in \mathcal{T}_h$

$$\begin{aligned}|\mathbf{u}|_{\mathbf{H}^s(T)} &\leq Ch_T^{1/2-s}|\hat{\mathbf{u}}|_{\mathbf{H}^s(\widehat{T})} \leq C(p,\widehat{T})h_T^{1/2-s}|\hat{\mathbf{u}}|_{\mathbf{H}^r(\widehat{T})}\\ &\leq C(p,\widehat{T})h_T^{1/2-s}\,h_T^{r-1/2}|\mathbf{u}|_{\mathbf{H}^r(T)} = C(p,\widehat{T})h_T^{r-s}|\mathbf{u}|_{\mathbf{H}^r(T)}\end{aligned}$$

and

$$\begin{aligned}|\,\mathbf{curl}\,\mathbf{u}|_{\mathbf{H}^s(T)} &\leq Ch_T^{-1/2-s}|\widehat{\mathbf{curl}}\,\hat{\mathbf{u}}|_{\mathbf{H}^s(\widehat{T})} \leq C(p,\widehat{T})h_T^{-1/2-s}|\widehat{\mathbf{curl}}\,\hat{\mathbf{u}}|_{\mathbf{H}^r(\widehat{T})}\\ &\leq C(p,\widehat{T})h_T^{-1/2-s}\,h_T^{r+1/2}|\,\mathbf{curl}\,\mathbf{u}|_{\mathbf{H}^r(T)} = C(p,\widehat{T})h_T^{r-s}|\,\mathbf{curl}\,\mathbf{u}|_{\mathbf{H}^r(T)}.\end{aligned}$$

Summing over all elements yields the result. □

An interpolation operator associated to the moments

Associated to the moments (4.1) we can define an interpolation operator Π_p^h onto the space $\mathcal{ND}_p(\mathcal{T}_h)$. First of all, we consider the reference cube $\widehat{T}$.

$$\begin{aligned}&\int_e (\mathbf{u}-\Pi_p\mathbf{u})\cdot\mathbf{t}\,q\,ds = 0 \qquad \forall\, q \in \mathbb{Q}_{p-1},\\ &\int_F ((\mathbf{u}-\Pi_p\mathbf{u})\times\mathbf{n})\cdot\mathbf{q}\,dS = 0 \quad \forall\,\mathbf{q} \in \mathbb{Q}_{p-2,p-1}\times\mathbb{Q}_{p-1,p-2},\\ &\int_{\widehat{T}} (\mathbf{u}-\Pi_p\mathbf{u})\cdot\mathbf{q}\,d\mathbf{x} = 0 \quad \forall\,\mathbf{q} \in \mathbb{Q}_{p-1,p-2,p-2}\times\mathbb{Q}_{p-2,p-1,p-2}\times\mathbb{Q}_{p-2,p-2,p-1},\end{aligned} \tag{4.10}$$

compare, e.g., Monk [78]. Note that these integral moments are not well defined for all functions in $\mathbf{H}(\mathbf{curl},\Omega)$, see Remark 4.1.1 and Amrouche *et al.* [10, Lemma 4.7]. Using the transformation (4.4) we then can define the operator Π_p^h which maps into the space $\mathcal{ND}_p(\mathcal{T}_h)$.

For the interpolant one can prove the following error estimate for the h-version, see Monk [78, Theorem 5.41 and Theorem 6.6].

Theorem 4.1.4 *Let $\mathcal{T}_h$ be a regular mesh on Ω. For $\mathbf{u} \in \mathbf{H}^s(\mathbf{curl},\Omega)$, $1/2+\delta \leq s \leq p$, $\delta > 0$, there holds dependent only on s, p, and the shape regularity of $\mathcal{T}_h$ such that*

$$\|\mathbf{u}-\Pi_p^h\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq Ch^s\|\mathbf{u}\|_{\mathbf{H}^s(\mathbf{curl},\Omega)}.$$

A more detailed description about interpolation operators and the regularities is given in Section 4.8.

4.2 A numerical experiment with Nédélec functions: FEM for the eddy current problem

In order to show the efficiency of the p-version with Nédélec basis functions we consider the following eddy current problem. Let Ω be a bounded domain with a simply connected, polyhedral boundary and $\Gamma := \partial\Omega$. Ω models the conductor. In Ω the magnetic permeability $\chi \in L^\infty(\Omega)$ is given. It which is uniformly bounded, i.e. there are constants $\chi_1, \chi_2 > 0$ such that $\chi_1 \geq \chi \geq \chi_2$, Furthermore, we have the conductivity β which is also uniformly bounded by constants $\sigma_1, \sigma_2 > 0$, i.e. $\sigma_1 \geq \sigma \geq \sigma_2$. In Ω we assume a solenoidal current $\mathbf{J}_0 \in \mathbf{H}(\mathrm{div}, \Omega)$ with $\mathrm{div}\, \mathbf{J}_0 = 0$ which induces an electric field $\mathbf{E}$. The **eddy current problem** in a bounded domain then reads, cf., e.g., Beck *et al.* [15]:
Find the electric field $\mathbf{E}$ *with*

$$\mathbf{curl}\, \chi\, \mathbf{curl}\, \mathbf{E} + i\beta\mathbf{E} = \mathbf{J}_0 \qquad \text{in } \Omega, \tag{4.11}$$

$$\mathbf{E} \times \mathbf{n} = 0 \qquad \text{on } \Gamma := \partial\Omega. \tag{4.12}$$

We set $\mathbf{u} := \mathbf{E}$ and consider the space $\mathbf{H}_0(\mathbf{curl}, \Omega) := \{\mathbf{u} \in \mathbf{L}^2(\Omega) \,:\, \mathbf{curl}\, \mathbf{u} \in \mathbf{L}^2(\Omega),\ \mathbf{u} \times \mathbf{n} = 0\}$. Multiplying (4.11) with $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ and partial integration leads to the variational formulation:
Find $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ *such that*

$$(\chi\, \mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{v})_{\mathbf{L}^2(\Omega)} + i(\beta\mathbf{u}, \mathbf{v})_{\mathbf{L}^2(\Omega)} = (\mathbf{J}_0, \mathbf{v})_{\mathbf{L}^2(\Omega)}$$

for all $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$.
The $\mathbf{L}^2(\Omega)$-scalar product is defined as $(\mathbf{u}, \mathbf{v})_{\mathbf{L}^2(\Omega)} := \int_\Omega \mathbf{u} \cdot \overline{\mathbf{v}}\, d\mathbf{x}$.

For the Galerkin finite element method we construct a quasi-uniform mesh $\mathcal{T}_h$ on Ω with mesh size h. As finite dimensional subspace we take $\mathcal{ND}_p^0(\mathcal{T}_h) := \mathcal{ND}_p(\mathcal{T}_h) \cap \mathbf{H}_0(\mathbf{curl}, \Omega)$ and the Galerkin method reads:
Find $\mathbf{u}_h \in \mathcal{ND}_p^0(\mathcal{T}_h)$ *such that*

$$a(\mathbf{u}_h, \mathbf{v}) := (\chi\, \mathbf{curl}\, \mathbf{u}_h, \mathbf{curl}\, \mathbf{v})_{\mathbf{L}^2(\Omega)} + i(\beta\mathbf{u}_h, \mathbf{v})_{\mathbf{L}^2(\Omega)} = (\mathbf{J}_0, \mathbf{v})_{\mathbf{L}^2(\Omega)}$$

for all $\mathbf{v} \in \mathcal{ND}_p^0(\mathcal{T}_h)$.

Due to the coercivity of the bilinear form we get the quasi-optimality of the Galerkin error

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}(\mathbf{curl}, \Omega)} \leq C \inf_{\boldsymbol{\xi} \in \mathcal{ND}_p^0(\mathcal{T}_h)} \|\mathbf{u} - \boldsymbol{\xi}\|_{\mathbf{H}(\mathbf{curl}, \Omega)}.$$

Using Lemma 4.1.4 this yields the a posteriori error estimate for sufficiently smooth $\mathbf{u}$

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq Ch^p |\mathbf{u}|_{\mathbf{H}^{p+1}(\Omega)}.$$

Thus, the expected convergence rate is p for the space $\mathcal{ND}^0_p(\mathcal{T}_h)$.

Numerical example

We consider the unit cube $\Omega := [-1,1]^3$ and set $\beta \equiv 1$ and $\chi \equiv 1$. Moreover a given exact solution is

$$\mathbf{u} := \sin\frac{\pi}{2}(y+1) \cdot \sin\frac{\pi}{2}(z+1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This function is smooth and should be approximated very well by the p-version. The right hand side can be calculated using (4.11)

$$\mathbf{J}_0 := \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} + i\mathbf{u}.$$

We mesh the unit cube with smaller cubes of mesh size h and we perform a uniform refinement. The resulting linear system is solved using a Gaussian solver. The error is measured in the $\mathbf{H}(\mathbf{curl},\Omega)$-norm which is defined by $\|\mathbf{u}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} := \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\,\mathbf{curl}\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}$.
In Figure 4.4 we consider the h-version of the FEM for different polynomial degrees. In Figure 4.5 we compare the uniform p-version with the uniform h-version with polynomials of order 1. While the h-version converges only algebraically, the p-version converges exponentially. The convergence rates for the polynomial degrees are given in Table 4.5. Here, we consider the convergence rate with respect to the mesh size h and also with respect to the degrees of freedom. For the lower polynomial degrees and this corresponds very well to the approximation result in Lemma 4.1.4. For higher polynomial degrees we are still in the pre-asymptotic region.

p	1	2	3	4	5	6
$\alpha(h)$	0.941	1.711	2.449	2.830	3.541	3.498
$\alpha(\text{DOF})$	0.320	0.634	0.958	1.264	1.598	1.900

Table 4.5: Convergence rates with respect to the mesh size h and the degrees of freedom.

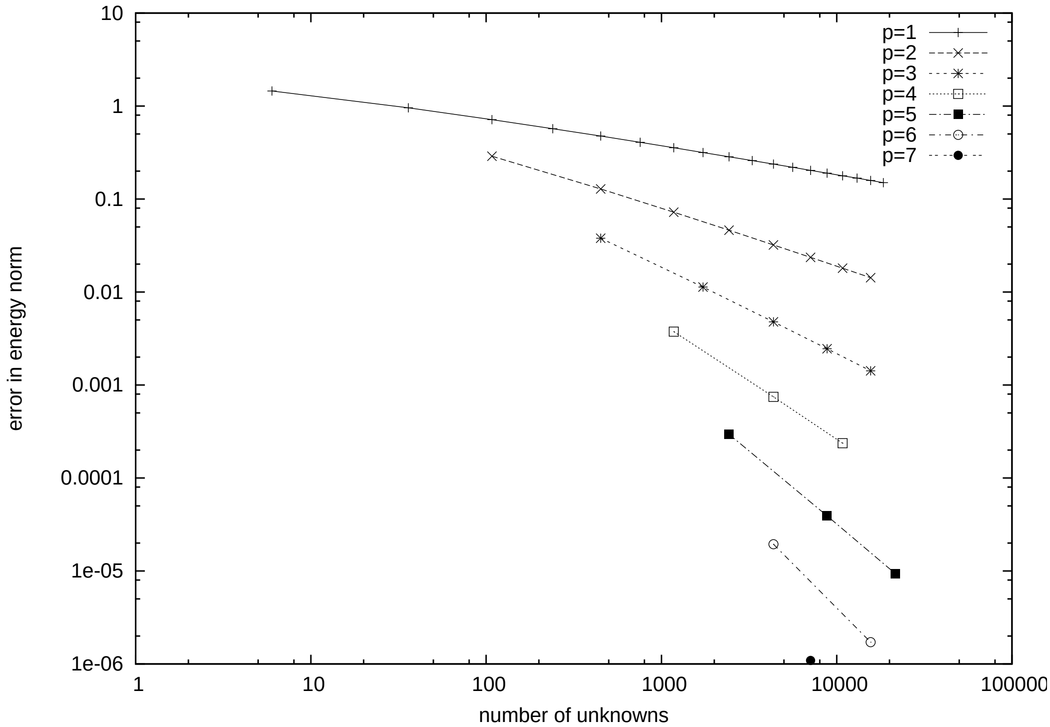

Figure 4.4: Uniform h-version for different polynomial degrees, error in energy norm.

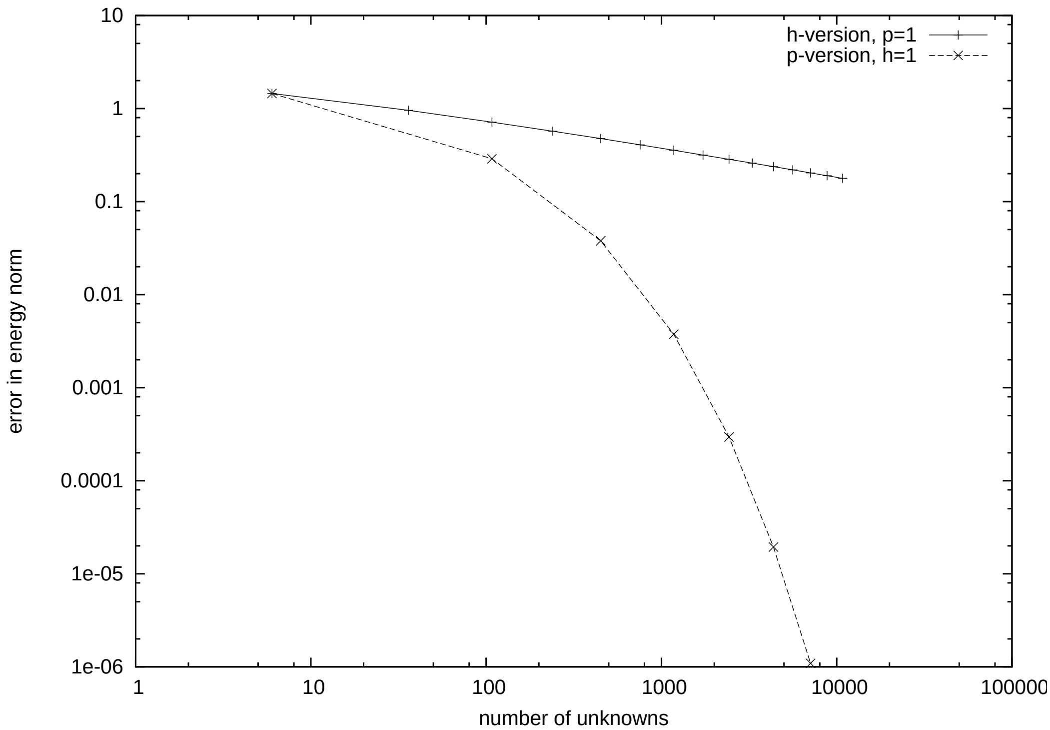

Figure 4.5: Uniform p-version, $h = 1$.

4.3 Raviart-Thomas basis functions for the approximation in $\mathbf{H}(\mathrm{div},\Omega)$

In this section we analyze the space $\mathbf{H}(\mathrm{div},\Omega)$. The constraint for $\mathbf{H}(\mathrm{div},\Omega)$-conformity is that the normal component, i.e. $\mathbf{u}\cdot\mathbf{n}$ is continuous between adjacent elements, cf. Nédélec [82].

Let $\widehat{T} := \{\mathbf{x}\in\mathbb{R}^3 : x_1, x_2, x_3 \geq 0,\ x_1+x_2+x_3 \leq 1\}$ denote the reference tetrahedron and the space of Raviart-Thomas functions of degree p is defined by

$$\begin{aligned}\mathcal{RT}_p(\widehat{T}) :=& (\mathbb{P}_{p-1}(\widehat{T}))^3 \oplus \mathbf{x}\mathbb{P}^0_{p-1}(\widehat{T})\\ =& (\mathbb{P}_{p-1}(\widehat{T}))^3 \oplus \{\mathbf{p}\in(\mathbb{P}^0_p(\widehat{T}))^3,\ \mathbf{p}(\mathbf{x})\times\mathbf{x}=0\,\forall\,\mathbf{x}\in\widehat{T}\},\end{aligned}$$

where $\mathbb{P}^0_p(\widehat{T})$ denotes the space of all homogeneous polynomials of degree p on $\widehat{T}$. It follows that the dimension of $\mathcal{RT}_p(\widehat{T})$ is $\frac{1}{2}p(p+1)(p+3)$. The degrees of freedom are defined by, see Nédélec [82],

1. $\int_F \mathbf{u}\cdot\mathbf{n}\,q\,dS \qquad \forall\,\mathbf{q}\in\mathbb{P}_{p-1}$, F face of $\hat{T}$,
2. $\int_{\widehat{T}} \mathbf{u}\cdot\mathbf{q}\,d\mathbf{x} \qquad \forall\,\mathbf{q}\in(\mathbb{P}_{p-2})^3$.

On the reference cube $\widehat{T} := [-1,1]^3$ the Raviart-Thomas space is given by

$$\mathcal{RT}_p(\widehat{T}) := \mathbb{Q}_{p,p-1,p-1}\times\mathbb{Q}_{p-1,p,p-1}\times\mathbb{Q}_{p-1,p-1,p}.$$

The dimension of this space is $3p^2(p+1)$ and the degrees of freedom are defined by

1. $\int_F \mathbf{u}\cdot\mathbf{n}\,q\,dS \qquad \forall\,\mathbf{q}\in\mathbb{Q}_{p-1,p-1}$, F face of $\widehat{T}$
2. $\int_{\widehat{T}} \mathbf{u}\cdot\mathbf{q}\,d\mathbf{x} \qquad \forall\,\mathbf{q}\in\mathcal{ND}_{p-1}(\widehat{T})$.

Using these degrees of freedom we can define an interpolation operator $\Pi^{\mathcal{RT}}$ as in the case of the Nédélec basis functions.

Transformations

Let $T\in\mathcal{T}_h$ be an element with diameter h and $\widehat{T}$ the reference element. The affine transformation between these elements is given in (4.2). For functions $\hat{\mathbf{q}}:\widehat{T}\to\mathbb{R}^3$ and $\mathbf{q}:T\to\mathbb{R}^3$ the $\mathbf{H}(\mathrm{div})$-conforming Piola transformation is then given by, see e.g. [82]

$$\mathbf{q}(\mathbf{x}) = \frac{1}{\det B_T}\,B_T\,\hat{\mathbf{q}}(F_T^{-1}(\mathbf{x})). \tag{4.13}$$

It is easy to prove that there holds

$$\|\mathbf{q}\|_{\mathbf{L}^2(T)} \sim h^{-1/2}\|\hat{\mathbf{q}}\|_{\mathbf{L}^2(\widehat{T})}, \tag{4.14}$$

compare Section 4.4.3 for the 2d case.

As in the case of the Nédélec space the local basis function on T are given by

$$\mathbf{b}_j(\mathbf{x}) = \frac{1}{\det B_T} B_T\, \hat{\mathbf{b}}_j(\hat{\mathbf{x}}), \quad j = 1, \dots, n_p.$$

Glueing together the local basis belonging to a common face we obtain global basis functions. Therefore we can define the space $\mathcal{RT}_p(\mathcal{T}_h)$ which is invariant under the transformation (4.2) if we transform the basis functions by (4.13). As for the $\mathbf{H}(\mathbf{curl}, \Omega)$-conforming space we can define the global interpolation operator $\Pi_h^{\mathcal{RT}_p}$, compare (4.10). There holds the following approximation result, see e.g. Hiptmair [62].

Theorem 4.3.1 *For $\mathbf{u} \in \mathbf{H}^s(\Omega)$, $1/2 < s \leq p$, there holds*

$$\|\mathbf{u} - \Pi_h^{\mathcal{RT}_p}\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq Ch^s\|\mathbf{u}\|_{\mathbf{H}^s(\Omega)}.$$

4.4 Raviart-Thomas basis functions for the approximation in $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

In this section we consider the approximation in $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. From Chapter 3 we know that there holds $\gamma_t^\times(\mathbf{H}(\mathbf{curl}, \Omega)) = \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and the resulting finite element space should be the twisted tangential trace of the space $\mathcal{ND}_p(\mathcal{T}_h)$. It follows that this is exactly the space $\mathcal{RT}_p(\mathcal{K}_h)$ of so-called Raviart-Thomas functions. This space was first considered by Raviart & Thomas [92], see also Brezzi & Fortin [23] and Nédélec [82].

As in the three-dimensional case the constraint for $\mathbf{H}(\mathrm{div}, \Gamma)$-conformity is that the normal component $\mathbf{u} \cdot \mathbf{n}$ is continuous between adjacent elements.

The definition of the basis functions is again done locally and we use the transformation between different elements to construct the global space in the same way as for the Nédélec functions.

4.4.1 Definition on squares

We first consider the reference square $\widehat{K} = [-1, 1]^2$. Furthermore, $\mathbb{Q}_{l,m}$ denotes all polynomials with maximum degrees l in x- and m in y-direction. The local Raviart-Thomas space of order p is then defined by

$$\mathcal{RT}_p(\widehat{K}) := \mathbb{Q}_{p,p-1} \times \mathbb{Q}_{p-1,p}.$$

The dimension is then $2p(p+1)$. In literature (e.g. Brezzi & Fortin [23]) this space is sometimes denoted by $\mathcal{RT}_{p-1}(\widehat{K})$, but we use the same counting scheme as in Nédélec [82].

In order to ensure continuity of the normal component we can construct basis functions $\boldsymbol{\phi}_i$ using the following moments. Here, e denotes an edge of $\widehat{K}$ with unit normal vector $\mathbf{n}$ and $\mathbb{P}_p(e)$ the space of polynomials of degree up to p on edge e.

$$m_i(\mathbf{u}) := \begin{cases} \int_e \mathbf{u}\cdot\mathbf{n}\, q\, ds & \text{for all } q \in \mathbb{P}_p(e) \\ \int_{\widehat{K}} \mathbf{u}\cdot\mathbf{q}\, d\mathbf{x} & \text{for all } \mathbf{q} \in \mathbb{Q}_{p-2,p-1}\times\mathbb{Q}_{p-1,p-2} \end{cases}.$$

For the construction of a basis of $\mathcal{RT}_p(\widehat{K})$ we first use monomials and we consider

$$\boldsymbol{\psi}_i(x,y) := \begin{cases} x^r y^s \mathbf{e}_1, & r \le p,\ s \le p-1, \quad \text{if } i = 1,\dots,p(p+1) \\ x^r y^s \mathbf{e}_2, & r \le p-1,\ s \le p, \quad \text{if } i = p(p+1)+1,\dots,2p(p+1) \end{cases}$$

with the unit Cartesian vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and we get the local space by $\mathcal{RT}_p(\widehat{K}) = \mathrm{span}\,\{\boldsymbol{\psi}_i,\ i = 1,\dots,2p(p+1)\}$.

The basis functions are calculated the same way as the Nédélec basis functions, see Section 4.1.1. Another possibility is to use Legendre polynomials as test and trial functions.

Let $W(\widehat{K}) := \{\mathbf{u} \in (L^q(\widehat{K}))^2 \,|\, \mathrm{div}\,\mathbf{u} \in L^2(\widehat{K})\}$, $q > 2$. An interpolation operator $\pi_{\widehat{K}} : W(\widehat{K}) \to \mathcal{RT}_p(\widehat{K})$ can be defined by, see Brezzi & Fortin [23],

$$\int_{\partial\widehat{K}} (\mathbf{u} - \pi_{\widehat{K}}\mathbf{u})\cdot\mathbf{n}\, q\, dS \quad \forall q \in \mathbb{P}_{p-1}(\widehat{K}),$$
$$\int_{\widehat{K}} (\mathbf{u} - \pi_{\widehat{K}}\mathbf{u})\cdot\mathbf{q}\, d\mathbf{x} \quad \forall \mathbf{q} \in (\mathbb{P}_{p-2})^2.$$

Next, we give some examples of basis functions on $\widehat{K}$ for the polynomial degrees $p = 1$ and $p = 2$. For the numbering of the edges, see Figure 4.6.

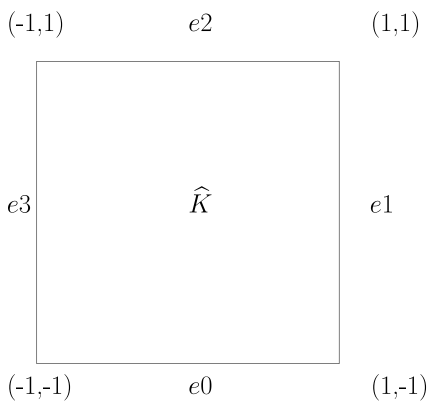

Figure 4.6: Numbering of the edges on the unit square $\widehat{K}$.

For the polynomial degree $p = 1$ we get.

$$\begin{aligned}
\boldsymbol{\lambda}^{(e0)} &:= \frac{1}{4}(y-1)\mathbf{e}_2, \quad \boldsymbol{\lambda}^{(e1)} := \frac{1}{4}(x+1)\mathbf{e}_1,\\
\boldsymbol{\lambda}^{(e2)} &:= \frac{1}{4}(y+1)\mathbf{e}_2, \quad \boldsymbol{\lambda}^{(e3)} := \frac{1}{4}(x-1)\mathbf{e}_1.
\end{aligned}$$

These basis functions are constant on the edge which they are associated to. On the other edges their normal components vanish.

Furthermore, the basis functions of $\mathcal{RT}_2(\widehat{K})$ are given by

$$\begin{aligned}
\boldsymbol{\lambda}_1^{(e0)} &= \frac{1}{8}(1+2y-3y^2)\mathbf{e}_2, & \boldsymbol{\lambda}_2^{(e0)} &= \frac{1}{8}(3x+5xy-9xy^2)\mathbf{e}_2,\\
\boldsymbol{\lambda}_1^{(e1)} &= \frac{1}{8}(-1+2x+3x^2)\mathbf{e}_1, & \boldsymbol{\lambda}_2^{(e1)} &= \frac{1}{8}(-3y+5xy+9x^2y)\mathbf{e}_1,\\
\boldsymbol{\lambda}_1^{(e2)} &= \frac{1}{8}(-1+2y+3y^2)\mathbf{e}_2, & \boldsymbol{\lambda}_2^{(e2)} &= \frac{1}{8}(-3x+5xy+9xy^2)\mathbf{e}_2,\\
\boldsymbol{\lambda}_1^{(e3)} &= \frac{1}{8}(1+2x-3x^2)\mathbf{e}_1, & \boldsymbol{\lambda}_2^{(e3)} &= \frac{1}{8}(3y+5xy-9x^2y)\mathbf{e}_1,\\
\boldsymbol{\lambda}_1^{(\widehat{K})} &= \frac{3}{8}(1-x^2)\mathbf{e}_1, & \boldsymbol{\lambda}_2^{(\widehat{K})} &= \frac{9}{8}(y-x^2y)\mathbf{e}_1,\\
\boldsymbol{\lambda}_3^{(\widehat{K})} &= \frac{3}{8}(1-y^2)\mathbf{e}_2, & \boldsymbol{\lambda}_4^{(\widehat{K})} &= \frac{9}{8}(x-xy^2)\mathbf{e}_2.
\end{aligned}$$

Now, the second edge basis functions are not constant any more on the edges but their normal component vanishes on all other edges which they are not associated to. The normal components of the interior functions vanish on all edges.

Let $\widehat{K}$ be associated to the face F_0 ($z=-1$) of the reference cube $\widehat{T}$. Comparing the degrees of freedom with the ones of $\mathcal{ND}_p(\widehat{T})$ one finds out that there holds

$$\gamma_t^\times\big(\mathcal{ND}_p(\widehat{T})\big) = \mathcal{RT}_p(\widehat{K}).$$

4.4.2 Definition on triangles

On the reference triangle $\widehat{K} := \{(x,y); x,y \geq 0,\, x+y \leq 1\}$ the Raviart-Thomas space is defined by

$$\begin{aligned}
\mathcal{RT}_p(\widehat{K}) &:= (\mathbb{P}_{p-1}(\widehat{K}))^2 \oplus \mathbf{x}\mathbb{P}_p^0(\widehat{K})\\
&= (\mathbb{P}_{p-1}(\widehat{K}))^2 \oplus \{\mathbf{p} \in (\mathbb{P}_p^0(\widehat{K}))^2,\ \mathbf{p}(\mathbf{x}) = 0 \forall \mathbf{x} \in \widehat{K}\}
\end{aligned}$$

where $\mathbb{P}_p^0(\widehat{K}) := \{\mathbf{p} \in \mathbb{P}_p(\widehat{K}) \,|\, p(\mathbf{x}) = 0 \,\forall \mathbf{x} \in \widehat{K}\}$ denotes the space of all homogeneous polynomials of degree p on $\widehat{K}$.

4.4.3 Transformations and an inverse inequality for Raviart-Thomas functions

As in the three dimensional case we consider here the $\mathbf{H}(\mathrm{div})$-conforming Piola transformation (4.13). For the ease of implementation we analyze the transformation for quadrilaterals in detail. The transformation for triangles is similar.

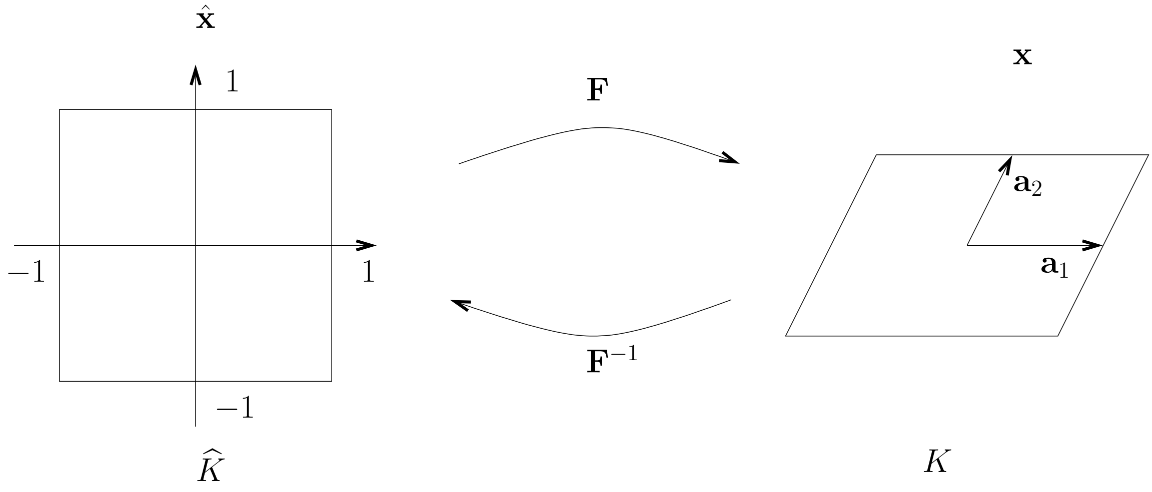

Figure 4.7: Transformation from the reference element $\widehat{K}$ to an arbitrary element K.

Let $\widehat{K} := [-1, 1]^2$ be the reference element in the xy-plane and K be an arbitrary parallelogram with directions $\mathbf{a}_1$ and $\mathbf{a}_2$, see Figure 4.7. The transformation $\mathbf{F} : \widehat{K} \to K$ takes the form

$$\mathbf{x} := \mathbf{F}(\hat{\mathbf{x}}) := B_K \hat{\mathbf{x}} + \mathbf{b}, \qquad \hat{\mathbf{x}} \in \mathbb{R}^3,$$

with the Jacobian matrix

$$B_K := \left(\mathbf{a}_1, \mathbf{a}_2, \mp \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} \right),$$

where the sign depends on the direction of the normal vector on K. Let $\mathbf{n} := (0,\ 0,\ -1)^\mathsf{T}$ be the unit normal vector of $\widehat{K}$ and there holds

$$B_K \mathbf{n} = \pm \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}$$

such that the normal vector on $\widehat{K}$ is mapped onto the normal vector of K.

For the transformation of $\hat{\mathbf{u}} \in \mathcal{RT}(\widehat{K})$ to $\mathbf{u} \in \mathcal{RT}(K)$ we use the **Piola transformation** which preserves the normal component

$$\mathbf{u} \circ F_K = \frac{1}{|\det B_K|} B_K \hat{\mathbf{u}}, \tag{4.15}$$

see Raviart & Thomas [92] or Brezzi & Fortin [23].

There holds

$$|\det B_K(\hat{\mathbf{x}})| = \|\mathbf{a}_1 \times \mathbf{a}_2\|.$$

If we only consider the case of two dimensions we get the transformation

$$\mathbf{u}(\mathbf{x}) = \frac{1}{|\mathbf{a}_1 \times \mathbf{a}_2|}(\mathbf{a}_1, \mathbf{a}_2)\tilde{\mathbf{u}}(\hat{\mathbf{x}}).$$

Now, we consider the transformation of the divergence. There holds, see Brezzi & Fortin [23, p.97]

$$\operatorname{div} \mathbf{v}(\mathbf{x}) = \frac{1}{|\det B_K|} \operatorname{div} \hat{\mathbf{v}}(\hat{\mathbf{x}}). \tag{4.16}$$

This is valid because of (here only for 2 d)

$$\begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} \end{pmatrix} = \frac{1}{|\det B_K|} B_K \begin{pmatrix} \frac{\partial \hat{v}_1}{\partial \hat{x}_1} & \frac{\partial \hat{v}_1}{\partial \hat{x}_2} \\ \frac{\partial \hat{v}_2}{\partial \hat{x}_1} & \frac{\partial \hat{v}_2}{\partial \hat{x}_2} \end{pmatrix} B_K^{-1}.$$

As the trace of a matrix is invariant under similarity transformations and we get (4.16).

Next, we consider the surface divergence $\operatorname{div}_\Gamma \mathbf{v} = \operatorname{div}(\gamma_D \mathbf{v})$. For $\mathbf{v} \in \mathcal{RT}(K)$ there holds $\gamma_D \mathbf{v} = \mathbf{v}$ and we get the transformation

$$\operatorname{div}_\Gamma \mathbf{v}(\mathbf{x}) = \operatorname{div}_{\mathbf{x}}(\mathbf{v}(\mathbf{x})) = \frac{1}{|\det B_K|} \operatorname{div}_{\hat{\mathbf{x}}} \hat{\mathbf{v}}(\hat{\mathbf{x}}) = \frac{1}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} \operatorname{div}_{\hat{\mathbf{x}}} \hat{\mathbf{v}}(\hat{\mathbf{x}}). \tag{4.17}$$

Using the Piola transformation and glueing together neighboring degrees of freedom we get the global space $\mathcal{RT}_p(\mathcal{K}_h)$ and also a global interpolation operator $\pi^{\mathcal{RT}_p}$.

Lemma 4.4.1 (Hiptmair [64, Lemma 2.4]) *The mapping*

$$\gamma_{\mathbf{t}}^{\times} : \mathcal{ND}_p(\mathcal{T}_h) \to \mathcal{RT}_p(\mathcal{K}_h), \quad \mathbf{u} \mapsto \mathbf{u} \times \mathbf{n}$$

is continuous and surjective. Furthermore, the degrees of freedom are transformed, i.e.

$$\gamma_{\mathbf{t}}^{\times} \Pi^{\mathcal{ND}_p} \mathbf{u} = \pi^{\mathcal{RT}_p} \gamma_{\mathbf{t}}^{\times} \mathbf{u} \quad \forall \mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega).$$

The space of the $\mathcal{RT}_p(\mathcal{K}_h)$-functions fulfills the following approximation property using a non-local projection, see Buffa & Hiptmair [31, Theorem 14].

Theorem 4.4.2 *Let* $\mathcal{P}_h : \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \to \mathcal{RT}_p(\mathcal{K}_h)$ *be the orthogonal projection with respect to the* $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$*-inner product. Then, for any* $-\frac{1}{2} \leq s \leq p$ *we have*

$$\|\mathbf{u} - \mathcal{P}_h \mathbf{u}\|_{\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)} \leq C\, h^{s+1/2} \, \|\mathbf{u}\|_{\mathbf{H}_{\|}^{s}(\operatorname{div}_\Gamma, \Gamma)}.$$

Next, we prove an inverse inequality for Raviart Thomas functions in two dimensions. Therefore, we have to examine the transformations between two elements of different size.

Lemma 4.4.3 *For $s \geq 0$ and a regular mesh $\mathcal{K}_h$ there holds for all functions $\mathbf{u}$ that are transformed using* (4.15)

$$\|\mathbf{u}\|_{\mathbf{L}^2(K)} \simeq \|\hat{\mathbf{u}}\|_{\mathbf{L}^2(\widehat{K})},$$
$$\|\operatorname{div}\mathbf{u}\|_{\mathbf{L}^2(K)} \simeq h^{-2}\|\operatorname{div}\hat{\mathbf{u}}\|_{\mathbf{L}^2(\widehat{K})}$$

and for the semi-norms

$$|\mathbf{u}|_{\mathbf{H}^s(K)} \simeq h_K^{-s}|\hat{\mathbf{u}}|_{\mathbf{H}^s(\widehat{K})},$$
$$|\operatorname{div}\mathbf{u}|_{\mathbf{H}^s(K)} \simeq h_K^{-1-s}|\operatorname{div}\hat{\mathbf{u}}|_{\mathbf{H}^s(\widehat{K})}$$

where the constant is independent of $\mathbf{u}$ and the mesh size h.

Proof. The proof is quite similar to the proof of Lemma 4.1.2, the results for the integer norm can also be found in Brezzi & Fortin [23, Lemma III.1.7]. In the two dimensional case there holds for the spectral norm $\|\cdot\|$ of the transformation matrix, compare Ciarlet [36],

$$\|B_k\| \simeq h, \qquad \|B_K^{-1}\| \simeq h^{-1}, \qquad |\det B_K| \simeq h^2.$$

Using the transformation rule, the integer norms become

$$\|\mathbf{u}\|^2_{\mathbf{L}^2(K)} \leq \frac{1}{|\det B_K|^2}\|B_K\|^2\,|\det B_K|\,\|\hat{\mathbf{u}}\|^2_{\mathbf{L}^2(\widehat{K})} \simeq \|\hat{\mathbf{u}}\|^2_{\mathbf{L}^2(\widehat{K})},$$
$$|\mathbf{u}|^2_{\mathbf{H}^m(K)} \leq \frac{1}{|\det B_K|^2}\|B_K\|^2\,|\det B_K|\,\|B_K^{-1}\|^2 m\,|\hat{\mathbf{u}}|^2_{\mathbf{H}^m(K)} \simeq h^{-2m}|\hat{\mathbf{u}}|^2_{\mathbf{H}^m(\widehat{K})}, \quad m \in \mathbb{N},$$

and

$$\|\operatorname{div}\mathbf{u}\|^2_{\mathbf{L}^2(K)} \leq \frac{1}{|\det B_K|^2}|\det B_K|\,\|\operatorname{div}\hat{\mathbf{u}}\|^2_{\mathbf{L}^2(\widehat{K})} \simeq h^{-2}\|\operatorname{div}\hat{\mathbf{u}}\|^2_{\mathbf{L}^2(\widehat{K})},$$
$$|\operatorname{div}\mathbf{u}|^2_{\mathbf{H}^m(K)} \leq \frac{1}{|\det B_K|^2}|\det B_K|\,\|B_K^{-1}\|^{2m}\,|\hat{\mathbf{u}}|^2_{\mathbf{H}^m(\widehat{K})} \simeq h^{-2-2m}|\hat{\mathbf{u}}|^2_{\mathbf{H}^m(\widehat{K})}, \quad m \in \mathbb{N}.$$

In order to estimate the non-integer semi-norms we get for $0 < s < 1$

$$\begin{aligned}
|\mathbf{u}|^2_{\mathbf{H}^s(K)} &= \int_K\int_K \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{2+2s}}\,d\mathbf{x}\,d\mathbf{y} \\
&= |\det B_K|^2 \int_{\widehat{K}}\int_{\widehat{K}} \frac{|(\det B_K)^{-1}B_K\big(\hat{\mathbf{u}}(\hat{\mathbf{x}}) - \hat{\mathbf{u}}(\hat{\mathbf{y}})\big)|^2}{|B_K(\hat{\mathbf{x}}-\hat{\mathbf{y}})|^{2+2s}}\,d\mathbf{x}\,d\mathbf{y} \\
&\leq \|B_K\|^{-2-2s}\int_{\widehat{K}}\int_{\widehat{K}} \frac{|B_K\big(\hat{\mathbf{u}}(\hat{\mathbf{x}}) - \hat{\mathbf{u}}(\hat{\mathbf{y}})\big)|^2}{|\hat{\mathbf{x}}-\hat{\mathbf{y}}|^{2+2s}}\,d\hat{\mathbf{x}}\,d\hat{\mathbf{y}} \\
&\leq \|B_K\|^{-2-2s}\|B_K\|^2|\hat{\mathbf{u}}|^2_{\mathbf{H}^s(\widehat{K})} \\
&\leq C\,h_K^{-2s}|\hat{\mathbf{u}}|^2_{\mathbf{H}^s(\widehat{K})}.
\end{aligned}$$

The proof for the norms of the divergence is analog. □

Following the idea of the proof of Lemma 4.1.3 we can show an inverse inverse inequality.

Lemma 4.4.4 *Let $\mathcal{K}_h$ be a regular triangulation of Γ with mesh size h. Then, there holds for $\mathbf{u} \in \mathcal{RT}_p(\mathcal{K}_h)$ and $s \geq r \geq 0$*

$$\|\mathbf{u}\|_{\mathbf{H}^s_{\|}(\mathrm{div}_\Gamma,\Gamma)} \leq Ch^{r-s}\|\mathbf{u}\|_{\mathbf{H}^r_{\|}(\mathrm{div}_\Gamma,\Gamma)} \tag{4.18}$$

with a constant $C > 0$ depending only on the polynomial degree p and the reference element $\widehat{K}$.

Remark 4.4.5 *This result also holds for negative norms. A general proof of this can be found in Babuška & Aziz [11].*

4.5 $\mathcal{TND}$-basis functions for the approximation in $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$

Finally, we analyze the approximation in the space $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. This is the tangential trace space of $\mathbf{H}(\mathbf{curl}, \Omega)$ and also the image of $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ under the map $R\mathbf{u} := \mathbf{n} \times \mathbf{u}$. We define the space $\mathcal{TND}_p(\mathcal{K}_h)$ as the tangential trace space of $\mathcal{ND}_p(\mathcal{T}_h)$, see also Teltscher [103],

$$\mathcal{TND}_p(\mathcal{K}_h) := \gamma_D(\mathcal{ND}_p(\mathcal{T}_h|_\Gamma)).$$

Hence, we easily see that for the reference square $\widehat{K} = [-1, 1]^2$ there holds

$$\mathcal{TND}_p(\widehat{K}) := \mathbb{Q}_{p-1,p} \times \mathbb{Q}_{p,p-1}$$

and that $\dim \mathcal{TND}_p(\widehat{K}) = 2p(p + 1)$. The basis functions can easily be calculated from the $\mathcal{ND}_p$-basis functions. For the lowest polynomial degree there holds

$$\begin{aligned} \boldsymbol{\phi}_0 &:= \frac{1}{4}(1-y)\mathbf{e}_1, \quad & \boldsymbol{\phi}_1 &:= \frac{1}{4}(1+x)\mathbf{e}_2, \\ \boldsymbol{\phi}_2 &:= \frac{1}{4}(1+y)\mathbf{e}_1, \quad & \boldsymbol{\phi}_3 &:= \frac{1}{4}(1-x)\mathbf{e}_2. \end{aligned}$$

4.5.1 Transformations for $\mathcal{TND}$-basis functions

In this section we examine the transformation of $\mathcal{TND}$-basis functions. As there also holds $\mathcal{TND}_p(\mathcal{K}_h) = R(\mathcal{RT}_p(\mathcal{K}_h))$ with $R\mathbf{u} = \mathbf{n} \times \mathbf{u}$ we use the Piola transformation.

Figure 4.8 shows the transformation of the basis functions. There holds $\boldsymbol{\lambda} = R\boldsymbol{\phi}(\mathbf{x})$ and $\hat{\boldsymbol{\phi}} = R^{-1}\hat{\boldsymbol{\lambda}}$. From (3.3) we have $R^{-1}\mathbf{u} = -R\mathbf{u}$. On the reference element we use the

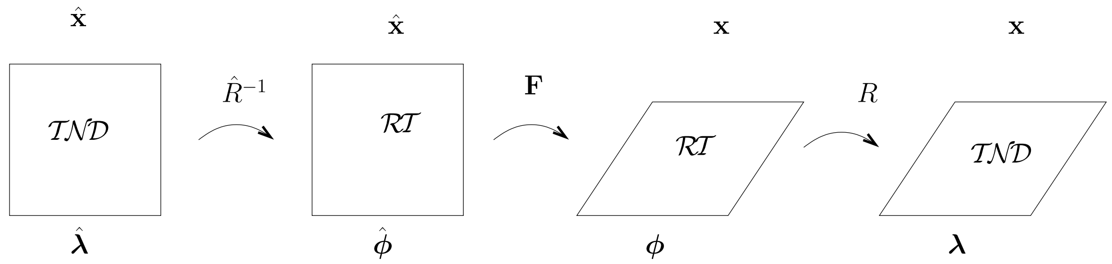

Figure 4.8: Transformation of $\mathcal{TND}$-basis functions.

unit normal vector $\hat{\mathbf{n}} = (0,\, 0,\, \pm 1)^\mathsf{T}$ and on the local element, which is spanned by the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, we have the normal vector $\mathbf{n} = \pm \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}$, the algebraic sign depends on the orientation of the element. We demand that the algebraic signs of $\mathbf{n}$ and $\hat{\mathbf{n}}$ are equal. Thus, there holds

$$
\begin{aligned}
\boldsymbol{\lambda}(\mathbf{x}) &= R\boldsymbol{\phi}(\mathbf{x}) \\
&= \frac{1}{\det B_K(\hat{\mathbf{x}})} R\big(B_K(\hat{\mathbf{x}})\hat{\boldsymbol{\phi}}(\hat{\mathbf{x}})\big) \\
&= \frac{1}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}\big((\mathbf{a}_1,\, \mathbf{a}_2, \mathbf{n})\hat{\boldsymbol{\phi}}\big) \times \mathbf{n} \\
&= \frac{1}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}\big((\mathbf{a}_1 \times \mathbf{n},\, \mathbf{a}_2 \times \mathbf{n}, 0)(-\hat{R}\hat{\boldsymbol{\lambda}}(\hat{\mathbf{x}}))\big) \\
&= \frac{1}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}\big((\mathbf{n} \times \mathbf{a}_1,\, \mathbf{n} \times \mathbf{a}_2, 0)(-\hat{R}\hat{\boldsymbol{\lambda}}(\hat{\mathbf{x}}))\big) \\
&= \frac{1}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}(\mathbf{n} \times \mathbf{a}_1,\, \mathbf{n} \times \mathbf{a}_2, 0)\left[\hat{\boldsymbol{\lambda}}(\hat{\mathbf{x}}) \times \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}\right] \\
&= \frac{1}{\|\mathbf{a}_1 \times \mathbf{a}_2\|^2}\big((\mathbf{a}_1 \times \mathbf{a}_2) \times \mathbf{a}_1,\, (\mathbf{a}_1 \times \mathbf{a}_2) \times \mathbf{a}_2, 0\big) \begin{pmatrix} \hat{\lambda}_2 \\ -\hat{\lambda}_1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\|\mathbf{a}_1 \times \mathbf{a}_2\|^2}\big(\mathbf{a}_2 \times (\mathbf{a}_1 \times \mathbf{a}_2),\, (\mathbf{a}_1 \times \mathbf{a}_2) \times \mathbf{a}_1, 0\big) \begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ 0 \end{pmatrix}.
\end{aligned}
$$

Finally, we have

Lemma 4.5.1 (Transformation for $\mathcal{TND}$ elements) *Let $\hat{\mathbf{x}}$ and $\hat{\boldsymbol{\lambda}}(\hat{\mathbf{x}})$ be defined on the reference element $[-1, 1]^2$ and $\mathbf{x}$ and $\boldsymbol{\lambda}(\mathbf{x})$ be defined on the local element which is spanned by the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. There holds*

$$
\boldsymbol{\lambda}(\mathbf{x}) = \frac{1}{\|\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}\|^2}\big(\mathbf{a}^{(2)} \times \big(\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}\big), (\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}) \times \mathbf{a}^{(1)}\big)\tilde{\boldsymbol{\lambda}}(\hat{\mathbf{x}}).
$$

Next, we calculate the transformation of the scalar surface curl operator $\operatorname{curl}_\Gamma$. This can be done using (4.17). In the beginning, we remark that there holds

$$\operatorname{div}_\Gamma(\boldsymbol{\psi} \times \mathbf{n}) = -\operatorname{curl}_\Gamma\big((\boldsymbol{\psi} \times \mathbf{n}) \times \mathbf{n}\big) = \operatorname{curl}_\Gamma(\gamma_D \boldsymbol{\psi}) = \operatorname{curl}_\Gamma \boldsymbol{\psi}.$$

As above let $\boldsymbol{\lambda}$ be a $\mathcal{TND}$-basis function on the local element and $\boldsymbol{\phi}$ be the related $\mathcal{RT}$-basis function. On the reference element there holds

$$\hat{\boldsymbol{\phi}} = \hat{R}^{-1}\hat{\boldsymbol{\lambda}} = -R\hat{\boldsymbol{\lambda}} = -\boldsymbol{\lambda} \times \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}.$$

Thus, we get with (4.16) and (4.17)

$$\begin{aligned} \operatorname{curl}_\Gamma \boldsymbol{\lambda} &= \operatorname{div}_\Gamma(\boldsymbol{\lambda} \times \mathbf{n}) = \operatorname{div}_\Gamma(R\boldsymbol{\phi} \times \mathbf{n}) = \operatorname{div}_\Gamma\big((\boldsymbol{\phi} \times \mathbf{n}) \times \mathbf{n}\big) \\ &= -\operatorname{div}_\Gamma(\gamma_D \boldsymbol{\phi}) = -\operatorname{div}(\gamma_D \gamma_D \boldsymbol{\phi}) = -\operatorname{div}(\boldsymbol{\phi}) \\ &= -\frac{1}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} \operatorname{div}_{\hat{\mathbf{x}}} \hat{\boldsymbol{\phi}} \\ &= \frac{\mp 1}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} \operatorname{div}_{\hat{\mathbf{x}}} \begin{pmatrix} -\hat{\lambda}_2 \\ \hat{\lambda}_1 \\ 0 \end{pmatrix} \\ &= \frac{\mp 1}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} \left(-\partial_{\hat{x}_1}\hat{\lambda}_2 + \partial_{\hat{x}_2}\hat{\lambda}_1\right) \\ &= \frac{\mp 1}{\|\mathbf{a}_1 \times \mathbf{a}_2\|} \left(\widehat{\operatorname{curl}}\hat{\boldsymbol{\lambda}}\right) \end{aligned} \tag{4.19}$$

with negative sign if $\mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}$ and positive sign if $\mathbf{n} = -\frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}$.

4.6 The de Rham diagram

In this subsection we consider the so-called de Rham diagram. It describes the mapping behavior of the differential operators **grad**, **curl** and div in the corresponding Sobolev spaces. Furthermore, we consider further properties of the canonical interpolation operators. Most of the results can be found in the articles of Hiptmair [64, 63, 66, 65].

For $\Omega \subset \mathbb{R}^3$ we consider the following **de Rham diagram**, see e.g. Monk [78]

$$H^1(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}(\mathbf{curl}, \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\operatorname{div}, \Omega) \xrightarrow{\operatorname{div}} L^2(\Omega).$$

A similar result holds for homogeneous boundary conditions

$$H^1_0(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}_0(\mathbf{curl}, \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}_0(\operatorname{div}, \Omega) \xrightarrow{\operatorname{div}} L^2(\Omega)/\mathbb{R}.$$

In these diagrams, the range of one operator is contained in the kernel of the following one. The range space of each operator is a closed subspace of the related operator with finite codimension, see Monk [78, Theorem 3.40].

The **discrete de Rham diagram** takes the following form

$$\mathcal{S}_p(\mathcal{T}_h) \xrightarrow{\textbf{grad}} \mathcal{ND}_p(\mathcal{T}_h) \xrightarrow{\textbf{curl}} \mathcal{RT}_p(\mathcal{T}_h) \xrightarrow{\text{div}} \mathcal{S}_{p-1}(\mathcal{T}_h),$$

see e.g. Hiptmair [64, 65].

There also hold the following commuting diagram property, see e.g. Hiptmair [64, 63], where I_p^h denotes the canonical interpolation operator for $\mathcal{S}_p(\mathcal{T}_h)$ and $\mathcal{D}(\cdot)$ denotes the domain of the interpolation operators.

Theorem 4.6.1 *For all $p \geq 1$ the following diagram commutes*

$$\begin{array}{ccccc} \mathcal{D}(I_p^h) \subset H^1(\Omega) & \xrightarrow{\textbf{grad}} & \mathcal{D}(\Pi_p^h) \subset \mathbf{H}(\textbf{curl}, \Omega) & \xrightarrow{\textbf{curl}} & \mathcal{D}(\pi_p^h) \subset \mathbf{H}(\text{div}, \Omega) \\ \downarrow I_K^h & & \downarrow \Pi_p^h & & \downarrow \pi_p^h \\ \mathcal{S}_p(\mathcal{T}_h) & \xrightarrow{\textbf{grad}} & \mathcal{ND}_p(\mathcal{T}_h) & \xrightarrow{\textbf{curl}} & \mathcal{RT}_p(\mathcal{T}_h) \end{array} .$$

This also holds true if we impose homogeneous boundary conditions.

Such that we have

$$\textbf{curl}\, \Pi_p^h \mathbf{u} = \pi_p^h \,\textbf{curl}\, \mathbf{u} \qquad \forall\, \mathbf{u} \in \mathbf{H}(\textbf{curl}, \Omega).$$

Furthermore, the kernels of the differential operators are preserved:

$$\begin{aligned} \mathbf{u} \in \mathcal{D}(\Pi_p^h), \textbf{curl}\, \mathbf{u} = 0 \quad &\Longrightarrow \quad \textbf{curl}\, \Pi_p^h \mathbf{u} = 0 \\ \mathbf{u} \in \mathcal{D}(\pi_p^h), \text{div}\, \mathbf{u} = 0 \quad &\Longrightarrow \quad \text{div}\, \pi_p^h \mathbf{u} = 0. \end{aligned}$$

Theorem 4.6.2 *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a simply connected domain. Then, there holds*

- *Let $\mathbf{u} \in \mathcal{ND}_p(\mathcal{T}_h)$ with $\textbf{curl}\, \mathbf{u} = 0$. There exists a $\phi \in \mathcal{S}_p(\mathcal{T}_h)$ with $\mathbf{u} = \textbf{grad}\, \phi$.*
- *Let $\mathbf{v} \in \mathcal{RT}_p^{(3)}(\mathcal{T}_h)$ with $\text{div}\, \mathbf{v} = 0$. There exists a $\mathbf{u} \in \mathcal{ND}_p(\mathcal{T}_h)$ with $\mathbf{v} = \textbf{curl}\, \phi$.*
- *Let $\mathbf{v} \in \mathcal{RT}_p^{(2)}(\mathcal{K}_h)$ with $\text{div}\, \mathbf{v} = 0$. There exists a $\phi \in \mathcal{S}_p(\mathcal{K}_h)$ with $\mathbf{v} = \textbf{curl}_\Gamma\, \phi$.*

Proof. See Hiptmair [63, 66]. □

4.7 An extension operator for $\mathcal{RT}_p(\mathcal{K}_h)$

In this section we construct continuous extension operators from $\mathcal{RT}_p(\mathcal{K}_h)$ to $\mathcal{ND}_p(\mathcal{T}_h)$ due to Alonso & Valli [5] and due to Hiptmair. These spaces are used for the discretization of the spaces $\mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$. We know that the mapping

$$\gamma_t^{\times}:\mathbf{H}(\mathbf{curl},\Omega)\to\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$$

mapping is linear, continuous and surjective and that there exists a continuous inverse mapping, see Lemma 3.1.3. Furthermore, we know that there holds

$$\gamma_t^{\times}\big(\mathcal{ND}_p(\mathcal{T}_h)\big)=\mathcal{RT}_p(\mathcal{K}_h).$$

The aim is to construct a continuous inverse for this mapping. A construction of such an inverse is given in Alonso & Valli [5, p. 617 ff.]. Therefore, the elliptic bilinear form

$$a(\mathbf{u},\mathbf{v}):=\int_\Omega(\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\overline{\mathbf{v}}+\mathbf{u}\cdot\overline{\mathbf{v}})$$

is considered. In the beginning, we consider for $\boldsymbol{\lambda}\in\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$ the problem of finding an extension $\mathbf{F}\boldsymbol{\lambda}\in\mathbf{H}(\mathbf{curl},\Omega)$ such that there holds

$$\begin{aligned} a(\mathbf{F}\boldsymbol{\lambda},\mathbf{v})&=0 \qquad \forall\,\mathbf{v}\in\mathbf{H}_0(\mathbf{curl},\Omega),\\ \mathbf{F}\boldsymbol{\lambda}\times\mathbf{n}&=\boldsymbol{\lambda}.\end{aligned}$$

From the Theorem of Lax-Milgram and the continuity of $\gamma_t^{\times}$ follows the existence of such an $\mathbf{F}\boldsymbol{\lambda}$ and the continuity of $\mathbf{F}$, such that there holds

$$\|\mathbf{F}\boldsymbol{\lambda}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\le C\,\|\boldsymbol{\lambda}\|_{\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)}$$

with a constant $C>0$.

The finite dimensional counterpart of $\mathbf{F}$ is the operator $\mathbf{F}_h:\mathcal{RT}_p(\mathcal{K}_h)\to\mathcal{ND}_p(\mathcal{T}_h)$ which is defined by solving the problem:
For $\boldsymbol{\lambda}_h\in\mathcal{RT}_p(\mathcal{K}_h)$ find $\mathbf{F}_h\boldsymbol{\lambda}_h\in\mathcal{ND}_p(\mathcal{T}_h)$ such that there holds

$$\begin{aligned} a(\mathbf{F}_h\boldsymbol{\lambda}_h,\mathbf{v})&=0 \qquad \forall\,\mathbf{v}\in\mathcal{ND}_p(\mathcal{T}_h)\cap\mathbf{H}_0(\mathbf{curl},\Omega),\\ \mathbf{F}_h\boldsymbol{\lambda}_h\times\mathbf{n}&=\boldsymbol{\lambda}_h.\end{aligned}$$

Thus, there exists a constant $C>0$, independent of h, such that

$$\|\mathbf{F}_h\boldsymbol{\lambda}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)}\le C\,\|\boldsymbol{\lambda}_h\|_{\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)}\quad\forall\,\boldsymbol{\lambda}\in\mathcal{RT}_p(\mathcal{K}_h),$$

cf. Alonso and Valli [5, p. 619]. There is an even weaker result proven in which Γ is only a part of the boundary $\partial\Omega$. On $\partial\Omega\backslash\Gamma$ the functions in $\mathbf{H}(\mathbf{curl},\Omega)$ are assumed to vanish. The functions in $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$ are extended by 0 on $\partial\Omega\backslash\Gamma$.

A further extension operator was communicated by Ralf Hiptmair. The main tool is the de Rham diagram from Section 4.6.

Initially, we construct an extension of a piecewise polynomial function on the boundary to a Raviart-Thomas function in the domain Ω, cf. Hiptmair [62].

Lemma 4.7.1 *Let $\xi_h \in \mathcal{S}_{p-1}(\mathcal{K}_h)$ be a piecewise polynomial function on Γ. Then, there exists an extension $\mathbf{v}_h \in \mathcal{RT}_p(\mathcal{T}_h)$ with $\operatorname{div} \mathbf{v}_h = 0$ and $\mathbf{v}_h \cdot \mathbf{n} = \xi_h$ on Γ such that*

$$\|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)} \le C\, \|\xi_h\|_{H^{-1/2}(\Gamma)} \tag{4.20}$$

with a constant $C > 0$ independent of h.

Proof. Let u be the solution of the Neumann problem

$$\begin{aligned} -\Delta u &= 0 \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= \xi_h \qquad \text{on } \Gamma. \end{aligned}$$

As $\xi_h \in L^2(\Gamma)$ and the domain Ω has a piecewise continuous boundary there holds $u \in H^{3/2+\varepsilon}(\Omega)$, $0 < \varepsilon < \varepsilon_0$, cf. Grisvard [52, Corollary 2.6.7] or Dauge [41]. It follows that $\nabla u \in \mathbf{H}^{1/2+\varepsilon}(\Omega)$ and $\operatorname{div} \nabla u = 0$. Let $\Pi^{\mathcal{RT}} : \mathbf{H}^{1/2+\varepsilon}(\Omega) \cap \mathbf{H}(\operatorname{div}, \Omega) \to \mathcal{RT}_p(\mathcal{T}_h)$ be the continuous interpolation operator defined in Section 4.3. We define

$$\mathbf{v}_h := \Pi^{\mathcal{RT}} \nabla u$$

and there holds

$$\operatorname{div} \mathbf{v}_h = \operatorname{div} \Pi^{\mathcal{RT}} \nabla u = 0$$

and

$$\|\mathbf{v}_h - \nabla u\|_{\mathbf{L}^2(\Omega)} \le C\, h^{1/2+\varepsilon} \|\nabla u\|_{\mathbf{H}^{1/2+\varepsilon}(\Omega)},$$

see Theorem 4.3.1 or Hiptmair & Schwab [68, Lemma 5.1], and

$$\mathbf{v}_h \cdot \mathbf{n} = \xi_h.$$

This holds because $\mathcal{RT}_p$ and $\Pi^{\mathcal{RT}}$ are constructed in such a way that the normal component on the element faces has to be continuous. Thus, there has to hold $\mathbf{n} \cdot \Pi^{\mathcal{RT}} \nabla u = \mathbf{n} \cdot \nabla u = \boldsymbol{\xi}_h$.
Using the inverse inequality

$$\|\xi_h\|_{H^{\varepsilon}(\Gamma)} \le C\, h^{-1/2-\varepsilon} \|\xi_h\|_{H^{-1/2}(\Gamma)}$$

we get

$$\begin{aligned} \|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)} &\le C\, h^{1/2+\varepsilon} \|\nabla u\|_{\mathbf{H}^{1/2+\varepsilon}(\Omega)} + \|\nabla u\|_{\mathbf{L}^2(\Omega)} \\ &\le C h^{1/2+\varepsilon} \|\xi_h\|_{H^{\varepsilon}(\Gamma)} + C\, \|\xi_h\|_{H^{-1/2}(\Gamma)} \\ &\le C\, \|\xi_h\|_{H^{-1/2}(\Gamma)}. \end{aligned}$$

□

Using this result, we can proof the following extension theorem.

Theorem 4.7.2 *For* $\boldsymbol{\lambda}_h \in \mathcal{RT}_p(\mathcal{K}_h) \subset \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *there exists* $\mathbf{w}_h \in \mathcal{ND}_p(\mathcal{T}_h)$ *with* $\mathbf{w}_h \times \mathbf{n} = \boldsymbol{\lambda}_h$ *and a constant* $C > 0$*, independent of* h*, such that*

$$\|\mathbf{w}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le C\,\|\boldsymbol{\lambda}_h\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}. \tag{4.21}$$

Proof. First of all, we define $\xi_h := \mathrm{div}\,\boldsymbol{\lambda}_h \in \mathcal{S}_{p-1}(\mathcal{K}_h)$. Using the constructed extension function of Lemma 4.7.1 we find $\mathbf{v}_h \in \mathcal{RT}_p(\mathcal{T}_h)$ with $\mathrm{div}\,\mathbf{v}_h = 0$ and $\mathbf{v}_h \cdot \mathbf{n} = \mathrm{div}\,\boldsymbol{\lambda}_h$. From the de Rham diagram we then know that there exists a $\mathbf{v}_h \in \mathcal{ND}_p(\mathcal{T}_h)$ with

$$\mathbf{curl}\,\mathbf{u}_h = \mathbf{v}_h$$

and

$$\|\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \le C\,\|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)}$$

such that there holds, using (4.20),

$$\|\mathbf{u}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le C\,\|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)} \le C\,\|\xi_h\|_{H^{-1/2}(\Gamma)} = C\,\|\,\mathrm{div}\,\boldsymbol{\lambda}_h\|_{H^{-1/2}(\Gamma)}. \tag{4.22}$$

Next, we define

$$\boldsymbol{\zeta}_h := \boldsymbol{\lambda}_h - (\mathbf{v}_h \times \mathbf{n}) \in \mathcal{RT}_p(\mathcal{K}_h)$$

and there holds

$$\begin{aligned}\mathrm{div}_\Gamma\,\boldsymbol{\zeta}_h &= \mathrm{div}\,\boldsymbol{\lambda}_h - \mathrm{div}(\mathbf{u}_h \times \mathbf{n})\\ &= \mathrm{div}\,\boldsymbol{\lambda}_h - \mathbf{n}\cdot\mathbf{curl}\,\mathbf{u}_h\\ &= \xi_h - \xi_h = 0.\end{aligned}$$

Using the de Rham diagram on the boundary there exists a $\phi_h \in \mathcal{S}_p(\mathcal{K}_h)$ with $\boldsymbol{\zeta}_h = \mathbf{curl}_\Gamma\,\phi_h$ and there holds

$$\|\phi_h\|_{H^{1/2}(\Gamma)} \le C\,\|\boldsymbol{\zeta}_h\|_{\mathbf{H}^{-1/2}(\Gamma)} = C\|\boldsymbol{\zeta}_h\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}.$$

Next, we use a discrete harmonic extension, see Chapter 2, to extend $\phi_h \in \mathcal{S}_p(\mathcal{K}_h)$ to $\Phi_h \in \mathcal{S}_p(\mathcal{T}_h)$ with

$$\|\Phi_h\|_{H^1(\Omega)} \le C\,\|\phi_h\|_{H^{1/2}(\Gamma)}. \tag{4.23}$$

Due to the de Rham diagram there holds $\mathbf{grad}\,\Phi_h \in \mathcal{ND}_p(\mathcal{T}_h)$ and we define

$$\mathbf{w}_h := \mathbf{u}_h + \mathbf{grad}\,\Phi_h$$

such that there holds

$$\begin{aligned}\mathbf{w}_h \times \mathbf{n} &= \mathbf{u}_h \times \mathbf{n} + \mathbf{curl}_\Gamma\,\Phi_h|_\Gamma = \mathbf{u}_h \times \mathbf{n} + \boldsymbol{\zeta}_h\\ &= \boldsymbol{\lambda}_h.\end{aligned} \tag{4.24}$$

Finally, we have to prove the stability of the extension (4.21). Using (4.22), (4.23) and (4.24) we get

$$\begin{aligned}
\|\mathbf{w}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)} &\le \|\mathbf{u}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\Phi_h\|_{H^1(\Omega)} \\
&\le C\,\left\{\|\operatorname{div}\lambda_h\|_{H^{-1/2}(\Gamma)} + \|\phi_h\|_{H^{1/2}(\Gamma)}\right\} \\
&\le C\,\left\{\|\operatorname{div}\boldsymbol{\lambda}_h\|_{H^{-1/2}(\Gamma)} + \|\boldsymbol{\zeta}_h\|_{H^{-1/2}(\Gamma)}\right\} \\
&\le C\,\left\{\|\boldsymbol{\lambda}_h\|_{\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)} + \|\mathbf{u}_h \times \mathbf{n}\|_{\mathbf{H}^{-1/2}(\Gamma)}\right\} \\
&\le C\,\|\boldsymbol{\lambda}_h\|_{\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)}.
\end{aligned}$$

□

Remark 4.7.3 *The above extension operators are only valid for the whole spaces $\mathcal{ND}_p(\mathcal{T}_h)$ and $\mathcal{RT}_p(\mathcal{K}_h)$. Although, we know that for every basis functions $\boldsymbol{\phi} \in \mathcal{RT}_p(\mathcal{K}_h)$ there exists a basis function $\mathbf{b} \in \mathcal{ND}_p(\mathcal{T}_h)$ with $\gamma_t^\times(\mathbf{b}) = \boldsymbol{\phi}$ it is not clear if the estimate*

$$\|\mathbf{b}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le C\,\|\boldsymbol{\phi}\|_{\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma,\Gamma)}$$

is independent of the mesh size h. This is the main problem in constructing a stable decomposition of the space $\mathcal{RT}_2(\mathcal{K}_h)$ in Section 4.10.

4.8 hp-Interpolation

The construction of residual error estimates involves the construction of suitable interpolation operators. In the previous sections we have considered several moment-based interpolation operators and have given estimates for the h-version. In this section we will introduce a quasi-optimal projection-based interpolation operator for the p-version and hp-version for $\mathbf{H}(\mathbf{curl}, \Omega)$. In Chapter 5 we use this result to derive an error estimator for the hp-version of the coupling of FEM and BEM for the eddy current problem.

For the moment-based interpolation operator, introduced in section 4.1, Monk [77] proves the following suboptimal p-interpolation error estimate.

Theorem 4.8.1 (Monk [77]) *Let* $\mathbf{u} \in \mathbf{H}^r(\Omega)$ *for some* $r > 1$ *and let* $\Pi_p^h : \mathbf{H}^r(\Omega) \to \mathcal{ND}_p(\mathcal{T}_h)$ *be the interpolation operator due to Nédélec , then there holds*

$$\|\mathbf{u} - \Pi_p^h \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq C h^{\min(p,r)} p^{-(r-1)} \|\mathbf{u}\|_{\mathbf{H}^r(\Omega)}.$$

In his proof he follows the ideas of Suri [102] and uses expansions into series of Legendre polynomials. Furthermore, we remark that he requires a strong regularity of $\mathbf{H}^r(\Omega)$, $r > 1$, for the interpolated function.

Another idea is to introduce projection-based interpolation operators, see e.g. Demkowicz *et al.* [46, 45] or Hiptmair [65]. Hiptmair [65, Theorem 3.18] proves the following estimate using the results from differential geometry for his interpolation operator Π_p^1.

Theorem 4.8.2 (Hiptmair) *Under certain assumptions on polynomial extensions the projection-based interpolation operators* Π_p^1, $p \in \mathbb{N}_0$, *satisfy*

$$\|\mathbf{u} - \Pi_p^1 \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq C\, p^{\frac{1}{2}-\epsilon} \left(\|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} + \|\,\mathbf{curl}\,\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \right),$$

for any $\epsilon > 0$, *and for all sufficiently smooth vector-fields* $\mathbf{u}$ *with a constant* $C = C(\varrho(\Omega_h), \epsilon) > 0$, *independent of* p. *For all* $r > 1$ *and* $\epsilon > 0$ *we have, with* $C = C(r, \epsilon, \varrho(\Omega_h)) > 0$,

$$\|\mathbf{u} - \Pi_p^1 \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq C\, p^{-(r-1-\epsilon)} \|\mathbf{u}\|_{\mathbf{H}^r(\Omega)}.$$

The exponent $-(r - 1 - \varepsilon)$ is still not optimal. The aim is to construct an interpolation operator with the factor $p^{-(r-\varepsilon)}$.

In two dimensions Babuška & Demkowicz [44] prove the following result for their interpolation operator operator Π^{curl} on a triangle T. They use extension operators and a discrete Friedrich's inequality.

Theorem 4.8.3 (Babuška/Demkowicz) *There exists a constant $C > 0$, independent of the polynomial degrees p and p_e, the polynomial degree on the edges e, such that*

$$\|\mathbf{u} - \Pi^{curl}\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},T)} \leq C p_{\min}^{-(r-\epsilon)} \left(\|\mathbf{u}\|^2_{\mathbf{H}^r(T)} + \|\,\mathbf{curl}\,\mathbf{u}\|^2_{\mathbf{H}^r(T)} \right)$$

for every $0 < r < 1$, and $0 < \epsilon < r$. Here, $p_{\min} = \min_e p_e$.

First of all, they prove the continuity of their extension operator on the space $\mathbf{H}^\epsilon(T) \cap \mathbf{H}(\mathbf{curl}, T)$, $\epsilon > 0$, and then use a best approximation result to get a stronger norm.

Using the ideas of Babuška and Demkowicz we construct an interpolation operator in three dimensions. In the following we consider a tetrahedron T. On this we define the polynomial space $\mathbf{P}^p_{p_e,p_f}(T)$, denoting the space of vector-valued polynomials of order p defined on T with traces of their tangential components on edges e of (possible lower) order p_e, and with traces of their tangential component on faces f of (possible lower) order p_f.

Extending the construction of Demkowicz & Babuška [44] we get the interpolation operator, compare also Demkowicz *et al.* [46, 45] and Hiptmair [65],

$$\tilde{\Pi}^1_p : \mathbf{H}^r(\mathbf{curl}, T) \longrightarrow \mathbf{P}^p_{p_e,p_f}(T), \quad r > 1/2 + \epsilon,$$

which is defined as the sum

$$\tilde{\Pi}^1_p \mathbf{u} := \mathbf{u}_1 + \mathbf{u}_2^{p_e} + \mathbf{u}_3^{p_F} + \mathbf{u}_4^{p_I}. \tag{4.25}$$

Therefore, we perform the following steps.

In **Step 1** we use Whitney's lowest order interpolant. Assuming that $\mathbf{u}$ fulfills the regularity demands in Remark4.1.1 we can perform the following construction.

For each edge e we define $\boldsymbol{\phi}^e \in \mathbf{P}^1(T)$ (the space of vector-valued linear polynomials on T) by the property

$$\boldsymbol{\phi}^e_{\mathbf{t}} := \mathbf{t} \cdot \boldsymbol{\phi}^e = \begin{cases} \frac{1}{|e|} & \text{on edge } e \\ 0 & \text{on the other edges.} \end{cases},$$

where $|e|$ denotes the length of e. Here, $\mathbf{t}$ is the unit tangent vector on e. Setting

$$\mathbf{u}_1 := \sum_e \left(\int_e \mathbf{t}_e \cdot \mathbf{u} \right) \boldsymbol{\phi}^e$$

then the tangential trace of $\mathbf{u} - \mathbf{u}_1$ has integral mean zero on each edge because of

$$\begin{aligned} \int_e \mathbf{t}_e \cdot (\mathbf{u} - \mathbf{u}_1) &= \int_e \mathbf{t}_e \cdot \left(\mathbf{u} - \sum_{e'} \left(\int_{e'} \mathbf{u} \cdot \mathbf{t}_{e'} \right) \boldsymbol{\phi}^{e'} \right) \\ &= \int_e \left(\mathbf{t}_e \cdot \mathbf{u} - \sum_{e'} \left(\int_{e'} \mathbf{u} \cdot \mathbf{t}_{e'} \right) \mathbf{t}_e \cdot \boldsymbol{\phi}^{e'} \right) \\ &= \int_e \left(\mathbf{t}_e \cdot \mathbf{u} - \frac{1}{|e|} \int_e \mathbf{t}_e \cdot \mathbf{u} \right) = 0. \end{aligned}$$

In **Step 2** we define the edge-interpolants. Firstly, we define a scalar-valued function ψ on the edges of T by setting

$$\frac{\partial \psi}{\partial s} = \mathbf{t} \cdot (\mathbf{u} - \mathbf{u}_1), \quad \psi = 0 \text{ at all vertices.} \tag{4.26}$$

This definition of ψ is equivalent to

$$\psi|_e = \int_{\hat{e}_1}^{x} \mathbf{t} \cdot (\mathbf{u} - \mathbf{u}_1)\, ds, \tag{4.27}$$

where $\hat{e}_1$ is a vertex of the edge e and x is a point on the edge. This construction is also referred to as lifting operator, see e.g. Hiptmair [65].
Next, we take the projection $\psi_{2,e}$ of $\psi|_e$ in the L^2-norm onto the the polynomial space $\mathcal{P}_{-1}^{p_e+1}(e)$ of polynomials of degree p_e+1 which vanish in the endpoints of e. Hence $\psi_{2,e} \in \mathcal{P}_{-1}^{p_e+1}(e)$ minimizes $\|\psi_{2,e} - \psi\|_{L^2(e)}$. We extend $\psi_{2,e}$ into the interior of the tetrahedron T such that the extended function vanishes on all other edges. This can be done using the extension presented in Chapter 1 and Chapter 2. Therefore we extend the functions on the edges into the interior of the triangle and use a discrete harmonic extension into the tetrahedron. Finally, we define $\mathbf{u}_{2,e}^{p_e} := \nabla \psi_{2,e} \in \mathbf{P}_{p_e}^{p}(T)$ and sum over all edges of the tetrahedron

$$\mathbf{u}_2^{p_e} := \sum_{e=1}^{6} \mathbf{u}_{2,e}^{p_e}.$$

In **Step 3** we define the face interpolants. Here, we consider the surface gradient $\nabla_F \phi := \gamma_D(\nabla \phi)$ and the scalar surface rotation $\operatorname{curl}_F \mathbf{u} := (\mathbf{curl}\,\mathbf{u}) \cdot \mathbf{n}$. We minimize the tangential trace of $\mathbf{u}$ in the $L^2(F)$-norm on the face F. Therefore, we compute $\mathbf{u}_{3,F}^{p_F} \in \mathbf{P}_{-1}^{p_F}(F)$ which minimizes

$$\| \operatorname{curl}_F \left(\mathbf{u}_{3,F}^{p_F} - (\mathbf{u} - \mathbf{u}_1 - \mathbf{u}_2^{p_e}) \right) \|_{L^2(F)} \tag{4.28}$$

and satisfies

$$\left(\mathbf{u}_{3,F}^{p} - (\mathbf{u} - \mathbf{u}_1 - \mathbf{u}_2^{p}), \nabla_F \phi \right) = 0 \quad \forall \phi \in \mathcal{P}_{-1}^{p+1}(F). \tag{4.29}$$

Here, $\mathbf{P}_{-1}^{p_F}(F)$ denotes the polynomials on the face F with tangential components vanishing on ∂F. Finally, we extend $\mathbf{u}_{3,F}^{p_F}$ into the tetrahedron T with tangential components vanishing on all other faces. Altogether we have

$$\mathbf{u}_3^{p_F} := \sum_{F=1}^{4} \mathbf{u}_{3,F}^{p_F}.$$

In **Step 4** we introduce the interior interpolant. Let $\mathbf{P}_{-1}^{p_I}(T)$ denote the space of vector-valued polynomials of degree p_I on T whose tangential components have vanishing trace on the boundary ∂T of T. Here, we compute $\mathbf{u}_4^{p_I} \in \mathbf{P}_{-1}^{p_I}(T)$ via the constraints

$$\| \mathbf{curl} \left(\mathbf{u}_4^{p_I} - (\mathbf{u} - \mathbf{u}_1 - \mathbf{u}_2^{p_e} - \mathbf{u}_3^{p_F}) \right) \|_{\mathbf{L}^2(T)} = \| \mathbf{curl} (\mathbf{u}_4^{p_I} - (\mathbf{u} - \mathbf{u}_1 - \mathbf{u}_3^{p_F})) \|_{\mathbf{L}^2(T)} \to \min$$
$$(\mathbf{u}_4^{p_I} - (\mathbf{u} - \mathbf{u}_1 - \mathbf{u}_2^{p_e} - \mathbf{u}_3^{p_F}), \nabla \phi) = 0 \quad \forall \phi \in \mathcal{P}_{-1}^{p+1}(T)$$

Finally, we get the interpolation operator

$$\tilde{\Pi}_p^1 \mathbf{u} := \mathbf{u}_1 + \mathbf{u}_2^{p_e} + \mathbf{u}_3^{p_F} + \mathbf{u}_4^{p_I}.$$

Using a different technique as in the proof of Theorem 4.8.3 and making certain assumptions on the existence of polynomial extension operators Demkowicz & Buffa [45] prove a similar result for three dimensions using the same construction above.

Theorem 4.8.4 (Demkowicz/Buffa) *Under the conjecture on the existence of polynomial preserving extension operators; for every $\epsilon > 0$, there exists a constant $C = \mathcal{O}(\epsilon^{-1})$ such that*

$$\|\mathbf{u} - \tilde{\Pi}_p^1 \mathbf{u}\|_{\mathbf{H}(\mathbf{curl},T)} \le C\, p^{-(r-\epsilon)} \|\mathbf{u}\|_{\mathbf{H}^r(\mathbf{curl},T)}$$

for every $\mathbf{u} \in \mathbf{H}^r(\mathbf{curl}, T)$, $r > \frac{1}{2} + \epsilon$. Here, $p = \min\{p_I, p_f, p_e\}$, where p_I, p_f and p_e are the polynomial degrees in the interior, on the faces f and on the edge e with $p_e = \min\{p_f\}$ for faces f neighboring edge e.

Remark 4.8.5 *Due to the demand that $\mathbf{u} \in \mathbf{H}^r(\mathbf{curl}, T)$, $r > \frac{1}{2} + \epsilon$ this result is still not optimal. This regularity assumption is due to Step 1 where we need strong regularity, compare Remark 4.1.1. One idea to get rid of this is the use of a non-local Clément type interpolation as introduced in §4.8.1.*

Using the twisted tangential trace $\gamma_t^\times$ we can also define an interpolation operator $\Pi_{p,\Gamma}^1$ on the boundary by

$$\Pi_{p,\Gamma}^1 \gamma_{\mathbf{t}}^\times \mathbf{u} = \gamma_{\mathbf{t}}^\times \tilde{\Pi}_p^1 \mathbf{u}. \tag{4.30}$$

Next, we consider the hp-version for the meshes $\mathcal{T}_h$ on Ω and the induced mesh $\mathcal{K}_h$ on Γ with suitable $\mathbf{H}(\mathbf{curl}, \Omega)$- and $\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$-conforming polynomial spaces $\mathcal{X}_{h,p}(\mathcal{T}_h)$ and $\mathcal{Y}_{h,p}(\mathcal{K}_h)$.

Using the approximation properties of the projection-based interpolation operators $\tilde{\Pi}_p^1$ defined in (4.25) we get the following approximation properties for $\tilde{\Pi}_p^1$ on $\mathcal{X}_{h,p}(\mathcal{T}_h)$ and $\Pi_{p,\Gamma}^1$ on $\mathcal{Y}_{h,p}(\mathcal{K}_h)$. As a corollary of Theorem 4.8.4 we obtain

Theorem 4.8.6 *For $\mathbf{u} \in \mathbf{H}^r(\mathbf{curl}, \Omega)$ and $\boldsymbol{\lambda} \in \mathbf{H}_\|^{r-1/2}(\mathrm{div}_\Gamma, \Gamma)$, $r > 1/2$. there exist constants $\tilde{c}$, c', $c > 0$, independent of h, p, $\mathbf{u}$ and $\boldsymbol{\lambda}$, such that for $k = \min\{r, p_{\min} + 1\}$, $\epsilon > 0$, (with $p_{\min}$ denoting minimal polynomial degree):*

$$\|\mathbf{u} - \tilde{\Pi}_p^1 \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \le \tilde{c} h^k p_{\min}^{-(r-\epsilon)} \|\mathbf{u}\|_{\mathbf{H}^r(\Omega)}, \tag{4.31}$$

$$\|\mathbf{u} - \tilde{\Pi}_p^1 \mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le c' h^k p_{\min}^{-(r-\epsilon)} \left(\|\mathbf{u}\|_{\mathbf{H}^r(\Omega)} + \|\mathbf{curl}\,\mathbf{u}\|_{\mathbf{H}^r(\Omega)} \right) \tag{4.32}$$

and

$$\|\boldsymbol{\lambda} - \Pi_{p,\Gamma}^1 \boldsymbol{\lambda}\|_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \le c h^k p_{\min}^{-(r-\epsilon)} \|\boldsymbol{\lambda}\|_{\mathbf{H}_\|^{r-1/2}(\mathrm{div}_\Gamma,\Gamma)}. \tag{4.33}$$

Using (4.30) the estimate (4.33) follows from (4.32) together with the continuity of the surjective mapping $\gamma_{\mathbf{t}}^{\times}$ from $\mathbf{H}(\mathbf{curl}, \Omega)$ onto $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and an extension argument. Furthermore, we need the inverse inequalities in Lemma 4.1.3 and Lemma 4.4.4.

Remark 4.8.7 *The above construction can similarly be applied to the case of hexahedrons.*

4.8.1 Non-local Clément type interpolation

One of the main difficulties in the construction of the interpolation operator is the regularity of the operator $\int_e \mathbf{t} \cdot \mathbf{u} \, ds$ where $\mathbf{t}$ denotes the tangential unit vector along the edge e, compare Remark 4.1.1.

In order to get rid of these constraints Schöberl [95] introduces the following non-local Clément-type interpolation operator for edge elements.

Let Ω be a polyhedral domain with Lipschitz boundary. On this we define a shape-regular mesh of tetrahedrons. We define $\mathcal{V} = \{V_i\}$ and $\mathcal{E} = \{E_{ij}\}$ as the sets of all vertices and edges, respectively.

In Section 4.1.3 we calculate the following interpolation operator for lowest order polynomials

$$(I\mathbf{q})(\mathbf{x}) := \sum_{E_{ij} \in \mathcal{E}} \phi_{ij}(\mathbf{x}) \int_{E_{ij}} \mathbf{t} \cdot \mathbf{q} \, ds.$$

Here, ϕ_{ij} denotes the Nédélec basis function related to edge E_{ij} with unit tangential vector $\mathbf{t}$.

In order to compute the integrals we consider the following Clément-type interpolation, compare Clément [37]. For every vertex V_i let $\omega_i \subset \Omega \cap B_{Ch}(V_i)$ be a set of non-zero measure, where $B_{Ch}(V_i)$ denotes the ball around V_i of radius $C \cdot h$, $C > 0$. It is not necessary that $V_i \in \omega_i$. Let $p \in \mathbb{N}$ and define the weight function $f_i \in L^\infty(\omega_i)$ such that there holds

$$\int_{\omega_i} f_i \, q \, dx = q(V_i)$$

for all polynomials q of degree up to p. We assume $\|f_i\|_{L^\infty(\omega_i)} \simeq h_i^{-3}$. If we set $q = 1$ on ω_i we immediately get $\|f_i\|_{L^1(\omega_i)} \simeq 1$ and there also holds $\|f_i\|_{L^2(\omega_i)} \simeq h_i^{-3/2}$. Hence, we can define

$$\psi_{ij}(\mathbf{q}) := \int_{\omega_i} \int_{\omega_j} \left[f_i(y_1) f_j(y_2) \left(\int_{[y_1, y_2]} \mathbf{t} \cdot \mathbf{q} \, ds \right) \right] dy_1 \, dy_2.$$

We are now in the position to define the interpolation operator Π^Q by

$$(\Pi^Q \mathbf{q})(x) := \sum_{E_{ij} \in \mathcal{E}} \psi_{ij}(\mathbf{q}) \phi_{ij}(x).$$

From the construction it follows directly that the interpolation operator is linear and we get the following result.

Lemma 4.8.8 (Schöberl [95]) *The operator* Π^Q *is well defined on* L^2*. Its norm is independent of the local mesh size* h *and there holds*

$$\|\Pi^Q \mathbf{q}\|_{L^2(T)} \lesssim \|\mathbf{q}\|_{L^2(\tilde{T})},$$
$$\|\operatorname{\mathbf{curl}} \Pi^Q \mathbf{q}\|_{L^2(T)} \lesssim \|\operatorname{\mathbf{curl}} \mathbf{q}\|_{L^2(\tilde{T})}$$

where $\tilde{T}$ *is the smallest patch of elements containing the* ω_i*.*

In the following we consider the tetrahedron T and use the Clément type interpolation operator Π^Q instead of the integration in Step 1 for the construction of $\tilde{\Pi}_p^1$, compare page 99.

There holds $\int_e 1 = |e|$ for the interpolation scheme because of

$$\begin{aligned}
\psi_{ij} &= \int_{\omega_i} \int_{\omega_j} \left[f_i(y_1) f_j(y_2) \left(\int_{[y_1,y_2]} 1 \, ds \right) \right] dy_2 \, dy_1 \\
&= \int_{\omega_i} \int_{\omega_j} f_i(y_1) f_j(y_2) \, |[y_1, y_2]| \, dy_1 \, dy_2 \\
&= |V_1 - V_2|
\end{aligned}$$

where $|V_1 - V_2|$ denotes the distance between V_1 and V_2. The last step is due to $\int_{\omega_i} f_i q \, dx = q(V_i)$. This only holds if $\omega_i \cap \omega_j = \emptyset$.
Therefore, we can use the operator Π^Q instead of the integration. Using Lemma 4.8.8 we get the estimate

$$\|\mathbf{u}_1\|_{\mathbf{H}(\mathbf{curl},T)} \lesssim \|\operatorname{\mathbf{curl}} \mathbf{u}\|_{\mathbf{H}(\mathbf{curl},T)}$$

which is the first step in proving the continuity of the interpolation operator.

Remark 4.8.9 *Using this construction we lose the locality of the interpolation operator such that this interpolation operator does not preserve piecewise polynomials. This makes the extension of p-estimates to hp-estimates difficult.*

4.9 Stable decompositions of $\mathcal{ND}_p(\mathcal{T}_h)$

In this section we construct an $\mathbf{H}(\mathbf{curl},\Omega)$-stable decomposition of the spaces $\mathcal{ND}_p(\mathcal{T}_h)$, $p \geq 1$. This can be used for the construction of 2-level error estimates, compare Teltscher *et al.* [105, 104] and Beck *et al.* [16]. Furthermore, we can use these results for the construction of an additive Schwarz preconditioner for the $\mathbf{H}(\mathbf{curl},\Omega)$-bilinear form.

4.9.1 Decomposition of $\mathcal{ND}_2(\mathcal{T}_h)$

Let $\mathcal{T}_h$ be a regular grid on Ω with mesh width h, and denote by M the number of edges, N the number of faces and L the number of elements. Further, let $\mathcal{S}_p$ denote the finite element space of scalar, continuous and piecewise polynomial functions of order p and let $\widetilde{\mathcal{S}}_p := \mathcal{S}_p \setminus \mathcal{S}_{p-1}$ (the hierarchical surplus). For a tetrahedron T the dimension of $\mathcal{S}_p(T)$ is $\dim \mathcal{S}_p(T) = \frac{1}{6}(p+1)(p+2)(p+3)$ and $\dim \mathcal{S}_p(T) = (p+1)^3$ for a hexahedron T.

The aim is to find an $\mathbf{H}(\mathbf{curl},\Omega)$-stable decomposition of $\mathcal{ND}_2(\mathcal{T}_h)$ into the space $\mathcal{ND}_1(\mathcal{T}_h)$, the space of gradients of hat functions and a gradient free space. After having derived the decomposition we will prove the stability in Lemma 4.9.1. This decomposition can also be used to construct a preconditioner for the $\mathbf{H}(\mathbf{curl},\Omega)$-bilinear form $a(\mathbf{u},\mathbf{v}) := (\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_\Omega + (\mathbf{u},\mathbf{v})_\Omega$.

Tetrahedral meshes

First of all, we consider the case of tetrahedral meshes. For those the decomposition given in [14, 16] reads

$$\mathcal{ND}_2(\mathcal{T}_h) = \mathcal{ND}_1(\mathcal{T}_h) \oplus \mathbf{grad}\,\widetilde{\mathcal{S}}_2(\mathcal{T}_h) \oplus \widetilde{\mathcal{ND}}_2^{\perp}(\mathcal{T}_h) \tag{4.34}$$

where

$$\widetilde{\mathcal{ND}}_2^{\perp}(\mathcal{T}_h) := \left\{ \mathbf{u}_h \in \mathcal{ND}_2(\mathcal{T}_h) : \int_e \mathbf{u}_h \cdot \mathbf{t} q\, ds = 0,\ \forall q \in \mathbb{P}_1,\ e \text{ edge of } \mathcal{T}_h \right\},$$

i.e., $\widetilde{\mathcal{ND}}_2^{\perp}(\mathcal{T}_h)$ is spanned by face functions only.
Counting the degrees of freedom on an element T, one sees that (4.34) is a direct sum: the dimension of $\mathcal{ND}_1(T)$ equals the number of edges, i.e. six, and the dimension of $\widetilde{\mathcal{S}}_2(T)$ is $10-4=6$ (again equal to the number of edges). We write $\mathbf{grad}\,\widetilde{\mathcal{S}}_2(\mathcal{T}_h) = \operatorname{span}\{\mathbf{grad}\,\phi^{(e_1)},\dots,\mathbf{grad}\,\phi^{(e_M)}\}$. The space $\mathcal{ND}_2(T)$ has dimension 20, corresponding to two basis functions per edge and two per face of T. The basis functions on the faces span the space $\widetilde{\mathcal{ND}}_2^{\perp}(T)$, which thus has dimension 8. Accordingly, on a tetrahedral

mesh we write

$$\widetilde{\mathcal{ND}}_2^{\perp}(\mathcal{T}_h) = \operatorname{span}\{\mathbf{b}_1^{(F_1)}, \mathbf{b}_2^{(F_1)}, \ldots, \mathbf{b}_1^{(F_N)}, \mathbf{b}_2^{(F_N)}\}$$

for the space spanned by the face-oriented basis functions of $\mathcal{ND}_2(\mathcal{T}_h)$. The decomposition (4.34) can then be written as:

$$\mathcal{ND}_2(\mathcal{T}_h) = \mathcal{ND}_1(\mathcal{T}_h) \oplus \sum_{i=1}^{M} \operatorname{span}\{\mathbf{grad}\,\phi^{(e_i)}\} \oplus \sum_{j=1}^{N} \operatorname{span}\{\mathbf{b}_1^{(F_j)}, \mathbf{b}_2^{(F_j)}\}. \tag{4.35}$$

Hexahedral meshes

This construction cannot be extended to the hexahedral case, for the decomposition defined in (4.34) is then no longer a direct sum. Counting degrees of freedom, we see that $\mathbf{grad}\,\widetilde{\mathcal{S}}_2(T)$ and $\widetilde{\mathcal{ND}}_2^{\perp}(T)$ overlap: the dimension of $\mathcal{ND}_1(T)$ equals the number of edges, i.e. 12, the dimension of $\widetilde{\mathcal{S}}_2(T)$ is equal to $27 - 8 = 19$ (corresponding to one function per edge, one per face and one inner function), and the dimension of $\widetilde{\mathcal{ND}}_2^{\perp}(T)$ is 30 (four functions per face and six inner functions). But the dimension of $\mathcal{ND}_2(T)$ is 54, such that there must hold $\dim(\mathbf{grad}\,\widetilde{\mathcal{S}}_2(T) \cap \mathcal{ND}_2^{\perp}(T)) = 7$. Hence, if we are looking for a direct decomposition of $\mathcal{ND}_2(T)$ for hexahedra, we must determine 7 functions to eliminate from $\mathbf{grad}\,\widetilde{\mathcal{S}}_2(T) \cap \mathcal{ND}_2^{\perp}(T)$. Let us write

$$\widetilde{\mathcal{S}}_2(T) = \operatorname{span}\{\phi^{(e_1)}, \ldots, \phi^{(e_{12})}, \phi^{(F_1)}, \ldots, \phi^{(F_6)}, \phi^{(T)}\}$$

with edge based functions $\phi^{(e_i)}$, face based functions $\phi^{(F_i)}$ and bubble function $\phi^{(T)}$. Furthermore, with face based functions $\mathbf{b}_i^{(F_j)}$ and suitable 'bubble' functions $\mathbf{b}_i^{(T)}$ we can write

$$\widetilde{\mathcal{ND}}_2^{\perp}(T) = \operatorname{span}\{\mathbf{b}_1^{(F_1)}, \ldots, \mathbf{b}_4^{(F_1)}, \ldots, \mathbf{b}_1^{(F_6)}, \ldots, \mathbf{b}_4^{(F_6)}, \mathbf{b}_1^{(T)}, \ldots, \mathbf{b}_6^{(T)}\}.$$

By explicitly computing the basis functions of $\widetilde{\mathcal{ND}}_2^{\perp}(T)$ for the reference element $T = [-1,1]^3$ (according to the degrees of freedom given earlier), one ascertains that the face functions of $\mathbf{grad}\,\widetilde{\mathcal{S}}_2(T)$ can be described by functions of $\widetilde{\mathcal{ND}}_2^{\perp}(T)$, for example there holds on the reference cube, cf. Section 4.1.1,

$$\mathbf{grad}\,\phi^{(F_0)} = \mathbf{grad}(1-x^2)(1-y^2)(1-z) = \frac{32}{9}(\mathbf{b}_2^{(F_0)} - \mathbf{b}_4^{(F_0)} - \mathbf{b}_2^{(T)} - \mathbf{b}_4^{(T)} - \mathbf{b}_5^{(T)}),$$

and similarly for the other $\mathbf{grad}\,\phi^{(F_j)}$. There further holds

$$\mathbf{grad}\,\phi^{(T)} = \mathbf{grad}(1-x^2)(1-y^2)(1-z^2) = -\frac{64}{27}(\mathbf{b}_2^{(T)} + \mathbf{b}_4^{(T)} + \mathbf{b}_6^{(T)}).$$

With this information, there are now many ways to exchange the spaces $\mathbf{grad}\,\widetilde{\mathcal{S}}_2(T)$ and $\widetilde{\mathcal{ND}}_2^{\perp}(T)$ by reduced spaces $\mathbf{grad}\,\widetilde{\mathcal{S}}_2^{-}(T)$ and $\widetilde{\mathcal{ND}}_2^{\perp,-}(T)$ to obtain a direct sum. We propose:

1. leave $\widetilde{\mathcal{S}}_2(T)$ as it is,
2. substitute $\mathbf{b}_2^{(F_j)} + \mathbf{b}_4^{(F_j)}$ for the face functions $\mathbf{b}_2^{(F_j)}$, $\mathbf{b}_4^{(F_j)}$ $(j = 1, \dots, 6)$ and $\mathbf{b}_2^{(T)} - \mathbf{b}_4^{(T)}$ and $\mathbf{b}_4^{(T)} - \mathbf{b}_6^{(T)}$ for the interior functions $\mathbf{b}_2^{(T)}$, $\mathbf{b}_4^{(T)}$, $\mathbf{b}_6^{(T)}$; this changes $\widetilde{\mathcal{ND}}_2^{\perp}(T)$ to $\widetilde{\mathcal{ND}}_2^{\perp,-}(T)$.

We then obtain the global space

$$\begin{aligned}\widetilde{\mathcal{ND}}_2^{\perp,-}(\mathcal{T}_h) := \operatorname{span}\{&\mathbf{b}_1^{(F_j)}, \mathbf{b}_3^{(F_j)}, \mathbf{b}_2^{(F_j)} + \mathbf{b}_4^{(F_j)},\\ &\mathbf{b}_1^{(T_k)}, \mathbf{b}_3^{(T_k)}, \mathbf{b}_5^{(T_k)}, \mathbf{b}_2^{(T_k)} - \mathbf{b}_4^{(T_k)}, \mathbf{b}_4^{(T_k)} - \mathbf{b}_6^{(T_k)},\\ &j = 1, \dots, N,\ k = 1, \dots, L\}\end{aligned}$$

and the direct decomposition

$$\mathcal{ND}_2(\mathcal{T}_h) = \mathcal{ND}_1(\mathcal{T}_h) \oplus \mathbf{grad}\, \widetilde{\mathcal{S}}_2(\mathcal{T}_h) \oplus \widetilde{\mathcal{ND}}_2^{\perp,-}(\mathcal{T}_h) \tag{4.36}$$

for hexahedral grids, which can be broken down to:

$$\begin{aligned}\mathcal{ND}_2(\mathcal{T}_h) = \mathcal{ND}_1(\mathcal{T}_h) &\oplus \sum_{i=1}^{M} \operatorname{span}\{\mathbf{grad}\, \phi^{(e_i)}\}\\ &\oplus \sum_{j=1}^{N} \Big(\operatorname{span}\{\mathbf{grad}\, \phi^{(F_j)}\} \oplus \operatorname{span}\{\mathbf{b}_1^{(F_j)}, \mathbf{b}_3^{(F_j)}, \mathbf{b}_2^{(F_j)} + \mathbf{b}_4^{(F_j)}\} \Big)\\ \oplus \sum_{k=1}^{L} &\Big(\operatorname{span}\{\mathbf{grad}\, \phi^{(T_k)}\} \oplus \operatorname{span}\{\mathbf{b}_1^{(T_k)}, \mathbf{b}_3^{(T_k)}, \mathbf{b}_5^{(T_k)}, \mathbf{b}_2^{(T_k)} - \mathbf{b}_4^{(T_k)}, \mathbf{b}_4^{(T_k)} - \mathbf{b}_6^{(T_k)}\} \Big).\end{aligned} \tag{4.37}$$

In what follows the stability of the decompositions (4.35) and (4.37) is crucial for the derivation of hierarchical error indicators. Therefore, we define for tetrahedra the subspace projections

$$\begin{aligned}P_1 &: \mathcal{ND}_2(\mathcal{T}_h) \to \mathcal{ND}_1(\mathcal{T}_h),\\ P^{(F)} &: \mathcal{ND}_2(\mathcal{T}_h) \to \operatorname{span}\{\mathbf{b}_1^{(F)}, \mathbf{b}_2^{(F)}\},\\ R^{(e)} &: \mathcal{ND}_2(\mathcal{T}_h) \to \operatorname{span}\{\mathbf{grad}\, \phi^{(e)}\},\end{aligned}$$

and for hexahedra the projections

$$\begin{aligned}\tilde{P}_1 &: \mathcal{ND}_2(\mathcal{T}_h) \to \mathcal{ND}_1(\mathcal{T}_h),\\ \tilde{P}^{(F)} &: \mathcal{ND}_2(\mathcal{T}_h) \to \operatorname{span}\{\mathbf{b}_1^{(F)}, \mathbf{b}_3^{(F)}, \mathbf{b}_2^{(F)} + \mathbf{b}_4^{(F)}\},\\ \tilde{P}^{(T)} &: \mathcal{ND}_2(\mathcal{T}_h) \to \operatorname{span}\{\mathbf{b}_1^{(T)}, \mathbf{b}_3^{(T)}, \mathbf{b}_5^{(T)}, \mathbf{b}_2^{(T)} - \mathbf{b}_4^{(T)}, \mathbf{b}_4^{(T)} - \mathbf{b}_6^{(T)}\},\\ \tilde{R}^{(e)} &: \mathcal{ND}_2(\mathcal{T}_h) \to \operatorname{span}\{\mathbf{grad}\, \phi^{(e)}\},\\ \tilde{R}^{(F)} &: \mathcal{ND}_2(\mathcal{T}_h) \to \operatorname{span}\{\mathbf{grad}\, \phi^{(F)}\},\\ \tilde{R}^{(T)} &: \mathcal{ND}_2(\mathcal{T}_h) \to \operatorname{span}\{\mathbf{grad}\, \phi^{(T)}\},\end{aligned}$$

so that for $\mathbf{u}_2 \in \mathcal{ND}_2(\mathcal{T}_h)$ the decompositions (4.35) and (4.37) can be written as

$$\mathbf{u}_2 = P_1\mathbf{u}_2 + \sum_{i=1}^{M} R^{(e_i)}\mathbf{u}_2 + \sum_{j=1}^{N} P^{(F_j)}\mathbf{u}_2 \tag{4.38}$$

and

$$\mathbf{u}_2 = \tilde{P}_1\mathbf{u}_2 + \sum_{i=1}^{M} \tilde{R}^{(e_i)}\mathbf{u}_2 + \sum_{j=1}^{N} \left(\tilde{R}^{(F_j)}\mathbf{u}_2 + \tilde{P}^{(F_j)}\mathbf{u}_2\right) + \sum_{k=1}^{L} \left(\tilde{R}^{(T_k)}\mathbf{u}_2 + \tilde{P}^{(T_k)}\mathbf{u}_2\right). \tag{4.39}$$

The next lemma states the stability result. For the sake of clarity, we will denote the $\mathbf{H}(\mathbf{curl},\Omega)$-norm simply by $\|\cdot\|$.

Lemma 4.9.1 *The decompositions* (4.35) *and* (4.37) *are both stable with respect to the* $\mathbf{H}(\mathbf{curl},\Omega)$*-norm, i.e., for all* $\mathbf{u}_2 \in \mathcal{ND}_2(\mathcal{T}_h)$ *there holds*

$$\|\mathbf{u}_2\|^2 \simeq \|P_1\mathbf{u}_2\|^2 + \sum_{i=1}^{M}\|R^{(e_i)}\mathbf{u}_2\|^2 + \sum_{j=1}^{N}\|P^{(F_j)}\mathbf{u}_2\|^2 \tag{4.40}$$

and

$$\begin{aligned}\|\mathbf{u}_2\|^2 \simeq \|\tilde{P}_1\mathbf{u}_2\|^2 + \sum_{i=1}^{M}\|\tilde{R}^{(e_i)}\mathbf{u}_2\|^2 + \sum_{j=1}^{N}\left(\|\tilde{R}^{(F_j)}\mathbf{u}_2\|^2 + \|\tilde{P}^{(F_j)}\mathbf{u}_2\|^2\right)\\ + \sum_{k=1}^{L}\left(\|\tilde{R}^{(T_k)}\mathbf{u}_2\|^2 + \|\tilde{P}^{(T_k)}\mathbf{u}_2\|^2\right),\end{aligned} \tag{4.41}$$

respectively.

Proof. First of all, let us consider the case of hexahedra, i.e. (4.41). The proof for the tetrahedral case is similar (see Beck *et al.* [16, Lemma 3 and Lemma 4]).
We observe that due to the uniqueness of the decomposition (4.37) the mapping $|\!|\!|\cdot|\!|\!|$ is a norm where $|\!|\!|\cdot|\!|\!|$ is defined by

$$\begin{aligned}|\!|\!|\mathbf{u}_2|\!|\!|^2_{\mathbf{L}^2(\Omega)} :=& \|\tilde{P}_1\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \sum_{i=1}^{M}\|\tilde{R}^{(e_i)}\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \sum_{j=1}^{N}\left(\|\tilde{R}^{(F_j)}\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \|\tilde{P}^{(F_j)}\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)}\right)\\ &+ \sum_{k=1}^{L}\left(\|\tilde{R}^{(T_k)}\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \|\tilde{P}^{(T_k)}\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)}\right) =: \sum_{P}\|P\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

Since the L^2-Norm is local, we conclude with (4.6) that there holds

$$\begin{aligned}|\!|\!|\mathbf{u}_2|\!|\!|^2_{\mathbf{L}^2(\Omega)} &= \sum_{T\in\mathcal{T}_h}|\!|\!|\mathbf{u}_2|\!|\!|^2_{\mathbf{L}^2(T)} = \sum_{P}\sum_{T\in\mathcal{T}_h}\|P\mathbf{u}_2\|^2_{\mathbf{L}^2(T)}\\ &\sim \sum_{P}\sum_{T\in\mathcal{T}_h} h_T\|\widehat{P\mathbf{u}_2}\|^2_{\mathbf{L}^2(\widehat{T})} \sim \sum_{T\in\mathcal{T}_h} h_T \sum_{P_T}\|\widehat{P_T\mathbf{u}_2}\|^2_{\mathbf{L}^2(\widehat{T})}\\ &\sim \sum_{T\in\mathcal{T}_h} h_T \sum_{P_{\widehat{T}}}\|P_{\widehat{T}}\hat{\mathbf{u}}_2\|^2_{\mathbf{L}^2(\widehat{T})}.\end{aligned} \tag{4.42}$$

Here, P_T denotes a projection operator that is related to the element T. Furthermore, $\hat{\mathbf{v}}(\hat{\mathbf{x}}) = B^\mathsf{T}\mathbf{v}(\mathbf{x})$ is the transformation of $\mathbf{v}$ to the reference element $\widehat{T}$. The constant in the equivalence relation depends only on the shape regularity of the mesh and there holds for $\hat{\mathbf{u}}_2 \in \mathcal{ND}_2(\widehat{T})$

$$|||\hat{\mathbf{u}}_2|||_{\mathbf{L}^2(\widehat{T})} = \sum_{P_{\widehat{T}}} \|P_{\widehat{T}}\hat{\mathbf{u}}_2\|_{\mathbf{L}^2(\widehat{T})} \sim \|\hat{\mathbf{u}}_2\|_{\mathbf{L}^2(\widehat{T})},$$

since all norms are equivalent on a finite dimensional space and the number of projection operators on an element is bounded. Here, the constant in the equivalence relation depends only on the decomposition on $\widehat{T}$. With (4.42) and (4.6) we obtain

$$|||\mathbf{u}_2|||^2_{\mathbf{L}^2(\Omega)} \sim \sum_{T\in\mathcal{T}_h} h_T |||\hat{\mathbf{u}}_2|||^2_{\mathbf{L}^2(\widehat{T})} \sim \sum_{T\in\mathcal{T}_h} \|\mathbf{u}_2\|^2_{\mathbf{L}^2(T)} = \|\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)}.$$

We still have to show that there holds

$$|||\mathbf{curl}\,\mathbf{u}_2|||_{\mathbf{L}^2(\Omega)} := \sum_P \|\mathbf{curl}\,P\mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} \sim \|\mathbf{curl}\,\mathbf{u}_2\|_{\mathbf{L}^2(\Omega)}.$$

This follows from the same arguments as above, when we use relation (4.14) for the transformation to the reference element since $\mathbf{curl}\,\mathbf{u}_2 \in \mathcal{RT}_2(\mathcal{T}_h)$ for $\mathbf{u}_2 \in \mathcal{ND}_2(\mathcal{T}_h)$. Note that also the following decomposition is unique:

$$\begin{aligned}\mathbf{curl}\,\mathbf{u}_2 = \mathbf{curl}\,\tilde{P}_1\mathbf{u}_2 + \sum_{i=1}^{M} \mathbf{curl}\,\tilde{R}^{(e_i)}\mathbf{u}_2 + \sum_{j=1}^{N} \left(\mathbf{curl}\,\tilde{R}^{(F_j)}\mathbf{u}_2 + \mathbf{curl}\,\tilde{P}^{(F_j)}\mathbf{u}_2\right)\\ + \sum_{k=1}^{L} \left(\mathbf{curl}\,\tilde{R}^{(T_k)}\mathbf{u}_2 + \mathbf{curl}\,\tilde{P}^{(T_k)}\mathbf{u}_2\right).\end{aligned}$$

This can be seen as follows.

Let $\mathbf{curl}\,\mathbf{u}_2 = 0$. Thus, we have $\mathbf{curl}\,\tilde{P}_1\mathbf{u}_2 = \mathbf{curl}\,\mathbf{\Pi}^{\mathcal{ND}_1}\mathbf{u}_2 = \mathbf{\Pi}^{\mathcal{RT}_1}\,\mathbf{curl}\,\mathbf{u}_2 = 0$, furthermore there holds $\mathbf{curl}\,\tilde{R}^{(e_i)}\mathbf{u}_2 = \mathbf{curl}\,\tilde{R}^{(F_j)}\mathbf{u}_2 = \mathbf{curl}\,\tilde{R}^{(T_k)}\mathbf{u}_2 = 0$ due to $\mathbf{curl}\,\mathbf{grad} \equiv 0$. The decomposition (4.39) yields $\mathbf{curl}\left(\sum_{j=1}^{N} \tilde{P}^{(F_j)}\mathbf{u}_2 + \sum_{k=1}^{L} \tilde{P}^{(T_k)}\mathbf{u}_2\right) = 0$. Therefore, there exists $\psi_2 \in \mathcal{S}_2(\mathcal{T}_h)$ with $\sum_{j=1}^{N} \tilde{P}^{(F_j)}\mathbf{u}_2 + \sum_{k=1}^{L} \tilde{P}^{(T_k)}\mathbf{u}_2 = \mathbf{grad}\,\psi_2$. There holds $\mathcal{S}_2 = \mathcal{S}_1 \oplus \widetilde{\mathcal{S}}_2$ and $\mathbf{grad}\,\mathcal{S}_2(\mathcal{T}_h) = \mathbf{grad}\,\mathcal{S}_1(\mathcal{T}_h) \oplus \mathbf{grad}\,\widetilde{\mathcal{S}}_2(\mathcal{T}_h)$. As $\mathbf{grad}\,\mathcal{S}_1(\mathcal{T}_h) \subset \mathcal{ND}_1(\mathcal{T}_h)$ and $\left(\mathcal{ND}_1(\mathcal{T}_h) \cup \mathbf{grad}\,\widetilde{\mathcal{S}}_2(\mathcal{T}_h)\right) \cap \widetilde{\mathcal{ND}}_2^{\perp,-}(\mathcal{T}_h) = \{0\}$ due to the construction of $\widetilde{\mathcal{ND}}_2^{\perp,-}(\mathcal{T}_h)$ and the direct sum in (4.36) there must hold $\mathbf{grad}\,\psi_2 = 0$. Hence, $\tilde{P}^{(F_j)}\mathbf{u}_2 = 0$ for all j and $\tilde{P}^{(T_k)}\mathbf{u}_2 = 0$ for all k. Especially there holds $\mathbf{curl}\,\tilde{P}^{(F_j)}\mathbf{u}_2 = 0$ for all j and $\mathbf{curl}\,\tilde{P}^{(T_k)}\mathbf{u}_2 = 0$ for all k. Thus, $\mathbf{curl}\,\mathbf{u}_2 = 0$ implies $\mathbf{curl}\,P\mathbf{u}_2 = 0$ for all projections P. Altogether there holds, independently of the mesh size h,

$$\begin{aligned}|||\mathbf{u}_2|||_{\mathbf{L}^2(\Omega)} &\sim \|\mathbf{u}_2\|_{\mathbf{L}^2(\Omega)},\\ |||\mathbf{curl}\,\mathbf{u}_2|||_{\mathbf{L}^2(\Omega)} &\sim \|\mathbf{curl}\,\mathbf{u}_2\|_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

This gives the assertion of the lemma in case of a hexahedral mesh for the $\mathbf{H}(\mathbf{curl},\Omega)$-norm and the equivalent energy norm. □

4.9.2 A stable decomposition of $\mathcal{ND}_p(\mathcal{T}_h)$

Similarly to the previous section, we examine stable decompositions of $\mathcal{ND}_p$ for arbitrary $p \in \mathbb{N}$ on a hexahedral mesh.
First of all, we remark that there holds the decomposition

$$\mathcal{ND}_p(\mathcal{T}_h) = \mathcal{ND}_{p-1}(\mathcal{T}_h) \oplus \Big(\mathbf{grad}\,\tilde{\mathcal{S}}_p(\mathcal{T}_h) + \widetilde{\mathcal{ND}}_p^{\perp}(\mathcal{T}_h)\Big) \tag{4.43}$$

with

$$\tilde{\mathcal{S}}_p(\mathcal{T}_h) := \mathcal{S}_p(\mathcal{T}_h)\backslash\mathcal{S}_{p-1}(\mathcal{T}_h)$$

and

$$\begin{aligned}\widetilde{\mathcal{ND}}_p^{\perp}(\mathcal{T}_h) := \Big\{&\mathbf{u}_h \in \mathcal{ND}_p(\mathcal{T}_h) : \int_e \mathbf{u}_h \cdot \mathbf{t}\, q\, ds = 0,\ \forall q \in \mathbb{P}_{p-1},\ e\,\text{edge of } \mathcal{T}_h,\\ &\int_F (\mathbf{u}_h \times \mathbf{n}) \cdot \mathbf{q}\, dS = 0, \forall \mathbf{q} \in \mathbb{P}_{p-3,p-2} \times \mathbb{P}_{p-2,p-3},\ F\,\text{face of } \mathcal{T}_h,\\ &\int_{\widehat{T}} \mathbf{u} \times \mathbf{q}\, d\hat{\mathbf{x}} = 0\, \forall \mathbf{q} \in \mathbb{P}_{p-2,p-3,p-3} \times \mathbb{P}_{p-3,p-2,p-3} \times \mathbb{P}_{p-3,p-3,p-2},\ T \in \mathcal{T}_h\Big\},\end{aligned}$$

compare the definition of the integral moments in (4.1). It follows that $\widetilde{\mathcal{ND}}_p(\mathcal{T}_h)$ consists of all those functions of $\mathcal{ND}_p(\mathcal{T}_h)$ that don't belong to the degrees of freedom that are associated to the edges and to the lower polynomial degrees. Due to the de Rham diagram (Section 4.6) there holds $\mathbf{grad}\,\tilde{\mathcal{S}}_p(\mathcal{T}_h) \subset \mathcal{ND}_p(\mathcal{T}_h)$ and the equality in (4.43) is ensured. Next, we count the dimensions of the used spaces on the reference hexahedron $\tilde{T}$. There holds with separation to edge, face and interior functions

$$\begin{array}{llllll}
\dim \mathcal{ND}_p(\widehat{T}) & = & & 12p & +12p^2 - 12p & +3p^3 - 6p^2 + 3p\\
\dim \mathcal{ND}_{p-1}(\widehat{T}) & = & & 12p - 12 & +12p^2 - 36p + 24 & +3p^3 - 15p^2 + 24p - 12\\
\dim \widetilde{\mathcal{ND}}_p^{\perp}(\widehat{T}) & = & & & +24p - 24 & +9p^2 - 21p + 12\\
\dim \mathcal{S}_p(\widehat{T}) & = & 8 + & 12p - 12 & +6(p-1)^2 & +(p-1)^3\\
\dim \tilde{\mathcal{S}}_p(\widehat{T}) & = & & 12 & +12p - 18 & +3p^2 - 9p + 7
\end{array}$$

Counting the degrees of freedom in (4.43), we thus get

$$\begin{aligned}\dim \mathcal{ND}_{p-1}(\widehat{T}) + \dim \tilde{\mathcal{S}}_p(\widehat{T}) + \dim \widetilde{\mathcal{ND}}_p^{\perp}(\widehat{T})&\\ &= 12p \quad + 12p^2 - 18 \quad + 3p^3 - 3p^2 - 6p + 7\\ &= 3p^3 + 9p^2 + 6p - 11.\end{aligned} \tag{4.44}$$

As the sum in (4.43) is direct, there holds

$$\dim\Big(\tilde{\mathcal{S}}_p(\widehat{T}) \cap \dim \widetilde{\mathcal{ND}}_p^{\perp}(\widehat{T})\Big) = 12p - 18 \quad + 3p^2 - 9p + 7.$$

This is the difference between (4.44) and $\dim \mathcal{ND}_p(\widehat{T})$. Thus, we have to change the spaces $\tilde{\mathcal{S}}_p(\widehat{T})$ and $\widetilde{\mathcal{ND}}_p^{\perp}(\widehat{T})$ in such a way that we get a direct sum.

First of all, we change the interior functions. Considering the dimensions and the direct sum in (4.43) it follows that the interior functions of $\tilde{\mathcal{S}}_p(\widehat{T})$ are contained in those of $\widetilde{\mathcal{ND}}(\widehat{T})$. Therefore, we reduce $\widetilde{\mathcal{ND}}_p^{\perp}(\widehat{T})$ by those interior functions.
We are repeating this for the face functions and reduce the face functions of $\widetilde{\mathcal{ND}}_p^{\perp}(\widehat{T})$ by $2p-3$ basis functions.

Hence, we define the space $\widetilde{\mathcal{ND}}_p^{\perp,-}(\widehat{T})$ as the subspace of $\widetilde{\mathcal{ND}}_p^{\perp}(\widehat{T})$ reduced by $3p^2-9p+7$ interior and $12p-18$ face functions. Thus, $\widetilde{\mathcal{ND}}_p^{\perp,-}(\widehat{T})$ consists of $12p+6$ face functions and $6p^2-12p+5$ interior functions. The advantage of this decomposition is that the subspaces are quite small and the preconditioning matrix is easier to invert.

Hence, we get the following decomposition

$$
\begin{aligned}
\mathcal{ND}_p(\mathcal{T}_h) = \mathcal{ND}_{p-1}(\mathcal{T}_h) &\oplus \sum_{i=1}^{M} \operatorname{span}\left\{\mathbf{grad}\,\phi^{(e_i)}\right\} \\
&\oplus \sum_{j=1}^{N}\left(\operatorname{span}\left\{\mathbf{grad}\,\phi_1^{(F_j)},\dots,\mathbf{grad}\,\phi_{2p-3}^{(F_j)}\right\} \oplus \operatorname{span}\left\{\tilde{\mathbf{b}}_1^{(F_j)},\dots,\tilde{\mathbf{b}}_{2p+1}^{(F_j)}\right\}\right) \\
&\oplus \sum_{k=1}^{L}\left(\operatorname{span}\left\{\mathbf{grad}\,\phi_1^{(T_k)},\dots,\mathbf{grad}\,\phi_{3p^2-9p+7}^{(T_k)}\right\} \oplus \left\{\tilde{\mathbf{b}}_1^{(T_k)},\dots,\tilde{\mathbf{b}}_{6p^2-12p+5}^{(T_k)}\right\}\right).
\end{aligned}
\tag{4.45}
$$

Here, M denotes the number of edges, N denotes the number of faces and L denotes the number of elements of the triangulation $\mathcal{T}_h$.
Furthermore, we introduce the following projection operators

$$
\begin{aligned}
P_{p-1} &: \mathcal{ND}_p(\mathcal{T}_h) \to \mathcal{ND}_{p-1}(\mathcal{T}_h), \\
P^{(F)} &: \mathcal{ND}_p(\mathcal{T}_h) \to \operatorname{span}\left\{\tilde{\mathbf{b}}_1^{(F)},\dots,\tilde{\mathbf{b}}_{2p+1}^{(F)}\right\} \subset \widetilde{\mathcal{ND}}_p^{\perp,-}(T), \\
P^{(T)} &: \mathcal{ND}_p(\mathcal{T}_h) \to \operatorname{span}\left\{\tilde{\mathbf{b}}_1^{(T)},\dots,\tilde{\mathbf{b}}_{6p^2-12p+5}^{(T)}\right\} \subset \widetilde{\mathcal{ND}}_p^{\perp,-}(T), \\
R^{(e)} &: \mathcal{ND}_p(\mathcal{T}_h) \to \operatorname{span}\left\{\mathbf{grad}\,\phi_1^{(F)}\right\}, \\
R^{(F)} &: \mathcal{ND}_p(\mathcal{T}_h) \to \operatorname{span}\left\{\mathbf{grad}\,\phi_1^{(F)},\dots,\mathbf{grad}\,\phi_{2p-3}^{(F)}\right\}, \\
R^{(T)} &: \mathcal{ND}_p(\mathcal{T}_h) \to \operatorname{span}\left\{\mathbf{grad}\,\phi_1^{(T)},\dots,\mathbf{grad}\,\phi_{3p^2-9p+7}^{(T)}\right\}.
\end{aligned}
$$

Another possibility would be to leave $\widetilde{\mathcal{ND}}_p^{\perp}(\widehat{T})$ as it is and to take only the 12 edge functions from $\tilde{\mathcal{S}}_p(\widehat{T})$. Thus, we get the following decomposition

$$
\begin{aligned}
\mathcal{ND}_p(\mathcal{T}_h) =& \mathcal{ND}_{p-1}(\mathcal{T}_h) \oplus \sum_{i=1}^{M} \operatorname{span}\left\{\mathbf{grad}\,\phi^{(e_i)}\right\} \oplus \sum_{j=1}^{N} \operatorname{span}\left\{\mathbf{b}_1^{(F_j)},\dots\right. \\
&\left.\dots,\mathbf{b}_{2(p-1)(p-2)}^{(F_j)}\right\} \oplus \sum_{l=1}^{L} \operatorname{span}\left\{\mathbf{b}_1^{(T_l)},\dots,\mathbf{b}_{3(p-1)(p-2)^2}^{(T_l)}\right\}.
\end{aligned}
\tag{4.46}
$$

Furthermore, we introduce the following projection operators

$$\begin{aligned} P_{p-1} &: \mathcal{ND}_p(\mathcal{T}_h) \to \mathcal{ND}_{p-1}(\mathcal{T}_h), \\ P^{(F)} &: \mathcal{ND}_p(\mathcal{T}_h) \to \operatorname{span}\left\{\mathbf{b}_1^{(F)}, \ldots, \mathbf{b}_{2(p-1)(p-2)}^{(F)}\right\}, \\ P^{(T)} &: \mathcal{ND}_p(\mathcal{T}_h) \to \operatorname{span}\left\{\mathbf{b}_1^{(T)}, \ldots, \mathbf{b}_{3(p-1)(p-2)^2}^{(T)}\right\}, \\ R^{(e)} &: \mathcal{ND}_p(\mathcal{T}_h) \to \operatorname{span}\left\{\mathbf{grad}\, \phi^{(e_i)}\right\}. \end{aligned}$$

It follows that the decomposition (4.46) for $\mathbf{u} \in \mathcal{ND}_p(\mathcal{T}_h)$ can also be written as

$$\mathbf{u} = P_{p-1}\mathbf{u} + \sum_{i=1}^{M} R^{(e_i)}\mathbf{u} + \sum_{j=1}^{N} P^{(F_j)}\mathbf{u} + \sum_{k=1}^{L} P^{(T_k)}\mathbf{u} \tag{4.47}$$

and there holds the following lemma.

Lemma 4.9.2 *The decomposition (4.46) is stable with respect to the* $\mathbf{H}(\mathbf{curl}, \Omega)$*-norm, i.e. for all* $\mathbf{u} \in \mathcal{ND}_p(\mathcal{T}_h)$ *there holds*

$$\begin{aligned} \|\mathbf{u}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \simeq \|P_{p-1}\mathbf{u}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} &+ \sum_{i=1}^{M} \|R^{(e_i)}\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \\ &+ \sum_{j=1}^{N} \|P^{(F_j)}\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)} + \sum_{k=1}^{L} \|P^{(T_k)}\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)}. \end{aligned}$$

Proof. The proof is similar to the proof of Lemma 4.9.1. □

Remark 4.9.3 *A similar result holds for the decomposition* (4.45).

Remark 4.9.4 *The space* $\mathcal{ND}_{p-1}(\mathcal{T}_h)$ *can also be split recursively to get smaller subspaces.*

4.9.3 A preconditioner for the $\mathbf{H}(\mathbf{curl}, \Omega)$-bilinear form, h-version

Here, we consider the $\mathbf{H}(\mathbf{curl}, \Omega)$-bilinear form

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_\Omega + (\mathbf{u}, \mathbf{v})_\Omega.$$

Let A denote the system matrix of $a(\cdot,\cdot)$ with respect to the space $\mathcal{ND}_2(\mathcal{T}_h)$. Using the stable decomposition (4.37) of $\mathcal{ND}_2(\mathcal{T}_h)$ and Lemma 4.9.1 we can construct an additive Schwarz preconditioner as in Chapter 2 such that the condition number of the preconditioned system is bounded, independent of the mesh size h. Here, we only

consider hexahedra but the case of a tetrahedral mesh can be analyzed similarly.
The resulting matrix is block diagonal where the blocks have the following form. For the meaning of the projections, see page 106.

$$
\begin{aligned}
V(\tilde{P}_1) &:= \big(a(\mathbf{b}^{(e_i)}, \mathbf{b}^{(e_j)})\big)_{i,j=1,\ldots,M}, \qquad \mathbf{b}^{(e_i)} \in \mathcal{ND}_1(\mathcal{T}_h),\\
V(\tilde{P}^{(F_j)}) &:= \big(a(\hat{\mathbf{b}}_i^{(F_j)}, \hat{\mathbf{b}}_j^{(F_j)})\big)_{i,j=1,\ldots,3},\\
&\qquad\qquad \hat{\mathbf{b}}_i^{(F_j)} \in \operatorname{span}\{\mathbf{b}_1^{(F_j)}, \mathbf{b}_3^{(F_j)}, \mathbf{b}_2^{(F_j)} + \mathbf{b}_4^{(F_j)}\}, \quad j = 1,\ldots N,\\
V(\tilde{P}^{(T_k)}) &:= \big(a(\hat{\mathbf{b}}_i^{(T_k)}, \hat{\mathbf{b}}_j^{(T_k)})\big)_{i,j=1,\ldots,5},\\
&\qquad \hat{\mathbf{b}}_i^{(T_k)} \in \operatorname{span}\{\mathbf{b}_1^{(T_k)}, \mathbf{b}_3^{(T_k)}, \mathbf{b}_5^{(T_k)}, \mathbf{b}_2^{(T_k)} - \mathbf{b}_4^{(T_k)}, \mathbf{b}_4^{(T_k)} - \mathbf{b}_6^{(T_k)}\}, \quad k = 1,\ldots,L,\\
V(\tilde{R}^{(e_i)}) &:= a(\mathbf{grad}\,\phi^{(e_i)}, \mathbf{grad}\,\phi^{(e_i)}), \quad i = 1,\ldots,M,\\
V(\tilde{R}^{(F_i)}) &:= a(\mathbf{grad}\,\phi^{(F_i)}, \mathbf{grad}\,\phi^{(F_i)}), \quad i = 1,\ldots,N,\\
V(\tilde{R}^{(T_i)}) &:= a(\mathbf{grad}\,\phi^{(T_i)}, \mathbf{grad}\,\phi^{(T_i)}), \quad i = 1,\ldots,L.
\end{aligned}
$$

Most of the matrices are quite small, the matrices related to the gradients are only one-dimensional, and easy to invert. The largest matrix is the one related to $\mathcal{ND}_1(\mathcal{T}_h)$ whose size depends on the number of edges.

Using these matrices we can construct the block-diagonal preconditioning matrix as

$$
\begin{aligned}
V := \Big(&\operatorname{diag}\big(V(\tilde{P}_1),\, V(\tilde{P}^{(F_1)}),\ldots,\, V(\tilde{P}^{(F_N)}),\, V(\tilde{P}^{(T_1)}),\ldots,\, V(\tilde{P}^{(T_L)}),\, V(\tilde{R}^{(e_i)}),\\
&V(\tilde{R}^{(e_1)}),\ldots,\, V(\tilde{R}^{(e_M)}),\, V(\tilde{R}^{(F_1)}),\ldots,\, V(\tilde{R}^{(F_N)}),\, V(\tilde{R}^{(T_1)}),\ldots,\, V(\tilde{R}^{(T_L)})\big)\Big)^{-1},
\end{aligned}
$$

where the blocks can be inverted separately. From Lemma 4.9.1 we get for the h-version

Theorem 4.9.5 *The condition number of the preconditionened system is bounded, i.e. there exists a constant $C > 0$, independent of the mesh size h, such that*

$$\operatorname{cond}(VA) \leq C.$$

Remark 4.9.6 *This result holds also for the decomposition of $\mathcal{ND}_p(\mathcal{T}_h)$ for higher polynomial degrees $p \geq 2$ as described in §4.9.2. Where the block related to $\mathcal{ND}_{p-1}(\mathcal{T}_h)$ can also be split inductively.*

4.10 Stable decompositions of $\mathcal{RT}_2(\mathcal{K}_h)$

Using the results of Section 4.9 we construct a stable decomposition of the Raviart-Thomas space $\mathcal{RT}_2(\mathcal{K}_h)$. Using this we construct a preconditioner to the $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$-bilinear form$b(\boldsymbol{\lambda}, \boldsymbol{\zeta}) := \langle V(\mathrm{div}_\Gamma \boldsymbol{\lambda}), \mathrm{div}_\Gamma \boldsymbol{\zeta}\rangle_\Gamma + \langle \mathcal{V}\boldsymbol{\lambda}, \boldsymbol{\zeta}\rangle$, see Subsection 4.10.2.

4.10.1 A stable decomposition

From Chapter 3 we know that the space $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is just the twisted tangential trace of $\mathbf{H}(\mathbf{curl}, \Omega)$. It is thus obvious to discretize $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ using the twisted tangential trace of the space of Nédélec elements. It is well known (see e.g. Hiptmair [64] and Lemma 4.4.1) that this yields the two dimensional $\mathbf{H}(\mathrm{div}, \Omega)$-conforming space of Raviart-Thomas, i.e.

$$\gamma_t^\times : \mathcal{ND}_p(\mathcal{T}_h) \to \mathcal{RT}_p(\mathcal{K}_h). \tag{4.48}$$

Also, the degrees of freedom carry over [64], i.e. for an element $T \in \mathcal{T}_h$, a face K of T and $\mathbf{u} \in (\mathcal{C}^\infty(\overline{T}))^3$ we have the identity

$$\gamma_t^\times \Pi^{\mathcal{ND}_p(T)} \mathbf{u} = \Pi^{\mathcal{RT}_p(K)} \gamma_t^\times \mathbf{u}. \tag{4.49}$$

We now aim to find a $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$-stable decomposition of $\mathcal{RT}_2(\mathcal{K}_h)$ using the results of section 4.9. Let m denote the number of edges and n the number of elements in $\mathcal{K}_h$, the triangular or quadrilateral trace mesh of $\mathcal{T}_h$. We apply the trace mapping (4.48) to decomposition (4.34) for tetrahedra and obtain the decomposition

$$\mathcal{RT}_2(\mathcal{K}_h) = \mathcal{RT}_1(\mathcal{K}_h) \oplus \mathbf{curl}_\Gamma \widetilde{\mathcal{S}}_2(\mathcal{K}_h) \oplus \widetilde{\mathcal{RT}}_2^\perp(\mathcal{K}_h) \tag{4.50}$$

for triangles, where

$$\widetilde{\mathcal{RT}}_2^\perp(\mathcal{K}_h) := \{\boldsymbol{\lambda}_h \in \mathcal{RT}_2(\mathcal{K}_h) : \int_e \boldsymbol{\lambda}_h \cdot \mathbf{n}\, q\, ds = 0, \forall q \in \mathbb{P}_1,\, e \text{ side of } \mathcal{K}_h\}$$

and $\widetilde{\mathcal{S}}_k(\mathcal{K}_h) := \mathcal{S}_k(\mathcal{K}_h) \setminus \mathcal{S}_{k-1}(\mathcal{K}_h)$. Here, $\mathcal{S}_k(\mathcal{K}_h)$ is the space of piecewise polynomials in two dimensions of degree k.

For $K \in \mathcal{K}_h$ there holds $|\mathcal{S}_p(K)| = \frac{1}{2}(p+1)(p+2)$, and the dimension of $\mathcal{RT}_2(K)$ is $|\mathcal{RT}_2(K)| = 8$, corresponding to two basis functions per side and two inner functions. If $K \in \mathcal{K}_h$ is the face of the element $T \in \mathcal{T}_h$, then its three sides are three edges of T, so that the three basis functions spanning $\mathcal{RT}_1(K)$ are the images of the the three basis functions of $\mathcal{ND}_1(T)$ corresponding to those edges under the mapping $\gamma_t^\times$. The three basis functions of $\widetilde{\mathcal{S}}_2(K)$ are the images of the three basis functions of $\widetilde{\mathcal{S}}_2(T)$ corresponding to those edges and the two basis functions spanning $\widetilde{\mathcal{RT}}_2^\perp(K)$ are the images of the two basis functions of $\widetilde{\mathcal{ND}}_1^\perp(T)$ corresponding to the face K. Counting the basis functions yields that (4.50) is a direct sum. We write $\widetilde{\mathcal{S}}_2(\mathcal{K}_h) = \operatorname{span}\{\varphi^{(e_1)}, \dots, \varphi^{(e_m)}\}$ and $\widetilde{\mathcal{RT}}_2^{\perp}(\mathcal{K}_h) = \operatorname{span}\{\boldsymbol{\lambda}_1^{(K_1)}, \boldsymbol{\lambda}_2^{(K_1)}, \dots, \boldsymbol{\lambda}_1^{(K_n)}, \boldsymbol{\lambda}_2^{(K_n)}\}$. Localization as before yields

$$\mathcal{RT}_2(\mathcal{K}_h) = \mathcal{RT}_1(\mathcal{K}_h) \oplus \sum_{i=1}^{m} \operatorname{span}\{\mathbf{curl}_\Gamma\, \varphi^{(e_i)}\} \oplus \sum_{j=1}^{n} \operatorname{span}\{\boldsymbol{\lambda}_1^{(K_j)}, \boldsymbol{\lambda}_2^{(K_j)}\}. \tag{4.51}$$

For the trace mesh of a hexahedral grid we obtain the decomposition

$$\mathcal{RT}_2(\mathcal{K}_h) = \mathcal{RT}_1(\mathcal{K}_h) \oplus \mathbf{curl}_\Gamma\, \widetilde{\mathcal{S}}_2(\mathcal{K}_h) \oplus \widetilde{\mathcal{RT}}_2^{\perp,-}(\mathcal{K}_h), \tag{4.52}$$

with $\mathbf{curl}_\Gamma\, \widetilde{\mathcal{S}}_2(K) = \{\mathbf{curl}_\Gamma\, \varphi^{(e_1)}, \dots, \mathbf{curl}_\Gamma\, \varphi^{(e_4)}, \mathbf{curl}_\Gamma\, \varphi^{(K)}\}$ (with a suitable bubble function $\varphi^{(K)}$) and $\widetilde{\mathcal{RT}}_2^{\perp,-}(K) = \{\boldsymbol{\lambda}_1^{(K)}, \boldsymbol{\lambda}_3^{(K)}, \boldsymbol{\lambda}_2^{(K)} - \boldsymbol{\lambda}_4^{(K)}\}$ where $\boldsymbol{\lambda}_i^{(K)}$ $(i = 1, \dots, 4)$ are the images of the basis functions $\mathbf{b}_i^{(F)}$ in $\widetilde{\mathcal{ND}}_2^{\perp}(T)$ corresponding to the face K. Again, (4.52) constitutes a direct sum (there holds $\dim \mathcal{RT}_2(K) = 12$ for quadrilateral elements), and its localization reads

$$\begin{aligned} \mathcal{RT}_2(\mathcal{K}_h) =& \mathcal{RT}_1(\mathcal{K}_h) \oplus \sum_{i=1}^{m} \operatorname{span}\{\mathbf{curl}_\Gamma\, \varphi^{(e_i)}\} \\ & \oplus \sum_{j=1}^{n} \Big(\operatorname{span}\{\mathbf{curl}_\Gamma\, \varphi^{(K_j)}\} \oplus \operatorname{span}\{\boldsymbol{\lambda}_1^{(K_j)}, \boldsymbol{\lambda}_3^{(K_j)}, \boldsymbol{\lambda}_2^{(K_j)} - \boldsymbol{\lambda}_4^{(K_j)}\} \Big). \end{aligned} \tag{4.53}$$

The task at issue is to show the stability of (4.51) and (4.53), respectively. Therefore, we define for the tetrahedral case the projection operators

$$\begin{aligned} p_1 &: \mathcal{RT}_2(\mathcal{K}_h) \to \mathcal{RT}_1(\mathcal{K}_h), \\ p^{(K)} &: \mathcal{RT}_2(\mathcal{K}_h) \to \operatorname{span}\{\boldsymbol{\lambda}_1^{(K)}, \boldsymbol{\lambda}_2^{(K)}\}, \\ r^{(e)} &: \mathcal{RT}_2(\mathcal{K}_h) \to \operatorname{span}\{\mathbf{curl}_\Gamma\, \varphi^{(e)}\} \end{aligned}$$

and for quadrilaterals the projections

$$\begin{aligned} \tilde{p}_1 &: \mathcal{RT}_2(\mathcal{K}_h) \to \mathcal{RT}_1(\mathcal{K}_h), \\ \tilde{p}^{(K)} &: \mathcal{RT}_2(\mathcal{K}_h) \to \operatorname{span}\{\boldsymbol{\lambda}_1^{(K)}, \boldsymbol{\lambda}_3^{(K)}, \boldsymbol{\lambda}_2^{(K)} - \boldsymbol{\lambda}_4^{(K)}\}, \\ \tilde{r}^{(e)} &: \mathcal{RT}_2(\mathcal{K}_h) \to \operatorname{span}\{\mathbf{curl}_\Gamma\, \varphi^{(e)}\}, \\ \tilde{r}^{(K)} &: \mathcal{RT}_2(\mathcal{K}_h) \to \operatorname{span}\{\mathbf{curl}_\Gamma\, \varphi^{(K)}\}, \end{aligned}$$

such that the decompositions (4.51) and (4.53) can be written as

$$\boldsymbol{\lambda}_2 = p_1 \boldsymbol{\lambda}_2 + \sum_{i=1}^{m} r^{(e_i)} \boldsymbol{\lambda}_2 + \sum_{j=1}^{n} p^{(K_j)} \boldsymbol{\lambda}_2 \tag{4.54}$$

and

$$\boldsymbol{\lambda}_2 = \tilde{p}_1 \boldsymbol{\lambda}_2 + \sum_{i=1}^{m} \tilde{r}^{(e_i)} \boldsymbol{\lambda}_2 + \sum_{j=1}^{n} \left(\tilde{r}^{(K_j)} \boldsymbol{\lambda}_2 + \tilde{p}^{(K_j)} \boldsymbol{\lambda}_2 \right), \tag{4.55}$$

respectively. Now, the stability of these $\mathcal{RT}_2$-decompositions can be proven via the stability of the $\mathcal{ND}_2$-decompositions, as we will show in the following lemma. Here, we will denote the $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$-norm simply by $\|\cdot\|$ in the statement of the lemma.

Lemma 4.10.1 *The decompositions* (4.51) *and* (4.53) *are stable with respect to the* $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$*-norm, i.e. for all* $\boldsymbol{\lambda}_2 \in \mathcal{RT}_2(\mathcal{K}_h)$ *there holds*

$$\|\boldsymbol{\lambda}_2\|^2 \simeq \|p_1\boldsymbol{\lambda}_2\|^2 + \left\| \sum_{i=1}^{m} p^{(e_i)}\boldsymbol{\lambda}_2 + \sum_{j=1}^{n} p^{(K_j)}\boldsymbol{\lambda}_2 \right\|^2 \tag{4.56}$$

and

$$\|\boldsymbol{\lambda}_2\|^2 \simeq \|\tilde{p}_1\boldsymbol{\lambda}_2\|^2 + \left\| \sum_{i=1}^{m} \tilde{r}^{(e_i)}\boldsymbol{\lambda}_2 + \sum_{j=1}^{n} \left(\tilde{r}^{(K_j)}\boldsymbol{\lambda}_2 + \tilde{p}^{(K_j)}\boldsymbol{\lambda}_2\right) \right\|^2, \tag{4.57}$$

respectively.

Proof. Take an arbitrary $\boldsymbol{\lambda}_2 \in \mathcal{RT}_2(\mathcal{K}_h)$. We decompose $\boldsymbol{\lambda}_2$ as

$$\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1 + \tilde{\boldsymbol{\lambda}}_2 \tag{4.58}$$

with $\boldsymbol{\lambda}_1 \in \mathcal{RT}_1(\mathcal{K}_h)$ and $\tilde{\boldsymbol{\lambda}}_2 \in \mathcal{RT}_2(\mathcal{K}_h) \setminus \mathcal{RT}_1(\mathcal{K}_h)$. From the extension theorems in §4.7 we know that there exist $\mathbf{u}_1 \in \mathcal{ND}_1(\mathcal{T}_h)$ and $\mathbf{u}_2 \in \mathcal{ND}_2(\mathcal{T}_h)$ with $\gamma_t^\times \mathbf{u}_1 = \boldsymbol{\lambda}_1$, $\gamma_t^\times \mathbf{u}_2 = \boldsymbol{\lambda}_2$ and $\|\mathbf{u}_1\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le C\,\|\boldsymbol{\lambda}_1\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}$, $\|\mathbf{u}_2\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le C\,\|\boldsymbol{\lambda}_2\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}$ with $C > 0$, independent of h. Due to the continuity of $\gamma_t^\times : \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ there holds

$$\|\mathbf{u}_1\|_{\mathbf{H}(\mathbf{curl},\Omega)} \simeq \|\boldsymbol{\lambda}_1\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)},$$
$$\|\mathbf{u}_2\|_{\mathbf{H}(\mathbf{curl},\Omega)} \simeq \|\boldsymbol{\lambda}_2\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}.$$

This, together with the stability result from Lemma 4.9.1 proves the statement of the lemma. □

Actually, we would like to prove the following result. The problem is the existence of stable extension operators from the basis functions on the boundary to the basis functions in the interior. From §4.7 we only know the existence of a global extension operator from $\mathcal{RT}_p(\mathcal{K}_h)$ to $\mathcal{ND}_p(\mathcal{T}_h)$. Under the assumption that there exists such an extension operator we could prove the following result.

Lemma 4.10.2 *Under the assumption that there exists a continuous extension from* $\mathcal{RT}_p(\mathcal{K}_h)$ *to* $\mathcal{ND}_p(\mathcal{K}_h)$ *which also preserves the basis functions, the decompositions* (4.51) *and* (4.53) *are stable with respect to the* $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$*-norm, i.e. for all* $\boldsymbol{\lambda}_2 \in \mathcal{RT}_2(\mathcal{K}_h)$ *there holds*

$$\|\boldsymbol{\lambda}_2\|^2 \simeq \|p_1\boldsymbol{\lambda}_2\|^2 + \sum_{i=1}^{m} \|p^{(e_i)}\boldsymbol{\lambda}_2\|^2 + \sum_{j=1}^{n} \|p^{(K_j)}\boldsymbol{\lambda}_2\|^2 \tag{4.59}$$

and

$$\|\boldsymbol{\lambda}_2\|^2 \simeq \|\tilde{p}_1\boldsymbol{\lambda}_2\|^2 + \sum_{i=1}^{m} \|\tilde{r}^{(e_i)}\boldsymbol{\lambda}_2\|^2 + \sum_{j=1}^{n} \left(\|\tilde{r}^{(K_j)}\boldsymbol{\lambda}_2\|^2 + \|\tilde{p}^{(K_j)}\boldsymbol{\lambda}_2\|^2\right), \tag{4.60}$$

respectively.

Proof. We take an arbitrary $\boldsymbol{\lambda}_2 \in \mathcal{RT}_2(\mathcal{K}_h)$. We decompose $\boldsymbol{\lambda}_2$ according to (4.54) or (4.55) into $\boldsymbol{\lambda}_2 = \sum_{i=0}^{r} \boldsymbol{\lambda}_{2,i}$ (where $r = m + n$ for a triangular mesh and $r = m + 2n$ for a quadrilateral mesh). We now consider the the extension $\mathbf{F}_h$ from §4.7. We know that there exists a $\mathbf{u}_2 \in \mathcal{ND}_2(\mathcal{K}_h)$ with $\gamma_t^\times \mathbf{u}_2 = \boldsymbol{\lambda}_2$ and $\|\mathbf{u}_2\|_{\mathbf{H}(\mathbf{curl},\Omega)} \lesssim \|\boldsymbol{\lambda}_2\|_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}$. Thus, $\mathbf{u}_2$ owns a stable decomposition according to Lemma 4.9.1 $\mathbf{u}_2 = \sum_{j=0}^{K} \mathbf{u}_{2,j}$ with $K = M + 2N + 2L$. We now assume that for every $\boldsymbol{\lambda}_{2,i}$ there exists a $\mathbf{u}_{2,j}$ of the decomposition with $\gamma_t^\times \mathbf{u}_{2,j} = \boldsymbol{\lambda}_{2,i}$ and $\|\mathbf{u}_{2,j}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \lesssim \|\boldsymbol{\lambda}_{2,j}\|_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}$. Using the continuity of $\gamma_t^\times$ we then obtain the equivalences

$$
\begin{aligned}
\|\mathbf{u}_2\|_{\mathbf{H}(\mathbf{curl},\Omega)} &\simeq \|\boldsymbol{\lambda}_2\|_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)},\\
\|\mathbf{u}_{2,i}\|_{\mathbf{H}(\mathbf{curl},\Omega)} &\simeq \|\boldsymbol{\lambda}_{2,i}\|_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)},\ i = 1,\dots,r.
\end{aligned}
$$

This, together with the $\mathcal{ND}_2$-stability $\sum_{i=0}^{r} \|\mathbf{u}_{2,i}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \simeq \|\mathbf{u}_2\|_{\mathbf{H}(\mathbf{curl},\Omega)}$ in Lemma 4.9.1 proves the statement of the lemma. □

Remark 4.10.3 *Such a continuous extension exists locally on one element. Therefore, the construction of a p-hierarchical error estimator (compare Teltscher [103] and Teltscher et al. [104, 105]) still works, although an extension as assumed in the lemma should not exist.*

Now, let $\mathcal{V}$ (V) denote the vectorial (scalar) single layer potential operator for the Laplace equation defined for vector (scalar) functions $\boldsymbol{\lambda}$ (λ), cf. Chapter 3 with the Laplace-kernel $\Phi(\mathbf{x}, \mathbf{y}) := \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$. We can define on $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ a continuous sesquilinear form b by

$$
b(\boldsymbol{\lambda}, \mathbf{w}) = \langle V \,\mathrm{div}_\Gamma \boldsymbol{\lambda}, \mathrm{div}_\Gamma \mathbf{w}\rangle_\Gamma + \langle \mathcal{V}\boldsymbol{\lambda}, \mathbf{w}\rangle_\Gamma \tag{4.61}
$$

and will consider the energy norm induced by b,

$$
\|\boldsymbol{\lambda}\|_{\mathfrak{e}} := |b(\boldsymbol{\lambda}, \boldsymbol{\lambda})|^{1/2},
$$

which is equivalent to the $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$-norm.

4.10.2 A preconditioner for the single layer potential

Using the result of §4.10.1 we can construct a preconditioner for the matrix related the $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$-bilinear form (4.61).

We consider the decomposition into two spaces according to Lemma 4.10.1. For this case we have proven the stability of the decomposition of $\mathcal{RT}_2(\mathcal{K}_h)$ into the space $\mathcal{RT}_1(\mathcal{K}_h)$

and the space $\mathcal{RT}_2(\mathcal{K}_h) \setminus \mathcal{RT}_1(\mathcal{K}_h)$ with respect to the $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$-norm. Thus, we define the two block matrices by

$$V_1 := \big(b(\boldsymbol{\lambda}^{(e_i)}, \boldsymbol{\lambda}^{(e_j)})\big)_{i,j=1,\dots,m}, \quad \boldsymbol{\lambda}^{(e_i)} \in \mathcal{RT}_1(\mathcal{K}_h),$$
$$V_2 := \big(b(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j)\big)_{i,j=1,\dots,m+2n}, \quad \boldsymbol{\lambda}^{(e_i)} \in \mathcal{RT}_2(\mathcal{K}_h) \setminus \mathcal{RT}_1(\mathcal{K}_h),$$

and the preconditioning matrix takes the following form

$$V := \begin{pmatrix} V_1^{-1} & 0 \\ 0 & V_2^{-1} \end{pmatrix}.$$

From Lemma 4.10.1 we then get

Theorem 4.10.4 *Let B denote the system matrix related to the bilinear form $b(\cdot,\cdot)$ on $\mathcal{RT}_2(\mathcal{K}_h) \times \mathcal{RT}_2(\mathcal{K}_h)$. It follows that the preconditioned system VB has a bounded condition number, i.e. there exists a constant $C > 0$, independent on the mesh size h, such that*

$$\mathrm{cond}\ (VB) \le C.$$

Numerical experiment

Here, we test our 2-block preconditioner for the bilinear form (4.61) $\boldsymbol{\lambda}$, $\boldsymbol{\zeta} \in \mathcal{RT}_2(\mathcal{K}_h)$. We consider the unit square $\Omega := [-1,1]^2$ with a uniform mesh of squares of size h. We calculate the condition number of the matrix and the condition number of the preconditioned matrix. The results are given in Table 4.6, see also Figure 4.9. While the condition number of the matrix increases rapidly with h the condition number of the preconditioned system stays low.

On the implementation

Using the program package *maiprogs* , see Maischak [75], we first calculate the Galerkin matrix to the bilinear form (4.61) with respect to a basis of $\mathcal{RT}_2(\mathcal{K}_h)$. The next task is to find a representation of the basis functions of the spaces $\mathcal{RT}_1(\mathcal{K}_h)$ in basis functions of $\mathcal{RT}_2(\mathcal{K}_h)$. Therefore, we consider the reference square $\widehat{K}$ and the associated basis functions, cf. §4.4.1. There holds

$$\begin{aligned}
\boldsymbol{\lambda}^{(e0)} =& \boldsymbol{\lambda}_1^{(e0)} - \boldsymbol{\lambda}_3^{(\widehat{K})},\\
\boldsymbol{\lambda}^{(e1)} =& \boldsymbol{\lambda}_1^{(e1)} + \boldsymbol{\lambda}_1^{(\widehat{K})},\\
\boldsymbol{\lambda}^{(e2)} =& \boldsymbol{\lambda}_1^{(e2)} + \boldsymbol{\lambda}_3^{(\widehat{K})},\\
\boldsymbol{\lambda}^{(e3)} =& \boldsymbol{\lambda}_1^{(e3)} - \boldsymbol{\lambda}_1^{(\widehat{K})}.
\end{aligned}$$

j	h	DOF	cond(B)	cond(VB)
2	1.0	40	257.3933	22.0932
3	0.667	84	545.9814	22.8748
4	0.5	144	947.6450	24.1816
5	0.4	220	1458.1811	25.6662
6	0.333	312	2095.8985	26.9247
7	0.286	420	2841.5957	27.9643
8	0.25	544	3708.7599	28.8534
9	0.222	684	4689.9097	29.6349
10	0.2	840	5786.7439	30.3333
11	0.182	1012	7000.7135	30.9646
12	0.167	1200	8329.0456	31.5406
13	0.154	1404	9774.3916	32.0700
14	0.143	1624	.1133E+05	32.5599

Table 4.6: Condition numbers for the linear system related to the bilinear form b, with and without preconditioning.

Using this we re-order the elements of the Galerkin matrix such that one diagonal block of the matrix only belongs to the space $\mathcal{RT}_1(\mathcal{K}_h)$ and the other diagonal block belongs to $\mathcal{RT}_2(\mathcal{K}_h) \setminus \mathcal{RT}_1(\mathcal{K}_h)$.

For completeness, we give here the representation of the surface curls of hat functions in the basis of $\mathcal{RT}_2(\widehat{K})$. This helps us to calculate an 2-level error indicator as introduced by Teltscher [103] and Teltscher *et al.* [104, 105].

$$\begin{aligned}
\mathbf{curl}_\Gamma\,\varphi^{(e0)} &= \mathbf{curl}_\Gamma(1-x^2)(1-y) &&= \begin{pmatrix} -1+x^2 \\ 2x-2xy \end{pmatrix} &&= -\tfrac{44}{15}\boldsymbol{\lambda}_2^{(e0)} - \tfrac{4}{15}\boldsymbol{\lambda}_2^{(e2)} - \tfrac{8}{3}\boldsymbol{\lambda}_1^{(\widehat{K})} + \tfrac{8}{3}\boldsymbol{\lambda}_4^{(\widehat{K})}, \\
\mathbf{curl}_\Gamma\,\varphi^{(e1)} &= \mathbf{curl}_\Gamma(1+x)(1-y^2) &&= \begin{pmatrix} -2y-2xy \\ -1+y^2 \end{pmatrix} &&= -\tfrac{44}{15}\boldsymbol{\lambda}_2^{(e1)} - \tfrac{4}{15}\boldsymbol{\lambda}_2^{(e3)} - \tfrac{8}{3}\boldsymbol{\lambda}_2^{(\widehat{K})} - \tfrac{8}{3}\boldsymbol{\lambda}_3^{(\widehat{K})}, \\
\mathbf{curl}_\Gamma\,\varphi^{(e2)} &= \mathbf{curl}_\Gamma(1-x^2)(1+y) &&= \begin{pmatrix} 1-x^2 \\ -2x-2xy \end{pmatrix} &&= -\tfrac{4}{15}\boldsymbol{\lambda}_2^{(e0)} - \tfrac{44}{15}\boldsymbol{\lambda}_2^{(e2)} + \tfrac{8}{3}\boldsymbol{\lambda}_1^{(\widehat{K})} - \tfrac{8}{3}\boldsymbol{\lambda}_4^{(\widehat{K})}, \\
\mathbf{curl}_\Gamma\,\varphi^{(e3)} &= \mathbf{curl}_\Gamma(1-x)(1-y^2) &&= \begin{pmatrix} -2y+2xy \\ 1-y^2 \end{pmatrix} &&= \tfrac{4}{15}\boldsymbol{\lambda}_2^{(e1)} - \tfrac{44}{15}\boldsymbol{\lambda}_2^{(e3)} - \tfrac{8}{3}\boldsymbol{\lambda}_2^{(\widehat{K})} + \tfrac{8}{3}\boldsymbol{\lambda}_3^{(\widehat{K})}, \\
\mathbf{curl}_\Gamma\,\varphi^{(\widehat{K})} &= \mathbf{curl}_\Gamma(1-x^2)(1-y^2) &&= \begin{pmatrix} -2y+2x^2y \\ 2x-2xy^2 \end{pmatrix} &&= -\tfrac{4}{9}\left(\boldsymbol{\lambda}_2^{(\widehat{K})} + \boldsymbol{\lambda}_4^{(\widehat{K})}\right).
\end{aligned}$$

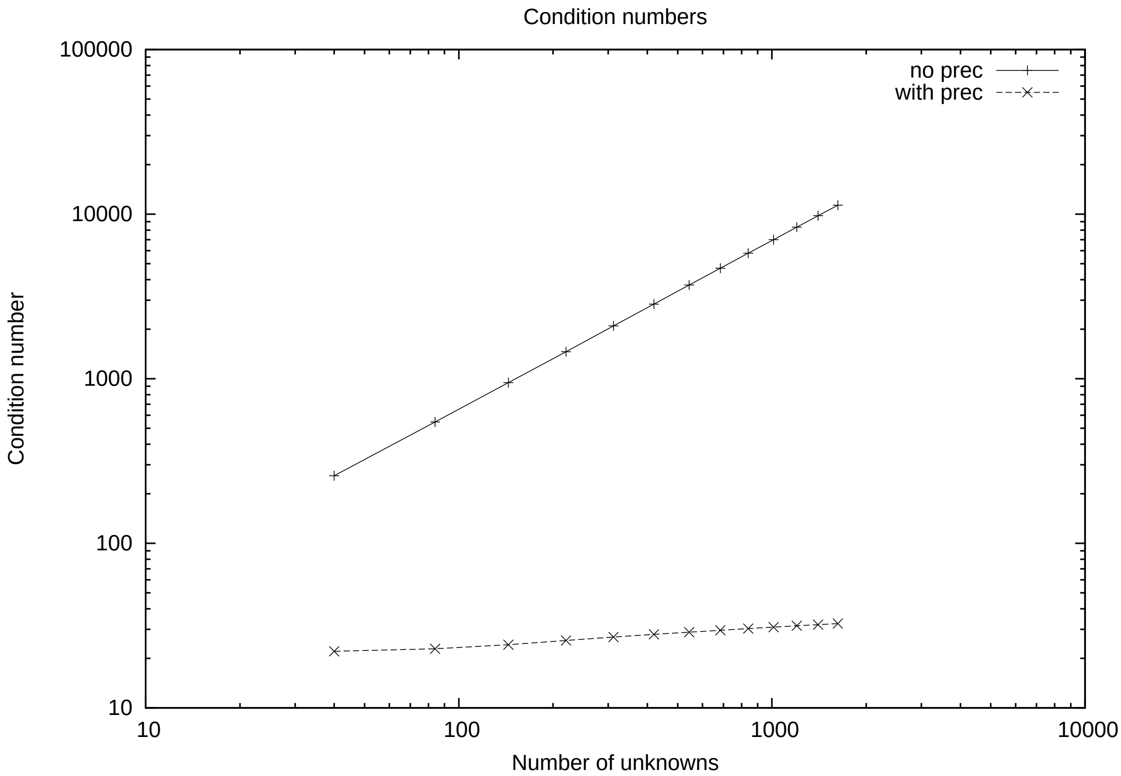

Figure 4.9: Condition numbers of the linear system related to bilinear form b, with and without preconditioning.

4.11 Hanging nodes / hanging edges

The above described basis functions and their transformations have been implemented in the program package *maiprogs* [75] and most of the experiments have been done on uniform refined meshes. In order to introduce an adaptive algorithm using certain error indicators one has to consider so-called hanging nodes. In the following we describe the implementation of hanging nodes for Nédélec and Raviart-Thomas functions. For the implementation of hanging nodes for standard hat functions of degree 1, see the PhD-thesis of Oestmann [85]. As the degrees of freedom of Nédélec and Raviart-Thomas spaces are edge based we are talking about "hanging edges" instead of hanging nodes. First of all, we describe the construction of hanging edges for functions of degree $p = 1$. This has been implemented in *maiprogs* so far. The implementation for higher polynomial degrees still has to be done.

In the beginning, we consider the construction for an $\mathbf{H}(\mathbf{curl}, \Omega)$-conforming piecewise polynomial. As described in §4.1 the constraint is that the tangential component of the function has to be continuous. For the case $p = 1$ the basis functions are calculated on page 71. We find out that the basis functions are constant on the corresponding edges.

Definition 4.11.1 *An edge e of the triangulation $\mathcal{T}_h$ is called* **regular** *if all elements, which are connected to e, have this as an edge of the same size.*
An edge e is called **hanging** *or* **dependent** *if this edge is adjacent to a longer neighbor edge or if e is adjacent to a neighboring face, see Figure 4.10 for the three possible situations.*

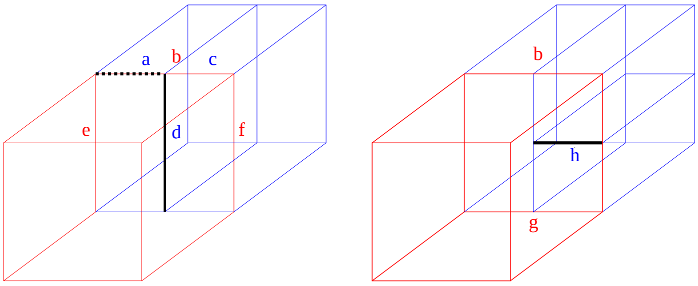

Figure 4.10: The edges a, d and h are hanging.

For the refinement we have to ensure the **one-constraint rule**, see e.g. Demkowicz *et al.* [47] or Oestmann [85]. This means that only one hanging node on an edge is allowed. For the case of edges this means that one edge has at most two smaller neighboring edges on the other element. The **refinement algorithm** is the following

1. Initialize a regular mesh without hanging nodes.
2. Calculate the local error indicators on every element.
3. Mark the elements to be refined.
4. Check if the one-constraint rule is fulfilled. If there is more than one hanging node on one edge mark the neighboring elements for refinement.
5. Go to 4. until no more extra refinements are necessary.
6. Initialize the new mesh.
7. Calculate the new approximation and go to 2.

For the construction of an $\mathbf{H}(\mathbf{curl}, \Omega)$-conforming function we have to ensure continuity of the tangential component over the element faces. Therefore we have to represent the basis function on the dependent edge by basis functions on the independent edges. In order to do so, we consider the following three cases, see Figure 4.10.

1. One edge (a) has a bigger neighbor (b).
   Due to the transformation $\mathbf{u} = (B_T^{\mathsf{T}})^{-1}\hat{\mathbf{u}}$ and $\|B_T\| \simeq h_T$, the (constant) value

of the basis function on the shorter edge is twice the value on the longer edge. In order to ensure continuity we have to halve the value on the shorter edge. It follows that the degree of freedom is copied and half-valued: a=$\frac{1}{2}$b.

2. Mid-edge situation on a face (the edge d in Figure 4.10, left).
   As the edge functions are linear in orthogonal direction to the edge we have to take the mean value between the two neighboring edges. Thus, the degree of freedom is a linear combination of the two neighboring edges: d=$\frac{1}{2}$e+$\frac{1}{2}f$.

3. Half-mid-edge situation.
   The edge d in Figure 4.10, right, is a small edge between two longer regular edges b and g. Again, the degree of freedom is a linear combination of the two neighboring longer edges: h=$\frac{1}{4}$b+$\frac{1}{4}$g.

These are the only possible cases. It may happen that one degree of freedom is dependent from a dependent degree of freedom and the degrees of freedom are then transfered multiplicatively. For further details see Oestmann [85].

Numerical experiments for meshes with hanging edges of polynomial degree $p = 1$ are given in Chapter 4.

In the case of Raviart-Thomas functions on the boundary we use the same strategy. But in two dimensions we only have to consider the case that an edge has a bigger neighbor, see Figure 4.11. Therefore, the degree of freedom on the dependent edge is half the value of the bigger edge. This is due to the Piola transformation $\boldsymbol{\lambda} = \frac{1}{\det B_K} B_k \widehat{\boldsymbol{\lambda}}$ and $\frac{1}{|\det B_K|} \|B_K\| \simeq h_K$.

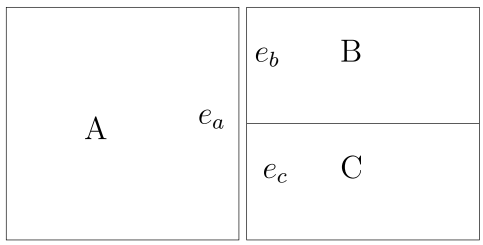

Figure 4.11: Hanging edge in two dimensions.

4.11.1 Hanging edges for higher polynomial degrees

Here we consider the construction of hanging nodes for the case of higher polynomial degrees. This hasn't been implemented yet in *maiprogs* . Therefore, there are no numerical experiments on this topic.

Raviart-Thomas elements

Here, we examine the two-dimensional case for Raviart-Thomas functions. There is only one case possible, i.e. one small edge has a bigger neighbor, see Figure 4.11. Thus the smaller edges e_b and e_c are dependent on the bigger edge e_a. In order to guarantee an $\mathbf{H}(\mathrm{div}, \Omega)$-conforming piecewise polynomial we have to ensure the continuity of the normal component. Therefore, we only have to examine those basis functions with a non-vanishing normal component on the considered edges.

As in the case of a polynomial degree $p = 1$ we have to describe the basis polynomials on the smaller edge e_b as a linear combination of the relevant polynomials on the adjacent edge e_a Those are the basis functions which are associated to edge e_a.

Every of the p basis functions on the edge e_b has to be the linear combination of the p edge functions on the edge e_a. In order to get the coefficients for the basis functions on e_b we have to solve a linear system. For the assembling of the linear system it is enough to choose p points on the edge e_b on which the function on e_b should be the same as the polynomial on e_a. Thus, the basis functions on e_b inherit the value of the linear combinations of the basis functions on e_a.

Nédélec elements

We now consider the space $\mathcal{ND}_p(\mathcal{T}_h)$ in three dimensions. Here we have to ensure continuity of the tangential trace. We know that the relevant Nédélec basis functions on a face F are those face functions which are associated to F and those edge functions which belong to the edges adjacent to F. All other basis function have vanishing tangential components, see §4.1. Due to the existence of face functions we have to consider hanging faces.

Definition 4.11.2 *A* **face** f *is called* **hanging** *or* **dependent** *if this face is adjacent to a bigger neighboring face.*

The principle is quite similar to the two dimensional case with Raviart-Thomas elements. Again the dependent basis functions on the hanging edge and the hanging face have to be a linear combination of all the basis functions on the neighboring regular face. Hence, one hanging basis function is now dependent on $2p + 4 \cdot 2$ basis functions. Again we have to assemble a linear system by checking the continuity of the tangential component in $2p + 8$ points of the smaller face.

5 The eddy current problem

In this chapter we analyze the eddy current problem for low frequencies. We first derive a weak formulation using the coupling of finite and boundary elements and present a priori estimates. In Section 5.3 we develop a reliable a posteriori estimator for the *hp*-version and prove the efficiency for the *h*-version in Section 5.4. Finally, we present different numerical experiments which underline the theoretical results.

5.1 The eddy current problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected open polyhedral domain and let the boundary $\Gamma = \partial\Omega$ be Lipschitz continuous and both Ω and Γ be simply connected. Furthermore, we denote the exterior domain by $\Omega_E := \mathbb{R}^3 \backslash \overline{\Omega}$ and the unit normal vector $\mathbf{n}$ on Γ pointing into Ω_E. The domain Ω represents the conductor with a conductivity $\sigma \in L^\infty(\mathbb{R}^3)$, $\sigma_1 \geq \sigma(\mathbf{x}) \geq \sigma_0 > 0$ and magnetic permeability $\mu \in L^\infty(\mathbb{R}^3)$, $\mu_1 \geq \mu(\mathbf{x}) \geq \mu_0 > 0$ with positive constants σ_0, σ_1, μ_0, μ_1. The exterior domain Ω_E represents the air. Therefore, we set $\sigma = 0$ and by scaling $\mu = 1$ in Ω_E. We consider a current $\mathbf{J}_0$ with frequency ω in the conductor Ω which induces electric and magnetic fields. As Ω_E is air there holds $\mathbf{J}_0 = 0$ in Ω_E and $\mathbf{J}_0 \cdot \mathbf{n} = 0$ on Γ, i.e. no current flows through Γ.

Then, the eddy current problem for low frequencies is given by, compare Ammari *et al.* [6]:
Find a magnetic field $\mathbf{H}(\mathbf{x})$ *and an electric field* $\mathbf{E}(\mathbf{x})$ *with*

$$\mathbf{curl}\,\mathbf{E} = -i\omega\mu\mathbf{H} \quad \textit{in } \mathbb{R}^3, \tag{5.1}$$

$$\mathbf{curl}\,\mathbf{H} = \sigma\mathbf{E} + \mathbf{J}_0 \quad \textit{in } \mathbb{R}^3, \tag{5.2}$$

$$\operatorname{div}\mathbf{E} = 0 \quad \textit{in } \Omega_E, \tag{5.3}$$

$$\int_{\Gamma_i} \mathbf{E}\cdot\mathbf{n} \;=\; 0 \quad \forall\,\Gamma_i,\; i = 1,\ldots,N_C, \tag{5.4}$$

$$[\mathbf{E}\times\mathbf{n}]_\Gamma = [\mathbf{H}\times\mathbf{n}]_\Gamma = 0 \quad \textit{on } \Gamma = \bar{\Omega}\cap\bar{\Omega}_E, \tag{5.5}$$

$$\mathbf{E}(\mathbf{x}) = \mathcal{O}\left(|\mathbf{x}|^{-1}\right),\; \mathbf{H}(\mathbf{x}) = \mathcal{O}\left(|\mathbf{x}|^{-1}\right) \quad \textit{for } |\mathbf{x}| \to \infty. \tag{5.6}$$

Here, N_C denotes the number of the finitely many connected components of Γ. We remark that (5.4) is only necessary if Ω is not simply connected.

In Ω_E (5.2) becomes $\mathbf{curl}\,\mathbf{H} = 0$. Therefore, $\mathbf{E}$ cannot be uniquely determined there and requires the further gauging condition $\operatorname{div}\mathbf{E} = 0$, known as Coulomb gauge. The transmission conditions (5.5) result from requiring $\mathbf{E}$, $\mathbf{H} \in \mathbf{L}^2_{loc}(\mathbb{R}^3)$ and the radiation conditions (5.6) follow from the Silver Müller conditions ((3.20) and (3.21)), see Colton & Kress [38, (6.19)].

Setting $\mathbf{u} := \mathbf{E}$, we observe from (5.1), (5.2) and (5.3) that $\operatorname{div}\mathbf{u} = 0$ and $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = 0$ in the exterior domain Ω_E yielding $\mathbf{\Delta u} = \mathbf{grad}\operatorname{div}\mathbf{u} - \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = 0$.

Therefore, $\mathbf{u}$ is given in Ω_E by the Stratton-Chu formula, see (3.25),

$$\begin{aligned}\mathbf{u}(\mathbf{x}) = &+ \mathbf{curl}_\mathbf{x}\int_\Gamma(\mathbf{n}\times\mathbf{u})(\mathbf{y})\Phi(\mathbf{x},\mathbf{y})\,ds(\mathbf{y})\\ &+\int_\Gamma(\mathbf{n}\times\mathbf{curl}\,\mathbf{u})(\mathbf{y})\Phi(\mathbf{x},\mathbf{y})\,ds(\mathbf{y})\\ &-\mathbf{grad}_\mathbf{x}\int_\Gamma(\mathbf{n}\cdot\mathbf{u})(\mathbf{y})\Phi(\mathbf{x},\mathbf{y})\,ds(\mathbf{y}),\quad \mathbf{x}\in\Omega_E,\end{aligned}$$

with Laplace kernel $\Phi(\mathbf{x},\mathbf{y}) = \dfrac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$.
When the point $\mathbf{x}$ moves to Γ from Ω_E we obtain jump relations

$$\gamma_D^+\mathbf{u} = \mathcal{K}(\gamma_D^+\mathbf{u}) - \mathcal{V}(\gamma_N^+\mathbf{u}) - \gamma_D^+\,\mathbf{grad}_\mathbf{x}\int_\Gamma(\mathbf{n}\cdot\mathbf{u})\Phi(\mathbf{x},\mathbf{y})\,ds(\mathbf{y}), \tag{5.7}$$

$$\gamma_N^+\mathbf{u} = \mathcal{W}(\gamma_D^+\mathbf{u}) - \tilde{\mathcal{K}}(\gamma_N^+\mathbf{u}). \tag{5.8}$$

here + denotes the limit from Ω_E. The appearing boundary integral operators $\mathcal{V}$, $\mathcal{K}$, $\tilde{\mathcal{K}}$ and $\mathcal{W}$ are defined for $\mathbf{x}\in\Gamma$ in Chapter 3.

In the interior domain Ω we obtain by setting $\mathbf{u} := \mathbf{E}$ in (5.1) and (5.2) and integrating by parts for suitable $\mathbf{v}$

$$\int_\Omega \mu^{-1}\,\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\,d\mathbf{x} + \int_\Omega i\omega\sigma\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x} - \langle\gamma_N^-\mathbf{u},\gamma_D^-\mathbf{v}\rangle = -\int_\Omega i\omega\mathbf{J}_0\cdot\mathbf{v}\,d\mathbf{x} \tag{5.9}$$

where γ_D^-, γ_N^- are the traces on Γ from Ω. Next we use the interface conditions (5.5), i.e. $[\gamma_N\mathbf{u}] = [\gamma_D\mathbf{u}] = 0$ on Γ. Inserting (5.8) into (5.9) and adding the weak form of (5.7) we obtain with $\mathbf{u} := \mathbf{E}_{|\Omega}$, $\boldsymbol{\lambda} := \mathbf{n}\times\mathbf{curl}\,\mathbf{E}_{|\Gamma}$ the **coupling formulation**:
Find $\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega)$, $\boldsymbol{\lambda}\in\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma 0,\Gamma)$ *such that*

$$\begin{aligned}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_\Omega + i\omega(\sigma\mathbf{u},\mathbf{v})_\Omega - \langle\mathcal{W}\gamma_D\mathbf{u},\gamma_D\mathbf{v}\rangle_\Gamma + \langle\tilde{\mathcal{K}}\boldsymbol{\lambda},\gamma_D\mathbf{v}\rangle_\Gamma &= -i\omega(\mathbf{J}_0,\mathbf{v})_\Omega,\\ \langle(I-\mathcal{K})\gamma_D\mathbf{u},\boldsymbol{\zeta}\rangle_\Gamma + \langle\mathcal{V}\boldsymbol{\lambda},\boldsymbol{\zeta}\rangle_\Gamma &= 0\end{aligned} \tag{5.10}$$

for all $\mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega)$, $\boldsymbol{\zeta}\in\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma 0,\Gamma)$.

We abbreviate (5.10) by

$$\mathcal{A}(\mathbf{u},\boldsymbol{\lambda};\mathbf{v},\boldsymbol{\zeta}) = \mathcal{L}(\mathbf{v},\boldsymbol{\zeta}) \tag{5.11}$$

and note that $\mathcal{A}$ is continuous and elliptic, in the sense that there holds

$$|\mathcal{A}(\mathbf{u},\boldsymbol{\lambda};\mathbf{u},\boldsymbol{\lambda})| \geq C\big(\|\mathbf{u}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\boldsymbol{\lambda}\|^2_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\big)$$

for all $\mathbf{u} \in \mathbf{H}(\mathbf{curl},\Omega)$ and $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0,\Gamma)$, see Hiptmair [66]. Hence, (5.11) has a unique solution.

5.2 A FEM/BEM coupling formulation

Next, we introduce the *hp*-version of the finite element / boundary element coupling procedure for the formulation (5.10). We define a regular mesh $\mathcal{T}_h$ (with tetrahedral elements) on Ω, inducing a mesh $\mathcal{K}_h$ on Γ. Let $\mathcal{X}_{h,p}(\mathcal{T}_h)$ and $\mathcal{Y}_{h,p}(\mathcal{K}_h)$ denote suitable finite element and boundary element spaces of piecewise polynomials of degree p on the meshes $\mathcal{T}_h$ and $\mathcal{K}_h$. The **hp-version** of the **Galerkin method** reads:
Find $\mathbf{u}_{h,p} \in \mathcal{X}_{h,p}(\mathcal{T}_h) \subset \mathbf{H}(\mathbf{curl},\Omega)$, $\boldsymbol{\lambda}_{h,p} \in \mathcal{Y}_{h,p}(\mathcal{K}_h) \subset \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0,\Gamma)$ *such that*

$$\begin{aligned}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p},\mathbf{curl}\,\mathbf{v})_\Omega + i\omega(\sigma\mathbf{u}_{h,p},\mathbf{v})_\Omega - \langle\mathcal{W}\gamma_D\mathbf{u}_{h,p},\mathbf{v}_\Gamma\rangle_\Gamma + \langle\tilde{\mathcal{K}}\boldsymbol{\lambda}_{h,p},\mathbf{v}_\Gamma\rangle_\Gamma &= -i\omega(\mathbf{J}_0,\mathbf{v})_\Omega,\\ \langle(I-\mathcal{K})\gamma_D\mathbf{u}_{h,p},\boldsymbol{\zeta}\rangle_\Gamma + \langle\mathcal{V}\boldsymbol{\lambda}_{h,p},\boldsymbol{\zeta}\rangle_\Gamma &= 0\end{aligned} \tag{5.12}$$

for all $\mathbf{v} \in \mathcal{X}_{h,p}(\mathcal{T}_h), \boldsymbol{\zeta} \in \mathcal{Y}_{h,p}(\mathcal{K}_h)$.

Let $(\mathbf{u},\boldsymbol{\lambda})$ and $(\mathbf{u}_{h,p},\boldsymbol{\lambda}_{h,p})$ denote the solutions of (5.10) and (5.12), respectively. Then, there holds for the Galerkin error

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_{h,p}\|_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}_{h,p}\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\\ &\qquad\leq C\inf\left\{\|\mathbf{u}-\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\boldsymbol{\lambda}-\boldsymbol{\zeta}\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\right\}\end{aligned} \tag{5.13}$$

where the infimum is taken over all finite elements $\mathbf{v} \in \mathcal{X}_{h,p}(\mathcal{T}_h)$ and boundary elements $\boldsymbol{\zeta} \in \mathcal{Y}_{h,p}(\mathcal{K}_h)$ and C is a positive constant, independent of $\mathbf{u}$, $\boldsymbol{\lambda}$, h and p. The estimate (5.13) follows from the ellipticity and continuity of the form $\mathcal{A}$ in combination with the analysis of conforming Galerkin schemes for general strongly elliptic systems (see Stephan & Wendland [97] and MacCamy & Stephan [71]).

5.3 A residual error estimator for the hp-version

In the following we give **a priori** estimates and **a posteriori** estimates for the Galerkin error. These estimates can be derived by using approximation properties of projection-based interpolation operators $\tilde{\Pi}_p^1$ and $\Pi_{p,\Gamma}^1$ on $\mathcal{X}_{h,p}(\mathcal{T}_h)$ and $\mathcal{Y}_{h,p}(\mathcal{K}_h)$, defined in (4.25) and (4.30). Inserting (4.32) and (4.33) into (5.13) yields the **a priori estimate**

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_{h,p}\|_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}_{h,p}\|_{\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma,\Gamma)} \\ \leq C h^k p_{\min}^{-(r-\epsilon)} \big(\|\mathbf{u}\|_{\mathbf{H}^r(\Omega)} + \|\boldsymbol{\lambda}\|_{\mathbf{H}_\parallel^{r-1/2}(\operatorname{div}_\Gamma,\Gamma)}\big)\end{aligned}$$

with a positive constant C, independent of $\mathbf{u}$, $\boldsymbol{\lambda}$, h and p, and arbitrary $\epsilon > 0$, where $k = \min\{r, p_{\min}+1\}$.

In order to derive the a posteriori error estimate we define the set of faces $\mathcal{F}_h$ of $\mathcal{T}_h$, the set of exterior faces $\mathcal{F}_h^\Gamma := \{F \in \mathcal{F}_h : F \subset \Gamma\}$ and the set of interior faces $\mathcal{F}_h^C := \mathcal{F}_h \setminus \mathcal{F}_h^\Gamma$ and $\mathcal{F}_h(T)$ as the set of faces of the element $T \in \mathcal{T}_h$. We further use h_T to denote the maximal diameter of an element $T \in \mathcal{T}_h$ and h_F for the maximal diameter of a face $F \in \mathcal{F}_h$. We assume that the mesh is regular, i.e. there holds

$$\begin{aligned} h_{T'} &\lesssim h_T \quad \forall\, T, T' \in \mathcal{T}_h,\ T \cap T' \neq \emptyset, \\ h_F &\lesssim h_T \quad \forall\, F \in \mathcal{F}_h(T). \end{aligned}$$

Next, we define the jumps. For $F \in \mathcal{F}_h^C$ a common face of two elements T_1, T_2 and the normal $\mathbf{n}$ pointing into T_2 we define the jump by

$$[\mathbf{n}\cdot\mathbf{q}]_F := \mathbf{n}\cdot\mathbf{q}_{|F\subset T_1} - \mathbf{n}\cdot\mathbf{q}_{|F\subset T_2}.$$

For $F \in \mathcal{F}_h^\Gamma$ we define

$$[\mathbf{n}\cdot\mathbf{q}]_F := \mathbf{n}\cdot\mathbf{q}_{|F}.$$

Analogously, we define the jumps

$$\begin{aligned} [\mathbf{n}\times\mathbf{q}]_F &:= \mathbf{n}\times\mathbf{q}_{|F\subset T_1} - \mathbf{n}\times\mathbf{q}_{|F\subset T_2}, && F \in \mathcal{F}_h^C, \\ [\mathbf{n}\times\mathbf{q}]_F &:= \mathbf{n}\times\mathbf{q}_{|F}, && F \in \mathcal{F}_h^\Gamma. \end{aligned}$$

Finally, the energy norms are given by

$$\|\mathbf{v}\|_{\mathfrak{E}}^2 := (\mu^{-1}\,\mathbf{curl}\,\mathbf{v}, \mathbf{curl}\,\mathbf{v})_\Omega + \omega(\sigma\mathbf{v}, \mathbf{v})_\Omega \simeq \|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2, \tag{5.14}$$

$$\|\boldsymbol{\zeta}\|_{\mathfrak{e}}^2 := \langle \mathcal{V}\boldsymbol{\zeta}, \boldsymbol{\zeta}\rangle_\Gamma \simeq \|\boldsymbol{\zeta}\|_{H^{-1/2}(\Gamma)}^2 \tag{5.15}$$

on $\mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma 0, \Gamma)$, resp., corresponding to the variational formulation.

As for the pure h-version (see Teltscher *et al.* [106] and Stephan & Maischak [99]), we now obtain for the hp-version of the coupling (5.10) the following reliable **a posteriori error estimate** with local error indicators of residual type:

Theorem 5.3.1 *Let* $(\mathbf{u}, \boldsymbol{\lambda})$ *denote the solution of the coupling formulation* (5.10) *and* $(\mathbf{u}_{h,p}, \boldsymbol{\lambda}_{h,p})$ *the solution of the Galerkin system* (5.12)*. Then, there holds*

$$\|\mathbf{u} - \mathbf{u}_{h,p}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{h,p}\|^2_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \lesssim (\eta_0^{\mathcal{T}})^2 + (\eta_1^{\mathcal{T}})^2 + (\eta_0^{\mathcal{F},C})^2 + (\eta_1^{\mathcal{F},C})^2 + (\eta_0^{\mathcal{F},\Gamma})^2 + (\eta_1^{\mathcal{F},\Gamma})^2 + (\eta_2^{\mathcal{F},\Gamma})^2$$

with $(j = 0, 1)$, $(k = 0,\ 1,\ 2)$

$$\left(\eta_j^{\mathcal{T}}\right)^2 := \sum_{T\in\mathcal{T}_h} (\eta_j^T)^2, \quad \left(\eta_j^{\mathcal{F},C}\right)^2 := \sum_{F\in\mathcal{F}_h^C} (\eta_j^{F,C})^2, \quad \left(\eta_k^{\mathcal{F},\Gamma}\right)^2 := \sum_{F\in\mathcal{F}_h^\Gamma} (\eta_k^{F,\Gamma})^2$$

and local error indicators (arbitrary $\epsilon > 0$*)*

$$\begin{aligned}
\eta_0^T &:= p^{-1+\epsilon}\, h_T\, \sqrt{\omega}\, \|\sqrt{\sigma}^{-1}(\mathrm{div}\,\mathbf{J}_0 + \mathrm{div}\,\sigma\mathbf{u}_{h,p})\|_{0,T},\\
\eta_1^T &:= p^{-1+\epsilon}\, h_T\, \|\sqrt{\mu}(i\omega\mathbf{J}_0 + i\omega\sigma\mathbf{u}_{h,p} + \mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p})\|_{0,T},\\
\eta_0^{F,C} &:= p^{-1/2+\epsilon/2}\, h_F^{1/2}\, \sqrt{\omega}\, \|\sqrt{\sigma_A}^{-1}[\sigma\,\mathbf{u}_{h,p}\cdot\mathbf{n}]_F\|_{0,F},\\
\eta_1^{F,C} &:= p^{-1/2+\epsilon/2}\, h_F^{1/2}\, \|\sqrt{\mu_A}[\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}\times\mathbf{n}]_F\|_{0,F},\\
\eta_0^{F,\Gamma} &:= p^{-1/2+\epsilon/2}\, h_F^{1/2}\, \sqrt{\omega}\, \|\sqrt{\sigma}\,\mathbf{u}_{h,p}\cdot\mathbf{n}\|_{0,F},\\
\eta_1^{F,\Gamma} &:= p^{-1/2+\epsilon/2}\, h_F^{1/2}\, \|\sqrt{\mu}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}\times\mathbf{n} - \mathcal{W}\gamma_D\mathbf{u}_{h,p} + \tilde{\mathcal{K}}\boldsymbol{\lambda}_{h,p})\|_{0,F},\\
\eta_2^{F,\Gamma} &:= p^{-1/2+\epsilon/2}\, h_F^{1/2}\, \|\mathrm{curl}_\Gamma(\mathcal{I} - \mathcal{K})\gamma_D\mathbf{u}_{h,p} + \mathcal{V}\boldsymbol{\lambda}_{h,p}\|_{0,F}.
\end{aligned}$$

Here σ_A *and* μ_A *denote the average of* σ *and* μ *on a face* F.

Proof. First of all, let us abbreviate the **Galerkin system** (5.12) by:
Find $(\mathbf{u}_{h,p}, \boldsymbol{\lambda}_{h,p}) \in \mathcal{X}_{h,p}(\mathcal{T}_h) \times \mathcal{Y}_{h,p}(\mathcal{K}_h)$ *such that*

$$\mathcal{A}(\mathbf{u}_{h,p}, \boldsymbol{\lambda}_{h,p}; \mathbf{v}, \boldsymbol{\zeta}) = \mathcal{L}(\mathbf{v}, \boldsymbol{\zeta}) \tag{5.16}$$

for all $(\mathbf{v}, \boldsymbol{\zeta}) \in \mathcal{X}_{h,p}(\mathcal{T}_h) \times \mathcal{Y}_{h,p}(\mathcal{K}_h)$.
Setting $\mathbf{e} := \mathbf{u} - \mathbf{u}_{h,p}$, $\boldsymbol{\varepsilon} := \boldsymbol{\lambda} - \boldsymbol{\lambda}_{h,p}$, we derive for arbitrary $(\mathbf{e}_{h,p}, \boldsymbol{\varepsilon}_{h,p}) \in \mathcal{X}_{h,p}(\mathcal{T}_h) \times \mathcal{Y}_{h,p}(\mathcal{K}_h)$ that

$$\begin{aligned}
&\|\mathbf{e}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\boldsymbol{\varepsilon}\|^2_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \lesssim |\mathcal{A}(\mathbf{e}, \boldsymbol{\varepsilon}; \mathbf{e}, \boldsymbol{\varepsilon})|\\
&= |\mathcal{L}(\mathbf{e}, \boldsymbol{\varepsilon}) - \mathcal{A}(\mathbf{u}_{h,p}, \boldsymbol{\lambda}_{h,p}; \mathbf{e}, \boldsymbol{\varepsilon})|\\
&= |\mathcal{L}(\mathbf{e} - \mathbf{e}_{h,p}, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{h,p}) - \mathcal{A}(\mathbf{u}_{h,p}, \boldsymbol{\lambda}_{h,p}; \mathbf{e} - \mathbf{e}_{h,p}, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{h,p})|\\
&= |-i\omega(\mathbf{J}_0 + \sigma\mathbf{u}_{h,p}, \mathbf{e} - \mathbf{e}_{h,p})_\Omega - (\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}, \mathbf{curl}(\mathbf{e} - \mathbf{e}_{h,p}))_\Omega\\
&\quad + \langle \mathcal{W}\gamma_D\mathbf{u}_{h,p} - \tilde{\mathcal{K}}\boldsymbol{\lambda}_{h,p}, \mathbf{e} - \mathbf{e}_{h,p}\rangle_\Gamma + \langle (\mathcal{K} - I)\gamma_D\mathbf{u}_{h,p} - \mathcal{V}\boldsymbol{\lambda}_{h,p}, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{h,p}\rangle_\Gamma|\\
&=: |\mathcal{R}(\mathbf{e} - \mathbf{e}_{h,p},\ \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{h,p})|.
\end{aligned} \tag{5.17}$$

Next, we assume Ω to be convex and use the Helmholtz decomposition

$$\mathbf{H}(\mathbf{curl}, \Omega) = \mathbf{M}(\Omega) \oplus \mathbf{grad}\, H^1(\Omega)/\mathbb{C}$$

with $\mathbf{M}(\Omega) = \mathbf{H}_0(\operatorname{div} 0, \Omega) \cap \mathbf{H}(\mathbf{curl}, \Omega)$. This follows from the $\mathbf{L}^2$-orthogonal decomposition

$$\mathbf{L}^2(\Omega) := \mathbf{H}_0(\operatorname{div} 0, \Omega) \oplus \mathbf{grad}\, H^1(\Omega)/\mathbb{C}$$

for connected Lipschitz domains, see Dautray and Lions [42, Chap. IX, §1, Prop. 1]. We split $\mathbf{e} \in \mathbf{H}(\mathbf{curl}, \Omega)$ as

$$\mathbf{e} = \mathbf{e}^\perp + \mathbf{grad}\, \psi \tag{5.18}$$

with $\mathbf{e}^\perp \in \mathbf{M}(\Omega)$ and $\psi \in H^1(\Omega)$ and there holds

$$\|\mathbf{e}^\perp\|_{\mathbf{H}^1(\Omega)} \lesssim \|\mathbf{curl}\, \mathbf{e}\|_{\mathbf{L}^2(\Omega)}, \tag{5.19}$$

$$\|\mathbf{grad}\, \psi\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{e}\|_{\mathbf{H}(\mathbf{curl},\Omega)}. \tag{5.20}$$

The first estimate is due to the fact that $\mathbf{M}(\Omega)$ is continuously embedded in $\mathbf{H}^1(\Omega)$, see Amrouche *et al.* [10, Theorem 2.17]. The second one follows with the definition of the $\mathbf{H}(\mathbf{curl}, \Omega)$-norm.
We then set

$$\mathbf{e}_{h,p} := \tilde{\Pi}^1_p \mathbf{e}^\perp + \mathbf{grad}\, \Pi_p \psi \in \mathcal{X}_{h,p}(\mathcal{T}_h).$$

Here $\tilde{\Pi}^1_p : \mathbf{H}^1(\Omega) \to \mathcal{Y}_{h,p}(\mathcal{T}_h)$ is the projection-based interpolation operator in (4.31) and $\Pi_p : H^1(\Omega) \to \mathcal{S}_p(\mathcal{T}_h)$ is the standard interpolation operator onto piecewise polynomials on $\mathcal{T}_h$ (cf. Schwab [96]).

There hold the following approximation properties, where D_T (D_F) denotes the set of elements containing at least one vertex of an element T (a face F) and D^1_T (D^1_F) denotes the set of elements containing at least one edge of T (F).

$$\|\mathbf{e}^\perp - \tilde{\Pi}^1_p \mathbf{e}^\perp\|_{\mathbf{L}_2(T)} \lesssim h_T p^{-1+\epsilon} |\mathbf{e}^\perp|_{\mathbf{H}^1(D^1_T)}, \tag{5.21}$$

$$\|\mathbf{e}^\perp - \tilde{\Pi}^1_p \mathbf{e}^\perp\|_{\mathbf{L}_2(F)} \lesssim h_F^{1/2} p^{-1/2+\epsilon/2}\, |\mathbf{e}^\perp|_{\mathbf{H}^1(D^1_F)}, \tag{5.22}$$

$$\|\psi - \Pi_p \psi\|_{L_2(T)} \lesssim h_T p^{-1+\epsilon} |\psi|_{H^1(D_T)}, \tag{5.23}$$

$$\|\psi - \Pi_p \psi\|_{L_2(F)} \lesssim h_F^{1/2} p^{-1/2+\epsilon/2}\, |\psi|_{H^1(D_F)}, \tag{5.24}$$

$$\|\phi - \pi_p \phi\|_{L_2(F)} \lesssim h_F^{1/2} p^{-1/2+\epsilon/2} \|\mathbf{curl}_\Gamma\, \phi\|_{\mathbf{H}_\parallel^{-1/2}(\Gamma)}. \tag{5.25}$$

The first two estimates are due to the properties of $\tilde{\Pi}^1_p$, compare Theorem 4.8.6. (5.23) and (5.24) can be found in [96]. And the last is proven in the following Lemma, see Teltscher *et al.* [106].

Lemma 5.3.2 *There exists a linear operator $\pi_p : H^{1/2}(\Gamma) \to \mathcal{S}_p(\mathcal{K}_h)$, such that for all $\phi \in H^{1/2}(\Gamma)$ there holds*

$$\|\phi - \pi_p \phi\|_{L^2(F)} \lesssim h_F^{1/2}\, p^{-1/2+\varepsilon/2}\, |v|_{H^1(D_F)}$$

for $v \in H^1(\Omega)$ with $\mathbf{grad}_\Gamma\, v = \mathbf{grad}_\Gamma\, \phi$ and

$$\|\mathbf{grad}\, v\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{curl}_\Gamma\, \phi\|_{\mathbf{H}^{-1/2}(\Gamma)}.$$

The constant in the estimate depends only the regularity of the mesh.

Proof. First of all, we show that $\gamma_t^\times : \mathbf{grad}\, H^1(\Omega) \to \mathbf{curl}_\Gamma H^{1/2}(\Gamma)$ is surjective and continuous. For $\phi \in H^{1/2}(\Gamma)$ there exists a continuous extension $w \in H^1(\Omega)$ of ϕ. As $\gamma_t^\times \mathbf{grad}\, w = \mathbf{curl}_\Gamma w_{|\Gamma} = \mathbf{curl}_\Gamma \phi$, the mapping is surjective. According to Lemma 3.1.3, $\gamma_t^\times : \mathbf{H}(\mathbf{curl}, \Omega_C) \to \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is also continuous and there holds $\mathbf{grad}\, H^1(\Omega) \subset \mathbf{H}(\mathbf{curl}, \Omega_C)$ and $\mathbf{curl}_\Gamma H^{1/2}(\Gamma) \subset \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. Due to the open mapping theorem there exists for every $\phi \in H^{1/2}(\Gamma)$ a $v_\phi \in H^1(\Omega)$ with

$$\|\mathbf{grad}\, v_\phi\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{curl}_\Gamma \phi\|_{\mathbf{H}^{-1/2}(\Gamma)},$$

and

$$\gamma_t^\times \mathbf{grad}\, v_\phi = \mathbf{curl}_\Gamma v_\phi = \mathbf{curl}_\Gamma \phi. \tag{5.26}$$

From (5.26) there follows $\mathbf{grad}_\Gamma(\phi - v_\phi) = 0$, such that $\phi - v_\phi = c_{\phi,F}$ on Γ with a constant $c_{\phi,F} \in \mathbb{C}$ on the face F.

Now we define the operator $\pi_p\phi := (\Pi_p v_\phi)_{|F} + c_{\phi,F}$ with Π_p as defined above. There holds with (5.24):

$$\|\phi - \pi_p\phi\|_{L^2(F)} = \|v_\phi + c_{\phi,F} - \Pi_p(v_\phi + c_{\phi,F})\|_{L^2(F)} \lesssim p^{-1/2+\varepsilon/2}\, h_F^{1/2} \|\mathbf{grad}\, v_\phi\|_{\mathbf{L}^2(D_F)}.$$

□

Since $\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma) = \mathbf{curl}_\Gamma H^{1/2}(\Gamma)/\mathbb{C}$, we set $\boldsymbol{\varepsilon} = \mathbf{curl}_\Gamma \phi$ with $\phi \in H^{1/2}(\Gamma)$. We take

$$\boldsymbol{\varepsilon}_{h,p} := \mathbf{curl}_\Gamma \pi_p\phi \in \mathbf{curl}_\Gamma S_{h,p}(\mathcal{K}_h),$$

where $S_{h,p}(\mathcal{K}_h)$ denotes the space of continuous, piecewise polynomials of degree p on $\mathcal{K}_h$, and $\pi_p : H^{1/2}(\Gamma) \to \mathcal{S}_p(\mathcal{K}_h)$ is the standard nodal interpolation on $\mathcal{K}_h$ [96].

With (5.18) and the above definitions of $\mathbf{e}_{h,p}$ and $\boldsymbol{\varepsilon}_{h,p}$, we obtain in place of (5.17) the residual estimate

$$\begin{aligned}
\|(\mathbf{u}-\mathbf{u}_{h,p}, \boldsymbol{\lambda} - \boldsymbol{\lambda}_{h,p})\|^2 &:= \|\mathbf{e}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\boldsymbol{\varepsilon}\|^2_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \\
&\lesssim |\mathcal{R}(\mathbf{e} - \mathbf{e}_{h,p}, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{h,p})| \\
&\leq |\mathcal{R}(\mathbf{e}^\perp - \tilde{\Pi}_p^1 \mathbf{e}^\perp, 0)| \\
&\quad + |\mathcal{R}(\mathbf{grad}\, \psi - \mathbf{grad}\, \Pi_p\psi, 0)| \\
&\quad + |\mathcal{R}(0, \mathbf{curl}_\Gamma \phi - \mathbf{curl}_\Gamma \pi_p\phi)| \\
&= \Big| -i\omega(\mathbf{J}_0 + \sigma \mathbf{u}_{h,p}, \mathbf{e}^\perp - \tilde{\Pi}_p^1 \mathbf{e}^\perp)_\Omega - (\mu^{-1} \mathbf{curl}\, \mathbf{u}_{h,p}, \mathbf{curl}(\mathbf{e}^\perp - \tilde{\Pi}_p^1 \mathbf{e}^\perp))_\Omega \\
&\quad + \langle \mathcal{W}\gamma_D \mathbf{u}_{h,p} - \tilde{\mathcal{K}}\boldsymbol{\lambda}_{h,p}, \gamma_D \mathbf{e}^\perp - \gamma_D \tilde{\Pi}_p^1 \mathbf{e}^\perp \rangle_\Gamma \Big| \\
&\quad + \Big| -i\omega(\mathbf{J}_0 + \sigma \mathbf{u}_{h,p}, \mathbf{grad}(\psi - \Pi_p\psi))_\Omega + \langle \mathcal{W}\gamma_D \mathbf{u}_{h,p} - \tilde{\mathcal{K}}\boldsymbol{\lambda}_{h,p}, \mathbf{grad}_\Gamma(\psi - \Pi_p\psi)\rangle_\Gamma \Big| \\
&\quad + |\langle (\mathcal{K} - I)\gamma_D \mathbf{u}_{h,p} - \mathcal{V}\boldsymbol{\lambda}_{h,p}, \mathbf{curl}_\Gamma(\phi - \pi_p\phi)\rangle_\Gamma| \,.
\end{aligned} \tag{5.27}$$

Next, we have to perform partial integrations in (5.27).

We start with the term $(\mathbf{J}_0 + \sigma \mathbf{u}_{h,p}, \mathbf{grad}\,\psi - \mathbf{grad}\,\Pi_p\psi)_\Omega$ and use the Green's formula (3.9) and observe that $\mathbf{u}_{h,p}$ is elementwise in $\mathbf{H}(\mathrm{div},\Omega)$. We then get

$$\begin{aligned}(\mathbf{J}_0 + \sigma \mathbf{u}_{h,p}, &\mathbf{grad}\,\psi - \mathbf{grad}\,\Pi_p\psi)_\Omega \\ &= -(\mathrm{div}\,\mathbf{J}_0 + \mathrm{div}\,\sigma\mathbf{u}_{h,p}, \psi - \Pi_p\psi)_\Omega + \sum_{F\in\mathcal{F}_h} \langle [\sigma \mathbf{u}_{h,p}\cdot \mathbf{n}]_F, \psi - \Pi_p\psi\rangle_F. \end{aligned} \tag{5.28}$$

Here we have interpreted the $H^{-1/2}(\partial T) - H^{1/2}(\partial T)$-duality as a $\mathbf{L}^2(\partial T)$-duality due to the regularity of $\mathbf{u}_{h,p}$. There appear no jumps of $\mathbf{J}_0\cdot\mathbf{n}$ over Γ due to the assumption that there is no flow of $\mathbf{J}_0$ through Γ.

Next, we consider the term $(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}, \mathbf{curl}(\mathbf{e}^\perp - \tilde{\Pi}_p^1\mathbf{e}^\perp))_\Omega$, for which we use the Green's formula (3.8). Since $\mathbf{u}_{h,p}$ is only elementwise in $\mathbf{H}(\mathbf{curl}\,\mathbf{curl},\Omega)$, we obtain

$$\begin{aligned}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}, \mathbf{curl}(\mathbf{e}^\perp - \tilde{\Pi}_p^1\mathbf{e}^\perp))_\Omega &= \sum_{T\in\mathcal{T}_h} (\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}, \mathbf{curl}(\mathbf{e}^\perp - \tilde{\Pi}_p^1\mathbf{e}^\perp))_T \\ &= \sum_{T\in\mathcal{T}_h}\Big((\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}), \mathbf{e}^\perp - \tilde{\Pi}_p^1\mathbf{e}^\perp)_T + \langle \mu^{-1}\gamma_N\mathbf{u}_{h,p}, \gamma_D\mathbf{e}^\perp - \gamma_D\tilde{\Pi}_p^1\mathbf{e}^\perp\rangle_{\partial T}\Big) \\ &= \sum_{T\in\mathcal{T}_h} (\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}), \mathbf{e}^\perp - \tilde{\Pi}_p^1\mathbf{e}^\perp)_\Omega + \sum_{F\in\mathcal{F}_h} \langle [\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}\times\mathbf{n}]_F, \gamma_D\mathbf{e}^\perp - \gamma_D\tilde{\Pi}_p^1\mathbf{e}^\perp\rangle_F.\end{aligned} \tag{5.29}$$

We have used the fact that the terms $\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}\times\mathbf{n}$ and $\gamma_D\mathbf{e}^\perp - \gamma_D\tilde{\Pi}_p^1\mathbf{e}^\perp$ are in $\mathbf{L}^2(\partial T)$ (since $\mathbf{u}_{h,p|T}$ is a polynomial and $\mathbf{e}^\perp, \tilde{\Pi}_p^1\mathbf{e}^\perp \in \mathbf{H}^1(T)$), such that we can consider the $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \partial T) - \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \partial T)$-duality $\langle\cdot,\cdot\rangle_{\partial T}$ as a $\mathbf{L}^2(\partial T)$-duality.

Next, we consider the terms with the boundary integral operators. In the beginning, we examine the term $\langle \mathcal{W}\gamma_D\mathbf{u}_{h,p} - \tilde{\mathcal{K}}\boldsymbol{\lambda}_{h,p}, \mathbf{grad}_\Gamma\,\psi - \mathbf{grad}_\Gamma\,\Pi_p\psi\rangle_\Gamma$, which constitutes a $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma) - \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$-duality pairing (the left hand side is in $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ due to Lemma 3.2.2, the right hand side is in $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ due to Lemma 3.1.1). We can use the integration by parts formula given in Buffa & Ciarlet [27] and obtain together with Lemma 3.2.5.

$$\begin{aligned}\langle \mathcal{W}\gamma_D\mathbf{u}_{h,p} - \tilde{\mathcal{K}}\boldsymbol{\lambda}_{h,p}, &\mathbf{grad}_\Gamma(\psi - \Pi_p\psi)\rangle_\Gamma \\ &= -\langle \mathrm{div}_\Gamma\,\mathcal{W}\gamma_D\mathbf{u}_{h,p} - \mathrm{div}_\Gamma\,\tilde{\mathcal{K}}\boldsymbol{\lambda}_{h,p}, \psi - \Pi_p\psi\rangle_\Gamma = 0.\end{aligned} \tag{5.30}$$

The last term from (5.27) to consider is $\langle (\mathcal{K} - I)\gamma_D\mathbf{u}_{h,p} - \mathcal{V}\boldsymbol{\lambda}_{h,p}, \mathbf{curl}_\Gamma\,\phi - \mathbf{curl}_\Gamma\,\pi_p\phi\rangle_\Gamma$, which is again a duality pairing between $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ and $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ (the left hand side is in $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ due to Lemma 3.2.2, the right hand side is in $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ due to Lemma 3.1.1). Using again the integration by parts formula [27] we obtain

$$\begin{aligned}\langle (\mathcal{K} - I)\gamma_D\mathbf{u}_{h,p} - \mathcal{V}\boldsymbol{\lambda}_{h,p}, &\mathbf{curl}_\Gamma(\phi - \pi_p\phi)\rangle_\Gamma \\ &= \langle \mathrm{curl}_\Gamma(\mathcal{K} - I)\gamma_D\mathbf{u}_{h,p} - \mathrm{curl}_\Gamma\,\mathcal{V}\boldsymbol{\lambda}_{h,p}, \phi - \pi_p\phi\rangle_\Gamma.\end{aligned} \tag{5.31}$$

We now use equations (5.28)–(5.31) to transform the estimate (5.27) and obtain

$$
\begin{aligned}
\|\mathbf{e}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}+\|\boldsymbol{\varepsilon}\|^2_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} &\lesssim \sum_{T\in\mathcal{T}_h} |(-i\omega\mathbf{J}_0 - i\omega\sigma\mathbf{u}_{h,p} - \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}), \mathbf{e}^\perp - \tilde{\Pi}^1_p\mathbf{e}^\perp)_T| \\
&+ \sum_{F\in\mathcal{F}_h^C} |\langle[\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}\times\mathbf{n}]_F, \gamma_D\mathbf{e}^\perp - \gamma_D\tilde{\Pi}^1_p\mathbf{e}^\perp\rangle_F| \\
&+ \sum_{F\in\mathcal{F}_h^\Gamma} |\langle\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}\times\mathbf{n} - \mathcal{W}\gamma_D\mathbf{u}_{h,p} + \tilde{\mathcal{K}}\boldsymbol{\lambda}_{h,p}, \gamma_D\mathbf{e}^\perp - \gamma_D\tilde{\Pi}^1_p\mathbf{e}^\perp\rangle_F| \\
&+ \sum_{T\in\mathcal{T}_h} |\omega\,\mathrm{div}\,\mathbf{J}_0 + \omega\,\mathrm{div}\,\sigma\mathbf{u}_{h,p}, \psi - \Pi_p\psi)_T| \\
&+ \sum_{F\in\mathcal{F}_h^C} |\omega\langle[\sigma\mathbf{u}_{h,p}\cdot\mathbf{n}]_F, \psi-\Pi_p\psi\rangle_F| + \sum_{F\in\mathcal{F}_h^\Gamma} |\omega\langle\sigma\mathbf{u}_{h,p}\cdot\mathbf{n}, \psi-\Pi_p\psi\rangle_F| \\
&+ \sum_{F\in\mathcal{F}_h^\Gamma} |\langle\mathrm{curl}_\Gamma(I-\mathcal{K})\gamma_D\mathbf{u}_{h,p} + \mathrm{curl}_\Gamma\,\mathcal{V}\boldsymbol{\lambda}_{h,p}, \phi - \pi_p\phi\rangle_F|.
\end{aligned}
$$

The Cauchy-Schwarz inequality (all scalar products are interpreted as L^2-products) and the approximation properties (5.21) – (5.25) then yield

$$
\begin{aligned}
&\|\mathbf{e}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\boldsymbol{\varepsilon}\|^2_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \\
&\lesssim \Bigg\{\Big(\sum_{T\in\mathcal{T}_h} p^{-2+2\epsilon}\,h_T^2\|\sqrt{\mu}\,(i\omega\mathbf{J}_0 + i\,\omega\sigma\mathbf{u}_{h,p} + \mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p})\|^2_{0,T}\Big)^{1/2} \\
&+ \Big(\sum_{F\in\mathcal{F}_h^C} p^{-1+\epsilon}h_F\|\sqrt{\mu_A}[\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}\times\mathbf{n}]_F\|^2_{0,F}\Big)^{1/2} \\
&+ \Big(\sum_{F\in\mathcal{F}_h^\Gamma} p^{-1+\epsilon}\,h_F\|\sqrt{\mu}^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}\times\mathbf{n} - \sqrt{\mu}\mathcal{W}\gamma_D\mathbf{u}_{h,p} + \sqrt{\mu}\tilde{\mathcal{K}}\boldsymbol{\lambda}_{h,p}\|^2_{0,F}\Big)^{1/2}\Bigg\}\,|\tfrac{1}{\sqrt{\mu}}\mathbf{e}^\perp|_{\mathbf{H}^1(\Omega)} \\
&+ \Bigg\{\Big(\sum_{T\in\mathcal{T}_h} p^{-2+2\epsilon}\,h_T^2\omega\|\sqrt{\sigma}^{-1}(\mathrm{div}\,\mathbf{J}_0 + \mathrm{div}\,\sigma\mathbf{u}_{h,p})\|^2_{0,T}\Big)^{1/2} \\
&+ \Big(\sum_{F\in\mathcal{F}_h^C} p^{-1+\epsilon}\,h_F\omega\|\sqrt{\sigma_A}^{-1}[\sigma\mathbf{u}_{h,p}\cdot\mathbf{n}]_F\|^2_{0,F}\Big)^{1/2} \\
&+ \Big(\sum_{F\in\mathcal{F}_h^\Gamma} p^{-1+\epsilon}\,h_F\omega\|\sqrt{\sigma}\mathbf{u}_{h,p}\cdot\mathbf{n}\|^2_{0,F}\Big)^{1/2}\Bigg\}\,\sqrt{\omega}\,\|\sqrt{\sigma}\;\mathbf{grad}\,\psi\|_{\mathbf{L}^2(\Omega)} \\
&+ \Big(\sum_{F\in\mathcal{F}_h^\Gamma} p^{-1+\epsilon}h_F\,\|\mathrm{curl}_\Gamma(I-\mathcal{K})\mathbf{u}_{h,p} + \mathrm{curl}_\Gamma\,\mathcal{V}\boldsymbol{\lambda}_{h,p}\|^2_{0,F}\Big)^{1/2}\|\mathbf{curl}_\Gamma\,\phi\|_{\mathbf{H}^{-1/2}(\Gamma)}.
\end{aligned}
$$

With σ,μ on Γ we always mean the interior σ,μ, i.e. the trace from Ω. Due to (5.20) there holds $\sqrt{\omega}\,\|\sqrt{\sigma}\;\mathbf{grad}\,\psi\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{e}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$ and $\|\mathbf{curl}_\Gamma\,\phi\|_{\mathbf{H}^{-1/2}(\Gamma)} \simeq \|\boldsymbol{\varepsilon}\|_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}$ (see Page 129). Furthermore, $|\frac{1}{\sqrt{\mu}}\mathbf{e}^\perp|_{\mathbf{H}^1(\Omega)}$ can be estimated from above by $\|\mathbf{e}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$ due to (5.19), and this concludes the proof of Theorem 5.3.1. □

Furthermore, we remark that the integral operators fulfill the following mapping properties, the proof can be found in Teltscher [103, Lemma 4.3.3.], see also Mitrea *et al.* [76, p. 16 and Theorem 5.1].

Lemma 5.3.3 *The mappings (with the Laplace kernel)*

$$\begin{aligned} \mathcal{W} : \mathcal{TND}_p(\mathcal{K}_h) &\to \mathbf{L}_t^2(\Gamma), \\ \operatorname{curl}_\Gamma \mathcal{K} : \mathcal{TND}_p(\mathcal{K}_h) &\to L^2(\Gamma), \\ \tilde{\mathcal{K}} : \mathbf{L}_t^2(\Gamma) &\to \mathbf{L}_t^2(\Gamma), \\ \operatorname{curl}_\Gamma \mathcal{V} : \mathbf{L}_t^2(\Gamma) &\to L^2(\Gamma) \end{aligned}$$

are continuous.

This guarantees that all the indicators are well defined.

5.3.1 A three-fold adaptive algorithm

The local error indicators can be used to steer a **three-fold adaptive algorithm**: Let *tol* denote the error tolerance and $0 < \theta < \delta < 1$.

1. Compute the Galerkin solution.
2. For each element $T \in \mathcal{T}_h$ calculate the local error indicators η_0^T, η_1^T, $\eta_0^{F,C}$, $\eta_1^{F,C}$, $\eta_0^{F,\Gamma}$, $\eta_1^{F,\Gamma}$, $\eta_2^{F,\Gamma}$ and
$$\eta^T := \left((\eta_0^T)^2 + (\eta_1^T)^2 + (\eta_0^{F,C})^2 + (\eta_1^{F,C})^2 + (\eta_0^{F,\Gamma})^2 + (\eta_1^{F,\Gamma})^2 + (\eta_2^{F,\Gamma})^2 \right)^{1/2}$$
and $\eta_{\max} := \max_{T\in\mathcal{T}_h} \eta^T$. Stop if $\eta_{\max} \le tol$.
3. If $\theta \cdot \eta_{\max} \le \eta^T < \delta \cdot \eta_{\max}$, increase the polynomial degree on element T by 1.
If $\delta \cdot \eta_{\max} \le \eta^T$, perform an h-refinement of element T.
Do nothing on element T if $\eta^T \le \theta \cdot \eta_{\max}$.
4. If necessary refine adjacent elements.

Numerical results for this algorithm are presented in Heuer [59] and Heuer *et al.* [60] for a hypersingular integral equation on a surface piece modeling a scalar screen problem in $\mathbb{R}^3$; the resulting *hp*-refinements give suitably refined meshes together with appropriate distributions of polynomial degrees. The corresponding implementation for the above eddy current problem can be performed with the program package *maiprogs* [75]; for the pure h-version corresponding numerical experiments are given in Section 5.6. The above *hp*-adaptive algorithm requires the implementation of hanging nodes for high order polynomials, see Chapter 4. The numerical experiments in Section 5.6 show the performance of the error indicators for the uniform h- and p-version.

5.4 Efficiency of the residual error estimator

In this section we prove the efficiency of the residual error estimator for the eddy current problem on quasi-uniform meshes on the boundary. We assume that there holds

$$1 \leq \frac{h_{\Gamma,\max}}{h_{\Gamma,\min}} \leq C$$

for a certain constant $C > 0$, independent of the mesh. Here, we solely examine the h-version for the lowest polynomial degree. The ideas of this proof can be found in the article of Beck *et al.* [15] for the FEM part. For the indicators with the boundary integral operators we use some ideas of Carstensen [34]. Furthermore, we use the spaces $\mathcal{ND}_1(\mathcal{T}_h)$ for the Galerkin approximation in $\mathbf{H}(\mathbf{curl},\Omega)$ and $\mathcal{RT}_k^0(\mathcal{K}_h) := \{\boldsymbol{\lambda} \in \mathcal{RT}_k(\mathcal{K}_h) : \operatorname{div}_\Gamma \boldsymbol{\lambda}_h = 0\}$ for the space $\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma 0,\Gamma)$.

We will bound the following error indicators. Here, T denotes a tetrahedron.

$$\begin{aligned}
\eta_0^T &:= h_T\sqrt{\omega}\|\sqrt{\sigma}^{-1}(\operatorname{div}\mathbf{J}_0 + \operatorname{div}\sigma\mathbf{u}_h)\|_{0,T},\\
\eta_1^T &:= h_T\|i\sqrt{\mu}\omega\mathbf{J}_0 + i\omega\sqrt{\mu}\sigma\mathbf{u}_h + \sqrt{\mu}\,\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h)\|_{0,T},\\
\eta_0^{F,C} &:= \sqrt{h_F}\sqrt{\omega}\|\sqrt{\sigma_A}^{-1}[\sigma\mathbf{u}_h\cdot\mathbf{n}]_F\|_{0,F},\\
\eta_1^{F,C} &:= \sqrt{h_F}\|\sqrt{\mu_A}[\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h\times\mathbf{n}]_F\|_{0,F},\\
\eta_0^{F,\Gamma} &:= \sqrt{h_F}\sqrt{\omega}\|\sqrt{\sigma}\mathbf{u}_h\cdot\mathbf{n}\|_{0,F},\\
\eta_1^{F,\Gamma} &:= \sqrt{h_F}\|\sqrt{\mu}^{-1}\,\mathbf{curl}\,\mathbf{u}_h\times\mathbf{n} - \sqrt{\mu}\mathcal{W}\gamma_D\mathbf{u}_h + \sqrt{\mu}\tilde{\mathcal{K}}\boldsymbol{\lambda}_h\|_{0,F},\\
\eta_2^{F,\Gamma} &:= \sqrt{h_F}\|\operatorname{curl}_\Gamma\mathbf{u}_h - \operatorname{curl}_\Gamma\mathcal{K}\gamma_D\mathbf{u}_h + \operatorname{curl}_\Gamma\mathcal{V}\boldsymbol{\lambda}_h\|_{0,F}.
\end{aligned}$$

The indicators η_0^T and η_1^T are bounded in Lemma 5.4.1, while in Lemma 5.4.2 we estimate the indicators for the jumps $\eta_0^{F,C}$ and $\eta_1^{F,C}$. Finally, the indicators on the boundary $\eta_0^{\mathcal{F},\Gamma}$, $\eta_1^{\mathcal{F},\Gamma}$ and $\eta_2^{\mathcal{F},\Gamma}$ are considered in Theorem 5.4.3.

The FEM-indicators in the interior

Here we consider the indicators η_0^T, η_1^T and estimate them locally from above by the energy norm of the error.
As in (5.18) we split the error $\mathbf{e} = \mathbf{e}^\perp + \mathbf{e}^0$ with $\mathbf{e}^0 := \mathbf{grad}\,\psi$.

Lemma 5.4.1 *Assuming that* $\operatorname{div}\mathbf{J}_0 = 0$ *and* σ *as piecewise constant, there is a constant* $C > 0$*, independent of the mesh size* h*, such that there holds*

$$\eta_0^T + \eta_1^T \leq C\left(\|\mathbf{e}^0\|_{\mathbf{L}^2(T)} + \|\mathbf{e}\|_{\mathfrak{E},T} + \eta_{1,2}^T\right)$$

with $\eta_{1,2}^T := h_T\|\sqrt{\mu}\omega(\mathbf{J}_0 - \Pi_1^h\mathbf{J}_0)\|_{\mathbf{L}^2(T)}$*, where* Π_1^h *is an interpolation operator into* $\mathcal{ND}_1(T)$*.*

Proof. First of all, we consider the estimator η_0^T and we assume that the current $\mathbf{J}_0$ is solenoidal, i.e. $\operatorname{div}\mathbf{J}_0 = 0$. There holds

$$\begin{aligned}\eta_0^T :=& h_T\sqrt{\omega}\|\sqrt{\sigma^{-1}}(\operatorname{div}\mathbf{J}_0 + \operatorname{div}\sigma\mathbf{u}_h)\|_{L^2(T)}\\ =& h_T\sqrt{\omega}\|\sqrt{\sigma^{-1}}\operatorname{div}\sigma\mathbf{u}_h\|_{L^2(T)}.\end{aligned}$$

On the tetrahedron T we define the bubble function

$$\lambda_T := 256\prod_{l=1}^{4}\lambda_{l,T}$$

with the barycentric coordinates $\lambda_{l,T}$ related to the vertex p_l of the tetrahedron T. Due to the scaling factor we get $0 \le \lambda_T \le 1$ and $\max_{x\in T}\lambda_T = 1$. There holds the following norm equivalence, see e.g. Verfürth [107],

$$\|\lambda_T^{1/2}\phi\|_{L^2(T)} \le \|\phi\|_{L^2(T)} \le C\|\lambda_T^{1/2}\phi\|_{L^2(T)} \quad \text{for all } \phi \in P_p(T). \tag{5.32}$$

Furthermore, we define the residual in the interior of the domain by

$$r(\mathbf{v}) := (\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v})_\Omega + i\omega(\sigma\mathbf{u}_h, \mathbf{v})_\Omega + i\omega(\mathbf{J}_0, \mathbf{v})_\Omega$$

for all $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$.

Using the norm equivalence (5.32) and partial integration, we get

$$\begin{aligned}\frac{(\eta_0^T)^2}{\omega h_T^2} &= \|\sqrt{\sigma^{-1}}\operatorname{div}\sigma\mathbf{u}_h\|^2_{L^2(T)}\\ &\le C\,\|\operatorname{div}\sigma\mathbf{u}_h\|^2_{L^2(T)}\\ &\le C\int_T(\operatorname{div}\sigma\mathbf{u}_h)^2\lambda_T\,d\mathbf{x}\\ &\le C\int_T\sigma\mathbf{u}_h\cdot\big(\mathbf{grad}(\lambda_T\operatorname{div}(\mathbf{J}_0+\sigma\mathbf{u}_h))\big)\,d\mathbf{x}\\ &= C\,|r\big(\mathbf{grad}(\lambda_T\operatorname{div}\sigma\mathbf{u}_h)\big)|\\ &\le C\,\big(\mathbf{e}^0, \mathbf{grad}(\lambda_T\operatorname{div}\sigma\mathbf{u}_h)\big)_\Omega\\ &\le C\,\|\mathbf{e}^0\|_{\mathbf{L}^2(T)}\,\|\mathbf{grad}(\lambda_T\operatorname{div}\sigma\mathbf{u}_h)\|_{\mathbf{L}^2(T)}.\end{aligned}$$

The second factor is estimated using an inverse inequality for polynomials

$$\|\mathbf{grad}(\lambda_T\operatorname{div}\sigma\mathbf{u}_h)\|_{\mathbf{L}^2(T)} \le Ch_T^{-1}\|\lambda_T\operatorname{div}\sigma\mathbf{u}_h\|_{\mathbf{L}^2(T)} \le C\,\frac{\eta_0^T}{h_T^2}.$$

Hence, we finally get

$$\eta_0^T \le C\|\mathbf{e}^0\|_{\mathbf{L}^2(T)}. \tag{5.33}$$

Next, we estimate the solenoidal part of the indicator. Therefore, we examine

$$\begin{aligned}\eta_1^T :=& h_T\|i\sqrt{\mu}\omega\mathbf{J}_0 + i\omega\sqrt{\mu}\sigma\mathbf{u}_h + \sqrt{\mu}\,\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h)\|_{\mathbf{L}^2(T)}\\ \le& C\,h_T\|i\sqrt{\mu}\omega\Pi_1^h\mathbf{J}_0 + i\omega\sqrt{\mu}\sigma\mathbf{u}_h + \sqrt{\mu}\,\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h)\|_{\mathbf{L}^2(T)}\\ &+ C\,h_T\|\sqrt{\mu}\omega(\mathbf{J}_0 - \Pi_1^h\mathbf{J}_0)\|_{\mathbf{L}^2(T)}\end{aligned}$$

with the projection operator Π_1^h into $\mathcal{ND}_1(\mathcal{T}_h)$. Moreover, we define

$$\eta_{1,1}^T := h_T \| i\sqrt{\mu}\omega\Pi_1^h \mathbf{J}_0 + i\omega\sqrt{\mu}\sigma\mathbf{u}_h + \sqrt{\mu}\,\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h)\|_{\mathbf{L}^2(T)}$$
$$\eta_{1,2}^T := h_T \| \sqrt{\mu}\omega(\mathbf{J}_0 - \Pi_1^h\mathbf{J}_0)\|_{\mathbf{L}^2(T)}$$

and we examine $\eta_{1,1}^T$. We abbreviate $\mathbf{j}_h := i\sqrt{\mu}\omega\Pi_1^h\mathbf{J}_0 + i\omega\sqrt{\mu}\sigma\mathbf{u}_h + \sqrt{\mu}\,\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h)$. As above we can estimate

$$\begin{aligned}\frac{(\eta_{1,1}^T)^2}{h_T^2} &\le C\int_T \big(i\sqrt{\mu}\omega\Pi_1^h\mathbf{J}_0 + i\omega\sqrt{\mu}\sigma\mathbf{u}_h + \sqrt{\mu}\,\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h)\big)\cdot\big(\mathbf{j}_h\lambda_T\big)\,d\mathbf{x}\\ &= C\left(r(\lambda_T\mathbf{j}_h) - \int_T \sqrt{\mu}\omega(\mathbf{J}_0 - \Pi_1^h\mathbf{J}_0)\cdot(\lambda_T\mathbf{j}_h)\,d\mathbf{x}\right)\\ &\le C\|\mathbf{e}\|_{\mathfrak{E},T}\,\|\lambda_T\mathbf{j}_h\|_{\mathfrak{E},T} + C\|\sqrt{\mu}\omega(\mathbf{J}_0 - \Pi_1^h\mathbf{J}_0)\|_{\mathbf{L}^2(T)}\,\|\lambda_T\mathbf{j}_h\|_{\mathbf{L}^2(T)}.\end{aligned}$$

where the energy norm $\|\cdot\|_{\mathfrak{E},T}$ is defined in (5.14). There holds due to an inverse inequality

$$\|\lambda_T\mathbf{j}_h\|_{\mathfrak{E},T} \le C\,h_T^{-1}\|\mathbf{j}_h\|_{\mathbf{L}^2(T)}$$

and we get

$$\begin{aligned}\frac{(\eta_{1,1}^T)^2}{h_T^2} &\le Ch_T^{-1}\|\mathbf{e}\|_{\mathfrak{E},T}\,\|\mathbf{j}_h\|_{\mathbf{L}^2(T)} + Ch_t^{-1}\eta_{1,2}^T\,\|\mathbf{j}_h\|_{\mathbf{L}^2(T)}\\ &= Ch_T^{-2}\,\|\mathbf{e}\|_{\mathfrak{E},T}\eta_{1,1}^T + Ch_T^{-2}\,\eta_{1,2}^T\end{aligned}$$

Finally,

$$\eta_1^T \le C\left(\eta_{1,1}^T + \eta_{1,2}^T\right) \le C\left(\|\mathbf{e}\|_{\mathfrak{E},T} + \eta_{1,2}^T\right). \tag{5.34}$$

The estimates in (5.33) and (5.34) complete the proof. □

The FEM-indicators for the jumps

Next, we estimate the indicators $\eta_0^{F,C}$ and $\eta_1^{F,C}$ related to the jumps on a face $F = \partial T_1 \cap \partial T_2$ which belongs to two adjacent tetrahedrons T_1 and T_2.

Lemma 5.4.2 *There is a constant $C > 0$, independent of the mesh size h, such that there holds*

$$\eta_0^{F,C} + \eta_1^{F,C} \le C\,\left(\|\mathbf{e}^0\|_{\mathbf{L}^2(T_1\cup T_2)} + \|\mathbf{e}\|_{\mathfrak{E},T_1} + \|\mathbf{e}\|_{\mathfrak{E},T_2} + \eta_{1,2}^{T_1} + \eta_{1,2}^{T_2}\right)$$

with $\eta_{1,2}^T$ defined in Lemma 5.4.1.

Proof. We define a bubble function λ_F on the face F by

$$\lambda_F := 27\prod_{l=1}^{3}\lambda_{l,F,T}$$

with the barycentric coordinates $\lambda_{l,F,T}$ with respect to the vertices of F. Due to the scaling factor we get $0 \leq \lambda_F \leq 1$ and $\max_{x\in F} \lambda_F = 1$. There holds the following norm equivalence, cf. (5.32),

$$\|\lambda_F^{1/2}\phi\|_{L^2(F)} \leq \|\phi\|_{L^2(F)} \leq C\|\lambda_F^{1/2}\phi\|_{L^2(F)} \qquad \text{for all } \phi \in P_p(F). \tag{5.35}$$

First of all, we consider the indicator

$$\eta_1^{F,C} := \sqrt{h_F}\|\sqrt{\mu_A}[\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h \times \mathbf{n}]_F\|_{0,F}$$

We extend the jump $[\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h \times \mathbf{n}]_F$ to a piecewise polynomial defined on T_1 and T_2 such that there holds

$$\|[\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h \times \mathbf{n}]_{F,T_i}\|_{\mathbf{L}^2(T_i)} \leq C h_{T_i}^{1/2}\,\|[\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h \times \mathbf{n}]_F\|_{\mathbf{L}^2(F)}, \quad i = 1,\, 2.$$

We define $\mathbf{j}_{h|T_i} := [\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h \times \mathbf{n}]_{F,T_i}$ and use (5.35) and Green's formula to get

$$\begin{aligned}
\frac{(\eta_1^{F,C})^2}{\mu_A\,h_F} &= \|[\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h \times \mathbf{n}]_F\|^2_{\mathbf{L}^2(F)} \\
&\leq C\int_F \big([\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h \times \mathbf{n}]_F\big) \cdot \big(\lambda_F \mathbf{j}_h\big)\,ds \\
&\leq C\int_{T_1\cup T_2} \big\{\big(-\,\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h\big)\cdot\big(\lambda_F\mathbf{j}_h\big) + \big(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h\big)\cdot\big(\mathbf{curl}\,\lambda_F\mathbf{j}_h\big)\big\}\,d\mathbf{x} \\
&= C\left\{r(\lambda_F\mathbf{j}_h) - \int_{T_1\cup T_2}\big(\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h + i\omega\sigma\mathbf{u}_h + i\omega\mathbf{J}_0\big)\cdot\big(\lambda_F\mathbf{j}_h\big)\,d\mathbf{x}\right\} \\
&\leq C\|\mathbf{e}\|_{\mathfrak{E},T_1\cup T_2}\|\lambda_F\mathbf{j}_h\|_{\mathfrak{E},T_1\cup T_2} + C\,\|\lambda_F\mathbf{j}_h\|_{\mathbf{L}^2(T_1\cup T_2)}\,\big(h_{T_1}^{-1}\,\eta_1^{T_1} + h_{T_2}^{-1}\,\eta_1^{T_2}\big) \\
&\leq C\,\big(h_{T_1}^{-1}(\|\mathbf{e}\|_{\mathfrak{E},T_1} + \eta_1^{T_1}) + h_{T_2}^{-1}(\|\mathbf{e}\|_{\mathfrak{E},T_2} + \eta_1^{T_2})\big)\,\eta_1^{F,C}.
\end{aligned}$$

Using (5.34), we finally get

$$\eta_1^{F,C} \leq C\,\big(\|\mathbf{e}\|_{\mathfrak{E},T_1} + \|\mathbf{e}\|_{\mathfrak{E},T_2} + \eta_{1,2}^{T_1} + \eta_{1,2}^{T_2}\big)\,. \tag{5.36}$$

Next, we estimate the indicator

$$\eta_0^{F,C} := \sqrt{h_F}\sqrt{\omega}\|\sqrt{\sigma_A}^{-1}[\sigma\mathbf{u}_h\cdot\mathbf{n}]_F\|_{L^2(F)}.$$

We extend $[\sigma\mathbf{u}_h \cdot \mathbf{n}]_F$ by a continuous piecewise polynomial function $[\sigma\mathbf{u}_h \cdot \mathbf{n}]_{F,T_i}$ onto $T_1 \cup T_2$ such that

$$\|[\mathbf{n}\cdot\sigma\mathbf{u}_h]_{F,T_i}\|_{L^2(T_i)} \leq C h_{T_i}^{1/2}\,\|[\mathbf{n}\cdot\sigma\mathbf{u}_h]_F\|_{L^2(F)}, \qquad i = 1,\, 2. \tag{5.37}$$

Using the norm equivalence (5.35) and partial integration we get

$$\begin{aligned}
\frac{(\eta_0^{F,C})^2}{\omega h_F} &= \|\sqrt{\sigma_A^{-1}}[\sigma \mathbf{u}_h \cdot \mathbf{n}]_F\|^2_{L^2(F)} \\
&\leq C \int_F \frac{1}{\sigma_A}[\sigma \mathbf{u}_h \cdot \mathbf{n}]_F^2 \lambda_F \, ds \\
&\leq \frac{C}{\sigma_A} \sum_{i=1}^2 \int_{T_i} \left\{ \sigma \mathbf{u}_h \cdot \big(\mathbf{grad}([\mathbf{n} \cdot \sigma \mathbf{u}_h]_{F,T_i} \lambda_F)\big) + (\operatorname{div} \sigma \mathbf{u}_h)[\mathbf{n} \cdot \sigma \mathbf{u}_h]_{F,T_i} \lambda_F \right\} d\mathbf{x} \\
&\leq \frac{C}{\sigma_A} \sum_{i=1}^2 \big(\|\sqrt{\sigma}\mathbf{e}^0\|_{\mathbf{L}^2(T_i)} \, \|\mathbf{grad}([\mathbf{n} \cdot \sigma \mathbf{u}_h]_{F,T_i} \lambda_F)\|_{\mathbf{L}^2(T_i)} \\
&\qquad\qquad + \|\operatorname{div} \sqrt{\sigma}\mathbf{u}_h\|_{L^2(T_i)} \|[\mathbf{n} \cdot \sigma \mathbf{u}_h]_{F,T_i}\|_{L^2(T_i)} \big) \\
&\leq \frac{C}{\sigma_A} \sum_{i=1}^2 \left(h_{T_i}^{-1/2} \|\sqrt{\sigma}\mathbf{e}^0\|_{\mathbf{L}^2(T_i)} + h_{T_i}^{1/2} \|\operatorname{div} \sqrt{\sigma}\mathbf{u}_h\|_{L^2(T_i)} \right) \frac{\eta_0^{F,C}}{h_F^{1/2}}.
\end{aligned}$$

The last steps are due to an inverse inequality and (5.37).
Finally, we get with (5.33)

$$\eta_0^{F,C} \leq C \sum_{i=1}^2 \|\sqrt{\sigma}\mathbf{e}^0\|_{\mathbf{L}^2(T_i)} + \eta_0^{T_i} \leq C \, \|\mathbf{e}^0\|_{\mathbf{L}^2(T_1 \cup T_2)}. \tag{5.38}$$

Combining (5.36) and (5.38) completes the proof. □

The indicators on the boundary

The main idea in this section is the use of the Poincaré-Steklov operator as used in Carstensen [34]. Due to the variational formulation of the eddy current problem (5.10) there holds

$$\langle (\mathcal{I} - \mathcal{K})\gamma_D \mathbf{u}, \boldsymbol{\zeta} \rangle_\Gamma + \langle \mathcal{V}\boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle_\Gamma = 0 \tag{5.39}$$

for all $\boldsymbol{\zeta} \in \mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma 0, \Gamma)$. As the single layer potential is invertible on the space $\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma 0, \Gamma)$ (cf. Lemma 3.2.3) it follows that

$$\boldsymbol{\lambda} = \mathcal{V}^{-1}(\mathcal{K} - \mathcal{I})\gamma_D \mathbf{u}.$$

Furthermore, we define the Poincaré-Steklov operator by

$$\mathcal{S} := -\mathcal{W} + \tilde{\mathcal{K}}\mathcal{V}^{-1}(\mathcal{K} - \mathcal{I}).$$

Thus, there holds

$$\mathcal{S} : \mathbf{H}_\perp^{-1/2}(\operatorname{curl}_\Gamma, \Gamma) \to \mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$$

and

$$\begin{aligned}\langle \mathcal{S}\gamma_D \mathbf{u}, \gamma_D \mathbf{v}\rangle_\Gamma &\leq \|\mathcal{S}\gamma_D \mathbf{u}\|_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \|\gamma_D \mathbf{v}\|_{\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} \\ &\leq C\,\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\Omega)}\,\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)}.\end{aligned}$$

Theorem 5.4.3 *For* $\mathbf{u} \in \mathbf{H}^{1/2+\delta}(\mathbf{curl},\Omega)$*,* $\delta > 0$*, and* $\boldsymbol{\lambda} \in \mathbf{H}_\|^0(\mathrm{div}_\Gamma,\Gamma)$ *there exits a constant* $C > 0$*, independent of the mesh size* h*, such that there holds*

$$\begin{aligned}\left(\eta_0^{\mathcal{F},\Gamma}\right)^2 + \left(\eta_1^{\mathcal{F},\Gamma}\right)^2 + \left(\eta_2^{\mathcal{F},\Gamma}\right)^2 \leq C\,\|\mathbf{u}-\mathbf{u}_h\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} &+ \|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|^2_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \\ &+ \sum_{T\in\mathcal{T}_\Gamma}\left(\|\mathbf{u}-\mathbf{u}_h\|_{\mathfrak{E},T} + \eta_{1,2}^T\right) \\ &+ h_F\|\mathbf{u}-\mathbf{u}_E\|^2_{\mathbf{H}^{1/2}(\mathbf{curl},\Omega)} + h_F^{1+2\delta}\|\mathbf{u}-\mathbf{u}_E\|^2_{\mathbf{H}^{1/2+\delta}(\mathbf{curl},\Omega)} \\ &+ h_F\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_E\|^2_{\mathbf{H}_\|^0(\mathrm{div}_\Gamma,\Gamma)} \\ &+ h_F\|\sqrt{\mu}\mathcal{S}\gamma_D\mathbf{u}\|^2_{\mathbf{L}^2(F)}\end{aligned}$$

with the interpolate $\mathbf{u}_E := \Pi_1^h\mathbf{u} \in \mathcal{ND}_1(\mathcal{T}_h)$ *and* $\boldsymbol{\lambda}_E \in \mathcal{RT}_1(\mathcal{K})$ *the orthogonal projection of* $\boldsymbol{\lambda}$ *with respect to the* $\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ *inner product. Furthermore,* $\mathcal{T}_\Gamma$ *denotes the set of elements which have at least one face on the boundary* Γ*.*

Proof. In the beginning, we examine $\eta_1^{F,\Gamma}$ on a face $F \subset \Gamma$. Here, we use $\boldsymbol{\lambda} = \mathcal{V}^{-1}(\mathcal{K}-\mathcal{I})\gamma_D\mathbf{u}$ and the Poincaré-Steklov operator to estimate

$$\begin{aligned}h_F\|\sqrt{\mu}^{-1}\,\mathbf{curl}\,\mathbf{u}_h\times\mathbf{n} - \sqrt{\mu}\mathcal{W}\gamma_D\mathbf{u}_h + \sqrt{\mu}\tilde{\mathcal{K}}\boldsymbol{\lambda}_h\|^2_{L^2(F)} \\ = h_F\|\sqrt{\mu}^{-1}\,\mathbf{curl}\,\mathbf{u}_h\times\mathbf{n} + \sqrt{\mu}\mathcal{S}\gamma_D\mathbf{u}_h\|^2_{L^2(F)} \\ \leq 2h_F\|\sqrt{\mu}^{-1}\,\mathbf{curl}\,\mathbf{u}_h\times\mathbf{n} - \sqrt{\mu}\mathcal{S}\gamma_D\mathbf{u}\|^2_{L^2(F)} \\ + 2h_F\|\sqrt{\mu}(\mathcal{S}\gamma_D\mathbf{u}_h - \mathcal{S}\gamma_D\mathbf{u})\|^2_{L^2(F)}.\end{aligned} \tag{5.40}$$

We have to estimate

$$\begin{aligned}h_F\|\sqrt{\mu}^{-1}\,\mathbf{curl}\,\mathbf{u}_h\times\mathbf{n} - \sqrt{\mu}\mathcal{S}\gamma_D\mathbf{u}\|^2_{L^2(F)} \\ \leq C\left(h_F\|\sqrt{\mu^{-1}}\,\mathbf{curl}\,\mathbf{u}_h\times\mathbf{n}\|^2_{\mathbf{L}^2(F)} + h_F\|\sqrt{\mu}\mathcal{S}\gamma_D\mathbf{u}\|^2_{\mathbf{L}^2(F)}\right).\end{aligned}$$

The first part can be estimated like $\eta_1^{F,C}$, the second one is just summed over all elements on Γ.

Next, we have to examine the term (5.40). For simplicity, we first assume μ to be constant. Thus, there holds

$$\begin{aligned}h_F\|\sqrt{\mu}(\mathcal{S}\gamma_D\mathbf{u}_h - \mathcal{S}\gamma_D\mathbf{u})\|^2_{L^2(F)} &= h_F\|\sqrt{\mu}\mathcal{W}(\gamma_D\mathbf{u}-\gamma_D\mathbf{u}_h) - \sqrt{\mu}\tilde{\mathcal{K}}(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|^2_{L^2(F)} \\ &\leq Ch_F\|\mathcal{W}(\gamma_D\mathbf{u}-\gamma_D\mathbf{u}_h)\|^2_{L^2(F)} + Ch_F\|\tilde{\mathcal{K}}(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|^2_{L^2(F)}.\end{aligned}$$

Summing over all elements, we have to estimate

$$h_{\Gamma,\max}\|\sqrt{\mu}\mathcal{W}(\gamma_D\mathbf{u}-\gamma_D\mathbf{u}_h)\|^2_{L^2(\Gamma)}+h_{\Gamma,\max}\|\tilde{\mathcal{K}}(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|^2_{L^2(\Gamma)}.$$

The first term is estimated in Lemma 5.4.4, the second one in Lemma 5.4.6.

Next, we have to estimate the second indicator

$$\eta_2^{F,\Gamma}=:\sqrt{h_F}\|\operatorname{curl}_\Gamma(\mathcal{I}-\mathcal{K})\gamma_D\mathbf{u}_h+\operatorname{curl}_\Gamma\mathcal{V}\boldsymbol{\lambda}_h\|_{L^2(F)}.$$

Here, we use the equation $\mathcal{V}\boldsymbol{\lambda}=(\mathcal{K}-\mathcal{I})\gamma_D\mathbf{u}$ and the triangle inequality to get

$$\begin{aligned}&\|\operatorname{curl}_\Gamma(\mathcal{I}-\mathcal{K})\gamma_D\mathbf{u}_h+\operatorname{curl}_\Gamma\mathcal{V}\boldsymbol{\lambda}_h\|_{L^2(F)}\\&\qquad\le\|\operatorname{curl}_\Gamma(\mathcal{K}-\mathcal{I})\gamma_D(\mathbf{u}-\mathbf{u}_h)\|_{L^2(F)}+\|\operatorname{curl}_\Gamma\mathcal{V}(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|_{L^2(F)}.\end{aligned}$$

Summing over all elements, we have to estimate the following terms

$$\begin{aligned}&h\|\operatorname{curl}_\Gamma(\mathcal{K}-\mathcal{I})\gamma_D(\mathbf{u}-\mathbf{u}_h)\|^2_{L^2(\Gamma)},\\&h\|\operatorname{curl}_\Gamma\mathcal{V}(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|^2_{L^2(\Gamma)}.\end{aligned}$$

The first term is estimated in Lemma 5.4.5, the second one in Lemma 5.4.7.

Finally, we have to estimate the indicator $\eta_0^{F,\Gamma}$ on the boundary of Ω

$$\eta_0^{F,\Gamma}:=\sqrt{h_F}\sqrt{\omega}\|\sqrt{\sigma}\mathbf{u}_h\cdot\mathbf{n}\|_{0,F}.$$

This can be done by using the estimate for $\eta_0^{F,C}$ and the above estimates. □

In the following, we consider the details of the proof of Theorem 5.4.3.

Lemma 5.4.4 *For* $\mathbf{u}\in\mathbf{H}^{1/2+\delta}(\mathbf{curl},\Omega)$*,* $\delta>0$*, there is a constant* $C>0$*, independent of* h*, such that there holds*

$$\begin{aligned}&h\|\mathcal{W}\gamma_D(\mathbf{u}-\mathbf{u}_h)\|^2_{\mathbf{L}^2(\Gamma)}\\&\qquad\le C\big(\|\mathbf{u}-\mathbf{u}_h\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}+h\|\mathbf{u}-\mathbf{u}_E\|^2_{\mathbf{H}^{1/2}(\mathbf{curl},\Omega)}+h^{1+2\delta}\|\mathbf{u}-\mathbf{u}_E\|^2_{\mathbf{H}^{1/2+\delta}(\mathbf{curl},\Omega)}\big)\end{aligned}$$

with the interpolate $\mathbf{u}_E:=\Pi_1^h\mathbf{u}\in\mathcal{ND}_1(\mathcal{T}_h)$*.*

Proof. From Theorem 3.3.3 we know that

$$\mathcal{W}:\mathbf{H}^0_\perp(\operatorname{curl}_\Gamma,\Gamma)\to\mathbf{H}^0_\parallel(\operatorname{div}_\Gamma,\Gamma)$$

is a continuous mapping and we thus get

$$\begin{aligned}\|\mathcal{W}\gamma_D(\mathbf{u}-\mathbf{u}_h)\|^2_{\mathbf{L}^2(\Gamma)}&\le C\|\gamma_D(\mathbf{u}-\mathbf{u}_h)\|_{\mathbf{H}^0_\perp(\operatorname{curl}_\Gamma,\Gamma)}\\&\le C\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}^{1/2}(\mathbf{curl},\Omega)}.\end{aligned}$$

The last step is due to the continuity of the mapping

$$\gamma_D : \mathbf{H}^s(\mathbf{curl}, \Omega) \to \mathbf{H}_\perp^{s-1/2}(\mathrm{curl}_\Gamma, \Gamma), \qquad 0 \le s < 1,$$

see Lemma 3.3.2.
Next, we to consider $\mathbf{u}_E := \Pi_1^h \mathbf{u} \in \mathcal{ND}_1(\mathcal{K}_h)$, the moment based interpolant of $\mathbf{u}$. Therefore, we have to assume that $\mathbf{u} \in \mathbf{H}^{1/2+\delta}(\mathbf{curl}, \Omega)$. We get

$$\|\mathcal{W}\gamma_D(\mathbf{u} - \mathbf{u}_h)\|^2_{\mathbf{L}^2(\Gamma)} \le C\|\mathbf{u} - \mathbf{u}_E\|^2_{\mathbf{H}^{1/2}(\mathbf{curl},\Omega)} + C\|\mathbf{u}_E - \mathbf{u}_h\|^2_{\mathbf{H}^{1/2}(\mathbf{curl},\Omega)}. \tag{5.41}$$

As $\mathbf{u}_h$, $\mathbf{u}_E \in \mathcal{ND}_1(\mathcal{T}_h)$, we can use the inverse inequality in Lemma 4.1.3 and we have

$$\begin{aligned} \|\mathbf{u}_h - \mathbf{u}_E\|^2_{\mathbf{H}^{1/2}(\mathbf{curl},\Omega)} &\le Ch^{-1}\|\mathbf{u}_h - \mathbf{u}_E\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \\ &\le Ch^{-1}\|\mathbf{u}_h - \mathbf{u}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + Ch^{-1}\|\mathbf{u} - \mathbf{u}_E\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}. \end{aligned} \tag{5.42}$$

For the interpolant $\mathbf{u}_E$ we know that there holds

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_E\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} &= \|\mathbf{u} - \Pi_1^h\mathbf{u} + \Pi_1^h\mathbf{u} - \Pi_1^h\mathbf{u}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \\ &= \|\mathbf{u} - \Pi_1^h\mathbf{u} - \Pi_1^h(\mathbf{u} - \Pi_1^h\mathbf{u})\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \\ &\le Ch^{1+2\delta}\|\mathbf{u} - \mathbf{u}_E\|^2_{\mathbf{H}^{1/2+\delta}(\mathbf{curl},\Omega)}, \end{aligned} \tag{5.43}$$

see e.g. Monk [78, Theorem 5.4.1], with $\delta > 0$.
Combining (5.41), (5.42) and (5.43) finishes the proof. □

Using the same ideas we get

Lemma 5.4.5 *For $\mathbf{u} \in \mathbf{H}^{1/2+\delta}(\mathbf{curl}, \Omega)$, $\delta > 0$, there exists a constant $C > 0$, independent of h, such that there holds*

$$\begin{aligned} &h\|\,\mathrm{curl}_\Gamma(\mathcal{K} - \mathcal{I})\gamma_D(\mathbf{u} - \mathbf{u}_h)\|^2_{L^2(\Gamma)} \\ &\qquad \le C\left(\|\mathbf{u} - \mathbf{u}_h\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + h\|\mathbf{u} - \mathbf{u}_E\|^2_{\mathbf{H}^{1/2}(\mathbf{curl},\Omega)} + h^{1+2\delta}\|\mathbf{u}\|^2_{\mathbf{H}^{1/2+\delta}(\mathbf{curl},\Omega)}\right) \end{aligned}$$

with the interpolate $\mathbf{u}_E := \Pi_1^h\mathbf{u} \in \mathcal{ND}_1(\mathcal{T}_h)$.

Proof. The proof is quite similar to the proof of Lemma 5.4.4. From Theorem 3.3.6 we know that

$$\mathcal{K} : \mathbf{H}_\perp^s(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_\perp^s(\mathrm{curl}_\Gamma, \Gamma)$$

is a continuous mapping.

$$\begin{aligned} h\|\,\mathrm{curl}_\Gamma(\mathcal{K} - \mathcal{I})\gamma_D(\mathbf{u} - \mathbf{u}_h)\|^2_{L^2(\Gamma)} &\le C\,h\|(\mathcal{K} - \mathcal{I})\gamma_D(\mathbf{u} - \mathbf{u}_h)\|^2_{\mathbf{H}_\perp^0(\mathrm{curl}_\Gamma,\Gamma)} \\ &\le C\,h\|\gamma_D(\mathbf{u} - \mathbf{u}_h)\|^2_{\mathbf{H}_\perp^0(\mathrm{curl}_\Gamma,\Gamma)} \\ &\le C\,h\|(\mathbf{u} - \mathbf{u}_h)\|^2_{\mathbf{H}^{1/2}(\mathbf{curl},\Omega)} \\ &\le C\,h\|\mathbf{u} - \mathbf{u}_E\|^2_{\mathbf{H}^{1/2}(\mathbf{curl},\Omega)} + C\,h\|\mathbf{u}_E - \mathbf{u}_h\|^2_{\mathbf{H}^{1/2}(\mathbf{curl},\Omega)}. \end{aligned}$$

Finally, we can estimate as in (5.42) and (5.43). □

The next task is to estimate the adjoint double layer potential.

Lemma 5.4.6 *For $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{0}(\mathrm{div}_{\Gamma}, \Gamma)$ there exists a constant $C > 0$, independent of h, such that there holds*

$$h\|\tilde{\mathcal{K}}(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|_{\mathbf{L}^2(\Gamma)}^2 \leq C\,\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma)}^2 + h\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_E\|_{\mathbf{H}_{\|}^{0}(\mathrm{div}_{\Gamma},\Gamma)}^2,$$

where $\boldsymbol{\lambda}_E \in \mathcal{RT}_1(\mathcal{K})$ is the orthogonal projection of $\boldsymbol{\lambda}$ with respect to the $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma)$ inner product.

Proof. Due to Theorem 3.3.6,

$$\tilde{\mathcal{K}} : \mathbf{H}_{\|}^{0}(\mathrm{div}_{\Gamma}, \Gamma) \to \mathbf{H}_{\|}^{0}(\mathrm{div}_{\Gamma}, \Gamma)$$

is continuous. We take $\boldsymbol{\lambda}_E \in \mathcal{RT}_1(\mathcal{K}_h)$ as the orthogonal projection of $\boldsymbol{\lambda}$ with respect to the $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma)$ inner product.

$$\begin{aligned} h\|\tilde{\mathcal{K}}(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|_{\mathbf{L}^2(\Gamma)}^2 &\leq C\,h\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{\mathbf{H}_{\|}^{0}(\mathrm{div}_{\Gamma},\Gamma)} \\ &\leq C\,h\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_E\|_{\mathbf{H}_{\|}^{0}(\mathrm{div}_{\Gamma},\Gamma)}^2 + C\,h\|\boldsymbol{\lambda}_E-\boldsymbol{\lambda}_h\|_{\mathbf{H}_{\|}^{0}(\mathrm{div}_{\Gamma},\Gamma)}^2. \end{aligned}$$

As $\boldsymbol{\lambda}_E - \boldsymbol{\lambda}_h \in \mathcal{RT}_1(\mathcal{K}_h)$, we can use the inverse inequality of Lemma 4.4.4.

$$\begin{aligned} \|\boldsymbol{\lambda}_h-\boldsymbol{\lambda}_E\|_{\mathbf{H}_{\|}^{0}(\mathrm{div}_{\Gamma},\Gamma)}^2 &\leq Ch^{-1}\|\boldsymbol{\lambda}_h-\boldsymbol{\lambda}_E\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma)}^2 \\ &\leq Ch^{-1}\|\boldsymbol{\lambda}_h-\boldsymbol{\lambda}\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma)}^2 + Ch^{-1}\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_E\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma)}^2. \end{aligned}$$

There also holds the estimate, see Theorem 4.4.2,

$$\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_E\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma)}^2 \leq Ch\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_E\|_{\mathbf{H}_{\|}^{0}(\mathrm{div}_{\Gamma},\Gamma)}^2.$$

Combining all these estimates gives the desired result. □

Lemma 5.4.7 *For $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{0}(\mathrm{div}_{\Gamma}, \Gamma)$ there holds*

$$h\|\operatorname{curl}_{\Gamma}\mathcal{V}(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|_{L^2(\Gamma)}^2 \leq C\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma},\Gamma)}^2 + C\,h\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_E\|_{\mathbf{H}_{\|}^{0}(\mathrm{div}_{\Gamma},\Gamma)}^2,$$

where $\boldsymbol{\lambda}_E \in \mathcal{RT}_1(\mathcal{K})$ is the orthogonal projection of $\boldsymbol{\lambda}$ with respect to the $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma}, \Gamma)$ inner product.

Proof. From Remark 3.3.5 we get that

$$\mathcal{V} : \mathbf{H}_{\|}^{s}(\Gamma) \to \mathbf{H}_{\|}^{s+1}(\Gamma)$$

is a continuous mapping. Furthermore, we know that

$$\operatorname{curl}_{\Gamma} : \mathbf{H}_{\|}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is a continuous mapping. There also holds the continuity of

$$\operatorname{curl}_\Gamma : \mathbf{H}^1_\parallel(\Gamma) \to L^2(\Gamma)$$

and we can estimate

$$\begin{aligned} h\|\operatorname{curl}_\Gamma \mathcal{V}(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|^2_{L^2(\Gamma)} &\le C\,h\|\mathcal{V}\gamma_D(\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|^2_{\mathbf{H}^1_t(\Gamma)} \\ &\le C\,h\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|^2_{\mathbf{H}^0_\parallel(\operatorname{div}_\Gamma,\Gamma)} \\ &\le C\,h\|\boldsymbol{\lambda}-\boldsymbol{\lambda}_E\|^2_{\mathbf{H}^0_\parallel(\operatorname{div}_\Gamma,\Gamma)} + C\,h\|\boldsymbol{\lambda}_E-\boldsymbol{\lambda}_h\|^2_{\mathbf{H}^0_\parallel(\operatorname{div}_\Gamma,\Gamma)}. \end{aligned}$$

The rest of the proof is the same as in Lemma 5.4.6. □

5.5 On the implementation of the indicators

For the implementation of the error indicators $\eta_1^{F,\Gamma}$ and $\eta_2^{F,\Gamma}$ we have to examine the behavior of the boundary integral operators on the finite element solution. Here, we only consider the Laplace kernel $\Phi(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$.

Let $\mathbf{u}_h \in \mathcal{TND}_p(\mathcal{K}_h)$. There holds with (3.15)

$$\begin{aligned} \mathcal{W}\gamma_D\mathbf{u}_h(\mathbf{x}) &= \mathbf{n}(\mathbf{x}) \times \mathbf{grad}_\Gamma V(\operatorname{curl}_\Gamma \mathbf{u}_h)(\mathbf{x}) \\ &= \int_\Gamma \big(\mathbf{n}(\mathbf{x}) \times \mathbf{grad}_\mathbf{x}\,\Phi(\mathbf{x},\mathbf{y})\big) \operatorname{curl}_\Gamma \mathbf{u}_h(\mathbf{y})\,dS_\mathbf{y}. \end{aligned}$$

From (3.12) we get

$$\begin{aligned} \operatorname{curl}_\Gamma \mathcal{K}\gamma_D\mathbf{u}_h(\mathbf{x}) &= \gamma_n \cdot \mathbf{curl}\,\gamma_D\mathbf{K}\gamma_D\mathbf{u}_h(\mathbf{x}) \\ &= -\gamma_n\,\mathbf{grad}\,V(\operatorname{curl}_\Gamma \mathbf{u}_h)(\mathbf{x}) \\ &= -\int_\Gamma \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{x})} \operatorname{curl}_\Gamma \mathbf{u}_h(\mathbf{y})\,dS_\mathbf{y} + \frac{1}{2}\operatorname{curl}_\Gamma \mathbf{u}_h(\mathbf{x}). \end{aligned}$$

The last step is due to the jump relations of the single layer potential.

Let $\boldsymbol{\lambda}_h \in \mathcal{RT}_p(\mathcal{K}_h)$. We can calculate, using the jump relation (3.13),

$$\begin{aligned} \tilde{\mathcal{K}}\boldsymbol{\lambda}_h(\mathbf{x}) &= -\mathbf{n}(\mathbf{x}) \times \mathbf{curl}_\mathbf{x} \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\,\boldsymbol{\lambda}_h(\mathbf{y})\,dS_\mathbf{y} - \frac{1}{2}\boldsymbol{\lambda}_h(\mathbf{x}) \\ &= \int_\Gamma \frac{\partial\Phi(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}}\boldsymbol{\lambda}_h(\mathbf{y})\,dS_\mathbf{y} - \int_\Gamma \mathbf{grad}_\mathbf{x}\,\Phi(\mathbf{x},\mathbf{y})\big(\boldsymbol{\lambda}_h(\mathbf{y})\cdot\mathbf{n}(\mathbf{x})\big)\,dS_\mathbf{y} - \frac{1}{2}\boldsymbol{\lambda}_h(\mathbf{x}). \end{aligned}$$

Due to the jump relation $[\gamma\,\mathbf{curl}\,\mathbf{V}\boldsymbol{\lambda}]_\Gamma = -\mathbf{n}\times\boldsymbol{\lambda}$, cf. Mitrea *et al.* [76, Sect. 3], there also holds $[\gamma_n\,\mathbf{curl}\,\mathbf{V}\boldsymbol{\lambda}]_\Gamma = 0$. Hence, we get

$$\begin{aligned} \operatorname{curl}_\Gamma \mathcal{V}\boldsymbol{\lambda}_h &= \mathbf{n}(\mathbf{x}) \cdot \mathbf{curl}_\mathbf{x} \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}_h(\mathbf{y})\,dS_\mathbf{y} \\ &= \int_\Gamma \big(\mathbf{n}(\mathbf{x}) \times \mathbf{grad}_\mathbf{x}\,\Phi(\mathbf{x},\mathbf{y})\big)\cdot\mathbf{u}_h(\mathbf{y})\,dS_\mathbf{y}. \end{aligned}$$

Finally, all potentials have been transformed into integrals with kernel Φ or $\mathbf{grad}\,\Phi$ and can be calculated using analytical formulas, see Maischak [74].

5.6 Numerical experiments

On Ω we construct a regular mesh $\mathcal{T}_h$ of tetrahedrons or hexahedrons of mesh size h. This mesh induces a mesh $\mathcal{K}_h$ of triangles and quadrilaterals on the boundary Γ. As described in Chapter 4, we use Nédélec elements of order p for the approximation in the space $\mathbf{H}(\mathbf{curl}, \Omega)$. For the discretization of the boundary function $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma 0, \Gamma)$ one could use divergence free Raviart-Thomas functions $\mathcal{RT}_p^0(\mathcal{K}_h) := \{\boldsymbol{\lambda} \in \mathcal{RT}_p(\mathcal{K}_h) : \operatorname{div}_\Gamma \boldsymbol{\lambda}_h = 0\}$. For the implementation it is more convenient to use the space $\mathbf{curl}_\Gamma\, \mathcal{S}_p(\mathcal{K}_h)$ where $\mathcal{S}_p(\mathcal{K}_h)$ denotes the space of piecewise polynomials on the triangulation $\mathcal{K}_h$, see Lemma 4.6.2. Instead of $\boldsymbol{\lambda}_h \in \mathcal{RT}_p^0(\mathcal{K}_h)$ we seek a function $\phi_p \in \mathcal{S}_p(\mathcal{K}_h)/\mathbb{C}$ and then set $\boldsymbol{\lambda}_h := \mathbf{curl}_\Gamma\, \phi_h$. In order to get a unique ϕ_h we demand that there holds $\int_\Gamma \phi_h(\mathbf{x})\, ds(\mathbf{x}) = 0$. In the calculations we achieve this constraint by introducing the bilinear form

$$\mathcal{P}(\phi_h, \tau_h) := \left(\int_\Gamma \phi_h(\mathbf{x})\, ds(\mathbf{x}) \right) \overline{\left(\int_\Gamma \tau_h(\mathbf{x})\, ds(\mathbf{x}) \right)}.$$

Then, the Galerkin systems reads:
Find $\mathbf{u}_h \in \mathcal{ND}_p(\mathcal{T}_h)$ *and* $\phi_h \in \mathcal{S}(\mathcal{K}_h)$ *such that*

$$\begin{aligned} (\mu^{-1}\, \mathbf{curl}\, \mathbf{u}_h, \mathbf{curl}\, \mathbf{v})_\Omega + i\omega(\sigma \mathbf{u}_h, \mathbf{v})_\Omega - \langle \mathcal{W} \gamma_D \mathbf{u}_h, \gamma_D \mathbf{v} \rangle_\Gamma & \\ + \langle \tilde{\mathcal{K}}\, \mathbf{curl}_\Gamma\, \phi_h, \gamma_D \mathbf{v} \rangle_\Gamma = -i\omega(\mathbf{J}_0, \mathbf{v})_\Omega, & \quad (5.44) \\ \langle (I - \mathcal{K}) \gamma_D \mathbf{u}_h, \mathbf{curl}_\Gamma\, \tau_h \rangle_\Gamma + \langle \mathcal{V}\, \mathbf{curl}_\Gamma\, \tau_h, \mathbf{curl}_\Gamma\, \tau_h \rangle_\Gamma + \mathcal{P}(\phi_h, \tau_h) = 0 & \end{aligned}$$

for all $\mathbf{v} \in \mathcal{ND}_p(\mathcal{T}_h)$, $\tau_h \in \mathcal{S}_p(\mathcal{K}_h)$.

An a priori estimate for this problem is given in Section 5.3.

5.6.1 Remarks on the experiments

The following experiments are performed using the program package *maiprogs*, cf. Maischak [75], which has been extended by the calculation of the Galerkin matrices for the eddy current problem and the residual error estimators and hanging edges. The implementation of hanging nodes for higher polynomial degrees still has to be done, so we present here only suitably refined meshes for the h-version with lowest polynomial degree.

For the assembling of the matrix and the right hand side see the remarks in Section 6.2 where the case of the scattering problem is explained.

For large linear systems iterative solvers as the GMRES are the best choice. But, this does not work here properly for our problem because of the high condition number of the whole system. Therefore, efficient preconditioners are needed. For the $\mathbf{H}(\mathbf{curl}, \Omega)$-bilinear form we have discussed a new preconditioner in §4.9.3 which has not been implemented yet. Other preconditioners using multigrid techniques are discussed by Hiptmair [64]. This algorithm also has to be implemented. In our implementation, just the Gauss-algorithm is used. For small matrices up to 5000 degrees of freedom the assembling of the Galerkin matrix takes longer than the solution using the Gauss algorithm. But for more degrees of freedom another solver is necessary. Thus, our computations are only up to 10000 degrees of freedom.

We consider a simply connected polyhedral domain and we compute the solution to the Galerkin system on a series of uniform meshes, obtained by dividing each edge of the domain Ω into n equal parts with a mesh-width of $h = \frac{2}{n}$. In most of the examples we compare the error in energy norm

$$e := \sqrt{\|\mathbf{u} - \mathbf{u}_h\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|^2_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}}.$$

with the value of the residual error estimator

$$\eta := \Big((\eta_0^{\mathcal{T}})^2 + (\eta_1^{\mathcal{T}})^2 + (\eta_0^{\mathcal{F},C})^2 + (\eta_1^{\mathcal{F},C})^2 + (\eta_0^{\mathcal{F},\Gamma})^2 + (\eta_1^{\mathcal{F},\Gamma})^2 + (\eta_2^{\mathcal{F},\Gamma})^2 \Big)^{1/2}.$$

In Examples 5.6.3 and 5.6.4 we use the error estimator to perform adaptive mesh refinements using the three-fold algorithm from §5.3.1 but only performing an h-refinement. The convergence rate α is calculated by evaluating the errors and the degrees of freedom of two successive meshes by

$$\alpha := \frac{\log(e_2/e_1)}{\log(N_1/N_2)}.$$

The effectivity index q is the quotient of the error estimator η and the real error e,

$$q := \frac{\eta}{e}.$$

5.6.2 The p-version

In the following two examples we calculate the h-version of the Galerkin coupling scheme (5.10) for different fixed polynomial degrees and the p-version. We define the moments for the Nédélec basis functions via Legendre polynomials. Therefore, we can perform our computations for the hp-version with moderate p, up to degree 7, compare §4.1.1.

Example 5.6.1 *We take* $\Omega = [-1,1]^3$ *and* $\mu = \sigma = \omega = 1$ *and the exact solution*

$$\mathbf{u}(\mathbf{x}) := \mathbf{grad} \int_\Omega \Phi(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\,d\mathbf{y}, \quad \mathbf{x} \in \Omega,$$

with density function

$$\rho(\mathbf{x}) := \big((1-x_1^2)(1-x_2^2)(1-x_3^2)\big)^2\, x_1x_2x_3 \quad \text{in } \Omega.$$

From (5.1) *and* (5.2) *the current* $\mathbf{J}_0$ *is given by*

$$\begin{aligned}
&\mathbf{J}_0 = -\sigma\mathbf{u} + i\omega^{-1}\,\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}) = -\sigma\mathbf{u} \quad \text{in } \Omega,\\
&\mathbf{J}_0 = 0 \quad \text{in } \Omega_E,
\end{aligned}$$

and on the boundary Γ *there holds*

$$\boldsymbol{\lambda} = \mu^{-1}\,\mathbf{curl}\,\mathbf{u} \times \mathbf{n} = 0.$$

In Figure 5.1 the h-version with different polynomial degrees is presented. We plot the error in energy norm and the indicators versus the degrees of freedom. The p-version (with constant mesh size $h=1$) and the h-version with $p=1$ are compared in Figure 5.2. One can see that the residual error estimator behaves like the error. The effectivity indices for the polynomial degrees of $p = 1, 3, 5$ are given in Tables 5.1, 5.2 and 5.3. The effectivity indices are stable, also for the p-version, see Table 5.5. This underlines the result about the reliability and efficiency of the residual error estimator. In Table 5.4 we consider the convergence rates. Due to the a priori estimate in Section 5.3 we expect a convergence rate of p with respect to h for a smooth solution. If we consider the degrees of freedom we expect a convergence rate of $\frac{p}{3}$. This is quite good fulfilled for most of the polynomial degrees. For $p = 7$ we could only calculate one refinement step and thus we are still in the pre-asymptotic region.

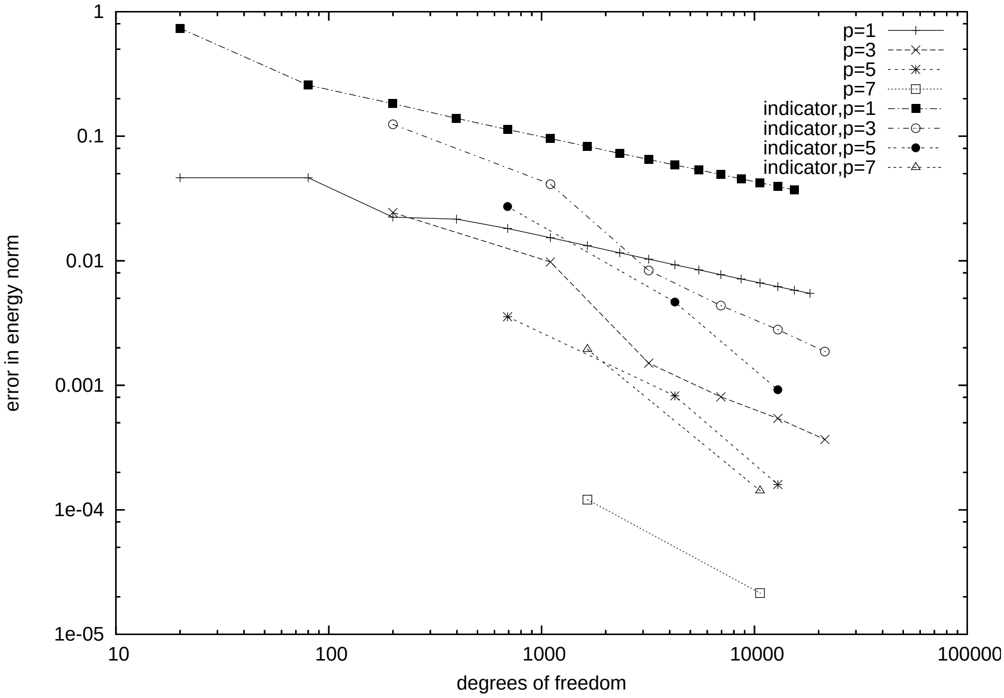

Figure 5.1: Example 5.6.1: Uniform h-version for different polynomial degrees, error in energy norm and error indicators.

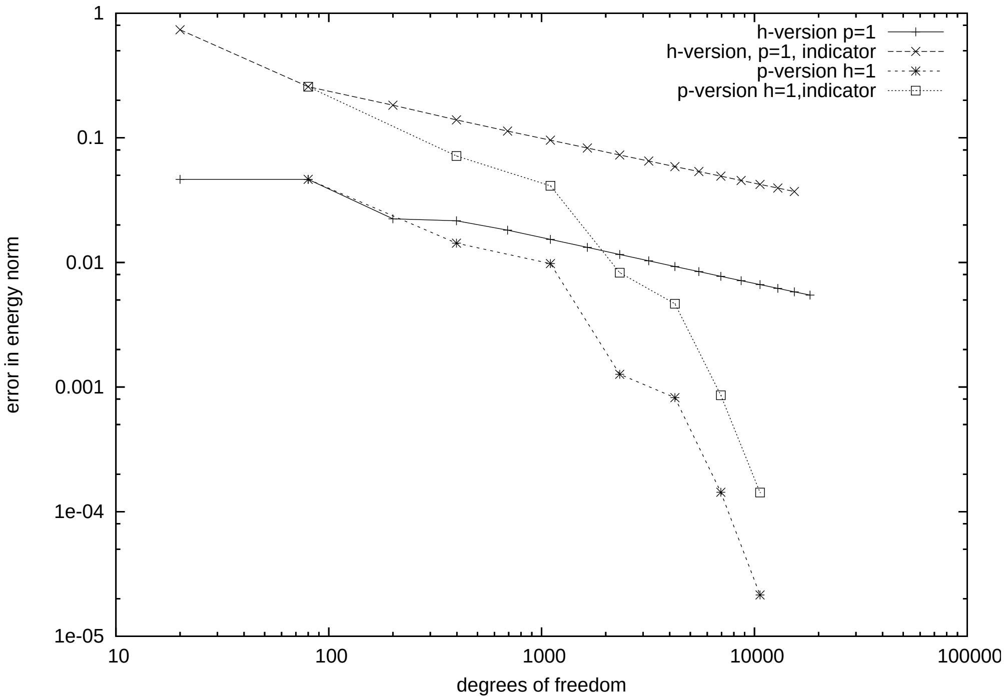

Figure 5.2: Example 5.6.1: Uniform p-version compared with uniform h-version ($p = 1$).

n	DOF	e	η	$q = \frac{\eta}{e}$
1	20	0.0464225	0.7378909	15.8951
2	80	0.0464241	0.2570505	5.5370
3	200	0.0223903	0.1828806	8.1679
4	398	0.0216248	0.1393502	6.4440
5	692	0.0181489	0.1134473	6.2509
6	1100	0.0153305	0.0957510	6.2459
7	1640	0.0132047	0.0828089	6.2715
8	2330	0.0115795	0.0729206	6.2974
9	3188	0.0103052	0.0651211	6.3224
10	4232	0.0092817	0.0588145	6.3366
11	5480	0.0084424	0.0536119	6.3503
12	6950	0.0077420	0.0492484	6.3628
13	8660	0.0071487	0.0455371	6.3690
14	10628	0.0066397	0.0455371	6.9583
15	12872	0.0061984	0.0423427	6.8312
16	15410	0.0058120	0.0395648	6.8074
17	18260	0.0054709	0.0371272	6.7850

Table 5.1: Example 5.6.1: Uniform h-version, $p = 1$.

n	DOF	e	η	$q = \frac{\eta}{e}$
1	200	0.0243608	0.1246737	5.1178
2	1100	0.0097928	0.0412815	4.2155
3	3188	0.0015046	0.0083536	5.5520
4	6950	0.0008064	0.0043726	5.4345
5	12872	0.0005414	0.0028039	5.1790
6	21440	0.0003668	0.0018682	5.0932

Table 5.2: Example 5.6.1: Uniform h-version, $p = 3$.

n	DOF	e	η	$q = \frac{\eta}{e}$
1	692	0.0035520	0.0272416	7.6694
2	4232	0.0008211	0.0046594	5.6746
3	12872	0.0001595	0.0009217	5.7787

Table 5.3: Example 5.6.1: Uniform h-version, $p = 5$.

p	1	2	3	4	5	6	7
α	0.3564	0.7326	0.7631	1.0859	1.4731	1.7670	0.9243

Table 5.4: Example 5.6.1: Convergence rate α with respect to the degrees of freedom of the h-version for different polynomial degrees.

p	DOF	e	η	$q = \frac{\eta}{e}$
1	80	0.046424	0.257051	5.53702
2	398	0.014293	0.071491	5.00182
3	110	0.009793	0.041282	4.21546
4	2330	0.001265	0.008281	6.54625
5	4232	0.000821	0.004659	5.67479
6	6950	0.000143	0.000858	6.00000
7	10628	.2142E-04	0.000142	6.62932

Table 5.5: Example 5.6.1: p-version, $h = 1$, energy norm error e, error estimator η, effectivity index $q = \frac{\eta}{e}$.

Example 5.6.2 *In this example we take an exact solution with non-vanishing* **curl**. *We set*

$$\mathbf{u}(\mathbf{x}) = \mathbf{curl} \int_\Omega \Phi(\mathbf{x}, \mathbf{y}) \boldsymbol{\rho}(\mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in \Omega,$$

with

$$\boldsymbol{\rho}(\mathbf{x}) := \rho(\mathbf{x})(1, 1, 1)^\mathsf{T}$$

and ρ as in Example 5.6.1. The current $\mathbf{J}_0$ is computed by

$$\mathbf{J}_0 = -\sigma \mathbf{u} + i\omega^{-1}\mu^{-1}\, \mathbf{curl}\, \mathbf{curl}\, \mathbf{u}.$$

Furthermore, we set

$$\boldsymbol{\lambda} := \mathbf{curl}\, \mathbf{u} \times \mathbf{n} \qquad \text{on } \Gamma.$$

The exact energy norm of $\boldsymbol{\lambda}$ is extrapolated using the sequence of uniformly refined meshes.

In Figure 5.3 and Figure 5.4 we list the plots of the h-version and the p-version for this example. The results are quite similar to the results of Example 5.6.1.

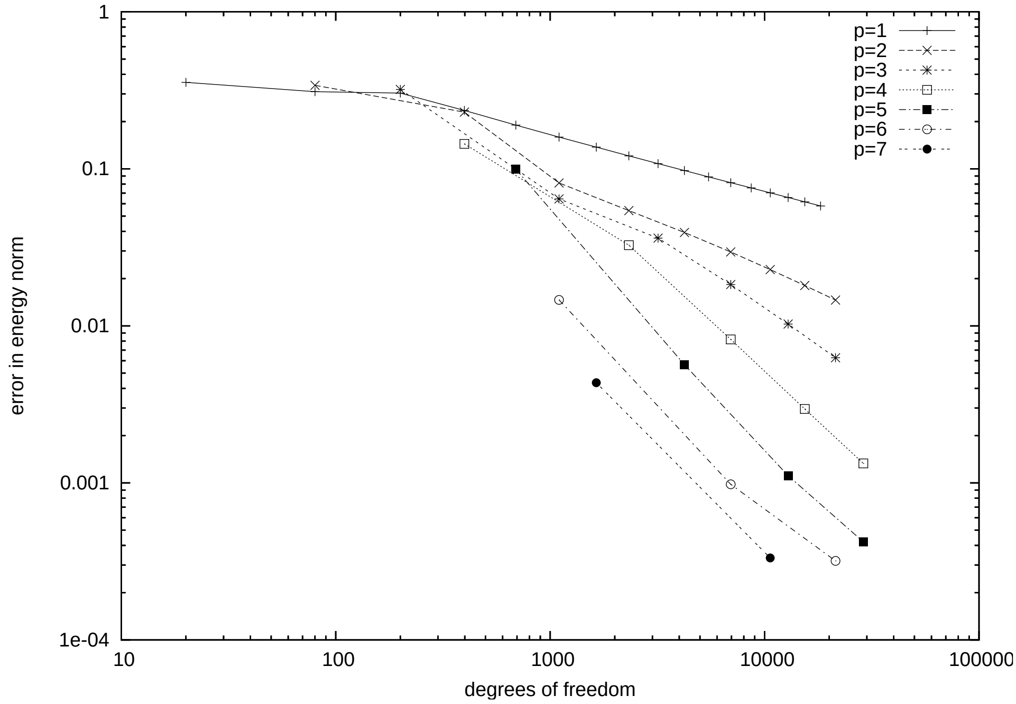

Figure 5.3: Example 5.6.2: Uniform h-version for different polynomial degrees.

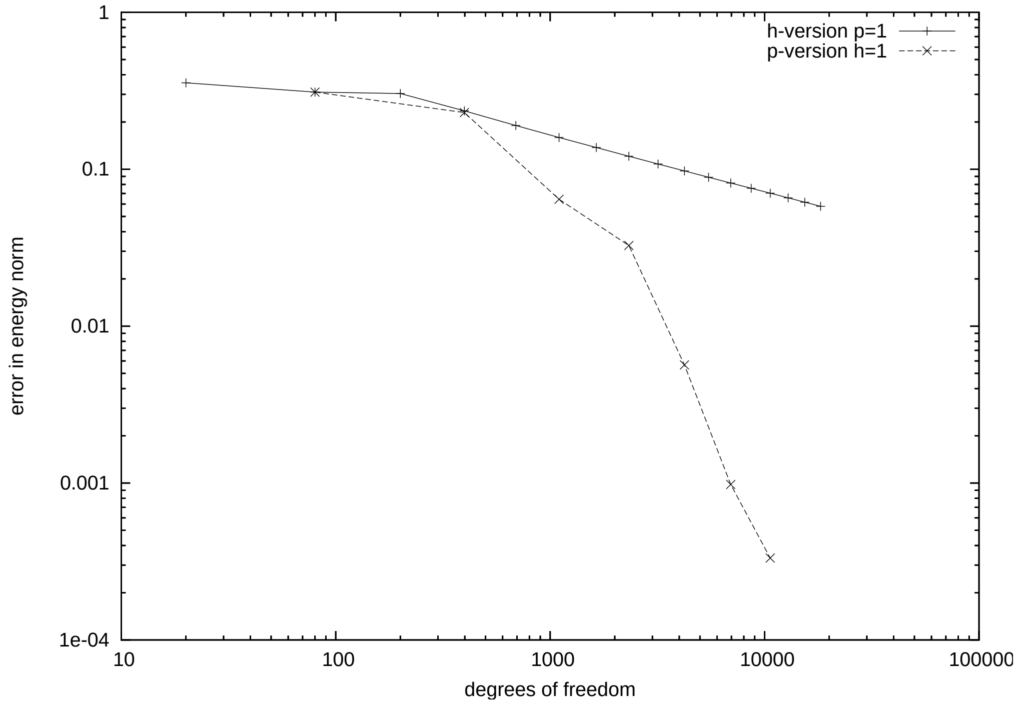

Figure 5.4: Example 5.6.2: Uniform p-version ($h = 1$) compared with uniform h-version ($p = 1$).

5.6.3 The h-version

In this section we use the residual error estimator to steer an adaptive refinement using hanging nodes as described in Section 4.11.

Example 5.6.3 *In this example we use the residual error estimator to construct an adaptive mesh. We use hexahedral elements with hanging nodes on the unit cube* $\Omega = [-1,1]^3$*. We set* $\mu \equiv 1$ *in* Ω *and choose a discontinuous conductivity* σ*, namely*

$$\sigma = \begin{cases} 0.1, & \frac{1}{3} < x_1, x_2, x_3 < 1 \\ 1, & else \end{cases}.$$

For the right hand side we choose the function

$$\mathbf{J}_0 = (1,1,1) \quad in\ \Omega$$

and $\mathbf{J} = 0$ *in* Ω_E*. Note, that we violate the (physical but no technical) assumption* $\mathbf{J}\cdot\mathbf{n} = 0$ *on* Γ*. But this creates no difficulty, we must only substitute the error estimator term* $\eta_0^{F,\Gamma} = \sqrt{h_F\omega}\,\|\sqrt{\sigma}\mathbf{u}_{h,p}\cdot\mathbf{n}\|_{0,F}$ *by* $\sqrt{h_F\omega}\,\|(\sqrt{\sigma}\mathbf{u}_{h,p} + \sqrt{\sigma}^{-1}\mathbf{J}_0)\cdot\mathbf{n}\|_{0,F}$*. We start by computing the Galerkin solution for the uniform mesh with* $n = 3$*. The refinement algorithm then proceeds by first refining the 20% of the elements on which the local contributions of the residual error estimator are the greatest and by then further refining in order to eliminate hanging nodes that violate the one-constraint rule. We expect the algorithm to refine the mesh near the* σ*-discontinuity interface between* $\Omega^{(1)} = (\frac{1}{3},1)^3$ *and* $\Omega^{(0)} = \Omega \setminus \Omega^{(1)}$*, and especially close to the vertex* $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$*. Figure 5.5 shows the series of adaptively generated meshes. One observes that the mesh is refined on the poorly conducting cube* $\Omega^{(1)}$ *and in its direct surroundings.*

We compute the same problem with uniform refinement, and then extrapolated the energy error for both series of meshes. The comparison between uniform and adaptive refinement as displayed in Figure 5.6 shows that the adaptive mesh yields the better approximation.

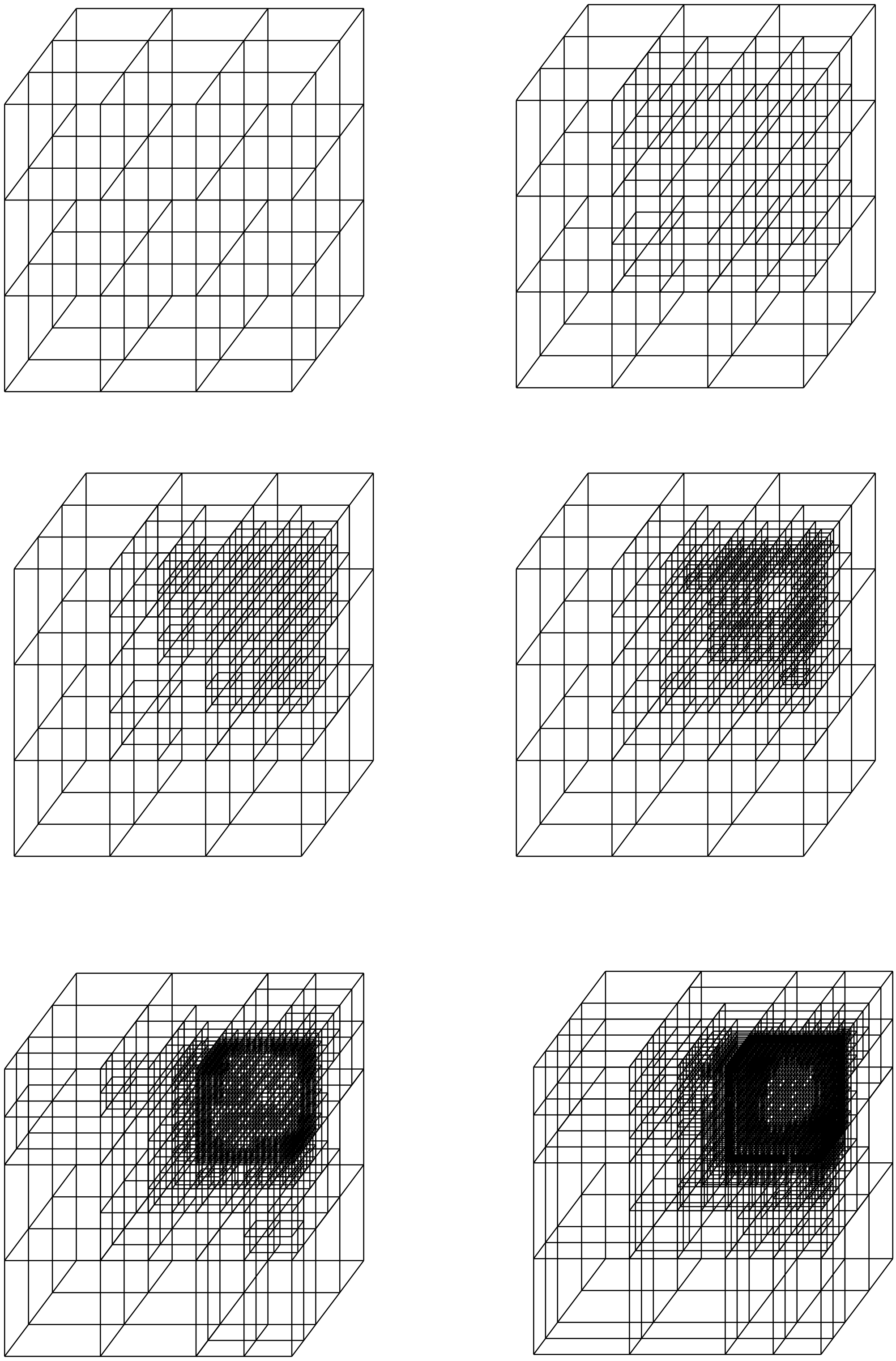

Figure 5.5: The adaptive meshes for Example 5.6.3, using the residual error estimator.

	Uniform refinement					
n	DOF	e	α	η	α	q
3	200	3.8706		2.7572		0.7123
6	1100	2.8400	0.1816	1.8880	0.2221	0.6648
9	3188	2.1252	0.2752	1.5032	0.2142	0.7073
12	6950	1.7016	0.2852	1.2471	0.2397	0.7329
	adaptive refinement					
	DOF	e	α	η	α	q
	200	3.8706		2.7572		0.7123
	377	2.8417	0.4874	1.8922	0.5939	0.6659
	797	1.7236	0.6679	1.3030	0.4984	0.7560
	1709	1.0483	0.6519	0.8783	0.5171	0.8379
	3835	0.7519	0.4111	0.6661	0.3422	0.8858
	9930	0.5392	0.2927	0.4940	0.3142	0.9162

Table 5.6: Values and convergence rates with respect to the total degrees of freedom DOF of the Galerkin error e and of the residual error estimator η and the effectivity indices $q := \frac{\eta}{e}$ for Example 5.6.3.

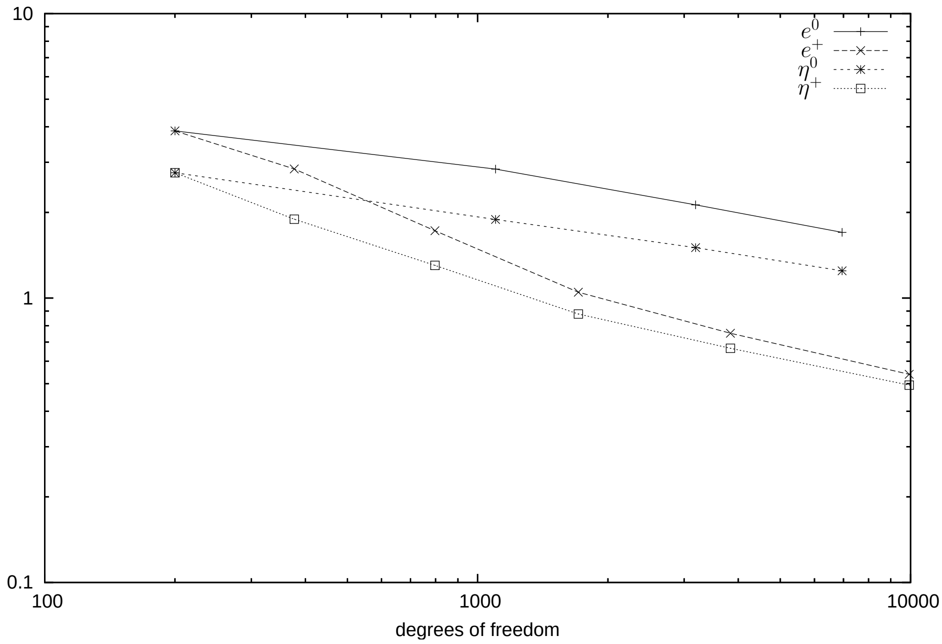

Figure 5.6: Energy norm e of the Galerkin error and the residual error estimator η of Example 5.6.3. The superscript 0 indicates uniform refinement, + indicates adaptive refinement.

Example 5.6.4 *The geometry in this example is the L-block* $\Omega := [-1,1]^3 \setminus \big([0,1]^3 \cup [0,1]^2 \times [-1,0]\big)$*. Here, we consider a singularity function as given current.*

$$\mathbf{J}_0 := \mathbf{grad}\,\big(r^{2/3}\sin(\tfrac{2}{3}\phi)\big) \quad \textit{in the L-Block,}$$

where r *and* ϕ *are polar coordinates. Hence, one expects an adaptive refinement towards the re-entrant edge.*

The energy norm of the unknown exact solution is extrapolated by the energy norms on the sequence of uniform meshes. We perform an adaptive refinement (10% of elements) using hanging nodes. The resulting meshes can be found in Figure 5.7 and the error in Figure 5.8. Due of the $2/3$*-singularity in the interior domain we expect a convergence rate of* $\alpha = \frac{2}{3}$ *with respect to the mesh size* h *and a convergence rate of* $\alpha = \frac{2}{9}$ *with respect to the degrees of freedom. This correspondents to the results in Table 5.7. For the adaptive refinement using the residual error indicators we get a better convergence rate of about* 0.4*. The effectivity indices are quite constant which underlines the reliability and efficiency of the error estimator.*

	Uniform refinement					
n	DOF	e	α	η	α	q
2	70	0.4186472		1.0506895		2.509725
4	334	0.2869302	0.241762	0.7324853	0.2308640	2.552834
6	902	0.2246235	0.246421	0.5789936	0.2366966	2.577618
8	1882	0.1881433	0.240962	0.4870686	0.2350675	2.588817
10	3382	0.1638018	0.236375	0.4248354	0.2332292	2.593594
12	5510	0.1462253	0.232553	0.3794070	0.2317002	2.594674
	adaptive refinement					
	DOF	e	α	η	α	q
	70	0.4186472		1.0506895		2.50972537
	152	0.3661693	0.172731	0.9203291	0.1708448	2.51339776
	231	0.3528255	0.088695	0.8116177	0.3003362	2.30033742
	362	0.2749754	0.554936	0.6961152	0.3417302	2.53155446
	526	0.2319625	0.455246	0.5897867	0.4435999	2.54259503
	778	0.1853135	0.573613	0.4921377	0.4624187	2.65570344
	1306	0.1501191	0.406604	0.4074256	0.3646738	2.71401574
	2229	0.1306073	0.260452	0.3577131	0.2434174	2.73884461
	3648	0.1056281	0.430896	0.2965062	0.3809444	2.80707690
	5615	0.0943108	0.262784	0.2627426	0.2803241	2.78592272

Table 5.7: Values and convergence rates with respect to the total degrees of freedom DOF of the Galerkin error e and of the residual error estimator η and the effectivity indices $q := \frac{\eta}{e}$ for Example 5.6.4 (the L-block).

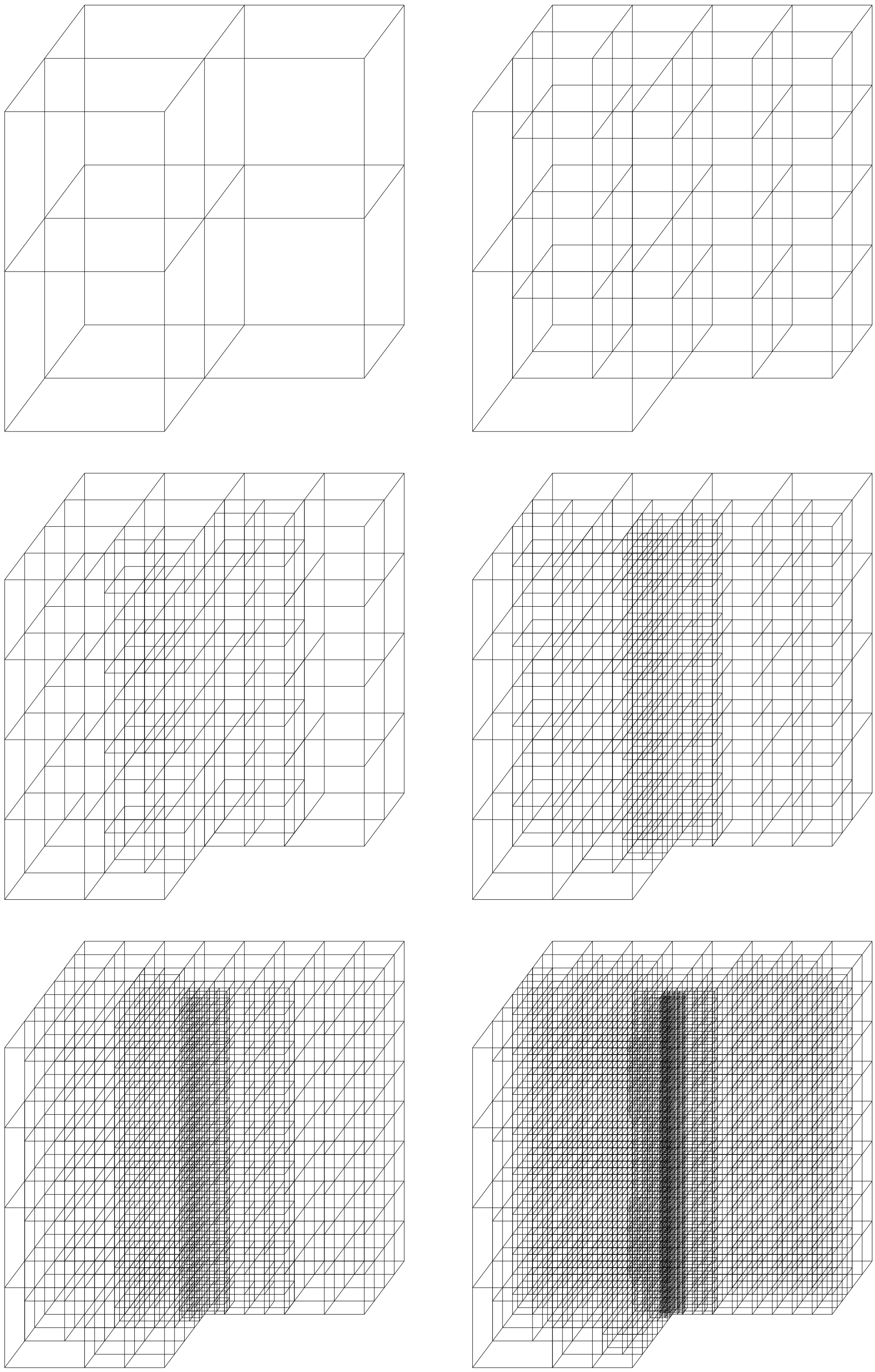

Figure 5.7: The adaptive meshes (levels of refinement: 1, 3, 5, 7, 9, 11) for Example 5.6.4 using the residual error estimator.

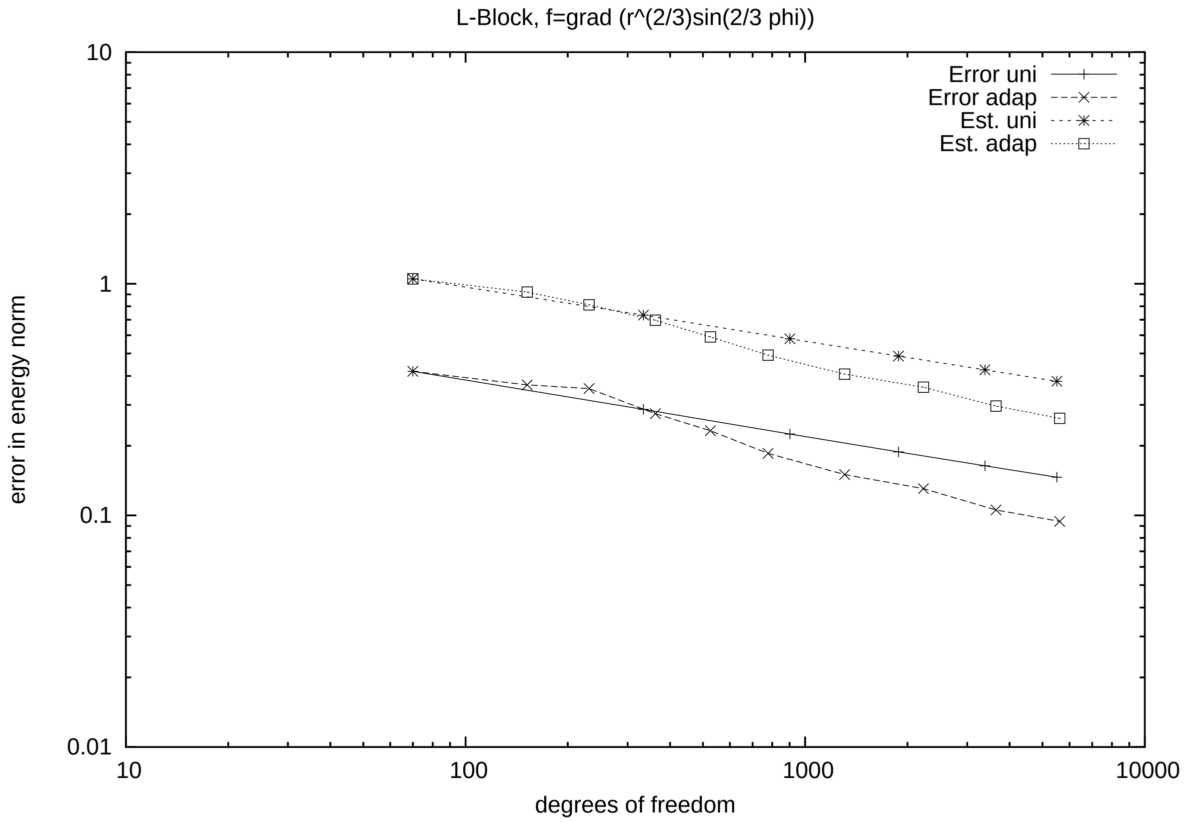

Figure 5.8: Energy norm e of the Galerkin error and the residual error estimator η of Example 5.6.4 (L-Block).

5.6.4 A 2-level hierarchical error estimator

The following hierarchical error estimator was developed by Teltscher [103], see also Teltscher *et al.* [105], using ideas of Beck *et al.* [14]. The error estimator is based on the p-hierarchical decomposition of the Nédélec space and the Raviart-Thomas space, see §4.9 and §4.10. We repeat here the central result for the eddy current problem, but use another decomposition than Teltscher.

Let $\mathcal{X} := \mathbf{H}(\mathbf{curl}, \Omega) \times \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma 0, \Gamma)$ and $\mathcal{X}_h := \mathcal{ND}_1(\mathcal{T}_h) \times \mathbf{curl}_\Gamma \, \mathcal{S}_1(\mathcal{K}_h)$ the finite element space as described above. Furthermore, we denote by $\mathcal{X}_2 := \mathcal{ND}_2(\mathcal{T}_h) \, \mathbf{curl}_\Gamma \, \mathcal{S}_1(\mathcal{K}_h)$ the higher order finite element space. Here, we just consider a mesh of hexahedrons. But there is a similar estimator for tetrahedrons.

As above, we define the energy norms on $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma 0, \Gamma)$ by

$$\|\mathbf{v}\|_{\mathfrak{E}}^2 := \big|(\mu^{-1} \, \mathbf{curl} \, \mathbf{u}, \mathbf{curl} \, \mathbf{u})_\Omega + i\omega(\sigma \mathbf{u}, \mathbf{u})_\Omega\big|, \qquad \|\boldsymbol{\lambda}\|_{\mathfrak{e}}^2 := |\langle \mathcal{V}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_\Gamma|.$$

Let $(\mathbf{u}_h, \boldsymbol{\lambda}_h) \in \mathcal{X}_h$ and $(\mathbf{u}_2, \boldsymbol{\lambda}_2) \in \mathcal{X}_2$ be the solutions of the Galerkin system (5.44). We assume that there holds the following saturation assumption

$$\|(\mathbf{u}-\mathbf{u}_2,\boldsymbol{\lambda}-\boldsymbol{\lambda}_2)\|_{\mathcal{X}} \le \delta_h \|(\mathbf{u}-\mathbf{u}_h,\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|_{\mathcal{X}} \tag{5.45}$$

for a $\delta_h > 0$ with $\delta_h \le \delta < 0$. Then, we can proof the following result.

Theorem 5.6.5 *In the case that the saturation assumption* (5.45) *is satisfied, there holds*

$$\eta \lesssim \|(\mathbf{u}-\mathbf{u}_h,\boldsymbol{\lambda}-\boldsymbol{\lambda}_h)\|_{\mathcal{X}} \lesssim \frac{1}{1-\delta}\eta$$

with the local a posteriori estimator

$$\eta^2 := \sum_{i=1}^{M}\left(\Theta^{(e_i)}\right)^2 + \sum_{j=1}^{N}\left(\left(\Theta_1^{(F_j)}\right)^2 + \left(\Theta_2^{(F_j)}\right)^2\right) + \sum_{k=1}^{L}\left(\Theta_1^{(T_k)}\right)^2 + \left(\Theta_2^{(T_k)}\right)^2 + \sum_{i=1}^{m}\left(\vartheta^{(e_i)}\right)^2 + \sum_{j=1}^{n}\left(\vartheta^{(F_j)}\right)^2.$$

We use the abbreviations $\mathcal{A}$ and $\mathcal{L}$ as in (5.11) and define the local error indicators by

$$\begin{aligned}
\Theta^{(e)} &:= \frac{|\mathcal{L}(\mathbf{grad}\,\phi^{(e)},0) - \mathcal{A}(\mathbf{u}_h,\boldsymbol{\lambda}_h;\mathbf{grad}\,\phi^{(e)},0)|}{\|\mathbf{grad}\,\phi^{(e)}\|_{\mathfrak{E}}},\\
\Theta_1^{(F)} &:= \frac{|\mathcal{L}(\mathbf{grad}\,\phi^{(F)},0) - \mathcal{A}(\mathbf{u}_h,\boldsymbol{\lambda}_h;\mathbf{grad}\,\phi^{(F)},0)|}{\|\mathbf{grad}\,\phi^{(F)}\|_{\mathfrak{E}}},\\
\Theta_1^{(T)} &:= \frac{|\mathcal{L}(\mathbf{grad}\,\phi^{(T)},0) - \mathcal{A}(\mathbf{u}_h,\boldsymbol{\lambda}_h;\mathbf{grad}\,\phi^{(T)},0)|}{\|\mathbf{grad}\,\phi^{(T)}\|_{\mathfrak{E}}},\\
\Theta_2^{(F)} &:= \|\kappa_1\mathbf{b}_1^{(F)} + \kappa_2\widetilde{\mathbf{b}}_2^{(F)} + \kappa_3\mathbf{b}_3^{(F)}\|_{\mathfrak{E}},
\end{aligned}$$

where $\widetilde{\mathbf{b}}_2^{(F)} := \mathbf{b}_2^{(F)} + \mathbf{b}_4^{(F)}$ and $(\kappa_1,\kappa_2,\kappa_3)^{\mathsf{T}}$ is the solution of the linear system

$$\begin{pmatrix}
a(\mathbf{b}_1^{(F)},\mathbf{b}_1^{(F)}) & a(\widetilde{\mathbf{b}}_2^{(F)},\mathbf{b}_1^{(F)}) & a(\mathbf{b}_3^{(F)},\mathbf{b}_1^{(F)})\\
a(\mathbf{b}_1^{(F)},\widetilde{\mathbf{b}}_2^{(F)}) & a(\widetilde{\mathbf{b}}_2^{(F)},\widetilde{\mathbf{b}}_2^{(F)}) & a(\mathbf{b}_3^{(F)},\widetilde{\mathbf{b}}_2^{(F)})\\
a(\mathbf{b}_1^{(F)},\mathbf{b}_3^{(F)}) & a(\widetilde{\mathbf{b}}_2^{(F)},\mathbf{b}_3^{(F)}) & a(\mathbf{b}_3^{(F)},\mathbf{b}_3^{(F)})
\end{pmatrix}
\begin{pmatrix}\kappa_1\\ \kappa_2\\ \kappa_3\end{pmatrix}
= \begin{pmatrix}
\mathcal{L}(\mathbf{b}_1^{(F)},0) - \mathcal{A}(\mathbf{u}_h,\boldsymbol{\lambda}_h;\mathbf{b}_1^{(F)},0)\\
\mathcal{L}(\widetilde{\mathbf{b}}_2^{(F)},0) - \mathcal{A}(\mathbf{u}_h,\boldsymbol{\lambda}_h;\widetilde{\mathbf{b}}_2^{(F)},0)\\
\mathcal{L}(\mathbf{b}_3^{(F)},0) - \mathcal{A}(\mathbf{u}_h,\boldsymbol{\lambda}_h;\mathbf{b}_3^{(F)},0)
\end{pmatrix}.$$

Furthermore,

$$\Theta_2^{(T)} := \|\sum_{\ell=1}^{5}\kappa_\ell\widetilde{\mathbf{b}}_\ell^{(T)}\|_{\mathfrak{E}},$$

where $\widetilde{\mathbf{b}}_1^{(T)} := \mathbf{b}_1^{(T)}$, $\widetilde{\mathbf{b}}_2^{(T)} := \mathbf{b}_2^{(T)} - \mathbf{b}_4^{(T)}$, $\widetilde{\mathbf{b}}_3^{(T)} := \mathbf{b}_3^{(T)}$, $\widetilde{\mathbf{b}}_4^{(T)} := \mathbf{b}_4^{(T)} - \mathbf{b}_6^{(T)}$, $\widetilde{\mathbf{b}}_5^{(T)} := \mathbf{b}_5^{(T)}$, and $(\kappa_1,\ldots,\kappa_5)^{\mathsf{T}}$ is the solution of the algebraic system

$$\left(a(\widetilde{\mathbf{b}}_k^{(T)},\widetilde{\mathbf{b}}_\ell^{(T)})\right)_{k,\ell=1,\ldots,5}\left(\kappa_\ell\right)_{\ell=1,\ldots,5} = \left(\mathcal{L}(\widetilde{\mathbf{b}}_k^{(T)},0) - \mathcal{A}(\mathbf{u}_h,\boldsymbol{\lambda}_h;\widetilde{\mathbf{b}}_k^{(T)},0)\right)_{k=1,\ldots,5},$$

and

$$\vartheta^{(e)} := \frac{|\mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; 0, \mathbf{curl}_\Gamma \varphi^{(e)})|}{\|\mathbf{curl}_\Gamma \varphi^{(e)}\|_{\mathfrak{e}}},$$
$$\vartheta^{(F)} := \frac{|\mathcal{A}(\mathbf{u}_h, \boldsymbol{\lambda}_h; 0, \mathbf{curl}_\Gamma \varphi^{(F)})|}{\|\mathbf{curl}_\Gamma \varphi^{(F)}\|_{\mathfrak{e}}}.$$

Example 5.6.6 *The numerical experiments in [103] have only been performed using the indicators which belong to the interior functions. Here we present numerical experiments using all indicators.*

The experiment has only been performed on uniform meshes because an adaptive refinement requires the implementation of hanging edges for higher polynomial degrees, compare §4.11, which hasn't been done yet.

We again consider the Example 5.6.2, where we have given the exact solution

$$\mathbf{u}(\mathbf{x}) = \mathbf{curl} \int_\Omega \Phi(\mathbf{x}, \mathbf{y}) \boldsymbol{\rho}(\mathbf{y}) \, d\mathbf{y}, \quad \mathbf{x} \in \Omega,$$

with $\boldsymbol{\rho}(\mathbf{x}) = ((1 - x_1^2)(1 - x_2^2)(1 - x_3^2))^2 x_1 x_2 x_3 (1, 1, 1)^\top$ *on the unit cube* $\Omega := [-1, 1]^3$.

In Figure 5.9 one sees that the error indicator η *behaves nearly the same as the error in energy norm, the effectivity indices* $q = \frac{\eta}{e}$*, calculated in Table 5.8, are nearly constant.*

n	h	DOF	e	η	$q = \frac{\eta}{e}$
2	1	80	0.30987	0.15081	0.4867
3	0.667	200	0.30369	0.06440	0.2121
4	0.5	398	0.23548	0.05420	0.2302
5	0.4	692	0.18994	0.03879	0.2042
6	0.333	1100	0.15938	0.02969	0.1863
7	0.143	1640	0.13748	0.02410	0.1753
8	0.25	2330	0.12095	0.02037	0.1684
9	0.222	3188	0.10800	0.01770	0.1639
10	0.2	4232	0.09755	0.01568	0.1607
11	0.091	5480	0.08894	0.01409	0.1584
12	0.083	6950	0.08172	0.01281	0.1568
13	0.077	8660	0.07558	0.01175	0.1555

Table 5.8: Energy norm e of the Galerkin error, the 2-level hierarchical error estimator η and the effectivity indices $q = \frac{\eta}{e}$ of Example 5.6.6.

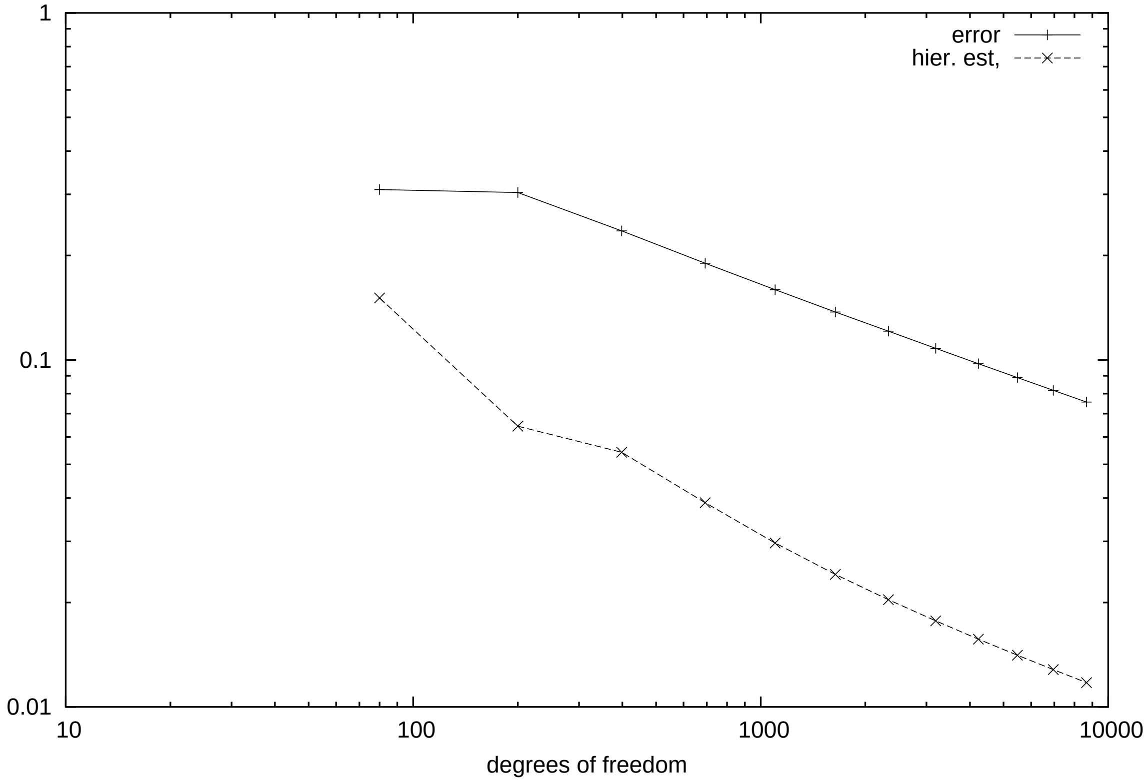

Figure 5.9: Energy norm e of the Galerkin error and the 2-level hierarchical error estimator η of Example 5.6.6.

6 The time-harmonic scattering problem

In this chapter we examine the time-harmonic scattering problem. Here, an incident wave is scattered at a dielectric body. After having formulated the problem we derive a coupling formulation using finite elements and boundary elements. Our coupling formulation is quite similar to the one derived by Hiptmair [67], but he uses different boundary integral operators. One can show that the formulations are equivalent.

We consider a bounded domain $\Omega \subset \mathbb{R}^3$ with simply connected Lipschitz boundary $\Gamma = \partial\Omega$. This domain represents a dielectric scatterer, while the exterior domain is representing air and is supposed to be dielectric and homogeneous. This means that the conductivity σ satisfies $\sigma = 0$ in $\mathbb{R}^3$. Furthermore, we assume that the material parameters ϵ (electric permittivity) and μ (magnetic permeability) satisfy $\epsilon, \mu \in L^\infty(\mathbb{R}^3)$ with $\epsilon_1 \geq \epsilon(\mathbf{x}) \geq \epsilon_2 > 0$ and $\mu_1 \geq \mu(\mathbf{x}) \geq \mu_2 > 0$ in Ω, where ϵ_1, ϵ_2, μ_1, μ_2 are constants. The outer domain $\Omega_E := \mathbb{R}^3 \setminus \bar{\Omega}$ consists of air with $\mu = \mu_0$ and $\epsilon = \epsilon_0$. Due to scaling we can assume that $\mu = 1$ and $\epsilon = 1$ in Ω_E.

Here, we consider an incident wave with electric field $\mathbf{E}^{in}$ and magnetic field $\mathbf{H}^{in}$ which is scattered at Ω. Some part of the wave is absorbed by Ω and the other part is reflected, see Figure 6.1.

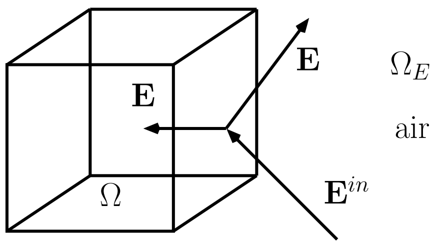

Figure 6.1: The time-harmonic scattering problem.

The problem is governed by the following two Maxwell's equations which are valid in the whole $\mathbb{R}^3$.

$$\mathbf{curl}\,\mathbf{E} = -i\omega\mu\mathbf{H}, \tag{6.1}$$

$$\mathbf{curl}\,\mathbf{H} = i\omega\epsilon\mathbf{E}. \tag{6.2}$$

On the boundary $\Gamma = \partial\Omega$ we assume the following jump conditions

$$(\mathbf{E} \times \mathbf{n})^+ + (\mathbf{E}^{in} \times \mathbf{n})^+ = (\mathbf{E} \times \mathbf{n})^-, \tag{6.3}$$

$$(\mathbf{H} \times \mathbf{n})^+ + (\mathbf{H}^{in} \times \mathbf{n})^+ = (\mathbf{H} \times \mathbf{n})^-. \tag{6.4}$$

Furthermore, we assume the Silver-Müller radiation condition at infinity, cf. Colton & Kress [38],

$$\lim_{|\mathbf{x}|\to\infty} (\mathbf{curl}\,\mathbf{E} \times \mathbf{x}) - i\kappa|\mathbf{x}|\mathbf{E} = 0$$

uniformly in all directions $\frac{\mathbf{x}}{|\mathbf{x}|}$. Here, we have defined the wave number by $\kappa := \omega\sqrt{\epsilon\mu}$. If we insert (6.2) into (6.1) we get the following equation for $\mathbf{E}$ in $\mathbb{R}^3$

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{E} - \kappa^2\mathbf{E} = 0 \qquad \text{in } \mathbb{R}^3 \tag{6.5}$$

and the jump relations

$$[\gamma_t^\times \mathbf{E}]_\Gamma = -\mathbf{E}^{in} \times \mathbf{n}, \qquad [\mu^{-1}\gamma_N \mathbf{E}]_\Gamma = -\gamma_N \mathbf{E}^{in}.$$

The main difference between this formulation and the eddy current formulation in Chapter 5 is the second Maxwell equation. In the scattering problem there is no current $\mathbf{J}_0$ and the conductivity satisfies $\sigma \equiv 0$ in $\mathbb{R}^3$. Nevertheless, we have the extra term $i\omega\epsilon\mathbf{E}$. Using solely the two Maxwell equations the eddy current problem is not unique in Ω_E and we had to demand the gauge condition $\operatorname{div}\mathbf{E} = 0$ and have got $\Delta\mathbf{E} = 0$ in Ω_E. Thus, we could use the boundary integral operators with Laplace kernel for the eddy current problem. Here, we will see that we have to use boundary integral operators with Helmholtz kernel.

6.1 A symmetric FEM/BEM-coupling method

A symmetric FEM/BEM-coupling formulation of the scattering problem, using finite elements in Ω and boundary elements on Γ for the exterior domain Ω_E, was derived for a smooth boundary by Ammari & Nédélec [7, 8] and generalized for non-smooth boundaries by Hiptmair [67]. We present here a slightly different derivation using different boundary integral operators than Hiptmair, but one can show that the formulations are equivalent.

First of all, we set $\mathbf{u} := \mathbf{E}$ and consider the exterior domain Ω_E. There holds

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \kappa^2\mathbf{u} = 0$$

in Ω_E and we apply the Stratton-Chu formula with Helmholtz kernel $\Phi(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$, compare (3.25),

$$\begin{aligned}\mathbf{u}(\mathbf{x}) = &+ \mathbf{curl}_\mathbf{x} \int_\Gamma (\mathbf{n} \times \mathbf{u})(\mathbf{y})\Phi(\mathbf{x},\mathbf{y})\,dS(\mathbf{y}) + \int_\Gamma (\mathbf{n} \times \mathbf{curl}\,\mathbf{u})(\mathbf{y})\Phi(\mathbf{x},\mathbf{y})\,dS(\mathbf{y}) \\ &- \mathbf{grad}_\mathbf{x} \int_\Gamma (\mathbf{n} \cdot \mathbf{u})(\mathbf{y})\Phi(\mathbf{x},\mathbf{y})\,dS(\mathbf{y}), \quad \mathbf{x} \in \Omega_E.\end{aligned}$$

We remark that there holds

$$\begin{aligned}\operatorname{div}_\Gamma(\mathbf{curl}\,\mathbf{u}\times\mathbf{n}) &= -\operatorname{curl}_\Gamma((\mathbf{curl}\,\mathbf{u}\times\mathbf{n})\times\mathbf{n}) = \operatorname{curl}_\Gamma(\gamma_D\,\mathbf{curl}\,\mathbf{u})\\ &= (\mathbf{curl}\,\mathbf{curl}\,\mathbf{u})\cdot\mathbf{n} = \kappa^2\mathbf{u}\cdot\mathbf{n}.\end{aligned}$$

Taking traces on Γ from Ω_E we get the following boundary integral equations

$$\gamma_D^+\mathbf{u} = \mathcal{K}(\gamma_D^+\mathbf{u}) - \mathcal{V}(\gamma_N^+\mathbf{u}) - \gamma_D^+\,\mathbf{grad}_{\mathbf{x}}\int_\Gamma \frac{1}{\kappa^2}\operatorname{div}_\Gamma(\mathbf{curl}\,\mathbf{u}\times\mathbf{n})\Phi(\mathbf{x},\mathbf{y})\,dS(\mathbf{y}), \tag{6.6}$$

$$\gamma_N^+\mathbf{u} = \mathcal{W}(\gamma_D^+\mathbf{u}) - \tilde{\mathcal{K}}(\gamma_N^+\mathbf{u}). \tag{6.7}$$

Next, we multiply (6.5) with test functions $\mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega)$ in Ω, integrate over Ω and perform a partial integration. Thus, we get

$$\int_\Omega \mu^{-1}\,\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\bar{\mathbf{v}}\,d\mathbf{x} - \int_\Omega \omega^2\epsilon\mu\mathbf{u}\cdot\bar{\mathbf{v}}\,d\mathbf{x} - \langle\mu^{-1}\gamma_N^-\mathbf{u},\gamma_D^-\mathbf{v}\rangle_\Gamma = 0.$$

In this equation we insert the jump condition $\gamma_N^-\mathbf{u} = \gamma_N^+\mathbf{u} + \gamma_N\mathbf{E}^{in}$, the relation (6.7) and $\gamma_D^+\mathbf{u} = \gamma_D^-\mathbf{u} + \gamma_D\mathbf{E}^{in}$ and have

$$\begin{aligned}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_\Omega - \omega^2(\epsilon\mu\mathbf{u},\mathbf{v})_\Omega - \langle\mathcal{W}\gamma_D^-\mathbf{u},\gamma_D^-\rangle_\Gamma + \langle\tilde{\mathcal{K}}\gamma_N^+\mathbf{u},\gamma_D^-\mathbf{v}\rangle_\Gamma&\\ = \langle\gamma_N\mathbf{E}^{in},\gamma_D^-\mathbf{v}\rangle_\Gamma + \langle\mathcal{W}\gamma_D\mathbf{E}^{in},\gamma_D^-\mathbf{v}\rangle_\Gamma&.\end{aligned}$$

We introduce a second variable $\boldsymbol{\lambda} := \mathbf{curl}\,\mathbf{E}_{|\Gamma}\times\mathbf{n} = \gamma_N^+\mathbf{u}$ representing the magnetic field on the boundary. Afterwards, we add the weak form of (6.6) and perform a partial integration. Finally, we get the following **coupling formulation**:
Find $\mathbf{u}\in\mathbf{H}(\mathbf{curl},\Omega)$, $\boldsymbol{\lambda}\in\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$ *such that*

$$\begin{aligned}&(\mu^{-1}\,\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_\Omega - \omega^2(\epsilon\mu\mathbf{u},\mathbf{v})_\Omega\\ &\quad-\langle\mathcal{W}\gamma_D^-\mathbf{u},\gamma_D^-\mathbf{v}\rangle_\Gamma + \langle\tilde{\mathcal{K}}\boldsymbol{\lambda},\gamma_D^-\mathbf{v}\rangle_\Gamma = \langle\gamma_N\mathbf{E}^{in},\gamma_D^-\mathbf{v}\rangle_\Gamma + \langle\mathcal{W}\gamma_D\mathbf{E}^{in},\gamma_D^-\mathbf{v}\rangle_\Gamma,\\ &\langle(I-\mathcal{K})\gamma_D^-\mathbf{u},\boldsymbol{\zeta}\rangle_\Gamma + \langle\mathcal{V}\boldsymbol{\lambda},\boldsymbol{\zeta}\rangle_\Gamma - \frac{1}{\kappa^2}\langle V\operatorname{div}_\Gamma\boldsymbol{\lambda},\operatorname{div}_\Gamma\boldsymbol{\zeta}\rangle_\Gamma = \langle(I-\mathcal{K})\gamma_D\mathbf{E}^{in},\boldsymbol{\zeta}\rangle_\Gamma\end{aligned} \tag{6.8}$$

for all $\mathbf{v}\in\mathbf{H}(\mathbf{curl},\Omega),\boldsymbol{\zeta}\in\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$.

We abbreviate this system by

$$\mathcal{A}(\mathbf{u},\boldsymbol{\lambda};\mathbf{v},\boldsymbol{\zeta}) = \mathcal{L}(\mathbf{v},\boldsymbol{\zeta})$$

and define the space $\mathcal{X} := \mathbf{H}(\mathbf{curl},\Omega)\times\mathbf{H}_\parallel^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$.

Assumption 6.1.1 *We assume that* κ *is not an eigenvalue of the interior Dirichlet problem in* Ω.

Theorem 6.1.2 (Hiptmair [67]) *$\mathcal{A}$ can be written as*

$$\mathcal{A} = \mathcal{D} + \mathcal{C},$$

where $\mathcal{D}$ is a positive definite and $\mathcal{C}$ a compact operator on X. Furthermore, there exists one $(\mathbf{u}, \boldsymbol{\lambda}) \in X$ such that

$$\mathcal{A}(\mathbf{u}, \boldsymbol{\lambda}; \mathbf{v}, \boldsymbol{\zeta}) = \mathcal{L}(\mathbf{v}, \boldsymbol{\zeta})$$

for all $(\mathbf{v}, \boldsymbol{\zeta}) \in X$.

On Ω we define a mesh of hexahedrals $\mathcal{T}_h$ that induces a mesh $\mathcal{K}_h$ on Γ. We use Nédélec elements $\mathcal{ND}_p(\mathcal{T}_h)$ for the Galerkin approximation in $\mathbf{H}(\mathbf{curl}, \Omega)$ and Raviart-Thomas elements $\mathcal{RT}_p(\mathcal{K}_h)$ for the approximation in $\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. Thus, the ***hp*-version of the FEM/BEM-coupling** reads as:
Find $\mathbf{u}_{h,p} \in \mathcal{ND}_p(\mathcal{T}_h)$, $\boldsymbol{\lambda}_{h,p} \in \mathcal{RT}_p(\mathcal{K}_h)$ such that

$$\begin{aligned}
(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_{h,p}, \mathbf{curl}\,\mathbf{v})_\Omega - \omega^2(\epsilon\mu\mathbf{u}_{h,p}, \mathbf{v})_\Omega& \\
-\langle \mathcal{W}\mathbf{u}_{h,p}, \mathbf{v}\rangle_\Gamma + \langle \tilde{\mathcal{K}}\boldsymbol{\lambda}_{h,p}, \mathbf{v}\rangle_\Gamma &= \langle \gamma_N \mathbf{E}^{in}, \mathbf{v}\rangle_\Gamma + \langle \mathcal{W}\mathbf{E}^{in}, \mathbf{v}\rangle_\Gamma \\
\langle (I-\mathcal{K})\mathbf{u}_{h,p}, \boldsymbol{\zeta}\rangle_\Gamma + \langle \mathcal{V}\boldsymbol{\lambda}_{h,p}, \boldsymbol{\zeta}\rangle_\Gamma - \frac{1}{\kappa^2}\langle V\,\mathrm{div}_\Gamma\,\boldsymbol{\lambda}_{h,p}, \mathrm{div}_\Gamma\,\boldsymbol{\zeta}\rangle_\Gamma &= \langle (I-\mathcal{K})\mathbf{E}^{in}, \boldsymbol{\zeta}\rangle_\Gamma
\end{aligned} \tag{6.9}$$

for all $\mathbf{v} \in \mathcal{ND}_p(\mathcal{T}_h), \boldsymbol{\zeta} \in \mathcal{RT}_p(\mathcal{K}_h)$.

Here, we abbreviate $\mathcal{X}_{h,p} := \mathcal{ND}_p(\mathcal{T}_h) \times \mathcal{RT}_p(\mathcal{K}_h)$. Thus, there holds the following theorem.

Theorem 6.1.3 *There exists $h_0 \in \mathbb{R}_+$ such that for all $h \leq h_0$ there exists exactly one solution $(\mathbf{u}_{h,p}, \boldsymbol{\lambda}_{h,p}) \in \mathcal{X}_{h,p}$ such that*

$$\mathcal{A}(\mathbf{u}_{h,p}, \boldsymbol{\lambda}_{h,p}; \tilde{\mathbf{v}}, \tilde{\boldsymbol{\zeta}}) = \mathcal{L}(\tilde{\mathbf{v}}, \tilde{\boldsymbol{\zeta}})$$

for all $(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\lambda}}) \in \mathcal{X}_{h,p}$. Furthermore, there exists a constant $C > 0$, independent of h and p such that

$$\|(\mathbf{u}, \boldsymbol{\lambda}) - (\mathbf{u}_{h,p}, \boldsymbol{\lambda}_{h,p})\|_{\mathcal{X}} \leq C \inf \|(\mathbf{u}, \boldsymbol{\lambda}) - (\tilde{\mathbf{v}}, \tilde{\boldsymbol{\zeta}})\|_{\mathcal{X}}$$

where the infimum is taken over all $(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\lambda}}) \in \mathcal{X}_{h,p}$.

Using the interpolation operators as in Section 5.3 we get the **a priori estimate**

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_{h,p}\|_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{h,p}\|_{\mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}& \\
\leq C h^k p_{\min}^{-(r-\epsilon)} \big(\|\mathbf{u}\|_{\mathbf{H}^r(\Omega)} + \|\boldsymbol{\lambda}\|_{\mathbf{H}_{\parallel}^{r-1/2}(\mathrm{div}_\Gamma,\Gamma)}\big)&
\end{aligned} \tag{6.10}$$

with a positive constant C, independent of $\mathbf{u}$, $\boldsymbol{\lambda}$, h and p, and arbitrary $\epsilon > 0$ where $k = \min\{r, p_{\min} + 1\}$. Similar to Section 5.3 we could also derive a residual a posteriori estimate, but the technique is similar, compare Teltscher [103], and we do not consider this here.

6.2 Implementation

In this section we consider the implementation of the Galerkin elements for the coupling formulation of the scattering problem (6.9). Here, we only examine quadrilateral elements.

Using Lemma 3.2.4 and the representation of the integral operators $\mathcal{K}$ and $\tilde{\mathcal{K}}$ from (3.15) we get the following formulation of the scattering problem:
Find $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ *and* $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *such that*

$$\begin{aligned}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_\Omega - \omega^2(\epsilon\mu\mathbf{u}, \mathbf{v})_\Omega \\ + \langle V \,\mathrm{curl}_\Gamma\, \mathbf{u}, \mathrm{curl}_\Gamma\, \mathbf{v}\rangle_\Gamma - \kappa^2\langle \mathcal{V}(\mathbf{u}\times\mathbf{n}), \mathbf{v}\times\mathbf{n}\rangle_\Gamma \\ + \langle(-\tfrac{1}{2}I - \mathbf{n}\times\mathcal{M})\boldsymbol{\lambda}, \gamma_D^-\mathbf{v}\rangle_\Gamma = \langle\gamma_N\mathbf{E}^{in}, \gamma_D^-\mathbf{v}\rangle_\Gamma - \langle\mathcal{W}\gamma_D\mathbf{E}^{in}, \gamma_D^-\mathbf{v}\rangle_\Gamma,\end{aligned}$$

$$\begin{aligned}\langle(\tfrac{1}{2}I - \mathcal{M}(\mathbf{n}\times\cdot))\gamma_D^-\mathbf{u}, \boldsymbol{\zeta}\rangle_\Gamma + \langle\mathcal{V}\boldsymbol{\lambda}, \boldsymbol{\zeta}\rangle_\Gamma - \frac{1}{\kappa^2}\langle V \,\mathrm{div}_\Gamma\, \boldsymbol{\lambda}, \mathrm{div}_\Gamma\, \boldsymbol{\zeta}\rangle_\Gamma \\ = \langle(\tfrac{1}{2}I - \mathcal{M}(\mathbf{n}\times\cdot))\gamma_D\mathbf{E}^{in}, \boldsymbol{\zeta}\rangle_\Gamma\end{aligned}$$

for all $\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$ *and all* $\boldsymbol{\zeta} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.
For the approximation in $\mathbf{H}(\mathbf{curl}, \Omega)$ we use the space $\mathcal{ND}_p(\mathcal{T}_h)$, for $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ we use $\mathcal{RT}_p(\mathcal{K}_h)$ and for $\gamma_D(\mathbf{H}(\mathbf{curl}, \Omega)) = \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ we use the space $\mathcal{TND}_p(\mathcal{K}_h)$ as introduced in Chapter 4.

The calculation of the boundary element matrices is performed via determining calculate double integrals on one or two elements over the product of two basis functions multiplied with the kernel $\Phi(\mathbf{x}, \mathbf{y})$. For the analytical computation of the integration of local monomials, see Maischak [74].

In the program package *maiprogs* , cf. Maischak [75], the boundary integrals are calculated in terms of local monomials x_1, x_2, y_1, y_2 and we thus have matrices of the form $(\langle V y_1^m y_2^n, x_1^k x_2^l\rangle_\Gamma)_{k,l,m,n=0,\ldots,p}$ for the single layer potential or $(\langle K y_1^m y_2^n, x_1^k x_2^l\rangle_\Gamma)_{k,l,m,n=0,\ldots,p}$ for the double layer potential. These integrals have to be reordered to Nédélec functions, $\mathcal{TND}$-functions and Raviart-Thomas functions and also transformed to the local elements Γ_x and Γ_y. Therefore, we use the transformation formulas for $\mathcal{TND}_p(\mathcal{K}_h)$ and $\mathcal{RT}_p(\mathcal{K}_h)$ from Chapter 4.

On the reference element $[-1, 1]^2$ an $\mathcal{RT}$-basis function of degree p can be written as linear combination of monomials

$$\hat{\boldsymbol{\lambda}} = \mathbf{e}_1 \sum_{m=0}^{p}\sum_{n=0}^{p-1} r_{mn}^{(1)} \hat{x}_1^m \hat{x}_2^n \;+\; \mathbf{e}_2 \sum_{m=0}^{p-1}\sum_{n=0}^{p} r_{mn}^{(2)} \hat{x}_1^m \hat{x}_2^n \tag{6.11}$$

with suitable coefficients $r_{mn}^{(1)}$ and $r_{mn}^{(2)}$.

If the local element Γ_i is spanned by the vectors $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$ then there holds for a transformed $\mathcal{RT}$-basis function $\boldsymbol{\lambda}_r$

$$\boldsymbol{\lambda} = \frac{1}{|\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}|} \left[\mathbf{a}^{(1)} \sum_{m=0}^{p} \sum_{n=0}^{p-1} r_{mn}^{(1)} x_1^m x_2^n \;+\; \mathbf{a}^{(2)} \sum_{m=0}^{p-1} \sum_{n=0}^{p} r_{mn}^{(2)} x_1^m x_2^n \right]. \tag{6.12}$$

On the reference element $[-1,1]^2$ a $\mathcal{TND}$-basis function has the following form as linear combination of monomials

$$\hat{\mathbf{u}} = \mathbf{e}_1 \sum_{m=0}^{p-1} \sum_{n=0}^{p} t_{mn}^{(1)} \hat{x}_1^m \hat{x}_2^n + \mathbf{e}_2 \sum_{m=0}^{p} \sum_{n=0}^{p-1} t_{mn}^{(2)} \hat{x}_1^m \hat{x}_2^n$$

with suitable coefficients $t_{mn}^{(1)}$ and $t_{mn}^{(2)}$.
On a local element Γ_i which is spanned by the vectors $\mathbf{a}^{(1)}$ and $\mathbf{a}^{(2)}$ there holds for a transformed $\mathcal{TND}$-basis function

$$\mathbf{u} = \frac{1}{|\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}|^2} \left[\mathbf{c}^{(1)} \sum_{m=0}^{p-1} \sum_{n=0}^{p} t_{mn}^{(1)} x_1^m x_2^n \;+\; \mathbf{c}^{(2)} \sum_{m=0}^{p} \sum_{n=0}^{p-1} t_{mn}^{(2)} x_1^m x_2^n \right], \tag{6.13}$$

where $\mathbf{c}^{(1)} := (\mathbf{a}^{(2)} \times (\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}))$ and $\mathbf{c}^{(2)} := ((\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}) \times \mathbf{a}^{(1)})$.

We are now in the position to examine the transformation of the Galerkin elements.

Calculation of $\langle \mathcal{V}(\mathbf{u} \times \mathbf{n}), \mathbf{v} \times \mathbf{n} \rangle$ with $\mathbf{u}$, $\mathbf{v} \in \mathcal{TND}_p(\mathcal{K}_h)$.

First of all, we know from (3.15) that there holds

$$\mathcal{V}(\mathbf{u} \times \mathbf{n})(\mathbf{y}) = [\mathbf{n}_\mathbf{y} \times \mathcal{L}(\mathbf{u} \times \mathbf{n}_\mathbf{x})] \times \mathbf{n}_\mathbf{y} \quad \text{with} \quad \mathcal{L}\boldsymbol{\lambda}(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x},\mathbf{y}) \boldsymbol{\lambda}(\mathbf{y})\, dS(\mathbf{y}).$$

The unit normal vectors $\mathbf{n}(\mathbf{y}) = \mathbf{n}_\mathbf{x}$ and $\mathbf{n}(\mathbf{y}) = \mathbf{n}_\mathbf{y}$ are constant on the elements $\Gamma_\mathbf{x}$ and $\Gamma_\mathbf{y}$ and we get for the scalar product between functions on $\Gamma_\mathbf{x}$ and $\Gamma_\mathbf{y}$

$$\langle \mathcal{V}(\mathbf{u}\times\mathbf{n}), \mathbf{v}\times\mathbf{n} \rangle = \int_{\Gamma_\mathbf{y}} (\mathbf{v}(\mathbf{y})\times\mathbf{n}_\mathbf{y}) \cdot \left(\left[\mathbf{n}_\mathbf{y} \times \int_{\Gamma_\mathbf{x}} \Phi(\mathbf{x},\mathbf{y})(\mathbf{u}(\mathbf{x}) \times \mathbf{n}_\mathbf{x})\, dS(\mathbf{x}) \right] \times \mathbf{n}_\mathbf{y} \right) dS(\mathbf{y})$$

and we get for the scalar product

$$\begin{aligned} \mathcal{V}(\mathbf{u}(\mathbf{x}) \times \mathbf{n}_\mathbf{x})(\mathbf{y}) \cdot (\mathbf{v}(\mathbf{y}) \times \mathbf{n}_\mathbf{y}) &= [\mathbf{n}_\mathbf{y} \times [\mathcal{L}(\mathbf{u}(\mathbf{x}) \times \mathbf{n}_\mathbf{x})] \times \mathbf{n}_\mathbf{y}] \cdot (\mathbf{v}(\mathbf{y}) \times \mathbf{n}_\mathbf{y}) \\ &= \Big[\underbrace{(\mathbf{n}_\mathbf{y} \cdot \mathbf{n}_\mathbf{y})}_{=1} \mathcal{L}(\mathbf{u}(\mathbf{x}) \times \mathbf{n}_\mathbf{x}) - \underbrace{(\mathbf{n}_\mathbf{y} \cdot \mathcal{L}(\mathbf{u}(\mathbf{x}) \times \mathbf{n}_\mathbf{x}))\mathbf{n}_\mathbf{y}}_{\perp(\mathbf{v}(\mathbf{y})\times\mathbf{n}_\mathbf{y})} \Big] \cdot (\mathbf{v}(\mathbf{y}) \times \mathbf{n}_\mathbf{y}) \\ &= (\mathcal{L}(\mathbf{u}(\mathbf{x}) \times \mathbf{n}_\mathbf{x}) \cdot (\mathbf{v}(\mathbf{y}) \times \mathbf{n}_\mathbf{y}). \end{aligned}$$

We now consider the basis functions $\mathbf{u}$ on $\Gamma_{\mathbf{x}}$ and $\mathbf{v}$ on $\Gamma_{\mathbf{y}}$ that are transformed as in (6.13). It follows that

$$\begin{aligned}\mathcal{V}(\mathbf{u}(\mathbf{x}) \times \mathbf{n}_{\mathbf{x}})(\mathbf{y}) \cdot (\mathbf{v}(\mathbf{y}) \times \mathbf{n}_{\mathbf{y}}) = \frac{1}{\left|\mathbf{a}_{\mathbf{x}}^{(1)} \times \mathbf{a}_{\mathbf{x}}^{(2)}\right|^2} \frac{1}{\left|\mathbf{a}_{\mathbf{y}}^{(1)} \times \mathbf{c}_{\mathbf{a}}^{(2)}\right|^2} \Bigg\{ \\ \sum_{k=0}^{p-1}\sum_{l=0}^{p}\sum_{m=0}^{p-1}\sum_{n=0}^{p} t_{kl}^{(1),\mathbf{y}} t_{mn}^{(1),\mathbf{x}} \Big[(V(x_1^m x_2^n) y_1^k y_2^l) \big[(\mathbf{c}_{\mathbf{x}}^{(1)} \times \mathbf{n}_{\mathbf{x}}) \cdot (\mathbf{c}_{\mathbf{y}}^{(1)} \times \mathbf{n}_{\mathbf{y}}) \big] \Big] \\ + \sum_{k=0}^{p}\sum_{l=0}^{p-1}\sum_{m=0}^{p}\sum_{n=0}^{p-1} t_{kl}^{(2),\mathbf{y}} t_{mn}^{(2),\mathbf{x}} \Big[(V(x_1^m x_2^n) y_1^k y_2^l) \big[(\mathbf{c}_{\mathbf{x}}^{(2)} \times \mathbf{n}_{\mathbf{x}}) \cdot (\mathbf{c}_{\mathbf{y}}^{(2)} \times \mathbf{n}_{\mathbf{y}}) \big] \Big] \Bigg\}.\end{aligned}$$

The integrals for the single layer potential $(V(x_1^m x_2^n) y_1^k y_2^l)$ can be calculated analytically using the formulas by Maischak [74].

Calculation of $\langle V \operatorname{curl}_\Gamma \mathbf{u}, \operatorname{curl}_\Gamma \mathbf{v} \rangle$ with $\mathbf{u}, \mathbf{v} \in \mathcal{TND}_p(\mathcal{K}_h)$.

There holds the following transformation, cf. (4.19),

$$\operatorname{curl}_\Gamma \boldsymbol{\psi}(\mathbf{x}) = \frac{\pm 1}{|\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}|} \widehat{\operatorname{curl}} \hat{\boldsymbol{\psi}}(\hat{\mathbf{x}})$$

with positive sign if $\mathbf{n} = -\frac{\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}}{|\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}|}$ and negative sign if $\mathbf{n} = \frac{\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}}{|\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}|}$. The vectors $\mathbf{n}$ and $\mathbf{a}^{(1)} \times \mathbf{a}^{(2)}$ are collinear and we can write

$$\operatorname{curl}_\Gamma \boldsymbol{\psi}(\mathbf{x}) = \frac{-1}{\mathbf{n} \cdot (\mathbf{c}^{(1)} \times \mathbf{c}^{(2)})} \widehat{\operatorname{curl}} \hat{\boldsymbol{\psi}}(\hat{\mathbf{x}}).$$

Thus, we only have to calculate the curl using the local monomials and multiply the terms with the above factor.

Calculation of $\langle \mathcal{V}\boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle$ with $\boldsymbol{\lambda}, \boldsymbol{\zeta} \in \mathcal{RT}_p(\mathcal{K}_h)$.

There holds

$$\langle \mathcal{V}\boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle = \int_{\Gamma_{\mathbf{y}}} \left[\left(\mathbf{n}_{\mathbf{y}} \times \int_{\Gamma_{\mathbf{x}}} \Phi(\mathbf{x}, \mathbf{y}) \boldsymbol{\lambda}(\mathbf{x}) \, dS(\mathbf{x}) \right) \times \mathbf{n}_{\mathbf{y}} \right] \cdot \boldsymbol{\zeta}(\mathbf{y}) \, dS(\mathbf{y}).$$

In general, for an arbitrary vector $\boldsymbol{\zeta}$ with $\boldsymbol{\zeta} \cdot \mathbf{n} = 0$ there holds

$$\gamma_D \boldsymbol{\zeta} = (\mathbf{n} \times \boldsymbol{\zeta}) \times \mathbf{n} = (\mathbf{n} \cdot \mathbf{n}) \boldsymbol{\zeta} - \underbrace{\boldsymbol{\zeta} \cdot \mathbf{n}}_{=0} \, \mathbf{n} = \boldsymbol{\zeta}$$

and for an arbitrary vector $\mathbf{a}$ we get

$$\begin{aligned}\gamma_D \mathbf{a} \cdot \boldsymbol{\zeta} &= [(\mathbf{n} \times \mathbf{a}) \times \mathbf{n}] \cdot \boldsymbol{\zeta} = [\mathbf{n} \times \mathbf{a}] \cdot [\mathbf{n} \times \boldsymbol{\zeta}] = [\mathbf{n} \times \boldsymbol{\zeta}] \cdot [\mathbf{n} \times \mathbf{a}] \\ &= [(\mathbf{n} \times \boldsymbol{\zeta}) \times \mathbf{n}] \cdot \mathbf{a} = \mathbf{a} \cdot \boldsymbol{\zeta}.\end{aligned}$$

Thus, we get

$$\langle \mathcal{V}\boldsymbol{\lambda}, \boldsymbol{\zeta}\rangle = \int_{\Gamma_{\mathbf{y}}} \left[\int_{\Gamma_{\mathbf{x}}} \Phi(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}(\mathbf{x})\, dS(\mathbf{x})\right] \cdot \boldsymbol{\zeta}(\mathbf{y})\, dS(\mathbf{y})$$

and for Raviart-Thomas functions $\boldsymbol{\lambda}$ on $\Gamma_{\mathbf{x}}$ and $\boldsymbol{\zeta}$ on $\Gamma_{\mathbf{y}}$ we end up

$$\begin{aligned}\langle \mathcal{V}(\boldsymbol{\lambda}(\mathbf{x}))(\mathbf{y}), \boldsymbol{\zeta}(\mathbf{y})\rangle = \frac{1}{\left|\mathbf{a}_{\mathbf{x}}^{(1)} \times \mathbf{a}_{\mathbf{x}}^{(2)}\right|}\frac{1}{\left|\mathbf{a}_{\mathbf{y}}^{(1)} \times \mathbf{a}_{\mathbf{y}}^{(2)}\right|}\Bigg\{ & \\ \sum_{k=0}^{p}\sum_{l=0}^{p-1}\sum_{m=0}^{p}\sum_{n=0}^{p-1} r_{kl}^{(1),\mathbf{y}} r_{mn}^{(1),\mathbf{x}} \Big[(V(x_1^m x_2^n) y_1^k y_2^l) \big[\mathbf{a}_{\mathbf{x}}^{(1)} \cdot \mathbf{a}_{\mathbf{y}}^{(1)}\big]\Big] & \\ + \sum_{k=0}^{p-1}\sum_{l=0}^{p}\sum_{m=0}^{p-1}\sum_{n=0}^{p} r_{kl}^{(2),\mathbf{y}} r_{mn}^{(2),\mathbf{x}} \Big[(V(x_1^m x_2^n)) y_1^k y_2^l \big[\mathbf{a}_{\mathbf{x}}^{(2)} \cdot \mathbf{a}_{\mathbf{y}}^{(2)}\big]\Big]\Bigg\} &\end{aligned}$$

where we have to compute the single layer potential in local coordinates.

Calculation of $\langle V \operatorname{div}_\Gamma \boldsymbol{\lambda}, \operatorname{div}_\Gamma \boldsymbol{\zeta}\rangle$ with $\boldsymbol{\lambda}$, $\boldsymbol{\zeta} \in \mathcal{RT}_p(\mathcal{K}_h)$.

There holds the following transformation, compare (4.17),

$$\operatorname{div}_\Gamma \boldsymbol{\varphi}(\mathbf{x}) = \frac{1}{|\mathbf{c}^{(1)} \times \mathbf{c}^{(2)}|}\, \widehat{\operatorname{div}}\hat{\varphi}(\hat{\mathbf{x}}).$$

Therefore, only a multiplication with the factor $\frac{1}{|\mathbf{c}^{(1)} \times \mathbf{c}^{(2)}|}$ is necessary.

Calculation of $\langle \mathcal{M}(\mathbf{n} \times \mathbf{u}), \boldsymbol{\lambda}\rangle$ with $\mathbf{u} \in \mathcal{TND}_p(\mathcal{K}_h)$ and $\boldsymbol{\lambda} \in \mathcal{RT}_p(\mathcal{K}_h)$.

First of all, we remark that there holds

$$\langle \boldsymbol{\lambda}, \mathcal{M}(\mathbf{n} \times \boldsymbol{\zeta})\rangle_\Gamma = -\langle \mathbf{n} \times \mathcal{M}\boldsymbol{\lambda}, \boldsymbol{\zeta}\rangle_\Gamma \tag{6.14}$$

for all $\boldsymbol{\lambda}$, $\boldsymbol{\zeta} \in \mathbf{L}_t^2(\Gamma)$. This holds due to $\mathbf{grad}_{\mathbf{x}}\, \Phi(\mathbf{x},\mathbf{y}) = -\,\mathbf{grad}_{\mathbf{y}}\, \Phi(\mathbf{x},\mathbf{y})$ and

$$\boldsymbol{\lambda}(\mathbf{y}) \cdot \big(\mathbf{grad}_{\mathbf{x}}\, \Phi(\mathbf{x},\mathbf{y}) \times (\mathbf{n}(\mathbf{x}) \times \boldsymbol{\zeta}(\mathbf{c}))\big) = \big(\mathbf{n}(\mathbf{x}) \times (\mathbf{grad}_{\mathbf{x}}\, \Phi(\mathbf{x},\mathbf{y}) \times \boldsymbol{\lambda}(\mathbf{y}))\big) \cdot \boldsymbol{\zeta}(\mathbf{x}).$$

Thus, this case also covers the matrix $\langle \mathbf{n} \times \mathcal{M}\boldsymbol{\lambda}, \mathbf{u}\rangle$. There holds

$$\mathcal{M}(\mathbf{n}(\mathbf{y}) \times \mathbf{u}(\mathbf{y}))(\mathbf{x}) = \int_\Gamma \mathbf{grad}_{\mathbf{x}}\, \Phi(\mathbf{x},\mathbf{y}) \times \big(\mathbf{n}(\mathbf{y}) \times \mathbf{u}(\mathbf{y})\big)\, dS(\mathbf{y}).$$

For the calculation of the vector product it is often useful to consider the ϵ_{ijk}-tensor.

$$\epsilon_{ijk} := \begin{cases} 1, & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1, & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ 0, & \text{at least two indices are equal} \end{cases}, \quad i,j,k = 1,2,3.$$

Furthermore, we adopt Einstein's sum convention which means the summation from 1 to 3 over two equal indices. For example, there holds for the i-th component of the vector product of two vectors $(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$.

Using the transformation for $\mathcal{TND}$-functions (6.13) we get on the local element $\Gamma_{\mathbf{y}}$ for the i-th component

$$\left(\int_{\Gamma_{\mathbf{y}}} \mathbf{grad_x}\, \Phi(\mathbf{x}, \mathbf{y}) \times (\mathbf{n_y} \times \mathbf{u}(\mathbf{y}))\, dS(\mathbf{y}) \right)_i = \frac{1}{|\mathbf{a}_{\mathbf{y}}^{(1)} \times \mathbf{a}_{\mathbf{y}}^{(2)}|^2} \Bigg\{ \sum_{m=0}^{p-1} \sum_{n=0}^{p} t_{mn}^{(1)} \epsilon_{ijk} \left(\int_{\Gamma_{\mathbf{y}}} \mathbf{grad_x}\, \Phi(\mathbf{x}, \mathbf{y}) y_1^m y_2^n \right)_j \left(\mathbf{n_y} \times \mathbf{c}_{\mathbf{y}}^{(1)} \right)_k + \sum_{m=0}^{p} \sum_{n=0}^{p-1} t_{mn}^{(2)} \epsilon_{ijk} \left(\int_{\Gamma_{\mathbf{y}}} \mathbf{grad_x}\, \Phi(\mathbf{x}, \mathbf{y}) y_1^m y_2^n \right)_j \left(\mathbf{n_y} \times \mathbf{c}_{\mathbf{y}}^{(2)} \right)_k \Bigg\}.$$

For the second integration we use the usual transformation for $\mathcal{RT}$-functions. The double integrals in local coordinates

$$\int_{\Gamma_{\mathbf{x}}} \int_{\Gamma_{\mathbf{y}}} \mathbf{grad_x}\, \Phi(\mathbf{x}, \mathbf{y}) x_1^k x_2^l y_1^m y_2^n$$

can also be calculated analytically, see Maischak [74].

Calculation of the right hand side $\langle I - \mathcal{M}(\mathbf{n} \times (\gamma_D \mathbf{E}^{in})), \boldsymbol{\zeta} \rangle$ with $\boldsymbol{\zeta} \in \mathcal{RT}_p(\mathcal{K}_h)$.

Using (6.14) it follows that

$$\langle -\mathcal{M}(\mathbf{n} \times (\gamma_D \mathbf{E}^{in})), \boldsymbol{\zeta} \rangle = \langle \gamma_D \mathbf{E}^{in}, \mathbf{n} \times \mathcal{M}\boldsymbol{\zeta} \rangle = \langle \gamma_D \mathbf{E}^{in} \times \mathbf{n}, \mathcal{M}\boldsymbol{\zeta} \rangle.$$

The last equation follows from $\mathbf{a} \cdot (\mathbf{n} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{n}) \cdot \mathbf{b}$ and there holds for the right hand side

$$\langle (I - \mathcal{M}(\mathbf{n} \times \cdot)) \gamma_D \mathbf{E}^{in}, \boldsymbol{\zeta} \rangle_\Gamma = \langle \gamma_D \mathbf{E}^{in}, \boldsymbol{\zeta} \rangle + \langle \gamma_D \mathbf{E}^{in} \times \mathbf{n}, \mathcal{M}\boldsymbol{\zeta} \rangle.$$

The functions $\gamma_D \mathbf{E}^{in} \times \mathbf{n} =: \mathbf{E}$ and $\gamma_D \mathbf{E}^{in}$ have to be given in the program and the term $\langle \mathbf{E}, \mathcal{M}\boldsymbol{\zeta} \rangle$ is computed using the transformation (6.12) for $\boldsymbol{\zeta}$, and we get with Einstein's summing convention

$$\begin{aligned} \mathbf{E} \cdot \mathcal{M}\boldsymbol{\zeta} &= \mathbf{E} \cdot \int_{\Gamma_{\mathbf{y}}} \mathbf{grad}\, \Phi(\mathbf{x}, \mathbf{y}) \times \boldsymbol{\zeta}(\mathbf{y})\, dS(\mathbf{y}) \\ &= \frac{1}{|\mathbf{c}^{(1)} \times \mathbf{c}^{(2)}|} \Bigg\{ \sum_{m=0}^{p} \sum_{n=0}^{p-1} r_{mn}^{(1)} \epsilon_{ijk} E_i (\mathbf{grad_x}\, \Phi y_1^m y_2^n)_j c_k^{(1)} \\ &\qquad + \sum_{m=0}^{p-1} \sum_{n=0}^{p} r_{mn}^{(2)} \epsilon_{ijk} E_i (\mathbf{grad_x}\, \Phi y_1^m y_2^n)_j c_k^{(2)} \Bigg\}. \end{aligned}$$

The integrals $(\mathbf{grad_x}\, \Phi y_1^m y_2^n)$ are evaluated using analytical formulas, cf. Maischak [74].

Calculation of the right hand side $\langle \mathcal{W}\gamma_D \mathbf{E}^{in}, \gamma_D \mathbf{v}\rangle$ with $\mathbf{v} \in \mathcal{ND}_p(\mathcal{T}_h)$.

We remark that there holds

$$\langle \mathcal{W}\gamma_D \mathbf{E}^{in}, \mathbf{v}\rangle = -\langle V \operatorname{curl}_\Gamma \gamma_D \mathbf{E}^{in}, \operatorname{curl}_\Gamma \mathbf{v}\rangle + \omega^2 \langle \mathcal{V}(\gamma_D \mathbf{E}^{in} \times \mathbf{n}), \mathbf{v} \times \mathbf{n}\rangle.$$

The transformation for the first term is similar to the transformation of $\langle V \operatorname{curl}_\Gamma \mathbf{u}, \operatorname{curl}_\Gamma \mathbf{v}\rangle$. For the second term there holds

$$\begin{aligned}\mathcal{V}(\gamma_D \mathbf{E}^{in} \times \mathbf{n}(\mathbf{x}))(\mathbf{y}) &= \mathcal{V}(\mathbf{E}^{in}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}))(\mathbf{y}) \\ &= (\mathbf{n}(\mathbf{y}) \times \mathcal{L}(\mathbf{E}^{in} \times \mathbf{n}(\mathbf{x}))(\mathbf{y})) \times \mathbf{n}(\mathbf{y}).\end{aligned}$$

$\mathbf{E} := \mathbf{E}^{in} \times \mathbf{n}(\mathbf{x})$ has to be given in the program and doesn't have to be transformed. Similarly to the transformation of $\langle \mathcal{V}(\mathbf{u} \times \mathbf{n}), \mathbf{v} \times \mathbf{n}\rangle$ we get

$$\begin{aligned}\mathcal{V}(\mathbf{E})(\mathbf{y}) \cdot (\mathbf{v}(\mathbf{y}) \times \mathbf{n}_\mathbf{y}) = \frac{1}{\left|\mathbf{c}_\mathbf{y}^{(1)} \times \mathbf{c}_\mathbf{y}^{(2)}\right|^2} &\Bigg\{ \sum_{k=0}^{p-1} \sum_{l=0}^{p} r_{kl}^{(1)} \left[\sum_{i=1}^{3} V(E_i, y_1^k y_2^l)(\mathbf{a}_\mathbf{y}^{(1)} \times \mathbf{n}_\mathbf{y})_i \right] \\ &+ \sum_{k=0}^{p} \sum_{l=0}^{p-1} r_{kl}^{(2)} \left[\sum_{i=1}^{3} V(E_i, y_1^k y_2^l)(\mathbf{a}_\mathbf{y}^{(2)} \times \mathbf{n}_\mathbf{y})_i \right] \Bigg\}.\end{aligned}$$

The integrals $V(E_i, y_1^k y_2^l)$ are calculated using numerical quadrature.

6.2.1 Regularization of single layer and double layer potential

In the following, we explain the evaluation of the single layer potential and the double layer potential applied to a given function.

The single layer potential

The Maxwell single layer potential $\mathcal{V}$ is the the Dirichlet trace of the vectorial single layer potential $\mathcal{L}$

$$\mathcal{V}\mathbf{u}(\mathbf{x}) = (\mathbf{n}_\mathbf{x} \times \mathcal{L}\mathbf{u}(\mathbf{x})) \times \mathbf{n}_\mathbf{x}.$$

First of all, we consider the expansion of the fundamental solution of the Helmholtz equation into a Taylor series. There holds

$$\frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} = \frac{1}{|\mathbf{x}-\mathbf{y}|} + i\kappa - \kappa^2 |\mathbf{x}-\mathbf{y}| - \dots.$$

Thus, the fundamental solution consists of the fundamental solution of the Laplace equation and regular terms.

The implementation in *maiprogs* can be done by evaluating the usual single layer potential and we write

$$\begin{aligned}\mathcal{L}\mathbf{u}(\mathbf{x}) &= \frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\mathbf{u}(\mathbf{y})\,dS_\mathbf{y}\\ &= \frac{1}{4\pi}\int_\Gamma \frac{1}{|\mathbf{x}-\mathbf{y}|}\mathbf{u}(\mathbf{y})\,dS_\mathbf{y} + \frac{1}{4\pi}\int_\Gamma \left(\frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\mathbf{u}(\mathbf{y})\,dS_\mathbf{y}.\end{aligned}$$

The first term is the single layer potential for the Laplace equation which can be evaluated analytically, see Maischak [74]. For the regularized kernel in the second integral we use a numerical quadrature.

Internally, in *maiprogs* we have to evaluate

$$\frac{e^{i\kappa d}}{d} - \frac{1}{d} = \frac{\cos(\kappa d) - 1}{d} + \frac{\sin(\kappa d)}{d}i,$$

where d denotes the distance between two points. For small κd ($< 10^{-6}$) we use the approximation

$$\frac{\cos(\kappa d) - 1}{d} + \frac{\sin(\kappa d)}{d}i \approx -\frac{\kappa^2 d}{2} + \frac{\kappa^4 d^3}{24} + i\left(\kappa - \frac{\kappa^3 d^2}{6}\right)$$

which is numerically more stable because of the avoidance of erasements.

The double layer potential

In order to evaluate the double layer potential with the Helmholtz fundamental solution

$$K\psi(\mathbf{x}) = \frac{1}{4\pi}\int_\Gamma \frac{\partial}{\partial n_\mathbf{y}}\frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\,\psi(\mathbf{y})\,dS_\mathbf{y}$$

we first consider the derivative with respect to one component y_j

$$\frac{\partial}{\partial y_j}\frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} = \left(\frac{i\kappa(y_j - x_j)}{|\mathbf{x}-\mathbf{y}|^2} - \frac{(y_j - x_j)}{|\mathbf{x}-\mathbf{y}|^3}\right)e^{i\kappa|\mathbf{x}-\mathbf{y}|}.$$

It follows that we have to evaluate the kernel for a given distance d by

$$\begin{aligned}\frac{e^{i\kappa d}}{d^2}i\kappa - \frac{e^{i\kappa d}}{d^3} &= \left(\frac{i\kappa}{d^2} - \frac{\kappa^2}{d} + \dots\right) - \left(\frac{1}{d^3} + \frac{i\kappa}{d^2} - \frac{\kappa^2}{2}\frac{1}{d} + \dots\right)\\ &= -\frac{1}{d^3} - \frac{\kappa^2}{2d} + \text{ smoother terms.}\end{aligned}$$

The first two terms can be evaluated analytically using the formulas of Maischak [74]. For the analytical terms we use a numerical quadrature. The advantage of this regularization is, that we got rid of the $\frac{1}{d^2}$ term.

6.3 Numerical experiments

6.3.1 The scattering problem

Here, we consider one example to test the implementation of the scattering problem. As domain we take the unit cube $\Omega = [-1, 1]^3$ and we choose a plane wave as the incident wave

$$\mathbf{E}^{in} := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (\cos \kappa y + i \sin \kappa y).$$

Furthermore, we set $\mu \equiv 1$, $\epsilon \equiv 1$ in $\mathbb{R}^3$ such that the plane wave goes through the obstacle without being scattered. Thus, the exact solution is given by

$$\mathbf{E}(\mathbf{x}) = \begin{cases} \mathbf{E}^{in} & , \mathbf{x} \in \Omega \\ 0 & , \mathbf{x} \in \Omega_E \end{cases} .$$

For our investigation we use different κ. In Table 6.1 we present the results for $\kappa = 0.5$ for the uniform h-version with polynomial degree $p = 1$. Due to the a priori estimate (6.10), we expect a convergence rate of $\frac{1}{3}$ with respect to the degrees of freedom. This correspondents to the results in Table 6.1. In Figure 6.2 we compare the h-version with different κ for $p = 1$. The errors behave the same way. Finally, in Figures 6.3 and 6.4 we consider the h-version for different polynomial degrees p. The behavior is the same we already got from the eddy current problem in Chapter 5.

h	DOF	$\|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl})(\Omega)}$	$\|\boldsymbol{\lambda}\|_{\mathbf{L}^2(\Gamma)}$	$\|\boldsymbol{\lambda}\|_{H^{-1/2}(\Gamma)}$	$e_{\mathbf{H}(\mathbf{curl}),H^{-1/2}}$	α
2	24	0.4224329	0.2108881	0.0153619	0.4227122	
1	102	0.2065661	0.0632124	0.0013874	0.2065708	0.4948780
2/3	252	0.1368602	0.0354160	0.0003384	0.1368607	0.4551680
1/2	492	0.1024041	0.0224634	0.0001177	0.1024042	0.4335043
2/5	840	0.0818297	0.0160496	.5182E-04	0.0818297	0.4192891
1/3	1314	0.0681478	0.0121257	.2630E-04	0.0681478	0.4089168
2/7	1932	0.0583893	0.0095922	.1480E-04	0.0583893	0.4009191
1/4	2712	0.0510773	0.0078259	.8972E-05	0.0510773	0.3945161
2/9	3672	0.0453938	0.0065431	.5765E-05	0.0453938	0.3892579
1/5	4830	0.0408490	0.0055753	.3877E-05	0.0408490	0.3848571

Table 6.1: Scattering problem: Errors in L^2-norm and energy norm and convergence rate α with respect to the degrees of freedom for $\kappa = 0.5$, h-version, polynomial degree $p = 1$.

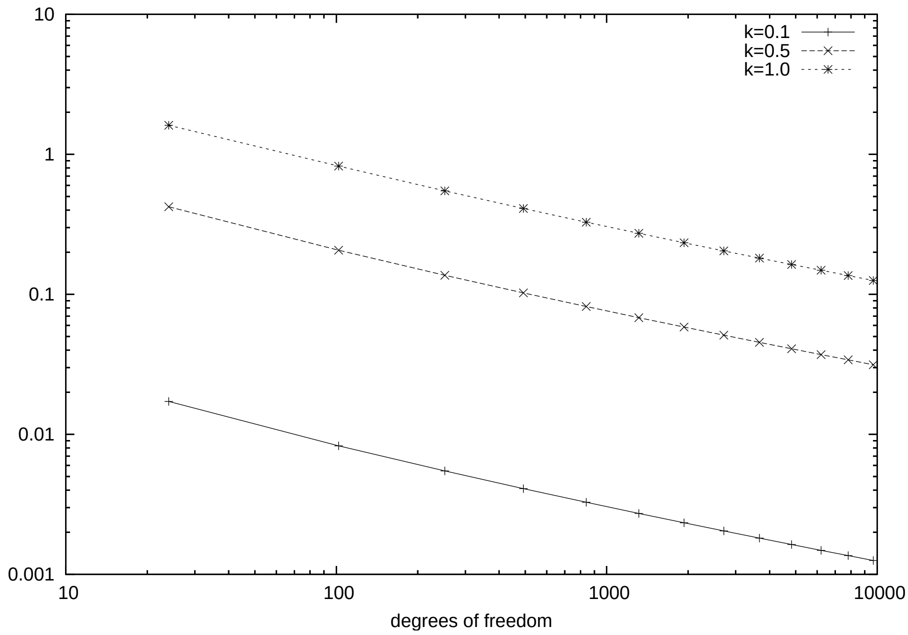

Figure 6.2: Scattering problem: Energy norm e of the Galerkin error $(\mathbf{E} - \mathbf{u}_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h)$ for different κ, h-version, $p = 1$.

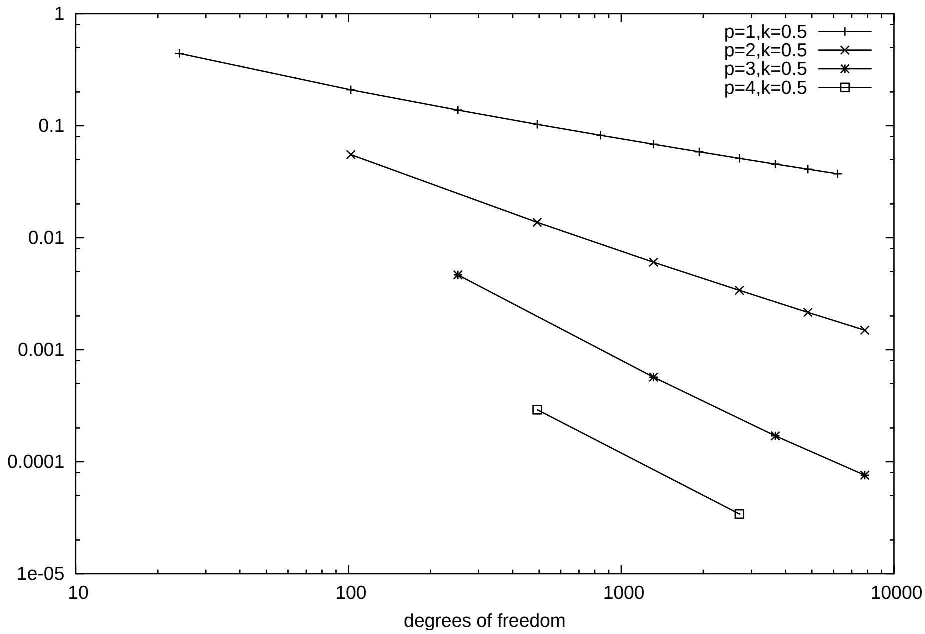

Figure 6.3: Scattering problem: Energy norm e of the Galerkin error $(\mathbf{E} - \mathbf{u}_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h)$ for $\kappa = 0.5$, h-version, different p.

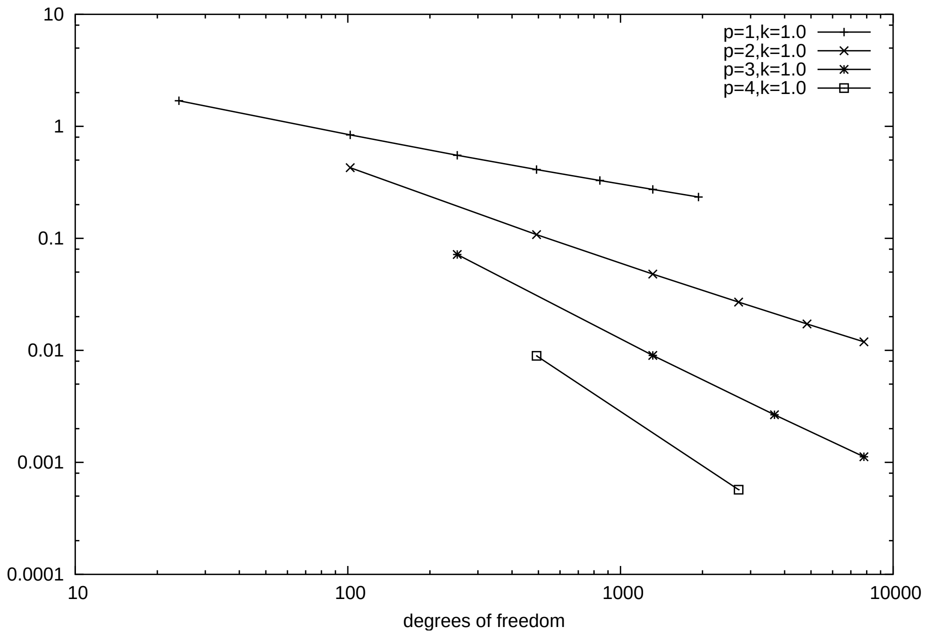

Figure 6.4: Scattering problem: Energy norm e of the Galerkin error $(\mathbf{E} - \mathbf{u}_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h)$ for $\kappa = 1$, h-version, different p.

6.3.2 The electric field integral equation

As part of the coupling formulation in (6.8) we get the electric field boundary integral equation (EFIE):
Find $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *such that*

$$\langle \mathcal{V}\boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle_\Gamma - \frac{1}{\kappa^2} \langle V \,\mathrm{div}_\Gamma \,\boldsymbol{\lambda}, \mathrm{div}_\Gamma \,\boldsymbol{\zeta} \rangle_\Gamma = f(\mathbf{v})$$

for all $\boldsymbol{\zeta} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, compare Hiptmair & Schwab [68]. In the following we present first results of the discretization of this equation. We use Raviart-Thomas functions on Γ.

For our numerical experiment we choose the reference cube $\Omega = [-1, 1]^3$ and consider the right hand side $f \equiv 1$. There is no exact solution known and so we have to extrapolate the exact solution using a sequence of uniform meshes. In Figures 6.5 and 6.6 the results for the h and the p-version are presented. One sees that the procedure convergences and that the p-version converges faster than the h-version.

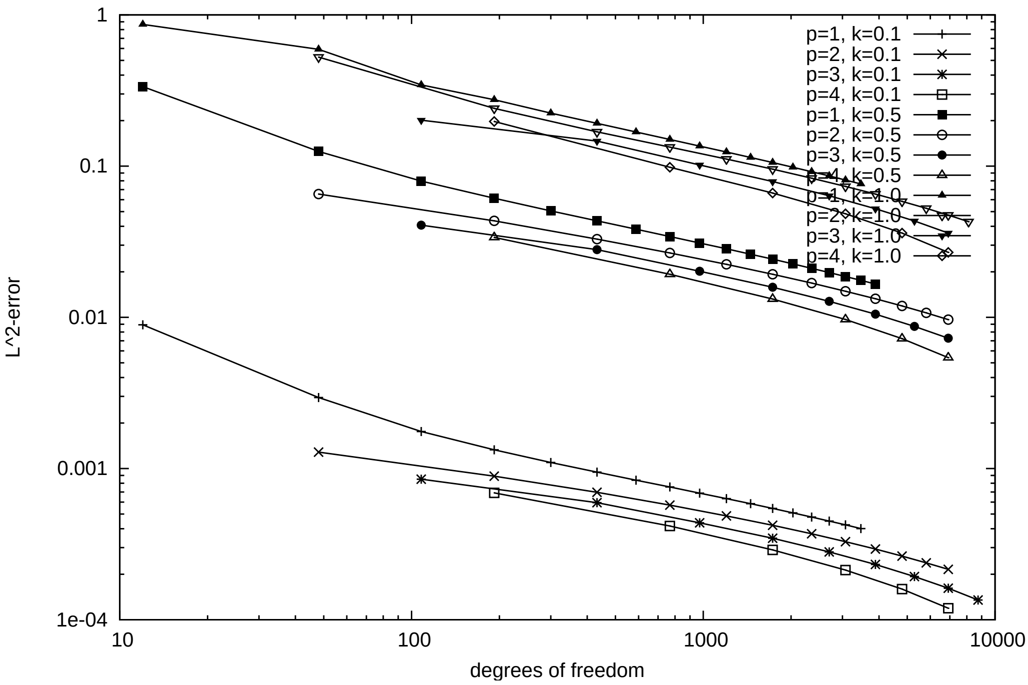

Figure 6.5: EFIE: uniform h-version with different p, different κ, error in L^2-norm.

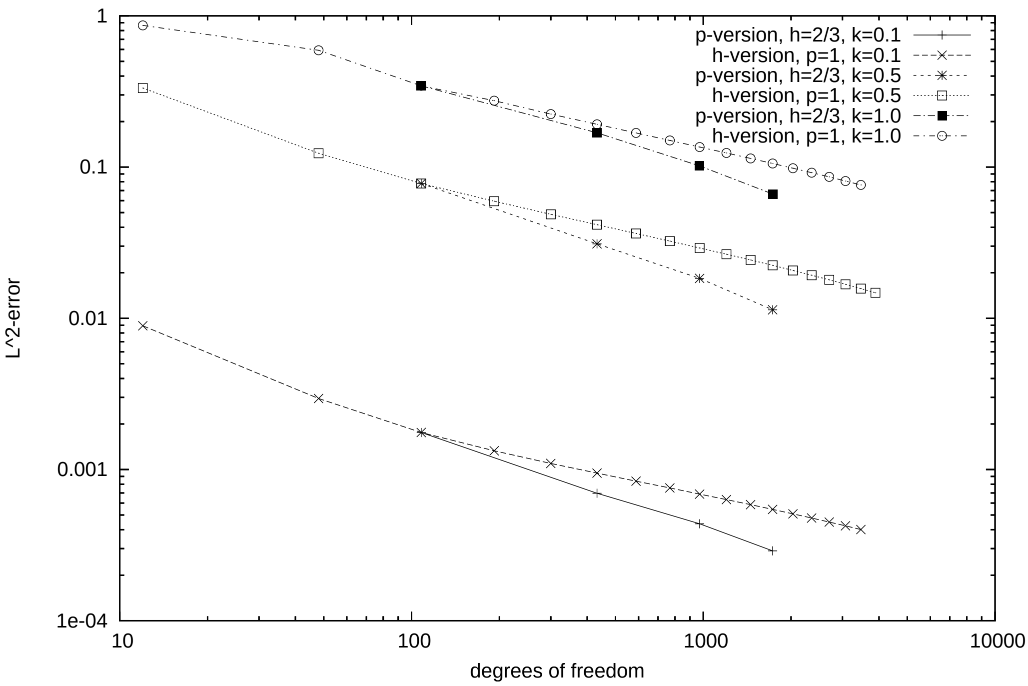

Figure 6.6: EFIE: uniform p-version, $h = 2/3$, error in L^2-norm.

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Curriculum vitae

15.06.1974	Born in Frankfurt/Main, Germany
July 1994	Abitur at Gymnasium Langenhagen
09/1994 – 11/1995	Alternative civilian service ("Zivildienst") at hospital Oststadtkrankenhaus, Hannover
11/1995 – 05/2001	Mathematics studies with minor subject Physics, Hannover University
10/1997	Intermediate diploma in Mathematics
04/1999 – 02/2000	Teaching assistant at the Institute of Mathematical Stochastics, Hannover University
09/2000 – 11/2000	Teaching assistant at the Institute of Mathematics, Hannover University
05/2001	Diploma in Mathematics, topic of the thesis: "Iteration of entire functions"
since June 2001	Scientific assistant and Ph.D. student at the Institute of Applied Mathematics, Hannover University