Optimal investment

in

time inhomogeneous

Poisson models

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Dipl.-Math. Mirko Kötter

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Referent: Prof. Dr. N. Bäuerle, Universität Karlsruhe

Korreferent: Prof. Dr. H. Schmidli, Universität zu Köln

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Zusammenfassung

Die Risikotheorie beschäftigt sich allgemein mit dem Versicherungsgeschäft und insbesondere mit Fragen der Solvenz. Das grundlegende Modell der Risikotheorie ist das klassische Cramér-Lundberg-Modell. Schadenankünfte sowie Schadenhöhen sind in diesem Modell zeitlich homogen und das Versicherungsunternehmen erhält kontinuierlich Prämien gemäß einer konstanten Prämienrate. In einigen Versicherungsbereichen stellte sich diese zeitliche Homogenität jedoch als unrealistisch heraus. Verschiedene zeitlich nicht homogene Modelle wie etwa das Markov-modulierte Poisson-Modell oder das periodische Poisson-Modell wurden infolgedessen eingeführt. Beide Modelle werden von einem Umweltprozess beeinflusst, der im ersten Modell durch einen Markov-Prozess und im zweiten Modell durch eine periodische Funktion gegeben ist.
Das klassische Cramér-Lundberg-Modell wurde vor kurzem unter der zusätzlichen Annahme untersucht, dass der Versicherer in einen Aktienindex investieren kann, der durch eine geometrische Brownsche Bewegung modelliert wird. Unglücklicherweise ist die zugehörige Ruinwahrscheinlichkeit $\Psi(u)$ in Abhängigkeit vom Anfangskapital u des Versicherungsunternehmens schwer zu bestimmen. Deshalb konzentriert man sich auf den Anpassungskoeffizienten des Modells, der als die größtmögliche Konstante R definiert ist, so dass $\Psi(u) \leq C\, e^{-Ru}$ mit $C > 0$ für alle u gilt. Es stellte sich heraus, dass der Anpassungskoeffizient unter allen, ausschließlich vom aktuellen Guthaben abhängenden Investitionsstrategien durch eine Strategie maximiert wird, die einen konstanten Betrag in den Aktienindex investiert.
Diese Arbeit zielt darauf ab, entsprechende Aussagen für die beiden oben genannten zeitlich nicht homogenen Poisson-Modelle herzuleiten. Die beiden Modelle werden daher unter der Möglichkeit untersucht, in einen Aktienindex zu investieren, der durch eine geometrische Brownsche Bewegung modelliert wird. Da im klassischen Cramér-Lundberg-Modell eine konstante Investitionsstrategie optimal ist, sind nur solche Strategien zugelassen, die ausschließlich vom jeweiligen Umweltprozess abhängen. In beiden Modellen ergibt sich mit Hilfe von Martingalmethoden, dass der zugehörige Anpassungskoeffizient unter allen zugelassenen Strategien wiederum durch ein konstantes Investment maximiert wird.
Das Markov-modulierte Poisson-Modell mit Investment wird in dieser Arbeit außerdem durch ein Markov-moduliertes Poisson-Modell ohne Investment approximiert. Es wird gezeigt, wie mit Hilfe dieser Approximation eine Darstellung für den Anpassungskoeffizienten des Markov-modulierten Poisson-Modells unter einer fest gewählten Investitionsstrategie gefunden werden kann. Schließlich gelingt ein direkter Vergleich der Ruinwahrscheinlichkeiten des Markov-modulierten Poisson-Modells und des zugehörigen klassischen Cramér-Lundberg-Modells mit gemittelten Parametern unter derselben konstanten Investitionsstrategie.

Schlagwörter: Ruinwahrscheinlichkeit, Anpassungskoeffizient, optimales Investment, Markovscher Umweltprozess, periodischer Umweltprozess, Martingalmethoden, Diffusionsapproximation.

Abstract

Risk theory in general is concerned with the business of insurance companies and in particular with aspects of solvency. The basic model in risk theory is the classical Cramér-Lundberg model. In this model claim arrivals as well as claim sizes are homogeneous in time and the insurance company receives premiums at a constant rate. However, it turned out that time homogeneity is not a realistic assumption for certain areas of insurance. Different time inhomogeneous models as for example the Markov-modulated Poisson model or the periodic Poisson model were therefore introduced. Both models are governed by an environmental process which is a Markov process in the first model and a periodic function in the second model.

Recently, the classical Cramér-Lundberg model was studied under the additional assumption that the insurer has the opportunity to invest into a stock index which is modelled by some geometric Brownian motion. Unfortunately, it is difficult to determine the corresponding ruin probability $\Psi(u)$ with respect to the initial reserve u of the insurance company. Hence, one concentrates on the adjustment coefficient of the model which is defined as the largest constant R fulfilling $\Psi(u) \leq C\,e^{-Ru}$ with $C > 0$ for all u. It was discovered that amongst all investment strategies which exclusively depend on the current wealth the adjustment coefficient is maximized by a strategy which invests a constant amount into the stock index.

This work aims to derive corresponding assertions for the two time inhomogeneous Poisson models mentioned above. The two models are consequently considered with the additional opportunity to invest into a stock index which is modelled by a geometric Brownian motion. Since a constant investment strategy is optimal in the classical Cramér-Lundberg model only investment strategies which exclusively depend on the respective environmental process are admitted. Using martingale methods it follows for both models that amongst all admissible strategies the corresponding adjustment coefficient is again maximized by a constant investment.

Further, the Markov-modulated Poisson model with investment is approximated by some Markov-modulated Poisson model without investment in this work. Using this approximation it is shown how to find a representation for the adjustment coefficient of the Markov-modulated Poisson model under some fixed investment strategy. Eventually, a pointwise comparison between the ruin probabilities of the Markov-modulated Poisson model and its associated classical Cramér-Lundberg model with averaged parameters under the same constant investment strategy is given.

Keywords: Ruin probability, adjustment coefficient, optimal investment, Markovian environment, periodic environment, martingale methods, diffusion approximation.

Contents

Chapter 1

Introduction

In general risk theory is concerned with the business of insurance companies and in particular with questions of solvency. The study of risk theory was initiated in the first half of the last century. It started with the basic model in risk theory, the so-called classical Cramér-Lundberg model. In this model claim arrivals as well as claim sizes are homogeneous in time and the insurance company receives premiums at a constant rate. Later, it turned out that time homogeneity is not a realistic assumption for certain areas of insurance. Different time inhomogeneous risk models were therefore introduced during the second half of the last century. Amongst them models in a stochastic Markovian environment and respectively in a deterministic periodic environment. Recently, the classical Cramér-Lundberg model was studied under the additional assumption that the insurer has the opportunity to invest into a stock index which is modelled by some geometric Brownian motion. It was discovered that the ruin probability is minimized by investing a certain constant amount into the stock index if the initial reserve is sufficiently large. This work now deals with optimal investment strategies for risk models in a Markovian and respectively a periodic environment.

1.1 Risk theory

The basic model for the time evolution of the reserves of an insurance company is the risk reserve process. Depending on the initial reserve of the insurance company, denoted by $u \geq 0$, the risk reserve process $R(u) := \{R_t(u), t \geq 0\}$ is defined by

$$R_t(u) := u + ct - \sum_{k=1}^{N_t} U_k$$

where $c > 0$ is the premium rate over time, N_t is the number of claims which occur until time $t \geq 0$ and U_k is the claim size of the k^{th} occurring claim. Note that the process $R(0)$ is often called the surplus process whereas the process $S = \{S_t, t \geq 0\}$ defined by $S_t := \sum_{k=1}^{N_t} U_k - ct$ is known as the claim surplus process.

In this work we assume that the claim sizes have exponential moments which means that for every claim U_k the expectation $\mathbb{E}(e^{rU_k})$ is finite for some $r > 0$. This case is often referred to as the small claim or respectively the light-tailed case. The large claim or respectively the heavy-tailed case where $\mathbb{E}(e^{rU_k})$ is infinite for all $r > 0$ is omitted in this work. Hence, we only summarize results for the small claim case in that what follows.

In risk theory the ruin probability in infinite time $\Psi(u)$, or ruin probability for short, is defined as the probability that the risk reserve process ever drops below zero provided that the initial reserve is given by $u \geq 0$, i.e.

$$\Psi(u) := \mathbb{P}\Big(\inf_{t \geq 0} R_t(u) < 0\Big).$$

The ruin probability is obviously of huge interest for the insurance company. However, only in certain cases we are able to calculate the ruin probability explicitly. Hence, the so-called Lundberg inequality is often considered which means that we choose R as large as possible such that

$$\Psi(u) \leq C\, e^{-Ru}$$

holds for all $u \geq 0$ where $C < \infty$ is some constant. The right hand side of this inequality is then called the Lundberg bound for the ruin probability $\Psi(u)$ and R is called the adjustment coefficient of the model.

The classical model in risk theory is the compound Poisson model which is broadly known as the classical Cramér-Lundberg model. In this model the claim arrival process $N := \{N_t, t \geq 0\}$ is a standard Poisson process and the claim sizes are independent and identically distributed with some common distribution concentrated on $(0, \infty)$.

Some of the main ideas were introduced by Lundberg [Lun1903] whereas the first mathematically substantial results were given in Lundberg [Lun26] and respectively Cramér [Cra30]. Meanwhile, it is well known for the small claim case that the ruin probability in this model decreases exponentially fast with the initial reserve of the insurer.

Since it turned out that the time homogeneity of the compound Poisson model is not realistic for certain areas of insurance, as for example car insurance where weather conditions play a major role for the occurrence of accidents, the Markov-modulated Poisson model has become more and more popular over the last decades. In this model the claims are not assumed to be homogeneous in time but determined by an irreducible Markov process on some finite state space, the so-called environmental Markov process. It is assumed that the intensity of the arrival process and the claim size distribution vary depending on the current state of the environmental Markov process. The Markov-modulated Poisson model was first introduced by Janssen [Jan80] and Reinhard [Rei84]. A more comprehensive treatment as well as a comparison with the classical compound Poisson model can be found in Asmussen [Asm89] and Asmussen et al [AFR+95], respectively.

Another possibility to get away from the time homogeneity of the classical compound Poisson model is to consider a deterministic periodic environment instead of the stochastic Markov-modulated environment. In such a periodic Poisson model the claim arrival process is a Poisson process whose intensity is given by a deterministic periodic function. Also the claim size distribution is assumed to depend periodically on its arrival time where the period is the same as for the intensity function. The periodic Poisson model has for example been studied in Beard et al [BPP84], Dassios and Embrechts [DE89] or Asmussen and Rolski ([AR92] and [AR94]).

There are of course other time inhomogeneous models in risk theory as for example the general Cox model which covers the Poisson models mentioned above. This model where the claims arrive according to a Cox process is due to Ammeter [Amm48]. Another time inhomogeneous model is the so called Sparre-Andersen model where the occurrence of the claims is described by a renewal process as introduced by Andersen [And57]. Good references for this model are Thorin [Tho74] and a review from the same author [Tho82].

However, in this work we concentrate on the Markov-modulated and respectively the periodic Poisson model. Note that we can compare each of these two models with an associated compound Poisson model by averaging over the environment. In this regard we refer the reader to the books by Gerber [Ger79], Grandell [Grl91], Rolski et al [RSS+99] or Asmussen [Asm00] which provide a good survey of risk theory in general and the Poisson models mentioned above in particular. In these sources one can also find results for the large claim case.

All the Poisson models introduced above can certainly be expanded by adding a further stochastic process to the underlying risk reserve process. Gerber [Ger70] for example kept the time homogeneity of the classical compound Poisson model but enlarged the corresponding risk reserve process by some diffusion component, namely a Brownian motion. A somewhat more detailed study of this model can be found in Dufresne and Gerber [DG91].

Later, Furrer and Schmidli [FS94] considered a risk reserve process which is also perturbed by a Brownian motion but where the claim arrival process is either a renewal process or a Cox process with a so-called independent jump intensity. Schmidli [Schm95] expanded this considerations to the Markov-modulated Poisson model which is perturbed by diffusion.

It has only been recently that the compound Poisson model was studied under the additional assumption that the insurer has the opportunity to invest into a risky asset. To the best of our knowledge, Paulsen et al ([GP97] and [Pau98]) were the first who incorporated a stochastic rate of return on investments. However, in their model the entire wealth of the insurance company is invested into the risky asset whose price process is modelled by another classical surplus process which is assumed to be independent of the original risk reserve process. Frolova, Kabanov and Pergamenshchikov [FKP02] investigated the same model but where the insurer invests into a stock index whose price process is determined by some geometric Brownian motion like in the classical Black-Scholes setting.

Later, Hipp and Plum [HP00] considered the case where the insurance company may invest parts of its wealth into a stock index whose price process is given by some geometric Brownian motion. They dealt with the question how to invest into the stock index in order to minimize the probability of ruin. Using the

Hamilton-Jacobi-Bellman equation a non-linear integro-differential equation for the minimal ruin probability was derived and the existence of a solution as well as a verification theorem was proved. For the case with exponential claim size distribution and special parameter values they gave an explicit solution.

Using an exponential martingale method Gaier, Grandits and Schachermayer [GGS03] showed that amongst all investment strategies which depend on the current wealth it is asymptotically optimal to invest a certain constant amount into a stock index in the sense that the corresponding adjustment coefficient is maximized. At this, the price process of the stock index was again modelled by some geometric Brownian motion. Eventually, Grandits [Grt04] as well as Hipp and Schmidli [HS04] specified an asymptotic approximation for the minimal ruin probability, the so-called Cramér-Lundberg approximation.

1.2 Outline of this work

After this introductory chapter we consider the Markov-modulated Poisson model in the small claim case. In our model the insurer has the opportunity to invest into a stock index whose price process is modelled by some geometric Brownian motion. This model of course implies the corresponding compound Poisson model which was treated in Gaier, Grandits and Schachermayer [GGS03]. They found out that the adjustment coefficient of the compound Poisson model with investment is maximized by a constant investment strategy. In this connection, the invested amount was allowed to be larger than the actual wealth or even negative in their work. For the Markov-modulated Poisson model we consequently admit investment strategies which only depend on the environmental Markov-process and which allow to invest an arbitrarily large amount into the stock index even if it is negative.

After introducing the Markov-modulated Poisson model with investment we initially determine the adjustment coefficient of this model when using any fixed investment strategy. Our methods are based on an exponential martingale technique given in Björk and Grandell [BG88] which is similar to the one used in Gaier, Grandits and Schachermayer [GGS03]. The obtained adjustment coefficient is then maximized with respect to the applied investment strategy. It turns out that the maximum is attained for a certain constant investment strategy.

Note that the Markov-modulated Poisson model under any constant investment strategy becomes a Markov-modulated Poisson model which is perturbed by some Brownian motion. We thus compare our assertions with a result for the Markov-modulated Poisson model perturbed by diffusion given in Schmidli [Schm95]. Thereafter, we prove that the obtained constant investment strategy is indeed asymptotically optimal in the sense that it minimizes the corresponding ruin probability for a sufficiently large initial reserve. Eventually, the Markov-modulated Poisson model and its associated compound Poisson model under the respective optimal investment strategy are compared in terms of their adjustment coefficients.

In the third chapter it is shown how to approximate the Markov-modulated Poisson model with investment by some Markov-modulated Poisson model without investment. The idea is based on the fact that a diffusion arises as the limit of properly scaled classical claim surplus processes where the claims are very small and frequent as for example given in Grandell [Grl77].

In the second part of the third chapter we use the obtained approximation in order to deduce results from what is known for the Markov-modulated Poisson model without investment. On the one hand we derive an approximation for the adjustment coefficient of the Markov-modulated Poisson model under an arbitrarily fixed investment strategy. On the other hand the ruin probabilities of the Markov-modulated Poisson model and its associated compound Poisson model are compared directly when using the same constant investment strategy in both models. For this comparison some additional assumptions on the model are needed in order to apply a result in Asmussen et al [AFR$^+$95].

In the fourth and final chapter of this work the deterministic periodic Poisson model is considered in the small claim case. As before the insurer has the opportunity to invest into a stock index whose price process is modelled by some geometric Brownian motion. The chapter is organized analogously to the chapter about the Markov-modulated Poisson model with investment. Initially, we again admit investment strategies which only depend on the periodic environment and which provide to invest an arbitrary amount into the stock index.

After introducing the periodic Poisson model with investment the corresponding adjustment coefficient is determined when using any fixed investment strategy. We then maximize this adjustment coefficient with respect to the applied investment strategy. As in the Markov-modulated environment the maximum is attained for some constant strategy. Later, we verify that this strategy is asymptotically optimal even amongst a broader class of investment strategies. The periodic Poisson model and its associated compound Poisson model are finally compared and it turns out that the optimal investment strategies and the associated adjustment coefficients coincide for both models.

1.3 Notation

The underlying probability space is generally denoted by $(\Omega, \mathcal{A}, \mathbb{P})$ and supposed to be sufficiently large. Thus, almost surely or respectively a.s. means $\mathbb{P}$-a.s. and $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$.

Further, let $\mathcal{G}$ be any σ-algebra. The probability measure $\mathbb{P}^{\mathcal{G}}$ then denotes the probability measure $\mathbb{P}$ conditioned under $\mathcal{G}$. With respect to the probability measure $\mathbb{P}^{\mathcal{G}}$ we consequently denote the expectation and the variance by $\mathbb{E}^{\mathcal{G}}$ and $\mathrm{Var}^{\mathcal{G}}$, respectively.

For the Markov-modulated Poisson model, $\mathbb{P}_j$ denotes the probability measure $\mathbb{P}$ conditioned under the event that the environmental Markov process starts in state $j \in E$. Thus, $\mathbb{E}_j$ denotes the expectation with respect to $\mathbb{P}_j$.

If two random elements X and Y have the same distribution we write $X \overset{\mathcal{D}}{=} Y$. At this, the distribution or respectively the law of a random element X is the image probability measure $P \circ X^{-1}$. Furthermore, we say that a sequence of random elements $\big(X_n\big)_{n\in\mathbb{N}}$ of some metric space (S, m) converges in distribution to a random element X of (S, m) as $n \to \infty$, denoted by $X_n \Rightarrow X$, if

$$\lim_{n\to\infty} \mathbb{E}\Big(f\big(X_n\big)\Big) = \mathbb{E}\Big(f\big(X\big)\Big)$$

for all real-valued, continuous, bounded functions f on S. In this definition the metric m apparently determines which functions on S are continuous. For random variables $X, X_1, X_2, \ldots$ with values in $(\mathbb{R}, |\cdot|)$ it turns out that $X_n \Rightarrow X$ denotes the commonly known convergence in distribution. The convergence of stochastic processes is defined in chapter 3 in terms of an adequate metric space.

Moreover note that the same symbol, say F, is used for a distribution F and its cumulative distribution function $F(x)$, i.e. $F(x) = \int_{-\infty}^{x} dF(t)$.

Finally, the following notation is used throughout this work:

$\mathbb{N}$	strictly positive integers $\{1, 2, \ldots\}$
$\mathbb{N}_0$	non-negative integers $\mathbb{N} \cup \{0\}$
$\mathbb{R}$	real line $(-\infty, \infty)$
$\mathbb{R}_+$	strictly positive real line $(0, \infty)$
$f(x^-)$	left limit $\lim_{t \uparrow x} f(t)$
$\overline{F}(x)$	tail of $F(x)$, i.e. $\overline{F}(x) = 1 - F(x)$
μ_F	mean of F, i.e. $\mu_F = \int x \, dF(x)$
$I(A) : \Omega \to \{0, 1\}$	indicator function of the event $A \in \mathcal{A}$,
	i.e. $I(A)(\omega) = 1$ if and only if $\omega \in A$
$\delta_B : \mathbb{R} \to \{0, 1\}$	indicator function of the set $B \subseteq \mathbb{R}$,
	i.e. $\delta_B(x) = 1$ if and only if $x \in B$
δ_{ij}	Kronecker's symbol, i.e. $\delta_{ij} = \delta_{\{i\}}(j)$
I_d	$d \times d$ identity matrix
$\mathrm{diag}\Big(a_i \, ; i \in \{1, \ldots, d\}\Big)$	$d \times d$ diagonal matrix with diagonal
	elements a_i for $i = 1, \ldots, d$
$\square$	marks the end of a proof
$\diamond$	marks the end of a remark or an example

Chapter 2

The Markov-modulated Poisson model with investment

In this chapter we consider the risk reserve process of an insurance company in a Markov-modulated environment where the insurer additionally has the opportunity to invest into a stock index. The price process of this stock index is modelled by a geometric Brownian motion and the invested amount of money only depends on the current state of the environmental Markov process. It is assumed that the claims have exponential moments.

After introducing the model we determine the adjustment coefficient of the Markov-modulated Poisson model under any fixed investment strategy in section 2.2. In the following two sections this adjustment coefficient is maximized with respect to the investment strategy. In section 2.5 we then show that the resulting adjustment coefficient and the corresponding investment strategy are indeed optimal. Finally, the adjustment coefficients of the Markov-modulated Poisson model and its associated compound Poisson model are compared under the respective optimal strategy.

2.1 The model

In the Markov-modulated Poisson model the premium rate and claim arrivals are not homogeneous in time but determined by a Markov-modulated environment. This environment is described by a continuous-time Markov process which is defined on some finite state space $E = \{1, \ldots, d\}$. We denote this environmental Markov process by J and its intensity matrix by $Q = \big(q_{ij}\big)_{i,j \in E}$. It is generally assumed that the environmental Markov process is irreducible. Since the state space E is finite this implies that J has a stationary distribution which is denoted by π. Unless otherwise stated the initial distribution of J is arbitrary.

The premium rate and the claim arrivals are influenced by the environmental Markov process in the following way. At time $t \geq 0$ the premium rate is given by c_{J_t} where $c_i > 0$ for $i \in E$, i.e. in time intervals when the environmental Markov process is in state $i \in E$ we have a linear income at constant rate c_i. Further, the claim arrival process $N := \{N_t, t \geq 0\}$ is assumed to be a Markov-modulated Poisson process. This means that N has intensity $\{\lambda_{J_t}, t \geq 0\}$ with $\lambda_i > 0$ for $i \in E$.

Moreover, a claim U_k which occurs at time $t \geq 0$ has distribution B_{J_t} where B_i is some distribution concentrated on $(0, \infty)$ for $i \in E$. Conditioned under the environmental Markov-process J, the claims $(U_k)_{k \in \mathbb{N}}$ are as usual assumed to be mutually independent and also to be independent of the Markov-modulated Poisson process N. The corresponding Markov-modulated risk reserve process $R(u) := \{R_t(u), t \geq 0\}$ is finally given by

$$R_t(u) = u + \int_0^t c_{J_s}\, ds - \sum_{k=1}^{N_t} U_k \tag{2.1}$$

where $u \geq 0$ is the initial reserve of the insurance company.

Furthermore, let the insurer have the opportunity to invest into a stock index or say some portfolio. The price process $S := \{S_t, t \geq 0\}$ of this portfolio is modelled by a geometric Brownian motion with dynamics

$$dS_t = S_t \left(a\, dt + b\, dW_t \right) ,\ t \geq 0\,.$$

Here, W is a standard Brownian motion independent of J as well as $R(u)$ and $a \in \mathbb{R}$, $b > 0$ are fixed constants. Let K_t be the amount of money which the insurer invests into the portfolio at time $t \geq 0$. We then call the process $K := \{K_t, t \geq 0\}$ the investment strategy of the insurer. Note that K_t can also be negative or even larger than the actual wealth for any $t \geq 0$. This fact can respectively be interpreted as the possibility to sell the portfolio short or to borrow an arbitrary amount of money from the bank. Further, $K = 0$ means that nothing is invested into the portfolio, i.e. $K_t \equiv 0$ for $t \geq 0$.

It is assumed throughout this chapter that the invested amount of money only depends on the current state of the environmental Markov process. This means that there exists some function $k : E \to \mathbb{R}$ such that $K_t = k(J_t)$ for $t \geq 0$. As a shorthand notation for this fact we write $K = k(J)$. If at time $t \geq 0$ the insurer invests the amount K_t into the portfolio and the remaining part of his reserve into a bond which yields no interest, the wealth process $Y(u, K) := \{Y_t(u, K), t \geq 0\}$ is given by

$$Y_t(u, K) = R_t(u) + \int_0^t \frac{K_v}{S_v}\, dS_v = R_t(u) + \int_0^t K_v\, dW_{a,b}(v) ,\ t \geq 0\,. \tag{2.2}$$

Here, $W_{a,b}$ denotes the Brownian motion with drift defined by $W_{a,b}(t) := at + bW_t$ for $t \geq 0$ where $a \in \mathbb{R}$ is called the drift parameter and $b > 0$ the volatility of the process. For notational reasons, let the surplus process $X(K) := \{X_t(K), t \geq 0\}$ be defined by $X_t(K) = Y_t(u, K) - u$ for $t \geq 0$.

Note that the investment strategy K should of course be defined by $K_t = k(J_{t-})$

for $t \geq 0$. Otherwise the insurer does not know how much to invest at a certain time $t \geq 0$ since K_t depends on the state of the Markov process at time $t \geq 0$. However, we see that the strategies $K = \{k(J_t), t \geq 0\}$ and $\{k(J_{t-}), t \geq 0\}$ coincide for a fixed function $k : E \to \mathbb{R}$ except at points of time where the environmental Markov process makes a jump to another state. Thus, the wealth process $Y(u, K)$ defined in (2.2) and the wealth process defined by (2.2) with K replaced by $\{k(J_{t-}), t \geq 0\}$ clearly coincide almost surely. It is therefore sufficient to consider the case where K is defined by $K_t = K(J_t)$ for $t \geq 0$.

Next, we define the time which the environmental Markov process J spends in some state $i \in E$ until time $t \geq 0$ by $\xi_i(t)$, i.e. $\xi_i(t) := \int_0^t \delta_{\{i\}}(J_s)\, ds$. Let us then consider independent standard Poisson processes $N^{(1)}, \ldots, N^{(d)}$ which are also independent of J. It is assumed that $N^{(i)} := \{N_t^{(i)}, t \geq 0\}$ has intensity λ_i for $i \in E$. Moreover, let $(U_k^{(1)})_{k\in\mathbb{N}}, \ldots, (U_k^{(d)})_{k\in\mathbb{N}}$ be independent sequences of random variables which are also independent of the processes $N^{(1)}, \ldots, N^{(d)}$ and J. It is further assumed that the random variables $(U_k^{(i)})_{k\in\mathbb{N}}$ are independent and identically distributed with distribution B_i for $i \in E$. Then,

$$R_t(u) \overset{\mathcal{D}}{=} u + \int_0^t c_{J_s}\, ds - \sum_{i\in E} \sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U_k^{(i)}\,, \; t \geq 0\,. \tag{2.3}$$

Furthermore, let $W^{(1)}, \ldots, W^{(d)}$ be independent standard Brownian motions which are also independent of the risk reserve process as given in (2.3) including the environmental Markov process J. We then have

$$\begin{aligned} Y_t(u, K) &= u + \int_0^t c_{J_s}\, ds - \sum_{k=1}^{N_t} U_k + a \int_0^t K_s\, ds + b \int_0^t K_s\, dW_s \\ &\overset{\mathcal{D}}{=} u + \int_0^t c_{J_s}\, ds - \sum_{i\in E} \sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U_k^{(i)} + a \sum_{i\in E} \int_0^{\xi_i(t)} k(i)\, ds + b \sum_{i\in E} \int_0^{\xi_i(t)} k(i)\, dW_s^{(i)} \end{aligned}$$

$$= u + \int_0^t c_{J_s}\, ds + a \sum_{i \in E} k(i)\, \xi_i(t) + b \sum_{i \in E} k(i)\, W^{(i)}_{\xi_i(t)} - \sum_{i \in E} \sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k \,,\ t \geq 0\,.$$

Without loss of generality we can furthermore assume that the premium rate is the same for all environmental states. Note that we are only interested in the ruin probability in infinite time. By applying the time transformation $\hat{Y}_t(u,K) := Y_{T(t)}(u,K)$ with $T(t) := \int_0^t \frac{c}{c_{J_s}}\, ds$ the structure of the model consequently does not change. We can therefore assume without loss of generality that $c_i = c$ for some $c > 0$ and all $i \in E$.

Nevertheless, the parameters of the Markov-modulated Poisson model change accordingly. The time transformed environmental Markov process $\hat{J}$ which is defined by $\hat{J}_t := J_{T(t)}$ for $t \geq 0$ has intensity matrix $\left(\frac{c}{c_i}\, q_{ij}\right)_{i,j \in E}$ and thus stationary distribution $\hat{\pi}$ where $\hat{\pi}_i := \frac{c_i\, \pi_i}{\sum_{j \in E} c_j\, \pi_j}$. We also have to notice that obviously $\hat{\xi}_i(t) := \int_0^{T(t)} \delta_{\{i\}}(\hat{J}_s)\, ds$ is equal to $\frac{c_i}{c}\, \xi_i(t)$ for all $t \geq 0$ and $i \in E$. Finally, the time transformed standard Poisson process $\hat{N}^{(i)}$ defined by $\hat{N}^{(i)}_t := N^{(i)}_{T(t)}$ for $t \geq 0$ has intensity $\frac{c}{c_i}\, \lambda_i$ for $i \in E$.

Hence, let us from now on consider the wealth process $Y(u,K)$ defined by

$$Y_t(u,K) := u + ct + a \sum_{i \in E} k(i)\, \xi_i(t) + b \sum_{i \in E} k(i)\, W^{(i)}_{\xi_i(t)} - \sum_{i \in E} \sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k \tag{2.4}$$

for $t \geq 0$. Note, if the investment strategy is constant over all $i \in E$, i.e. if $\hat{K}_t \equiv \hat{k}$ for all $t \geq 0$ and some $\hat{k} \in \mathbb{R}$, then (2.4) becomes

$$Y_t(u,\hat{K}) = u + (c + a\,\hat{k})\, t + b\,\hat{k} W_t - \sum_{i \in E} \sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k \,,\ t \geq 0\,. \tag{2.5}$$

Let the natural filtration of the wealth process $Y(u,K)$ be denoted by $\mathcal{F}^Y := \{\mathcal{F}^Y_t, t \geq 0\}$ and the natural filtration of the environmental Markov process J by $\mathcal{F}^J := \{\mathcal{F}_t^J, t \geq 0\}$. We then define the filtration $\mathcal{F} := \{\mathcal{F}_t, t \geq 0\}$ by $\mathcal{F}_t := \mathcal{F}_t^Y \vee \mathcal{F}_\infty^J$ for $t \geq 0$. Note that $\mathcal{F}_0 = \mathcal{F}_\infty^J$ so that K_t is apparently $\mathcal{F}_0$-measurable for all $t \geq 0$.

We generally suppose that the claims have exponential moments. This means that for every $i \in E$ there exists a possibly infinite constant $r_\infty^{(i)} > 0$ such that the centered moment generating function h_i defined by

$$h_i(r) := \int_0^\infty e^{rx}\, dB_i(x) - 1\,,\; r \geq 0\,,$$

is finite for every $r < r_\infty^{(i)}$. It is moreover assumed that $h_i(r) \to \infty$ as $r \to r_\infty^{(i)}$. Considering any fixed $i \in E$ this assumption implies that h_i is increasing, convex and continuous on $[0, r_\infty^{(i)})$ with $h_i(0) = 0$. The important part of this assumption is that $h_i(r) < \infty$ for some $r > 0$. Thus, the tail of the distribution B_i decreases at least exponentially fast. By this condition the lognormal and Pareto distribution are for example excluded. Further, the case when $\lim_{r \to r_\infty^{(i)}} h_i(r) < \infty$ and $h(r) = \infty$ for $r > r_\infty^{(i)}$ is not allowed. An example that such cases exist is for example given on page 3 of Grandell [Grl91].

In this work we are interested in the ruin probability in infinite time which is defined as

$$\Psi(u, K) := \mathbb{P}\Big(\inf_{t \geq 0} Y_t(u, K) < 0\Big)$$

depending on the initial reserve $u \geq 0$ and the investment strategy $K = k(J)$. Furthermore, let

$$\tau(u, K) := \inf \Big\{ t > 0\,; Y_t(u, K) < 0 \Big\}$$

be the corresponding time of ruin. It is obvious that $\Psi(u, K) = \mathbb{P}\big(\tau(u, K) < \infty\big)$. In this chapter we study the so-called Lundberg inequality

$$\Psi(u, K) \leq C\, e^{-Ru} \tag{2.6}$$

with $C < \infty$ for all $u \geq 0$ where $K = k(J)$ is some fixed investment strategy. At this, the right hand side of (2.6) is called Lundberg bound for the ruin probability $\Psi(u, K)$ and the largest possible R such that (2.6) holds is called the adjustment coefficient or Lundberg exponent of the model. The aim of this chapter is to maximize this adjustment coefficient with respect to the investment strategy $K = k(J)$. We then call such an optimal adjustment coefficient the adjustment coefficient of the Markov-modulated model under optimal investment and the corresponding strategy the optimal investment strategy. It turns out that this optimal investment strategy is constant over all environmental states $i \in E$.

Note that the classical compound Poisson model without investment coincides with the present Markov-modulated Poisson model with investment when $E = \{1\}$ and $K = 0$. As for example given in Asmussen [Asm00], the adjustment coefficient of the compound Poisson model is given as the unique strictly positive solution of the equation

$$\lambda h(r) = cr$$

where $\lambda := \lambda_1$ and $h(r) := h_1(r)$.

It can be found in the same book that the adjustment coefficient of the Markov modulated model without investment is given as the strictly positive solution of the equation $\tilde{\kappa}(r) = 0$ where $\tilde{\kappa}(r)$ is that eigenvalue of the matrix

$$Q + \mathrm{diag}\Big(\lambda_i h_i(r)\,; i \in E\Big) - crI_d\,,\; r \geq 0\,,$$

which has maximum real part. Here, $\mathrm{diag}\big(\lambda_i h_i(r)\,; i \in E\big)$ is the $d \times d$ diagonal matrix with diagonal elements $\lambda_i h_1(r), \ldots, \lambda_d h_d(r)$ and I_d is the $d \times d$ identity matrix.

Another very important value in risk theory is the so-called safety loading. The Markov-modulated Poisson model has the property that there exists a constant ς such that $\lim_{t\to\infty} \frac{1}{t}\sum_{k=1}^{N_t} U_k = \varsigma$ almost surely. The interpretation of ς is as the average amount of claim per unit time. Without investment it is intuitively clear that independently of the initial reserve the insurance company will get ruined if the premium rate does not exceed ς. In this context the relative safety loading is widely used. For the Markov-modulated model without investment it is defined as $\rho = \frac{c-\varsigma}{\varsigma}$. It is in fact well known for this model that independently of $u \geq 0$ the insurer gets almost surely ruined whenever $\rho \leq 0$, confer Asmussen [Asm00].

Let us now consider the Markov-modulated Poisson model with investment. For our purposes it suffices to consider the absolute safety loading $\rho^{(K)}$ with respect to some investment strategy $K = k(J)$. It is defined as the almost sure limit of $\frac{1}{t}X_t(u, K)$ as $t \to \infty$. If this limit almost surely exists we thus have

$$\lim_{t\to\infty} \frac{1}{t}\, Y_t(0, K) \overset{a.s.}{=} \rho^{(K)} \,.$$

We refer to $\rho^{(K)}$ as the safety loading with respect to $K = k(J)$ unless otherwise stated. Under some regularity assumptions it is later shown that $\rho^{(K)} > 0$ is a necessary and sufficient condition for an adjustment coefficient to exist where $K = k(J)$ is some fixed investment strategy.

As mentioned in the introductory chapter, Gaier, Grandits and Schachermeyer [GGS03] considered the compound Poisson model with the additional opportunity to invest into a stock index. The price process of the stock index was modelled by a geometric Brownian motion in the same way as described above. They found out that amongst all investment strategies which depend on the current wealth a constant strategy is optimal in the sense that it maximizes the corresponding adjustment coefficient. This optimal strategy K^* is defined by $K_t^* \equiv \frac{a}{R^* b^2}$ for $t \geq 0$. At this, R^* is the adjustment coefficient of the compound Poisson model under optimal investment and determined as the unique strictly positive solution of the equation

$$\lambda h(r) = cr + \frac{a^2}{2b^2}$$

where we again put $\lambda := \lambda_1$ and $h(r) := h_1(r)$.

The authors also showed that such a solution and therefore an adjustment coefficient exists as long as the drift parameter a of the Brownian motion with drift $W_{a,b}$ does not equal zero. Note that this is even the case if the safety loading of the underlying model without investment is not strictly positive. However, in the case where $a = 0$ the safety loading of the model without investment has to be strictly positive in order to assure that an adjustment coefficient exists.

2.2 The adjustment coefficient for any fixed investment strategy

Throughout this section we consider an arbitrary but fixed investment strategy $K = k(J)$. In order to obtain a Lundberg bound for the ruin probability $\Psi(u, K)$ we choose an exponential martingale technique as given in Björk and Grandell [BG88] which goes back to Gerber [Ger73]. An appropriate exponential martingale is given as follows.

Proposition 2.1. *Let the investment strategy $K = k(J)$ and $u, r \geq 0$ be fixed. Then, the process $M(u, K, r)$ defined by*

$$M_t(u, K, r) = \frac{\exp\Big(- rY_t(u, K)\Big)}{\exp\Big(\sum_{i\in E} \big[\lambda_i h_i(r) + \frac{1}{2} r^2 b^2 k(i)^2 - r(c + ak(i))\big]\xi_i(t)\Big)}$$

for $t \geq 0$ is a martingale with respect to $\mathcal{F}$.

Proof:

Let $t \geq 0$ and $i \in E$. Note, Proposition 2.13 in Yong and Zhou [YZ99], page 20, for example shows that under the probability measure $\mathbb{P}\big(\cdot \,\big|\mathcal{F}^J_\infty\big)(\omega)$, where $\omega \in \Omega$ is fixed, the random variable $\xi_i(t)$ is almost surely a deterministic constant $\xi_i(t)(\omega)$. This implies that $\mathbb{E}\big(e^{-\alpha W^{(i)}_{\xi_i(t)}}\big|\mathcal{F}^J_\infty\big) = e^{\frac{\alpha^2}{2}\xi_i(t)}$ a.s. for all $\alpha \in \mathbb{R}$ as well as $\mathbb{P}\big(N^{(i)}_{\xi_i(t)} = m\big|\mathcal{F}^J_\infty\big) = e^{-\lambda_i}\frac{(\lambda_i\xi_i(t))^m}{m!}$ a.s. for all $m \in \mathbb{N}$. At this, the first equality follows since the process $\big(e^{\alpha W_t - \frac{\alpha^2}{2}t}\big)_{t\geq 0}$ is a martingale whenever W is a standard Brownian motion. It therefore follows that almost surely

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}^J_\infty}\Big(\exp\big(- rb \sum_{i\in E} k(i)\, W^{(i)}_{\xi_i(t)}\big)\Big) &= \prod_{i\in E} \mathbb{E}^{\mathcal{F}^J_\infty}\Big(\exp\big(- rbk(i)\, W^{(i)}_{\xi_i(t)}\big)\Big) \\
&= \prod_{i\in E} \exp\Big(\frac{r^2 b^2 k(i)^2}{2}\,\xi_i(t)\Big)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}^J_\infty}\Big(\exp\big(r\sum_{i\in E}\sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k\big)\Big) &= \prod_{i\in E}\mathbb{E}^{\mathcal{F}^J_\infty}\Big(\exp\big(r\sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k\big)\Big)\\
&= \prod_{i\in E}\mathbb{E}^{\mathcal{F}^J_\infty}\Big(\sum_{m=0}^{\infty}\exp\Big(r\sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k\Big)\, I(N^{(i)}_{\xi_i(t)}=m)\Big)\\
&= \prod_{i\in E}\sum_{m=0}^{\infty}\mathbb{E}^{\mathcal{F}^J_\infty}\Big[\exp\Big(r\sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k\Big)\Big| N^{(i)}_{\xi_i(t)}=m\Big]\,\mathbb{P}^{\mathcal{F}^J_\infty}\big(N^{(i)}_{\xi_i(t)}=m\big)\\
&= \prod_{i\in E}\sum_{m=0}^{\infty}\mathbb{E}^{\mathcal{F}^J_\infty}\Big(e^{r\sum_{k=1}^m U^{(i)}_k}\Big)\,\mathbb{P}^{\mathcal{F}^J_\infty}\big(N^{(i)}_{\xi_i(t)}=m\big)\\
&= \prod_{i\in E}\sum_{m=0}^{\infty}\prod_{k=1}^{m}\mathbb{E}\Big(e^{rU^{(i)}_k}\Big)\,\mathbb{P}^{\mathcal{F}^J_\infty}\big(N^{(i)}_{\xi_i(t)}=m\big)\\
&= \prod_{i\in E}\sum_{m=0}^{\infty}\big(1+h_i(r)\big)^m\, e^{-\lambda_i\xi_i(t)}\,\frac{(\lambda_i\xi_i(t))^m}{m!}\\
&= \prod_{i\in E} e^{-\lambda_i\xi_i(t)}\sum_{m=0}^{\infty}\frac{\big((1+h_i(r))\lambda_i\xi_i(t)\big)^m}{m!}\\
&= \prod_{i\in E} e^{\lambda_i h_i(r)\xi_i(t)} = \exp\Big(\sum_{i\in E}\lambda_i\, h_i(r)\,\xi_i(t)\Big)\,.
\end{aligned}$$

Putting the things together we consequently obtain

$$\begin{aligned}
&\mathbb{E}\big[M_t(u,K,r)\big|\mathcal{F}_s\big]\\
&= M_s(u,K,r)\\
&\quad\cdot\mathbb{E}^{\mathcal{F}^J_\infty}\left[\frac{\exp\Big(-rb\sum_{i\in E}k(i)\,(W^{(i)}_{\xi_i(t)}-W^{(i)}_{\xi_i(s)})+r\sum_{i\in E}\sum_{k=N^{(i)}_{\xi_i(s)}+1}^{N^{(i)}_{\xi_i(t)}}U^{(i)}_k\Big)}{\exp\Big(\sum_{i\in E}\big(\lambda_i h_i(r)+\frac{1}{2}r^2b^2k(i)^2\big)\big(\xi_i(t)-\xi_i(s)\big)\Big)}\middle|\mathcal{F}^Y_s\right]\\
&= M_s(u,K,r)\\
&\quad\cdot\frac{\mathbb{E}^{\mathcal{F}^J_\infty}\Big(\exp\big(-rb\sum_{i\in E}k(i)\,W^{(i)}_{\xi_i(t)-\xi_i(s)}\big)\Big)}{\exp\Big(\sum_{i\in E}\frac{r^2b^2k(i)^2}{2}\big(\xi_i(t)-\xi_i(s)\big)\Big)}\,\frac{\mathbb{E}^{\mathcal{F}^J_\infty}\Big(\exp\big(r\sum_{i\in E}\sum_{k=1}^{N^{(i)}_{\xi_i(t)-\xi_i(s)}}U^{(i)}_k\big)\Big)}{\exp\Big(\sum_{i\in E}\lambda_i h_i(r)\big(\xi_i(t)-\xi_i(s)\big)\Big)}\\
&= M_s(u,K,r)
\end{aligned}$$

for $0 \leq s \leq t$. Since $M_t(u, K, r)$ is positive for all $t \geq 0$ this also implies that the process $M(u, K, r)$ is integrable. It is moreover easy to see that $M(u, K, r)$ is measurable with respect to $\mathcal{F}$. Thus, $M(u, K, r)$ is an exponential martingale with respect to the filtration $\mathcal{F}$.

□

If $r \leq 0$ the inequality $\Psi(u, K) \leq e^{-ru}$ is trivially fulfilled for all $u \geq 0$ where $K = k(J)$ is any fixed investment strategy. For $r > 0$ an upper bound for the ruin probability $\Psi(u, K)$ can now be found using the exponential martingale from Proposition 2.1.

Proposition 2.2. *Let the investment strategy $K = k(J)$ and $r > 0$ be fixed. Then, we have*

$$\Psi(u, K) \leq e^{-ru}\, C(K, r)$$

for all $u \geq 0$ where

$$C(K, r) := \mathbb{E}\left(\sup_{t \geq 0}\, \exp\left(\sum_{i \in E}\left[\lambda_i h_i(r) + \frac{r^2 b^2 k(i)^2}{2} - r\big(c + a k(i)\big)\right] \xi_i(t)\right)\right).$$

Proof:

For simplicity reasons let us denote the time of ruin by $\tau := \tau(u, K)$. We have already shown in Proposition 2.1 that the process $M(u, K, r)$ is a martingale with respect to the filtration $\mathcal{F}$. Hence, also the stopped process $\tilde{M}(u, K, r)$ defined by $\tilde{M}_t(u, K, r) := M_{t \wedge \tau}(u, K, r)$ is a martingale with respect to $\mathcal{F}$.

For $r > 0$ and $u \geq 0$ it therefore follows that

$$\begin{aligned} e^{-ru} &= \tilde{M}_0(u, K, r) = \mathbb{E}\Big[\tilde{M}_t(u, K, r)\Big|\mathcal{F}_0\Big] = \mathbb{E}^{\mathcal{F}^J_\infty}\Big(\tilde{M}_t(u, K, r)\Big) \\ &= \mathbb{E}^{\mathcal{F}^J_\infty}\Big(\tilde{M}_t(u, K, r)\, I(\tau \leq t)\Big) + \mathbb{E}^{\mathcal{F}^J_\infty}\Big(\tilde{M}_t(u, K, r)\, I(\tau > t)\Big) \end{aligned}$$

$$
\begin{aligned}
&\geq \mathbb{E}^{\mathcal{F}^J_\infty}\left[M_\tau(u,K,r)\middle|\tau \leq t\right]\,\mathbb{P}^{\mathcal{F}^J_\infty}(\tau \leq t)\\
&= \mathbb{E}^{\mathcal{F}^J_\infty}\left[\frac{\exp\left(-rY_\tau(u,K)\right)}{\exp\left(\sum_{i\in E}\left[\lambda_i h_i(r)+\frac{r^2b^2k(i)^2}{2}-r\big(c+ak(i)\big)\right]\xi_i(\tau)\right)}\middle|\tau \leq t\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot\,\mathbb{P}^{\mathcal{F}^J_\infty}(\tau \leq t)\\
&\geq \frac{\mathbb{P}^{\mathcal{F}^J_\infty}(\tau \leq t)}{\sup_{0\leq v\leq t}\ \exp\left(\sum_{i\in E}\left[\lambda_i h_i(r)+\frac{r^2b^2k(i)^2}{2}-r\big(c+ak(i)\big)\right]\xi_i(v)\right)}
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\mathbb{P}^{\mathcal{F}^J_\infty}(\tau \leq t)\\
&\qquad \leq e^{-ru}\sup_{0\leq v\leq t}\ \exp\left(\sum_{i\in E}\left[\lambda_i h_i(r)+\frac{r^2b^2k(i)^2}{2}-r\big(c+ak(i)\big)\right]\xi_i(v)\right)\,.
\end{aligned}
$$

Letting $t\to\infty$ and taking the expectation on both sides we obtain

$$
\Psi(u,K)=\mathbb{P}\big(\tau(u,K)<\infty\big)\leq e^{-ru}\,C(K,r)\,.
$$

□

We therefore have to maximize $r>0$ under the restriction that $C(K,r)<\infty$ in order to get the asymptotically best possible upper bound using Proposition 2.2. Hence, put

$$
R^{(K)} := \sup\Big\{r>0\,;C(K,r)<\infty\Big\}\,. \tag{2.7}
$$

We consequently say that $R^{(K)}$ does not exist if $C(K,r)=\infty$ for all $r>0$.

The way to find such a maximizing constant $R^{(K)}$ is similar to what Björk and Grandell [BG88] do for the ordinary Cox model. Let the time epoch of the n^{th} entry of the environmental Markov process to state $j\in E$ be denoted by $\tau_n^{(j)}$.

This means we recursively define

$$\tau_n^{(j)} := \inf\left\{t > \tau_{n-1}^{(j)}\,; J_{t-} \neq j, J_t = j\right\}$$

for $n \in \mathbb{N}$ where $\tau_0^{(j)} \equiv 0$. Since $\tau_1^{(j)}$ is often used we put $\tau^{(j)} := \tau_1^{(j)}$. For $j, k \in E$ we now have to consider the function $\phi_{kj}^{(K)}$ defined by

$$\phi_{kj}^{(K)}(r) := \mathbb{E}_k\left(\exp\left(\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2 b^2 k(i)^2}{2} - r\big(c + ak(i)\big)\right]\xi_i(\tau^{(j)})\right)\right)$$

where $r \geq 0$. Using these functions we are able to state a necessary condition for $C(K, r)$ being finite.

Proposition 2.3. *Let $K = k(J)$ and $r > 0$ be fixed. Then, $\phi_{jj}^{(K)}(r) < 1$ and $\phi_{kj}^{(K)}(r) < \infty$ for all $k, j \in E$ is a necessary condition for $C(K, r) < \infty$.*

Proof:

Let $r > 0$ and $K = k(J)$ be fixed. For any given $\omega \in \Omega$ the function $\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2b^2k(i)^2}{2} - r\big(c + ak(i)\big)\right]\xi_i(t)$ is piecewise linear in t. Hence, it suffices to examine $\exp\left(\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2b^2k(i)^2}{2} - r\big(c + ak(i)\big)\right]\xi_i(t)\right)$ at the jump times $\big(\tau_n^{(j)}\big)_{n\in\mathbb{N}}$, $j \in E$, of the environmental Markov process J. We obtain

$$\begin{aligned}
C(K, r) = \mathbb{E}&\left(\sup_{t\geq 0}\exp\left(\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2b^2k(i)^2}{2} - r\big(c + ak(i)\big)\right]\xi_i(t)\right)\right) < \infty \\
\Leftrightarrow \mathbb{E}&\left(\max_{j\in E}\sup_{n\in\mathbb{N}}\exp\left(\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2b^2k(i)^2}{2} - r\big(c + ak(i)\big)\right]\xi_i(\tau_n^{(j)})\right)\right) < \infty \\
\Leftrightarrow \mathbb{E}&\left(\sup_{n\in\mathbb{N}}\exp\left(\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2b^2k(i)^2}{2} - r\big(c + ak(i)\big)\right]\xi_i(\tau_n^{(j)})\right)\right) < \infty \\
&\qquad \forall j \in E \quad (2.8)
\end{aligned}$$

where the last equivalence follows since E is finite.

Without loss of generality we now assume that $J_0 = k$ and consider any fixed $j \in E$. For $n \in \mathbb{N}$ define

$$Z_n^{(j)}(K,r) := \sum_{i \in E} \left[\lambda_i h_i(r) + \frac{r^2 b^2 k(i)^2}{2} - r\big(c + ak(i)\big) \right] \Big(\xi_i(\tau_n^{(j)}) - \xi_i(\tau_{n-1}^{(j)}) \Big)$$

and

$$\begin{aligned} W_n^{(j)}(K,r) &:= \exp\left(\sum_{i \in E} \left[\lambda_i h_i(r) + \frac{r^2 b^2 k(i)^2}{2} - r\big(c + ak(i)\big) \right] \xi_i(\tau_n^{(j)}) \right) \\ &= \exp\left(\sum_{m=1}^{n} Z_m^{(j)}(K,r) \right) = \prod_{m=1}^{n} \exp\left(Z_m^{(j)}(K,r) \right) . \end{aligned}$$

For simplicity reasons put $Z_n^{(j)} := Z_n^{(j)}(K,r)$ and $W_n^{(j)} := W_n^{(j)}(K,r)$, $n \in \mathbb{N}$. Since the $Z_n^{(j)}$ are independent for all $n \in \mathbb{N}$ and also identically distributed for $n \geq 2$ we get

$$\mathbb{E}_k\left(W_n^{(j)}\right) = \mathbb{E}_k\left(e^{Z_1^{(j)}}\right) \prod_{m=2}^{n} \mathbb{E}\left(e^{Z_m^{(j)}}\right) = \phi_{kj}^{(K)}(r) \left(\phi_{jj}^{(K)}(r)\right)^{n-1} .$$

Thus, $\phi_{kj}^{(K)}(r) < \infty$ is clearly a necessary condition for $C(K,r) < \infty$. Moreover, $\phi_{jj}^{(K)}(r) > 1$ implies $\mathbb{E}_k\big(W_n^{(j)}\big) \to \infty$ as $n \to \infty$ and therefore $C(K,r) = \infty$.

Now suppose that $C(K,r) < \infty$ and $\phi_{jj}^{(K)}(r) = 1$. Recall that the $Z_n^{(j)}$ are independent and identically distributed for $n \geq 2$. $(W_n^{(j)})_{n \in \mathbb{N}}$ is therefore a martingale with respect to its natural filtration since

$$\mathbb{E}[W_{n+1}^{(j)} | W_n^{(j)}] = W_n^{(j)}\, \mathbb{E}\left(e^{Z_{n+1}^{(j)}}\right) = W_n^{(j)}\, \phi_{jj}^{(K)}(r) = W_n^{(j)} \ , \ n \in \mathbb{N} .$$

Jensen's inequality yields $\exp\big(\mathbb{E}(Z_n^{(j)})\big) < \mathbb{E}\big(e^{Z_n^{(j)}}\big) = \phi_{jj}^{(K)}(r) = 1$ and thus $\mathbb{E}\big(Z_n^{(j)}\big) < 0$ for $n \geq 2$. Now, $C(K,r) < \infty$ implies $\phi_{kj}^{(K)}(r) < \infty$ and therefore $Z_1^{(j)} \overset{a.s.}{<} \infty$ in particular. It hence follows that $\lim_{n\to\infty} \sum_{k=1}^{n} Z_k^{(j)} \overset{a.s.}{=} -\infty$ and consequently $\lim_{n\to\infty} W_n^{(j)} \overset{a.s.}{=} 0$.

We have already shown in (2.8) that $C(K,r) < \infty$ implies $\mathbb{E}(\sup_{n\in\mathbb{N}} W_n^{(j)}) < \infty$ which means that $(W_n^{(j)})_{n\in\mathbb{N}}$ is uniformly integrable. By standard martingale theory the existence of a random variable $W_\infty^{(j)}$ with $W_\infty^{(j)} \overset{a.s.}{=} \lim_{n\to\infty} W_n^{(j)}$ and $\mathbb{E}[W_\infty^{(j)}|W_n^{(j)}] = W_n^{(j)}$ for all $n \in \mathbb{N}$ follows.

Knowing that $\lim_{n\to\infty} W_n^{(j)} \overset{a.s.}{=} 0$ we conclude that $W_\infty^{(j)} \overset{a.s.}{=} 0$ and accordingly $W_n^{(j)} \overset{a.s.}{=} 0$ for all $n \in \mathbb{N}$ in contradiction to

$$W_2^{(j)} = \exp\left(\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2b^2k(i)^2}{2} - r\big(c + ak(i)\big)\right]\xi_i(\tau_2^{(j)})\right) \overset{a.s.}{>} 0$$

for example. Hence, we cannot have $C(K,r) < \infty$ and $\phi_{jj}^{(K)}(r) = 1$ for some $j \in E$ at the same time.

$\square$

Let us take a closer look at the environmental Markov process J. Remember that we denote its intensity matrix by $Q = (q_{ij})_{i,j\in E}$. Putting $q_i := -q_{ii}$ for $i \in E$ the corresponding embedded Markov chain has transition probability matrix $P = (p_{ij})_{i,j\in E}$ defined by $p_{ij} := (1 - \delta_{ij})\frac{q_{ij}}{q_i}$ where δ_{ij} is Kronecker's symbol.

For $n \in \mathbb{N}$ let $\sigma_n^{(j)}$ be the time which the environmental Markov process J spends in state $j \in E$ when the process makes its n^{th} visit to this state. It is well known that conditioned under the embedded Markov chain the $\sigma_n^{(j)}$ are independent for all $n \in \mathbb{N}$ and $j \in E$ and that the sequence $(\sigma_n^{(j)})_{n\in\mathbb{N}}$ is furthermore identically distributed with $\sigma_1^{(j)} \sim \text{Exp}(q_j)$, $j \in E$. For simplicity reasons put $\sigma^{(j)} := \sigma_1^{(j)}$.

We now define the function $\phi_j^{(K)}$ by

$$\phi_j^{(K)}(r) := \mathbb{E}\left(\exp\left(\left[\lambda_j h_j(r) + \frac{r^2b^2k(j)^2}{2} - r\big(c + ak(j)\big)\right]\sigma^{(j)}\right)\right)$$

for $r \geq 0$ and $j \in E$. It thus follows from what is mentioned above that

$$\phi_{kj}^{(K)}(r) = \phi_{k}^{(K)}(r)\, p_{kj} + \sum_{\substack{m \in E \\ m \neq j}} \phi_{k}^{(K)}(r)\, p_{km}\, \phi_{mj}^{(K)}(r) \tag{2.9}$$

and in particular

$$\phi_{jj}^{(K)}(r) = \sum_{m \in E} \phi_{j}^{(K)}(r)\, p_{jm}\, \phi_{mj}^{(K)}(r) \tag{2.10}$$

for all $r \geq 0$ and $j, k \in E$. Having Proposition 2.3 in mind we initially show that it suffices to consider $\phi_{jj}^{(K)}$.

Proposition 2.4. *Let $K = k(J)$, $r > 0$ and $j \in E$ be fixed. Then, $\phi_{jj}^{(K)}(r) < \infty$ implies $\phi_{kj}^{(K)}(r) < \infty$ for all $k \in E$.*

Proof:

Suppose that $\phi_{kj}^{(K)}(r) = \infty$ for some $k, j \in E$. It then follows from equation (2.9) that also $\phi_{mj}^{(K)} = \infty$ for all $m \in E$ with $p_{mk} > 0$. Since the environmental Markov process J is assumed to be irreducible we gradually get $\phi_{mj}^{(K)} = \infty$ for all $m \in E$. Using equation (2.10) we finally conclude that $\phi_{jj}^{(K)}(r) = \infty$.

□

Let $j \in E$. Recall, that the functions h_i are convex for all $i \in E$. It therefore follows that

$$\sum_{i \in E} \left[\lambda_i h_i(r) + \frac{r^2 b^2 k(i)^2}{2} - r\big(c + ak(i)\big) \right] \xi_i(\tau^{(j)})$$

and consequently also

$$\exp\left(\sum_{i \in E} \left[\lambda_i h_i(r) + \frac{r^2 b^2 k(i)^2}{2} - r\big(c + ak(i)\big) \right] \xi_i(\tau^{(j)}) \right)$$

are almost surely convex functions in r since the exponential function is convex and increasing. Taking the expectation preserves convexity. Thus, the function

$\phi_{jj}^{(K)}$ is convex and therefore continuous on the interior of its domain. Additionally, it directly follows from the definition of $\phi_{jj}^{(K)}$ that

$$\phi_{jj}^{(K)}(0) = 1\,. \tag{2.11}$$

According to the previous two propositions $\phi_{jj}^{(K)}(r) < 1$ for all $j \in E$ is a necessary condition for $C(K,r)$ to be finite. In order to show that this is also a sufficient condition for $C(K,r) < \infty$ we have to consider the following functions. For $r \geq 0$, $\delta \geq 0$ and $j,k \in E$ let $\phi_{kj}^{(K)}(r,\delta)$ be defined by

$$\begin{aligned}\phi_{kj}^{(K)}&(r,\delta)\\ &:= \mathbb{E}_k\left(\exp\left((1+\delta)\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2 b^2 k(i)^2}{2} - r\big(c + ak(i)\big)\right]\xi_i(\tau^{(j)})\right)\right)\,.\end{aligned}$$

Note that $\phi_{kj}^{(K)}(r,0) = \phi_{kj}^{(K)}(r)$. Using these functions we can also give a sufficient condition for $C(K,r) < \infty$.

Proposition 2.5. *Let $K = k(J)$ and $r > 0$ be fixed. The existence of a $\delta > 0$ such that $\phi_{jj}^{(K)}(r,\delta) < 1$ for all $j \in E$ is a sufficient condition for $C(K,r) < \infty$.*

Proof:

Recall the definition of the processes $(Z_n^{(j)})_{n\in\mathbb{N}}$ and $(W_n^{(j)})_{n\in\mathbb{N}}$ in the proof of Proposition 2.3. This time we consider the process $\big((W_n^{(j)})^{1+\delta}\big)_{n\in\mathbb{N}}$ for an arbitrary $\delta > 0$, i.e.

$$\begin{aligned}(W_n^{(j)})^{1+\delta} &= \exp\left((1+\delta)\sum_{i\in E}\left[\lambda_i h_i(r) + \frac{r^2 b^2 k(i)^2}{2} - r\big(c + ak(i)\big)\right]\xi_i(\tau_n^{(j)})\right)\\ &= \exp\left((1+\delta)\sum_{m=1}^{n} Z_m^{(j)}\right) = \prod_{m=1}^{n}\exp\big((1+\delta)Z_m^{(j)}\big)\,.\end{aligned}$$

Note that $\mathbb{E}_k\big((W_1^{(j)})^{1+\delta}\big) = \phi_{kj}^{(K)}(r,\delta)$ and $(W_{n+1}^{(j)})^{1+\delta} = (W_n^{(j)})^{1+\delta}\cdot e^{(1+\delta)Z_{n+1}^{(j)}}$ with $\mathbb{E}\big(e^{(1+\delta)Z_{n+1}^{(j)}}\big) = \phi_{jj}^{(K)}(r,\delta)$ for all $n\in\mathbb{N}$, $k,j\in E$ and $r\geq 0$.

Let $r>0$ and suppose that there exists a $\delta>0$ such that $\phi_{jj}^{(K)}(r,\delta)<1$ holds for all $j\in E$. For this $\delta>0$ and any given $j\in E$, $\big((W_n^{(j)})^{1+\delta}\big)_{n\in\mathbb{N}}$ is a positive supermartingale with respect to its natural filtration since

$$\begin{aligned}\mathbb{E}\Big[(W_{n+1}^{(j)})^{1+\delta}\Big|(W_n^{(j)})^{1+\delta}\Big] &= (W_n^{(j)})^{1+\delta}\,\mathbb{E}\left(e^{(1+\delta)Z_{n+1}^{(j)}}\right)\\ &= (W_n^{(j)})^{1+\delta}\,\phi_{jj}^{(K)}(r,\delta) < (W_n^{(j)})^{1+\delta}\,,\; n\in\mathbb{N}\,.\end{aligned}$$

A supermartingale inequality yields

$$\alpha\,\mathbb{P}\Big(\sup_{n\in\mathbb{N}}(W_n^{(j)})^{1+\delta}\geq\alpha\Big)\leq\mathbb{E}\Big((W_1^{(j)})^{1+\delta}\Big)+\sup_{n\in\mathbb{N}}\mathbb{E}\Big(\min\big\{0,(W_n^{(j)})^{1+\delta}\big\}\Big)$$

for $\alpha\geq 0$ as for example shown in Lemma 3.21 in Elliott [Ell82], page 23. Analogously to the proof of Proposition 2.4, one can show that $\phi_{jj}^{(K)}(r,\delta)<1$ implies $\phi_{kj}^{(K)}(r,\delta)<\infty$ for all $k\in E$. Hence, $\mathbb{E}\big((W_1^{(j)})^{1+\delta}\big)$ is finite under our assumptions. Since $(W_n^{(j)})^{1+\delta}$ is strictly positive for all $n\in\mathbb{N}$ we therefore have $\alpha\,\mathbb{P}\big(\sup_{n\in\mathbb{N}}(W_n^{(j)})^{1+\delta}\geq\alpha\big)\leq\mathbb{E}\big((W_1^{(j)})^{1+\delta}\big) =: D<\infty$ for all $\alpha\geq 0$. This implies

$$\begin{aligned}\mathbb{P}\left(\sup_{n\in\mathbb{N}}W_n^{(j)}\geq t\right) &= \mathbb{P}\left(\big(\sup_{n\in\mathbb{N}}W_n^{(j)}\big)^{1+\delta}\geq t^{1+\delta}\right)\\ &= \mathbb{P}\left(\sup_{n\in\mathbb{N}}(W_n^{(j)})^{1+\delta}\geq t^{1+\delta}\right)\leq D\,t^{-(1+\delta)}\end{aligned}$$

for all $t>0$ and therefore

$$\mathbb{E}\left(\sup_{n\in\mathbb{N}}W_n^{(j)}\right) = \int_0^\infty\mathbb{P}\left(\sup_{n\in\mathbb{N}}W_n^{(j)}>t\right)dt\leq 1+D\int_1^\infty t^{-(1+\delta)}\,dt<\infty\,.$$

Together with the fact that $C(K,r)<\infty$ if and only if $\mathbb{E}\big(\sup_{n\in\mathbb{N}}W_n^{(j)}\big)<\infty$ for all $j\in E$ the result follows.

$\square$

Using the previous propositions we can now state an alternative definition for $R^{(K)}$ based on the functions $\phi_{jj}^{(K)}$ for $j \in E$.

Proposition 2.6.

$$R^{(K)} = \sup\Big\{ r > 0\,; \phi_{jj}^{(K)}(r) < 1 \ \forall j \in E \Big\} \tag{2.12}$$

Proof:

At first, put $\tilde{R}^{(K)} := \sup\big\{ r > 0\,; \phi_{jj}^{(K)}(r) < 1 \ \forall j \in E \big\}$. If $\tilde{R}^{(K)}$ does not exist we must have some $j \in E$ such that $\phi_{jj}(r) \geq 1$ for all $r > 0$. It then follows by Proposition 2.3 that $C(K, r) = \infty$ for all $r > 0$ which means that also $R^{(K)}$ does not exist.

Now, let us assume that $\tilde{R}^{(K)}$ exists and consider any $0 < r < \tilde{R}^{(K)}$. Furthermore, choose some $\delta > 0$ sufficiently small such that $r' := (1+\delta)r < \tilde{R}^{(K)}$. Since $\phi_{jj}^{(K)}$ is convex with $\phi_{jj}^{(K)}(0) = 1$ it follows that $\phi_{jj}^{(K)}(r') < 1$, $j \in E$. We then get

$$\begin{aligned}
&\phi_{jj}^{(K)}(r') - \phi_{jj}^{(K)}(r,\delta)\\
&= \mathbb{E}_j \left(\exp\left(\sum_{i\in E} \left[\lambda_i h_i(r') + \frac{r'^2 b^2 k(i)^2}{2} - r'\big(c + ak(i)\big) \right] \xi_i(\tau^{(j)}) \right) \right.\\
&\qquad\quad \left. - \exp\left((1+\delta) \sum_{i\in E} \left[\lambda_i h_i(r) + \frac{r^2 b^2 k(i)^2}{2} - r\big(c + ak(i)\big) \right] \xi_i(\tau^{(j)}) \right) \right)\\
&= \mathbb{E}_j \left(\exp\left(-(1+\delta) \sum_{i\in E} r\big(c + ak(i)\big)\, \xi_i(\tau^{(j)}) \right) \right.\\
&\qquad\quad \cdot \left[\exp\left(\sum_{i\in E} \left[\lambda_i h_i\big((1+\delta)r\big) + (1+\delta)^2 \frac{r^2 b^2 k(i)^2}{2} \right] \xi_i(\tau^{(j)}) \right) \right.\\
&\qquad\qquad \left.\left. - \exp\left(\sum_{i\in E} \left[(1+\delta)\lambda_i h_i(r) + (1+\delta) \frac{r^2 b^2 k(i)^2}{2} \right] \xi_i(\tau^{(j)}) \right) \right] \right)\\
&\geq 0
\end{aligned}$$

since $(1+\delta)^2 \frac{r^2 b^2 k(i)^2}{2} \geq (1+\delta) \frac{r^2 b^2 k(i)^2}{2}$ and $h_i\big((1+\delta)r\big) \geq (1+\delta) h_i(r)$ for each $i \in E$. At this, the last inequality follows due to the fact that h_i ist convex with $h_i(0) = 0$ for all $i \in E$.

We therefore have $\phi_{jj}^{(K)}(r,\delta) \leq \phi_{jj}^{(K)}(r') < 1$ for all $j \in E$, i.e. there exists a $\delta > 0$ such that $\phi_{jj}^{(K)}(r,\delta) < 1$. This means that $\phi_{jj}^{(K)}(r) < 1$ for all $j \in E$ implies the existence of a $\delta > 0$ such that $\phi_{jj}^{(K)}(r,\delta) < 1$ for all $j \in E$. Since the latter is a sufficient condition for $C(K,r) < \infty$ we obtain $\tilde{R}^{(K)} \leq R^{(K)}$. On the other hand, it follows from Proposition 2.3 that $R^{(K)} \leq \tilde{R}^{(K)}$. Thus, $R^{(K)} = \tilde{R}^{(K)}$.

$\square$

From now on we use (2.12) as the definition of $R^{(K)}$. Consequently, $R^{(K)}$ does not exist if there is an environmental state $j \in E$ such that $\phi_{jj}^{(K)}(r) \geq 1$ for all $r > 0$. Note, if $R^{(K)}$ exists then $r \in (0, R^{(K)})$ particularly implies that $C(K,r) < \infty$.

We are now able to state the first important result of this work. In Proposition 2.2 we have already given an upper bound for the ruin probability $\Psi(u,K)$. Unfortunately, we have not been able to give any conditions ensuring that the given upper bound is finite so far. Using all the results above we can now make up for this.

Theorem 2.7. *Consider any fixed investment strategy $K = k(J)$ and suppose that $R^{(K)}$ defined by (2.12) exists. For any $r < R^{(K)}$ we then have*

$$\Psi(u,K) \leq e^{-ru}\, C(K,r)$$

with $C(K,r) < \infty$ for all $u \geq 0$.

Proof:

The inequality of interest is trivial for all $r \leq 0$. Recalling that $r \in (0, R^{(K)})$

implies $C(K, r) < \infty$ as shown in the proof of Proposition 2.6 the assertion follows.

$\square$

Note that the inequality in Theorem 2.7 holds for $r < R^{(K)}$. Theorem 2.21 in section 2.5 furthermore shows that under some mild regularity conditions we have

$$\lim_{u \to \infty} \frac{\Psi(u, K)}{e^{-ru}} = \infty$$

for all $r > R^{(K)}$. From now on we thus refer to $R^{(K)}$ as the adjustment coefficient of the Markov-modulated Poisson model with respect to the investment strategy K.

Recall, in this work we are interested in the investment strategy $K = k(J)$ which maximizes the adjustment coefficient $R^{(K)}$. Hence we do not investigate here if the Lundberg inequality given in Theorem 2.7 also holds for $R^{(K)}$ itself.

Before $R^{(K)}$ is maximized with respect to $K = k(J)$ in the following section, we conclude this section with stating conditions which ensure that the adjustment coefficient $R^{(K)}$ exists for a given investment strategy $K = k(J)$. In order to do so we need the following relation between the function $\phi_{jj}^{(K)}$ and the random variable $X_{\tau(j)}(K)$ for $j \in E$. It is also required in order to prove Theorem 2.21 in section 2.5.

Proposition 2.8. *Consider any fixed investment strategy $K = k(J)$. For all $j \in E$ and $r \geq 0$ we have*

$$\phi_{jj}^{(K)}(r) = \mathbb{E}_j\left(e^{-rX_{\tau(j)}(K)}\right).$$

Proof:

Consider any fixed investment strategy $K = k(J)$ and arbitrary $u, r \geq 0$, $j \in E$. It is shown in Proposition 2.1 that $M(u, K, r)$ is a martingale with respect to $\mathcal{F}$. Recall that $\tau^{(j)}$ is $\mathcal{F}_0$-measurable. Under the probability measure $P\big(\,\cdot\,\big|\mathcal{F}_0\big)(\omega)$, where $\omega \in \Omega$ is fixed, $\tau^{(j)}$ is therefore almost surely a deterministic constant $\tau^{(j)}(\omega)$ according to Proposition 2.13 in Yong and Zhou [YZ99], page 20. We thus have $\mathbb{E}\big[M_{\tau^{(j)}}(u, K, r)\big|\mathcal{F}_0\big] = M_0(u, K, r) = e^{-ru}$ almost surely and consequently

$$\begin{aligned}
\mathbb{E}_j\Big(e^{-rX_{\tau^{(j)}}(K)}\Big) &= \mathbb{E}_j\Big(e^{ru}\, e^{-rY_{\tau^{(j)}}(u,K)}\Big) = \mathbb{E}_j\left(e^{ru}\,\mathbb{E}\Big[e^{-rY_{\tau^{(j)}}(u,K)}\Big|\mathcal{F}_0\Big]\right) \\
&= \mathbb{E}_j\left(e^{\sum_{i\in E}\left[\lambda_i h_i(r)+\frac{1}{2}r^2b^2k(i)^2-r(c+ak(i))\right]\xi_i(\tau^{(j)})}\, e^{ru}\,\mathbb{E}\big[M_{\tau^{(j)}}(u, K, r)\big|\mathcal{F}_0\big]\right) \\
&= \phi_{jj}^{(K)}(r)
\end{aligned}$$

□

Let us from now on denote the ruin probability conditioned under the event that the environmental Markov process starts in state $j \in E$ by $\Psi_j(u, K)$, i.e.

$$\Psi_j(u, K) := \mathbb{P}_j\Big(\inf_{t\geq 0} Y_t(u, K) < 0\Big) = \mathbb{P}\Big(\inf_{t\geq 0} Y_t(u, K) < 0 \,\Big|\, J_0 = j\Big)$$

If the environmental Markov process has initial distribution $\nu = (\nu_i)_{i\in E}$ it thus follows that $\Psi(u, K) = \sum_{i\in E} \nu_i\, \Psi_i(u, K)$.

Certainly, the existence of an $r_0 > 0$ such that $\phi_{jj}^{(K)}(r_0) < \infty$ for all $j \in E$ is a necessary condition for the adjustment coefficient $R^{(K)}$ to exist. The following result shows us that we cannot find an adjustment coefficient if there does not exist such an r_0.

Proposition 2.9. *Consider any fixed investment strategy $K = k(J)$ and suppose that there exists an environmental state $j \in E$ such that $\phi_{jj}^{(K)}(r) = \infty$ for all $r > 0$. Then, for all $\epsilon > 0$ we have*

$$\limsup_{u\to\infty} \frac{\Psi_j(u,K)}{e^{-\epsilon u}} = \infty\,.$$

Proof:

Let us consider any fixed $j \in E$ and suppose that $\phi_{jj}^{(K)}(r) = \infty$ for all $r > 0$. Denoting the distribution function of $X_{\tau^{(j)}}(K)$ by G we recognize that

$$\Psi_j(u,K) \ge \mathbb{P}_j\big(Y_{\tau^{(j)}}(u,K) < 0\big) = \mathbb{P}_j\big(X_{\tau^{(j)}}(K) < -u\big) = G(-u)$$

for $u \ge 0$. Let us now assume that there exist an $\epsilon > 0$ such that

$$\limsup_{u\to\infty} \frac{G(-u)}{e^{-\epsilon u}} \le \limsup_{u\to\infty} \frac{\Psi_j(u,K)}{e^{-\epsilon u}} < \infty\,.$$

This means that we can find a constant $D_\epsilon < \infty$ such that $e^{\epsilon u}G(-u) \le D_\epsilon$ for all $u \ge 0$. Choose any $r \in (0,\epsilon)$. Using Proposition 2.8 we have

$$\begin{aligned}
\phi_{jj}^{(K)}(r) = \mathbb{E}_j\Big(e^{-rX_{\tau^{(j)}}(K)}\Big) &= \int_{-\infty}^{\infty} e^{-rx}\,dG(x) \\
&= \int_{-\infty}^{0} e^{-rx}\,dG(x) + \int_{0}^{\infty} e^{-rx}\,dG(x) \\
&\le \lim_{y\to\infty} \int_{-y}^{0} e^{-rx}\,dG(x) + 1 - G(0)\,.
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
\int_{-y}^{0} e^{-rx}\,dG(x) &= \Big[e^{-rx}G(x)\Big]_{x=-y}^{0} + r\int_{-y}^{0} e^{-rx}G(x)\,dx \\
&= G(0) - e^{ry}G(-y) + r\int_{-y}^{0} e^{-rx}G(x)\,dx \\
&= G(0) - e^{\epsilon y}G(-y)\,e^{-(\epsilon-r)y} + r\int_{-y}^{0} e^{-\epsilon x}G(x)\,e^{(\epsilon-r)x}\,dx
\end{aligned}$$

Since $\epsilon - r > 0$ it follows that

$$0 \le \lim_{y\to\infty} e^{\epsilon y} G(-y)\, e^{-(\epsilon - r)y} \le \lim_{y\to\infty} D_\epsilon\, e^{-(\epsilon-r)y} = 0$$

as well as

$$\lim_{y\to\infty} r \int_{-y}^{0} e^{-\epsilon x} G(x)\, e^{(\epsilon - r)x}\, dx \le r D_\epsilon \int_{-\infty}^{0} e^{(\epsilon-r)x}\, dx = \frac{r}{\epsilon - r}\, D_\epsilon\,.$$

Putting the pieces together we have $\phi_{jj}(r) \le 1 + \frac{r}{\epsilon - r}\, D_\epsilon < \infty$ for all $r \in (0, \epsilon)$ in contradiction to our assumption that $\phi_{jj}^{(K)}(r) = \infty$ for all $r > 0$.

□

However, since we have $\phi_{jj}^{(K)}(0) = 1$ for all $j \in E$ it clearly does not suffice to assume that there exists an $r_0 > 0$ such that $\phi_{jj}^{(K)}(r_0) < \infty$ for all $j \in E$ in order to ensure that $R^{(K)}$ exists. Recall that for the Markov-modulated Poisson model without investment the adjustment coefficient only exists if and only if the safety loading $\rho^{(0)}$ is strictly positive. As mentioned in section 2.1, the safety loading for this model is given by $\rho^{(0)} = c - \sum_{i\in E} \pi_i \lambda_i \mu_{B_i}$.

For the Markov-modulated Poisson model with investment we get a similar representation of the safety loading $\rho^{(K)}$. Recall that the safety loading with respect to a given investment strategy $K = k(J)$ is defined as the constant $\rho^{(K)}$ for which

$$\lim_{t\to\infty} \frac{1}{t}\, Y_t(0, K) \overset{a.s.}{=} \rho^{(K)}\,. \tag{2.13}$$

Proposition 2.10. *Consider any fixed investment strategy $K = k(J)$ and let the corresponding safety loading $\rho^{(K)}$ be defined by (2.13). Then,*

$$\rho^{(K)} = c + a \sum_{i\in E} \pi_i k(i) - \sum_{i \in E} \pi_i \lambda_i \mu_{B_i}\,.$$

Proof:

We have

$$Y_t(0,K) = ct + a\sum_{i\in E} k(i)\xi_i(t) + b\sum_{i\in E} k(i)W^{(i)}_{\xi_i(t)} - \sum_{i\in E}\sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k\,,\ t\geq 0\,.$$

Note that $\lim_{t\to\infty}\frac{\xi_i(t)}{t}\overset{a.s.}{=}\pi_i$ by the Ergodic Theorem. Further, it is well known that $\lim_{t\to\infty}\frac{W^{(i)}_t}{t}\overset{a.s.}{=}0$ as well as $\lim_{t\to\infty}\frac{N^{(i)}_t}{t}\overset{a.s.}{=}\lambda_i$ for $i\in E$. For every environmental state $i\in E$ we thus get

$$\lim_{t\to\infty}\frac{W^{(i)}_{\xi_i(t)}}{t} = \lim_{t\to\infty}\frac{\xi_i(t)}{t}\frac{W^{(i)}_{\xi_i(t)}}{\xi_i(t)} = 0 \text{ a.s.}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i\in E}\sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k = \sum_{i\in E}\lim_{t\to\infty}\frac{\xi_i(t)}{t}\frac{N^{(i)}_{\xi_i(t)}}{\xi_i(t)}\frac{1}{N^{(i)}_{\xi_i(t)}}\sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k = \sum_{i\in E}\pi_i\lambda_i\mu_{B_i} \text{ a.s.}$$

where $\lim_{t\to\infty}\frac{1}{N^{(i)}_{\xi_i(t)}}\sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k \overset{a.s.}{=} \mu_{B_i}$ follows from the law of large numbers. Putting the things together we see that almost surely

$$\lim_{t\to\infty}\frac{1}{t}Y_t(0,K) = c + \sum_{i\in E}\pi_i a k(i) - \sum_{i\in E}\pi_i\lambda_i\mu_{B_i}\,.$$

□

Now, consider the environmental Markov process J and recall that the time epoch of the n^{th} entry of the Markov process to state $j\in E$ is denoted by $\tau^{(j)}_n$ for $n\in\mathbb{N}$. Let us now assume that the environmental Markov process J starts in state $j\in E$, i.e. $J_0=j$. For a given investment strategy $K=k(J)$ it follows from the definition of the process $X(K)$ that the sequence $(\tilde{X}^{(j)}_n(K))_{n\in\mathbb{N}}$ defined by $\tilde{X}^{(j)}_n(K) := X_{\tau^{(j)}_n}(K) - X_{\tau^{(j)}_{n-1}}(K)$ is independent and identically distributed. Thus,

$$\left(X_{\tau^{(j)}_n}(K)\right)_{n\in\mathbb{N}} = \left(\sum_{k=1}^{n}\tilde{X}^{(j)}_k(K)\right)_{n\in\mathbb{N}}$$

is a random walk with $X_{\tau^{(j)}}(K)$ as the generic random variable for the steps.

Analogous to the Markov-modulated model without investment we then get the following result.

Proposition 2.11. *Consider any fixed investment strategy $K = k(J)$ and suppose that the corresponding safety loading $\rho^{(K)} \leq 0$. Then,*

$$\Psi_j(u, K) = 1$$

for all $u \geq 0$ and $j \in E$.

Proof:

The case when $\rho^{(K)} < 0$ is obvious since $\frac{1}{t} X_t(K)$ almost surely converges to a strictly negative limit by the definition of the safety loading. Hence, we have $\inf_{t\geq 0} Y_t(u, K) = -\infty$ a.s. from which our assertion directly follows. Now let $\rho^{(K)} = 0$ and assume that $J_0 = j$ for some environmental state $j \in E$. As described above, $\left(X_{\tau_n^{(j)}}(K)\right)_{n\in\mathbb{N}}$ is then a random walk with generic random variable $X_{\tau^{(j)}}(K)$.

Theorem 4.2 in Asmussen [Asm03], page 51, shows that $\mathbb{E}_j\left(\xi_i(\tau^{(j)})\right) = \pi_i \mathbb{E}_j(\tau^{(j)})$. Thus,

$$\begin{aligned}
\mathbb{E}_j\left(\sum_{i\in E}\sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U_k^{(i)}\right) &= \sum_{i\in E}\mathbb{E}_j\left(\mathbb{E}\left[\sum_{k=1}^{N^{(i)}_{\xi_i(\tau^{(j)})}} U_k^{(i)}\middle| N^{(i)}_{\xi_i(\tau^{(j)})}\right]\right) = \sum_{i\in E}\mu_{B_i}\mathbb{E}_j\left(N^{(i)}_{\xi_i(\tau^{(j)})}\right) \\
&= \sum_{i\in E}\mu_{B_i}\mathbb{E}_j\left(\mathbb{E}\left[N^{(i)}_{\xi_i(\tau^{(j)})}\middle|\xi_i(\tau^{(j)})\right]\right) = \sum_{i\in E}\mu_{B_i}\lambda_i\mathbb{E}_j\left(\xi_i(\tau^{(j)})\right) \\
&= \sum_{i\in E}\pi_i\lambda_i\mu_{B_i}\mathbb{E}_j\left(\tau^{(j)}\right).
\end{aligned}$$

Since the environmental Markov process is assumed to be irreducible with stationary distribution π we have $\mathbb{E}_j\left(\tau^{(j)}\right) < \infty$. Together with the fact that

$\mathbb{E}_j\big(W_{\xi_i(\tau^{(j)})}\big) = \mathbb{E}_j\big(\mathbb{E}\big[W_{\xi_i(\tau^{(j)})}\big|\xi_i(\tau^{(j)})\big]\big) = 0$ we therefore get

$$\mathbb{E}\Big(X_{\tau^{(j)}}(K)\Big) = \Big(c + \sum_{i\in E}\pi_i a k(i) - \sum_{i\in E}\pi_i\lambda_i\mu_{B_i}\Big)\,\mathbb{E}_j\big(\tau^{(j)}\big) = 0\,.$$

Hence, $\big(X_{\tau_n^{(j)}}(K)\big)_{n\in\mathbb{N}}$ is a random walk with zero mean. This implies that $\big(X_{\tau_n^{(j)}}(K)\big)_{n\in\mathbb{N}}$ oscillates between ∞ and $-\infty$, as for example Theorem 4.2 in Asmussen [Asm03], page 224, shows. Therefore, also in the case when $\rho^{(K)} = 0$ we have $\inf_{t\geq 0} Y_t(u,K) \overset{a.s.}{=} -\infty$ and consequently $\Psi_j(u,K) = 1$ for all $u \geq 0$ and $j \in E$.

□

This means that we cannot find an adjustment coefficient for the Markov-modulated Poisson model under any investment strategy $K = k(J)$ unless $\rho^{(K)} > 0$.

Using the previous results we can eventually give conditions which ensure that the adjustment coefficient $R^{(K)}$ exists.

Proposition 2.12. *Consider any fixed investment strategy $K = k(J)$ and suppose there exists an $r_0 > 0$ such that $\phi_{jj}^{(K)}(r_0) < \infty$ for all $j \in E$. Then, $\rho^{(K)} > 0$ implies that the adjustment coefficient $R^{(K)}$ defined by (2.12) exists.*

Proof:

Suppose there exists an $r_0 > 0$ such that $\phi_{jj}^{(K)}(r_0) < \infty$ for all $j \in E$ and let $\Delta \in (0, \frac{r_0}{2})$ be arbitrarily chosen. It is well known that for continuously differentiable, convex functions f we have

$$\left|\frac{f(\Delta) - f(0)}{\Delta}\right| \leq |f'(\Delta)| + |f'(0)|\,.$$

For any fixed $\omega \in \Omega$ it thus follows that

$$\left| \frac{e^{-\Delta X_{\tau^{(j)}}(K)} - 1}{\Delta} \right| \leq \left| X_{\tau^{(j)}}(K) \right| \left(e^{-\Delta X_{\tau^{(j)}}(K)} + 1 \right)$$
$$\leq \left| X_{\tau^{(j)}}(K) \right| \left(e^{-\frac{r_0}{2} X_{\tau^{(j)}}(K)} + 2 \right) .$$

Next, consider any $j \in E$ and assume that $J_0 = j$. Obviously, $\left| X_{\tau^{(j)}}(K) \right|$ is integrable since $\mathbb{E}_j(\tau^{(j)}) < \infty$. Choosing $\alpha \geq 0$ large enough, i.e. such that $\alpha + e^{\frac{r_0}{2} x} \geq x$ for all $x \geq 0$, we have

$$\begin{aligned}
&\mathbb{E}_j \Big(\big| X_{\tau^{(j)}}(K) \big| \, e^{-\frac{r_0}{2} X_{\tau^{(j)}}(K)} \Big) \\
&\quad = \mathbb{E}_j \Big(X_{\tau^{(j)}}(K) \, e^{-\frac{r_0}{2} X_{\tau^{(j)}}(K)} \, I\big(X_{\tau^{(j)}}(K) \geq 0 \big) \Big) \\
&\qquad\qquad + \mathbb{E}_j \Big(- X_{\tau^{(j)}}(K) \, e^{-\frac{r_0}{2} X_{\tau^{(j)}}(K)} \, I\big(X_{\tau^{(j)}}(K) < 0 \big) \Big) \\
&\quad \leq \mathbb{E}_j \Big(X_{\tau^{(j)}}(K) \, I\big(X_{\tau^{(j)}}(K) \geq 0 \big) \Big) \\
&\qquad\qquad + \mathbb{E}_j \Big(\big(\alpha + e^{-\frac{r_0}{2} X_{\tau^{(j)}}(K)} \big) \, e^{-\frac{r_0}{2} X_{\tau^{(j)}}(K)} \, I\big(X_{\tau^{(j)}}(K) < 0 \big) \Big) \\
&\quad \leq \mathbb{E}_j \Big(\big| X_{\tau^{(j)}}(K) \big| \Big) + \alpha \, \mathbb{E}_j \Big(e^{-\frac{r_0}{2} X_{\tau^{(j)}}(K)} \Big) + \mathbb{E}_j \Big(e^{-r_0 X_{\tau^{(j)}}(K)} \Big) \\
&\quad = \mathbb{E}_j \Big(\big| X_{\tau^{(j)}}(K) \big| \Big) + \alpha \, \phi_{jj}^{(K)} \left(\frac{r_0}{2} \right) + \phi_{jj}^{(K)}(r_0) < \infty
\end{aligned}$$

where we make use of Proposition 2.8. We have therefore found an integrable upper bound for $\left| \frac{e^{-\Delta X_{\tau^{(j)}}(K)} - 1}{\Delta} \right|$.

Using the same proposition again, it follows by dominated convergence that

$$\begin{aligned}
\frac{d}{dr} \, \phi_{jj}^{(K)}(r) \bigg|_{r=0} &= \lim_{\Delta \to 0} \frac{\phi_{jj}^{(K)}(\Delta) - \phi_{jj}^{(K)}(0)}{\Delta} = \lim_{\Delta \to 0} \frac{\mathbb{E}_j \left(e^{-\Delta X_{\tau^{(j)}}(K)} \right) - 1}{\Delta} \\
&= \lim_{\Delta \to 0} \mathbb{E}_j \left(\frac{e^{-\Delta X_{\tau^{(j)}}(K)} - 1}{\Delta} \right) = \mathbb{E}_j \left(\lim_{\Delta \to 0} \frac{e^{-\Delta X_{\tau^{(j)}}(K)} - 1}{\Delta} \right) \\
&= \mathbb{E}_j \left(\frac{d}{dr} \, e^{-r X_{\tau^{(j)}}(K)} \bigg|_{r=0} \right) = \mathbb{E}_j \Big(- X_{\tau^{(j)}}(K) \Big) = -\rho^{(K)} < 0 \, .
\end{aligned}$$

Under our assumptions, $\phi_{jj}^{(K)}$ is therefore continuous on $[0, r_0]$ with $\phi_{jj}^{(K)}(0) = 1$

and $\frac{d}{dr}\,\phi_{jj}^{(K)}(r)\Big|_{r=0} < 0$ for all $j \in E$. This means that

$$R^{(K)} = \sup\left\{r > 0\,;\phi_{jj}^{(K)}(r) < 1\ \forall j \in E\right\}$$

exists.

$\square$

Considering a certain investment strategy $K = k(J)$, Proposition 2.12 yields that $\rho^{(K)} > 0$ and the existence of an $r_0 > 0$ such that $\phi_{jj}^{(K)}(r_0) < \infty$ for all $j \in E$ are sufficient conditions for the adjustment coefficient $R^{(K)}$ to exist. On the other hand, it is shown in Proposition 2.9 and Proposition 2.11 that these two conditions are necessary for the existence of an adjustment coefficient. Thus, in the Markov-modulated Poisson model with investment an adjustment coefficient with respect to any fixed investment strategy $K = k(J)$ exists if and only if $\rho^{(K)} > 0$ and $\phi_{jj}^{(K)}(r_0) < \infty$ for all $j \in E$ and some $r_0 > 0$.

2.3 Maximizing the adjustment coefficient

In the previous section the Markov-modulated Poisson model with investment is considered under any fixed investment strategy $K = k(J)$. We have shown so far that

$$\Psi(u, K) \leq e^{-ru} C(K, r) \tag{2.14}$$

with $C(K, r) < \infty$ for all $u \geq 0$ whenever $r < R^{(K)}$. Here, $R^{(K)}$ is defined by $R^{(K)} := \sup \{r > 0\,; \phi_{jj}^{(K)}(r) < 1\ \forall j \in E\}$. Recall that we want to find the optimal investment strategy $K = k(J)$ which maximizes the adjustment coefficient $R^{(K)}$. Thus, let us concentrate on the functions $\phi_{jj}^{(K)}(r)$ for $j \in E$ in the definition of $R^{(K)}$. For any $r > 0$ we have

$$\begin{aligned}
\phi_{jj}^{(K)}(r) &= \mathbb{E}_j \left(\exp \left(\sum_{i \in E} \left[\lambda_i h_i(r) + \frac{r^2 b^2 k(i)^2}{2} - r\big(c + ak(i)\big) \right] \xi_i(\tau^{(j)}) \right) \right) \\
&= \mathbb{E}_j \left(\exp \left(\sum_{i \in E} \left[\lambda_i h_i(r) - \Big(rc + \frac{a^2}{2b^2}\Big) + \frac{r^2 b^2}{2} \Big(k(i) - \frac{a}{rb^2}\Big)^2 \right] \xi_i(\tau^{(j)}) \right) \right) .
\end{aligned} \tag{2.15}$$

Defining the constant investment strategy $K^{(r)}$ by $K_t^{(r)} \equiv \frac{a}{rb^2}$ for $t \geq 0$ and some $r > 0$ we consequently obtain

$$\phi_{jj}^{(K^{(r)})}(r) = \mathbb{E}_j \left(\exp \left(\sum_{i \in E} \left[\lambda_i h_i(r) - \Big(rc + \frac{a^2}{2b^2}\Big) \right] \xi_i(\tau^{(j)}) \right) \right) .$$

Motivated through this let us examine the functions ϕ_{kj} defined by

$$\phi_{kj}(r) := \mathbb{E}_k \left(\exp \left(\sum_{i \in E} \lambda_i h_i(r) \xi_i(\tau^{(j)}) - \big(rc + \frac{a^2}{2b^2}\big) \tau^{(j)} \right) \right) \tag{2.16}$$

for $r \geq 0$ and $j, k \in E$. Recall, we show around (2.11) in the previous section that $\phi_{jj}^{(K)}$ is convex and consequently continuous on the interior of its domain for every investment strategy $K = k(J)$. Using exactly the same arguments as there we obtain that also ϕ_{jj} is convex and therefore continuous on the interior of its domain. This time, we obviously have

$$\phi_{jj}(0) = \mathbb{E}_j\Big(\exp\Big(-\frac{a^2}{2b^2}\,\tau^{(j)}\Big)\Big) \leq 1 \tag{2.17}$$

with strict inequality if the drift parameter a of the Brownian motion with drift $W_{a,b}$ does not equal zero.

Analogously to the definition of $R^{(K)}$ in (2.12) we now define R by

$$R := \sup\Big\{ r > 0\,;\, \phi_{jj}(r) < 1\ \forall j \in E \Big\} \tag{2.18}$$

and say that R does not exist if there is an environmental state $j \in E$ such that $\phi_{jj}(r) \geq 1$ for all $r > 0$. Comparing ϕ_{jj} and $\phi_{jj}^{(K)}$ as given in (2.15) above, we see that $\phi_{jj}^{(K)}(r) \geq \phi_{jj}(r)$ for all $r \geq 0$, $j \in E$ and $K = k(J)$. This implies that $R^{(K)} \leq R$ for every investment strategy $K = k(J)$.

The connection between ϕ_{jj} and $\phi_{jj}^{(K^{(s)})}$ for some fixed $s > 0$ and $j \in E$ is illustrated in Figure 2.1 on the next page. As mentioned above we have $\phi_{jj}^{(K^{(s)})}(r) \geq \phi_{jj}(r)$ for all $r \geq 0$ with equality for $r = s$.

The following result now shows us that R is indeed a sharp upper bound for all $R^{(K)}$ with $K = k(J)$ since it can almost be attained by choosing an appropriate constant investment strategy.

Theorem 2.13. *Suppose that R defined by (2.18) exists. For any fixed $r \in (0, R)$ and the corresponding investment strategy $K^{(r)}$ defined by $K_t^{(r)} \equiv \frac{a}{rb^2}$ for $t \geq 0$ it follows that*

$$\Psi(u, K^{(r)}) \leq e^{-ru}\, C(K^{(r)}, r)$$

with $C(K^{(r)}, r) < \infty$ for all $u \geq 0$.

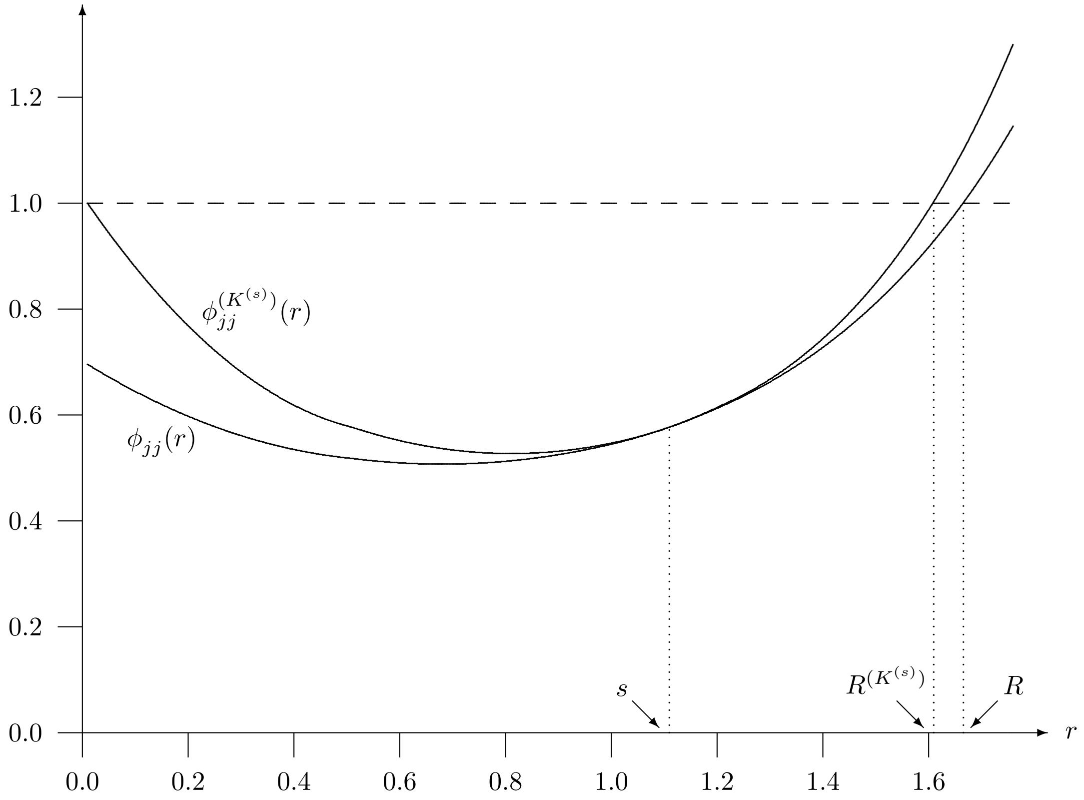

Figure 2.1: Plot of $\phi_{jj}(r)$ and $\phi_{jj}^{(K^{(s)})}(r)$

Proof:

Choose $r \in (0, R)$ and consider the associated constant investment strategy $K^{(r)}$. We then have $\phi_{jj}^{(K^{(r)})}(r) = \phi_{jj}(r) < 1$ for all $j \in E$ as shown above. Hence, we get $C(K^{(r)}, r) < \infty$ and consequently the desired result using Theorem 2.7.

$\square$

Let us consider any $0 < r < R$ and the investment strategy $K^{(r)}$ defined by $K_t^{(r)} \equiv \frac{a}{rb^2}$ for $t \geq 0$. Without making any assumptions about the safety loading $\rho^{(K^{(r)})}$ with respect to this investment strategy we have

$$\Psi(u, K^{(r)}) \leq e^{-ru} C(K^{(r)}, r)$$

with $C(K^{(r)}, r) < \infty$ for all $u \geq 0$ according to Theorem 2.13. Recall, it is shown in Proposition 2.11 that for any investment strategy $K = k(J)$ with $\rho^{(K)} \leq 0$ we get $\Psi_j(u, K) = 1$ for all $u \geq 0$ and $j \in E$. Comparing these two results we see that $\rho^{(K^{(r)})}$ must be strictly positive. This means that the absolute value $\frac{|a|}{rb^2}$ which the strategy $K^{(r)}$ provides to invest is sufficiently large so that the safety loading

$$\rho^{(K^{(r)})} = c + a \sum_{i \in E} \pi_i \frac{a}{rb^2} - \sum_{i \in E} \pi_i \lambda_i \mu_{B_i} = c + \frac{a^2}{rb^2} - \sum_{i \in E} \pi_i \lambda_i \mu_{B_i}$$

becomes positive.

But note that the inequality given in Theorem 2.13 only holds for $r < R$. Certainly, the next question is what we get for R itself. Does there also exist an investment strategy $K = k(J)$ and a finite constant C such that $\Psi(u, K) \leq C\, e^{-Ru}$ for all $u \geq 0$? Before we get to this problem in section 2.4 let us look at the definition of R again.

Initially, we want to derive conditions under which R defined by (2.18) exists as done for $R^{(K)}$ in section 2.2. If R does not exist it certainly follows that we cannot find an investment strategy $K = k(J)$ for which $R^{(K)}$ exists since $\phi_{jj}^{(K)}(r) \geq \phi_{jj}(r)$ for all $r > 0$ and $K = k(J)$. As shown at the end of the previous section we therefore do not get an adjustment coefficient for any investment strategy $K = k(J)$ unless R exists.

A first necessary condition for R to exist is obviously the existence of an $r_0 > 0$ such that $\phi_{jj}(r_0) < \infty$ for all $j \in E$. Taking this for granted we have to distinguish between two cases. Firstly, consider the case where the drift parameter a of the Brownian motion with drift $W_{a,b}$ does not equal zero. We then have $\phi_{jj}(0) < 1$ for all $j \in E$ and it is easy to see that R exists. Secondly, let us assume that $a = 0$. Then, we apparently have $\phi_{jj} = \phi_{jj}^{(0)}$ for all $j \in E$ and it consequently follows that R is equal to $R^{(0)}$. According to Proposition 2.12 this implies that R exist if $\rho^{(0)} > 0$.

Always provided that there exists an $r_0 > 0$ such that $\phi_{jj}(r_0) < \infty$ for all $j \in E$ we can therefore choose the constant investment strategy $K^{(r)}$ with $0 < r < R$ and get an adjustment coefficient for the corresponding Markov-modulated Poisson model with investment as long as $a \neq 0$. In the case where $a = 0$ we have the existence of an adjustment coefficient if the safety loading of the corresponding Markov-modulated Poisson model without investment is strictly positive, i.e. if there exists an adjustment coefficient for the Markov-modulated Poisson model without investment.

Recall that R is defined as the supremum of all $r > 0$ such that $\phi_{jj}(r) < 1$ for all $j \in E$. Unfortunately, this supremum is generally not easy to determine since only in very few cases we know ϕ_{jj} as an explicit function of r, confer Example 2.22 in section 2.5. In that what follows we thus give an alternative definition for R.

In order to find such a definition, some matrix notation need to be introduced. Let $A = (a_{ij})_{i,j \in E} \in \mathbb{R}^{d \times d}$ be a non-negative matrix with eigenvalues $\kappa_1, \ldots, \kappa_d$. The spectral radius of A is then defined as

$$\operatorname{spr}(A) := \max\left\{|\kappa_1|, \ldots, |\kappa_d|\right\}.$$

We denote the n^{th} power of A by $A^n = \left(a_{ij}^{(n)}\right)_{i,j \in E}$ for $n \in \mathbb{N}$. A non-negative matrix A is called irreducible if the pattern of zero and non-zero elements is the same as for an irreducible transition probability matrix. This means, for each $i, j \in E$ there has to exist an $n \in \mathbb{N}$ such that $a_{ij}^{(n)} > 0$. For an irreducible and non-negative matrix A it follows from the Perron-Frobenius-Theorem that

$\mathrm{spr}(A)$ itself is a strictly positive and simple eigenvalue of A. Moreover, the corresponding left and right eigenvectors can be chosen with strictly positive elements. If the matrix A has some infinite element we put as a convention $\mathrm{spr}(A) = \infty$.

In that what follows we refer to the $d \times d$ identity matrix by I_d. Furthermore, let the $d \times d$ diagonal matrix where the k^{th} diagonal element is given by a_k for $k \in E$ be denoted by

$$\mathrm{diag}\Big(a_k\,;k \in E\Big)\,.$$

After introducing some matrix notation let us get back to the definition of the environmental Markov process J which has intensity matrix $Q = (q_{ij})_{i,j \in E}$. As shown in section 2.2, the transition probability matrix $P = (p_{ij})_{i,j \in E}$ of the embedded Markov chain is given by $p_{ij} := (1 - \delta_{ij})\frac{q_{ij}}{q_i}$ where $q_i := -q_{ii}$ and δ_{ij} is Kronecker's symbol. Thus,

$$P = \mathrm{diag}\big(q_i\,;i \in E\big)^{-1}\Big(Q + \mathrm{diag}\big(q_i\,;i \in E\big)\Big) = diag\big(q_i\,;i \in E\big)^{-1}\,Q + I_d\,.$$

The time which the environmental Markov process J spends in state $j \in E$ when the process makes its n^{th} visit to this state is denoted by $\sigma_n^{(j)}$ for $n \in \mathbb{N}$. Recall that conditioned under the embedded Markov chain the $\sigma_n^{(j)}$ are independent for all $n \in \mathbb{N}$ and $j \in E$ and that the sequence $(\sigma_n^{(j)})_{n \in \mathbb{N}}$ is furthermore identically distributed with $\sigma^{(j)} := \sigma_1^{(j)} \sim \mathrm{Exp}(q_j)$ for $j \in E$.

Analogously to the definition of the functions $\phi_j^{(K)}$ for any investment strategy $K = k(J)$ and $j \in E$ in section 2.2 we now let the function ϕ_j be given by

$$\phi_j(r) := \mathbb{E}\left(\exp\left(\big(\lambda_j h_j(r) - (rc + \frac{a^2}{2b^2})\big)\,\sigma^{(j)}\right)\right)$$

for $r \geq 0$ and $j \in E$. Using these functions let the matrix $B(r) \in \mathbb{R}^{d \times d}$ be defined by

$$B(r) := \operatorname{diag}\Big(\phi_j(r)\,;\, j \in E\Big) \cdot P\,. \tag{2.19}$$

We denote the matrix elements of $B(r)$ by $b_{ij}(r)$, i.e. $B(r) = \big(b_{ij}(r)\big)_{i,j \in E}$. Note that the diagonal elements of $B(r)$ are all zero by the definition of P.

Proposition 2.14. *Let the matrix $B(r)$ be defined by (2.19). Then,*

$$R = \sup\Big\{r > 0;\ \operatorname{spr}\big(B(r)\big) < 1\Big\}\,. \tag{2.20}$$

Proof:

Firstly, we are going to show that $\operatorname{spr}\big(B(r)\big) < 1$ implies $\phi_{jj}(r) < 1$ for all $j \in E$. Let $r > 0$ be fixed. Similarly to what is shown around (2.9) and (2.10) we now get

$$\begin{aligned}
\phi_{kj}(r) &= \phi_k(r)\, p_{kj} + \sum_{\substack{m \in E \\ m \neq j}} \phi_k(r)\, p_{km}\, \phi_{mj}(r) \\
&= \phi_k(r)\, p_{kj} + \sum_{m \in E} \phi_k(r)\, p_{km}\, \phi_{mj}(r) - \phi_k(r)\, p_{kj}\, \phi_{jj}(r) \\
&= \sum_{m \in E} \phi_k(r)\, p_{km}\, \phi_{mj}(r) + \phi_k(r)\, p_{kj}\, \big(1 - \phi_{jj}(r)\big)
\end{aligned} \tag{2.21}$$

and in particular

$$\phi_{jj}(r) = \sum_{m \in E} \phi_j(r)\, p_{jm}\, \phi_{mj}(r) \tag{2.22}$$

for $k, j \in E$. Now, put $\Phi(r) := \big(\phi_{kj}(r)\big)_{k,j \in E}$. In matrix notation we therefore have

$$\Phi(r) = B(r)\,\Phi(r) + B(r)\operatorname{diag}\big(I_d - \Phi(r)\big) \tag{2.23}$$

or equivalently

$$\big(I_d - B(r)\big)\,\Phi(r) = B(r)\operatorname{diag}\big(I_d - \Phi(r)\big)\,. \tag{2.24}$$

Since $\operatorname{spr}\big(B(r)\big) < 1$ it follows that $\lim_{n\to\infty} B(r)^n = 0$ and consequently

$$\big(I_d - B(r)\big)\sum_{n=0}^{N} B(r)^n = \sum_{n=0}^{N} B(r)^n - \sum_{n=0}^{N} B(r)^{n+1} = I_d - B(r)^{N+1} \longrightarrow I_d$$

as $N \to \infty$. Thus, the invers of $(I_d - B(r))$ exists and is equal to $\sum_{n=0}^{\infty} B(r)^n$. From (2.24) we therefore have $\Phi(r) = (I_d - B(r))^{-1}\, B(r)\operatorname{diag}(I_d - \Phi(r))$ and hence

$$\begin{aligned}\operatorname{diag}\big(\Phi(r)\big) &= \operatorname{diag}\Big(\big(I_d - B(r)\big)^{-1} B(r)\operatorname{diag}\big(I_d - \Phi(r)\big)\Big)\\ &= \operatorname{diag}\Big(\big(I_d - B(r)\big)^{-1} B(r)\Big)\cdot\operatorname{diag}\big(I_d - \Phi(r)\big)\,.\end{aligned}$$

Put $A := \operatorname{diag}\big((I_d - B(r))^{-1}\, B(r)\big)$. All entries of $B(r)$ are non-negative. It thus follows from $\big(I_d - B(r)\big)^{-1} = \sum_{n=0}^{\infty} B(r)^n$ that all entries of $\big(I_d - B(r)\big)^{-1}$ are non-negative as well. Hence, also the diagonal matrix $A = \big(a_{ij}\big)_{i,j\in E}$ is non-negative, i.e. $a_{jj} \geq 0$ for all $j \in E$. Recalling that the diagonal elements of $\Phi(r)$ are given by $\phi_{jj}(r) = a_{jj}(1 - \phi_{jj}(r))$ we can therefore conclude that

$$\phi_{jj}(r) = \frac{a_{jj}}{1 + a_{jj}} < 1\,,\; j \in E\,.$$

This means that indeed $\operatorname{spr}\big(B(r)\big) < 1$ implies $\phi_{jj}(r) < 1$ for all $j \in E$.

In order to prove the other direction we assume without loss of generality that $\phi_{11}(r) < 1$ for any given $r > 0$. Let $j \in E$ be arbitrarily chosen. The environmental Markov process J is assumed to be irreducible. Hence, there exists a sequence of states $i_1, \ldots, i_N \in \{2, \ldots, d\}$ with $N \in \mathbb{N}$ such that $p_{1i_1}p_{i_1i_2}\cdot\ldots\cdot p_{i_{N-1}i_N}p_{i_N1} > 0$ and $i_n = j$ for some $1 \leq n \leq N$. Having equations (2.21) and (2.22) in mind we thus have

$$\phi_{11}(r) \geq \underbrace{\phi_1(r)p_{1i_1}\phi_{i_1}(r)\cdot\ldots\cdot p_{i_{n-1}j}}_{>0}\,\phi_j(r)\,\underbrace{p_{ji_{n+1}}\cdot\ldots\cdot\phi_{i_N}(r)p_{i_N1}}_{>0}$$

for $r \geq 0$ since $p_{kj} \geq 0$, $\phi_j(r) > 0$ and $\phi_{kj}(r) > 0$. Therefore, $\phi_{11}(r) < 1$ particularly implies $\phi_j(r) < \infty$ for all $j \in E$, i.e. all entries of the matrix $B(r)$ are finite. From (2.23) we furthermore get

$$\begin{aligned}
\Phi(r) &= B(r)\,\Phi(r) + B(r)\operatorname{diag}\big(I_d - \Phi(r)\big)\\
&= B(r)^2\,\Phi(r) + B(r)^2\operatorname{diag}\big(I_d - \Phi(r)\big) + B(r)\operatorname{diag}\big(I_d - \Phi(r)\big)\\
&= \ldots = B(r)^N\,\Phi(r) + \sum_{n=1}^{N} B(r)^n\operatorname{diag}\big(I_d - \Phi(r)\big)\\
&\geq \sum_{n=1}^{N} B(r)^n\operatorname{diag}\big(I_d - \Phi(r)\big)
\end{aligned}$$

for $N \in \mathbb{N}$ since the matrices $B(r)$ and $\Phi(r)$ are non-negative. Now, this yields $\operatorname{spr}\big(B(r)\big) < 1$ exactly as in the proof of Lemma 8 in Björk and Grandell [BG88].

□

Note, the proof of Proposition 2.14 also shows us that $\phi_{jj}(r) < 1$ for some $j \in E$ already implies $\operatorname{spr}\big(B(r)\big) < 1$ and therefore $\phi_{jj}(r) < 1$ for all $j \in E$. Thus, for a given $r > 0$ we either have $\phi_{jj}(r) < 1$ for all $j \in E$ or there does not exist any $j \in E$ with $\phi_{jj}(r) < 1$.

Finally, let us have a closer look at the matrix $B(r)$. For $i \in E$ let $\hat{r}_\infty^{(i)}$ be the strictly positive solution of the equation $q_i + rc + \frac{a^2}{2b^2} - \lambda_i h_i(r) = 0$ and put

$$\hat{r}_\infty := \min_{i \in E} \hat{r}_\infty^{(i)}\,. \tag{2.25}$$

From the assumptions on $h_i(r)$ it follows that the $\hat{r}_\infty^{(i)}$ and consequently also $\hat{r}_\infty$ are uniquely defined and that $q_i + rc + \frac{a^2}{2b^2} - \lambda_i h_i(r) > 0$ for all $0 \leq r < \hat{r}_\infty^{(i)}$ and $i \in E$. We get the following final result of this section.

Proposition 2.15. *Let $\hat{r}_\infty$ be defined by (2.25). Then,*

$$B(r) = \left(\frac{q_{ij}\,(1-\delta_{ij})}{q_i + rc + \frac{a^2}{2b^2} - \lambda_i h_i(r)} \right)_{i,j\in E} \tag{2.26}$$

for $r < \hat{r}_\infty$ where δ_{ij} denotes Kronecker's symbol. For all $r \geq \hat{r}_\infty$, $B(r)$ has at least one infinite element.

Proof:

The moment generating function $\hat{m}(t)$ of a random variable which ist exponentially distributed with parameter $\lambda > 0$ is equal to $\frac{\lambda}{\lambda - t}$ for $t < \lambda$ and infinite otherwise. Since $\sigma_j \sim \mathrm{Exp}(q_j)$ it follows from what is shown around (2.25) that

$$\phi_j(r) = \begin{cases} \frac{q_j}{q_j + rc + \frac{a^2}{2b^2} - \lambda_j h_j(r)} & , \ r < \hat{r}_\infty^{(j)} \\ \infty & , \ r \geq \hat{r}_\infty^{(j)} \end{cases} . \tag{2.27}$$

Recall, for a given intensity matrix Q we can compute the corresponding transition probability matrix P via

$$P = \mathrm{diag}\big(q_i\,; i \in E\big)^{-1} \Big(Q + \mathrm{diag}\big(q_i\,; i\in E\big)\Big).$$

For any $r < \hat{r}_\infty$ we consequently get

$$\begin{aligned} B(r) &= \mathrm{diag}\big(\phi_j(r)\,; j \in E\big)\, P \\ &= \mathrm{diag}\left(\frac{q_i}{q_i + rc + \frac{a^2}{2b^2} - \lambda_i h_i(r)}\,; i \in E \right) P \\ &= \mathrm{diag}\Big(q_i + rc + \frac{a^2}{2b^2} - \lambda_i h_i(r)\,; i\in E\Big)^{-1} \Big(Q + \mathrm{diag}\big(q_i\,; i \in E\big)\Big) \\ &= \left(\frac{q_{ij}\,(1-\delta_{ij})}{q_i + rc + \frac{a^2}{2b^2} - \lambda_i h_i(r)} \right)_{i,j\in E} \end{aligned}$$

where δ_{ij} is Kronecker's symbol. For any $r \geq \hat{r}_\infty$ we have $\phi_j(r) = \infty$ for some $j \in E$, i.e. $B(r)$ has some infinite element.

□

Recall that by the definition of the spectral radius we have $\mathrm{spr}\big(B(r)\big) = \infty$ if $B(r)$ has some infinite element. Thus, R is given by

$$R = \sup\left\{ 0 < r < \hat{r}_\infty \ ; \ \mathrm{spr}\left(\left(\frac{q_{ij}\,(1-\delta_{ij})}{q_i + rc + \frac{a^2}{2b^2} - \lambda_i h_i(r)}\right)_{i,j\in E}\right) < 1 \right\} .$$

2.4 The Markov-modulated Poisson model under some constant investment strategy

We have already proved that for any $0 < r < R$ there exists an investment strategy $K = k(J)$, namely $K^{(r)}$, and a finite constant C, namely $C(K^{(r)}, r)$, such that

$$\Psi(u, K) \leq C\, e^{-ru}$$

for all $u \geq 0$. Now, the question arises whether we can find a similar upper bound where r is equal to R. Recall that $C(K, r)$ is finite if and only if $\phi_{jj}^{(K)}(r) < 1$ for all $j \in E$. It therefore follows from what is shown in the previous section that $C(K^{(R)}, R) = \infty$. This means that we have to consider other methods in order to determine a Lundberg bound with adjustment coefficient R.

Thus, in this section we initially consider another model, namely the Markov-modulated Poisson model perturbed by diffusion. In this model the wealth process $Y^{(\eta)}(u)$ is defined by

$$Y_t^{(\eta)}(u) := R_t(u) + \eta W_t = u + ct - \sum_{k=1}^{N_t} U_k + \eta W_t \tag{2.28}$$

where $R(u)$ is the risk reserve process from the Markov-modulated Poisson model as defined before, W is a Brownian motion independent of $R(u)$ as well as J and $\eta \in \mathbb{R}$. We refer to η as the diffusion volatility.

Without Markov-modulation this model was introduced by Gerber [Ger70]. He derived a Cramér-Lundberg approximation for the case where the counting process N is a standard Poisson process. A somewhat more detailed study of the same model can be found in Dufresne and Gerber [DG91]. Later, Furrer and Schmidli [FS94] determined Lundberg inequalities for the case where the counting process N is either a renewal process or a Cox process with a so-called independent jump intensity. Using their method it unfortunately could not be proved that they determined the best possible exponential upper bounds for the ruin probabilities. However, Schmidli [Schm95] made up for this and also stated a Cramér-Lundberg approximation for the renewal case. Furthermore, he considered the Markov-modulated Poisson model perturbed by diffusion as defined above.

In order to adapt the results in Schmidli [Schm95] to our model the matrix $H(r) \in \mathbb{R}^{d\times d}$ given by

$$H(r) := Q + \mathrm{diag}\Big(\lambda_i h_i(r)\,;\, i \in E\Big) - \Big(cr + \frac{a^2}{2b^2}\Big) I_d \tag{2.29}$$

for $r \geq 0$ is needed. In that what follows the spectral radius of the matrix $e^{H(r)}$ is denoted by $e^{\theta(r)}$. The next result is then due to Schmidli [Schm95].

Theorem 2.16. *Let $e^{\theta(r)}$ be the spectral radius of $e^{H(r)}$ as defined above. If $\theta(r) = 0$ for some $r > 0$ then the constant investment strategy $K^{(r)}$ defined by $K^{(r)}_t \equiv \frac{a}{rb^2}$ for $t \geq 0$ yields*

$$\Psi(u, K^{(r)}) \leq C\, e^{-ru}$$

with $C < \infty$ for all $u \geq 0$.

Proof:

Firstly, let us examine the Markov-modulated Poisson model with investment when using the constant investment strategy $K^{(r)}$. Since we only consider strictly positive solutions r of the equation $\theta(r) = 0$ the investment strategy $K^{(r)}$ is well defined and

$$Y_t(u, K^{(r)}) = u + (c + \frac{a^2}{rb^2})t - \sum_{k=1}^{N_t} U_k + \frac{a}{rb} W_t \,.$$

Comparing this with (2.28) we see that the Markov-modulated Poisson model under the constant investment strategy $K^{(r)}$ coincides with the Markov-modulated Poisson model perturbed by diffusion where the premium rate is given by $\tilde{c} := c + \frac{a^2}{rb^2}$ and the diffusion volatility by $\tilde{\eta} := \frac{a}{rb}$.

Let $\Psi^{(\eta)}(u)$ denote the ruin probability of the Markov-modulated Poisson model perturbed by diffusion whose wealth process is defined by (2.28), i.e.

$$\Psi^{(\eta)}(u) := \mathbb{P}\left(\inf_{t\geq 0} Y_t^{(\eta)}(u) < 0\right) \,.$$

According to Theorem 4 in Schmidli [Schm95] it then follows that

$$\Psi^{(\eta)}(u) \leq C\, e^{-ru}$$

with $C < \infty$ for all $u \geq 0$ if the equation $\tilde{\theta}(r) = 0$ is fulfilled for some $r > 0$. At this, $e^{\tilde{\theta}(r)}$ is defined as the spectral radius of the matrix $e^{L(r)}$ where

$$L(r) := Q + \operatorname{diag}\Big(\lambda_i h_i(r)\,; i \in E\Big) + \left(\frac{\eta^2 r^2}{2} - cr\right) I_d \ , \ r \geq 0 \,. \tag{2.30}$$

Now, suppose that $\theta(r) = 0$ for some $r > 0$. It thus suffices to show that $L(r) = H(r)$ where the parameters $\tilde{c}$ and $\tilde{\eta}$ have to be plugged into the definition of $L(r)$ above. After all, we indeed get

$$\begin{aligned} L(r) &= Q + \operatorname{diag}\Big(\lambda_i h_i(r)\,; i \in E\Big) + \left(\frac{\tilde{\eta}^2 r^2}{2} - \tilde{c}r\right) I_d \\ &= Q + \operatorname{diag}\Big(\lambda_i h_i(r)\,; i \in E\Big) + \left(\frac{\left(\frac{a}{rb}\right)^2 r^2}{2} - \left(c + \frac{a^2}{rb^2}\right) r\right) I_d \\ &= Q + \operatorname{diag}\Big(\lambda_i h_i(r)\,; i \in E\Big) - \left(cr + \frac{a^2}{2b^2}\right) I_d = H(r) \end{aligned}$$

□

Using this result we thus have to prove that $\theta(R) = 0$ in order to get the desired upper bound for $\Psi(u, K^{(R)})$. First of all, we need the following result about the spectral radius of $B(R)$.

Proposition 2.17. *Suppose that R defined by (2.20) exists. Then,*

$$\operatorname{spr}\big(B(R)\big) = 1\,.$$

Proof:

Using the same notation as in the previous chapter we recall from Proposition 2.15 that

$$B(r) = \left(\frac{q_{ij}\,(1-\delta_{ij})}{q_i + rc + \frac{a^2}{2b^2} - \lambda_i h_i(r)} \right)_{i,j\in E}$$

for $r < \hat{r}_\infty$ where δ_{ij} denotes Kronecker's symbol and that $B(r)$ has at least one infinite element for all $r \geq \hat{r}_\infty$. It obviously follows from this representation of $B(r) = \big(b_{ij}(r)\big)_{i,j\in E}$ that all the matrix elements $b_{ij}(r)$ are continuous functions in $r \in (0, \hat{r}_\infty)$. Since the spectral radius of a matrix is a continuous mapping with respect to the matrix elements also $\operatorname{spr}\big(B(r)\big)$ is a continuous function in $r \in (0, \hat{r}_\infty)$.

It is furthermore shown in the proof of Proposition 2.15 that

$$\phi_j(r) = \begin{cases} \frac{q_j}{q_j + rc + \frac{a^2}{2b^2} - \lambda_j h_j(r)} & , \ r < \hat{r}_\infty^{(j)} \\ \infty & , \ r \geq \hat{r}_\infty^{(j)} \end{cases}.$$

There consequently exists an environmental state $m \in E$, namely the one which satisfies $\hat{r}_\infty = \hat{r}_\infty^{(m)}$, such that $\phi_m(r) \to \infty$ as $r \to \hat{r}_\infty$. Recall from the proof of Proposition 2.14 that $\phi_{jj}(r) \geq \alpha\, \phi_m(r)$ for all $r \geq 0$ where α is some strictly positive constant. Hence, we get

$$\phi_{jj}(r) \to \infty \text{ as } r \to \hat{r}_\infty$$

for all $j \in E$. This implies that $R \in (0, \hat{r}_\infty)$. Together with the continuity of $\operatorname{spr}\big(B(r)\big)$ on $(0, \hat{r}_\infty)$ it follows that $\operatorname{spr}\big(B(R)\big)$ equals 1.

□

Using the proposition above it can now be shown that we indeed have $\theta(R) = 0$.

Proposition 2.18. *Suppose that R defined by (2.20) exists and let $e^{\theta(r)}$ be the spectral radius of $e^{H(r)}$. Then,*

$$\theta(R) = 0\,.$$

Proof:

The environmental Markov process J and therefore the transition probability matrix P of its embedded Markov chain are assumed to be irreducible. Thus, also the matrix $B(R)$ is irreducible. We have already mentioned that the matrix $B(r)$ is moreover non-negative for all $r \geq 0$. Applying the Perron-Frobenius-Theorem we therefore know that $\operatorname{spr}\big(B(R)\big)$ itself is an eigenvalue of $B(R)$ and that the corresponding right eigenvector g can be chosen with strictly positive elements which is denoted by $g > 0$.

Together with $\operatorname{spr}\big(B(R)\big) = 1$ from Proposition 2.17 we thus get

$$\begin{aligned}
&B(R)\, g = g \\
&\Leftrightarrow \operatorname{diag}\Big(q_i + Rc + \frac{a^2}{2b^2} - \lambda_i h_i(R)\,;i \in E\Big)^{-1} \Big(Q + \operatorname{diag}\big(q_i\,;i \in E\big)\Big)\, g = g \\
&\Leftrightarrow \Big(Q + \operatorname{diag}\big(q_i\,;i \in E\big)\Big)\, g = \operatorname{diag}\Big(q_i + Rc + \frac{a^2}{2b^2} - \lambda_i h_i(R)\,;i \in E\Big)\, g \\
&\Leftrightarrow Q\, g = \Big(Rc + \frac{a^2}{2b^2}\Big) I_d\, g - \operatorname{diag}\Big(\lambda_i h_i(R)\,;i \in E\Big)\, g \\
&\Leftrightarrow H(R)\, g = 0
\end{aligned}$$

for this vector $g > 0$, i.e. 0 is an eigenvalue of $H(R)$ with right eigenvector $g > 0$.

Note that we have

$$e^{H(R)}v = \sum_{k=0}^{\infty} \frac{H(R)^k}{k!}\, v = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}\, v = e^{\alpha} v \tag{2.31}$$

for any eigenvalue α of $H(R)$ with right eigenvector v. This means that α is an eigenvalue of $H(R)$ with right eigenvector v if and only if e^{α} is an eigenvalue of $e^{H(R)}$ with the same right eigenvector v. We have already shown that zero is an eigenvalue of $H(R)$ with right eigenvector $g > 0$. This consequently implies that one is an eigenvalue of $e^{H(R)}$ with the same right eigenvector $g > 0$.

It is easy to see that the matrix $H(r)$ has non-negative off-diagonal elements which implies that $e^{H(r)}$ is a non-negative matrix for all $r \geq 0$. Since the matrix $e^{H(r)}$ is moreover irreducible for $r \geq 0$ it follows that $e^{\theta(R)} := \operatorname{spr}\big(e^{H(R)}\big) = 1$ using the Subinvariance Theorem which can for example be found in Seneta [Sen81], page 23.

□

Remark 2.19. Let us fix some $r \geq 0$ and recall that the matrix $e^{H(r)}$ is non-negative and irreducible. It thus follows by the Perron-Frobenius-Theorem that the spectral radius of $e^{H(r)}$ itself is a simple, real and strictly positive eigenvalue of $e^{H(r)}$. Hence, we have $\operatorname{spr}\big(e^{H(r)}\big) = e^{\alpha}$ for some $\alpha \in \mathbb{R}$.

At this, the last assertion is equivalent to the fact that α is the eigenvalue of $H(r)$ which has maximum real part. This can be seen as follows. Obviously, we obtain from (2.31) above that α is an eigenvalue of the matrix $H(r)$. It thus suffices to show that $\alpha \geq \operatorname{Re}(\tilde{\alpha})$ where $\tilde{\alpha}$ is an arbitrary eigenvalue of $H(R)$. Here, $\operatorname{Re}(\tilde{\alpha})$ denotes the real part of $\tilde{\alpha}$. Again, we use the fact that $\tilde{\alpha}$ is an eigenvalue of $H(R)$ if and only if $e^{\tilde{\alpha}}$ is an eigenvalue of $e^{H(r)}$. By the definition of the spectral radius we thus have $e^{\alpha} = \operatorname{spr}\big(e^{H(r)}\big) \geq \big|e^{\tilde{\alpha}}\big| = e^{\operatorname{Re}(\tilde{\alpha})}$ and hence $\alpha \geq \operatorname{Re}(\tilde{\alpha})$.

◇

Motivated through Remark 2.19 we try to find an alternative definition for R which is defined by (2.20) in section 2.3. As mentioned in Schmidli [Schm95], the function $\theta(r)$ is strictly convex in r. Note that this follows from a theorem in Kingman [Kin61]. Furthermore, it is easy to see that the matrix $H(0) = Q - \frac{a^2}{2b^2}\, I_d$ has $-\frac{a^2}{2b^2}$ as an eigenvalue for which the right eigenvector can be chosen such that each of its components is equal to one. Thus, $e^{-\frac{a^2}{2b^2}}$ is an eigenvalue of the matrix $e^{H(0)}$ with exactly the same right eigenvector. Recalling that $e^{H(0)}$ is non-negative and irreducible it hence follows from the Subinvariance Theorem which is for example given in Seneta [Sen81] that the matrix $e^{H(0)}$ has spectral radius $e^{-\frac{a^2}{2b^2}}$. This in turn implies $\theta(0) = -\frac{a^2}{2b^2}$ and consequently that the equation $\theta(r) = 0$ has at most one strictly positive solution.

On the one hand Proposition 2.18 therefore yields that if R defined by (2.20) exists it is the unique strictly positive solution of the equation $\theta(r) = 0$. On the other hand it is shown in the previous section that we cannot find an adjustment coefficient for the Markov-modulated Poisson model under any investment strategy $K = k(J)$ if R does not exist. But recall that according to Theorem 2.16 there exists an adjustment coefficient for the Markov-modulated Poisson model when using the investment strategy $K^{(r)}$ if $r > 0$ solves the equation $\theta(r) = 0$. In the case that $\theta(r) = 0$ for some strictly positive r it hence follows that R exists and consequently that $r = R$. We can therefore define R as the strictly positive solution of the equation $\theta(r) = 0$ and say that R does not exist if no such solution can be found.

Using Remark 2.19 we can alternatively define R as the strictly positive solution of the equation $\kappa(r) = 0$ where $\kappa(r)$ is that eigenvalue of the matrix

$$H(r) := Q + \mathrm{diag}\Big(\lambda_i h_i(r)\,;\, i \in E\Big) - \Big(cr + \frac{a^2}{2b^2}\Big) I_d \tag{2.32}$$

which has maximum real part and say that R does not exist if the equation $\kappa(r) = 0$ has no strictly positive solution.

Combining these assertions with Theorem 2.16 we directly get the following final result of this section.

Corollary 2.20. *Let $\kappa(r)$ be the eigenvalue of $H(r)$ which has maximum real part as defined above. If there exists a solution $R > 0$ of the equation $\theta(r) = 0$ then the constant investment strategy $K^{(R)}$ defined by $K_t^{(R)} \equiv \frac{a}{Rb^2}$ for $t \geq 0$ yields*

$$\Psi(u, K^{(R)}) \leq C\, e^{-Ru}$$

with $C < \infty$ for all $u \geq 0$.

A verification that R is indeed the optimal adjustment coefficient of the Markov-modulated Poisson model with investment can be found in the following section. At last note, it follows from Corollary 2.20 that $\rho^{(K^{(R)})}$ must be strictly positive whenever R exists since otherwise $\Psi_j(u, K^{(R)}) = 1$ for all $u \geq 0$ and every $j \in E$ according to Proposition 2.11.

2.5 Optimality

Recall that R is originally defined by $R = \sup\{r > 0\,;\phi_{jj}(r) < 1\ \forall j \in E\}$ in section 2.3. Under the assumption that R exists we have proved that

$$\Psi(u, K^{(R)}) \le C\, e^{-Ru}$$

with $C < \infty$ for all $u \ge 0$ where the constant investment strategy $K^{(R)}$ is defined by $K_t^{(R)} \equiv \frac{a}{Rb^2}$ for $t \ge 0$. Let us now show that the investment strategy $K^{(R)}$ is indeed optimal amongst all investment strategies $K = k(J)$ in the sense that we cannot find an investment strategy $K = k(J)$, a constant $\hat{C} < \infty$ and some $r > R$ such that

$$\Psi(u, K) \le \hat{C}\, e^{-ru}$$

for all $u \ge 0$. Note that we can restrict ourselves to the case where R exists. If R does not exist it particularly follows that $R^{(K)}$ does not exist for any $K = k(J)$. As shown at the end of section 2.2 this implies that we cannot find an adjustment coefficient for the Markov-modulated Poisson model under any investment strategy $K = k(J)$ if R does not exist.

Theorem 2.21. *Suppose that R defined by (2.18) exists and consider any fixed investment strategy $K = k(J)$. For this investment strategy K we furthermore assume that $R^{(K)}$ exists and that we can find a constant $\delta > 0$ such that $\phi_{jj}^{(K)}(R^{(K)} + \delta) < \infty$ for some $j \in E$. We then have*

$$\lim_{u\to\infty} \frac{\Psi_j(u, K)}{e^{-ru}} = \infty$$

for all $r > R^{(K)}$ and thus in particular for all $r > R$.

Proof:

Consider any fixed environmental state $j \in E$ such that $\phi_{jj}^{(K)}(R^{(K)} + \delta) < \infty$ for some $\delta > 0$ and suppose throughout this proof that $J_0 = j$. Since $\phi_{jj}^{(K)}$ is convex and therefore continuous on the interior of its domain it follows that $\phi_{jj}^{(K)}$ is finite in the δ-neighborhood of $R^{(K)}$ with $\phi_{jj}^{(K)}(R^{(K)}) = 1$.

As shown in section 2.2, $\big(X_{\tau_n^{(j)}}(K)\big)_{n\in\mathbb{N}}$ is a random walk under the assumption that $J_0 = j$. Conditioned under $J_0 = j$ we define the ruin probability of the shifted random walk $\big(Y_{\tau_n^{(j)}}(u,K)\big)_{n\in\mathbb{N}} = \big(u + X_{\tau_n^{(j)}}(K)\big)_{n\in\mathbb{N}}$ by

$$\Psi_j^{rw}(u,K) := \mathbb{P}_j\Big(\inf_{n\in\mathbb{N}} Y_{\tau_n^{(j)}}(u,K) < 0\Big)$$

for $u \geq 0$. It is obvious that $\Psi_j^{rw}(u,K) \leq \Psi_j(u,K)$ for all $u \geq 0$. Now, Proposition 2.8 yields

$$\mathbb{E}_j\Big(e^{-R^{(K)}X_{\tau^{(j)}}(K)}\Big) = \phi_{jj}^{(K)}(R^{(K)}) = 1\,.$$

Note that the distribution of $X_{\tau^{(j)}}(K)$, i.e. the distribution of the generic random variable for the steps, is clearly non-lattice. Since the existence of $R^{(K)}$ moreover implies that $\rho^{(K)} > 0$ it follows from Theorem 6.5.7 and the associated remark in Rolski et al. [RSS$^+$99] that

$$\lim_{u\to\infty} \frac{\Psi_j^{rw}(u,K)}{e^{-R^{(K)}u}} = \tilde{C}$$

for some constant $\tilde{C} > 0$. From $\Psi_j(u,K) \geq \Psi_j^{rw}(u,K)$ for all $u \geq 0$ it thus follows that

$$\lim_{u\to\infty} \frac{\Psi_j(u,K)}{e^{-ru}} \geq \lim_{u\to\infty} \frac{\Psi_j^{rw}(u,K)}{e^{-R^{(K)}u}}\, e^{(r-R^{(K)})u} = \infty$$

for all $r > R^{(K)}$ and therefore in particular for all $r > R$ since $R \geq R^{(K)}$.

□

Hence, R is the optimal adjustment coefficient of the Markov-modulated Poisson model with investment and $K^{(R)}$ is the corresponding optimal investment strategy in the sense that it minimizes the ruin probability $\Psi(u,K)$ amongst all investment strategies $K = k(J)$ if the initial reserve $u \geq 0$ is sufficiently large.

We carry on with an example for which the condition $\phi_{jj}^{(K)}(R^{(K)}+\delta) < \infty$ is fulfilled for some $\delta > 0$, $j \in E$ and for all investment strategies $K = k(J)$.

Example 2.22. In this example let the Markov process J be periodic in the sense that its intensity matrix is without loss of generality given by

$$Q := \begin{pmatrix} -q_1 & q_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -q_2 & q_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & q_{-3} & q_3 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & 0 & -q_{d-2} & q_{d-2} & 0 \\ 0 & \cdots & 0 & 0 & 0 & -q_{d-1} & q_{d-1} \\ q_d & 0 & \cdots & 0 & 0 & 0 & -q_d \end{pmatrix}$$

where $q_1, \ldots, q_d \in \mathbb{R}_+$. Thus, if the Markov process jumps to state $j \in E$, it stays there for a stochastic time $\sigma^{(j)} \sim \mathrm{Exp}(q_j)$ and then jumps almost surely to state k where $k = 1$ if $j = d$ and $k = j + 1$ otherwise. In state k the Markov process then stays the stochastic time $\sigma^{(k)} \sim \mathrm{Exp}(q_k)$ and so on.

Let us now consider any arbitrary investment strategy $K = k(J)$. It follows from the choice of J that

$$\phi_{jj}^{(K)}(r) = \phi_j^{(K)}(r) \cdot \phi_{j+1}^{(K)}(r) \cdot \ldots \cdot \phi_d^{(K)}(r) \cdot \phi_1^{(K)}(r) \cdot \ldots \cdot \phi_{j-1}^{(K)}(r) \tag{2.33}$$

for all $r \geq 0$ and $j \in E$. Note that the function $\phi_i^{(K)}$ is given by

$$\phi_i^{(K)}(r) = \mathbb{E}\left(e^{\left[\lambda_i h_i(r)+\frac{1}{2}r^2b^2k(i)^2-r(c+ak(i))\right]\sigma^{(i)}}\right)$$
$$= \begin{cases} \frac{q_i}{q_i+r(c+ak(i))-\lambda_i h_i(r)-\frac{1}{2}r^2b^2k(i)^2} & , \quad q_i + r(c+ak(i)) - \lambda_i h_i(r) - \frac{1}{2}r^2b^2k(i)^2 > 0 \\ \infty & , \quad \text{otherwise} \end{cases}$$

for $r \geq 0$ and $i \in E$. It thus follows from our assumptions on the centered moment generating function h_i that $\phi_i^{(K)}$ continuously converges to infinity for every $i \in E$. Hence, (2.33) implies that the same is true for the function $\phi_{jj}^{(K)}$ for $j \in E$. This in turn means that $R^{(K)}$ exists and that we can find a $\delta > 0$ such that $\phi_{jj}^{(K)}(R^{(K)} + \delta)$ is finite for all $j \in E$. In this setup it thus follows from Theorem 2.21 that

$$\lim_{u \to \infty} \frac{\Psi_j(u, K)}{e^{-ru}} = \infty$$

for all $r > R$ and every $j \in E$.

$\diamond$

Now that we have verified R as the optimal adjustment coefficient of the Markov-modulated Poisson model with investment let us suppose for the moment that R exists. As mentioned in the introductory section of this chapter, the adjustment coefficient of the Markov-modulated Poisson model without investment is given as the strictly positive solution of the equation $\tilde{\kappa}(r) = 0$ where $\tilde{\kappa}(r)$ is the eigenvalue of the matrix

$$Q + \operatorname{diag}\Big(\lambda_i h_i(r)\,;i \in E\Big) - crI_d\,,\; r \geq 0\,,$$

which has maximum real part. Recall from the previous section that we have a similar result for the optimal adjustment coefficient of the Markov-modulated Poisson model with investment. The optimal adjustment coefficient R can be defined as the strictly positive solution of the equation $\kappa(r) = 0$ where $\kappa(r)$ is the eigenvalue of the matrix

$$H(r) = Q + \operatorname{diag}\Bigg(\lambda_i h_i(r)\,;i \in E\Bigg) - \left(cr + \frac{a^2}{2b^2}\right)I_d$$

which has maximum real part.

It is obvious that these two matrices coincide if the drift parameter a of the Brownian motion with drift $W_{a,b}$ equals zero. In this case the optimal adjustment coefficient of the Markov-modulated Poisson model with investment is therefore equal to the adjustment coefficient of the Markov-modulated Poisson model without investment, i.e. $R = R^{(0)}$. But note that this is not surprising since the optimal investment strategy $K^{(R)}$ defined by $K_t^{(R)} \equiv \frac{a}{Rb^2}$ for $t \geq 0$ provides not to invest into the portfolio if $a = 0$.

2.6 A comparison with the compound Poisson model

In the final section of this chapter we compare the adjustment coefficients of the Markov-modulated Poisson model and its associated compound Poisson model under the respective optimal investment strategy. Recall from the previous sections that the optimal adjustment coefficient of the Markov-modulated Poisson model with investment is defined by

$$R = \sup \Big\{ r > 0 \,;\, \phi_{jj}(r) < 1 \;\forall j \in E \Big\} \tag{2.34}$$

and that the corresponding optimal investment strategy is given by $K^{(R)}$.

As mentioned in the introductory chapter, it is intuitively clear that we can associate a compound Poisson model to the Markov-modulated Poisson model in a natural way by averaging over the environment, confer for example Asmussen [Asm00], page 148. More precisely, we consider a compound Poisson model with investment where the intensity of the claim arrival process and respectively the claim size distribution are defined by

$$\lambda^* = \sum_{i \in E} \pi_i \lambda_i \quad \text{and} \quad B^* = \sum_{i \in E} \frac{\pi_i \lambda_i}{\lambda^*} \, B_i \,.$$

We refer to this model as the associated compound Poisson model. Note, that its claims have exponential moments since

$$\begin{aligned} h^*(r) &:= \int_0^\infty e^{rx} \, dB^*(x) - 1 = \left(\sum_{i \in E} \frac{\pi_i \lambda_i}{\lambda^*} \int_0^\infty e^{rx} \, dB_i(x) \right) - 1 \\ &= \left(\sum_{i \in E} \frac{\pi_i \lambda_i}{\lambda^*} \Big(h_i(r) + 1 \Big) \right) - 1 = \left(\sum_{i \in E} \frac{\pi_i \lambda_i}{\lambda^*} \, h_i(r) \right) + \left(\sum_{i \in E} \frac{\pi_i \lambda_i}{\lambda^*} \right) - 1 \\ &= \sum_{i \in E} \frac{\pi_i \lambda_i}{\lambda^*} \, h_i(r) \,. \end{aligned} \tag{2.35}$$

Recall from Proposition 2.10 that the safety loading of the Markov-modulated Poisson model without investment is given by $\rho^{(0)} = c - \sum_{i\in E} \pi_i \lambda_i \mu_{B_i}$. Since $\mu_{B^*} = \sum_{i\in E} \frac{\pi_i \lambda_i}{\lambda^*} \mu_{B_i}$ it is thus obvious that the safety loadings of the Markov-modulated Poisson model without investment and its associated compound Poisson model without investment coincide.

As mentioned in the first section of this chapter, it can be found in Gaier, Grandits and Schachermayer [GGS03] that the optimal adjustment coefficient R^* of this associated compound Poisson model with investment is defined as the strictly positive solution of the equation

$$\lambda^* h^*(r) = rc + \frac{a^2}{2b^2} \tag{2.36}$$

and that the corresponding optimal investment strategy is given by $K^{(R^*)}$.

Without investment it is known that the adjustment coefficient of the Markov-modulated Poisson model does not exceed the adjustment coefficient of its associated compound Poisson model, confer Remark 2.24. Under optimal investment we get exactly the same result.

Theorem 2.23.

(i) Let R be defined by (2.34) and let R^* be the strictly positive solution of equation (2.36), i.e. R and R^* are the adjustment coefficients of the Markov-modulated Poisson model and respectively its associated compound Poisson model under optimal investment. Then,

$$R \leq R^* .$$

(ii) *Let the investment strategy $K^{(r)}$ be defined by $K_t^{(r)} \equiv \frac{a}{rb^2}$ for all $t \geq 0$ and $r > 0$, i.e. the optimal investment strategies for the Markov-modulated Poisson model and its associated compound Poisson model with investment are given by $K^{(R)}$ and $K^{(R^*)}$, respectively. For $t \geq 0$ we then have*

$$\left| K_t^{(R)} \right| \geq \left| K_t^{(R^*)} \right| .$$

Proof:

(i) Consider any fixed $r > R^*$ and recall that $\mathbb{E}_j\big(\xi_i(\tau^{(j)})\big) = \pi_i \mathbb{E}_j(\tau^{(j)})$ for all $i \in E$ as for example given in Asmussen [Asm03], page 51. Using Jensen's inequality we thus get

$$\begin{aligned}
\phi_{jj}(r) &= \mathbb{E}_j\left(\exp\left(\sum_{i\in E}\lambda_i h_i(r)\xi_i(\tau^{(j)}) - \Big(rc + \frac{a^2}{2b^2}\Big)\tau^{(j)}\right)\right)\\
&\geq \exp\left(\mathbb{E}_j\left(\sum_{i\in E}\lambda_i h_i(r)\xi_i(\tau^{(j)}) - \Big(rc + \frac{a^2}{2b^2}\Big)\tau^{(j)}\right)\right)\\
&\geq \exp\left(\left(\sum_{i\in E}\pi_i\lambda_i h_i(r) - \Big(rc + \frac{a^2}{2b^2}\Big)\right)\mathbb{E}_j\big(\tau^{(j)}\big)\right) .
\end{aligned}$$

Now, we have $\lambda^* h^*(r) = \sum_{i\in E}\pi_i\lambda_i h_i(r)$ as shown around (2.35). Since R^* solves equation (2.36) it consequently follows for $r > R^*$ that

$$\sum_{i\in E}\pi_i\lambda_i h_i(r) - \Big(rc + \frac{a^2}{2b^2}\Big) = \lambda^* h^*(r) - \Big(rc + \frac{a^2}{2b^2}\Big) \geq 0 .$$

This implies

$$\phi_{jj}(r) \geq \exp\left(\left(\sum_{i\in E}\pi_i\lambda_i h_i(r) - \Big(rc + \frac{a^2}{2b^2}\Big)\right)\mathbb{E}_j\big(\tau^{(j)}\big)\right) \geq 1$$

for $r > R^*$ and consequently $R \leq R^*$ according to definition (2.34). Note that the last inequality also implies that the existence of R^* is a necessary condition for R to exist.

(ii) Noting that the investment strategy $K^{(r)}$ is defined by $K_t^{(r)} \equiv \frac{a}{rb^2}$ for $t \geq 0$ it is obvious that part (ii) directly follows from part (i).

$\square$

Remark 2.24. In the special case where the drift parameter a of the Brownian motion with drift $W_{a,b}$ is equal to zero the investment strategy $K^{(r)}$ provides not to invest into the portfolio for all $r > 0$. In this case Theorem 2.23 thus coincides with Theorem 3 in Asmussen and O'Cinneide [AO02] where it is shown that the adjustment coefficient of the Markov-modulated Poisson model without investment does not exceed the adjustment coefficient of its associated compound Poisson model without investment.

$\diamond$

Using the optimal investment strategy in the respective model we therefore get a smaller adjustment coefficient in the Markov-modulated Poisson model than in its associated compound Poisson model. Moreover, in the Markov-modulated Poisson model the optimal investment strategy provides to invest a larger amount of money into the portfolio than in its associated compound Poisson model if the drift parameter a of the Brownian motion with drift $W_{a,b}$ is positive. If the drift parameter is negative we have to obtain a larger amount of money in the Markov-modulated Poisson model than in its associated compound Poisson model by selling the portfolio short. In both models we do not invest into the portfolio if the drift parameter equals zero.

Finally note that under some additional assumptions a pointwise comparison of the ruin probabilities of the Markov-modulated Poisson model and its associated compound Poison model under the same constant investment strategy can be found in the following chapter.

Chapter 3

Diffusion approximation

In this chapter we consider the same Markov-modulated Poisson model with investment as before. However, this time the model is approximated by a certain Markov-modulated Poisson model without investment. We then try to deduce assertions for the Markov-modulated Poisson model with investment from well known results for the Markov-modulated Poisson model without investment.

After stating what is meant by the convergence of stochastic processes we initially introduce the basic ideas in order to approximate a diffusion process. Then, a Markov-modulated Poisson model without investment is determined which approximates the original Markov-modulated Poisson model with investment. We further show that the ruin probability as well as the adjustment coefficient of the approximating model converge to the ruin probability and respectively the adjustment coefficient of the model of interest. Finally, the ruin probabilities of the Markov-modulated Poisson model and its associated compound Poisson model under the same constant investment strategy are compared under some additional assumptions.

3.1 An approximation for the Markov-modulated Poisson model with investment

As in the previous chapter we consider the Markov-modulated Poisson model with investment. Using the same notation as before, the wealth process for any investment strategy $K = k(J)$ is given by

$$\begin{aligned} Y_t(u,K) &= R_t(u) + a\int_0^t K_s\,ds + b\int_0^t K_s\,dW_s \\ &= u + ct - \sum_{k=1}^{N_t} U_k + a\sum_{i\in E} k(i)\,\xi_i(t) + b\sum_{i\in E} k(i)\,W^{(i)}_{\xi_i(t)} \\ &= u + ct - \sum_{i\in E}\sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k + a\sum_{i\in E} k(i)\,\xi_i(t) + b\sum_{i\in E} k(i)\,W^{(i)}_{\xi_i(t)}\,,\ t\geq 0\,. \end{aligned}$$

Recall that the standard Brownian motions $W^{(1)},\ldots,W^{(d)}$ are assumed to be independent.

In chapter 2 we directly determine the adjustment coefficient of the Markov-modulated Poisson model with respect to any fixed investment strategy $K = k(J)$. An alternative way is to approximate the diffusion part of the wealth process $Y(u,K)$. It is well known that a diffusion arises as the limit of properly scaled classical risk processes where the claims are very small and frequent. We thus might be able to deduce assertions for the Markov-modulated Poisson model with investment from well known results for the Markov-modulated Poisson model without investment. This fact was for example also exploited in Sarkar and Sen [SaSe05] for the classical Poisson model without Markov-modulation.

First of all, we certainly have to define what is meant by the convergence of stochastic processes. In this work the convergence of stochastic processes is defined as the weak convergence of their distributions with respect to the commonly used Skorohod topology. We only give a short sketch of the definition which can for example be found in the books by Billingsley [Bil99] and Whitt [Whi02], respectively.

For any subinterval I of the real line let $D(I, \mathbb{R}^k)$ be the space of all cadlag functions $x : I \to \mathbb{R}^k$. We initially consider the space $D_T := D\big([0,T], \mathbb{R}\big)$ where $T > 0$. In order to define a metric on D_T let Υ_T be the set of all strictly increasing functions υ mapping the domain $[0,T]$ onto itself such that υ as well as its inverse υ^{-1} are continuous. Furthermore, let id be the identity mapping on $[0,T]$, i.e. $id(t) = t$ for all $t \in [0,T]$. We now endow the space D_T with the commonly used J_1 topology, the so-called Skorohod topology. Then, the standard J_1 metric on D_T is defined by

$$d_{J_1}(x_1, x_2) := \inf_{\upsilon \in \Upsilon_T} \Big\{ \max \big\{ \|x_1 \circ \upsilon - x_2\|, \|\upsilon - id\| \big\} \Big\} \quad \text{for} \quad x_1, x_2 \in D_T \tag{3.1}$$

where the uniform metric $\|\cdot\|$ on D_T is given by

$$\|x\| := \sup_{0 \le t \le T} \big\{ |x(t)| \big\} \quad \text{for} \quad x \in D_T\,. \tag{3.2}$$

By using the standard J_1 metric d_{J_1} instead of the uniform metric $\|\cdot\|$ functions are close in the metric space (D_T, d_{J_1}) if they are uniformly close over $[0,T]$ after allowing small perturbations of time. Examples of functions which converge in (D_T, d_{J_1}) but not in $(D_T, \|\cdot\|)$ can for example be found in the books mentioned above.

But note that the wealth process $Y(u, K)$ has infinite time horizon. Furthermore, we have to deal with the convergence of multidimensional stochastic processes in this chapter. Hence, the space D_T has to be modified in the following two ways. Firstly, let us extend the range of the functions from $\mathbb{R}$ to $\mathbb{R}^k$ with $k \in \mathbb{N}$. The standard J_1 metric defined in (3.1) extends directly to $D_T^k := D\big([0,T], \mathbb{R}^k\big)$ when the norm $|\cdot|$ on $\mathbb{R}$ in (3.2) is replaced by a corresponding norm on $\mathbb{R}^k$ as for example the maximum norm. Using the maximum norm on $\mathbb{R}^k$ we obtain the so-called standard J_1 metric on D_T^k.

For a fixed $k \in \mathbb{N}$ let us secondly extend the domain of the functions and consider the space $D^k := D\big([0,\infty), \mathbb{R}^k\big)$ of all cadlag functions $x : [0,\infty) \to \mathbb{R}^k$. It is now natural to define the convergence of a sequence $\big(x_n\big)_{n\in\mathbb{N}}$ in D^k in terms of the associated convergence of the restrictions of x_n to the subintervals $[0,T]$ in the space D_T^k for all $T > 0$. However, as described in Whitt [Whi02] this causes problems if the right endpoint T is a discontinuity point of the prospective limit function x. In the space D^k a sequence $\big(x_n\big)_{n\in\mathbb{N}}$ is thus said to converge to x as $n \to \infty$ if the restrictions of x_n to $[0,T]$ converge to the restriction of x to $[0,T]$ in D_T^k for all continuity points $T > 0$ of x. In order to ease notation we put $D := D^1$.

Let now $C(D^k)$ be the space of all functions $f : D^k \to \mathbb{R}$ which are bounded and continuous with respect to the standard J_1 metric on D^k. A sequence of k-dimensional stochastic processes $\big(X^{(n)}\big)_{n\in\mathbb{N}}$ with $X^{(n)} := \big\{X_t^{(n)}, t \geq 0\big\}$ is then said to converge to a k-dimensional stochastic process $X := \big\{X_t, t \geq 0\big\}$ which is denoted by $X^{(n)} \Rightarrow X$ if

$$\lim_{n\to\infty} \mathbb{E}\Big(f\big(X^{(n)}\big)\Big) = \mathbb{E}\Big(f\big(X\big)\Big) \quad \text{for all} \quad f \in C(D^k)\,.$$

The idea for the diffusion approximation considered in this chapter is based on the following result in Grandell [Grl77] which can also be found in the appendix of Grandell [Grl91]. Let $\tilde{N}$ be a standard Poisson process with intensity α and $(\tilde{U}_k)_{k\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with expectation $\tilde{\mu}$ and variance $\tilde{\sigma}^2$. It then follows that

$$\left\{ \frac{\sum_{k=1}^{\tilde{N}_{nt}} \tilde{U}_k - n\alpha\tilde{\mu}t}{\sqrt{n}}, t \geq 0 \right\} \Rightarrow \left\{ \sqrt{\alpha(\tilde{\sigma}^2 + \tilde{\mu}^2)}\, W_t, t \geq 0 \right\} \tag{3.3}$$

as $n \to \infty$ where W is a standard Brownian motion.

Let us now fix any investment strategy $K = k(J)$ and $n \in \mathbb{N}$. In that what follows we consider independent standard Poisson processes $\tilde{N}^{(1,n)}, \dots, \tilde{N}^{(d,n)}$ which are also independent of the risk reserve process $R(u)$ as well as the environmental Markov process J. Each of these processes have intensity nb^2. Further, let $(\tilde{U}_k^{(1)})_{k\in\mathbb{N}}, \dots, (\tilde{U}_k^{(d)})_{k\in\mathbb{N}}$ be independent sequences of strictly positive random variables which are also independent of the processes $\tilde{N}^{(1,n)}, \dots, \tilde{N}^{(d,n)}$, $R(u)$ and J. For each $i \in E$ it is moreover assumed that the random variables $(\tilde{U}_k^{(i)})_{k\in\mathbb{N}}$ are independent and identically distributed with expectation $\tilde{\mu}$ and second moment $k(i)^2$. We denote the corresponding distribution by $\tilde{B}_i$. The process $Y^{(n)}(u, K)$ is then defined by

$$Y_t^{(n)}(u, K) = R_t(u) + a \sum_{i\in E} k(i)\, \xi_i(t) + \sqrt{n}\, b^2 \tilde{\mu}\, t - \sum_{i\in E} \sum_{k=1}^{\tilde{N}^{(i,n)}_{\xi_i(t)}} \frac{\tilde{U}_k^{(i)}}{\sqrt{n}}, t \geq 0. \tag{3.4}$$

It turns out that we can use the processes $Y^{(n)}(u, K)$, $n \in \mathbb{N}$, in order to approximate the wealth process $Y(u, K)$ as the following result shows.

Proposition 3.1. *Let the process $Y^{(n)}(u, K)$ for $n \in \mathbb{N}$ be defined by (3.4). For any investment strategy $K = k(J)$ and any initial reserve $u \geq 0$ we then have*

$$Y^{(n)}(u, K) \Rightarrow Y(u, K) \quad \text{as} \quad n \to \infty.$$

Proof:

Let us consider the independent standard Brownian motions $W^{(1)}, \dots, W^{(d)}$

and initially fix some $i \in E$. As already mentioned in (3.3) it follows from Grandell [Grl77] that

$$\Big\{\frac{\sum_{k=1}^{\tilde{N}_t^{(i,n)}} \tilde{U}_k^{(i)} - nb^2\tilde{\mu}t}{\sqrt{n}}\,, t \geq 0\Big\} \Rightarrow \Big\{b\,k(i)\,W_t^{(i)}\,, t \geq 0\Big\} \quad \text{as} \quad n \to \infty\,.$$

Next, we want to prove that this already implies

$$\Bigg(\Big\{\sum_{k=1}^{\tilde{N}_t^{(i,n)}} \frac{\tilde{U}_k^{(i)}}{\sqrt{n}} - \sqrt{n}\,b^2\tilde{\mu}t\,, t \geq 0\Big\}\,, i \in E\Bigg) \Rightarrow \Bigg(\Big\{b\,k(i)\,W_t^{(i)}\,, t \geq 0\Big\}\,, i \in E\Bigg) \tag{3.5}$$

as $n \to \infty$. The proof is similar to the proof of the multidimensional Donsker FCLT as for example given in Whitt [Whi02]. However, we have to be more careful here since the sums contain a random number of terms. The convergence of the one-dimensional marginal processes is mentioned above. By Corollary 11.6.2 in Whitt [Whi02], a corollary of Prohorov's Theorem, it thus follows that these marginal processes are tight. This in turn implies the tightness of the d-dimensional process

$$\Bigg(\Big\{\frac{\sum_{k=1}^{\tilde{N}_t^{(i,n)}} \tilde{U}_k^{(i)} - nb^2\tilde{\mu}t}{\sqrt{n}}\,, t \geq 0\Big\}\,, i \in E\Bigg)\,,$$

confer Theorem 11.6.7 in Whitt [Whi02].

For $\alpha_1, \ldots, \alpha_d \in \mathbb{R}$ it moreover follows by general marked point process theory that

$$\begin{aligned}
&\Big\{\sum_{i \in E} \alpha_i\,\Big(\sum_{k=1}^{\tilde{N}_t^{(i,n)}} \frac{\tilde{U}_k^{(i)}}{\sqrt{n}} - \sqrt{n}b^2\tilde{\mu}t\Big)\,, t \geq 0\Big\} \\
&\qquad = \Big\{\frac{1}{\sqrt{n}}\Big(\sum_{i \in E}\sum_{k=1}^{\tilde{N}_t^{(i,n)}} \alpha_i\,\tilde{U}_k^{(i)} - nb^2\tilde{\mu}\big(\sum_{i \in E}\alpha_i\big)t\Big)\,, t \geq 0\Big\} \\
&\qquad \stackrel{\mathcal{D}}{=} \Big\{\frac{1}{\sqrt{n}}\Big(\sum_{k=1}^{\tilde{N}_t^{(0,n)}} \tilde{U}_k^{(0)} - nb^2 d\,\frac{\tilde{\mu}}{d}\big(\sum_{i \in E}\alpha_i\big)t\Big)\,, t \geq 0\Big\}
\end{aligned}$$

where $\tilde{N}^{(0,n)}$ is a standard Poisson process with intensity $nb^2 d$ and where the random variables $(\tilde{U}_k^{(0)})_{k\in\mathbb{N}}$ are independent and identically distributed with distribution $\sum_{i\in E}\frac{1}{d}\tilde{B}_i(\frac{x}{\alpha_i})$. This means that $\tilde{U}_1^{(0)}$ has expectation $\frac{\tilde{\mu}}{d}\sum_{i\in E}\alpha_i$ and second moment $\frac{1}{d}\sum_{i\in E}\alpha_i^2 k(i)^2$. Using the convergence result (3.3) in Grandell [Grl77] again it therefore follows that

$$\Big\{\frac{1}{\sqrt{n}}\Big(\sum_{k=1}^{\tilde{N}_t^{(0,n)}}\tilde{U}_k^{(0)} - nb^2 d\,\frac{\tilde{\mu}}{d}\big(\sum_{i\in E}\alpha_i\big)t\Big)\,, t\geq 0\Big\} \Rightarrow \Big\{b\Big(\sum_{i\in E}\alpha_i^2 k(i)^2\Big)^{\frac{1}{2}}\,W_t^{(0)}\,, t\geq 0\Big\}$$

where $W^{(0)}$ is some standard Brownian motion. Finally, since the standard Brownian motions $W^{(1)},\ldots,W^{(d)}$ are assumed to be independent we have

$$\Big\{b\Big(\sum_{i\in E}\alpha_i^2 k(i)^2\Big)^{\frac{1}{2}}\,W_t^{(0)}\,, t\geq 0\Big\} \stackrel{\mathcal{D}}{=} \Big\{b\sum_{i\in E}\alpha_i\,k(i)\,W_t^{(i)}\,, t\geq 0\Big\}$$

and consequently obtain

$$\Big\{\sum_{i\in E}\alpha_i\,\Big(\sum_{k=1}^{\tilde{N}_t^{(i,n)}}\frac{\tilde{U}_k^{(i)}}{\sqrt{n}} - \sqrt{n}b^2\tilde{\mu}t\Big)\,, t\geq 0\Big\} \Rightarrow \Big\{\sum_{i\in E}\alpha_i\,b\,k(i)\,W_t^{(i)}\,, t\geq 0\Big\}$$

as $n\to\infty$. Applying the Cramér-Wold device which is given as Theorem 4.3.3 in Whitt [Whi02] we thus obtain the convergence of all finite dimensional distributions of the process of interest. This together with the tightness finally yields the convergence in (3.5) according to Corollary 11.6.2 in Whitt [Whi02].

Recalling that $W^{(i)} \stackrel{\mathcal{D}}{=} -W^{(i)}$ and applying the time transformation $t\mapsto\xi_i(t)$ we consequently obtain

$$\Big(\Big\{\sum_{k=1}^{\tilde{N}_{\xi_i(t)}^{(i,n)}}\Big(\frac{\tilde{U}_k^{(i)}}{\sqrt{n}} - \sqrt{n}\,b^2\tilde{\mu}\,\xi_i(t)\Big)\,, t\geq 0\Big\}\,, i\in E\Big)$$
$$\Rightarrow \Big(\Big\{-b\,k(i)\,W_{\xi_i(t)}^{(i)}\,, t\geq 0\Big\}\,, i\in E\Big) \quad \text{as} \quad n\to\infty\,.$$

Note that addition on $D\times\ldots\times D$ is measurable and continuous at limits with respect to the standard J_1 metric if the limiting functions have no common discontinuity points, confer Whitt [Whi02]. In our case, it thus follows from the

Continuous Mapping Theorem which is for example given as Theorem 3.4.3 in Whitt [Whi02] that as $n \to \infty$ we have

$$\begin{aligned} Y^{(n)}(u,K) =& \Big\{ R_t(u) + a \sum_{i\in E} k(i)\, \xi_i(t) + \sqrt{n}\, b^2 \tilde{\mu}\, t - \sum_{i\in E} \sum_{k=1}^{\tilde{N}^{(i,n)}_{\xi_i(t)}} \frac{\tilde{U}^{(i)}_k}{\sqrt{n}}\, , t \geq 0 \Big\} \\ & \Rightarrow \Big\{ R_t(u) + a \sum_{i\in E} k(i)\, \xi_i(t) + b \sum_{i\in E} k(i)\, W^{(i)}_{\xi_i(t)}\, , t \geq 0 \Big\} = Y(u,K)\,. \end{aligned}$$

$\square$

Proposition 3.1 thus gives us an approximation for the wealth process $Y(u, K)$. But we are certainly still interested in the ruin probability $\Psi(u, K)$. Recall from Proposition 2.11 that $\Psi(u, K) = 1$ if the safety loading

$$\rho^{(K)} = c + a \sum_{i\in E} \pi_i k(i) - \sum_{i\in E} \pi_i \lambda_i \mu_{B_i}$$

is not strictly positive. We can consequently restrict ourselves to the case where $\rho^{(K)} > 0$.

Unfortunately, the mapping which takes the infimum of a function $x \in D$ over an infinite domain is not continuous at limits with respect to the standard J_1 metric. Thus, in order to show that $Y^{(n)}(u, K) \Rightarrow Y(u, K)$ implies

$$\inf_{t\geq 0} Y^{(n)}_t(u,K) \Rightarrow \inf_{t\geq 0} Y_t(u,K)$$

as $n \to \infty$ we cannot directly use the Continuous Mapping Theorem which is for example Theorem 3.4.3 in Whitt [Whi02]. Nevertheless, we can use

$$\Psi^{(n)}(u,K) := \mathbb{P}\Big(\inf_{t\geq 0}\ Y^{(n)}_t(u,K) < 0 \Big) \tag{3.6}$$

with $n \to \infty$ in order to approximate the ruin probability $\Psi(u, K)$ as the following result shows.

Theorem 3.2. *Consider any investment strategy $K = k(J)$ and suppose that $\rho^{(K)} > 0$. Further, let the ruin probability $\Psi^{(n)}(u,K)$ for $n \in \mathbb{N}$ be defined by (3.6). We then have*

$$
\begin{aligned}
&(i) \quad \sup_{t\geq 0} -Y_t^{(n)}(0,K) \Rightarrow \sup_{t\geq 0} -Y_t(0,K) \quad \text{as} \quad n \to \infty\,,\\
&(ii) \quad \Psi^{(n)}(u,K) \longrightarrow \Psi(u,K) \quad \text{as} \quad n \to \infty \quad \text{for all} \quad u \geq 0\,.
\end{aligned}
$$

Proof:

(i) Firstly, note that we have

$$-Y^{(n)}(0,K) \Rightarrow -Y(0,K)$$

as $n \to \infty$ using Proposition 3.1. According to Theorem 6 and respectively Theorem 8 in Grandell [Grl77] this yields

$$\sup_{t\geq 0} -Y_t^{(n)}(0,K) \Rightarrow \sup_{t\geq 0} -Y_t(0,K)$$

as $n \to \infty$ if we can prove that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{t\geq m} -Y_t^{(n)}(0,K) > 0 \Big) = 0\,. \tag{3.7}$$

It thus remains to show that condition (3.7) is fulfilled. From the Ergodic Theorem as for example given in Brémaud [Bré99] we know that

$$\lim_{t\to\infty} \frac{1}{t}\xi_i(t) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \delta_{\{i\}}(J_s)\, ds \overset{a.s.}{=} \pi_i$$

and consequently

$$\lim_{t\to\infty} \frac{1}{t}\Big(ct + a \sum_{i\in E} k(i)\, \xi_i(t) - \sum_{i\in E} \lambda_i \mu_{B_i} \xi_i(t) \Big) \overset{a.s.}{=} \rho^{(K)}\,.$$

In particular, we obtain

$$\begin{aligned}
&\liminf_{t\to\infty}\ c + a\sum_{i\in E} k(i)\,\frac{\xi_i(t)}{t} - \sum_{i\in E}\lambda_i\mu_{B_i}\frac{\xi_i(t)}{t} \\
&\quad = \lim_{m\to\infty}\ \inf_{t\geq m}\ c + a\sum_{i\in E} k(i)\,\frac{\xi_i(t)}{t} - \sum_{i\in E}\lambda_i\mu_{B_i}\frac{\xi_i(t)}{t} \overset{a.s.}{=} \rho^{(K)}\,.
\end{aligned}$$

Since almost sure convergence implies convergence in probability this yields

$$\begin{aligned}
&\lim_{m\to\infty}\ \mathbb{P}\Big(ct + a\sum_{i\in E} k(i)\,\xi_i(t) - \sum_{i\in E}\lambda_i\,\mu_{B_i}\,\xi_i(t) \geq \frac{\rho^{(K)}}{2}\,t \ \ \forall t\geq m\Big) \\
&= \lim_{m\to\infty}\ \mathbb{P}\Big(\inf_{t\geq m}\ c + a\sum_{i\in E} k(i)\,\frac{\xi_i(t)}{t} - \sum_{i\in E}\lambda_i\mu_{B_i}\frac{\xi_i(t)}{t} \geq \frac{\rho^{(K)}}{2}\Big) \\
&\geq \lim_{m\to\infty}\ \mathbb{P}\Big(\Big|\inf_{t\geq m}\ c + a\sum_{i\in E} k(i)\,\frac{\xi_i(t)}{t} - \sum_{i\in E}\lambda_i\mu_{B_i}\frac{\xi_i(t)}{t} - \rho^{(K)}\Big| \leq \frac{\rho^{(K)}}{2}\Big) = 1\,.
\end{aligned}$$

Now, let us define the process $M^{(n)}(K)$ by

$$M_t^{(n)}(K) := \sum_{i\in E}\sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U_k^{(i)} - \sum_{i\in E}\lambda_i\,\mu_{B_i}\,\xi_i(t) + \sum_{i\in E}\sum_{k=1}^{\tilde{N}^{(i,n)}_{\xi_i(t)}} \frac{\tilde{U}_k^{(i)}}{\sqrt{n}} - \sqrt{n}\,b^2\tilde{\mu}\,t\,.$$

Putting

$$A_m := \Big\{ct + a\sum_{i\in E} k(i)\,\xi_i(t) - \sum_{i\in E}\lambda_i\,\mu_{B_i}\,\xi_i(t) \geq \frac{\rho^{(K)}}{2}\,t \ \ \forall t\geq m\Big\}$$

we have

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\geq m} -Y_t^{(n)}(0,K) > 0\Big) \\
&= \mathbb{P}\Big(\sup_{t\geq m} M_t^{(n)}(K) - \big(ct + a\sum_{i\in E} k(i)\,\xi_i(t) - \sum_{i\in E}\lambda_i\,\mu_{B_i}\,\xi_i(t)\big) > 0\Big) \\
&= \mathbb{P}\Big(\Big\{\sup_{t\geq m} M_t^{(n)}(K) - \big(ct + a\sum_{i\in E} k(i)\,\xi_i(t) - \sum_{i\in E}\lambda_i\,\mu_{B_i}\,\xi_i(t)\big) > 0\Big\} \cap A_m\Big) \\
&\qquad + \mathbb{P}\Big(\Big\{\sup_{t\geq m} M_t^{(n)}(K) - \big(ct + a\sum_{i\in E} k(i)\,\xi_i(t) - \sum_{i\in E}\lambda_i\,\mu_{B_i}\,\xi_i(t)\big) > 0\Big\} \cap A_m^c\Big) \\
&\leq \mathbb{P}\Big(\Big\{\sup_{t\geq m} M_t^{(n)}(K) - \frac{\rho^{(K)}}{2}\,t > 0\Big\} \cap A_m\Big) + 1 - \mathbb{P}\big(A_m\big) \\
&= \mathbb{E}\Big(I(A_m)\ \mathbb{P}^{\mathcal{F}^J_\infty}\Big(\sup_{t\geq m} M_t^{(n)}(K) - \frac{\rho^{(K)}}{2}\,t > 0\Big)\Big) + 1 - \mathbb{P}\big(A_m\big)\,.
\end{aligned}$$

As done in Grandell [Grl78] we intend to bound this probability using the Hájek-Rényi inequality in the version as given in Theorem 2 of Frank [Fra66]. For any $h \in (0,1)$ we obtain

$$\mathbb{P}^{\mathcal{F}^J_\infty}\Big(\sup_{j \geq \lfloor \frac{m}{h} \rfloor + 1} M^{(n)}_{jh}(K) - \frac{\rho^{(K)}}{2}\, jh > 0\Big) = \mathbb{P}^{\mathcal{F}^J_\infty}\Big(\sup_{j \geq \lfloor \frac{m}{h} \rfloor + 1} \frac{M^{(n)}_{jh}(K)}{\frac{\rho^{(K)}}{2}\, jh} > 1\Big)$$
$$\leq \mathbb{P}^{\mathcal{F}^J_\infty}\Big(\sup_{j \geq \lfloor \frac{m}{h} \rfloor + 1} \left| \frac{M^{(n)}_{jh}(K)}{\frac{\rho^{(K)}}{2}\, jh} \right| > 1\Big)$$

Next, let us check if the conditions for the Hájek-Rényi inequality are fulfilled. Firstly, we note that the sequence $\big(\Delta M^{(n)}_j(K)\big)_{j \in \mathbb{N}}$ defined by $\Delta M^{(n)}_j(K) := M^{(n)}_{jh}(K) - M^{(n)}_{(j-1)h}(K)$ satisfies

$$\mathbb{E}^{\mathcal{F}^J_\infty}\Big[\Delta M^{(n)}_j(K) \,\Big|\, \Delta M^{(n)}_{j-1}(K), \ldots, \Delta M^{(n)}_1(K)\Big] = \mathbb{E}^{\mathcal{F}^J_\infty}\Big(\Delta M^{(n)}_j(K)\Big) = 0$$

for every $j \in \mathbb{N}$. Secondly, we have

$$\mathbb{E}^{\mathcal{F}^J_\infty}\Big(\big(\Delta M^{(n)}_j(K)\big)^2\Big)$$
$$= \mathbb{E}^{\mathcal{F}^J_\infty}\Bigg(\Big(\sum_{i \in E} \sum_{k = N^{(i)}_{\xi_i((j-1)h)}+1}^{N^{(i)}_{\xi_i(jh)}} U^{(i)}_k - \sum_{i \in E} \lambda_i\, \mu_{B_i}\, \big(\xi_i(jh) - \xi_i((j-1)h)\big)\Big)^2\Bigg)$$
$$+ \mathbb{E}^{\mathcal{F}^J_\infty}\Bigg(\Big(\sum_{i \in E} \sum_{k = \tilde{N}^{(i,n)}_{\xi_i((j-1)h)}+1}^{\tilde{N}^{(i,n)}_{\xi_i(jh)}} \frac{\tilde{U}^{(i)}_k}{\sqrt{n}} - \sqrt{n}\, b^2 \tilde{\mu}\, h\Big)^2\Bigg)$$

for every $j \in \mathbb{N}$ where the first part of this sum is equal to

$$\mathrm{Var}^{\mathcal{F}^J_\infty}\Big(\sum_{i \in E} \sum_{k = N^{(i)}_{\xi_i((j-1)h)}+1}^{N^{(i)}_{\xi_i(jh)}} U^{(i)}_k - \sum_{i \in E} \lambda_i\, \mu_{B_i}\, \big(\xi_i(jh) - \xi_i((j-1)h)\big)\Big)$$
$$= \sum_{i \in E} \mathrm{Var}^{\mathcal{F}^J_\infty}\Big(\sum_{k = N^{(i)}_{\xi_i((j-1)h)}+1}^{N^{(i)}_{\xi_i(jh)}} U^{(i)}_k\Big)$$
$$= \sum_{i \in E} \Bigg(\mathbb{E}^{\mathcal{F}^J_\infty}\Big(N^{(i)}_{\xi_i(jh) - \xi_i((j-1)h)}\Big)\, \mathrm{Var}^{\mathcal{F}^J_\infty}\big(U^{(i)}_1\big)$$
$$+ \mathrm{Var}^{\mathcal{F}^J_\infty}\Big(N^{(i)}_{\xi_i(jh) - \xi_i((j-1)h)}\Big)\, \Big(\mathbb{E}^{\mathcal{F}^J_\infty}\big(U^{(i)}_1\big)\Big)^2\Bigg)$$

$$
\begin{aligned}
&= \sum_{i\in E} \lambda_i \Big(\xi_i(jh) - \xi_i((j-1)h)\Big) \Big(\mathrm{Var}\big(U_1^{(i)}\big) + \Big(\mathbb{E}\big(U_1^{(i)}\big)\Big)^2\Big) \\
&\le h \max_{i\in E} \lambda_i \, \mathbb{E}\Big(\big(U_1^{(i)}\big)^2\Big)\,.
\end{aligned}
$$

Analogously, we can show that the second part is less or equal than

$$
h\, nb^2 \max_{i\in E} \mathbb{E}\Big(\big(\frac{\tilde{U}_1^{(i)}}{\sqrt{n}}\big)^2\Big) = h\, b^2 \max_{i\in E} k(i)^2\,.
$$

Hence,

$$
\mathbb{E}^{\mathcal{F}^J_\infty}\Big(\big(\Delta M_j^{(n)}(K)\big)^2\Big) \le h \underbrace{\Big(\max_{i\in E} \lambda_i \, \mathbb{E}\Big(\big(U_1^{(i)}\big)^2\Big) + b^2 \max_{i\in E} k(i)^2\Big)}_{=:C}\,.
$$

Applying the Hájek-Rényi inequality we consequently obtain

$$
\begin{aligned}
&\mathbb{P}^{\mathcal{F}^J_\infty}\Big(\sup_{j\ge \lfloor \frac{m}{h}\rfloor+1} \left|\frac{M_{jh}^{(n)}(K)}{\frac{\rho^{(K)}}{2}\, jh}\right| > 1\Big) \\
&\le \frac{1}{\big(\lfloor \frac{m}{h}\rfloor h \, \frac{\rho^{(K)}}{2}\big)^2} \sum_{j=1}^{\lfloor \frac{m}{h}\rfloor} C\, h + \sum_{j=\lfloor \frac{m}{h}\rfloor+1}^{\infty} \frac{1}{\big(j\, h \, \frac{\rho^{(K)}}{2}\big)^2}\, C\, h \\
&= \frac{4C}{\rho^{(K)2}}\Big(\frac{1}{\lfloor \frac{m}{h}\rfloor h} + \frac{1}{h} \sum_{j=\lfloor \frac{m}{h}\rfloor+1}^{\infty} \frac{1}{j^2}\Big) \le \frac{4C}{\rho^{(K)2}}\Big(\frac{1}{\lfloor \frac{m}{h}\rfloor h} + \frac{1}{h} \int_{\lfloor \frac{m}{h}\rfloor}^{\infty} \frac{1}{x^2}\, dx\Big) \\
&= \frac{4C}{\rho^{(K)2}}\, \frac{2}{\lfloor \frac{m}{h}\rfloor h} \le \frac{8C}{(m-1)\rho^{(K)2}}\,.
\end{aligned}
$$

Since this bound is independent of $h \in (0,1)$ and $n \in \mathbb{N}$ it follows that

$$
\mathbb{P}^{\mathcal{F}^J_\infty}\Big(\sup_{t\ge m+1} M_t^{(n)}(K) - \frac{\rho^{(K)}}{2}\, t > 0\Big) \le \frac{8C}{(m-1)\rho^{(K)2}}\,.
$$

Plugging all things together we thus have

$$
\begin{aligned}
&\lim_{m\to\infty} \limsup_{n\to\infty} \; \mathbb{P}\Big(\sup_{t\ge m} -Y_t^{(n)}(0,K) > 0\Big) \\
&\le \lim_{m\to\infty} \limsup_{n\to\infty} \; \mathbb{E}\Big(I(A_m)\, \mathbb{P}^{\mathcal{F}^J_\infty}\Big(\sup_{t\ge m} M_t^{(n)}(K) - \frac{\rho^{(K)}}{2}\, t > 0\Big)\Big) + 1 - \mathbb{P}\big(A_m\big) \\
&\le \lim_{m\to\infty} \limsup_{n\to\infty} \; \frac{8C}{(m-2)\rho^{(K)2}}\, \mathbb{E}\big(I(A_m)\big) + 1 - P(A_m) = 0\,.
\end{aligned}
$$

Therefore, condition (3.7) is fulfilled and the assertion follows.

(ii) Noting that

$$\Psi(u,K) = \mathbb{P}\Big(\inf_{t\geq 0}\ Y_t(u,K) < 0\Big) = \mathbb{P}\Big(\sup_{t\geq 0} -Y_t(0,K) > u\Big)\,.$$

and

$$\Psi^{(n)}(u,K) = \mathbb{P}\Big(\inf_{t\geq 0}\ Y_t^{(n)}(u,K) < 0\Big) = \mathbb{P}\Big(\sup_{t\geq 0} -Y_t^{(n)}(0,K) > u\Big)$$

for $n \in \mathbb{N}$ it is obvious that part (ii) directly follows from part (i).

□

Finally note that the approximating wealth process $Y^{(n)}(u,K)$ can be regarded as a risk reserve process from the Markov-modulated Poisson model without investment. Recall that for any $t \geq 0$ we have

$$\begin{aligned} Y_t^{(n)}(u,K) &= R_t(u) + a\sum_{i\in E} k(i)\,\xi_i(t) + \sqrt{n}\,b^2\tilde{\mu}\,t - \sum_{i\in E}\sum_{k=1}^{\tilde{N}^{(i,n)}_{\xi_i(t)}} \frac{\tilde{U}_k^{(i)}}{\sqrt{n}} \\ &= u + ct + a\sum_{i\in E} k(i)\,\xi_i(t) + \sqrt{n}\,b^2\tilde{\mu}\,t - \sum_{i\in E}\left(\sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U_k^{(i)} + \sum_{k=1}^{\tilde{N}^{(i,n)}_{\xi_i(t)}} \frac{\tilde{U}_k^{(i)}}{\sqrt{n}}\right)\,. \end{aligned}$$

Let us now consider independent standard Poisson processes $\hat{N}^{(1,n)},\ldots,\hat{N}^{(d,n)}$ which are also independent of the environmental Markov process J. For $i \in E$ the process $\hat{N}^{(i,n)}$ have intensity $\lambda_i + nb^2$. Further, let $\big(\hat{U}_k^{(1,n)}\big)_{k\in\mathbb{N}},\ldots,\big(\hat{U}_k^{(d,n)}\big)_{k\in\mathbb{N}}$ be independent sequences of random variables which are also independent of the processes $\hat{N}^{(1,n)},\ldots,\hat{N}^{(d,n)}$ and J. For each $i \in E$ the random variables $(\hat{U}_k^{(i)})_{k\in\mathbb{N}}$ are assumed to be independent and identically distributed with distribution

$$\hat{B}_i^{(n)}(x) := \frac{\lambda_i}{\lambda_i + nb^2}\,B_i(x) + \frac{nb^2}{\lambda_i + nb^2}\,\tilde{B}_i(\sqrt{n}x)\,.$$

It then follows by general marked point process theory that

$$Y_t^{(n)}(u,K) \overset{\mathcal{D}}{=} u + \sum_{i\in E}\big(c + a\,k(i) + \sqrt{n}\,b^2\tilde{\mu}\big)\,\xi_i(t) - \sum_{i\in E}\sum_{k=1}^{\hat{N}^{(i,n)}_{\xi_i(t)}} \hat{U}_k^{(i,n)}\,,\ t\geq 0\,.$$

Thus, $Y^{(n)}(u, K)$ can be regarded as a risk reserve process from the Markov-modulated Poisson model without investment. However, note that the premium rate of the resulting model obviously depends on the environmental Markov process J. Therefore, we have to apply the time transformation

$$T(t) := \int_0^t \frac{c}{c + a\,k(J_s) + \sqrt{n}\,b^2\,\tilde{\mu}}\,ds$$

in order to obtain a model with constant premium rate c but where all other parameters are also changed accordingly, confer section 2.1.

3.2 Two applications

3.2.1 Approximating the adjustment coefficient

We can use the diffusion approximation obtained in the previous section in order to approximate the adjustment coefficient of the Markov-modulated Poisson model with respect to some fixed investment strategy. Recall from chapter 2 that the adjustment coefficient $R^{(K)}$ with respect to any fixed investment strategy $K = k(J)$ is given by

$$R^{(K)} = \sup\Big\{r > 0\,;\phi_{jj}^{(K)}(r) < 1\ \forall j \in E\Big\}$$

where

$$\phi_{jj}^{(K)}(r) = \mathbb{E}_j\Bigg(\exp\Bigg(\sum_{i\in E}\Big[\lambda_i h_i(r) + \frac{r^2b^2k(i)^2}{2} - r\big(c + ak(i)\big)\Big]\xi_i(\tau^{(j)})\Bigg)\Bigg).$$

As mentioned at the end of the previous section we can regard $Y^{(n)}(u, K)$ as the risk reserve process from a certain Markov-modulated Poisson model without investment. Applying an appropriate time transformation the adjustment coefficient of our approximating Markov-modulated Poisson model without investment is therefore given by $\hat{R}^{(K,n)} := \sup\big\{r > 0\,;\hat{\phi}_{jj}^{(K,n)}(r) < 1\ \forall j \in E\big\}$ where

$$\hat{\phi}_{jj}^{(K,n)}(r) := \mathbb{E}_j\Bigg(\exp\Bigg(\sum_{i\in E}\Big[\big(\lambda_i + nb^2\big)\hat{h}_i^{(n)}(r) - r\big(c + ak(i) + \sqrt{n}b^2\tilde{\mu}\big)\Big]\xi_i(\tau^{(j)})\Bigg)\Bigg).$$

At this, $\hat{h}_i^{(n)}$ denotes the centered moment generating function of the distribution $\hat{B}_i^{(n)}$ for $i \in E$. Thus,

$$\begin{aligned}\hat{h}_i^{(n)}(r) &= \int_0^\infty e^{rx}\,d\hat{B}_i^{(n)}(x) - 1\\ &= \frac{\lambda_i}{\lambda_i + nb^2}\left(\int_0^\infty e^{rx}\,dB_i(x) - 1\right) + \frac{nb^2}{\lambda_i + nb^2}\left(\int_0^\infty e^{r\frac{x}{\sqrt{n}}}\,d\tilde{B}_i(x) - 1\right)\\ &= \frac{\lambda_i}{\lambda_i + nb^2}\,h_i(r) + \frac{nb^2}{\lambda_i + nb^2}\left(\mathbb{E}\big(e^{r\frac{\tilde{U}_1^{(i)}}{\sqrt{n}}}\big) - 1\right).\end{aligned}$$

Using a Taylor series expansion we therefore obtain

$$
\begin{aligned}
&\big(\lambda_i + nb^2\big)\hat{h}_i^{(n)}(r) - r\big(c + ak(i) + \sqrt{n}b^2\tilde{\mu}\big) \\
&= \lambda_i h_i(r) + nb^2\Big(\mathbb{E}\big(e^{r\frac{\tilde{U}_1^{(i)}}{\sqrt{n}}}\big) - 1\Big) - r\big(c + ak(i) + \sqrt{n}b^2\tilde{\mu}\big) \\
&= \lambda_i h_i(r) + nb^2\Big(1 + \frac{r\,\tilde{\mu}}{\sqrt{n}} + \frac{r^2\,k(i)^2}{2n} + \mathcal{O}\big(n^{-\frac{3}{2}}\big) - 1\Big) - r\big(c + ak(i) + \sqrt{n}b^2\tilde{\mu}\big) \\
&= \lambda_i h_i(r) + \frac{r^2 b^2 k(i)^2}{2} - r\big(c + ak(i)\big) + \mathcal{O}\big(\frac{1}{\sqrt{n}}\big)\,,\ r \geq 0\,.
\end{aligned}
$$

It consequently follows that $\hat{\phi}_{jj}^{(K,n)}(r) \longrightarrow \phi_{jj}^{(K)}(r)$ for all $r \geq 0$ and hence $\hat{R}^{(K,n)} \longrightarrow R^{(K)}$ as $n \to \infty$. Instead of computing the adjustment coefficient for the Markov-modulated Poisson model with investment as described in the second chapter it is thus possible to approximate it by the obtained adjustment coefficient for the Markov-modulated Poisson model without investment sufficiently close to the limit.

3.2.2 Another comparison with the compound Poisson model

We consider the Markov-modulated Poisson model with investment and its associated compound Poisson model with investment once again in this section. Recall that the intensity of the claim arrival process and respectively the claim size distribution of the associated compound Poisson model are given by

$$\lambda^* = \sum_{i\in E} \pi_i\lambda_i \quad \text{and} \quad B^* = \sum_{i\in E} \frac{\pi_i\lambda_i}{\lambda^*}\, B_i\,.$$

It is shown in chapter 2 that a constant investment strategy is optimal in both models in the sense that it maximizes the corresponding adjustment coefficient.

Moreover, we have already shown that the optimal adjustment coefficient of the Markov-modulated Poisson model with investment is smaller or equal to the optimal adjustment coefficient of the associated compound Poisson model with investment.

Let us now compare the Markov-modulated Poisson model and its associated compound Poisson model when using the same constant investment strategy $\hat{K}$ in both models. We define $\hat{K}$ by $\hat{K}_t \equiv \hat{k}$ for $t \geq 0$ where $\hat{k}$ is any real constant. Recall that the wealth process $Y(u, \hat{K})$ from the Markov-modulated Poisson model is then given by

$$Y_t(u, \hat{K}) := u + (c + a\,\hat{k})\,t - \sum_{i \in E} \sum_{k=1}^{N^{(i)}_{\xi_i(t)}} U^{(i)}_k + b\,\hat{k}W_t\,,\; t \geq 0\,.$$

The wealth process $Y^*(u, \hat{K})$ from the associated compound Poisson model is defined by

$$Y^*_t(u, \hat{K}) := u + (c + a\,\hat{k})\,t - \sum_{k=1}^{N^*_t} U^*_k + b\,\hat{k}W_t\,,\; t \geq 0\,,$$

where the random variables $(U^*_k)_{k \in \mathbb{N}}$ are independent and identically distributed with distribution B^* and where the standard Poisson process N^* has intensity λ^*.

We know from what is shown in the second chapter of this work that the Markov-modulated Poisson model under investment strategy $\hat{K}$ has the adjustment coefficient $R^{(\hat{K})} := \sup\left\{r > 0\,; \phi^{(\hat{K})}_{jj}(r) < 1\ \forall j \in E\right\}$ where

$$\phi^{(\hat{K})}_{jj}(r) := \mathbb{E}_j\left(\exp\left(\sum_{i \in E} \lambda_i h_i(r) \xi_i(\tau^{(j)}) + \left[\frac{r^2 b^2 \hat{k}^2}{2} - r\big(c + a\hat{k}\big)\right]\tau^{(j)}\right)\right)\,.$$

As given at the beginning of the second chapter, the adjustment coefficient $R^{(\hat{K})^*}$ of the associated compound Poisson model under the same constant investment strategy $\hat{K}$ is the strictly positive solution of the equation

$$\lambda^* h^*(r) + \frac{r^2 b^2 \hat{k}^2}{2} - r\big(c + a\hat{k}\big) = \sum_{i \in E} \pi_i \, \lambda_i \, h_i(r) + \frac{r^2 b^2 \hat{k}^2}{2} - r\big(c + a\hat{k}\big) = 0\,.$$

Analogously to the proof of Theorem 2.23 it thus follows that $R^{(\hat{K})} \leq R^{(\hat{K})^*}$ provided that both adjustment coefficients exist.

In that what follows we directly compare the ruin probability $\Psi(u, \hat{K}) = \mathbb{P}\big(\inf_{t \geq 0} Y_t(u, \hat{K}) < 0\big)$ of the considered Markov-modulated Poisson model and the ruin probability $\Psi^*(u, \hat{K}) = \mathbb{P}\big(\inf_{t \geq 0} Y_t^*(u, \hat{K}) < 0\big)$ of the associated compound Poisson model for any given $u \geq 0$. However, in order to apply a result in Asmussen at al [AFR+95] we have to make the following additional assumptions.

Besides $\lambda_1 \leq \ldots \leq \lambda_d$ which can without loss of generality be assumed we need that $B_1 \leq_{st} \ldots \leq_{st} B_d$ where $\leq_{st}$ denotes the usual univariate stochastic order as for example defined in Müller and Stoyan [MS02], i.e. $B_i \leq_{st} B_j$ holds if

$$B_i(x) \geq B_j(x) \quad \text{for all} \quad x \in \mathbb{R}\,.$$

Further, the environmental Markov process J has to be monotone in the sense that

$$\sum_{k=l}^{d} q_{ik} \leq \sum_{k=l}^{d} q_{i+1,k}$$

for all $i = 1, \ldots, d-1$ and $l \neq i+1$ and its initial distribution has to be its stationary distribution π. Considering any $t \geq 0$ and $i, j \in E$ with $i < j$ the monotonicity of the Markov process J implies that J_t given $J_0 = i$ is smaller than J_t given $J_0 = j$ with respect to the usual univariate stochastic order, i.e. $\mathbb{P}_i\big(J_t \in \{1, \ldots, k\}\big) \geq \mathbb{P}_j\big(J_t \in \{1, \ldots, k\}\big)$ for all $k \in E$, confer

Theorem 5.2.19 in Müller and Stoyan [MS02]. Note that the monotonicity condition is automatically fulfilled if the environmental Markov process only has two states.

We then get the following result which compares the ruin probabilities of the Markov-modulated Poisson model and its associated compound Poisson model under the same constant investment strategy.

Theorem 3.3. *Let the Markov-modulated Poisson model and its associated compound Poisson model under the same constant investment strategy $\hat{K}$ be given. Further assume that*

$$
\begin{aligned}
&(i) && \lambda_1 \le \ldots \le \lambda_d\,; \\
&(ii) && B_1 \le_{st} \ldots \le_{st} B_d\,; \\
&(iii) && J \text{ is monotone and has stationary initial distribution } \pi\,.
\end{aligned}
$$

Denoting the ruin probability of the Markov-modulated Poisson model by $\Psi(u, \hat{K})$ and the ruin probability of the associated compound Poisson model by $\Psi^*(u, \hat{K})$ it follows that

$$\Psi^*(u, \hat{K}) \le \Psi(u, \hat{K}) \quad \textit{for all} \quad u \ge 0\,.$$

Proof:

Let us fix any $u \ge 0$ and consider the investment strategy $\hat{K}$ defined by $\hat{K}_t \equiv \hat{k}$ for $t \ge 0$. If $\hat{k} = 0$ we are in the case without investment and the assertion follows directly from Theorem 1.1 in Asmussen et al [AFR$^+$95]. Hence, let us suppose that $\hat{k} \neq 0$.

It is shown at the end of section 3.1 that the ruin probability $\Psi(u, \hat{K})$ of the Markov-modulated Poisson model under the investment strategy $\hat{K}$ can be approximated by the ruin probability of an adequate Markov-modulated Poisson model without investment. Before we define the wealth process of the latter model let us initially consider its claim size distributions. Recall from section 3.1 that in the approximating model without investment a claim which occurs when the environmental Markov process is in state $i \in E$ has distribution

$$\hat{B}_i^{(n)}(x) := \frac{\lambda_i}{\lambda_i + nb^2}\, B_i(x) + \frac{nb^2}{\lambda_i + nb^2}\, \tilde{B}_i(\sqrt{n}x)\,. \tag{3.8}$$

However, since we consider the constant investment strategy $\hat{K}$ we can this time suppose that $B_i = \tilde{B}$ for every $i \in E$ where $\tilde{B}$ is some distribution concentrated on $(0, \infty)$ with second moment $\hat{k}^2$. Note that the expectation of the distribution $\tilde{B}$, denoted by $\tilde{\mu}$, is arbitrary but certainly has to be considered in the definition of the wealth process of the approximating model. For this proof we moreover have to choose $\tilde{B}$ such that there exists a sufficiently large $n \in \mathbb{N}$ with $\tilde{B}(\sqrt{n}x) \geq B_i(x)$ for all $x \in \mathbb{R}_+$ and every $i \in E$.

In that what follows we show that such a choice of $\tilde{B}$ is possible. Recall that all random elements in this work are defined on the same probability space. The random variable $\tilde{U}' := \min\left\{U_1^{(1)}, \ldots, U_1^{(d)}\right\}$ is therefore well defined and we denote its expectation by $\tilde{\mu}'$ and its second moment by $\tilde{\mu}^{(2)'}$. Note, that the distribution, say $\tilde{B}'$, of this random variable $\tilde{U}'$ is by definition concentrated on $(0, \infty)$. Consequently, $\alpha := \frac{\hat{k}^2}{\tilde{\mu}^{(2)'}}$ is a strictly positive finite constant.

We now define $\tilde{B}$ as the distribution of the random variable $\tilde{U} := \sqrt{\alpha}\,\tilde{U}'$. It hence follows that also the distribution $\tilde{B}$ is concentrated on $(0, \infty)$ and that it has second moment $\mathbb{E}\big(\tilde{U}^2\big) = \alpha\,\mathbb{E}\big((\tilde{U}')^2\big) = \frac{\hat{k}^2}{\tilde{\mu}^{(2)'}}\,\tilde{\mu}^{(2)'} = \hat{k}^2$. Furthermore, we have

$$\frac{\tilde{U}}{\sqrt{n}} = \frac{\sqrt{\alpha}}{\sqrt{n}}\,\tilde{U}' = \frac{\sqrt{\alpha}}{\sqrt{n}}\,\min\left\{U_1^{(1)}, \ldots, U_1^{(d)}\right\} \leq \min\left\{U_1^{(1)}, \ldots, U_1^{(d)}\right\}$$

for all $n \in \mathbb{N}$ with $n \geq \alpha$. For these integers n we thus obtain $\tilde{B}(\sqrt{n}x) \geq B_i(x)$

for all $x \in \mathbb{R}_+$ and every $i \in E$. Eventually note that the expectation of the distribution $\tilde{B}$ is given by $\tilde{\mu} := \mathbb{E}\big(\tilde{U}\big) = \sqrt{\alpha}\,\tilde{\mu}'$.

Using the same notation as at the end of section 3.1, the ruin probability $\Psi(u, \hat{K})$ can be approximated by $\Psi^{(n)}(u, \hat{K}) = \mathbb{P}\big(\inf_{t\geq 0} Y_t^{(n)}(u, \hat{K}) < 0\big)$ with $n \to \infty$ where

$$Y_t^{(n)}(u, \hat{K}) = u + \big(c + a\,\hat{k} + \sqrt{n}\,b^2\tilde{\mu}\big)\,t - \sum_{i\in E} \sum_{k=1}^{\hat{N}^{(i,n)}_{\xi_i(t)}} \hat{U}_k^{(i,n)}\,,\; t \geq 0\,.$$

At this, the random variables $\big(\hat{U}_k^{(i,n)}\big)_{k\in\mathbb{N}}$ are independent and identically distributed with distribution $\hat{B}_i^{(n)}(x)$ as defined above and the standard Poisson process $\hat{N}^{(i,n)}$ has intensity $\lambda_i + nb^2$ for $i \in E$.

Later in this proof we want to use a result for the Markov-modulated Poisson model without investment and with constant premium rate one. In order to obtain such a premium rate we thus have to apply the time transformation given by $T(t) := \frac{t}{c+a\hat{k}+\sqrt{n}\,b^2\tilde{\mu}}$. As described in section 2.1 we therefore consider a Markov-modulated Poisson model without investment whose wealth process $\breve{Y}^{(n)}(u, \hat{K})$ is given by

$$\breve{Y}_t^{(n)}(u, \hat{K}) := Y_{T(t)}^{(n)}(u, \hat{K}) \overset{\mathcal{D}}{=} u + t - \sum_{i\in E} \sum_{k=1}^{\breve{N}^{(i,n)}_{\breve{\xi}_i(t)}} \hat{U}_k^{(i,n)}\,,\; t \geq 0\,.$$

At this, $\breve{N}^{(i,n)}$ is a standard Poisson process with intensity $\frac{\lambda_i + nb^2}{c+a\hat{k}+\sqrt{n}\,b^2\tilde{\mu}}$ for $i \in E$. The environmental Markov process after time transformation has intensity matrix $\big(\frac{q_{ij}}{c+a\hat{k}+\sqrt{n}\,b^2\tilde{\mu}}\big)_{i,j\in E}$ and thus still stationary initial distribution π. Furthermore, $\breve{\xi}_i(t) = \big(c + a\hat{k} + \sqrt{n}\,b^2\tilde{\mu}\big)\,\xi_i(t)$ for all $t \geq 0$ and $i \in E$. Recall that this time transformation clearly does not effect the ruin probability which means that $\Psi^{(n)}(u, \hat{K}) = \mathbb{P}\big(\inf_{t\geq 0} Y_t^{(n)}(u, \hat{K}) < 0\big) = \mathbb{P}\big(\inf_{t\geq 0} \breve{Y}_t^{(n)}(u, \hat{K}) < 0\big)$.

Analogously, the ruin probability of the associated compound Poisson model $\Psi^*(u, \hat{K})$ under the same investment strategy $\hat{K}$ can be approximated by the ruin probabilities $\Psi^{(n)^*}(u, \hat{K}) = \mathbb{P}\big(\inf_{t\geq 0} \breve{Y}_t^{(n)^*}(u, \hat{K}) < 0\big)$ for $n \in \mathbb{N}$ where the process $\breve{Y}^{(n)^*}(u, \hat{K})$ is defined by

$$\breve{Y}_t^{(n)^*}(u, \hat{K}) = u + t - \sum_{k=1}^{\breve{N}_t^{(n)^*}} \hat{U}_k^{(n)^*}\,,\ t \geq 0\,.$$

Here, $\breve{N}^{(n)^*}$ is a standard Poisson process with intensity $\frac{\lambda^*+nb^2}{c+a\hat{k}+\sqrt{n}\,b^2\tilde{\mu}}$ and the independent and identically distributed random variables $\big(\hat{U}_k^{(n)^*}\big)_{k\in\mathbb{N}}$ have distribution

$$\hat{B}^{(n)^*}(x) := \frac{\lambda^*}{\lambda^*+nb^2}\, B^*(x) + \frac{nb^2}{\lambda^*+nb^2}\, \tilde{B}(\sqrt{n}x)\,.$$

We have already mentioned that $\breve{Y}^{(n)}(u, \hat{K})$ can be regarded as the risk reserve process of a Markov-modulated Poisson model without investment. It is easy to verify that $\breve{Y}^{(n)^*}(u, \hat{K})$ is the risk reserve process of the associated compound Poisson model without investment.

Let us now consider any $i, j \in E$ with $i < j$ and choose $n \in \mathbb{N}$ sufficiently large such that $\tilde{B}(\sqrt{n}x) \geq B_j(x)$ for all $x \in \mathbb{R}_+$. We want to show that $\hat{B}_i^{(n)} \leq_{st} \hat{B}_j^{(n)}$, i.e. that

$$\begin{aligned}\hat{B}_i^{(n)}(x) &= \frac{\lambda_i}{\lambda_i+nb^2}\, B_i(x) + \frac{nb^2}{\lambda_i+nb^2}\, \tilde{B}(\sqrt{n}x)\\ &\geq \frac{\lambda_j}{\lambda_j+nb^2}\, B_j(x) + \frac{nb^2}{\lambda_j+nb^2}\, \tilde{B}(\sqrt{n}x) = \hat{B}_j^{(n)}(x)\end{aligned}$$

for all $x \in \mathbb{R}_+$. If $B_j(x) = 0$ the inequality is obviously fulfilled since $\lambda_i \leq \lambda_j$. Hence, consider $x \in \mathbb{R}_+$ with $B_j(x) > 0$. Note that $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{\lambda_i+x}{\lambda_j+x}$ is an increasing function for $\lambda_i \leq \lambda_j$. Since we have $\tilde{B}(\sqrt{n}x) \geq B_j(x)$ and $B_i(x) \geq B_j(x)$ from our assumptions it thus follows that

$$\frac{\lambda_i\, B_i(x) + nb^2 \tilde{B}(\sqrt{n}x)}{\lambda_j\, B_j(x) + nb^2 \tilde{B}(\sqrt{n}x)} = \frac{\lambda_i \frac{B_i(x)}{B_j(x)} + nb^2 \frac{\tilde{B}(\sqrt{n}x)}{B_j(x)}}{\lambda_j + nb^2 \frac{\tilde{B}(\sqrt{n}x)}{B_j(x)}} \geq \frac{\lambda_i + nb^2\, \frac{\tilde{B}(\sqrt{n}x)}{B_j(x)}}{\lambda_j + nb^2\, \frac{\tilde{B}(\sqrt{n}x)}{B_j(x)}} \geq \frac{\lambda_i + nb^2}{\lambda_j + nb^2}$$

which is equivalent to $\hat{B}_i^{(n)}(x) \geq \hat{B}_j^{(n)}(x)$.

Further, it follows from our assumptions that $\frac{\lambda_1+nb^2}{c+a\hat{k}+\sqrt{n}\,b^2\tilde{\mu}} \leq \ldots \leq \frac{\lambda_d+nb^2}{c+a\hat{k}+\sqrt{n}\,b^2\tilde{\mu}}$. Since the environmental Markov process of the time transformed model is still monotone with stationary initial distribution π we can thus apply Theorem 1.1 in Asmussen et al [AFR+95]. This yields

$$\Psi^{(n)^*}(u,\hat{K}) \leq \Psi^{(n)}(u,\hat{K})\,.$$

for all $u \geq 0$ and sufficiently large $n \in \mathbb{N}$. Letting $n \to \infty$ it thus follows from Theorem 3.2 that $\Psi^*(u,\hat{K}) \leq \Psi(u,\hat{K})$ holds for all $u \geq 0$.

$\square$

Chapter 4

The periodic Poisson model with investment

In this chapter we consider the risk reserve process of an insurance company in a deterministic periodic environment. As before, the insurer has the opportunity to invest into a stock index whose price process is modelled by a geometric Brownian motion. Initially, the invested amount only depends on the current state of the environment. Later in this chapter also a broader class of investment strategies is permitted. The claims again have exponential moments.

The outline of this chapter is similar to that of the second chapter. After introducing the actual model the adjustment coefficient with respect to any fixed investment strategy is determined in section 4.2. In the following section this adjustment coefficient is maximized with respect to the investment strategy. We then prove the optimality of the obtained investment strategy in section 4.4. At this, we do not restrict ourselves to investment strategies which only depend on the environment. Finally, the periodic Poisson model and its associated compound Poisson model are compared under optimal investment.

4.1 The model

As in the Markov-modulated Poisson model the premium rate and the claim arrivals in the periodic Poisson model are inhomogeneous in time. However, instead of an stochastic environment as in the previous chapters we this time consider the following deterministic, periodic environment.

The premium rate at time $t \geq 0$ is given by $c_t := c(t)$ where $c : [0, \infty) \to \mathbb{R}_+$ is a bounded and periodic function. We denote the period of this function c by $T > 0$. This means that $c(t) = c(t+T)$ for all $t \geq 0$. Further let $\lambda : [0, \infty) \to \mathbb{R}_+$ also be a bounded and periodic function with the same period $T > 0$. The claim arrival process $N := \{N_t, t \geq 0\}$ is then assumed to be a Poisson process with intensity process $\{\lambda_t, t \geq 0\}$ where we put $\lambda_t := \lambda(t)$. Furthermore, a claim occurring at time $t \geq 0$ have some distribution B_t concentrated on $(0, \infty)$. In the periodic Poisson model it is assumed that also the claim size distribution periodically depends on the time parameter t with period T in the sense that the distributions B_t and B_{t+T} coincide for all $t \geq 0$. As a minimum requirement we further have to assume that λ_t and B_t are measurable functions in t. The corresponding risk reserve process $R(u) := \{R_t(u), t \geq 0\}$ is then given by

$$R_t(u) = u + \int_0^t c_s \, ds - \sum_{k=1}^{N_t} U_k \tag{4.1}$$

where as before $u \geq 0$ is the initial reserve of the insurance company.

Again, let the insurer have the opportunity to invest into a stock index or say some portfolio. The price process $S := \{S_t, t \geq 0\}$ of this portfolio is modelled in the same way as in chapter 2 by a geometric Brownian motion with dynamics

$$dS_t = S_t \left(a \, dt + b \, dW_t\right), \; t \geq 0 \, .$$

Apart from section 4.4 it is assumed throughout this chapter that the invested amount at time $t \geq 0$ only depends on the current state of the environment. This means that the investment strategy $K := \{K_t, t \geq 0\}$ is determined by some periodic function $k : [0, \infty) \to \mathbb{R}$ with period $T > 0$ such that $K_t = k(t)$. At this, we furthermore suppose that the integral $\int_0^T K_s^2 \, ds$ is finite. Note that this is a necessary and sufficient for the stochastic integral $\int_0^t K_s \, dW_s$ to exist for all $t \geq 0$. In that what follows we denote the class of such investment strategies by $\mathcal{K}$.

As in chapter 2 we can assume without loss of generality that the premium rate is constant over time, i.e. $c_t = c$ for all $t \geq 0$ and some $c > 0$. However, this time the appropriate time transformation is given by $T(t) := \int_0^t \frac{c}{c_s} \, ds$. Also in this model we certainly have to take into account that the parameters of the model change accordingly. In that what follows we consequently consider the wealth process $Y(u, K)$ given by

$$Y_t(u, K) = u + ct + a \int_0^t K_s \, ds + b \int_0^t K_s \, dW_s - \sum_{k=1}^{N_t} U_k \, , \, t \geq 0 \, . \tag{4.2}$$

As before, let the process $X(K)$ by defined by $X_t(K) = Y_t(u, K) - u$ for $t \geq 0$ and let $\mathcal{F}^Y := \{\mathcal{F}_t, t \geq 0\}$ be the natural filtration of the wealth process $Y(u, K)$.

For the present model we also suppose that the claims have exponential moments. This means that for every $t \in [0, T)$ there exists a possibly infinite constant $r_\infty^{(t)} \in (0, \infty]$ such that the centered moment generating function h_t defined by

$$h_t(r) := \int_0^\infty e^{rx} \, dB_t(x) \; - 1 \, , \, r \geq 0 \, ,$$

is finite for every $r < r_\infty^{(t)}$ with $h_t(r) \to \infty$ as $r \to r_\infty^{(t)}$. As already mentioned this assumption implies that h_t is increasing, convex and continuous on $[0, r_\infty^{(t)})$ with $h_t(0) = 0$ for any fixed $t \in [0, T)$.

Denoting the ruin probability by $\Psi(u,K)$ and the time of ruin by $\tau(u,K)$ the aim of this chapter is the same as of the second chapter. We want to find the optimal investment strategy in the sense that it maximizes the corresponding adjustment coefficient R. Recall that R is defined as the largest possible value such that the Lundberg inequality $\Psi(u,K) \leq C\, e^{-Ru}$ with $C < \infty$ is fulfilled for all $u \geq 0$. Note that the compound Poisson model fits into the actual framework by choosing the same claim size distribution B_t and putting $\lambda_t = \lambda$ for all $t \in [0,T)$.

Let us now consider what has been shown for the periodic Poisson model without investment. As for example given in Asmussen [Asm00], the adjustment coefficient for this model is given as the strictly positive solution of the equation $\lambda^* h^*(r) = cr$. At this, h^* is the centered moment generating function of the distribution B^* where

$$B^* := \frac{1}{T}\int_0^T \frac{\lambda_t}{\lambda^*}\, B_t\, dt \quad \text{with } \lambda^* := \frac{1}{T}\int_0^T \lambda_t\, dt\,.$$

Note, if we associate a classical compound Poisson model to the periodic Poisson model by averaging over the environment the corresponding Poisson process has intensity λ^* and the corresponding claim size distribution is given by B^*, confer section 4.5. It thus follows that without investment the adjustment coefficients of the periodic Poisson model and its associated compound Poisson model coincide.

We have already mentioned that an adjustment coefficient of the classical compound Poisson model without investment exists if and only if the corresponding absolute safety loading is strictly positive. Consequently, an adjustment coefficient of the periodic Poisson model exists if and only if

$$c - \lambda^* \mu_{B^*} = c - \lambda^* \int_0^\infty x\, dB^*(x) > 0\,.$$

Analogously to the Markov-modulated Poisson model with investment we define the absolute safety loading of the periodic Poisson model with respect to some given investment strategy K as the constant $\rho^{(K)}$ for which

$$\lim_{t\to\infty} \frac{1}{t} Y_t(0,K) \overset{a.s.}{=} \rho^{(K)} .$$

As before, we refer to $\rho^{(K)}$ as the safety loading with respect to K unless otherwise stated. It is later shown that $\rho^{(K)} > 0$ is a necessary and sufficient condition for an adjustment coefficient to exist when using investment strategy K.

4.2 The adjustment coefficient for any fixed investment strategy

Throughout this section we consider any fixed investment strategy $K \in \mathcal{K}$. As in the second chapter of this work we use a martingale method in order to obtain a Lundberg bound for the ruin probability $\Psi(u, K)$. However, this time our exponential martingale slightly differs from the martingale used for the Markov-modulated Poisson model. Besides some obvious changes we consider an exponential martingale process which is stopped at the time of ruin $\tau(u, K)$.

Proposition 4.1. *Consider any investment strategy $K \in \mathcal{K}$ and let $u, r \geq 0$ be fixed. Define the process $M(u, K, r)$ by*

$$M_t(u, K, r) := \frac{\exp\Big(-rY_t(u, K)\Big)}{\exp\Big(\int_0^t \lambda_s h_s(r) + \frac{1}{2}r^2 b^2 K_s^2 - r(c + aK_s)\,ds\Big)}\,,\; t \geq 0\,.$$

The stopped process $\tilde{M}(u, K, r)$ given by

$$\tilde{M}_t(u, K, r) := M_{t \wedge \tau(u,K)}(u, K, r)\,,\; t \geq 0\,,$$

is then a martingale with respect to $\mathcal{F}^Y$.

Proof:

For simplicity reasons put $\tau := \tau(u, K)$. Note that $\sum_{k=1}^{N_t} U_k = \int_0^t U_{N_s}\,dN_s$ for all $t \geq 0$. It thus follows that $M_t(u, K, r) = \exp(V_t)$ for $t \geq 0$ where the process $\{V_t, t \geq 0\}$ has dynamics

$$dV_t = -\big(\lambda_t h_t(r) + \frac{1}{2}r^2 b^2 K_t^2\big)\,dt - rbK_t\,dW_t + rU_{N_t}\,dN_t\,.$$

Itô's Formula as given in Protter [Pro04], page 78, then yields

$$\begin{aligned}
&M_t(u,K,r)\\
&\quad= M_0(u,K,r) + \int_{0^+}^{t} M_{s^-}(u,K,r)\,dV_s + \frac{1}{2}\int_{0^+}^{t} M_{s^-}(u,K,r)\,r^2b^2K_s^2\,ds\\
&\qquad+ \sum_{0<s\le t}\Big(M_s(u,K,r) - M_{s^-}(u,K,r) - (V_s - V_{s^-})\,M_{s^-}(u,K,r)\Big)\\
&\quad= M_0(u,K,r) - \int_{0^+}^{t} M_{s^-}(u,K,r)\,\big(\lambda_s h_s(r) + \frac{1}{2}r^2b^2K_s^2\big)\,ds\\
&\qquad- \int_{0^+}^{t} M_{s^-}(u,K,r)\,rbK_s\,dW_s + \int_{0^+}^{t} M_{s^-}(u,K,r)\,rU_{N_s}\,dN_s\\
&\qquad+ \frac{1}{2}\int_{0^+}^{t} M_{s^-}(u,K,r)\,r^2b^2K_s^2\,ds\\
&\qquad+ \int_{0^+}^{t}\Big(M_s(u,K,r) - M_{s^-}(u,K,r) - rU_{N_s}\,M_{s^-}(u,K,r)\Big)\,dN_s\\
&\quad= M_0(u,K,r) - \int_{0^+}^{t} rbK_s\,M_{s^-}(u,K,r)\,dW_s\\
&\qquad+ \int_{0^+}^{t}\Big(M_s(u,K,r) - M_{s^-}(u,K,r)\Big)\,dN_s - \int_{0^+}^{t}\lambda_s h_s(r)\,M_{s^-}(u,K,r)\,ds\,.
\end{aligned} \tag{4.3}$$

Recall, we want to show that $\tilde{M}(u,K,r)$ is a martingale with respect to $\mathcal{F}^Y$. Since $K \in \mathcal{K}$ and $0 \le Y_{t^-}(u,K) \le 1$ for all $t \le \tau$ it follows by the definition of the Itô integral that the process

$$\left\{\int_{0^+}^{t\wedge\tau} rbK_s\,M_{s^-}(u,K,r)\,dW_s,\, t \ge 0\right\}$$

is an $\mathcal{F}^Y$-martingale. In order to complete the proof it thus suffices to show that also the process

$$\left\{\int_{0^+}^{t\wedge\tau}\Big(M_s(u,K,r) - M_{s^-}(u,K,r)\Big)\,dN_s - \int_{0^+}^{t\wedge\tau}\lambda_s h_s(r)\,M_{s^-}(u,K,r)\,ds,\, t \ge 0\right\}$$

is an $\mathcal{F}^Y$-martingale. Recall, the claim arrival process N has intensity $\{\lambda_t, t \ge 0\}$ with respect to $\mathcal{F}^Y$. Further, $\{M_{t^-}(u,K,r)h_t(r), t \ge 0\}$ is an $\mathcal{F}^Y$-predictable process with

$$\mathbb{E}\left(\int_{0+}^{t\wedge\tau} \Big|M_{s^-}(u,K,r)h_s(r)\Big|\lambda_s\,ds\right) < \infty$$

for all $t \geq 0$ since $0 \leq Y_{t^-}(u,K) \leq 1$ for $t \leq \tau$. It therefore follows from Theorem T8 in Brémaud [Bré81], page 27, that the process

$$\left\{\int_{0+}^{t\wedge\tau} M_{s^-}(u,K,r)h_s(r)\,dN_s - \int_{0+}^{t\wedge\tau} M_{s^-}(u,K,r)h_s(r)\lambda_s\,ds, t \geq 0\right\}$$

is a martingale with respect to $\mathcal{F}^Y$. Using this fact in the last equality below we conclude that

$$\begin{aligned}
&\mathbb{E}\left[\int_{(v\wedge\tau)+}^{t\wedge\tau} \Big(M_s(u,K,r) - M_{s^-}(u,K,r)\Big)\,dN_s\Big|\mathcal{F}^Y_{v\wedge\tau}\right]\\
&= \mathbb{E}\left[\int_{(v\wedge\tau)+}^{t\wedge\tau} M_{s^-}(u,K,r)\left(e^{rU^{(s)}_{N_s}}-1\right)dN_s\Big|\mathcal{F}^Y_{v\wedge\tau}\right]\\
&= \mathbb{E}\left[\sum_{k=1}^{\infty} M_{\nu_k^-}(u,K,r)\left(e^{rU^{(\nu_k)}_{N_{\nu_k}}}-1\right)\delta_{(v\wedge\tau,t\wedge\tau]}(\nu_k)\Big|\mathcal{F}^Y_{v\wedge\tau}\right]\\
&= \sum_{k=1}^{\infty}\mathbb{E}\left[\mathbb{E}\left[M_{\nu_k^-}(u,K,r)\left(e^{rU^{(\nu_k)}_k}-1\right)\delta_{(v\wedge\tau,t\wedge\tau]}(\nu_k)\Big|\mathcal{F}^Y_{\nu_k^-}\right]\Big|\mathcal{F}^Y_{v\wedge\tau}\right]\\
&= \sum_{k=1}^{\infty}\mathbb{E}\left[M_{\nu_k^-}(u,K,r)\,\mathbb{E}\left[e^{rU^{(\nu_k)}_k}-1\Big|\mathcal{F}^Y_{\nu_k^-}\right]\delta_{(v\wedge\tau,t\wedge\tau]}(\nu_k)\Big|\mathcal{F}^Y_{v\wedge\tau}\right]\\
&= \mathbb{E}\left[\sum_{k=1}^{\infty} M_{\nu_k^-}(u,K,r)\,h_{\nu_k}(r)\,\delta_{(v\wedge\tau,t\wedge\tau]}(\nu_k)\Big|\mathcal{F}^Y_{v\wedge\tau}\right]\\
&= \mathbb{E}\left[\int_{(v\wedge\tau)+}^{t\wedge\tau} M_{s^-}(u,K,r)\,h_s(r)\,dN_s\Big|\mathcal{F}^Y_{v\wedge\tau}\right]\\
&= \mathbb{E}\left[\int_{(v\wedge\tau)+}^{t\wedge\tau} M_{s^-}(u,K,r)\,h_s(r)\,\lambda_s\,ds\Big|\mathcal{F}^Y_{v\wedge\tau}\right]
\end{aligned}$$

for $v \leq t$ where ν_k denotes the k^{th} jump epoch of the claim arrival process N for $k \in \mathbb{N}$. Using the representation of $M_t(u,K,r)$ given in (4.3), the integrability of the stopped process $\tilde{M}(u,K,r)$ can easily be shown. This finally completes the proof since $\tilde{M}(u,K,r)$ is obviously $\mathcal{F}^Y$-measurable.

□

Using the exponential martingale from Proposition 4.1 we can now determine an upper bound for the ruin probability $\Psi(u, K)$ in the same way as done in chapter 2.

Proposition 4.2. *Consider any investment strategy $K \in \mathcal{K}$. For $r > 0$ we then have*

$$\Psi(u, K) \leq C(K, r)\, e^{-ru}$$

for all $u \geq 0$ where

$$C(K, r) := \sup_{t \geq 0} \exp\left(\int_0^t \lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s)\, ds \right).$$

Proof:

Let us again denote the time of ruin by $\tau := \tau(u, K)$. We have already shown that the process $\tilde{M}(u, K, r)$ is a martingale with respect to the filtration $\mathcal{F}^Y$. Analogously to the respective result in chapter 2, it therefore follows for $r > 0$ and $u \geq 0$ that

$$\begin{aligned}
e^{-ru} &= \tilde{M}_0(u, K, r) = \mathbb{E}\Big(\tilde{M}_t(u, K, r)\Big) \\
&= \mathbb{E}\Big(\tilde{M}_t(u, K, r)\, I(\tau \leq t)\Big) + \mathbb{E}\Big(\tilde{M}_t(u, K, r)\, I(\tau > t)\Big) \\
&\geq \mathbb{E}\Big[M_\tau(u, K, r)\Big|\tau \leq t\Big]\ \mathbb{P}(\tau \leq t) \\
&\geq \frac{\mathbb{P}(\tau \leq t)}{\sup_{0 \leq v \leq t} \exp\left(\int_0^v \lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s)\, ds\right)}
\end{aligned}$$

and hence

$$\mathbb{P}(\tau \leq t) \leq e^{-ru} \sup_{0 \leq v \leq t} \exp\left(\int_0^v \lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s)\, ds \right)$$

Letting $t \to \infty$ we consequently obtain

$$\Psi(u, K) \leq e^{-ru} \sup_{v \geq 0} \exp\left(\int_0^v \lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s)\, ds \right).$$

□

Recall that the period of the periodic environment is denoted by T. It is thus obvious that $C(K, r)$ is finite if and only if

$$\exp\left(\int_0^T \lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s)\, ds \right) \leq 1\,.$$

For a given investment strategy $K \in \mathcal{K}$ let us therefore define $R^{(K)}$ as

$$\begin{aligned} R^{(K)} &:= \sup\left\{ r > 0\,; \exp\left(\int_0^T \lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s)\, ds \right) \leq 1 \right\} \\ &= \sup\left\{ r > 0\,; \int_0^T \lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s)\, ds \leq 0 \right\} \qquad (4.4) \end{aligned}$$

It now follows from our assumptions on the functions h_t, $t \in [0, T)$, that $R^{(K)}$ is the strictly positive solution of the equation

$$\int_0^T \lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s)\, ds = 0\,. \qquad (4.5)$$

We consequently say that $R^{(K)}$ does not exist if equation (4.5) does not have a strictly positive solution. If $R^{(K)}$ exists it is uniquely determined.

Using $R^{(K)}$ we can now give a Lundberg bound for the periodic Poisson model with respect to some fixed investment strategy $K \in \mathcal{K}$.

Theorem 4.3. *Let $K \in \mathcal{K}$ and assume that a strictly positive solution $R^{(K)}$ of equation (4.5) exists. For any $r \leq R^{(K)}$ we then have*

$$\Psi(u, K) \leq C(K, r)\, e^{-ru}$$

with

$$C(K,r) = \sup_{0 \le t < T} \exp\left(\int_0^t \lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s)\, ds \right) < \infty$$

for all $u \ge 0$.

Proof:

Recall that the inequality of interest is trivial for $r \le 0$. As described above, $C(K,r)$ is finite if $r \le R^{(K)}$. Noting that $\int_0^t \lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s)\, ds$ has its supremum on the interval $[0,T)$ if $r \le R^{(K)}$ the assertion follows.

$\square$

Note that we are interested in the investment strategy $K \in \mathcal{K}$ which minimizes the Lundberg bound for the ruin probability $\Psi(u,K)$. Hence we do not investigate here if there exists some constant $C < \infty$ such that the Lundberg inequality given in Theorem 4.3 also holds for some $r > R^{(K)}$ and C instead of $C(K,r)$. After maximizing $R^{(K)}$ with respect to $K \in \mathcal{K}$ in the following section we then verify in section 4.4 that the resulting R is indeed the optimal adjustment coefficient of the periodic Poisson model with investment. Nevertheless, we refer to $R^{(K)}$ as the adjustment coefficient with respect to some fixed investment strategy $K \in \mathcal{K}$ in this work.

As for the Markov-modulated Poisson model we conclude this section with the study of conditions which ensure that $R^{(K)}$ exists for a given investment strategy $K \in \mathcal{K}$. However, this time it is less complicated since we have a deterministic environment. Recall from section 4.1 that the safety loading with respect to some investment strategy $K \in \mathcal{K}$ is defined as the constant $\rho^{(K)}$ for which

$$\lim_{t \to \infty} \frac{Y_t(u,K) - u}{t} \overset{a.s.}{=} \rho^{(K)} . \tag{4.6}$$

We thus get the following representation of the safety loading for the periodic Poisson model with investment.

Proposition 4.4. *Consider any fixed investment strategy $K \in \mathcal{K}$ and let the corresponding safety loading $\rho^{(K)}$ be defined by (4.6). Then,*

$$\rho^{(K)} = c + \frac{a}{T}\int_0^T K_s\,ds - \frac{1}{T}\int_0^T \lambda_s \mu_{B_s}\,ds\,.$$

Proof:

We have

$$Y_t(u,K) - u = X_t(K) = ct + a\int_0^t K_s\,ds + b\int_0^t K_s\,dW_s - \sum_{k=1}^{N_t} U_k\,, t \geq 0\,.$$

Noting that $\lim_{t\to\infty} \frac{1}{t}\big(X_t(K) - X_{\lfloor \frac{t}{T}\rfloor T}(K)\big) \stackrel{a.s.}{=} 0$ we almost surely have

$$\lim_{t\to\infty}\frac{1}{t}\,X_t(K) = \lim_{t\to\infty}\frac{1}{t}\,X_{\lfloor \frac{t}{T}\rfloor T}(K) = \lim_{t\to\infty}\frac{\lfloor \frac{t}{T}\rfloor T}{t}\,\frac{1}{\lfloor \frac{t}{T}\rfloor T}\,X_{\lfloor \frac{t}{T}\rfloor T}(K) = \frac{1}{T}\,\mathbb{E}\Big(X_T(K)\Big)$$

by the law of large numbers since $\big(X_{nT}(K)\big)_{n\in\mathbb{N}_0}$ is a random walk. We thus have to determine $\mathbb{E}\big(X_T(K)\big)$. At this,

$$\mathbb{E}\Big(\int_0^T K_s\,dW_s\Big) = 0$$

and

$$\mathbb{E}\Big(\sum_{k=1}^{N_T} U_k\Big) = \int_0^T \lambda_s\,\mu_{B_s}\,ds$$

where the latter is well known and can for example be found in section 12.4 of Rolski et al [RSS+99]. It therefore follows that

$$\begin{aligned}\rho^{(K)} &= \frac{1}{T}\,\mathbb{E}\Big(X_T(K)\Big) = \frac{1}{T}\Big(cT + a\int_0^T K_s\,ds + \int_0^T \lambda_s\,\mu_{B_s}\,ds\Big)\\ &= c + \frac{a}{T}\int_0^T K_s\,ds - \frac{1}{T}\int_0^T \lambda_s\,\mu_{B_s}\,ds\,.\end{aligned}$$

□

Analogously to the Poisson models we have considered so far the following result follows.

Proposition 4.5. *Consider any fixed investment strategy $K \in \mathcal{K}$ and suppose that the corresponding safety loading $\rho^{(K)} \leq 0$. Then,*

$$\Psi(u, K) = 1$$

for all $u \geq 0$.

Proof:

Let $K \in \mathcal{K}$. Recall from the proof of Proposition 4.4 that $\big(X_{nT}(K)\big)_{n\in\mathbb{N}_0}$ is a random walk with

$$\mathbb{E}\Big(X_T(K)\Big) = T\rho^{(K)}\,.$$

If $\rho^{(K)} < 0$ it follows that $X_{nT}(K)$ converges to $-\infty$ as $n \to \infty$. According to Theorem 4.2 in Asmussen [Asm03], page 224, the random walk $\big(X_{nT}\big)_{n\in\mathbb{N}_0}$ oscillates between ∞ and $-\infty$ if $\rho^{(K)} = 0$. In both cases we therefore have $\inf_{t\geq 0} Y_t(u, K) = -\infty$ almost surely and consequently $\psi(u, K) = 1$ for all $u \geq 0$.

□

This means that we cannot find an adjustment coefficient of the periodic Poisson model under some fixed investment strategy $K \in \mathcal{K}$ if $\rho^{(K)} \leq 0$. On the other hand it can be shown that the adjustment coefficient $R^{(K)}$ exist if $\rho^{(K)} > 0$.

Proposition 4.6. *Consider any fixed investment strategy $K \in \mathcal{K}$. Then, $R^{(K)}$ defined as the strictly positive solution of equation (4.5) exists if $\rho^{(K)} > 0$.*

Proof:

Let $K \in \mathcal{K}$ and suppose that $\rho^{(K)} > 0$. Now, $R^{(K)}$ is defined as the strictly positive solution of the equation

$$g^{(K)}(r) := \int_0^T \lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s)\, ds = 0\,.$$

It is well known that under our assumptions we have $h_t'(0) = \mu_{B_t}$ for all $t \in [0, T)$, confer Asmussen [Asm00]. We therefore obtain

$$\begin{aligned}
\frac{d}{dr}\, g^{(K)}(r)\Big|_{r=0} &= \frac{d}{dr} \int_0^T \lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s)\, ds \Big|_{r=0} \\
&= \int_0^T \frac{d}{dr}\Big(\lambda_s h_s(r) + \frac{1}{2} r^2 b^2 K_s^2 - r(c + aK_s) \Big)\Big|_{r=0} ds \\
&= \int_0^T \lambda_s h_s'(0) - (c + aK_s)\, ds \\
&= \int_0^T \lambda_s \mu_{B_s}\, ds - cT - a \int_0^T K_s\, ds = -\rho(K)\, T < 0
\end{aligned}$$

which implies that the equation $g^{(K)}(r) = 0$ must have a strictly positive solution.

□

4.3 Maximizing the adjustment coefficient

So far, it has been shown that for an arbitrarily chosen investment strategy $K \in \mathcal{K}$ we have

$$\Psi(u,K) \leq C(K,r)\, e^{-ru}$$

with $C(K,r) < \infty$ for all $u \geq 0$ whenever $r \leq R^{(K)}$. Analogously to chapter 2, we thus have to maximize $R^{(K)}$ with respect to the investment strategy $K \in \mathcal{K}$ under the constraint that $C(K, R^{(K)})$ is finite. Recall that $R^{(K)}$ is defined as the strictly positive solution of

$$\int_0^T \lambda_t h_t(r) + \frac{1}{2} r^2 b^2 K_t^2 - r(c + aK_t)\, dt = 0\,.$$

For $r > 0$ we have

$$\lambda_t h_t(r) + \frac{1}{2} r^2 b^2 K_t^2 - r(c + aK_t) = \lambda_t h_t(r) + \frac{r^2 b^2}{2}\big(K_t - \frac{a}{rb^2}\big)^2 - \big(rc + \frac{a^2}{2b^2}\big)\,.$$

Thus, $R^{(K)}$ is the strictly positive solution of the equation

$$\int_0^T \lambda_t h_t(r) + \frac{r^2 b^2}{2}\big(K_t - \frac{a}{rb^2}\big)^2 - \big(rc + \frac{a^2}{2b^2}\big)\, dt = 0\,. \tag{4.7}$$

As in the second chapter let the constant investment strategy $K^{(r)}$ be defined by $K_t^{(r)} \equiv \frac{a}{rb^2}$ for $t \geq 0$. It thus follows from equation (4.7) that the strategy $K^{(r)}$ maximizes $R^{(K)}$ for some $r > 0$.

Motivated by the definition of $R^{(K)}$ through equation (4.7) we define R as the strictly positive solution of

$$\int_0^T \lambda_t h_t(r) - \big(rc + \frac{a^2}{2b^2}\big)\, dt = \int_0^T \lambda_t h_t(r)\, dt - \big(rc + \frac{a^2}{2b^2}\big)\, T = 0\,. \tag{4.8}$$

Our assumptions on h_t for $t \in [0, T)$ imply that R is uniquely determined and that it exists whenever the drift parameter a of the Brownian motion with drift $W_{a,b}$ does not equal zero.

In the case where $a = 0$ we have

$$\frac{d}{dr} \int_0^T \lambda_t h_t(r)\, dt - rcT \bigg|_{r=0} = \int_0^T \lambda_t \mu_{B_t}\, dt - cT = -\rho^{(0)}\, T$$

as noted in the proof of Proposition 4.6. Hence, R exists if and only if $\rho^{(0)} > 0$.

As in the Markov-modulated Poisson model it therefore follows that R exists as long as $a \neq 0$. For $a = 0$ we have the existence of R if the safety loading of the periodic Poisson model without investment is strictly positive, i.e. if there exists an adjustment coefficient without investment.

It moreover follows from the respective definitions that $R^{(K)} \leq R$ for all investment strategies $K \in \mathcal{K}$ with equality if $K = K^{(R)}$. Further, $R^{(K)}$ apparently does not exist for any investment strategy $K \in \mathcal{K}$ if R does it exist.

Together with Theorem 4.3 in the previous section we finally get the following result.

Corollary 4.7. *Suppose that the strictly positive solution R of equation (4.8) exists. Under the investment strategy $K^{(R)}$ defined by $K_t^{(R)} \equiv \frac{a}{Rb^2}$ for $t \geq 0$ we have*

$$\Psi(u, K^{(R)}) \leq C(K^{(R)}, R)\, e^{-Ru}$$

with $C(K^{(R)}, R) < \infty$ for all $u \geq 0$.

4.4 Optimality

We have already mentioned that there does not exist an adjustment coefficient for the periodic Poisson model under any investment strategy $K \in \mathcal{K}$ if R which is defined in the previous section does not exist. Hence, let us from now on assume that R exists. It is also proved in the previous section that

$$\Psi(u, K^{(R)}) \leq C(K^{(R)}, R)\, e^{-Ru}$$

with $C(K^{(R)}, R) < \infty$ for all $u \geq 0$. In this section we are going to show that R is indeed the optimal adjustment coefficient for the periodic Poisson model with investment.

Our method to prove this optimality is taken from Gaier, Grandits and Schachermayer [GGS03]. As there, we do not have to restrict ourselves to investment strategies $K \in \mathcal{K}$. Throughout this section, investment strategies K are considered which are measurable and adapted to $\mathcal{F}^Y$. We further have to assume that the integral $\int_0^t K_s^2\, ds$ is almost surely finite for every $t \geq 0$. Note that this is a necessary and sufficient condition for the stochastic integral $\int_0^t K_s\, dW_s$ to exist for $t \geq 0$. In that what follows we denote the class of such strategies by $\mathcal{K}^*$.

As in Gaier, Grandits and Schachermayer [GGS03] we need the following assumption on the claim size distributions in order to prove the optimality of the investment strategy $K^{(R)}$. Let the random variable $U^{(t)}$ have distribution B_t for $t \in [0, T)$. We then assume that

$$\sup_{\substack{0 \leq t < T \\ y > 0}} \mathbb{E}\Big[e^{-R(y - U^{(t)})} \Big| U^{(t)} > y\Big] < \infty\,. \tag{4.9}$$

Recall from Proposition 4.1 that the stopped process $\tilde{M}(u, K, r)$ is a martingale with respect to $\mathcal{F}^Y$ for all $u, r \geq 0$ and any investment strategy $K \in \mathcal{K}$. Plugging in R and the corresponding investment strategy $K^{(R)} \in \mathcal{K}$ we observe that $\tilde{M}(u, K^{(R)}, R)$ is a martingale for all $u \geq 0$. At this,

$$M_t(u, K^{(R)}, R) = \frac{\exp\Big(- R\,Y_t(u, K^{(R)})\Big)}{\exp\Big(\int_0^t \lambda_s h_s(R)\,ds - \big(Rc + \frac{a^2}{2b^2}\big)\,t\Big)}\,, \; t \geq 0\,.$$

Motivated by this formula we define such a process for any arbitrary investment strategy $K \in \mathcal{K}^*$ and get the following result.

Proposition 4.8. *Suppose that R defined by (4.8) exists. For any investment strategy $K \in \mathcal{K}^*$ and any $u \geq 0$ let the process $M^*(u, K, R)$ be defined by*

$$M_t^*(u, K, R) := \frac{\exp\Big(- R\,Y_t(u, K)\Big)}{\exp\Big(\int_0^t \lambda_s h_s(R)\,ds - \big(Rc + \frac{a^2}{2b^2}\big)\,t\Big)}\,, \; t \geq 0\,.$$

With respect to $\mathcal{F}^Y$, the stopped process $\tilde{M}^*(u, K, R)$ given by

$$\tilde{M}_t^*(u, K, R) := M^*_{t\wedge\tau(u,K)}(u, K, R)\,, \; t \geq 0\,,$$

is then a submartingale for any $K \in \mathcal{K}^*$ and a martingale if $K = K^{(R)}$. Moreover, $\tilde{M}^*(u, K, R)$ is uniformly integrable for all $K \in \mathcal{K}^*$ if assumption (4.9) is fulfilled.

Proof:

Let $u \geq 0$ be fixed. As already mentioned, $\tilde{M}^*(u, K^{(R)}, R)$ is a martingale according to Proposition 4.1. Analogously to the proof of that proposition we can moreover show that $\tilde{M}^*(u, K, R)$ is a submartingale for all $K \in \mathcal{K}^*$. Comparing the processes $\tilde{M}(u, K, R)$ and $\tilde{M}^*(u, K, R)$ we recognize that $M_t^*(u, K, R) = \exp(V_t^*)$ where the process $\{V_t^*, t \geq 0\}$ has dynamics

$$dV_t^* = -\Big(\lambda_t h_t(R) - \frac{a^2}{2b^2}\Big)\,dt - RaK_t\,dt - RbK_t\,dW_t + RU_{N_t}\,dN_t\,.$$

Applying Itô's Formula in this case we consequently get

$$\begin{aligned} &M_t^*(u,K,R) \\ &= M_0^*(u,K,R) + \int_{0+}^t M_{s-}^*(u,K,R)\Big(\frac{a^2}{2b^2} - RaK_s + \frac{1}{2}R^2b^2K_s^2\Big)\,ds \\ &\quad - \int_{0+}^t RbK_s\, M_{s-}^*(u,K,R)\,dW_s \\ &\quad + \int_{0+}^t \Big(M_s^*(u,K,R) - M_{s-}^*(u,K,R)\Big)\,dN_s - \int_{0+}^t \lambda_s h_s(R)\, M_{s-}^*(u,K,R)\,ds\,. \end{aligned} \tag{4.10}$$

Analogously to the proof of Proposition 4.1 it can be shown that the processes

$$\Big\{\int_{0+}^{t\wedge\tau} RbK_s\, M_{s-}^*(u,K,R)\,dW_s, t \geq 0\Big\}$$

and

$$\Big\{\int_{0+}^{t\wedge\tau} \Big(M_s^*(u,K,R) - M_{s-}^*(u,K,R)\Big)\,dN_s - \int_{0+}^{t\wedge\tau} \lambda_s h_s(R)\, M_{s-}^*(u,K,R)\,ds\,,\ t \geq 0\Big\}$$

are martingales with respect to $\mathcal{F}^Y$. Further,

$$\begin{aligned} &\int_{0+}^t M_{s-}^*(u,K,R)\Big(\frac{a^2}{2b^2} - RaK_s + \frac{1}{2}R^2b^2K_s^2\Big)\,ds \\ &\qquad = \int_{0+}^t M_{s-}^*(u,K,R)\,\frac{R^2b^2}{2}\Big(K_s - \frac{a}{Rb^2}\Big)^2\,ds \geq 0 \end{aligned}$$

for all $t \geq 0$. Since $K \in \mathcal{K}^*$ it moreover follows from the representation of $M_t^*(u,K,R)$ in (4.10) that the stopped process $\tilde{M}^*(u,K,R)$ is integrable. Hence, $\tilde{M}^*(u,K,R)$ is a submartingale for all $K \in \mathcal{K}^*$ since the process is apparently $\mathcal{F}^Y$-measurable.

It thus remains to show that $\tilde{M}^*(u,K,R)$ is uniformly integrable for any $K \in \mathcal{K}^*$ if assumption (4.9) is fulfilled. Let us again put $\tau := \tau(u,K)$. It then follows that

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{t\geq 0}\big|\tilde{M}^*_t(u,K,R)\big|\Big) &\leq \frac{\mathbb{E}\Big(\sup_{t\geq 0} e^{-RY_{t\wedge\tau}(u,K)}\Big)}{\inf_{t\geq 0}\exp\Big(\int_0^t \lambda_s h_s(R)\,ds - \big(Rc+\frac{a^2}{2b^2}\big)\,t\Big)} \\
&\leq \frac{\mathbb{E}\Big[\sup_{t\geq 0} e^{-RY_{t\wedge\tau}(u,K)}\Big|\tau<\infty\Big]}{\inf_{t\geq 0}\exp\Big(\int_0^t \lambda_s h_s(R)\,ds - \big(Rc+\frac{a^2}{2b^2}\big)\,t\Big)} \\
&= \frac{\mathbb{E}\Big[e^{-RY_{\tau}(u,K)}\Big|\tau<\infty\Big]}{\inf_{0\leq t<T}\exp\Big(\int_0^t \lambda_s h_s(R)\,ds - \big(Rc+\frac{a^2}{2b^2}\big)\,t\Big)} \\
&\leq \frac{\mathbb{E}\Big[e^{-RY_{\tau}(u,K)}\Big|\tau<\infty, Y_{\tau^-}(u,K)>0\Big]}{\inf_{0\leq t<T}\exp\Big(\int_0^t \lambda_s h_s(R)\,ds - \big(Rc+\frac{a^2}{2b^2}\big)\,t\Big)} \\
&\leq \frac{\sup_{0\leq t<T}\mathbb{E}\Big[e^{-RY_{\tau}(u,K)}\Big|\tau<\infty, Y_{\tau^-}(u,K)>0, B_\tau=B_t\Big]}{\inf_{0\leq t<T}\exp\Big(\int_0^t \lambda_s h_s(R)\,ds - \big(Rc+\frac{a^2}{2b^2}\big)\,t\Big)} < \infty\,.
\end{aligned}
$$

□

The following result considers the fact that in the periodic Poisson model with investment the insurer either becomes infinitely rich or ruin occurs.

Proposition 4.9. *Suppose that R defined by (4.8) exists and that assumption (4.9) is fulfilled. For any $K\in\mathcal{K}^*$ and $u\geq 0$ the stopped wealth process $\tilde{Y}(u,K)$ given by $\tilde{Y}_t(u,K) := Y_{t\wedge\tau(u,K)}(u,K)$ for $t\geq 0$ then almost surely converges on $\{\tau(u,K)=\infty\}$ to ∞ as $t\to\infty$.*

Proof:

Let $K\in\mathcal{K}^*$ and $u\geq 0$. Recall from Proposition 4.8 that $\tilde{M}^*(u,K,R)$ is an uniformly integrable submartingale. Applying Doob's Supermartingale Convergence Theorem to $-\tilde{M}^*(u,K,R)$ it follows that $\lim_{t\to\infty}\tilde{M}^*_t(u,K,R)$ almost surely exists. Hence, also $\tilde{Y}_\infty(u,K) := \lim_{t\to\infty}\tilde{Y}_t(u,K)$ almost surely exists.

Now, note that the distribution B_t is concentrated on $(0, \infty)$ for every $t \in [0, T)$. As described in the proof of Lemma 5 in Gaier, Grandits and Schachermayer [GGS03], page 11, there exists some $\delta > 0$ such that the wealth process infinitely often has a jump of a size which is greater than δ. Apart from these downward jumps the wealth process is almost surely continuous. On the event $\{\tau(u, K) = \infty\}$, $\tilde{Y}_\infty(u, K)$ can consequently not be equal to a finite value with positive probability.

□

Having Proposition 4.8 and 4.9 we can finally prove the optimality of the investment strategy $K^{(R)}$.

Theorem 4.10. *Suppose that R defined by (4.8) exists and that assumption (4.9) is fulfilled. For any investment strategy $K \in \mathcal{K}^*$ we then have*

$$\Psi(u, K) \geq C^* \, e^{-Ru}$$

with $C^* > 0$ for all $u \geq 0$.

Proof:

Let $K \in \mathcal{K}^*$ and $u \geq 0$. We know from Proposition 4.8 that $\tilde{M}_t^*(u, K, R)$ is an uniformly integrable submartingale. Once again putting $\tau := \tau(u, K)$ it thus follows from Doob's Optional Sampling Theorem that

$$\begin{aligned} e^{-Ru} &= \tilde{M}_0^*(u, K, R) \leq \mathbb{E}\Big(\tilde{M}_\tau^*(u, K, R)\Big) \\ &= \mathbb{E}\Big[M_\tau^*(u, K, R)\Big|\tau < \infty\Big] \, \mathbb{P}\big(\tau < \infty\big) + \mathbb{E}\Big[\lim_{t\to\infty} M_t^*(u, K, R)\Big|\tau = \infty\Big] \, \mathbb{P}\big(\tau = \infty\big) \\ &= \mathbb{E}\Big[M_\tau^*(u, K, R)\Big|\tau < \infty\Big] \, \mathbb{P}\big(\tau < \infty\big) \end{aligned} \tag{4.11}$$

$$\leq \frac{\sup\limits_{\substack{0\leq t<T\\ y>0}} \mathbb{E}\Big[e^{-R(y-U^{(t)})}\Big|U^{(t)}>y\Big]}{\inf\limits_{0\leq t<T} \exp\Big(\int_0^t \lambda_s h_s(R)\,ds - \big(Rc+\frac{a^2}{2b^2}\big)\,t\Big)}\,\mathbb{P}\big(\tau<\infty\big) \tag{4.12}$$

where the equality in (4.11) follows from Proposition 4.9 and the inequality in (4.12) as in the proof of Proposition 4.8. This implies that $\Psi(u,K)\geq C^*\,e^{-Ru}$ where

$$C^* := \frac{\inf\limits_{0\leq t<T} \exp\Big(\int_0^t \lambda_s h_s(R)\,ds - \big(Rc+\frac{a^2}{2b^2}\big)\,t\Big)}{\sup\limits_{\substack{0\leq t<T\\ y>0}} \mathbb{E}\Big[e^{-R(y-U^{(t)})}\Big|U^{(t)}>y\Big]} > 0$$

according to assumption (4.9).

□

Thus, R is the optimal adjustment coefficient for the periodic Poisson model under any investment strategy $K\in\mathcal{K}^*$. Recall that R is given as the strictly positive solution of the equation

$$\int_0^T \lambda_t h_t(r)\,dt - \big(rc+\frac{a^2}{2b^2}\big)\,T = 0\,.$$

and that the corresponding optimal investment strategy $K^{(R)}$ is defined by $K_t^{(R)}\equiv\frac{a}{Rb^2}$ for $t\geq 0$.

4.5 A comparison with the compound Poisson model

In the final section of this chapter we compare the adjustment coefficients of the periodic Poisson model and its associated compound Poisson model under optimal investment. For the periodic Poisson model with investment we have already found out that the optimal adjustment coefficient R is given as the strictly positive solution of the equation

$$\int_0^T \lambda_t h_t(r)\,dt - \big(rc + \frac{a^2}{2b^2}\big)\,T = 0 \tag{4.13}$$

and that the corresponding optimal investment strategy $K^{(R)}$ is defined by $K_t^{(R)} \equiv \frac{a}{Rb^2}$ for $t \geq 0$.

We can now associate a compound Poisson model to the periodic Poisson model in a natural way by averaging over the environment, confer Asmussen [Asm00], page 176. As mentioned in the introductory section of this chapter, this yields a compound Poisson model with parameters

$$\lambda^* = \frac{1}{T}\int_0^T \lambda_t\,dt \quad \text{and} \quad B^* = \frac{1}{T}\int_0^T \frac{\lambda_t}{\lambda^*}\,B_t\,dt\,.$$

Note that the claims of this associated compound Poisson model have exponential moments since

$$\begin{aligned}
h^*(r) &= \int_0^\infty e^{rx}\,dB^*(x)\, - 1 = \bigg(\frac{1}{T}\int_0^T \frac{\lambda_t}{\lambda^*}\int_0^\infty e^{rx}\,dB_t(x)\,dt\bigg) - 1 \\
&= \bigg(\frac{1}{T}\int_0^T \frac{\lambda_t}{\lambda^*}\Big(h_t(r) + 1\Big)\,dt\bigg) - 1 \\
&= \bigg(\frac{1}{T}\int_0^T \frac{\lambda_t}{\lambda^*}\,h_t(r)\,dt\bigg) + \bigg(\frac{1}{T}\int_0^T \frac{\lambda_t}{\lambda^*}\,dt\bigg) - 1 \\
&= \frac{1}{T}\int_0^T \frac{\lambda_t}{\lambda^*}\,h_t(r)\,dt\,.
\end{aligned}$$

As mentioned at the beginning of the second chapter, it is due to Gaier, Grandits and Schachermeyer [GGS03] that the optimal adjustment coefficient R^* of the associated compound Poisson model with investment is given as the strictly positive solution of the equation

$$\lambda^* h^*(r) = cr + \frac{a^2}{2b^2}$$

and that the corresponding optimal investment strategy is given by $K^{(R^*)}$. Noting that

$$\begin{aligned}
\lambda^* h^*(r) - \big(rc + \frac{a^2}{2b^2}\big) &= \lambda^* \frac{1}{T} \int_0^T \frac{\lambda_t}{\lambda^*} h_t(r)\, dt - \big(rc + \frac{a^2}{2b^2}\big) \\
&= \frac{1}{T} \left(\int_0^T \lambda_t h_t(r)\, dt - \big(rc + \frac{a^2}{2b^2}\big)\, T \right) .
\end{aligned}$$

it is obvious that the adjustment coefficients of the periodic Poisson model and its associated compound Poisson model coincide under optimal investment. Consequently, also the optimal investment strategy is the same for both models.

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