

Stabilization of Flows Through Porous Media

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Zusammenfassung

Transportphänomene in porösen Medien treten in vielen ingenieurwissenschaftlichen Disziplinen auf. Im Bauingenieurwesen studiert man z.B. Stömungen in wasserführenden Schichten, die Befeuchtung technischer Strukturen oder den Transport von Schadstoffen im Boden. In der Agrartechnik wird der Transport von Wasser und Nährstoffkonzentrationen im Erdreich untersucht. Wärme- und Massenflüsse, sowie Trocknungsprozesse führen in der chemischen Verfahrenstechnik zu Transportphänomenen durch poröse Medien. Schließlich untersucht man in der Mineralölfördertechnik den Transport von Erdöl, Wasser und Gas in porösen Strukturen. Allgemein ist man an Modellen interessiert, in welchen verschiedene Größen wie Masse, chemische Konzentrationen oder Impuls durch ein poröses Medium transportiert werden. Oft enthält das poröse Medium mehrere Flüssigkeitsphasen, die gleichzeitig transportiert werden. Die Trennschichten zwischen den verschiedenen Phasen können sich frei bewegen, und sind deshalb als unbekannte Größe zu betrachten. Dies bedeutet, dass Transportphänomene in porösen Medien mathematisch durch freie Randwertaufgaben beschrieben werden, und die entsprechenden partiellen Differentialgleichungen in zeitlich sich verändernden Gebieten zu lösen sind. Außerdem erzeugen freie Ränder stets eine inhärente nicht-lineare Struktur, da für diese Systeme offensichtlich kein Superpositionsprinzip für Lösungen gelten kann.

In dieser Arbeit wird der Fluss einer inkompressiblen homogenen Newtonschen Flüssigkeit in einem starren porösen Medium betrachtet. In dem unbeschränkten Medium befindet sich eine fixierte Schicht, die für die Flüssigkeit undurchlässig ist und eine externen Quelle trägt, über welche das System gesteuert werden kann. Der obere Rand ist frei und bewegt sich aufgrund der Gravitationskraft. Das Hauptanliegen dieser Arbeit besteht in einer Stabilitätsanalyse der Gleichgewichtslage des oben beschriebenen freien Randwertproblems.

Im ersten Teil der Arbeit werden die mathematischen Grundlagen zusammengestellt, die zur Untersuchung des betrachteten Problems benötigt werden. Es werden geeignete Funktionenräume eingeführt und die Theorie der maximalen Hölderregularität erklärt. Danach werden die physikalischen Grundlagen der Modellierung von Flüssen durch poröse Medien erläutert und die daraus hervorgehenden Modelle vorgestellt. Die mathematische Untersuchung der entsprechenden freien Randwertaufgabe beginnt mit der Transformation auf ein festes Referenzgebiet. Eine Reduktion des vollen Systems führt dann auf eine nichtlineare Evolutionsgleichung, die nur noch den freien Rand beinhaltet. Schließlich werden die Theorien der Fouriermultiplikatoren, der analytischen Halbgruppen, der maximalen Regularität und das Prinzip der linearisierten Stabilität angewendet, um diese Evolutionsgleichung zu untersuchen. Im Hauptresultat dieser Arbeit wird gezeigt, dass die eindeutig bestimmte Gleichgewichtslage des Systems asymptotisch stabil ist, falls die Injektionsrate in der Nähe der Gleichgewichtslage monoton wächst.

Stichwörter: Freie Ränder, Fluss durch poröse Medien, Gesetz von Darcy, voll

nichtlineare Evolutionsgleichungen, analytische Halbgruppen, maximale Regularität, Stabilität.

Abstract

Phenomena of transport in porous media are encountered in many engineering disciplines. Civil engineering deals, for example, with the flow of water in aquifers, the movement of moisture through engineering structures, transport of pollutants in aquifers and the propagation of stresses under foundations of structures. Agricultural engineering deals with the movement of water and solutes in the root zone in the soil. Heat and mass transport in packed-bed reactor columns and drying processes are encountered in chemical engineering. Reservoir engineers deal with the flow of oil, water and gas in petroleum reservoirs. One is interested in models in which various quantities, such as mass or momentum, through a porous medium. Often the porous material contains several fluid phases and the various quantities are transported simultaneously throughout the multiphase system. In general, the interface between different phases can move freely and are thus unknown. This means that these transport problems in porous media are mathematically described by free boundary problems, and the corresponding partial differential equations have to be solved in non-constant time depending domains. Moreover, the free interface causes an inherent nonlinear structure of these systems, since it prevents the existence of a superposition principle of solution.

This thesis considers the motion of an incompressible homogeneous Newtonian fluid in a rigid porous medium of infinite extent. This fluid is bounded below by a fixed layer having an external source, and above by a free surface moving under the influence of gravity. The main focus of this thesis is to analyze the stability of the equilibria of the moving boundary problem.

In this thesis, we first introduce some preliminary material. We give a precise definition of all function spaces needed for our analysis, as well as an introduction in the theory of maximal Hölder regularity. Then we give a short introduction on modelling of flows through porous media and describe the problem which we consider. After this, we transform the considered problem into a nonlinear evolution equation for the free interface only. Finally, we mainly use the theory of Fourier multipliers, analytic semigroup, maximal regularity and principle of linearized stability to investigate the nonlinear evolution equation. Our main result shows that if the injection rate is strictly increasing in a neighborhood of the unique equilibrium, then this equilibrium is asymptotically stable.

Keywords: Moving Boundaries, Flow through Porous Media, Darcy's Law, Fully Nonlinear Evolution Equations, Analytic Semigroups, Maximal Regularity, Stability of equilibria, Stabilization.

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Chapter 1

Introduction

In the thesis, we shall investigate the motion of an incompressible homogeneous Newtonian fluid in a rigid porous medium of infinite extent. That fluid is bounded below by a fixed layer Γ_0 having an external source (with an injection rate b), and above by a free surface Γ_f moving under the influence of gravity, see **Figure 0**. Moreover, the fluid flow is governed by Darcy's law. More precisely, writing $u(t, x, y)$ for the pressure at the moment t and at the position (x, y) and $f(t, x)$ for the moving interface, we can describe it as the following problem:

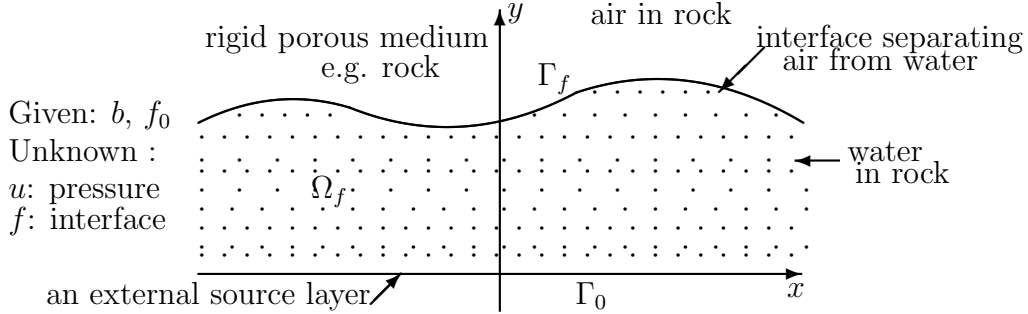


Figure 0

$$(P) \quad \left\{ \begin{array}{ll} \Delta u(t, x, y) = 0, & t \in J, (x, y) \in \Omega_{f(t)} \\ \frac{\partial u}{\partial y}(t, x, y)|_{y=0} = b(f(t, x)), & t \in J, x \in \mathbb{R} \\ u(t, x, y)|_{y=f(t, x)} = f(t, x), & t \in J, x \in \mathbb{R} \\ \lim_{|x|+|y| \rightarrow \infty} u(t, x, y) = c, & t \in J, (x, y) \in \Omega_{f(t)} \\ \partial_t f(t, x) + (-f_x, 1) \cdot \nabla u(t, x, y) = 0, & t \in \dot{J}, (x, y) \in \Gamma_{f(t)} \\ f(0, x) = f_0(x), & x \in \mathbb{R}, \end{array} \right.$$

where $T > 0$ is fixed, and $J = [0, T)$, $\dot{J} = J \setminus \{0\}$. Moreover,

$$\Omega_{f(t)} := \{(x, y) \in \mathbb{R} \times (0, \infty); 0 < y < f(t, x)\}$$

and

$$\Gamma_{f(t)} := \text{graph}(f) := \{(x, y) \in \mathbb{R} \times (0, \infty); y = f(t, x)\}.$$

For simplicity, we will use Ω_f and Γ_f to replace $\Omega_{f(t)}$ and $\Gamma_{f(t)}$, respectively. The injection rate b is a sufficiently smooth function from \mathbb{R} to \mathbb{R} . Since the injection rate can be controlled in advance, we can think it is a given smooth function. Finally, we are interested in classical solutions of the problem (P) . Therefore f has to satisfy at least the following conditions:

$$f \in \tilde{\mathfrak{A}} := \left\{ f \in BC^2(\mathbb{R}); \inf_{x \in \mathbb{R}} f(x) > 0 \right\}.$$

Otherwise, if $f(x) = 0$ for some x then the interface and the fluid domain have a singularity at this point.

In [12], J. Escher and G. Simonett studied the case of the fluid bounded below by a fixed impermeable layer, i.e. $b \equiv 0$. They proved that the problem (P) with $b \equiv 0$ is locally in time well-posed (cf. [1]). However, the question of the stability of the equilibrium solution $(u, f) \equiv (c, c)$ was not addressed in [12]. The main goal of this research is to specify classes of injection rate b such that the equilibrium $(u, f) \equiv (c, c)$ is stable. Clearly, if $(u, f) = (c, c)$ is an equilibrium of the problem (P) , it is necessary to assume that

$$b(c) = 0.$$

In order to state our results for problem (P) , we need some notations. Given $m \in \mathbb{N}$, we write $\mathcal{S}(\mathbb{R}^m)$ for the Schwartz space, that is, the Fréchet space of all rapidly decreasing smooth functions on \mathbb{R}^m . Moreover, assume that $k \in \mathbb{N}$, $\alpha \in (0, 1)$ and that U is an open subset of \mathbb{R}^m . Then $BUC^{k+\alpha}(U)$ denote the classical Hölder spaces of functions having bounded derivatives up to order k , and such that the k -th derivatives satisfy a uniformly α -Hölder condition, see Section 2.1 for a precise definition.

For $s > 0$, we define the closed subspaces $h^s(\mathbb{R}^m)$ of little Hölder spaces $buc^s(\mathbb{R}^m)$ by

$$h^s(\mathbb{R}^m) := \text{closure of } \mathcal{S}(\mathbb{R}^m) \text{ in } BUC^s(\mathbb{R}^m).$$

Furthermore, let γ_U denote the restriction operator with respect to U . Then we define

$$h^s(U) := \text{closure of } \gamma_U \mathcal{S}(\mathbb{R}^m) \text{ in } BUC^s(U).$$

For convenience, we also call $h^s(\mathbb{R}^m)$ to be *little Hölder spaces*. Finally, let

$$\mathfrak{A} := \left\{ g \in BC^2(\mathbb{R}); \inf_{x \in \mathbb{R}} \{c + g(x)\} > 0 \right\},$$

and for $s > 0$,

$$h^s := h^s(\mathbb{R}), \quad \text{and} \quad h_{\mathfrak{A}}^s := h^s \cap \mathfrak{A}.$$

We are now ready to state the following existence result for problem (P) .

Theorem 1.0.1. *Assume that $b(c) = 0$ and $b'(c) > 0$. Then there exists $\omega_0 > 0$, such that for any $\omega \in [0, \omega_0)$, there also exist positive constants r and $C = C(r)$. If $f_0 - c \in h_{\mathbb{R}^d}^{2+\alpha}$ with $\|f_0 - c\|_{h^{2+\alpha}} < r$, the problem (P) has a unique global solution $(u(t, x, y), f(t, x))$ such that*

$$u - c \in C([0, +\infty), h^{2+\alpha}(\Omega_f)) \quad \text{and} \quad f - c \in C([0, +\infty), h_{\mathbb{R}^d}^{2+\alpha}).$$

Furthermore, we get the following estimate

$$\|f(t) - c\|_{h^{2+\alpha}} + \|f'(t)\|_{h^{1+\alpha}} + \|u(t) - c\|_{h^{2+\alpha}(\Omega_{f(t)})} \leq Ce^{-\omega t} \|f_0 - c\|_{h^{2+\alpha}}, \quad \forall t \geq 0.$$

A fundamental and inherent difficulty of an analytic treatment of problem (P) is manifested in the fact that one has to work with variable domains. The nonlinear structure hidden behind this circumstance is clearly disclosed by the observation that there is no superposition principle for solution of the problem (P). This means problem (P) is a nonlinear partial differential problem. To prove our result, we first consider a transformed version of (P) which applies over a fixed domain. In fact, it turns out that the transformed problem consists of a nonlinear elliptic equation and a parabolic equation with a nonlinear nonlocal pseudo-differential operator of first order. Then we solve the nonlinear elliptic equation, and reduce the transformed problem to a nonlinear evolution equation. Finally, we mainly use the theory of Fourier multipliers, analytic semigroup and maximal regularity to investigate the nonlinear evolution equation.

We mention that the corresponding set of equations for incompressible fluids in porous media was investigated in [11, 13, 18, 22]. In particular, J. Escher and G. Prokert investigated spatially periodic flows in porous media in [16]. The moving boundary problem in deformable porous media was also investigated in [14, 15]. Moreover, there is a different approach in weighted Hölder spaces to that problem outlined in [7, 8]. In addition, a L_p -theory for that problem was proposed in [25, 26].

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Chapter 2

Preliminary I: Function Spaces

In the first part of this chapter we fix some notations by defining some basic function spaces. We then introduce Fourier multipliers on Lebesgue and Besov spaces. We close this chapter by discussing some results from interpolation theory which we need for our approach.

2.1 Some Function Spaces

2.1.1 Bounded Functions

Let E be a Banach space and suppose that X is a nonempty set. Then $B(X, E)$ is the Banach space of all **bounded functions** $u : X \rightarrow E$, equipped with the supremum norm

$$\|u\|_\infty := \|u\|_\infty^X := \sup_{x \in X} \|u(x)\|_E.$$

2.1.2 Continuous Functions

Let now X be a metric space. We denote by $BC(X, E)$ the closed subspace of $B(X, E)$ consisting of all **bounded and continuous** E -valued functions. Moreover, $BUC(X, E)$ is the closed subspace of $BC(X, E)$ consisting of all **bounded and uniformly continuous** E -valued functions on X .

Of course, $C(X, E)$ is given the topology of uniform convergence on compact subsets of X , the compact-open topology. If X is locally compact and separable, $C(X, E)$ is a Fréchet space.

If X is locally compact, $C_0(X, E)$ is the Banach space of all **continuous functions vanishing at infinity**, equipped with the supremum norm. Recall that $u : X \rightarrow E$ vanishes at infinity if, given $\varepsilon > 0$, there exists $K \subset\subset X$ such that $\|u(x)\|_E < \varepsilon$ for $x \in K^c := X \setminus K$. It is easily seen that $C_0(X, E)$ is a closed subspace of $BUC(X, E)$. Hence

$$C_0(X, E) \hookrightarrow BUC(X, E) \hookrightarrow BC(X, E) \hookrightarrow C(X, E) \quad (2.1.1)$$

and these spaces all coincide if X is compact.

2.1.3 Hölder Continuous Functions

Let d be a metric for X and let $0 < \rho < 1$. A function $u : X \rightarrow E$ is **uniformly ρ -Hölder continuous** if

$$[u]_\rho := [u]_\rho^X := \sup_{\substack{x, y \in X \\ x \neq y}} \frac{\|u(x) - u(y)\|_E}{[d(x, y)]^\rho} < \infty. \quad (2.1.2)$$

It is (locally) **ρ -Hölder continuous** if each point has a neighborhood Y such that $u|_Y$ is uniformly ρ -Hölder continuous. We write $u \in UC^\rho(X, E)$ if u is uniformly ρ -Hölder continuous, and

$$C^\rho(X, E) := \{u : X \rightarrow E; u \text{ is } \rho\text{-Hölder continuous}\}.$$

If ρ is replaced by 1, a [uniformly] 1-Hölder continuous function is said to be [**uniformly**] **Lipschitz continuous** and we write $UC^{1-}(X, E)$ and $C^{1-}(X, E)$ for the sets of all uniformly Lipschitz continuous functions and of all Lipschitz continuous functions, respectively, mapping X into E . In numerical calculations the symbol 1- will always be identified with the number 1.

If $u \in C^{1-}(X, E)$ and K is a compact subset of X , there exists a neighborhood Y of K in X such that $u|_Y$ is uniformly Lipschitz continuous (e.g., Proposition 6.4 in [3]). It is not difficult to see that d^ρ is an equivalent metric for X . Thus $u \in C^\rho(X, E)$ iff $u \in C^{1-}((X, d^\rho), E)$. This implies that, given $K \subset\subset X$ and $u \in C^\rho(X, E)$, there exists a neighborhood Y of K such that $u|_Y \in UC^\rho(Y, E)$.

Given $\rho \in (0, 1) \cup \{1-\}$, we equip $C^\rho(X, E)$ with the family of seminorms

$$\|\cdot\|_{C^\rho}^K := \|\cdot\|_\infty^K + [\cdot]_\rho^K, \quad K \subset\subset X.$$

Then $C^\rho(X, E)$ is a *LCS* (Locally Convex Space) such that

$$C^\rho(X, E) \hookrightarrow C(X, E). \quad (2.1.3)$$

If X is locally compact and separable, $C^\rho(X, E)$ is a Fréchet space.

For each $\rho \in (0, 1) \cup \{1-\}$, we denote by

$$BUC^\rho(X, E) := (BUC^\rho(X, E), \|\cdot\|_{C^\rho}) \quad (2.1.4)$$

the Banach space of all **bounded and uniformly ρ -Hölder continuous** E -valued functions on X , where $\|\cdot\|_{C^\rho} := \|\cdot\|_{C^\rho}^X$. Observe that

$$BUC^\rho(X, E) \hookrightarrow BUC^\sigma(X, E) \hookrightarrow BUC(X, E) \quad (2.1.5)$$

provided $\rho, \sigma \in (0, 1) \cup \{1-\}$ satisfy $\sigma < \rho$. In particular, we write $Lip(X, E)$ instead of $BUC^{1-}(X, E)$.

2.1.4 Smooth Functions

Now suppose that X is a nonempty open subset of \mathbb{R}^n , where $n \in \dot{\mathbb{N}}$. In the following $D_j := -i\partial_j$ with $\partial_j := \partial/\partial x_j$ for $j = 1, \dots, n$, and we use standard multiindex notation.

Of course, $C^m(X, E)$ has the usual meaning for $m \in \dot{\mathbb{N}}$, and $C^0 := C$. Given a nonempty open subset Y of X ,

$$\|u\|_{C^m}^Y := \max_{|\alpha| \leq m} \|\partial^\alpha u\|_\infty^Y \quad (2.1.6)$$

defines a seminorm on $C^m(X, E)$ and

$$C^m(X, E) := \left(C^m(X, E), \{ \|\cdot\|_{C^m}^K; K = \overset{\circ}{K} \subset\subset X \} \right)$$

is a Fréchet space. If $\rho \in (0, 1) \cup \{1-\}$, we put

$$C^{m+\rho}(X, E) := \{u \in C^m(X, E); \partial^\alpha u \in C^\rho(X, E), |\alpha| = m\},$$

equipped with the family of seminorms

$$u \mapsto \|u\|_{C^{m+\rho}}^K := \|u\|_{C^m}^K + \max_{|\alpha|=m} [\partial^\alpha u]_\rho^K, \quad K = \overset{\circ}{K} \subset\subset X. \quad (2.1.7)$$

It is not difficult to see that $C^{m+\rho}(X, E)$ is a Fréchet space and

$$C^{m+\rho}(X, E) \hookrightarrow C^{m+\sigma}(X, E) \hookrightarrow C^m(X, E) \hookrightarrow C^{m-}(X, E), \quad (2.1.8)$$

where $\rho, \sigma \in (0, 1) \cup \{1-\}$ satisfy $\sigma < \rho$ and $m- := (m-1) + 1-$ for $m \in \dot{\mathbb{N}}$. In fact, $C^m(X, E)$ is a closed linear subspaces of $C^{m-}(X, E)$.

Of course,

$$C^\infty(X, E) := \bigcap_{m=0}^{\infty} C^m(X, E) = \bigcap_{s \geq 0} C^s(X, E)$$

is a Fréchet space too (with respect to the obvious projective limit topology).

We write

$$\|\cdot\|_{C^{m+\rho}} := \|\cdot\|_{C^{m+\rho}}^X, \quad \rho \in [0, 1) \cup \{1-\}.$$

Then the Banach spaces $BC^m(X, E)$ and $BUC^m(X, E)$ are defined in the obvious way and are equipped with the norm $\|\cdot\|_{C^m}$. Moreover, given $\rho \in (0, 1) \cup \{1-\}$,

$$BUC^{m+\rho}(X, E) := (\{u \in C^{m+\rho}(X, E); \|u\|_{C^{m+\rho}} < \infty\}, \|\cdot\|_{C^{m+\rho}})$$

is a Banach space too. In addition, we introduce the Fréchet spaces

$$BC^\infty(X, E) := \bigcap_{m=0}^{\infty} BC^m(X, E)$$

and

$$BUC^\infty(X, E) := \bigcap_{m=0}^{\infty} BUC^m(X, E).$$

2.1.5 Little Hölder Spaces

Let X be a nonempty open subset of \mathbb{R}^n . It is well known that the injection

$$BUC^{m+\rho}(X, E) \hookrightarrow BUC^{m+\sigma}(X, E), \quad m \in \mathbb{N}, \quad 0 < \sigma < \rho < 1,$$

which is an obvious consequence of (2.1.5), is not dense. For this reason we introduce the **little Hölder spaces**:

$$buc^s(X, E) := \text{closure of } BUC^{m+1}(X, E) \text{ in } BUC^s(X, E)$$

for $m \in \mathbb{N}$ and $m < s < m + 1$. Then $u \in buc^{m+\rho}(\mathbb{R}^n, E)$ iff $u \in BUC^m(\mathbb{R}^n, E)$ and

$$\lim_{t \rightarrow 0} \sup_{\substack{x, y \in \mathbb{R}^n \\ 0 < |x-y| \leq t}} \frac{\|\partial^\alpha u(x) - \partial^\alpha u(y)\|_E}{|x-y|^\rho} = 0, \quad \alpha \in \mathbb{N}^n, \quad |\alpha| = m. \quad (2.1.9)$$

Further, $BUC^\infty(\mathbb{R}^n, E)$ is dense in $buc^{m+\rho}(\mathbb{R}^n, E)$ for $m \in \mathbb{N}$ and $\rho \in (0, 1)$.

Example 2.1.1. Let $\alpha \in (0, 1)$, then $x^\alpha \in BUC^\alpha(0, 1) \setminus buc^\alpha(0, 1)$.

Proof Clearly, x^α is bounded and continuous in $(0, 1)$. To prove $x^\alpha \in BUC^\alpha(0, 1)$, it is sufficient to prove that

$$|x^\alpha - y^\alpha| \leq |x - y|^\alpha \quad \text{for } x, y > 0. \quad (2.1.10)$$

Without loss of generality, we suppose that $x \geq y > 0$. Then we define

$$h(x) = x^\alpha - y^\alpha - (x - y)^\alpha \text{ for } x \geq y > 0.$$

By directly calculating, we know that $h(y) = 0$ and $h'(x) = \alpha(x^{\alpha-1} - (x - y)^{\alpha-1}) < 0$. This means $h(x) \geq h(y) = 0$, i.e.

$$x^\alpha - y^\alpha \leq (x - y)^\alpha \text{ for } x \geq y > 0.$$

Thus (2.1.10) holds. This implies $x^\alpha \in BUC^\alpha(0, 1)$.

For any $t \in (0, 1)$, we can take $x = t \in (0, 1)$ and $y = t/2 \in (0, 1)$, then we know that $|x - y| = t/2 < t$, and

$$\frac{|x^\alpha - y^\alpha|}{|x - y|^\alpha} = \frac{t^\alpha - (t/2)^\alpha}{(t/2)^\alpha} = 2^\alpha - 1 \neq 0.$$

Therefore $x^\alpha \notin buc^\alpha(0, 1)$. This completes the proof. \square

2.1.6 Functions With Compact Supports

If $m \in \mathbb{N} \cup \{\infty\}$ and $\rho \in [0, 1) \cup \{1-\}$, we denote by $C_c^{m+\rho}(X, E)$ the subspace of $C^{m+\rho}(X, E)$ consisting of all functions u whose supports, $\text{supp}(u)$, are compact in X , endowed with the usual inductive limit topology. In particular,

$$\mathcal{D}(X, E) := C_c^\infty(X, E)$$

is the space of E -valued **test functions** on X and $\mathcal{D}(X) := \mathcal{D}(X, \mathbb{K})$, where \mathbb{K} is either \mathbb{R} or \mathbb{C} , if no confusion seems likely. Lastly,

$$\mathring{C}^s(X, E) := \text{closure of } \mathcal{D}(X, E) \text{ in } BUC^s(X, E), \quad s \in \mathbb{R}^+.$$

It is not difficult to see that

$$\mathring{C}^m(X, E) = C_0^m(X, E) := \{u \in C^m(X, E); \partial^\alpha u \in C_0(X, E), |\alpha| \leq m\}.$$

2.1.7 Integrable Functions

Let X be a σ -compact metric space and let μ be a positive Radon measure on X . Given $q \in (0, \infty]$, a measurable subset Y of X , and a (strongly) measurable function $u : X \rightarrow E$, put

$$\|u\|_q^Y := \begin{cases} \left(\int_Y \|u(y)\|_E^q d\mu \right)^{1/q}, & 0 < q < \infty, \\ \text{ess-sup}_{y \in Y} \|u(y)\|_E, & q = \infty. \end{cases}$$

If $q \geq 1$ then $L_{q,loc}(X, \mu, E)$ is the Fréchet space of all (equivalence classes of strongly) measurable functions $u : X \rightarrow E$ such that

$$\|u\|_q^K < \infty, \quad K = \mathring{K} \subset\subset X, \quad (2.1.11)$$

topologized with the family of seminorms (2.1.11). We put $\|\cdot\|_q := \|\cdot\|_q^X$ and denote by

$$L_q(X, \mu, E) := (L_q(X, \mu, E), \|\cdot\|_q), \quad 1 \leq q \leq \infty,$$

the usual **Lebesgue spaces**, which are Banach spaces. (The fact that we use the symbol $\|\cdot\|_\infty$ in two different contexts will cause no confusion.) In particular,

$$l_p(E) := L_p(\mathbb{N}, \kappa, E), \quad 1 \leq p \leq \infty,$$

where κ is the counting measure, are Lebesgue spaces of E -valued sequences.

If $E = \mathbb{K}$ and no confusion seems possible, we put $L_p(X, \mu) := L_p(X, \mu, \mathbb{K})$.

If X is an open subset of \mathbb{R}^n (more generally, a Lebesgue measurable subset of \mathbb{R}^n) and μ is the n -dimensional Lebesgue measure on X , we simply put

$$L_p(X, E) := L_p(X, \mu, E), \quad 1 \leq p \leq \infty,$$

and $dx := d\mu$.

Let now X again be an open subset of \mathbb{R}^n . Recall that

$$\mathcal{D}(X, E) \hookrightarrow L_q(X, E) \hookrightarrow L_{q,loc}(X, E) \xrightarrow{d} L_{1,loc}(X, E), \quad 1 \leq q \leq \infty, \quad (2.1.12)$$

and that the first two injections are dense too if $q < \infty$.

2.1.8 Distributions and Sobolev Spaces

We write $\mathcal{D}'(X, E)$ for the spaces of E -valued distributions on X , that is,

$$\mathcal{D}'(X, E) := \mathcal{L}(\mathcal{D}(X), E),$$

equipped with the topology of uniform convergence on bounded sets, and

$$\mathcal{D}'(X) := \mathcal{D}'(X, \mathbb{K}).$$

We also use ∂_j (and D_j) to denote distributional partial derivatives. Recall that

$$L_{1,loc}(X, E) \hookrightarrow \mathcal{D}'(X, E), \quad (2.1.13)$$

by means of the identification

$$u(\phi) := \int_X \phi(x)u(x)dx, \quad \phi \in \mathcal{D}(X), \quad u \in L_{1,loc}(X, E). \quad (2.1.14)$$

Given $m \in \mathbb{N}$ and $q \in [1, \infty]$, the **Sobolev space**, $W_q^m(X, E)$, of order m over L_q consisting of E -valued distributions on X is defined by

$$W_q^m(X, E) := (\{u \in L_q(X, E); \partial^\alpha u \in L_q(X, E), |\alpha| \leq m\}, \|\cdot\|_{m,q}),$$

where

$$\|u\|_{m,q} := \|u\|_{W_q^m(X,E)} := \begin{cases} \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_q^q \right)^{1/q}, & 1 \leq q < \infty, \\ \max_{|\alpha| \leq m} \|\partial^\alpha u\|_\infty, & q = \infty. \end{cases} \quad (2.1.15)$$

For $q \in [1, \infty)$ and $s \in (0, 1)$ we put

$$[u]_{s,q} := [u]_{W_q^s(X,E)} := \left(\int_{X \times X} \left(\frac{\|u(x) - u(y)\|}{|x - y|^s} \right)^q \frac{d(x, y)}{|x - y|^n} \right)^{1/q}. \quad (2.1.16)$$

Moreover, we denote by $[t]$ the largest integer less or equal to $t \in \mathbb{R}$. Then, given $s \in \mathbb{R}^+ \setminus \mathbb{N}$ and $q \in [1, \infty)$, we define the **Slobodeckii space**, $W_q^s(X, E)$, of order s consisting of E -valued distributions on X by

$$W_q^s(X, E) := (\{u \in W_q^{[s]}(X, E); [\partial^\alpha u]_{s-[s],q} < \infty, |\alpha| = [s]\}, \|\cdot\|_{s,q}),$$

where

$$\|u\|_{s,q} := \|u\|_{W_q^s(X,E)} := \left(\|u\|_{[s],q}^q + \sum_{|\alpha|=[s]} [\partial^\alpha u]_{s-[s],q}^q \right)^{1/q}. \quad (2.1.17)$$

Further, $W_q^0(X, E) := L_q(X, E)$ so that $\|\cdot\|_{0,q} = \|\cdot\|_q$. It is well known that the Sobolev and the Slobodeckii spaces are Banach spaces and that

$$W_q^s(X, E) \hookrightarrow W_q^t(X, F), \quad 1 \leq q < \infty, \quad 0 \leq t < s < \infty, \quad (2.1.18)$$

provided $E \hookrightarrow F$.

As usual,

$$\mathring{W}_q^s(X, E) := \text{closure of } \mathcal{D}(X, E) \text{ in } W_q^s(X, E).$$

Then

$$\mathring{W}_q^s(\mathbb{R}^n, E) = W_q^s(\mathbb{R}^n, E), \quad s \in \mathbb{R}^+, \quad 1 \leq q < \infty. \quad (2.1.19)$$

2.2 Fourier Multipliers

2.2.1 Generalities

Let \mathcal{S} be the Schwartz space of rapidly decreasing functions on \mathbb{R}^n and \mathcal{S}' its dual, the space of tempered distributions, endowed with the strong topology. We denote by \mathcal{F} the Fourier transform $u \mapsto \hat{u}$ in \mathcal{S}' , defined by $\langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle$ and

$$\hat{\phi}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle \xi, x \rangle} \phi(x) dx, \quad \xi \in \mathbb{R}^n, \quad \phi \in \mathcal{S}.$$

Recall that

$$\mathcal{F} \in \mathcal{L}aut(\mathcal{S}) \cap \mathcal{L}aut(\mathcal{S}') \quad (2.2.1)$$

and that

$$\mathcal{F}^{-1}u = \check{u}, \quad u \in \mathcal{S}', \quad (2.2.2)$$

where $\langle \check{u}, \phi \rangle := \langle u, \check{\phi} \rangle$ for $u \in \mathcal{S}'$ and $\phi \in \mathcal{S}$ with $\check{\phi}(x) := \phi(-x)$ for $x \in \mathbb{R}^n$, and where $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\mathcal{S}}$.

Given $u \in \mathcal{S}'$ and $\phi \in \mathcal{S}$, the convolution $u * \phi$ is a well defined tempered distribution and $(u * \phi)^\wedge = (2\pi)^{n/2} \hat{u} \hat{\phi}$ by the convolution theorem. Thus, given $a \in \mathcal{S}'$, we define a linear operator, $a(D) : \mathcal{S} \rightarrow \mathcal{S}'$, a **pseudodifferential operator** with symbol a , by

$$a(D)u := \mathcal{F}^{-1}a\mathcal{F}u := \mathcal{F}^{-1}(a\hat{u}) = (2\pi)^{-n/2} \mathcal{F}^{-1}a * u, \quad u \in \mathcal{S}. \quad (2.2.3)$$

It is well known that the convolution $u * v$ can be defined for various classes of distributions u and v . Thus, given a Banach space with $E \hookrightarrow \mathcal{S}'$, an element $a \in \mathcal{S}'$ is said to be a **Fourier multiplier** for E (an E -multiplier) if $\mathcal{F}^{-1}a * u$ is a well defined element of E for each $u \in E$ and $(u \mapsto \mathcal{F}^{-1}a * u) \in \mathcal{L}(E)$. If this is the case, we again use the notations of (2.2.3). We denote by M_E the vector space of all E -multipliers equipped with the norm

$$\|a\|_{M_E} := \|a(D)\|_{\mathcal{L}(E)}.$$

In other words, $\|a\|_{M_E}$ is the infimum of all $c \in \mathbb{R}$ satisfying

$$\|\mathcal{F}^{-1}a * u\|_E \leq c(2\pi)^{n/2}\|u\|_E, \quad u \in E.$$

The notation $a(D)$ is, of course, justified by the well known formula

$$(D^\alpha u)^\wedge = \xi^\alpha \hat{u}, \quad \alpha \in \mathbb{N}^n, \quad u \in \mathcal{S}'. \quad (2.2.4)$$

Lastly, we recall the **Plancherel's Theorem** guarantees that

$$\mathcal{F} \in \mathcal{L}aut(L_2) \quad (2.2.5)$$

and

$$\|\hat{u}\|_2 = \|u\|_2, \quad u \in L_2. \quad (2.2.6)$$

2.2.2 L_p -Multipliers

We put $M_p := M_{L_p}$, $1 \leq p \leq \infty$. It is well known (e.g., [27]) that M_p is a Banach algebra with respect to pointwise multiplication and that the map $a \mapsto a(D)$ from M_p to $\mathcal{L}(L_p)$ is an algebra homomorphism. Moreover, (see Theorem 6.1.2 in [6])

$$M_1 \hookrightarrow M_p \hookrightarrow M_q \hookrightarrow M_2 = L_\infty, \quad 1 < p < q < 2, \quad (2.2.7)$$

and

$$M_{p'} = M_p, \quad 1 \leq p \leq 2, \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (2.2.8)$$

Set

$$\mathcal{M}_0 := (\{a \in L_\infty(\mathbb{R}^n); |\xi|^{|\alpha|} \partial^\alpha a \in L_\infty(\mathbb{R}^n), |\alpha| \leq [n/2] + 1\}, \|\cdot\|_{\mathcal{M}_0}),$$

where

$$\|a\|_{\mathcal{M}_0} := \max_{|\alpha| \leq [n/2] + 1} \|\xi^{|\alpha|} \partial^\alpha a\|_\infty.$$

Then, by the **Mikhlin's multiplier theorem** (cf. Theorem 7.9.5 in [21] or Theorem 6.1.6 in [6]), we have the following theorem.

Theorem 2.2.1. $\mathcal{M}_0 \hookrightarrow M_p$, $1 < p < \infty$.

2.2.3 L_1 -Multipliers

It is well known that Theorem 2.2.1 is optimal in the sense that the elements of \mathcal{M}_0 are not L_1 -multipliers, in general. The following corollary gives a simple sufficient condition for an element of L_∞ to be an L_1 -multiplier.

Consider the space

$$\mathcal{F}L_1 := (\{u \in \mathcal{S}'; \hat{u} \in L_1\}, \|u\|_{\mathcal{F}L_1} := \|\hat{u}\|_1), \quad (2.2.9)$$

and observe that, thanks to (2.2.1), $\mathcal{F}L_1$ is a well defined Banach space. Also observe that, thanks to (2.2.1) and (2.2.2), it follows from Young's inequality for convolutions that

$$\mathcal{F}L_1 \hookrightarrow M_1. \quad (2.2.10)$$

Since there is no good direct description of $\mathcal{F}L_1$ available, we have to restrict our considerations to suitable subspaces of $\mathcal{F}L_1$ to obtain useful multiplier theorems for L_1 .

Proposition 2.2.2. $W_2^{[\frac{n}{2}]+1}(\mathbb{R}^n) \hookrightarrow \mathcal{F}L_1$.

Proof It follows from (2.1.15) and Plancherel's theorem that, given $m \in \mathbb{N}$,

$$\|u\|_{m,2} = \left(\sum_{|\alpha| \leq m} \|\xi^\alpha \hat{u}\|_2^2 \right)^{1/2}, \quad u \in W_2^m(\mathbb{R}^n). \quad (2.2.11)$$

Let

$$\Lambda_1(\xi) := (1 + |\xi|^2)^{1/2}, \quad \xi \in \mathbb{R}^n, \quad (2.2.12)$$

and observe that (2.2.11) and the multinomial theorem imply

$$\|\cdot\|_{m,2} \sim \|\Lambda_1^m \mathcal{F} \cdot\|_2 \quad (2.2.13)$$

where \sim denotes equivalent norms. By the Cauchy-Schwarz inequality

$$\|u\|_{\mathcal{F}L_1} = \|\hat{u}\|_1 = \|\Lambda_1^m \hat{u} \Lambda_1^{-m}\|_1 \leq \|\Lambda_1^m \hat{u}\|_2 \|\Lambda_1^{-m}\|_2, \quad (2.2.14)$$

and $\Lambda_1^{-m} \in L_2$ iff $m > n/2$. Thus the assertion is a consequence of (2.2.13) and (2.2.14). \square

Corollary 2.2.3. $W_2^{[\frac{n}{2}]+1}(\mathbb{R}^n) \hookrightarrow M_q$, $1 \leq q \leq \infty$.

Proof Recall (2.2.7), (2.2.8) and (2.2.10), this is an immediate consequence of Proposition 2.2.2. \square

Unfortunately, the space $W_2^{[\frac{n}{2}]+1}(\mathbb{R}^n)$ is too small to be a useful space of multipliers for L_1 and L_∞ . For this reason we introduce now a Banach space \mathcal{M} — a subspace of \mathcal{M}_0 — which will turn out to be a fundamental space of multipliers for our purposes.

Set

$$\mathcal{M} := \left(\{a \in L_\infty(\mathbb{R}^n); \Lambda_1^{|\alpha|} \partial^\alpha a \in L_\infty(\mathbb{R}^n), |\alpha| \leq [n/2] + 1\}, \|\cdot\|_{\mathcal{M}} \right),$$

where

$$\|a\|_{\mathcal{M}} := \max_{|\alpha| \leq [\frac{n}{2}]+1} \|\Lambda_1^{|\alpha|} \partial^\alpha a\|_\infty,$$

and Λ_1 is defined in (2.2.12).

2.3 Besov Spaces

2.3.1 A Partition of Unity

Let $\psi \in \mathcal{D} := \mathcal{D}(\mathbb{R}^n, \mathbb{C})$ be fixed with $\text{supp}(\psi) \subset (3/2)\mathbb{B}$, where $\mathbb{B} := \mathbb{B}_{\mathbb{R}^n}$, and such that $0 \leq \psi \leq 1$ and $\psi|_{\mathbb{B}} = 1$. Put

$$\varphi_0(x) := \psi(x), \quad \varphi_1(x) := \psi(x/2) - \psi(x), \quad \varphi_k(x) := \varphi_1(2^{-k+1}x) \quad (2.3.1)$$

for $x \in \mathbb{R}^n$ and $k \in \dot{\mathbb{N}}$. Then

$$\begin{aligned} \sum_{j=0}^m \varphi_j(x) &= \psi(2^{-m}x), \quad m \in \mathbb{N}, \\ \sum_{j=0}^{\infty} \varphi_j(x) &= 1, \quad x \in \mathbb{R}^n, \\ \text{supp}(\varphi_j) &\subset \{x \in \mathbb{R}^n; 2^{j-1} \leq |x| \leq 3 \cdot 2^{j-1}\}, \quad j \in \dot{\mathbb{N}}, \end{aligned}$$

and

$$\varphi_j(x) = 1, \quad 3 \cdot 2^{j-2} \leq |x| \leq 2^j, \quad j \in \dot{\mathbb{N}}.$$

2.3.2 Definition of Besov Spaces

Given $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, the **Besov space**, $B_{p,q}^s$ of order s (and integrability orders p and q) is defined by

$$B_{p,q}^s := B_{p,q}^s(\mathbb{R}^n, \mathbb{C}) := (\{u \in \mathcal{S}' ; \|u\|_{B_{p,q}^s} < \infty\}, \|\cdot\|_{B_{p,q}^s}),$$

where

$$\|u\|_{B_{p,q}^s} := \|2^{sk} \varphi_k(D)u\|_{l_q(L_p)} = \left(\sum_{k=0}^{\infty} \left(\int_{\mathbb{R}^n} |2^{sk} (2\pi)^{-n/2} \mathcal{F}^{-1} \varphi_k * u(x)|^p dx \right)^{q/p} \right)^{1/q},$$

and where (φ_k) is the partition of unity of Subsection 2.3.1. It is well known that these Besov spaces are Banach spaces which — except for equivalent norms — are independent of the choice of ψ .

2.3.3 The Mihlin-Hörmander theorem on Besov spaces

It follows from the definition of the Besov spaces and known properties of spaces of distributions that

$$\mathcal{D} \xrightarrow{d} \mathcal{S} \hookrightarrow B_{p,1}^s \hookrightarrow B_{p,q}^s \hookrightarrow B_{p,\infty}^s \hookrightarrow B_{p,1}^t \xrightarrow{d} \mathcal{S}' \xrightarrow{d} \mathcal{D}', \quad s > t, \quad (2.3.2)$$

and that

$$\mathcal{S} \xrightarrow{d} B_{p,q}^s, \quad p \vee q < \infty,$$

where $p \vee q := \max\{p, q\}$. In particular,

$$B_{\infty, \infty}^s \doteq BUC^s, \quad s \in \mathbb{R}^+ \setminus \mathbb{N}, \quad (2.3.3)$$

where the symbol \doteq denote equivalent spaces (cf. Proposition 6 in Section V.4.1 in [27]).

It is well known that

$$M_{B_{p,q}^0} = M_{B_{p,q}^s}, \quad p, q \in [1, \infty], \quad s \in \mathbb{R}. \quad (2.3.4)$$

In order to prove **Mikhlin-Hörmander theorem** we need the following simple and well known facts about dilatations.

Given $t > 0$, define the **dilatation**, σ_t , by

$$\sigma_t \varphi(x) := \varphi(tx), \quad x \in \mathbb{R}^n, \quad \varphi \in \mathcal{S}, \quad (2.3.5)$$

and by

$$\langle \sigma_t u, \varphi \rangle := t^{-n} \langle u, \sigma_{1/t} \varphi \rangle, \quad u \in \mathcal{S}', \quad \varphi \in \mathcal{S}. \quad (2.3.6)$$

Then

$$\{\sigma_t; t > 0\} \text{ is a subgroup of } \mathcal{L}aut(\mathcal{S}) \text{ and of } \mathcal{L}aut(\mathcal{S}')$$

and

$$(\sigma_t)^{-1} = \sigma_{1/t}. \quad (2.3.7)$$

Furthermore,

$$\partial^\alpha \sigma_t = t^{|\alpha|} \sigma_t \partial^\alpha, \quad \alpha \in \mathbb{N}^n, \quad (2.3.8)$$

and

$$\mathcal{F} \sigma_t = t^{-n} \sigma_{1/t} \mathcal{F}. \quad (2.3.9)$$

Lastly,

$$\|\sigma_t u\|_p = t^{-n/p} \|u\|_p, \quad 1 \leq p \leq \infty. \quad (2.3.10)$$

Theorem 2.3.1 (Mikhlin-Hörmander Theorem).

$$\mathcal{M} \hookrightarrow M_{B_{p,q}^s}, \quad p, q \in [1, \infty], \quad s \in \mathbb{R}.$$

Proof Fix $\chi_j \in \mathcal{D}$, $j = 0, 1$, such that $\text{supp}(\chi_0) \subset 2\mathbb{B}$ and $\chi_0|_{(3/2)\mathbb{B}} = 1$, and such that $\text{supp}(\chi_1) \subset 4\mathbb{B} \setminus (1/2)\mathbb{B}$ and $\chi_1|_{(3\mathbb{B} \setminus \mathbb{B})} = 1$. Moreover, put

$$\chi_k := \sigma_{2^{-k+1}} \chi_1, \quad k \in \dot{\mathbb{N}}.$$

Given $a \in \mathcal{M}$, since $\varphi_k \chi_k = \varphi_k$ for $k \in \mathbb{N}$, then

$$\begin{aligned} \varphi_k(D)a(D)u &= \mathcal{F}^{-1} \varphi_k \mathcal{F} \mathcal{F}^{-1} a \mathcal{F} u = \mathcal{F}^{-1} \varphi_k a \mathcal{F} u = \mathcal{F}^{-1} a \chi_k \varphi_k \mathcal{F} u \\ &= \mathcal{F}^{-1} \chi_k a \mathcal{F} \mathcal{F}^{-1} \varphi_k \mathcal{F} u = \mathcal{F}^{-1} \chi_k a \mathcal{F} \varphi_k(D)u, \quad u \in \mathcal{S}', \end{aligned} \quad (2.3.11)$$

where φ_k has been defined in (2.3.1). Since $\chi_k a = (\sigma_{2^{-k+1}} \chi_1) a = \sigma_{2^{-k+1}} (\chi_1 \sigma_{2^{k-1}} a)$ for $k \in \dot{\mathbb{N}}$, it follows from (2.3.11), (2.3.9) and (2.2.2) that

$$\begin{aligned} \varphi_k(D)a(D)u &= \mathcal{F}^{-1} \{ \sigma_{2^{-k+1}} (\chi_1 \sigma_{2^{k-1}} a) \} \mathcal{F} \varphi_k(D)u \\ &= \mathcal{F}^{-1} \sigma_{2^{-k+1}} \{ \chi_1 (\sigma_{2^{k-1}} a) (\sigma_{2^{k-1}} \mathcal{F} \varphi_k(D)u) \} \\ &= 2^{(k-1)n} \sigma_{2^{k-1}} \left\{ \mathcal{F}^{-1} (\chi_1 a_k (2^{(1-k)n} \mathcal{F} (\sigma_{2^{-k+1}} (\varphi_k(D)u)))) \right\} \\ &= \sigma_{2^{k-1}} \mathcal{F}^{-1} \chi_1 a_k \mathcal{F} \sigma_{2^{-k+1}} (\varphi_k(D)u) \\ &= \sigma_{2^{k-1}} (\chi_1 a_k)(D) \sigma_{2^{-k+1}} (\varphi_k(D)u) \end{aligned} \quad (2.3.12)$$

for $k \in \dot{\mathbb{N}}$, where

$$a_k := \sigma_{2^{k-1}} a.$$

Observe that $\chi_0, \chi_1 \in \mathcal{D}$ and $a \in \mathcal{M}$, then

$$\chi_0 a, \chi_1 a_k \in W_2^m, \quad k \in \dot{\mathbb{N}},$$

where $m := [n/2] + 1$. Thus we deduce from (2.3.12), Corollary 2.2.3 and (2.3.10) that

$$\begin{aligned} \|\varphi_k(D)a(D)u\|_p &= 2^{(1-k)n/p} \|(\chi_1 a_k)(D) \sigma_{2^{-k+1}} (\varphi_k(D)u)\|_p \\ &\leq c 2^{(1-k)n/p} \|\chi_1 a_k\|_{m,2} \|\sigma_{2^{-k+1}} (\varphi_k(D)u)\|_p \\ &= c \|\chi_1 a_k\|_{m,2} \|\varphi_k(D)u\|_p \end{aligned} \quad (2.3.13)$$

for $1 \leq p \leq \infty$ and $k \in \dot{\mathbb{N}}$. Similarly,

$$\|\varphi_0(D)a(D)u\|_p \leq c \|\chi_0 a\|_{m,2} \|\varphi_0(D)u\|_p, \quad 1 \leq p \leq \infty. \quad (2.3.14)$$

It follows from (2.3.8) that

$$\partial^\alpha a_k = \partial^\alpha (\sigma_{2^{k-1}} a) = 2^{(k-1)|\alpha|} \sigma_{2^{k-1}} (\partial^\alpha a), \quad \alpha \in \mathbb{N}^n.$$

Thus $a \in \mathcal{M}$ implies

$$\begin{aligned} \operatorname{ess\,sup}_{1/2 \leq |x| \leq 4} |\partial^\alpha a_k(x)| &= 2^{(k-1)|\alpha|} \operatorname{ess\,sup}_{1/2 \leq |x| \leq 4} \left| \partial^\alpha a(y) \Big|_{y=2^{k-1}x} \right| \\ &\leq 2^{(k-1)|\alpha|} \|a\|_{\mathcal{M}} \operatorname{ess\,sup}_{1/2 \leq |x| \leq 4} (1 + |2^{k-1}x|^2)^{-|\alpha|/2} \\ &\leq 2^{(k-1)|\alpha|} \|a\|_{\mathcal{M}} 2^{-(k-2)|\alpha|} \\ &= 2^{|\alpha|} \|a\|_{\mathcal{M}} \end{aligned}$$

for $|\alpha| \leq m$ and $k \in \dot{\mathbb{N}}$. By means of these estimates, Leibniz' rule, and (2.3.13), (2.3.14), we deduce that

$$\|\varphi_k(D)a(D)u\|_p \leq c \|a\|_{\mathcal{M}} \|\varphi_k(D)u\|_p, \quad 1 \leq p \leq \infty, \quad k \in \dot{\mathbb{N}}.$$

Now it follows from the definition of the norm of $B_{p,q}^0$ that

$$\|a(D)u\|_{B_{p,q}^0} \leq c \|a\|_{\mathcal{M}} \|u\|_{B_{p,q}^0}, \quad 1 \leq p, q \leq \infty, \quad u \in B_{p,q}^0.$$

This means $\mathcal{M} \hookrightarrow M_{B_{p,q}^0}$. Thanks to (2.3.4), we get that

$$\mathcal{M} \hookrightarrow M_{B_{p,q}^s}, \quad p, q \in [1, \infty], \quad s \in \mathbb{R}.$$

□

Corollary 2.3.2. $\mathcal{M} \hookrightarrow M_{BUC^s}$, $s \in \mathbb{R}^+ \setminus \mathbb{N}$.

Proof This is a direct consequence of (2.3.3) and Theorem 2.3.1. □

2.4 Interpolation Theory

In this section we collect some facts on interpolation theory. Our representation follows closely the monograph [24] of Lunardi. If X, Y, D are Banach spaces such that

$$D \hookrightarrow Y \hookrightarrow X,$$

we say that Y is an **intermediate space** between X and D . If, in addition, for every linear operator $T \in \mathcal{L}(X)$ such that $T|_D \in \mathcal{L}(D)$ it holds $T|_Y \in \mathcal{L}(Y)$, then Y is called **interpolation space** between X and D .

2.4.1 Interpolatory Inclusion

Let X, D be Banach space, with norms $\|\cdot\|, \|\cdot\|_D$ respectively, and assume that D is continuously embedded in X .

Definition 2.4.1. Let $0 \leq \alpha \leq 1$. A Banach space Y such that $D \hookrightarrow Y \hookrightarrow X$ is said to belong to the class J_α between X and D if there is a constant $c > 0$ such that

$$\|x\|_Y \leq c\|x\|^{1-\alpha}\|x\|_D^\alpha, \quad \forall x \in D.$$

In this case we write $Y \in J_\alpha(X, D)$.

Some important examples are given in the next propositions.

Proposition 2.4.2. *Let k, m be positive integers such that $k < m$. Then:*

- (i) $BC^k(\mathbb{R}, X)$ belongs to the class $J_{k/m}$ between $BC(\mathbb{R}, X)$ and $BC^m(\mathbb{R}, X)$;
- (ii) $BC^k(\mathbb{R}^n)$ belongs to the class $J_{k/m}$ between $BC(\mathbb{R}^n)$ and $BC^m(\mathbb{R}^n)$; $BC^k(\mathbb{R}_+^n)$ belongs to the class $J_{k/m}$ between $BC(\mathbb{R}_+^n)$ and $BC^m(\mathbb{R}_+^n)$;
- (iii) if Ω is an open set in \mathbb{R}^n with uniformly C^m boundary, then $BC^k(\overline{\Omega})$ belongs to the class $J_{k/m}$ between $BC(\overline{\Omega})$ and $BC^m(\overline{\Omega})$.

Proof Let us prove statement (i). It is sufficient to show that for every $m \in \mathbb{N}$ there is $c_m > 0$ such that if $BC^m(\mathbb{R}, X)$ then

$$\|f^{(k)}\|_\infty \leq c_m(\|f\|_\infty)^{1-k/m}(\|f^{(m)}\|_\infty)^{k/m}, \quad \forall k = 1, \dots, m-1. \quad (2.4.1)$$

First we prove that (2.4.1) holds for $m = 2$ and $k = 1$. From the inequality

$$\|f(x+h) - f(x) - f'(x)h\| \leq \frac{1}{2}\|f''\|_\infty h^2, \quad \forall x \in \mathbb{R}, h > 0,$$

we get

$$\|f'(x)\| \leq \frac{\|f(x+h) - f(x)\|}{h} + \frac{1}{2}\|f''\|_\infty h, \quad \forall x \in \mathbb{R}, h > 0,$$

so that

$$\|f'\|_\infty \leq \frac{2\|f\|_\infty}{h} + \frac{1}{2}\|f''\|_\infty h, \quad \forall h > 0.$$

Taking the minimum on h over $(0, +\infty)$ we get

$$\|f'\|_\infty \leq 2(\|f\|_\infty)^{1/2}(\|f''\|_\infty)^{1/2}, \quad \forall f \in BC^2(\mathbb{R}, X). \quad (2.4.2)$$

Assume now by recurrence that (2.4.1) holds for some $m > 2$. Then for every $f \in BC^{m+1}(\mathbb{R}, X)$ we get

$$\begin{aligned} \|f'\|_\infty &\leq c_m(\|f\|_\infty)^{1-1/m}(\|f^{(m)}\|_\infty)^{1/m} \\ &\leq c_m(\|f\|_\infty)^{1-1/m} \left[c_m(\|f'\|_\infty)^{1-\frac{m-1}{m}} (\|f^{(m+1)}\|_\infty)^{\frac{m-1}{m}} \right]^{1/m}. \end{aligned}$$

It follows that

$$\|f'\|_\infty \leq c_m^{\frac{m+1}{m}} (\|f\|_\infty)^{1-\frac{1}{m+1}} (\|f^{(m+1)}\|_\infty)^{\frac{1}{m+1}},$$

and, for $k = 2, \dots, m$,

$$\begin{aligned} \|f^{(k)}\|_\infty = \|(f')^{(k-1)}\|_\infty &\leq c_m(\|f'\|_\infty)^{1-(k-1)/m} (\|f^{(m+1)}\|_\infty)^{(k-1)/m} \\ &\leq c_m^{\frac{1+m-k+1}{m-1}} (\|f\|_\infty)^{1-\frac{k}{m+1}} (\|f^{(m+1)}\|_\infty)^{\frac{k}{m+1}}. \end{aligned}$$

Therefore, (2.4.1) holds with m replaced by $m+1$, and statement (i) is proved.

Statement (ii) can be shown similarly, replacing h by he_i , where e_i is the vector whose k -th component is 0 if $k \neq i$, 1 if $k = i$.

Statement (iii) follows from (ii) by localizing and straightening the boundary. \square

Similar arguments lead to the following generalization.

Proposition 2.4.3. *Let $0 < \theta < \alpha$. Then:*

- (i) $BUC^\theta(\mathbb{R}, X)$ belongs to the class $J_{\theta/\alpha}$ between $BC(\mathbb{R}, X)$ and $BUC^\alpha(\mathbb{R}, X)$;
- (ii) $BUC^\theta(\mathbb{R}^n)$ belongs to the class $J_{\theta/\alpha}$ between $BC(\mathbb{R}^n)$ and $BUC^\alpha(\mathbb{R}^n)$; $BUC^\theta(\mathbb{R}_+^n)$ belongs to the class $J_{\theta/\alpha}$ between $BC(\mathbb{R}_+^n)$ and $BUC^\alpha(\mathbb{R}_+^n)$;
- (iii) if Ω is an open set in \mathbb{R}^n with uniformly C^α boundary, then $BUC^\theta(\overline{\Omega})$ belongs to the class $J_{\theta/\alpha}$ between $BC(\overline{\Omega})$ and $BUC^\alpha(\overline{\Omega})$.

Now we are able to state some interpolatory inclusions between spaces of functions defined in an interval I .

Proposition 2.4.4. *Let $X_\alpha \in J_\alpha(X, D)$. Then*

- (i) $B(I, D) \cap Lip(I, X) \hookrightarrow BUC^{1-\alpha}(I, X_\alpha)$,
- (ii) $B(I, D) \cap BUC^\theta(I, X) \hookrightarrow BUC^{\theta(1-\alpha)}(I, X_\alpha), \quad \forall \theta \in (0, 1)$,
- (iii) $B(I, D) \cap BC(I, X) \hookrightarrow BC(I, X_\alpha)$.

Proof Let us prove statement (i). For every $u \in B(I, D) \cap Lip(I, X)$ and for $s, t \in I$ we have

$$\begin{aligned} \|u(t) - u(s)\|_{X_\alpha} &\leq c\|u(t) - u(s)\|^{1-\alpha}\|u(t) - u(s)\|_D^\alpha \\ &\leq c[u]_{Lip(I, X)}^{1-\alpha}|t - s|^{1-\alpha}(2\|u\|_{B(I, D)})^\alpha, \end{aligned}$$

so that u is uniformly $(1 - \alpha)$ -Hölder continuous with values in X_α , and

$$\begin{aligned} \|u\|_{BUC^{1-\alpha}(I, X_\alpha)} &\leq c\|u\|_{B(I, X)}^{1-\alpha}\|u\|_{B(I, D)}^\alpha + 2^\alpha c[u]_{Lip(I, X)}^{1-\alpha}\|u\|_{B(I, D)}^\alpha, \\ &\leq (2^\alpha + 1)c\|u\|_{Lip(I, X)}^{1-\alpha}\|u\|_{B(I, D)}^\alpha. \end{aligned}$$

The proof of statements (ii) and (iii) is similar. \square

2.4.2 The K -method

Let X, Y be Banach spaces, with $Y \hookrightarrow X$, and let $c > 0$ be such that

$$\|y\|_X \leq c\|y\|_Y, \quad \forall y \in Y.$$

We describe briefly the construction of a family of intermediate spaces between X and Y , called **real interpolation spaces**, and denoted by $(X, Y)_{\theta, p}$, $(X, Y)_\theta$, with $0 < \theta \leq 1$, $1 \leq p \leq \infty$, see also [6, 29]. We follow the so-called K -method, and we always set $1/\infty = 0$.

Definition 2.4.5. For every $x \in X$ and $t > 0$, set

$$K(t, x, X, Y) = \inf_{x=a+b, a \in X, b \in Y} (\|a\|_X + t\|b\|_Y). \quad (2.4.3)$$

If there is no danger of confusion, we shall write $K(t, x)$ instead of $K(t, x, X, Y)$.

From Definition 2.4.5 it follows immediately that for every $t > 0$ and $x \in X$ we have

$$\begin{cases} (i) & \min\{1, t\}K(1, x) \leq K(t, x) \leq \max\{1, t\}K(1, x), \\ (ii) & K(t, x) \leq \|x\|_X. \end{cases} \quad (2.4.4)$$

Now we define a family of Banach spaces by means of the function K .

Definition 2.4.6. Let $0 < \theta \leq 1$, $1 \leq p \leq \infty$, and set

$$\begin{cases} (X, Y)_{\theta, p} = \{x \in X; t \mapsto t^{-\theta-1/p}K(t, x, X, Y) \in L_p(0, +\infty)\}, \\ \|x\|_{(X, Y)_{\theta, p}} = \|t^{-\theta-1/p}K(t, x, X, Y)\|_{L_p(0, +\infty)}; \end{cases} \quad (2.4.5)$$

$$(X, Y)_\theta = \{x \in X; \lim_{t \rightarrow 0} t^{-\theta}K(t, x, X, Y) = 0\}. \quad (2.4.6)$$

The mapping $x \mapsto \|x\|_{(X,Y)_{\theta,p}}$ is easily seen to be a norm in $(X,Y)_{\theta,p}$. Where there is no danger of confusion, we shall write $\|x\|_{\theta,p}$ instead of $\|x\|_{(X,Y)_{\theta,p}}$.

Since $t \mapsto K(t,x)$ is bounded, it is clear that only the behavior near $t = 0$ of $t^{-\theta}K(t,x)$ plays a role in the definition of $(X,Y)_{\theta,p}$ and of $(X,Y)_{\theta}$.

For $\theta = 1$, from the first inequality in (2.4.4)(i) we get

$$(X,Y)_1 = (X,Y)_{1,p} = \{0\}, \quad p < \infty.$$

Therefore, from now on we shall consider the case $(\theta,p) \in (0,1) \times [1,+\infty]$ and $(\theta,p) = (1,\infty)$.

If $X = Y$, then $K(t,x) = \min\{t,1\}\|x\|$. Therefore, as one can expect, $(X,X)_{\theta,p} = (X,X)_{1,\infty} = X$ for $0 < \theta < 1$, $1 \leq p \leq \infty$, and

$$\begin{aligned} \|x\|_{(X,X)_{\theta,p}} &= \left(\frac{1}{p\theta(1-\theta)} \right)^{1/p} \|x\|_X, \quad 0 < \theta < 1, p < \infty, \\ \|x\|_{(X,X)_{\theta,\infty}} &= \|x\|_X, \quad 0 < \theta \leq 1. \end{aligned}$$

Some inclusion properties are stated below.

Proposition 2.4.7. *For $0 < \theta < 1$, $1 \leq p_1 \leq p_2 \leq \infty$ we have*

$$Y \subset (X,Y)_{\theta,p_1} \subset (X,Y)_{\theta,p_2} \subset (X,Y)_{\theta} \subset (X,Y)_{\theta,\infty} \subset \bar{Y}. \quad (2.4.7)$$

For $0 < \theta_1 < \theta_2 \leq 1$ we have

$$(X,Y)_{\theta_2,\infty} \subset (X,Y)_{\theta_1,1}. \quad (2.4.8)$$

Proof From the inequality $K(t,x) \leq \min\{c,t\}\|x\|_Y$ for every $x \in Y$, it follows immediately that Y is continuously embedded in $(X,Y)_{1,\infty}$ and in $(X,Y)_{\theta,p}$ for $0 < \theta < 1$, $1 \leq p \leq \infty$.

Let us show that $(X,Y)_{\theta,\infty}$ is contained in \bar{Y} and it is continuously embedded in X . For $x \in (X,Y)_{\theta,\infty}$ and for every $n \in \mathbb{N}$ there are $a_n \in X, b_n \in Y$ such that $x = a_n + b_n$, and

$$n^\theta(\|a_n\|_X + \frac{1}{n}\|b_n\|_Y) \leq 2\|x\|_{\theta,\infty}.$$

In particular, $\|x - b_n\|_X = \|a_n\|_X \leq 2\|x\|_{\theta,\infty}n^{-\theta}$, so that the sequence $\{b_n\}$ goes to x in X as $n \rightarrow \infty$. This implies that $(X,Y)_{\theta,\infty}$ is contained in \bar{Y} . Moreover, from the inequality

$$\|x\|_X \leq \|a\|_X + \|b\|_X \leq \|a\|_X + c\|b\|_Y, \quad \text{if } x = a + b,$$

we get

$$\|x\|_X \leq K(c,x) \leq c^\theta\|x\|_{\theta,\infty}, \quad \forall x \in (X,Y)_{\theta,\infty},$$

so that $(X,Y)_{\theta,\infty}$ is continuously embedded in X .

The inclusion $(X,Y)_{\theta} \subset (X,Y)_{\theta,\infty}$ is trivial, since $K(\cdot,x)$ is bounded.

Let us show that $(X, Y)_{\theta, p}$ is contained in $(X, Y)_\theta$ and it is continuously embedded in $(X, Y)_{\theta, \infty}$ for $p < \infty$. Note that $K(\cdot, x)$ satisfies

$$K(t, x) \leq \frac{t}{s} K(s, x) \quad \text{for } x \in X, 0 < s < t.$$

Therefore, for each $x \in (X, Y)_{\theta, p}$ and $t > 0$,

$$\begin{aligned} t^{1-\theta} K(t, x) &= [(1-\theta)p]^{1/p} \left(\int_0^t s^{(1-\theta)p-1} ds \right)^{1/p} K(t, x) \\ &\leq [(1-\theta)p]^{1/p} \left(\int_0^t s^{-\theta p-1} t^p K(s, x)^p ds \right)^{1/p}, \end{aligned}$$

so that

$$t^{-\theta} K(t, x) \leq [(1-\theta)p]^{1/p} \left(\int_0^t s^{-\theta p-1} K(s, x)^p ds \right)^{1/p}.$$

Letting $t \rightarrow 0$, it follows that $x \in (X, Y)_\theta$. The same inequality yields

$$\|x\|_{\theta, \infty} \leq [(1-\theta)p]^{1/p} \|x\|_{\theta, p}. \quad (2.4.9)$$

Let us prove that $(X, Y)_{\theta, p_1} \subset (X, Y)_{\theta, p_2}$ for $p_1 < p_2$. For $x \in (X, Y)_{\theta, p_1}$ we have

$$\begin{aligned} \|x\|_{\theta, p_2} &= \left(\int_0^{+\infty} t^{-\theta p_2-1} K(t, x)^{p_2} dt \right)^{1/p_2} \\ &\leq \left(\int_0^{+\infty} t^{-\theta p_1-1} K(t, x)^{p_1} dt \right)^{1/p_2} \left(\sup_{t>0} t^{-\theta} K(t, x) \right)^{(p_2-p_1)/p_2} \\ &= (\|x\|_{\theta, p_1})^{p_1/p_2} (\|x\|_{\theta, \infty})^{1-p_1/p_2}, \end{aligned}$$

and using (2.4.9) we find

$$\|x\|_{\theta, p_2} \leq [(1-\theta)p_1]^{1/p_1-1/p_2} \|x\|_{\theta, p_1}. \quad (2.4.10)$$

Let us prove that (2.4.8) holds. If $0 < \theta_1 < \theta_2 \leq 1$ and $x \in (X, Y)_{\theta_2, \infty}$, we have

$$\begin{aligned} \|x\|_{\theta_1, 1} &= \int_0^1 t^{-\theta_1-1} K(t, x) dt + \int_1^{+\infty} t^{-\theta_1-1} K(t, x) dt \\ &\leq \int_0^1 t^{-\theta_1-1} \|x\|_{\theta_2, \infty} t^{\theta_2} dt + \int_1^{+\infty} t^{-\theta_1-1} \|x\|_X dt \\ &\leq \frac{1}{\theta_2 - \theta_1} \|x\|_{\theta_2, \infty} + \frac{1}{\theta_1} \|x\|_X. \end{aligned} \quad (2.4.11)$$

The statement is so completely proved. \square

Proposition 2.4.8. $(X, Y)_{\theta, p}$ is a Banach space.

Proof Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $(X, Y)_{\theta, p}$. Due to the continuous embedding of $(X, Y)_{\theta, p}$ in X , $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X too, so that it converges to an element $x \in X$.

Let us estimate $\|x_n - x\|_{\theta, p}$. Fix $\varepsilon > 0$, and let $\|x_n - x_m\|_{\theta, p} \leq \varepsilon$ for $n, m \geq n_\varepsilon$. Since $y \mapsto K(t, y)$ is a norm, for every $n, m \in \mathbb{N}$ and $t > 0$ we have

$$t^{-\theta} K(t, x_n - x) \leq t^{-\theta} K(t, x_n - x_m) + t^{-\theta} \|x_m - x\|_X. \quad (2.4.12)$$

Let $p = \infty$. Then for every $t > 0$ and $n, m \geq n_\varepsilon$

$$t^{-\theta} K(t, x_n - x) \leq \varepsilon + t^{-\theta} \|x_m - x\|_X.$$

Letting $m \rightarrow +\infty$, we find $t^{-\theta} K(t, x_n - x) \leq \varepsilon$ for every $t > 0$. This implies that $x \in (X, Y)_{\theta, \infty}$ and that $x_n \rightarrow x$ in $(X, Y)_{\theta, \infty}$.

Let now $p < \infty$. Then

$$\|x_n - x\|_{\theta, p} = \lim_{\delta \rightarrow 0} \left(\int_{\delta}^{1/\delta} t^{-\theta p - 1} K(t, x_n - x)^p dt \right)^{1/p}.$$

Due again to (2.4.12), for every $\delta \in (0, 1)$ we get, for $n, m \geq n_\varepsilon$,

$$\begin{aligned} & \left(\int_{\delta}^{1/\delta} t^{-\theta p - 1} K(t, x_n - x)^p dt \right)^{1/p} \\ & \leq \|x_n - x_m\|_{\theta, p} + \|x_m - x\|_X \left(\int_{\delta}^{1/\delta} t^{-\theta p - 1} dt \right)^{1/p} \\ & \leq \varepsilon + \|x_m - x\|_X \left(\frac{1}{\theta p \delta^{\theta p}} \right)^{1/p}. \end{aligned}$$

Letting $m \rightarrow \infty$ and then $\delta \rightarrow 0$ we get $x \in (X, Y)_{\theta, p}$ and $x_n \rightarrow x$ in $(X, Y)_{\theta, p}$. \square

Corollary 2.4.9. *For $0 < \theta \leq 1$, $(X, Y)_\theta$ is a Banach space, endowed with the norm of $(X, Y)_{\theta, \infty}$.*

Proof It is easy to see that $(X, Y)_\theta$ is a closed subspace of $(X, Y)_{\theta, \infty}$. By Proposition 2.4.8, $(X, Y)_{\theta, \infty}$ is complete, then $(X, Y)_\theta$ is also complete. \square

The spaces $(X, Y)_{\theta, p}$ and $(X, Y)_\theta$ enjoy an important interpolation property, stated in the next proposition. It implies that they are in fact interpolation spaces.

Proposition 2.4.10. *Let X_1, X_2, Y_1, Y_2 be Banach spaces, such that Y_i is continuously embedded in X_i , for $i = 1, 2$. If $T \in \mathcal{L}(X_1, X_2) \cap \mathcal{L}(Y_1, Y_2)$, then $T \in \mathcal{L}((X_1, Y_1)_{\theta, p}, (X_2, Y_2)_{\theta, p}) \cap \mathcal{L}((X_1, Y_1)_\theta, (X_2, Y_2)_\theta)$ for every $\theta \in (0, 1)$ and $p \in [1, \infty]$, and for $(\theta, p) = (1, \infty)$. Moreover,*

$$\|T\|_{\mathcal{L}((X_1, Y_1)_{\theta, p}, (X_2, Y_2)_{\theta, p})} \leq (\|T\|_{\mathcal{L}(X_1, X_2)})^{1-\theta} (\|T\|_{\mathcal{L}(Y_1, Y_2)})^\theta. \quad (2.4.13)$$

Proof If $T = 0$ the statement is trivial, so that we can assume that $T \neq 0$. Let $x \in (X_1, Y_1)_{\theta, p}$, then for every $a \in X_1, b \in Y_1$ such that $x = a + b$ and for every $t > 0$ we have

$$\|Ta\|_{X_2} + t\|Tb\|_{Y_2} \leq \|T\|_{\mathcal{L}(X_1, X_2)} \left(\|a\|_{X_1} + t \frac{\|T\|_{\mathcal{L}(Y_1, Y_2)}}{\|T\|_{\mathcal{L}(X_1, X_2)}} \|b\|_{Y_1} \right),$$

so that

$$K(t, Tx, X_2, Y_2) \leq \|T\|_{\mathcal{L}(X_1, X_2)} K \left(t \frac{\|T\|_{\mathcal{L}(Y_1, Y_2)}}{\|T\|_{\mathcal{L}(X_1, X_2)}}, x, X_1, Y_1 \right). \quad (2.4.14)$$

Setting $s = t \frac{\|T\|_{\mathcal{L}(Y_1, Y_2)}}{\|T\|_{\mathcal{L}(X_1, X_2)}}$ we get $Tx \in (X_2, Y_2)_{\theta, p}$, and

$$\|Tx\|_{(X_2, Y_2)_{\theta, p}} \leq \|T\|_{\mathcal{L}(X_1, X_2)} \left(\frac{\|T\|_{\mathcal{L}(Y_1, Y_2)}}{\|T\|_{\mathcal{L}(X_1, X_2)}} \right)^\theta \|x\|_{(X_1, Y_1)_{\theta, p}},$$

and (2.4.13) follows. From (2.4.14) it follows also that

$$\lim_{t \rightarrow 0} t^{-\theta} K(t, x, X_1, Y_1) = 0 \Rightarrow \lim_{t \rightarrow 0} t^{-\theta} K(t, Tx, X_2, Y_2) = 0,$$

that is, T maps $(X_1, Y_1)_\theta$ into $(X_2, Y_2)_\theta$. \square

Corollary 2.4.11. *For $0 < \theta < 1$, $1 \leq p \leq \infty$ and for $(\theta, p) = (1, \infty)$ there is $c(\theta, p)$ such that*

$$\|y\|_{(X, Y)_{\theta, p}} \leq c(\theta, p) \|y\|_X^{1-\theta} \|y\|_Y^\theta, \quad \forall y \in Y. \quad (2.4.15)$$

Proof Set $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, according to the fact that X is a real or a complex Banach space. Let $y \in Y$, and define $T : \mathbb{K} \rightarrow X$, by $T(\lambda) = \lambda y$ for each $\lambda \in \mathbb{K}$. Then $\|T\|_{\mathcal{L}(\mathbb{K}, X)} = \|y\|_X$, $\|T\|_{\mathcal{L}(\mathbb{K}, Y)} = \|y\|_Y$, and $\|T\|_{\mathcal{L}(\mathbb{K}, (X, Y)_{\theta, p})} = \|y\|_{(X, Y)_{\theta, p}}$. The statement follows now from Proposition 2.4.10, through the equality $(\mathbb{K}, \mathbb{K})_{\theta, p} = \mathbb{K}$. \square

The statement of Corollary 2.4.11 can be rephrased saying that every $(X, Y)_{\theta, p}$ belongs to $J_\theta(X, Y)$. In particular, $(X, Y)_{\theta, 1}$ belongs to $J_\theta(X, Y)$. We will see later (Proposition 2.4.17) that in fact a space E belongs to the class $J_\theta(X, Y)$ if and only if $(X, Y)_{\theta, 1}$ is continuously embedded in E .

2.4.3 The trace method

We describe now another construction of the real interpolation spaces, which is one of the most common in the literature, see [23], and which will be useful for proving other properties.

Definition 2.4.12. For $0 \leq \theta < 1$ and $1 \leq p \leq \infty$ set

$$\begin{aligned} V(p, \theta, Y, X) &= \left\{ u : \mathbb{R}_+ \rightarrow X; t \mapsto u_\theta(t) = t^{\theta-1/p} u(t) \in L_p(0, +\infty; Y), \right. \\ &\quad \left. t \mapsto v_\theta(t) = t^{\theta-1/p} u'(t) \in L_p(0, +\infty; X) \right\}, \\ \|u\|_{V(p, \theta, Y, X)} &= \|u_\theta\|_{L_p(0, +\infty; Y)} + \|v_\theta\|_{L_p(0, +\infty; X)}. \end{aligned} \quad (2.4.16)$$

Moreover, for $p = +\infty$ we define a subspace of $V(\infty, \theta, Y, X)$, by

$$V_0(\infty, \theta, Y, X) = \left\{ u \in V(\infty, \theta, Y, X); \lim_{t \rightarrow 0} \|t^\theta u(t)\|_Y = \lim_{t \rightarrow 0} \|t^\theta u'(t)\|_X = 0 \right\}. \quad (2.4.17)$$

It is not difficult to see that $V(p, \theta, Y, X)$ is a Banach space endowed with the norm $\|\cdot\|_{V(p, \theta, Y, X)}$, and that $V_0(\infty, \theta, Y, X)$ is a closed subspace of $V(\infty, \theta, Y, X)$. Moreover, if $\theta < 1$, any function belonging to $V(p, \theta, Y, X)$ has a X -valued continuous extension at $t = 0$. Indeed, for $0 < s < t$, from the equality $u(t) - u(s) = \int_s^t u'(\sigma) d\sigma$ it follows, for $1 < p < \infty$,

$$\begin{aligned} \|u(t) - u(s)\|_X &\leq \left(\int_s^t \|\sigma^{\theta-1/p} u'(\sigma)\|_X^p d\sigma \right)^{1/p} \left(\int_s^t \sigma^{-(\theta-1/p)q} d\sigma \right)^{1/q} \\ &\leq \|u\|_{V(p, \theta, Y, X)} [q(1-\theta)]^{-1/q} (t^{q(1-\theta)} - s^{q(1-\theta)})^{1/q}, \end{aligned}$$

with $q = p/(p-1)$. Arguing similarly, one sees that if $p = 1$ or $p = \infty$, then u is Lipschitz continuous (respectively, $(1-\theta)$ -Hölder continuous) near $t = 0$.

We shall use the Hardy-Young inequalities, which hold for every positive measurable function $\varphi : (0, a) \rightarrow \mathbb{R}$, $0 < a \leq \infty$, and every $\alpha > 0, p \geq 1$, (see [19], p.245–246),

$$\begin{cases} (i) & \int_0^a t^{-\alpha p} \left(\int_0^t \varphi(s) \frac{ds}{s} \right)^p \frac{dt}{t} \leq \frac{1}{\alpha^p} \int_0^a s^{-\alpha p} \varphi(s)^p \frac{ds}{s}, \\ (ii) & \int_0^a t^{\alpha p} \left(\int_0^t \varphi(s) \frac{ds}{s} \right)^p \frac{dt}{t} \leq \frac{1}{\alpha^p} \int_0^a s^{\alpha p} \varphi(s)^p \frac{ds}{s}. \end{cases} \quad (2.4.18)$$

Corollary 2.4.13. *Let u be a function such that $t \mapsto u_\theta(t) = t^{\theta-1/p} u(t)$ belongs to $L_p(0, a; X)$, with $0 < a \leq \infty$, $0 < \theta < 1$ and $1 \leq p \leq \infty$. Then also the mean value*

$$v(t) = \frac{1}{t} \int_0^t u(s) ds, \quad t > 0 \quad (2.4.19)$$

has the same property, and setting $v_\theta = t^{\theta-1/p} v(t)$ we have

$$\|v_\theta\|_{L_p(0, a; X)} \leq \frac{1}{1-\theta} \|u_\theta\|_{L_p(0, a; X)}. \quad (2.4.20)$$

Proof For $p < \infty$, according to (2.4.18)(i), we have

$$\begin{aligned} \int_0^a \|v_\theta(t)\|_X^p dt &= \int_0^a t^{-(1-\theta)p} \left\| \int_0^t u(s) ds \right\|_X^p \frac{dt}{t} \\ &\leq \int_0^a t^{-(1-\theta)p} \left(\int_0^t s \|u(s)\|_X \frac{ds}{s} \right)^p \frac{dt}{t} \\ &\leq \frac{1}{(1-\theta)^p} \int_0^a s^{-(1-\theta)p} (s \|u(s)\|_X)^p \frac{ds}{s} \\ &\leq \frac{1}{(1-\theta)^p} \|u_\theta\|_{L_p(0, a; X)}^p, \end{aligned}$$

i.e. $v_\theta(t) \in L_p(0, a; X)$, and

$$\|v_\theta\|_{L_p(0, a; X)} \leq \frac{1}{1-\theta} \|u_\theta\|_{L_p(0, a; X)}.$$

For $p = \infty$, we have

$$\begin{aligned} \|v_\theta\|_X &\leq t^{\theta-1} \left\| \int_0^t u(s) ds \right\|_X \\ &\leq t^{\theta-1} \int_0^t s^{-\theta} \|s^\theta u(s)\|_X ds \\ &\leq \frac{1}{1-\theta} \|u_\theta\|_{L_\infty(0, a; X)} \end{aligned}$$

This means $\|v_\theta\|_{L_\infty(0, a; X)} \leq \frac{1}{1-\theta} \|u_\theta\|_{L_\infty(0, a; X)}$. Thus, they complete the proof. \square

With the aid of Corollary 2.4.13, we are able to characterize the real interpolation spaces as trace spaces.

Proposition 2.4.14. *For $(\theta, p) \in (0, 1) \times [1, +\infty] \cup \{(1, \infty)\}$, $(X, Y)_{\theta, p}$ is the set of the traces at $t = 0$ of the functions in $V(p, 1 - \theta, Y, X)$, and the norm*

$$\|x\|_{\theta, p}^T = \inf \{ \|u\|_{V(p, 1-\theta, Y, X)}; x = u(0), u \in V(p, 1 - \theta, Y, X) \}$$

is an equivalent norm in $(X, Y)_{\theta, p}$. Moreover, for $0 < \theta < 1$, $(X, Y)_\theta$ is the set of the traces of $t = 0$ of the functions in $V_0(\infty, 1 - \theta, Y, X)$.

Proof Let $x \in (X, Y)_{\theta, p}$. For every $n \in \mathbb{N}$, let $a_n \in X, b_n \in Y$ be such that $a_n + b_n = x$, and

$$\|a_n\|_X + \frac{1}{n} \|b_n\|_Y \leq 2K(1/n, x).$$

For $t > 0$ set

$$u(t) = \sum_{n=1}^{\infty} b_{n+1} \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(t) = \sum_{n=1}^{\infty} (x - a_{n+1}) \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(t)$$

where χ_I is the characteristic function of the interval I , and

$$v(t) = \frac{1}{t} \int_0^t u(s) ds.$$

Thanks to Proposition 2.4.7, $(X, Y)_{\theta, p} \subset (X, Y)_{\theta, \infty}$, then $\lim_{t \rightarrow 0} K(t, x) = 0$. In particular,

$\lim_{n \rightarrow \infty} a_n = 0$ i.e. $x = \lim_{n \rightarrow \infty} b_n$, so that $x = \lim_{t \rightarrow 0} u(t) = \lim_{t \rightarrow 0} v(t)$. Moreover,

$$\begin{aligned} \|t^{1-\theta} u(t)\|_Y &\leq t^{1-\theta} \sum_{n=1}^{\infty} \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(t) 2(n+1) K(1/(n+1), x) \\ &\leq 2t^{-\theta} \sup_{n \in \mathbb{N}} \left\{ \frac{n+1}{n} \right\} K(t, x) \\ &\leq 4t^{-\theta} K(t, x), \end{aligned} \tag{2.4.21}$$

so that $t \mapsto t^{1-\theta-1/p}u(t) \in L_p(0, +\infty; Y)$. By Corollary 2.4.13, $t \mapsto t^{1-\theta-1/p}v(t)$ belongs to $L_p(0, +\infty; Y)$, and

$$\|t^{1-\theta-1/p}v\|_{L_p(0, +\infty; Y)} \leq \theta^{-1} \|t^{1-\theta-1/p}u\|_{L_p(0, +\infty; Y)} \leq 4\theta^{-1} \|x\|_{\theta, p}.$$

On the other hand,

$$v(t) = x - \frac{1}{t} \int_0^t \sum_{n=1}^{\infty} \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(s) a_{n+1} ds,$$

so that v is differentiable almost everywhere with values in X , and

$$v'(t) = \frac{1}{t^2} \int_0^t g(s) ds - \frac{1}{t} g(t),$$

where $g(t) = \sum_{n=1}^{\infty} \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(t) a_{n+1}$ is such that

$$\|g(t)\|_X \leq \sum_{n=1}^{\infty} \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(s) 2K(1/(n+1), x) \leq 2K(t, x).$$

It follows that

$$\|t^{1-\theta}v'(t)\|_X \leq t^{-\theta} \sup_{0 < s < t} \|g(s)\|_X + \|t^{-\theta}g(t)\|_X \leq 4t^{-\theta}K(t, x). \quad (2.4.22)$$

Then $t \mapsto t^{1-\theta-1/p}v'(t)$ belongs to $L_p(0, +\infty; X)$, and

$$\|t^{1-\theta-1/p}v'(t)\|_{L_p(0, +\infty; X)} \leq 4\|x\|_{\theta, p}.$$

Therefore, x is the trace at $t = 0$ of a function $v \in V(p, 1 - \theta, Y, X)$, and

$$\|x\|_{\theta, p}^T \leq 4(1 + 1/\theta)\|x\|_{\theta, p}.$$

If $x \in (X, Y)_{\theta}$, then, by (2.4.21), $\lim_{t \rightarrow 0} t^{1-\theta} \|u(t)\|_Y = 0$, so that $\lim_{t \rightarrow 0} t^{1-\theta} \|v(t)\|_Y = 0$. By (2.4.22), $\lim_{t \rightarrow 0} t^{-\theta} \|g(t)\|_X = 0$, so that $\lim_{t \rightarrow 0} t^{1-\theta} \|v'(t)\|_X = 0$. Then $v \in V_0(\infty, 1 - \theta, Y, X)$.

Conversely, let x be the trace of a function $u \in V(p, 1 - \theta, Y, X)$. Then

$$x = x - u(t) + u(t) = - \int_0^t u'(s) ds + u(t), \quad \forall t > 0,$$

so that

$$t^{-\theta}K(t, x) \leq t^{1-\theta} \left\| \frac{1}{t} \int_0^t u'(s) ds \right\|_X + t^{1-\theta} \|u(t)\|_Y. \quad (2.4.23)$$

Corollary 2.4.13 implies now that $t \mapsto t^{-\theta-1/p}K(t, x)$ belongs to $L_p(0, +\infty)$, so that $x \in (X, Y)_{\theta, p}$, and

$$\|x\|_{\theta, p} \leq \frac{1}{\theta} \|x\|_{\theta, p}^T.$$

If x is the trace of a function $u \in V_0(\infty, 1 - \theta, Y, X)$, then, by (2.4.23), $\lim_{t \rightarrow 0} t^{-\theta}K(t, x) = 0$, so that $x \in (X, Y)_{\theta}$. \square

Remark 2.4.15. By Proposition 2.4.14, if $x \in (X, Y)_{\theta, p}$ or $x \in (X, Y)_{\theta}$, then x is the trace at $t = 0$ of a function u belonging to $L_p(a, b; Y) \cap W^{1,p}(a, b; X)$ for $0 < a < b$. But it is possible to find a more regular function $v \in V(p, 1 - \theta, Y, X)$ (or $v \in V_0(\infty, 1 - \theta, Y, X)$) such that $v(0) = x$. For any $u \in V(p, 1 - \theta, Y, X)$ (or $u \in V_0(\infty, 1 - \theta, Y, X)$) such that $u(0) = x$, set

$$v(t) = \frac{1}{t} \int_0^t u(s) ds, \quad t > 0.$$

Then $v \in W^{1,p}(a, b; Y) \cap W^{2,p}(a, b; X)$ for $0 < a < b$, and $v(0) = x$. By Corollary 2.4.13, $t \mapsto t^{\theta-1/p}v(t)$ belongs to $L_p(0, +\infty; Y)$; moreover

$$v'(t) = -\frac{1}{t^2} \int_0^t (u(s) - u(t)) ds = -\frac{1}{t^2} \int_0^t ds \int_s^t u'(\sigma) d\sigma,$$

so that

$$\|v'(t)\|_X \leq \frac{1}{t} \sup_{0 < s < t} \left\| \int_s^t u'(\sigma) d\sigma \right\|_X \leq \frac{1}{t} \int_0^t \|u'(\sigma)\|_X d\sigma,$$

and again by Corollary 2.4.13, $t \mapsto t^{1-\theta-1/p}v'(t)$ belongs to $L_p(0, +\infty; X)$, and

$$\|v\|_{V(p, 1-\theta, Y, X)} \leq \frac{1}{\theta} \|u\|_{V(p, 1-\theta, Y, X)}. \quad (2.4.24)$$

Moreover, $v'(t) = (u(t) - v(t))/t$, so that $t \mapsto t^{2-\theta-1/p}v'(t)$ belongs to $L_p(0, +\infty; Y)$, and

$$\|t^{2-\theta-1/p}v'\|_{L_p(0, +\infty; Y)} \leq (1 + 1/\theta) \|t^{1-\theta-1/p}u\|_{L_p(0, +\infty; Y)}. \quad (2.4.25)$$

If $u \in V_0(\infty, 1 - \theta, Y, X)$, it is easy to see that $v \in V_0(\infty, 1 - \theta, Y, X)$, and that $\lim_{t \rightarrow 0} t^{2-\theta} \|v'(t)\|_Y = 0$. \square

By means of the trace method it is easy to prove some important density properties.

Proposition 2.4.16. *Let $0 < \theta < 1$. For $1 \leq p < \infty$, Y is dense in $(X, Y)_{\theta, p}$. For $p = \infty$, $(X, Y)_{\theta}$ is the closure of Y in $(X, Y)_{\theta, \infty}$.*

In the previous subsection we have seen that every $(X, Y)_{\theta, p}$ belongs to $J_{\theta}(X, Y)$. In particular, $(X, Y)_{\theta, 1}$ belongs to $J_{\theta}(X, Y)$. Now we can characterize all the spaces in the class $J_{\theta}(X, Y)$.

Proposition 2.4.17. *Let $0 < \theta < 1$, and let E be a Banach space such that $Y \subset E \subset X$. The following statements are equivalent:*

- (i) E belongs to the class J_{θ} between X and Y ,
- (ii) $(X, Y)_{\theta, 1} \subset E$.

Proof The implication (ii) \Rightarrow (i) is a straightforward consequence of Corollary 2.4.11, with $p = 1$. Let us show that (i) \Rightarrow (ii). For every $x \in (X, Y)_{\theta,1}$, let $u \in V(1, 1-\theta, Y, X)$ be such that $u(t) = 0$ for $t \geq 1$. $u(0) = x$, and set

$$v(t) = \frac{1}{t} \int_0^t u(s) ds.$$

Then $v(0) = x$, $v(+\infty) = 0$, so that

$$x = - \int_0^{+\infty} v'(t) dt.$$

Let c be such that $\|y\|_E \leq c\|y\|_X^{1-\theta}\|y\|_Y^\theta$ for every $y \in Y$. Then

$$\|v'(t)\|_E \leq c\|v'(t)\|_X^{1-\theta}\|v'(t)\|_Y^\theta = c\|t^{-\theta}v'(t)\|_X^{1-\theta}\|t^{1-\theta}v'(t)\|_Y^\theta.$$

By Remark 2.4.15, $t \mapsto t^{1-\theta}v'(t)$ belongs to $L_1(0, +\infty; Y)$, and $t \mapsto t^{-\theta}v'(t)$ belongs to $L_1(0, +\infty; X)$. By the Hölder inequality, v' belongs to $L_1(0, +\infty; E)$, and, by means of (2.4.24) and (2.4.25),

$$\begin{aligned} \|x\|_E &\leq \|v'(t)\|_{L_1(0, \infty; E)} \\ &\leq c(\|t^{-\theta}v'(t)\|_{L_1(0, \infty; X)})^{1-\theta}(\|t^{1-\theta}v'(t)\|_{L_1(0, \infty; Y)})^\theta \\ &\leq c\left(\frac{1}{\theta}\|u\|_{V(1, 1-\theta, Y, X)}\right)^{1-\theta}\left(\left(1 + \frac{1}{\theta}\right)\|u\|_{V(1, 1-\theta, Y, X)}\right)^\theta \\ &\leq \text{const.}\|u\|_{V(1, 1-\theta, Y, X)}. \end{aligned}$$

Since u is arbitrary and Proposition 2.4.14, we have $\|x\|_E \leq \text{const.}\|x\|_{\theta,1}$. \square

2.4.4 The Reiteration Theorem

We need some preliminaries about certain classes of intermediate spaces between X and Y . Now we define another class of intermediate spaces.

Definition 2.4.18. Let E be a Banach space such that $Y \subset E \subset X$, and let $0 \leq \theta \leq 1$. E is said to belong to the class K_θ between X and Y if there is $k > 0$ such that

$$K(t, x) \leq kt^\theta\|x\|_E, \quad \forall x \in E, t > 0.$$

In other words, E belongs to the class K_θ if and only if it is continuously embedded in $(X, Y)_{\theta, \infty}$. In this case, we write $E \in K_\theta(X, Y)$.

By Definition 2.4.18 and Proposition 2.4.17, a space E belongs to $K_\theta(X, Y) \cap J_\theta(X, Y)$ if and only if

$$(X, Y)_{\theta,1} \subset E \subset (X, Y)_{\theta, \infty}.$$

Now we are able to state the Reiteration Theorem.

Theorem 2.4.19. *Let $0 \leq \theta_0 \leq \theta_1 \leq 1$. Fix $\theta \in (0, 1)$ and set $\omega = (1 - \theta)\theta_0 + \theta\theta_1$. The following statements hold true.*

(i) *If E_i belong to the class K_{θ_i} ($i = 0, 1$) between X and Y , then*

$$(E_0, E_1)_{\theta, p} \subset (X, Y)_{\omega, p}, \quad \forall p \in [1, \infty], \quad (E_0, E_1)_{\theta} \subset (X, Y)_{\omega}.$$

(ii) *If E_i belong to the class J_{θ_i} ($i = 0, 1$) between X and Y , then*

$$(X, Y)_{\omega, p} \subset (E_0, E_1)_{\theta, p}, \quad \forall p \in [1, \infty], \quad (X, Y)_{\omega} \subset (E_0, E_1)_{\theta}.$$

Consequently, if E_i belong to $K_{\theta_i}(X, Y) \cap J_{\theta_i}(X, Y)$, then

$$(E_0, E_1)_{\theta, p} = (X, Y)_{\omega, p}, \quad \forall p \in [1, \infty], \quad (E_0, E_1)_{\theta} = (X, Y)_{\omega},$$

with equivalence of the respective norms.

Remark 2.4.20. By Proposition 2.4.7, $(X, Y)_{\theta, p}$ and $(X, Y)_{\theta}$ belong to $K_{\theta}(X, Y) \cap J_{\theta}(X, Y)$ for $0 < \theta < 1$ and $1 \leq p \leq \infty$. The Reiteration Theorem yields

$$\begin{aligned} ((X, Y)_{\theta_0, q_0}, (X, Y)_{\theta_1, q_1})_{\theta, p} &= (X, Y)_{(1-\theta)\theta_0 + \theta\theta_1, p}, \\ ((X, Y)_{\theta_0}, (X, Y)_{\theta_1, q})_{\theta, p} &= (X, Y)_{(1-\theta)\theta_0 + \theta\theta_1, p}, \\ ((X, Y)_{\theta_0, q}, (X, Y)_{\theta_1})_{\theta, p} &= (X, Y)_{(1-\theta)\theta_0 + \theta\theta_1, p}, \end{aligned}$$

for $0 < \theta_0, \theta_1 < 1$, $1 \leq p, q \leq \infty$. Moreover, since X belongs to $K_0(X, Y) \cap J_0(X, Y)$, and Y belongs to $K_1(X, Y) \cap J_1(X, Y)$ between X and Y , then

$$((X, Y)_{\theta_0, q}, Y)_{\theta, p} = (X, Y)_{(1-\theta)\theta_0 + \theta, p}, \quad ((X, Y)_{\theta_0}, Y)_{\theta} = (X, Y)_{(1-\theta)\theta_0 + \theta},$$

and

$$(X, (X, Y)_{\theta_1, q})_{\theta, p} = (X, Y)_{\theta_1, p}, \quad (X, (X, Y)_{\theta_1})_{\theta} = (X, Y)_{\theta_1, \theta},$$

for $0 < \theta_0, \theta_1 < 1$, $1 \leq p, q \leq \infty$. □

Chapter 3

Preliminary II: Maximal Hölder Regularity

3.1 Analytic Semigroup

Let X be a complex Banach space, with norm $\|\cdot\|$. This section deals with the solution of an initial value problem in X ,

$$u'(t) = Au(t), \quad t > 0; \quad u(0) = x,$$

where $A : D(A) \subset X \rightarrow X$ is a linear operator, with not necessarily dense domain.

Definition 3.1.1. Let $A : D(A) \subset X \rightarrow X$ be a linear operator. The resolvent set $\rho(A)$ and the spectrum $\sigma(A)$ of A are defined by

$$\rho(A) = \{\lambda \in \mathbb{C}; \exists (\lambda I - A)^{-1} \in \mathcal{L}(X)\}, \quad \sigma(A) = \mathbb{C} \setminus \rho(A).$$

The complex numbers $\lambda \in \sigma(A)$ such that $\lambda I - A$ is not one to one are called eigenvalue. The set $\sigma_p(A)$ consisting of all eigenvalues of A is called point spectrum.

If $\lambda \in \rho(A)$, we set

$$(\lambda I - A)^{-1} = R(\lambda, A).$$

The operator $R(\lambda, A)$ is called resolvent operator or simply resolvent.

We state below some properties of the spectrum and the resolvent set.

First, it is clear that if $A : D(A) \subset X \rightarrow X$ and $B : D(B) \subset X \rightarrow X$ are linear operators such that $R(\lambda_0, A) = R(\lambda_0, B)$ for some $\lambda_0 \in \mathbb{C}$, then $D(A) = D(B)$ and $A = B$. Indeed, $D(A) = \text{Range}(R(\lambda_0, A)) = \text{Range}(R(\lambda_0, B)) = D(B)$, and for every $x \in D(A) = D(B)$ we have

$$R(\lambda_0, A)(\lambda_0 x - Ax) = x = R(\lambda_0, B)(\lambda_0 x - Bx) = R(\lambda_0, A)(\lambda_0 x - Bx),$$

so that $\lambda_0 x - Ax = \lambda_0 x - Bx$, which implies $Ax = Bx$.

Next formula is called *resolvent identity*, its verification is straightforward:

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A), \quad \forall \lambda, \mu \in \rho(A). \quad (3.1.1)$$

Proposition 3.1.2. *If $\rho(A)$ is not void, then A is closed.*

Proof Let (x_i) be a sequence in $D(A)$, and

$$\begin{aligned} x_i &\longrightarrow x && \text{in } X, \\ Ax_i &\longrightarrow y && \text{in } X. \end{aligned}$$

Since $\rho(A) \neq \emptyset$, there exists some $\lambda \in \rho(A)$ such that $R(\lambda, A) \in \mathcal{L}(X)$. Then

$$x_i = R(\lambda, A)(\lambda x_i - Ax_i) \longrightarrow R(\lambda, A)(\lambda x - y).$$

By the uniqueness, we have

$$x = R(\lambda, A)(\lambda x - y).$$

This implies $x \in \text{Range}(R(\lambda, A)) = D(A)$. Moreover, $R(\lambda, A)$ is injective, we get

$$\lambda x - Ax = \lambda x - y.$$

That is, $Ax = y$. Thus, A is closed. □

Proposition 3.1.3. *Let $\lambda_0 \in \rho(A)$. Then the ball*

$$\left\{ \lambda \in \mathbb{C}; |\lambda - \lambda_0| < \|R(\lambda_0, A)\|_{\mathcal{L}(X)}^{-1} \right\}$$

is contained in $\rho(A)$, and

$$\begin{aligned} R(\lambda, A) &= \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n R^{n+1}(\lambda_0, A) \\ &= R(\lambda_0, A) [I + (\lambda - \lambda_0)R(\lambda_0, A)]^{-1}. \end{aligned} \quad (3.1.2)$$

Therefore, the resolvent set $\rho(A)$ is open in \mathbb{C} and $\lambda \mapsto R(\lambda, A)$ is analytic in $\rho(A)$.

Definition 3.1.4. A is said to be **sectorial** if there are constants $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$ and $M > 0$ such that

$$\left\{ \begin{array}{l} (i) \quad \rho(A) \supset S_{\theta, \omega} = \{\lambda \in \mathbb{C}; \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}, \\ (ii) \quad \|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|} \quad \forall \lambda \in S_{\theta, \omega}. \end{array} \right. \quad (3.1.3)$$

The domain $D(A)$, endowed with the graph norm

$$\|x\|_{D(A)} = \|x\| + \|Ax\|,$$

is a Banach space. For every $t > 0$, (3.1.3) allows us to define a linear bounded operator e^{tA} in X , by means of the Dunford integral

$$e^{tA} = \frac{1}{2\pi i} \int_{\omega + \gamma_{r, \eta}} e^{t\lambda} R(\lambda, A) d\lambda, \quad t > 0, \quad (3.1.4)$$

where $r > 0$, $\eta \in (\frac{\pi}{2}, \theta)$, and $\gamma_{r,\eta}$ is the curve $\{\lambda \in \mathbb{C}; |\arg \lambda| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C}; |\arg \lambda| \leq \eta, |\lambda| = r\}$, oriented counterclockwise. We also set

$$e^{0A}x = x, \quad \forall x \in X. \quad (3.1.5)$$

Since the function $\lambda \mapsto e^{t\lambda}R(\lambda, A)$ is holomorphic in $S_{\theta,\omega}$, the definition of e^{tA} is independent of the choice of r and η .

Definition 3.1.5. Let $A : D(A) \subset X \rightarrow X$ be a sectorial operator. The family $\{e^{tA}; t \geq 0\}$ defined by (3.1.4)–(3.1.5) is said to be the **analytic semigroup** generated by A in X .

We recall that a family of linear operators $\{T(t)\}_{t \geq 0} \subset \mathcal{L}(X)$ is said to be a **semigroup** if

$$\begin{cases} T(t)T(s) = T(t+s), & t, s \geq 0, \\ T(0) = I. \end{cases}$$

A semigroup $T(t)$ is said to be **analytic** if the function $t \mapsto T(t)$ is analytic in $(0, +\infty)$ with values in $\mathcal{L}(X)$. It is said to be **strongly continuous** if for each $x \in X$ the function $t \mapsto T(t)x$ is continuous on $[0, +\infty)$. We shall show that if A is sectorial, then $\{e^{tA}\}_{t \geq 0}$ is analytic, so that it is strongly continuous if and only if

$$\lim_{t \rightarrow 0} e^{tA}x = x, \quad \forall x \in X.$$

We shall see later in Proposition 3.1.9 that

$$\lim_{t \rightarrow 0} e^{tA}x = x \iff x \in \overline{D(A)}.$$

Therefore, $\{e^{tA}\}_{t \geq 0}$ is strongly continuous if and only if the domain $D(A)$ is dense in X .

3.1.1 Basic properties of e^{tA}

In this subsection, the proofs are omitted(cf. [24]).

Proposition 3.1.6. (i) $e^{tA}x \in D(A^k)$ for each $t > 0$, $x \in X$, $k \in \mathbb{N}$. If $x \in D(A^k)$, then

$$A^k e^{tA}x = e^{tA}A^k x, \quad \forall t \geq 0.$$

(ii) $e^{tA}e^{sA} = e^{(t+s)A}$, $\forall t, s \geq 0$.

(iii) There are constants M_0, M_1, M_2, \dots , such that

$$\begin{cases} (a) & \|e^{tA}\|_{\mathcal{L}(X)} \leq M_0 e^{\omega t}, \quad t > 0, \\ (b) & \|t^k (A - \omega I)^k e^{tA}\|_{\mathcal{L}(X)} \leq M_k e^{\omega t}, \quad t > 0, \end{cases} \quad (3.1.6)$$

where ω is the constant of assumption (3.1.3). In particular, from (3.1.6)(b), it follows that for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there is $C_{k,\varepsilon} > 0$ such that

$$\|t^k A^k e^{tA}\|_{\mathcal{L}(X)} \leq C_{k,\varepsilon} e^{(\omega+\varepsilon)t}, \quad t > 0. \quad (3.1.7)$$

(iv) The function $t \mapsto e^{tA}$ belongs to $C^\infty((0, +\infty); \mathcal{L}(X))$, and

$$\frac{d^k}{dt^k} e^{tA} = A^k e^{tA}, \quad t > 0, \quad (3.1.8)$$

moreover, it has an analytic extension in the sector

$$S \in \left\{ \lambda \in \mathbb{C}; \lambda \neq 0, |\arg \lambda| < \theta - \frac{\pi}{2} \right\}.$$

Proposition 3.1.7. For every integer $n > \omega$, set

$$A_n : X \rightarrow X, \quad A_n = nAR(n, A). \quad (3.1.9)$$

Then $\rho(A) \subset \rho(A_n)$, and $R(\lambda, A_n) \rightarrow R(\lambda, A)$ in $\mathcal{L}(X)$ as $n \rightarrow \infty$, for every $\lambda \in \rho(A)$. Moreover, $e^{tA_n} \rightarrow e^{tA}$ in $\mathcal{L}(X)$ as $n \rightarrow \infty$, for every $t > 0$.

The family $\{A_n; n \in \mathbb{N}\}$ is said to be the Yosida approximation of A .

Corollary 3.1.8. Let X be a real Banach space, and let $A : D(A) \subset X \rightarrow X$ be a linear operator such that the complexification

$$\tilde{A} : D(\tilde{A}) = D(A) + iD(A) \rightarrow \tilde{X} = X + iX, \quad \tilde{A}(x + iy) = Ax + iAy,$$

is a sectorial operator in \tilde{X} . Then $e^{t\tilde{A}}(X) \subset X$.

The following proposition deals with the behavior of $e^{tA}x$ near $t = 0$.

Proposition 3.1.9. The following statements hold true.

(i) If $x \in \overline{D(A)}$, then $\lim_{t \rightarrow 0^+} e^{tA}x = x$. Conversely, if there exists $y = \lim_{t \rightarrow 0^+} e^{tA}x$, then $x \in \overline{D(A)}$, and $y = x$.

(ii) For every $x \in X$ and $t \geq 0$, the integral $\int_0^t e^{sA}x ds$ belongs to $D(A)$, and

$$A \int_0^t e^{sA}x ds = e^{tA}x - x.$$

If in addition the function $s \mapsto Ae^{sA}x$ belongs to $L_1(0, t; X)$, then

$$e^{tA}x - x = \int_0^t Ae^{sA}x ds.$$

(iii) If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then $\lim_{t \rightarrow 0^+} (e^{tA}x - x)/t = Ax$. Conversely, if there exists $z = \lim_{t \rightarrow 0^+} (e^{tA}x - x)/t$, then $x \in D(A)$ and $z = Ax \in \overline{D(A)}$.

(iv) If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then $\lim_{t \rightarrow 0^+} Ae^{tA}x = Ax$. Conversely, if there exists $v = \lim_{t \rightarrow 0^+} Ae^{tA}x$, then $x \in D(A)$ and $v = Ax \in \overline{D(A)}$.

If X_0 is a subspace of X , the *part of A in X_0* is defined by

$$\begin{cases} D(A_0) = \{x \in D(A) \cap X_0; Ax \in X_0\}, \\ A_0 : D(A_0) \rightarrow X_0, \quad A_0x = Ax. \end{cases}$$

Remark 3.1.10. Let $X_0 = \overline{D(A)}$, and let A_0 be the part of A in X_0 . Then $D(A_0)$ is dense in X_0 . Moreover, A_0 is sectorial so that it generates the analytic semigroup e^{tA_0} in X_0 , and we have $e^{tA_0}x = e^{tA}x$ for $x \in X_0$. Due to Proposition 3.1.9(i), e^{tA_0} is strongly continuous in X_0 . \square

3.1.2 Identification of the generator

Now we consider the problem of identifying the generator of a given analytic semigroup.

Lemma 3.1.11. *Let $A : D(A) \subset X \rightarrow X$ satisfy (3.1.3). Then for every $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > \omega$ we have*

$$R(\lambda, A) = \int_0^{+\infty} e^{-\lambda t} e^{tA} dt. \quad (3.1.10)$$

Proof Let $0 < r < \operatorname{Re} \lambda - \omega$ and $\eta \in (\pi/2, \theta)$. Then

$$\begin{aligned} \int_0^{+\infty} e^{-\lambda t} e^{tA} dt &= \frac{1}{2\pi i} \int_{\omega + \gamma_{r, \eta}} R(z, A) \int_0^{+\infty} e^{-\lambda t + zt} dt dz \\ &= \frac{1}{2\pi i} \int_{\omega + \gamma_{r, \eta}} R(z, A) (z - \lambda)^{-1} dz = R(\lambda, A). \end{aligned}$$

\square

Corollary 3.1.12. *For every $t \geq 0$, e^{tA} is one to one.*

Corollary 3.1.13. *If $A : D(A) \subset X \rightarrow X$ and $B : D(B) \subset X \rightarrow X$ are sectorial operators such that $e^{tA} = e^{tB}$ for every $t > 0$, then $D(A) = D(B)$ and $A = B$.*

Proposition 3.1.14. *Let $\{T(t); t > 0\}$ be a family of linear bounded operators such that $t \mapsto T(t)$ is differentiable with values in $\mathcal{L}(X)$, and*

(i) $T(t)T(s) = T(t+s)$, for every $t, s > 0$;

(ii) there are $\omega \in \mathbb{R}$, $M_0, M_1 > 0$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq M_0 e^{\omega t}$, $\|tT'(t)\|_{\mathcal{L}(X)} \leq M_1 e^{\omega t}$ for $t > 0$;

(iii) either (a) there is $t > 0$ such that $T(t)$ is one to one, or (b) for every $x \in X$,

$$\lim_{t \rightarrow 0} T(t)x = x.$$

Then $t \mapsto T(t)$ is analytic in $(0, +\infty)$ with values in $\mathcal{L}(X)$, and there exists a unique sectorial operator $A : D(A) \subset X \rightarrow X$ such that $T(t) = e^{tA}$ for every $t \geq 0$.

Proof The proof can be found in [24] Proposition 2.1.9. \square

3.1.3 A sufficient condition to be a sectorial operator

Proposition 3.1.15. *Let $A : D(A) \subset X \rightarrow X$ be a linear operator such that $\rho(A)$ contains a half plane $\{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq \omega\}$, and*

$$\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \leq M, \quad \operatorname{Re}\lambda \geq \omega,$$

with $\omega \in \mathbb{R}$, $M > 0$. Then A is sectorial.

Proof By Proposition 3.1.3, for every $r > 0$ the resolvent set of A contains the open ball centered at $\omega + ir$ with radius $|\omega + ir|/M$. The union of such balls and $\rho(A)$ contains the sector $S = \{\lambda \in \mathbb{C}; \lambda \neq \omega, |\arg(\lambda - \omega)| < \pi - \arctan M\}$. Moreover, for $\lambda \in V = \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda < \omega, |\arg(\lambda - \omega)| \leq \pi - \arctan 2M\}$, $\lambda = \omega + ir - \theta r/M$ with $0 < \theta \leq 1/2$, formula (3.1.2) gives

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \sum_{n=0}^{\infty} |\lambda - (\omega + ir)|^n \frac{M^{n+1}}{(\omega^2 + r^2)^{(n+1)/2}} \leq \frac{2M}{r}.$$

On the other hand, for $\lambda = \omega + ir - \theta r/M$ it holds

$$r \geq \left(\frac{1}{4M^2} + 1 \right)^{-1/2} |\lambda - \omega|,$$

so that $\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq 2M \left(\frac{1}{4M^2} + 1 \right)^{1/2} |\lambda - \omega|^{-1}$. The statement follows. \square

Definition 3.1.16. Let $D \xrightarrow{d} X$, we define the following class

$$\mathcal{H}(D, X) := \left\{ A : D \rightarrow X; \exists \omega \in \mathbb{R}, \exists M > 0 \text{ such that} \right. \\ \left. \begin{array}{l} \omega - A \in \mathcal{L}is(D, X), \quad \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq \omega\} \subset \rho(A) \\ \text{and } |\lambda - \omega| \|R(\lambda, A)\|_{\mathcal{L}(X)} \leq M, \text{ for } \operatorname{Re}\lambda \geq \omega \end{array} \right\}.$$

Given $A \in \mathcal{H}(D, X)$, we know from Proposition 3.1.9 and Proposition 3.1.15 that A generates a strongly continuous analytic semigroup on X . In fact the converse is true as well: If A generates a strongly continuous semigroup then A belongs to $\mathcal{H}(D, X)$ (see [17] Section 4.2).

To prove the next proposition, we need the following lemma.

Lemma 3.1.17. *Let X be a connected metric space, let*

$$B \in C(X, \mathcal{L}(E, F)),$$

and suppose that there exists $\beta > 0$ such that

$$\|B(x)e\|_F \geq \beta\|e\|_E, \quad x \in X, \quad e \in E.$$

Then $B(X) \cap \mathcal{L}is(E, F) \neq \emptyset$ implies $B(X) \subset \mathcal{L}is(E, F)$.

Proof The proof can be found in [4] Proposition 1.1.1. □

Proposition 3.1.18. *$\mathcal{H}(D, X)$ is open in $\mathcal{L}(D, X)$.*

Proof For any $A \in \mathcal{H}(D, X)$, there exist $\omega = \omega(A) \in \mathbb{R}$ and $M = M(A) > 0$ such that $\omega - A \in \mathcal{L}is(D, X)$,

$$\{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq \omega\} \subset \rho(A) \text{ and } |\lambda - \omega|\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq M, \text{ for } \operatorname{Re}\lambda \geq \omega.$$

For $\operatorname{Re}\lambda \geq \omega$, we have

$$x = R(\omega, A)[(\lambda - A)x + (\omega - \lambda)x] \text{ for } x \in D.$$

Then for $x \in D$, we get

$$\begin{aligned} \|x\|_D &\leq \|R(\omega, A)\|_{\mathcal{L}(X, D)}(\|(\lambda - A)x\| + |\lambda - \omega|\|x\|) \\ &\leq \|R(\omega, A)\|_{\mathcal{L}(X, D)}(1 + M)\|(\lambda - A)x\|. \end{aligned}$$

Set $K = M + \|R(\omega, A)\|_{\mathcal{L}(X, D)}(1 + M)$, we know that for $x \in D$ and $\operatorname{Re}\lambda \geq \omega$,

$$|\lambda - \omega|\|x\| + \|x\|_D \leq K\|(\lambda - A)x\|.$$

For any $B \in \mathcal{L}(D, X)$ with $\|B\|_{\mathcal{L}(D, X)} \leq \frac{1}{2K}$, we see that

$$\begin{aligned} \|(\lambda - (A + B))x\| &\geq \|(\lambda - A)x\| - \|Bx\| \\ &\geq \frac{1}{K}(|\lambda - \omega|\|x\| + \|x\|_D) - \frac{1}{2K}\|x\|_D \\ &= \frac{1}{K}|\lambda - \omega|\|x\| + \frac{1}{2K}\|x\|_D, \end{aligned} \tag{3.1.11}$$

for $x \in D$ and $\operatorname{Re}\lambda \geq \omega$. Since $D \xrightarrow{d} X$, (3.1.11) implies that $\{\lambda; \operatorname{Re}\lambda \geq \omega\} \subset \rho(A + B)$ and

$$|\lambda - \omega|\|R(\lambda, A + B)\|_{\mathcal{L}(X)} \leq K \text{ for } \operatorname{Re}\lambda \geq \omega.$$

From (3.1.11), we deduce that

$$\frac{1}{2K}\|x\|_D \leq \|(\omega - (A + tB))x\|, \quad x \in D, \quad 0 \leq t \leq 1.$$

Thanks to Lemma 3.1.17, we know that $\omega - (A + B) \in \mathcal{L}is(D, X)$. Therefore $A + B \in \mathcal{H}(D, X)$. This completes the proof. □

3.1.4 Intermediate spaces

Through the whole subsection we set

$$C_n = \sup_{0 < t \leq 1} \|t^n A^n e^{tA}\|_{\mathcal{L}(X)}, \quad \forall n \in \mathbb{N}. \quad (3.1.12)$$

Then $C_n < +\infty$ for every n , thanks to estimates (3.1.7).

Proposition 3.1.19. *For $0 < k < n$, $D(A^k)$ belongs to the class $J_{k/n} \cap K_{k/n}$ between X and $D(A^n)$.*

Proof Since $D(A^k) = D((A - (\omega + 1)I)^k)$ for every k , we may assume without loss of generality that $\omega < 0$. Then the graph norm on $D(A^k)$ is equivalent to the norm $x \mapsto \|A^k x\|$, which will be used here.

Let us prove that $D(A^k) \in J_{k/n}(X, D(A^n))$. First we consider the case $k = 1$, $n = 2$. We claim that there is $C > 0$ such that

$$\|Ax\| \leq C\|x\|^{1/2}\|A^2x\|^{1/2}, \quad x \in D(A^2). \quad (3.1.13)$$

Let $x \in D(A^2)$. Since $\omega < 0$, then $0 \in \rho(A^2)$, so that if $A^2x = 0$ then $x = 0$, and in this case (3.1.13) holds. Moreover, for every $t > 0$ we have

$$Ae^{tA}x - Ax = A^2 \int_0^t e^{sA} x ds = \int_0^t e^{sA} A^2 x ds,$$

so that, according to (3.1.6), we get

$$\|Ax\| \leq \|Ae^{tA}x\| + \int_0^t \|e^{sA} A^2 x\| ds \leq \frac{M_1}{t}\|x\| + M_0 t \|A^2 x\|, \quad t > 0.$$

If $A^2x \neq 0$, taking the minimum of the right hand side for $t > 0$ we get (3.1.13) with constant $C = 2\sqrt{M_0 M_1}$. (3.1.13) means that $D(A) \in J_{1/2}(X, D(A^2))$. Arguing by recurrence as in the proof of Proposition 2.4.2(i) one can see that $D(A^k) \in J_{k/n}(X, D(A^n))$ for $0 < k < n$.

To prove that $D(A^k) \in K_{k/n}(X, D(A^n))$ we show first that $D(A) \in K_{1/n}(X, D(A^n))$. If $x \in D(A)$ split $x = e^{t^{1/n}A}x + (x - e^{t^{1/n}A}x)$, by means of Proposition 3.1.6, we have

$$\|A^n e^{t^{1/n}A}x\| \leq M_{n-1} t^{-1+1/n} \|Ax\|,$$

and

$$\|x - e^{t^{1/n}A}x\| = \left\| \int_0^{t^{1/n}} A e^{sA} x ds \right\| \leq M_0 t^{1/n} \|Ax\|.$$

It follows that

$$\begin{aligned} K(t, x; X, D(A^n)) &\leq \|x - e^{t^{1/n}A}x\| + t \|e^{t^{1/n}A}x\|_{D(A^n)} \\ &\leq M_0 t^{1/n} \|Ax\| + C_{eq,n} M_{n-1} t^{1/n} \|Ax\| \\ &\leq C_{eq,1} (M_0 + C_{eq,n} M_{n-1}) t^{1/n} \|x\|_{D(A)}, \end{aligned}$$

for $t > 0$, here, $C_{eq,k}$ is the coefficient concerned with equivalent norms between $\|\cdot\|_{D(A^k)}$ and $\|A^k \cdot\|$, i.e.

$$C_{eq,k}^{-1} \|A^k x\| \leq \|x\|_{D(A^k)} \leq C_{eq,k} \|A^k x\|, \quad \forall k \in \mathbb{N}, x \in D(A^k).$$

Therefore, $D(A)$ is in the class $K_{1/n}$ between X and $D(A^n)$.

Let us argue by recurrence. Assume that for some $n \geq 3$ we have $D(A^s) \subset (X, D(A^{n-1}))_{s/(n-1), \infty}$ for $s = 1, \dots, n-2$. We have just proved that this is true for $n = 3$. Set $Y = D(A)$. The part A_Y of A in Y is sectorial, so that $D(A_Y^s) \subset (Y, D(A_Y^{n-1}))_{s/(n-1), \infty}$ for $s = 1, \dots, n-2$. On the other hand, $D(A_Y^s) = D(A^{s+1})$, $D(A_Y^{n-1}) = D(A^n)$, so that $D(A^{s+1}) \subset (D(A), D(A^n))_{s/(n-1), \infty}$. Since $D(A)$ belongs to $K_{1/n}(X, D(A^n))$, by part (i) of the Reiteration Theorem 2.4.19, we get

$$(D(A), D(A^n))_{s/(n-1), \infty} \subset (X, D(A^n))_{(s+1)/n, \infty}$$

for $1 \leq s \leq n-2$. Setting $s+1 = k$ we get

$$D(A^k) \subset (X, D(A^n))_{k/n, \infty}, \quad 2 \leq k \leq n-1,$$

and the statement follows. \square

In the following, we shall define a class of intermediate spaces between X and $D(A)$ ($0 < \alpha < 1$, $1 \leq p \leq \infty$, and $(\alpha, p) = (1, \infty)$), by

$$\begin{cases} D_A(\alpha, p) = \{x \in X; t \mapsto v(t) = \|t^{1-\alpha-1/p} A e^{tA} x\| \in L_p(0, 1)\}, \\ \|x\|_{D_A(\alpha, p)} = \|x\| + [x]_{D_A(\alpha, p)} = \|x\| + \|v\|_{L_p(0, 1)}; \end{cases} \quad (3.1.14)$$

$$D_A(\alpha) = \left\{ x \in D_A(\alpha, \infty); \lim_{t \rightarrow 0} t^{1-\alpha} A e^{tA} x = 0 \right\}. \quad (3.1.15)$$

As easily seen, for every $x \in D_A(\alpha, p)$ and $T > 0$, the function $s \mapsto \|A e^{sA} x\|$ belongs to $L_1(0, T)$, so that by Proposition 3.1.9(ii) we have

$$e^{tA} x - x = \int_0^t A e^{sA} x ds \quad \forall t \geq 0, \quad x = \lim_{t \rightarrow 0} e^{tA} x.$$

In particular, all the spaces $D_A(\alpha, p)$ and $D_A(\alpha)$ are contained in the closure of $D(A)$. Moreover we have

$$D_A(\alpha, p) = D_{A_0}(\alpha, p), \quad D_A(\alpha) = D_{A_0}(\alpha),$$

where A_0 is the part of A in $\overline{D(A)}$. See Remark 3.1.10. In the following, we shall state several characterizations of the spaces $D_A(\alpha, p)$ and $D_A(\alpha)$.

First, the spaces $D_A(\alpha, p)$ and $D_A(\alpha)$ are real interpolation spaces between X and $D(A)$.

Proposition 3.1.20. *For $0 < \alpha < 1$ and $1 \leq p \leq \infty$, and for $(\alpha, p) = (1, \infty)$ we have*

$$D_A(\alpha, p) = (X, D(A))_{\alpha, p},$$

with equivalence of the respective norms. Moreover, for $0 < \alpha < 1$,

$$D_A(\alpha) = (X, D(A))_{\alpha}.$$

Proof Let $\varphi : [0, +\infty) \rightarrow \mathbb{R}$ be a C^∞ function such that

$$\begin{cases} 0 \leq \varphi(t) \leq 1, & |\varphi'(t)| \leq 2 \quad \forall t > 0, \\ \varphi(t) = 1 \text{ for } 0 \leq t \leq 1/3, & \varphi(t) = 0 \text{ for } t \geq 1. \end{cases} \quad (3.1.16)$$

Let $x \in D_A(\alpha, p)$. Then, choosing $u(t) = \varphi(t)e^{tA}x$, we get $x = u(0)$, $u(t) = 0$ for $t \geq 1$, and for $0 < t \leq 1$

$$\begin{cases} \|t^{1-\alpha}u(t)\|_{D(A)} \leq \|t^{1-\alpha}Ae^{tA}x\| + \|t^{1-\alpha}e^{tA}x\|, \\ \|t^{1-\alpha}u'(t)\| \leq \|t^{1-\alpha}Ae^{tA}x\| + \|\varphi'\|_\infty \|t^{1-\alpha}e^{tA}x\|, \end{cases}$$

so that $u \in V(p, 1 - \alpha, D(A), X)$, and

$$\|u\|_{V(p, 1-\alpha, D(A), X)} \leq 2[x]_{D_A(\alpha, p)} + 3\|t^{1-\alpha}e^{tA}x\|_{L_p(0, 1; X)}.$$

Due to Proposition 2.4.14, $x \in (X, D(A))_{\alpha, p}$, and

$$\|x\|_{\alpha, p}^T \leq 2[x]_{D_A(\alpha, p)} + 3C_0c_p\|x\|, \quad (3.1.17)$$

where c_p is a suitable constant. Therefore, $D_A(\alpha, p) \subset (X, D(A))_{\alpha, p}$. From the above considerations it is clear that if $x \in D_A(\alpha)$, then u belongs to $V_0(\infty, 1 - \alpha, D(A), X)$, so that $x \in (X, D(A))_\alpha$.

Conversely, let $x \in (X, D(A))_{\alpha, p}$. Then $x = u(0)$, with $u \in V(p, 1 - \alpha, D(A), X)$. It follows that

$$\begin{aligned} \|t^{1-\alpha}Ae^{tA}x\| &\leq \|t^{1-\alpha}Ae^{tA}u(t)\| + \left\| t^{1-\alpha}Ae^{tA} \int_0^t u'(s)ds \right\| \\ &\leq C_0\|t^{1-\alpha}Au(t)\| + C_1 \left\| t^{1-\alpha} \frac{1}{t} \int_0^t u'(s)ds \right\|. \end{aligned} \quad (3.1.18)$$

Due to Corollary 2.4.13, $t \mapsto \|t^{1-\alpha-1/p}Ae^{tA}x\|$ belongs to $L_p(0, 1)$, and

$$\begin{aligned} \|t^{1-\alpha-1/p}Ae^{tA}x\|_{L_p(0, 1)} &\leq C_0\|t^{1-\alpha-1/p}Au(t)\|_{L_p(0, 1)} \\ &\quad + \frac{C_1}{\alpha}\|t^{1-\alpha-1/p}u'(t)\|_{L_p(0, 1)} \\ &\leq \max\left\{C_0, \frac{C_1}{\alpha}\right\} \|u\|_{V(p, 1-\alpha, D(A), X)}. \end{aligned}$$

Since u is arbitrary, we have

$$\|t^{1-\alpha-1/p}Ae^{tA}x\|_{L_p(0, 1)} \leq C\|x\|_{\alpha, p}^T. \quad (3.1.19)$$

Estimate (3.1.19) also holds for $p = \infty$, if we set $1/\infty = 0$. Therefore, $D_A(\alpha, p)$ is continuously embedded in $(X, D(A))_{\alpha, p}$ for $1 \leq p \leq \infty$.

If $x \in (X, D(A))_\alpha$, then $\|t^{1-\alpha}u(t)\|_{D(A)}$ and $\|t^{1-\alpha}u'(t)\|$ go to 0 as $t \rightarrow 0$, and then it means that $\lim_{t \rightarrow 0} t^{1-\alpha}Ae^{tA}x = 0$, so that $x \in D_A(\alpha)$, and the statement is completely proved. \square

The above characterization yields immediately several properties of the spaces $D_A(\alpha, p)$ and $D_A(\alpha)$.

Corollary 3.1.21. *The following statements hold true.*

- (i) *The spaces $D_A(\alpha, p)$ and $D_A(\alpha)$ do not depend explicitly on the operator A , but only on $D(A)$ and the graph norm of A . Precisely, if $B : D(B) = D(A) \rightarrow X$ is a sectorial operator such that*

$$c^{-1}\|Ax\| \leq \|Bx\| \leq c\|Ax\| \quad \forall x \in D(A)$$

for some $c \geq 1$, then we have (with equivalence of the respective norms)

$$D_B(\alpha, p) = D_A(\alpha, p), \quad D_B(\alpha) = D_A(\alpha).$$

- (ii) *For $0 < \alpha_1 < \alpha_2 < 1$, $1 \leq p \leq \infty$, and for $(\alpha_2, p) = (1, \infty)$, we have*

$$D_A(\alpha_2, p) \subset D_A(\alpha_1, p).$$

For $0 < \alpha < 1$, $1 \leq p_1 < p_2 < \infty$,

$$D_A(1, \infty) \subset D_A(\alpha, p_1) \subset D_A(\alpha, p_2) \subset D_A(\alpha) \subset D_A(\alpha, \infty) \subset \overline{D(A)}.$$

- (iii) *The spaces $D_A(\alpha, p)$ and $D_A(\alpha)$ belong to the class J_α between X and $D(A)$. Moreover, if E is a Banach space such that $D(A) \subset E \subset X$, then E belongs to the class J_α between X and $D(A)$ if and only if $D_A(\alpha, 1) \subset E$.*

- (iv) *$D(A)$ is dense in $D_A(\alpha, p)$ for $p < \infty$. $D_A(\alpha)$ is the closure of $D(A)$ in $D_A(\alpha, \infty)$.*

Proof Statement (i) is an obvious consequence of Proposition 3.1.20. Statements (ii), (iii) and (iv) follow from Proposition 3.1.20 through Proposition 2.4.7, Corollary 2.4.11 with Proposition 2.4.17, and Proposition 2.4.16, respectively. \square

The next proposition gives a characterization of the spaces $D_A(\alpha, p)$ and $D_A(\alpha)$ in terms of the behavior of the function $u(t) = e^{tA}x$ near $t = 0$.

Proposition 3.1.22. *It holds*

$$D_A(\alpha, p) = \left\{ x \in X; t \mapsto w(t) = t^{-\alpha-1/p} \|e^{tA}x - x\| \in L_p(0, 1) \right\},$$

and, setting $[[x]]_{D_A(\alpha, p)} = \|w\|_{L_p(0, 1)}$, the norm

$$x \mapsto \|x\| + [[x]]_{D_A(\alpha, p)}$$

is equivalent to the norm of $D_A(\alpha, p)$. Moreover,

$$D_A(\alpha) = \left\{ x \in X; \lim_{t \rightarrow 0} t^{-\alpha} (e^{tA}x - x) = 0 \right\}.$$

Remark 3.1.23. We remark that, in the case $p = \infty$, Proposition 3.1.22 states that $x \in D_A(\alpha, \infty)$ if and only if the function $t \mapsto e^{tA}x$ belongs to $BUC^\alpha([0, 1]; X)$, if $0 < \alpha < 1$, or to $Lip([0, 1]; X)$, if $\alpha = 1$. This is clearly equivalent to $t \mapsto e^{tA}x \in BUC^\alpha([0, T]; X)$ (respectively, $Lip([0, T]; X)$) for all $T > 0$. \square

The following proposition gives a characterization of $D_A(\alpha, p)$ and $D_A(\alpha)$ in terms of the behavior of $AR(\lambda, A)x$ as $\lambda \rightarrow +\infty$.

Proposition 3.1.24. *Let $a > \max\{1, \omega\}$. Then*

$$D_A(\alpha, p) = \{x \in X; t \mapsto z(t) = \|t^{\alpha-1/p}AR(t, A)x\| \in L_p(a, +\infty)\},$$

and, setting $[x]_{D_A(\alpha, p)}^* = \|z\|_{L_p(a, +\infty)}$, the norm

$$x \mapsto \|x\| + [x]_{D_A(\alpha, p)}^*$$

is equivalent to the norm $D_A(\alpha, p)$. Moreover,

$$D_A(\alpha) = \left\{ x \in X; \lim_{t \rightarrow +\infty} t^\alpha AR(t, A)x = 0 \right\}.$$

For $k \in \mathbb{N}$, $\alpha \in (0, 1)$, $p \in [1, \infty]$, set

$$\begin{aligned} D_A(\alpha + k, p) &= \{x \in D(A^k); A^k x \in D_A(\alpha, p)\}, \\ \|x\|_{D_A(\alpha+k, p)} &= \|x\| + \|A^k x\|_{D_A(\alpha, p)}; \\ D_A(\alpha + k) &= \{x \in D(A^k); A^k x \in D_A(\alpha)\}. \end{aligned}$$

Thanks to Proposition 3.1.22, the function $t \mapsto e^{tA}x$ belongs to $BUC^{k+\alpha}([0, 1]; X)$ if and only if $x \in D_A(k + \alpha, \infty)$. Moreover, since $D(A)$ is dense in $D_A(\alpha, p)$ for $p < \infty$, then $D(A^{k+1})$ is dense in $D_A(k + \alpha, p)$ for $p < \infty$; since $D_A(\alpha)$ is the closure of $D(A)$ in $D_A(\alpha, \infty)$, then $D_A(k + \alpha)$ is the closure of $D(A^{k+1})$ in $D_A(k + \alpha, \infty)$.

A very important fact is that the parts of A in $D_A(\alpha, p)$ and in $D_A(\alpha)$, defined by

$$\begin{aligned} A_{\alpha, p} &: D_A(\alpha + 1, p) \rightarrow D_A(\alpha, p), & A_{\alpha, p}x &= Ax; \\ A_\alpha &: D_A(\alpha + 1) \rightarrow D_A(\alpha), & A_\alpha x &= Ax, \end{aligned}$$

are sectorial operators in $D_A(\alpha, p)$ and in $D_A(\alpha)$, respectively.

Proposition 3.1.25. *$\rho(A)$ is contained in $\rho(A_{\alpha, p})$, and for every $\lambda \in \rho(A)$,*

$$\|R(\lambda, A_{\alpha, p})\|_{\mathcal{L}(D_A(\alpha, p))} \leq \|R(\lambda, A)\|_{\mathcal{L}(X)}.$$

Consequently, $A_{\alpha, p}$ is a sectorial operator in $D_A(\alpha, p)$ and A_α is a sectorial operator in $D_A(\alpha)$.

We study now the behavior near $t = 0$ of the function $t \mapsto e^{tA}x$, when $x \in D_A(\alpha, p)$ or $x \in D_A(\alpha)$.

Proposition 3.1.26. For $0 < \alpha < 1$ and $1 \leq p < \infty$,

$$\lim_{t \rightarrow 0} \|e^{tA}x - x\|_{D_A(\alpha, p)} = 0, \quad \forall x \in D_A(\alpha, p). \quad (3.1.20)$$

For $0 < \alpha < 1$ and $x \in D_A(\alpha, \infty)$,

$$\lim_{t \rightarrow 0} \|e^{tA}x - x\|_{D_A(\alpha, \infty)} = 0 \iff x \in D_A(\alpha). \quad (3.1.21)$$

Now we give some estimates for the function $t \mapsto A^n e^{tA}$ when $t \rightarrow 0$ and $t \rightarrow \infty$. For convenience, in the next proposition we set

$$D_A(0, p) = X, \quad \forall p \in [1, \infty].$$

Proposition 3.1.27. Let $(\alpha, p), (\beta, p) \in (0, 1) \times [1, \infty] \cup \{(1, \infty)\}$, and let $n \in \mathbb{N}$. Then there are $C = C(n, p; \alpha, \beta)$, $C' = C'(n, p; \alpha, \beta)$ such that

$$\begin{cases} (i) & \|t^{n-\alpha+\beta} A^n e^{tA}\|_{\mathcal{L}(D_A(\alpha, p), D_A(\beta, p))} \leq C, \quad 0 < t \leq 1; \\ (ii) & \|t^n (A - \omega I)^n e^{tA}\|_{\mathcal{L}(D_A(\alpha, p), D_A(\beta, p))} \leq C' e^{\omega t}, \quad t \geq 1. \end{cases} \quad (3.1.22)$$

The statement holds also for $n = 0$, provided $\alpha \leq \beta$.

Remark 3.1.28. In fact, we can also know that

$$\sup_{0 < t \leq 1} \|t^{n-\alpha+\beta} A^n e^{tA}\|_{\mathcal{L}(D_A(\alpha, \infty), D_A(\beta, p))} < \infty, \quad \forall n \in \mathbb{N},$$

for $0 < \alpha, \beta < 1$, $1 \leq p \leq \infty$. In particular, for $\alpha = \beta$,

$$\sup_{0 < t \leq 1} \|t^n A^n e^{tA}\|_{\mathcal{L}(D_A(\alpha, \infty), D_A(\alpha, p))} < \infty, \quad \forall n \in \mathbb{N}, p \in [1, \infty].$$

This is not true in general for $n = 0$. □

Proposition 3.1.29. Let $0 < \alpha < 1$, and let I be a (possibly unbounded) interval. The following statements hold.

(i) If $u \in BUC^\alpha(I; D(A)) \cap BUC^{1+\alpha}(I; X)$, then $u'(t) \in D_A(\alpha, \infty)$ for every $t \in I$, and

$$\|u'\|_{D_A(\alpha, \infty)} \leq C (\|u\|_{BUC^\alpha(I; D(A))} + \|u\|_{BUC^{1+\alpha}(I; X)}), \quad \forall t \in I. \quad (3.1.23)$$

In addition, $BUC^\alpha(I; D(A)) \cap BUC^{1+\alpha}(I; X) \subset Lip(I; D_A(\alpha, \infty))$.

(ii) $buc^\alpha(I; D(A)) \cap buc^{1+\alpha}(I; X) \subset BUC^1(I, D_A(\alpha))$.

Proof (i) First we shall prove that (3.1.23) holds. Let $u \in BUC^\alpha(I; D(A)) \cap BUC^{1+\alpha}(I; X)$. For $t, t+h \in I$, split $u'(t)$ as

$$u'(t) = \int_0^1 (u'(t) - u'(t+\sigma h))d\sigma + \frac{u(t+h) - u(t)}{h}.$$

Then for every $\xi \in (0, 1]$, it holds

$$\begin{aligned} & \|\xi^{1-\alpha} Ae^{\xi A} u'(t)\| \\ & \leq \xi^{1-\alpha} \|Ae^{\xi A}\|_{\mathcal{L}(X)} \int_0^1 \|u'(t) - u'(t+\sigma h)\| d\sigma \\ & \quad + \xi^{1-\alpha} \|e^{\xi A}\|_{\mathcal{L}(X)} \left\| \frac{Au(t+h) - Au(t)}{h} \right\| \\ & \leq \frac{C_1}{\alpha+1} \xi^{-\alpha} [u']_\alpha^X |h|^\alpha + C_0 \xi^{1-\alpha} [u]_\alpha^{D(A)} |h|^{\alpha_1}. \end{aligned} \tag{3.1.24}$$

If I is unbounded, for every $\xi \in (0, 1]$ and $t \in I$, there is $h \in \mathbb{R}$ such that $|h| = \xi$ and $t+h \in I$. Replacing in (3.1.24), we find

$$\|\xi^{1-\alpha} Ae^{\xi A} u'(t)\| \leq \frac{C_1}{\alpha+1} [u']_\alpha^X + C_0 [u]_\alpha^{D(A)}, \tag{3.1.25}$$

and estimate (3.1.23) holds. If $I = [a, b]$, and $\xi \leq (b-a)/2$, for every $t \in (a, b]$, there is h such that $|h| = \xi$ and $t+h \in [a, b]$. For such value of ξ , (3.1.25) holds. On the other hand, if $\xi > (b-a)/2$, then

$$\|\xi^{1-\alpha} Ae^{\xi A} u'(t)\| \leq \xi^{-\alpha} C_1 \|u'(t)\| \leq \left(\frac{2}{b-a}\right)^\alpha C_1 \|u'\|_\infty,$$

so that $[u']_{D_A(\alpha, \infty)}$ is bounded, and (3.1.23) is proved.

Let us show now that u is Lipschitz continuous with values in $D_A(\alpha, \infty)$. For $0 < \xi \leq 1$ and $s < t \in I$, it holds

$$\begin{aligned} \|\xi^{1-\alpha} Ae^{\xi A} (u(t) - u(s))\| & \leq \int_s^t \|\xi^{1-\alpha} Ae^{\xi A} u'(\sigma)\| d\sigma \\ & \leq (t-s) \sup_{\sigma \in I} [u'(\sigma)]_{D_A(\alpha, \infty)}, \end{aligned}$$

so that

$$[u(t) - u(s)]_{D_A(\alpha, \infty)} \leq (t-s) \sup_{\sigma \in I} [u'(\sigma)]_{D_A(\alpha, \infty)}.$$

Let us prove that statement (ii) holds. If $u \in buc^\alpha(I; D(A)) \cap buc^{1+\alpha}(I; X)$, then there is a sequence of approximating functions $u_n \in BUC^\infty(I; D(A))$ and converge to u in $BUC^\alpha(I; D(A)) \cap BUC^{1+\alpha}(I; X)$. By statement (i), $\{u'_n\}_{n \in \mathbb{N}}$ converges to u' in $L_\infty(I; D_A(\alpha, \infty))$. Since $BUC(I, D_A(\alpha))$ is closed in $L_\infty(I; D_A(\alpha, \infty))$, then $u' \in BUC(I; D_A(\alpha))$. \square

3.2 Maximal Regularity

In the section we shall study the solvability of the initial value problem

$$u'(t) = Au(t) + f(t), \quad t > 0; \quad u(0) = u_0, \quad (3.2.1)$$

where A is a linear sectorial operator in general Banach space X . The function f is defined and continuous in a (possibly unbounded) interval I such that $\inf I = 0$. We prove several properties of the solution in a bounded interval $[0, T]$. We also show that, under reasonable assumptions on the data, any solution of (3.2.1) in an arbitrary interval $[0, T]$ is given by the variation of constants formula

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s)ds, \quad 0 \leq t \leq T. \quad (3.2.2)$$

In this situation, the study of the solution of (3.2.1) is reduced to the study of the representation formula (3.2.2). Again we follow here [24], see also [9, 10].

Through the whole section, we fix $T > 0$ and we set

$$M_k = \sup_{0 < t \leq T+1} \|t^k A^k e^{tA}\|_{\mathcal{L}(X)}, \quad k \in \mathbb{N} \cup \{0\}, \quad (3.2.3)$$

and, for $\alpha \in (0, 1]$,

$$M_{k,\alpha} = \sup_{0 < t \leq T+1} \|t^{k-\alpha} A^k e^{tA}\|_{\mathcal{L}(D_A(\alpha, \infty), X)}, \quad k \in \mathbb{N}, \quad (3.2.4)$$

$$K_{k,\alpha} = \sup_{0 < t \leq T+1} \|t^{k+\alpha} A^k e^{tA}\|_{\mathcal{L}(X, D_A(\alpha, 1))}, \quad k \in \mathbb{N} \cup \{0\}. \quad (3.2.5)$$

Due to estimate (3.1.22), we have $M_k, M_{k,\alpha}, K_{k,\alpha} < \infty$ for every k .

Due to the singular behavior near $t = 0$ of the derivatives of $t \mapsto e^{tA}x$, it will be sometimes convenient to work with weighted functional spaces. In particular, we shall consider spaces of functions defined in a bounded interval $(a, b]$, which are bounded or uniformly Hölder continuous in each interval $[a + \varepsilon, b]$ but that are not necessarily bounded (respectively, Hölder continuous) up to $t = a$. Let $\mu \in \mathbb{R}$, and set

$$\begin{aligned} B_\mu((a, b]; X) &= \left\{ f : (a, b] \rightarrow X; \|f\|_{B_\mu((a, b]; X)} \right. \\ &= \left. \sup_{a < t \leq b} (t-a)^\mu \|f(t)\|_X < \infty \right\}. \end{aligned} \quad (3.2.6)$$

$$\begin{aligned} C_\mu((a, b]; X) &= C((a, b]; X) \cap B_\mu((a, b]; X), \\ \|f\|_{C_\mu((a, b]; X)} &= \|f\|_{B_\mu((a, b]; X)}. \end{aligned} \quad (3.2.7)$$

For $0 < \alpha < 1, \beta > 0$, set

$$\begin{aligned} C_\beta^\alpha((a, b]; X) &= \left\{ f \in B_{\beta-\alpha}((a, b]; X) \cap BUC^\alpha([a + \varepsilon, b]; X), \forall \varepsilon \in (0, b-a); \right. \\ &\quad \left. [f]_{C_\beta^\alpha((a, b]; X)} = \sup_{0 < \varepsilon < b-a} \varepsilon^\beta [f]_{BUC^\alpha([a+\varepsilon, b]; X)} < +\infty \right\}; \\ \|f\|_{C_\beta^\alpha((a, b]; X)} &= \|f\|_{B_{\beta-\alpha}((a, b]; X)} + [f]_{C_\beta^\alpha((a, b]; X)}. \end{aligned} \quad (3.2.8)$$

In particular, for $\beta = \alpha$, the space $C_\alpha^\alpha((a, b]; X)$ is the set of the bounded functions $f : (a, b] \rightarrow X$ such that

$$\sup_{0 < \varepsilon < b-a} \varepsilon^\alpha [f]_{BUC^\alpha((a+\varepsilon, b]; X)} < +\infty.$$

It may be easily seen that it coincides with the set of the bounded functions $f : (a, b] \rightarrow X$ such that $t \mapsto g(t) = (t-a)^\alpha f(t)$ belongs to $BUC^\alpha((a, b]; X)$, and that the norm $[[f]]_\alpha = \|f\|_\infty + [g]_{BUC^\alpha((a, b]; X)}$ is equivalent to the C_α^α norm. More generally, one can show that $C_\beta^\alpha((a, b]; X)$ is the space of the functions $f : (a, b] \rightarrow X$ such that $h(t) = (t-a)^{\beta-\alpha} f(t)$ is bounded and $g(t) = (t-a)^\beta f(t)$ is uniformly α -Hölder continuous in $(a, b]$ with values in X , and that the norms $f \mapsto \|f\|_{C_\beta^\alpha((a, b]; X)}$ and

$$f \mapsto \|h\|_{B((a, b]; X)} + [g]_{BUC^\alpha((a, b]; X)}$$

are equivalent. However, in what follows it is more convenient to use the norm defined in (3.2.8).

3.2.1 Solutions of linear problems

We consider several types of solutions of problem (3.2.1).

Definition 3.2.1. Let $T > 0$, let $f : [0, T] \rightarrow X$ be a continuous function, and let $u_0 \in X$. Then:

- (i) A function $u \in C^1([0, T]; X) \cap C([0, T]; D(A))$ is said to be a **strict solution** of (3.2.1) in the interval $[0, T]$ if $u'(t) = Au(t) + f(t)$ for each $t \in [0, T]$, and $u(0) = u_0$.
- (ii) A function $u \in C([0, T]; X)$ is said to be a **strong solution** of (3.2.1) in the interval $[0, T]$ if there is a sequence $\{u_n\}_{n \in \mathbb{N}} \subset C^1([0, T]; X) \cap C([0, T]; D(A))$ such that

$$u_n \rightarrow u, \quad u_n' - Au_n \rightarrow f \text{ in } C([0, T]; X) \text{ as } n \rightarrow +\infty.$$

Let now $f : (0, T] \rightarrow X$ be continuous. Then

- (iii) A function $u \in C^1((0, T]; X) \cap C((0, T]; D(A)) \cap C([0, T]; X)$ is said to be a **classical solution** of (3.2.1) in the interval $[0, T]$ if $u'(t) = Au(t) + f(t)$ for each $t \in (0, T]$, and $u(0) = u_0$.

If f is defined in $[0, +\infty)$, a function $u : [0, +\infty) \rightarrow X$ is said to be a strict (respectively classical, strong) solution of (3.2.1) in $[0, +\infty)$ if for every $T > 0$ the restriction $u|_{[0, T]}$ is a strict (respectively classical, strong) solution of (3.2.1) in $[0, T]$.

From Definition 3.2.1 it follows easily that if problem (3.2.1) has a strict solution then

$$u_0 \in D(A), \quad Au_0 + f(0) \in \overline{D(A)}, \quad (3.2.9)$$

whereas if problem (3.2.1) has a classical or strong solution, then

$$u_0 \in \overline{D(A)}. \quad (3.2.10)$$

Moreover, any strict solution is also strong and classical.

We are going to show that if $f \in L_1(0, T; X)$ then any type of solution of (3.2.1) may be represented by the variation of constants formula (3.2.2). We begin with the classical solution.

Proposition 3.2.2. *Let $f \in L_1(0, T; X) \cap C((0, T]; X)$, and let $u_0 \in \overline{D(A)}$. If u is a classical solution of (3.2.1), then*

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s)ds, \quad 0 \leq t \leq T.$$

Proposition 3.2.2 implies that if $f \in L_1(0, T; X) \cap C((0, T]; X)$ then the classical solution of (3.2.1) is unique. In particular, if $f \in C([0, T]; X)$ the strict solution of (3.2.1) is unique and it is given by (3.2.2). As a consequence, also the strong solution of (3.2.1) is unique, and it is given by (3.2.2).

In the applications we shall find situations in which $D(A)$ is not dense in X , and the initial datum u_0 does not belong to $\overline{D(A)}$. Then the initial condition $u(0) = u_0$ has to be understood in a weak sense, such as

$$\lim_{t \rightarrow 0} R(\lambda, A)u(t) = R(\lambda, A)u_0$$

for some $\lambda \in \rho(A)$. Also in this case the representation formula (3.2.2) holds, as the next corollary states.

Corollary 3.2.3. *Let $f \in L_1(0, T; X) \cap C((0, T]; X)$ and $u_0 \in X$. If $u \in C^1((0, T]; X) \cap C((0, T]; D(A))$ satisfies $u'(t) = Au(t) + f(t)$ for $t > 0$, and $\lim_{t \rightarrow 0} R(\lambda, A)u(t) = R(\lambda, A)u_0$ for some $\lambda \in \rho(A)$, then u is given by (3.2.2).*

It is easy to see that (3.2.2) makes sense whenever $f \in L_1(0, T; X)$ and $u_0 \in X$. Therefore we give the following definition.

Definition 3.2.4. Let $f \in L_1(0, T; X)$, and let $u_0 \in X$. Then function u defined in (3.2.2) is call the **mild solution** of (3.2.1).

Due to estimate (3.2.3), with $k = 0$, the mild solution satisfies

$$\|u(t)\| \leq M_0 \left(\|u_0\| + \int_0^t \|f(s)\|ds \right), \quad 0 \leq t \leq T. \quad (3.2.11)$$

It is easy to see that if $u_0 \in \overline{D(A)}$ then u belongs to $C([0, T]; X)$.

The mild solution is also an *integral solution*, in the sense specified by the next proposition.

Proposition 3.2.5. *Let $f \in L_1(0, T; X)$, and let $u_0 \in X$. If u is defined by (3.2.2), then for every $t \in [0, T]$ the integral $\int_0^t u(s)ds$ belongs to $D(A)$, and*

$$u(t) = u_0 + A \int_0^t u(s)ds + \int_0^t f(s)ds, \quad 0 \leq t \leq T. \quad (3.2.12)$$

The result of Proposition 3.2.5 is used in the next lemma, where we give sufficient conditions in order that a mild solution be classical or strict.

Lemma 3.2.6. *Let $f \in L_1(0, T; X) \cap C((0, T]; X)$, let $u_0 \in \overline{D(A)}$, and let u be the mild solution of (3.2.1). The following conditions are equivalent.*

- (a) $u \in C((0, T]; D(A))$,
- (b) $u \in C^1((0, T]; X)$,
- (c) u is a classical solution of (3.2.1).

If in addition $f \in C([0, T]; X)$, then the following conditions are equivalent.

- (a') $u \in C([0, T]; D(A))$,
- (b') $u \in C^1([0, T]; X)$,
- (c') u is a strict solution of (3.2.1).

Now we discuss the relationship between mild and strong solutions. It is clear from the definition that if problem (3.2.1) has a strong solution, then $f \in C([0, T]; X)$ and $u_0 \in \overline{D(A)}$. Indeed, these conditions are also sufficient for the mild solution be strong, as the following proposition show.

Proposition 3.2.7. *Let $f \in C([0, T]; X)$, $u_0 \in \overline{D(A)}$. Then the mild solution of problem (3.2.1) is strong.*

3.2.2 Mild solutions

This subsection is devoted to the properties of the mild solution of (3.2.1) in the case where f belongs either to $L_\infty(0, T; X)$ or to $C((0, T]; X) \cap L_1(0, T; X)$. Since the properties of the function $t \mapsto e^{tA}u_0$ were studied in last section, we focus our attention on the function

$$v(t) = (e^{tA} * f)(t) = \int_0^t e^{(t-s)A} f(s)ds, \quad 0 \leq t \leq T. \quad (3.2.13)$$

Proposition 3.2.8. *Let $f \in L_\infty(0, T; X)$. Then, for every $\alpha \in (0, 1)$, $v \in BUC^\alpha([0, T]; X) \cap C([0, T]; D_A(\alpha, 1))$. Precisely, it belongs to $BUC^{1-\alpha}([0, T]; D_A(\alpha, 1))$, and there is C independent of f such that*

$$\|v\|_{BUC^{1-\alpha}([0, T]; D_A(\alpha, 1))} \leq C \|f\|_{L_\infty(0, T; X)}. \quad (3.2.14)$$

It follows that v belongs to $BUC^{1-\alpha}([0, T], X_\alpha)$ for every space $X_\alpha \in J_\alpha(X, D(A))$.

Proof Since $s \mapsto \|e^{(t-s)A}\|_{\mathcal{L}(X, D_A(\alpha, 1))}$ belongs to $L_1(0, t)$ for every $t \in (0, T]$, then $v(t)$ belongs to $D_A(\alpha, 1)$ for every $\alpha \in (0, 1)$, and

$$\begin{aligned} \|v(t)\|_{D_A(\alpha, 1)} &\leq K_{0, \alpha} \|f\|_{L_\infty(0, T; X)} \int_0^t (t-s)^{-\alpha} ds \\ &\leq K_{0, \alpha} (1-\alpha)^{-1} T^{1-\alpha} \|f\|_{L_\infty(0, T; X)}. \end{aligned} \quad (3.2.15)$$

Moreover, for $0 \leq s < t \leq T$,

$$\begin{aligned} v(t) - v(s) &= \int_0^s (e^{(t-\sigma)A} - e^{(s-\sigma)A}) f(\sigma) d\sigma + \int_s^t e^{(t-\sigma)A} f(\sigma) d\sigma \\ &= \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} A e^{\tau A} d\tau + \int_s^t e^{(t-\sigma)A} f(\sigma) d\sigma, \end{aligned}$$

which implies

$$\begin{aligned} \|v(t) - v(s)\|_{D_A(\alpha, 1)} &\leq K_{1, \alpha} \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{-1-\alpha} d\tau \|f\|_\infty \\ &\quad + K_{0, \alpha} \int_s^t (t-\sigma)^{-\alpha} d\sigma \|f\|_\infty \\ &\leq \left(\frac{K_{1, \alpha}}{\alpha(1-\alpha)} + \frac{K_{0, \alpha}}{1-\alpha} \right) (t-s)^{1-\alpha} \|f\|_\infty, \end{aligned} \quad (3.2.16)$$

so that v is uniformly $(1-\alpha)$ -Hölder continuous with values in $D_A(\alpha, 1)$. Estimate (3.2.14) follows now from (3.2.15) and (3.2.16).

Thanks to Proposition 2.4.17, we also get that v is uniformly $(1-\alpha)$ -Hölder continuous with values in X_α for $X_\alpha \in J_\alpha(X, D(A))$. \square

Corollary 3.2.9. *Let $f \in L_\infty(0, T; X)$, $u_0 \in X$, and let u be the mild solution of (3.2.1). Then $u \in L_\infty(0, T; X)$, and*

$$\|u\|_{L_\infty(0, T; X)} \leq M_0 (\|u_0\| + T \|f\|_{L_\infty(0, T; X)}). \quad (3.2.17)$$

For every $\alpha \in (0, 1)$, and $\varepsilon \in (0, T)$, $u \in BUC^{1-\alpha}([\varepsilon, T]; D_A(\alpha, 1))$. Consequently, u belongs to $BUC^{1-\alpha}([\varepsilon, T]; X_\alpha)$ for every space $X_\alpha \in J_\alpha(X, D(A))$. There is C independent of f such that

$$\begin{aligned} \|u\|_{BUC^\alpha([\varepsilon, T]; X)} + \|u\|_{B([\varepsilon, T]; D_A(\alpha, 1))} &\leq C(\varepsilon^{-\alpha} \|u_0\| + \|f\|_{L_\infty(0, T; X)}), \\ \|u\|_{BUC^{1-\alpha}([\varepsilon, T]; D_A(\alpha, 1))} &\leq C(\varepsilon^{-1} \|u_0\| + \|f\|_{L_\infty(0, T; X)}). \end{aligned} \quad (3.2.18)$$

Moreover, $u \in C([0, T]; X) \iff u_0 \in \overline{D(A)}$. For $0 < \alpha < 1$, $u \in BUC^\alpha([0, T], X) \iff u_0 \in D_A(\alpha, \infty)$.

Proof This is a consequence of Proposition 3.2.8 and Remark 3.1.23. \square

In the applications to nonlinear problems we shall consider frequently functions f which are bounded in every interval $[\varepsilon, T]$ with $\varepsilon \in (0, T)$, and blow up at $t = 0$. Next proposition deals with functions belonging to the weighted spaces defined in (3.2.6), (3.2.8).

Proposition 3.2.10. *Let $f \in L_1(0, T; X) \cap L_\infty(\varepsilon, T; X)$ for every $\varepsilon \in (0, T)$. Then $v = e^{tA} * f$ belong to $C([0, T]; X) \cap BUC^{1-\alpha}([\varepsilon, T]; D_A(\alpha, 1))$ for every $\alpha \in (0, 1)$, $\varepsilon \in (0, T)$.*

If in addition $t \mapsto t^\theta f(t)$ is bounded and measurable in $(0, T]$, for some $\theta \in (0, 1)$ (so that $f \in B_\theta((0, T]; X)$), then the following statements hold.

(i) $v \in BUC^{1-\theta}([0, T]; X)$, and

$$\|v\|_{BUC^{1-\theta}([0, T]; X)} \leq C \|f\|_{B_\theta((0, T]; X)}.$$

(ii) For every $\alpha \in (0, 1)$, $v \in C_\theta^{1-\alpha}((0, T]; D_A(\alpha, 1))$, and

$$\|v\|_{C_\theta^{1-\alpha}((0, T]; D_A(\alpha, 1))} \leq C \|f\|_{B_\theta((0, T]; X)}.$$

In particular, v is bounded with values in $D_A(1 - \theta, 1)$, and $t \mapsto t^{\theta+\alpha-1}v(t) \in B([0, T]; D_A(\alpha, 1))$, $t \mapsto t^\theta v(t) \in BUC^{1-\alpha}([0, T]; D_A(\alpha, 1))$ for every $\alpha \in (0, 1)$. If in addition $\lim_{t \rightarrow 0} \|t^\theta f(t)\| = 0$, then $v \in C([0, T]; D_A(1 - \theta, 1))$.

(iii) If $\alpha + \theta < 1$ then $v \in BUC^{1-\alpha-\theta}([0, T]; D_A(\alpha, 1))$, and

$$\|v\|_{BUC^{1-\alpha-\theta}([0, T]; D_A(\alpha, 1))} \leq C \|f\|_{B_\theta((0, T]; X)}.$$

Proof Obviously, v is continuous with values in X up to $t = 0$. For $0 < \varepsilon \leq t \leq T$, we have

$$v(t) = e^{(t-\varepsilon/2)A} \int_0^{\varepsilon/2} e^{(\varepsilon/2-s)A} f(s) ds + \int_{\varepsilon/2}^t e^{(t-s)A} f(s) ds = v_1(t) + v_2(t).$$

By Proposition 3.2.8, v_2 belongs to $BUC^{1-\alpha}([\varepsilon/2, T]; D_A(\alpha, 1))$ for every α . Moreover, v_1 belongs to $BUC^\infty([\varepsilon, T]; D(A^n))$ for every $n \in \mathbb{N}$, due to Proposition 3.1.6. Therefore, $v \in C([0, T]; X) \cap BUC^{1-\alpha}([\varepsilon, T]; D_A(\alpha, 1))$.

Let now $t \mapsto t^\theta f(t)$ be measurable and bounded. Then for $0 \leq r < t \leq T$

$$\begin{aligned} \|v(t) - v(r)\| &\leq \left\| \int_0^r (e^{(t-s)A} - e^{(r-s)A}) f(s) ds \right\| + \left\| \int_r^t e^{(t-s)A} f(s) ds \right\| \\ &\leq \left(M_1 \int_0^r \frac{ds}{s^\theta} \int_{r-s}^{t-s} \frac{d\sigma}{\sigma} + M_0 \int_r^t \frac{ds}{s^\theta} \right) \|f\|_{B_\theta((0, T]; X)} \\ &\leq \left(M_1 \int_0^r \frac{ds}{(r-s)^{1-\theta} s^\theta} \int_{r-s}^{t-s} \frac{d\sigma}{\sigma^\theta} + M_0 \int_r^t \frac{ds}{s^\theta} \right) \|f\|_{B_\theta((0, T]; X)} \\ &\leq \frac{1}{1-\theta} \left(M_1 \int_0^1 \frac{ds}{(1-s)^{1-\theta} s^\theta} + M_0 \right) (t-r)^{1-\theta} \|f\|_{B_\theta((0, T]; X)}, \end{aligned}$$

and statement (i) follows.

Let us prove that statement (ii) holds. For $0 < t \leq T$ we have

$$\begin{aligned} \|v(t)\|_{D_A(\alpha,1)} &\leq K_{0,\alpha} \int_0^t \frac{ds}{(t-s)^\alpha s^\theta} \|f\|_{B_\theta((0,T];X)} \\ &= K_{0,\alpha} t^{1-\alpha-\theta} \int_0^1 \frac{ds}{(1-s)^\alpha s^\theta} \|f\|_{B_\theta((0,T];X)}, \end{aligned} \quad (3.2.19)$$

so that $t \mapsto t^{\theta+\alpha-1}v(t)$ is bounded in $(0, T]$ with values in $D_A(\alpha, 1)$. Moreover, if N is such that $\|tAe^{tA}\|_{\mathcal{L}(D_A(\alpha,1))} \leq N$ for $0 < t \leq T$, then for $0 < \varepsilon \leq r < t \leq T$

$$\begin{aligned} \|v_1(t) - v_1(r)\|_{D_A(\alpha,1)} &= \left\| \int_{r-\varepsilon/2}^{t-\varepsilon/2} Ae^{\sigma A} d\sigma \int_0^{\varepsilon/2} e^{(\varepsilon/2-s)A} f(s) ds \right\|_{D_A(\alpha,1)} \\ &\leq \frac{NK_{0,\alpha}}{(r-\varepsilon/2)^{1-\alpha}} \int_{r-\varepsilon/2}^{t-\varepsilon/2} \frac{d\sigma}{\sigma^\alpha} \int_0^1 \frac{d\tau}{(1-\tau)^\alpha \tau^\theta} \left(\frac{\varepsilon}{2}\right)^{1-\alpha-\theta} \|f\|_{B_\theta((0,T];X)} \\ &\leq \frac{NK_{0,\alpha}}{1-\alpha} \int_0^1 \frac{d\sigma}{(1-\sigma)^\alpha \sigma^\theta} \left(\frac{\varepsilon}{2}\right)^{-\theta} (t-r)^{1-\alpha} \|f\|_{B_\theta((0,T];X)}, \end{aligned}$$

and by estimate (3.2.16) there is $C > 0$ independent of ε such that

$$\begin{aligned} \|v_2(t) - v_2(r)\|_{D_A(\alpha,1)} &\leq C(t-r)^{1-\alpha} \|f\|_{B([\varepsilon/2,T];X)} \\ &\leq C(t-r)^{1-\alpha} \left(\frac{\varepsilon}{2}\right)^{-\theta} \|f\|_{B_\theta((0,T];X)}, \end{aligned}$$

so that $v \in C_\theta^{1-\alpha}((0, T]; D_A(\alpha, 1))$. Choosing $\alpha = 1 - \theta$, we see that v is bounded up to $t = 0$ and continuous for $t > 0$ with values in $D_A(1 - \theta, 1)$. If in addition

$$\lim_{t \rightarrow 0} \|t^\theta f(t)\| = 0,$$

then v is continuous up to $t = 0$ with values in $D_A(1 - \theta, 1)$. Indeed, for every $\varepsilon > 0$ there is $\delta > 0$ such that for $0 < s \leq \delta$ we have $\|s^\theta f(s)\| \leq \varepsilon$, so that for $0 < t \leq \delta$ estimate (3.2.19) can be replaced by

$$\|v(t)\|_{D_A(1-\theta,1)} \leq K_{0,1-\theta} \varepsilon \int_0^1 \frac{d\sigma}{(1-\sigma)^{1-\theta} \sigma^\theta},$$

which implies that $\lim_{t \rightarrow 0} \|v(t)\|_{D_A(1-\theta,1)} = 0$.

The proof of statement (iii) is similar to the one of statement (i). \square

In the next Corollary we combine the above results with the results of last section about the function $t \mapsto e^{tA}u_0$.

Corollary 3.2.11. *Let $u_0 \in X$, $f \in L_1(0, T; X)$, and let u be the mild solution of (3.2.1). Then u belongs to $C((0, T]; X)$ and*

$$\sup_{0 \leq t \leq T} \|u(t)\| \leq C (\|u_0\| + \|f\|_{L_1(0,T;X)}).$$

Moreover, if $u_0 \in D_A(\theta, \infty)$ and $t \mapsto t^{1-\theta}f(t)$ is measurable and bounded, for some $\theta \in (0, 1)$, then $u \in BUC^\theta([0, T]; X) \cap C_{1-\theta}^{1-\alpha}([0, T]; D_A(\alpha, 1))$ for every $\alpha \in (0, 1)$ (so that in particular $t \mapsto t^{1-\theta}u(t) \in BUC^{1-\alpha}([0, T]; D_A(\alpha, 1))$), and

$$\begin{aligned} \|u\|_{BUC^\theta([0, T]; X)} &\leq C (\|u_0\|_{D_A(\theta, \infty)} + \|f\|_{B_{1-\theta}([0, T]; X)}), \\ \|u\|_{C_{1-\theta}^{1-\alpha}([0, T]; D_A(\alpha, 1))} &\leq C (\|u_0\|_{D_A(\theta, \infty)} + \|f\|_{B_{1-\theta}([0, T]; X)}). \end{aligned}$$

For $\alpha > \theta$, $t \mapsto t^{\alpha-\theta}u(t) \in B([0, T]; D_A(\alpha, 1))$, and

$$\sup_{0 \leq t \leq T} \|t^{\alpha-\theta}u(t)\|_{D_A(\alpha, 1)} \leq C (\|u_0\|_{D_A(\theta, \infty)} + \|f\|_{B_{1-\theta}([0, T]; X)}).$$

For $\alpha < \theta$, $u \in BUC^{\theta-\alpha}([0, T]; D_A(\alpha, 1))$, and

$$\|u\|_{BUC^{\theta-\alpha}([0, T]; D_A(\alpha, 1))} \leq C (\|u_0\|_{D_A(\theta, \infty)} + \|f\|_{B_{1-\theta}([0, T]; X)}).$$

Note that, if u_0 does not belong to the closure of $D(A)$, then u is not continuous up to $t = 0$. However, a mild continuity result at $t = 0$ holds. See next proposition.

Proposition 3.2.12. *Let $f \in L_1(0, T; X)$, $u_0 \in X$, and let u be the mild solution of (3.2.1). For every $\lambda \in \rho(A)$, it holds*

$$\lim_{t \rightarrow 0} \|R(\lambda, A)(u(t) - u_0)\|_{D_A(\theta, p)} = 0,$$

for each $\theta \in (0, 1)$, $p \geq 1$.

Proof The function $v = e^{tA} * f$ is continuous up to $t = 0$ with values in X , so that $\lim_{t \rightarrow 0} \|R(\lambda, A)v(t)\|_{D_A(\theta, p)} = 0$; moreover, since $R(\lambda, A)u_0 \in D(A)$, then

$$\lim_{t \rightarrow 0} \|R(\lambda, A)(e^{tA}u_0 - u_0)\|_{D_A(\theta, p)} = \lim_{t \rightarrow 0} \|(e^{tA} - I)R(\lambda, A)u_0\|_{D_A(\theta, p)} = 0.$$

□

3.2.3 Time regularity

The results of this subsection can be roughly grouped in two parts: the first part, concerning the case where f is uniformly Hölder continuous up to $t = 0$, and the second part, concerning the case where f is uniformly Hölder continuous in each interval $[\varepsilon, T]$, with $0 < \varepsilon < T$.

The case where f is uniformly Hölder continuous up to $t = 0$

Let u be mild solution of (3.2.1), and set $u = u_1 + u_2$, where

$$\begin{cases} u_1(t) = \int_0^t e^{(t-s)A} (f(s) - f(t)) ds, & 0 \leq t \leq T, \\ u_2(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s) ds, & 0 \leq t \leq T. \end{cases} \quad (3.2.20)$$

Theorem 3.2.13. *Let $0 < \alpha < 1$, $f \in BUC^\alpha([0, T]; X)$, $u_0 \in X$, and let u be the mild solution of (3.2.1). Then u belongs to $BUC^\alpha([\varepsilon, T]; D(A)) \cap BUC^{1+\alpha}([\varepsilon, T]; X)$ for every $\varepsilon \in (0, T)$, and*

(i) *if $u_0 \in \overline{D(A)}$, then u is a classical solution of (3.2.1);*

(ii) *if $u_0 \in D(A)$ and $Au_0 + f(0) \in \overline{D(A)}$, then u is a strict solution of (3.2.1), and there is C such that*

$$\|u\|_{C^1([0, T]; X)} + \|u\|_{C([0, T]; D(A))} \leq C (\|f\|_{BUC^\alpha([0, T]; X)} + \|u_0\|_{D(A)}); \quad (3.2.21)$$

(iii) *if $u_0 \in D(A)$ and $Au_0 + f(0) \in D_A(\alpha, \infty)$, then both u' and Au belong to $BUC^\alpha([0, T]; X)$, u' belongs to $B([0, T]; D_A(\alpha, \infty))$, and there is C such that*

$$\begin{aligned} & \|u\|_{BUC^{1+\alpha}([0, T]; X)} + \|Au\|_{BUC^\alpha([0, T]; X)} + \|u'\|_{B([0, T]; D_A(\alpha, \infty))} \\ & \leq C (\|f\|_{BUC^\alpha([0, T]; X)} + \|u_0\|_{D(A)} + \|Au_0 + f(0)\|_{D_A(\alpha, \infty)}). \end{aligned} \quad (3.2.22)$$

Proof Thanks to Lemma 3.2.6, to prove statement (i) and (ii), it is sufficient to show that u belongs to $C((0, T]; D(A))$ in the case where $u_0 \in \overline{D(A)}$, and to $C([0, T]; D(A))$ in the case where $u_0 \in D(A)$ and $Au_0 + f(0) \in \overline{D(A)}$.

Let u_1 and u_2 be defined by (3.2.20). Then $u_1(t) \in D(A)$ for $t \geq 0$, $u_2(t) \in D(A)$ for $t > 0$, and

$$\begin{cases} (i) & Au_1(t) = \int_0^t Ae^{(t-s)A}(f(s) - f(t))ds, \quad 0 \leq t \leq T, \\ (ii) & Au_2(t) = Ae^{tA}u_0 + (e^{tA} - 1)f(t), \quad 0 < t \leq T. \end{cases} \quad (3.2.23)$$

If $u_0 \in D(A)$, then (3.2.23)(ii) holds also for $t = 0$.

Let us show that Au_1 is uniformly Hölder continuous in $[0, T]$. For $0 \leq s < t \leq T$,

$$\begin{aligned} Au_1(t) - Au_1(s) &= \int_0^s A(e^{(t-\sigma)A} - e^{(s-\sigma)A})(f(\sigma) - f(s))d\sigma \\ &\quad + (e^{tA} - e^{(t-s)A})(f(s) - f(t)) \\ &\quad + \int_s^t Ae^{(t-\sigma)A}(f(\sigma) - f(t))d\sigma, \end{aligned} \quad (3.2.24)$$

so that

$$\begin{aligned} & \|Au_1(t) - Au_1(s)\| \\ & \leq M_2 \int_0^s (s - \sigma)^\alpha \int_{s-\sigma}^{t-\sigma} \tau^{-2} d\tau d\sigma [f]_\alpha + 2M_0(t - s)^\alpha [f]_\alpha \\ & \quad + M_1 \int_s^t (t - \sigma)^{\alpha-1} d\sigma [f]_\alpha \\ & \leq M_2 \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{\alpha-2} d\tau [f]_\alpha + (2M_0 + M_1\alpha^{-1})(t - s)^\alpha [f]_\alpha \\ & \leq \left(\frac{M_2}{\alpha(1 - \alpha)} + 2M_0 + \frac{M_1}{\alpha} \right) (t - s)^\alpha [f]_\alpha. \end{aligned} \quad (3.2.25)$$

Therefore, Au_1 is uniformly α -Hölder continuous in $[0, T]$. Moreover, Au_2 is obviously continuous in $(0, T]$: hence, if $u_0 \in \overline{D(A)}$, then $u \in C([0, T]; X)$, and $Au \in C((0, T]; X)$, so that, by Lemma 3.2.6, u is a classical solution of (3.2.1), and statement (i) is proved.

If $u_0 \in D(A)$ we have

$$Au_2(t) = e^{tA}(Au_0 + f(0)) + e^{tA}(f(t) - f(0)) - f(t), \quad 0 \leq t \leq T, \quad (3.2.26)$$

so that if $Au_0 + f(0) \in \overline{D(A)}$, then Au_2 is continuous also at $t = 0$, and statement (ii) follows.

In the case where $Au_0 + f(0) \in D_A(\alpha, \infty)$, from (3.2.26) we get, for $0 \leq s < t \leq T$,

$$\begin{aligned} & \|Au_2(t) - Au_2(s)\| \\ & \leq \|(e^{tA} - e^{sA})(Au_0 + f(0))\| + \|(e^{tA} - e^{sA})(f(s) - f(0))\| \\ & \quad + \|(e^{tA} - 1)(f(t) - f(s))\| \\ & \leq \int_s^t \|Ae^{\sigma A}\|_{\mathcal{L}(D_A(\alpha, \infty), X)} d\sigma \|Au_0 + f(0)\|_{D_A(\alpha, \infty)} \\ & \quad + s^\alpha \left\| A \int_s^t e^{\sigma A} d\sigma \right\|_{\mathcal{L}(X)} [f]_\alpha + (M_0 + 1)(t - s)^\alpha [f]_\alpha \\ & \leq \frac{M_{1, \alpha}}{\alpha} \|Au_0 + f(0)\|_{D_A(\alpha, \infty)} (t - s)^\alpha \\ & \quad + \left(\frac{M_1}{\alpha} + M_0 + 1 \right) (t - s)^\alpha [f]_\alpha, \end{aligned} \quad (3.2.27)$$

so that also Au_2 is uniformly α -Hölder continuous, and the estimate

$$\begin{aligned} & \|u\|_{BUC^{1+\alpha}([0, T]; X)} + \|Au\|_{BUC^\alpha([0, T]; X)} \\ & \leq C (\|f\|_{BUC^\alpha([0, T]; X)} + \|u_0\|_{D(A)} + \|Au_0 + f(0)\|_{D_A(\alpha, \infty)}) \end{aligned}$$

follows easily.

Since u' and Au are uniformly α -Hölder continuous, from Proposition 3.1.29(i), it follows that u' is bounded with values in $D_A(\alpha, \infty)$. However, the embedding constant given by Proposition 3.1.29(i) depends on the length T of the interval in such a way that it blows up as $T \rightarrow 0$. So we estimate $[u'(t)]_{D_A(\alpha, \infty)}$ directly. For $0 \leq t \leq T$ we have, by (3.2.23),

$$u'(t) = \int_0^t Ae^{(t-s)A}(f(s) - f(t))ds + e^{tA}(Au_0 + f(0)) + e^{tA}(f(t) - f(0)),$$

so that for $0 < \xi \leq 1$,

$$\begin{aligned} & \|\xi^{1-\alpha} Ae^{\xi A} u'(t)\| \\ & \leq \left\| \xi^{1-\alpha} \int_0^t A^2 e^{(t+\xi-s)A} (f(s) - f(t)) ds \right\| \\ & \quad + \|\xi^{1-\alpha} Ae^{(t+\xi)A} (Au_0 + f(0))\| + \|\xi^{1-\alpha} Ae^{(t+\xi)A} (f(t) - f(0))\| \\ & \leq M_2 \xi^{1-\alpha} \int_0^t (t-s)^\alpha (t+\xi-s)^{-2} ds [f]_\alpha \\ & \quad + M_0 [Au_0 + f(0)]_{D_A(\alpha, \infty)} + M_1 \xi^{1-\alpha} (t+\xi)^{-1} t^\alpha [f]_\alpha \\ & \leq M_2 \int_0^{+\infty} \sigma^\alpha (\sigma+1)^{-2} d\sigma [f]_\alpha + M_0 [Au_0 + f(0)]_{D_A(\alpha, \infty)} + M_1 [f]_\alpha. \end{aligned} \quad (3.2.28)$$

Therefore, $[u'(t)]_{D_A(\alpha, \infty)}$ is bounded in $[0, T]$, and the proof is complete. \square

Corollary 3.2.14. *Let $0 < \alpha < 1$, and let $f \in buc^\alpha([0, T]; X)$, $u_0 \in D(A)$. Assume moreover that*

$$Au_0 + f(0) \in D_A(\alpha). \quad (3.2.29)$$

Then the solution u of problem (3.2.1) belongs to $buc^{1+\alpha}([0, T]; X) \cap buc^\alpha([0, T]; D(A))$, and u' belongs to $C([0, T]; D_A(\alpha))$.

Proof Let $\{f_n\}_{n \in \mathbb{N}} \subset BUC^\infty([0, T]; X)$ be such that $f_n \rightarrow f$ in $BUC^\alpha([0, T]; X)$. Let $\lambda \in \rho(A)$ be fixed. Since $D(A^2)$ is dense in $D_A(\alpha + 1)$, and $u_0 + (A - \lambda I)^{-1}f(0) = (A - \lambda I)^{-1}(Au_0 + f(0) - \lambda u_0) \in D_A(\alpha + 1)$, thanks to (3.2.29), there is a sequence $\{y_n\}_{n \in \mathbb{N}} \subset D(A^2)$ such that $y_n \rightarrow u_0 + (A - \lambda I)^{-1}f(0)$ in $D_A(\alpha + 1, \infty)$. Set

$$x_n = y_n - (A - \lambda I)^{-1}f_n(0), \quad n \in \mathbb{N}.$$

Then $x_n \in D(A)$, and $Ax_n + f_n(0) = Ay_n - \lambda(A - \lambda I)^{-1}f_n(0) \in D(A)$. By means of Theorem 3.2.13, the solution u_n of problem

$$u'_n(t) = Au_n(t) + f_n(t), \quad 0 \leq t \leq T; \quad u_n(0) = x_n$$

belongs to $BUC^{1+\alpha+\varepsilon}([0, T]; X) \cap BUC^{\alpha+\varepsilon}([0, T]; D(A))$, and $u'_n \in B([0, T]; D_A(\alpha + \varepsilon, \infty))$ for every $\varepsilon \in (0, 1 - \alpha)$, so that $u_n \in buc^{1+\alpha}([0, T]; X) \cap buc^\alpha([0, T]; D(A))$ and $u'_n \in C([0, T]; D_A(\alpha))$ thanks to Proposition 3.1.29(ii). Moreover,

$$\begin{aligned} f_n &\rightarrow f \text{ in } BUC^\alpha([0, T]; X), & x_n &\rightarrow u_0 \text{ in } D(A), \\ Ax_n + f_n(0) &\rightarrow Au_0 + f(0) \text{ in } D_A(\alpha, \infty), \end{aligned}$$

thus, due again to Theorem 3.2.13, $u_n \rightarrow u$ in $BUC^{1+\alpha}([0, T]; X) \cap BUC^\alpha([0, T]; D(A))$ as $n \rightarrow \infty$, and $\{u'_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T], D_A(\alpha))$. The statement follows now easily. \square

Remark 3.2.15. From Proposition 3.1.29, it follows that if $f \in BUC^\alpha([0, T]; X)$ then the condition $Au_0 + f(0) \in D_A(\alpha, \infty)$ is necessary to get $u', Au \in BUC^\alpha([0, T]; X)$, and if $f \in buc^\alpha([0, T]; X)$ then the condition $Au_0 + f(0) \in D_A(\alpha)$ is necessary to get $u', Au \in buc^\alpha([0, T]; X)$. \square

The case where f is not uniformly Hölder continuous up to $t = 0$

In many applications to nonlinear or nonautonomous problems we have to deal with functions f that are not necessarily uniformly Hölder continuous up to $t = 0$.

Theorem 3.2.16. *Let $0 < \alpha < 1$, $f \in L_1(0, T; X) \cap BUC^\alpha([\varepsilon, T]; X)$ for every $\varepsilon \in (0, T)$, and let $u_0 \in X$. Then the mild solution u of problem (3.2.1) belongs to $C((0, T]; D(A)) \cap C^1((0, T]; X) \cap BUC^\alpha([\varepsilon, T]; D(A)) \cap BUC^{1+\alpha}([\varepsilon, T]; X)$, for every $\varepsilon \in (0, T)$. If in addition $u_0 \in \bar{D}(A)$, then u is a classical solution.*

Proof Let us split u as $u(t) = e^{tA}u_0 + v(t)$, where $v(t) = e^{tA} * f$. The first term is obviously continuous in $(0, T]$ with values in $D(A)$, and it belongs to $C([0, T]; X)$ if and only if $u_0 \in \overline{D(A)}$. So, we consider only the function v .

We showed in Proposition 3.2.10 that v is continuous in $[0, T]$ with values in X . Moreover, for $0 < \varepsilon \leq t \leq T$,

$$v(t) = e^{(t-\varepsilon/2)A}v(\varepsilon/2) + \int_{\varepsilon/2}^t e^{(t-s)A}f(s)ds = v_1(t) + v_2(t),$$

where $v_1 \in BUC^\infty([\varepsilon, T]; D(A^n))$ for every $n \in \mathbb{N}$, $v_2 \in BUC^\alpha([\varepsilon, T]; D(A)) \cap BUC^{1+\alpha}([\varepsilon, T]; X)$ thanks to Theorem 3.2.13. Since ε is arbitrary, $u \in C((0, T]; D(A)) \cap C^1((0, T]; X)$.

If in addition $u_0 \in \overline{D(A)}$, then $u \in C([0, T]; X)$ and it is a classical solution thanks to Lemma 3.2.6. The statement follows. \square

If we have more precise information on the behavior of the Hölder seminorm of f near $t = 0$, we can study the behavior of u near $t = 0$. In the next theorem we assume that f belongs to the weighted Hölder space $C_\alpha^\alpha((0, T]; X)$, or to $C([0, T]; X) \cap C_\alpha^\alpha((0, T]; X)$, with $0 < \alpha < 1$. We recall that $f \in C_\alpha^\alpha((0, T]; X)$ means that f is bounded and that $t \mapsto t^\alpha f(t)$ is α -Hölder continuous in $(0, T]$. Weighted Hölder spaces naturally arise in the study of parabolic evolution equations: for instance, the function

$$t \mapsto e^{tA}x, \quad 0 \leq t \leq T,$$

belongs to $C_\alpha^\alpha((0, T]; X)$ for all $x \in X$, and to $C([0, T], X) \cap C_\alpha^\alpha((0, T]; X)$ for all $x \in \overline{D(A)}$.

Theorem 3.2.17. *Let $0 < \alpha < 1$, $f \in C_\alpha^\alpha((0, T]; X)$, and set $v = e^{tA} * f$. Then $v \in C_\alpha^\alpha((0, T]; D(A))$, it is differentiable in $(0, T]$ with values in X , and $v' \in C_\alpha^\alpha((0, T]; X) \cap B_\alpha((0, T]; D_A(\alpha, \infty))$. There is $C > 0$ such that*

$$\|v'\|_{C_\alpha^\alpha(X)} + \|Av\|_{C_\alpha^\alpha(X)} + \|v'\|_{B_\alpha(D_A(\alpha, \infty))} \leq C\|f\|_{C_\alpha^\alpha(X)}. \quad (3.2.30)$$

Proof By Theorem 3.2.16, v belongs to $C((0, T]; D(A)) \cap C^1((0, T]; X)$. Let us show that it is bounded with values in $D(A)$. For $0 \leq t \leq T$ we have

$$\|v(t)\| \leq M_0 T \|f\|_{L^\infty(0, T; X)}.$$

Moreover, using the decomposition

$$v(t) = \int_0^t e^{(t-s)A}(f(s) - f(t))ds + \int_0^t e^{\sigma A}f(t)d\sigma,$$

we get

$$\begin{aligned} \|Av(t)\| &\leq M_1 \int_0^t (t-s)^{\alpha-1} s^{-\alpha} ds [f]_{C_\alpha^\alpha(X)} + \|(e^{tA} - I)f(t)\| \\ &\leq M_1 \int_0^1 (1-s)^{\alpha-1} s^{-\alpha} ds [f]_{C_\alpha^\alpha(X)} + (M_0 + 1)\|f\|_\infty. \end{aligned}$$

Thus,

$$\|v(t)\|_{D(A)} \leq C_1 \|f\|_{C_\alpha^\alpha((0,T];X)}, \quad 0 < t \leq T.$$

As a second step, we prove that $t^\alpha v'(t)$ is bounded with values in $D_A(\alpha, \infty)$, which means that $v' \in B_\alpha((0, T]; D_A(\alpha, \infty))$. Since

$$v'(t) = Av(t) + f(t) = \int_0^t Ae^{(t-s)A}(f(s) - f(t))ds + e^{tA}f(t),$$

then for each $t \in (0, T]$ and $\xi \in (0, 1]$ we have

$$\begin{aligned} & t^\alpha \xi^{1-\alpha} \|Ae^{\xi A} v'(t)\| \\ & \leq t^\alpha \left\| \xi^{1-\alpha} \int_0^{t/2} A^2 e^{(t+\xi-s)A} (f(s) - f(t)) ds \right\| \\ & \quad + t^\alpha \xi^{1-\alpha} \left(\left\| \int_{t/2}^t A^2 e^{(t+\xi-s)A} (f(s) - f(t)) ds \right\| + \|Ae^{(t+\xi)A} f(t)\| \right) \\ & \leq M_2 t^\alpha \xi^{1-\alpha} \int_0^{t/2} (t+\xi-s)^{-2} ds 2 \|f\|_\infty \\ & \quad + M_2 t^\alpha \xi^{1-\alpha} \int_{t/2}^t \frac{(t-s)^\alpha}{s^\alpha (t+\xi-s)^2} ds [f]_{C_\alpha^\alpha(X)} + M_1 t^\alpha \xi^{1-\alpha} (t+\xi)^{-1} \|f(t)\| \\ & \leq 2^{\alpha+1} M_2 \|f\|_\infty + 2^\alpha M_2 \int_0^{t/2} \sigma^\alpha (\sigma+1)^{-2} d\sigma [f]_{C_\alpha^\alpha(X)} + M_1 \|f\|_\infty. \end{aligned}$$

Hence, $t^\alpha v'(t)$ is bounded in $(0, T]$ with values in $D_A(\alpha, \infty)$, which implies that v' belongs to $B_\alpha((0, T]; D_A(\alpha, \infty))$. Further, there is C_2 such that

$$\|v'\|_{B_\alpha((0,T];D_A(\alpha,\infty))} \leq C_2 \|f\|_{C_\alpha^\alpha((0,T];X)}.$$

Let us prove that $v \in C_\alpha^\alpha((0, T]; D(A))$. For every $\varepsilon \in (0, T)$ it holds

$$v(t) = e^{(t-\varepsilon)A} v(\varepsilon) + \int_\varepsilon^t e^{(t-s)A} f(s) ds, \quad \varepsilon \leq t \leq T.$$

Since $f \in BUC^\alpha([\varepsilon, T]; X)$ and $v(\varepsilon) \in D(A)$, $Av(\varepsilon) + f(\varepsilon) = v'(\varepsilon) \in D_A(\alpha, \infty)$, then from Theorem 3.2.13(iii), applied in the interval $[\varepsilon, T]$ instead of $[0, T]$, it follows that $v \in BUC^\alpha([\varepsilon, T]; D(A)) \cap BUC^{1+\alpha}([\varepsilon, T]; X)$, and that $v' \in B([\varepsilon, T]; D_A(\alpha, \infty))$. From estimate (3.2.22) we get

$$\begin{aligned} & \|v\|_{BUC^\alpha([\varepsilon,T];D(A))} + \|v'\|_{BUC^\alpha([\varepsilon,T];X)} + \|v'\|_{B([\varepsilon,T];D_A(\alpha,\infty))} \\ & \leq C \|v(\varepsilon)\|_{D(A)} + \|v'(\varepsilon)\|_{D_A(\alpha,\infty)} + \|f\|_{BUC^\alpha([\varepsilon,T];X)} \\ & \leq C \left(C_1 \|f\|_{C_\alpha^\alpha((0,T];X)} + \frac{C_2}{\varepsilon^\alpha} \|f\|_{C_\alpha^\alpha((0,T];X)} + \frac{1}{\varepsilon^\alpha} \|f\|_{C_\alpha^\alpha((0,T];X)} \right), \end{aligned}$$

so that $v \in C_\alpha^\alpha((0, T]; D(A))$, and estimate (3.2.30) follows. \square

Corollary 3.2.18. *Let $0 < \alpha < 1$, $f \in C_\alpha^\alpha((0, T]; X)$, $u_0 \in X$, and let u be the mild solution of (3.2.1). The following statements hold.*

(i) If $u_0 \in \overline{D(A)}$, then u is a classical solution of (3.2.1);

(ii) if $u_0 \in D_A(1, \infty)$, then u' and Au belong to $C_\alpha^\alpha((0, T]; X)$, u' belongs to $B_\alpha((0, T]; D_A(\alpha, \infty))$, and there is $C > 0$ such that

$$\begin{aligned} & \|u'\|_{C_\alpha^\alpha((0, T]; X)} + \|Au\|_{C_\alpha^\alpha((0, T]; X)} + \|u'\|_{B_\alpha((0, T]; D_A(\alpha, \infty))} \\ & \leq C (\|f\|_{C_\alpha^\alpha((0, T]; X)} + \|u_0\|_{D_A(1, \infty)}); \end{aligned} \quad (3.2.31)$$

(iii) if $f \in C([0, T]; X)$, $u_0 \in D(A)$, and $Au_0 + f(0) \in \overline{D(A)}$, then $u', Au \in C([0, T]; X)$ and u is a strict solution of problem (3.2.1).

Proof Statement (i) follows obviously from Proposition 3.1.9(i) and Theorem 3.2.17.

To prove statement (ii) we have to check that for every $x \in D_A(1, \infty)$ the function $t \mapsto e^{tA}x$ belongs to $C_\alpha^\alpha((0, T]; D(A))$, and that $\|t^\alpha A e^{tA}x\|_{D_A(\alpha, \infty)}$ is bounded. For $0 < \varepsilon \leq r < t \leq T$ it holds

$$\begin{aligned} \|A(e^{tA}x - e^{rA}x)\| &= \left\| \int_r^t A^2 e^{\sigma A} x d\sigma \right\| \leq \frac{M_{2,1}}{\varepsilon^\alpha} \int_r^t \frac{d\sigma}{\sigma^{1-\alpha}} \|x\|_{D_A(1, \infty)} \\ &\leq \frac{M_{2,1}}{\alpha \varepsilon^\alpha} (t - r)^\alpha \|x\|_{D_A(1, \infty)}. \end{aligned}$$

Moreover, estimate (3.2.4) yields

$$\begin{aligned} \|t^\alpha A e^{tA}x\|_{D_A(\alpha, \infty)} &= \|t^\alpha A e^{tA}x\| + \sup_{0 < \xi \leq 1} \|t^\alpha \xi^{1-\alpha} A^2 e^{(t+\xi)A}x\| \\ &\leq T^\alpha M_{1,1} \|x\|_{D_A(1, \infty)} + M_{2,1} \|x\|_{D_A(1, \infty)}, \quad 0 < t \leq T. \end{aligned}$$

Statement (ii) follows now easily.

To prove statement (iii) we will show that Au is continuous up to $t = 0$, provided f is continuous up to $t = 0$ and $Au_0 + f(0) \in \overline{D(A)}$. Then the statement will follow from Lemma 3.2.6.

We know already that Au is continuous in $(0, T]$, so that we have only to prove that $Au(t) \rightarrow Au_0$ as $t \rightarrow 0$. Fix $\varepsilon > 0$, and let $\delta \in (0, 1)$ be such that

$$\int_{1-\delta}^1 (1 - \sigma)^{\alpha-1} \sigma^{-\alpha} d\sigma \leq \varepsilon.$$

Split again $u(t)$ as $u(t) = u_1(t) + u_2(t)$, where u_1 and u_2 are defined by (3.2.20). If t is so small that

$$\sup_{0 \leq s \leq t} \|f(s) - f(t)\| \leq \varepsilon |\log \delta|^{-1},$$

then

$$\begin{aligned} \|Au_1(t)\| &\leq \left\| \int_0^{t(1-\delta)} A e^{(t-s)A} (f(s) - f(t)) ds \right\| \\ &\quad + \left\| \int_{t(1-\delta)}^t A e^{(t-s)A} (f(s) - f(t)) ds \right\| \\ &\leq M_1 |\log \delta| \sup_{0 \leq s \leq t} \|f(s) - f(t)\| + M_1 \int_{1-\delta}^1 (1 - \sigma)^{\alpha-1} \sigma^{-\alpha} d\sigma [f]_{C_\alpha^\alpha(x)} \\ &\leq M_1 \varepsilon (1 + [f]_{C_\alpha^\alpha(x)}). \end{aligned}$$

Therefore, $Au_1(t) \rightarrow 0$ as $t \rightarrow 0$. Moreover, by (3.2.26) it follows easily that $Au_2(t) \rightarrow Au_0$ as $t \rightarrow 0$. Hence, Au is continuous up to $t = 0$. \square

Now we consider the case where f is unbounded near $t = 0$. The introduction of the weighted spaces $C_{\alpha+\mu}^\alpha((0, T]; X)$ and $C_{\alpha+\mu}^\alpha((0, T]; D(A))$ is motivated again by the behavior of $e^{tA}x$ as $t \rightarrow 0$: indeed, it is easy to see that if $x \in D_A(1-\mu, \infty)$, $0 < \mu < 1$, then $t \mapsto e^{tA}x$ belongs to $C_{\alpha+\mu}^\alpha((0, T]; D(A))$ for each $\alpha \in (0, 1)$. We recall that $f \in C_{\alpha+\mu}^\alpha((0, T]; X)$ means that $t \mapsto t^\mu f(t)$ is bounded and $t \mapsto t^{\alpha+\mu} f(t)$ is uniformly α -Hölder continuous in $(0, T]$ with values in X . Of course, since f is possibly unbounded, we will not get a strict solution but only a classical one.

Theorem 3.2.19. *Let $0 < \alpha, \mu < 1$, $f \in C_{\alpha+\mu}^\alpha((0, T]; X)$. Then $v = e^{tA} * f$ belongs to $C_{\alpha+\mu}^\alpha((0, T]; D(A))$, it is differentiable in $(0, T]$ with values in X , $v' \in C_{\alpha+\mu}^\alpha((0, T]; X) \cap B_{\alpha+\mu}((0, T]; D_A(\alpha, \infty))$, and there is C such that*

$$\|v\|_{C_{\alpha+\mu}^\alpha(D(A))} + \|v'\|_{C_{\alpha+\mu}^\alpha(X)} + \|v'\|_{B_{\alpha+\mu}(D_A(\alpha, \infty))} \leq C \|f\|_{C_{\alpha+\mu}^\alpha(X)}. \quad (3.2.32)$$

Consequently,

(i) if $u_0 \in \overline{D(A)}$, then the mild solution u of (3.2.1) is classical;

(ii) if $u_0 \in D_A(1-\mu, \infty)$, then u' and Au belong to $C_{\alpha+\mu}^\alpha((0, T]; X)$, u' belongs to $B_{\alpha+\mu}((0, T], D_A(\alpha, \infty))$, and there is C such that

$$\begin{aligned} & \|u\|_{C_{\alpha+\mu}^\alpha(D(A))} + \|u'\|_{C_{\alpha+\mu}^\alpha(X)} + \|u'\|_{B_{\alpha+\mu}(D_A(\alpha, \infty))} \\ & \leq C \left(\|u_0\|_{D_A(1-\mu, \infty)} + \|f\|_{C_{\alpha+\mu}^\alpha(X)} \right). \end{aligned} \quad (3.2.33)$$

Proof v belongs to $C((0, T]; D(A)) \cap C^1((0, T]; X)$ thanks to Theorem 3.2.16. To estimate $Av(t)$, we split it in three addenda, setting

$$Av(t) = \int_0^{t/2} Ae^{(t-s)A} f(s) ds + \int_{t/2}^t Ae^{(t-s)A} (f(s) - f(t)) ds + (e^{tA/2} - I)f(t),$$

and we get

$$\begin{aligned} t^\mu \|Av(t)\| & \leq M_1 t^\mu \int_0^{t/2} (t-s)^{-1} s^{-\mu} ds \|f\|_{B_\mu(X)} + (M_0 + 1) \|f\|_{B_\mu(X)} \\ & \quad + M_1 t^\mu \int_{t/2}^t (t-s)^{\alpha-1} s^{-\alpha-\mu} ds [f]_{C_{\alpha+\mu}^\alpha(X)} \\ & \leq M_1 \int_0^{1/2} (1-\sigma)^{-1} \sigma^{-\mu} d\sigma \|f\|_{B_\mu(X)} + (M_0 + 1) \|f\|_{B_\mu(X)} \\ & \quad + M_1 \int_{1/2}^1 (1-\sigma)^{\alpha-1} \sigma^{-\alpha-\mu} d\sigma [f]_{C_{\alpha+\mu}^\alpha(X)}. \end{aligned}$$

So, Av belongs to $B_\mu((0, T]; X)$. Now we estimate $[v']_{D_A(\alpha, \infty)}$. For $0 < t \leq T$ and $0 < \xi \leq 1$,

$$\begin{aligned} t^{\alpha+\mu} \|\xi^{1-\alpha} A e^{\xi A} v'(t)\| &\leq t^{\alpha+\mu} \xi^{1-\alpha} \left\| \int_0^{t/2} A^2 e^{(t+\xi-s)A} (f(s) - f(t)) ds \right\| \\ &\quad + t^{\alpha+\mu} \xi^{1-\alpha} \left\| \int_{t/2}^t A^2 e^{(t+\xi-s)A} (f(s) - f(t)) ds \right\| \\ &\quad + t^{\alpha+\mu} \|\xi^{1-\alpha} A e^{(t+\xi)A} f(t)\| \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &\leq t^{\alpha+\mu} \xi^{1-\alpha} \left\| \int_0^{t/2} A^2 e^{(t+\xi-s)A} f(s) ds \right\| \\ &\quad + t^{\alpha+\mu} \xi^{1-\alpha} \|(Ae^{(t+\xi)A} - Ae^{(t/2+\xi)A})f(t)\| \\ &\leq M_2 t^{\alpha+\mu} \xi^{1-\alpha} \int_0^{t/2} (t+\xi-s)^{-2} s^{-\mu} ds \|f\|_{B_\mu(X)} \\ &\quad + t^{\alpha+\mu} \xi^{1-\alpha} \|(Ae^{(t+\xi)A} - Ae^{(t/2+\xi)A})f(t)\| \\ &\leq M_2 \int_0^{1/2} (1-\sigma)^{-2} \sigma^{-\mu} d\sigma \|f\|_{B_\mu(X)} + M_2 (1+2^\alpha) \|f\|_{B_\mu(X)}; \\ I_2 &\leq M_2 t^{\alpha+\mu} \xi^{1-\alpha} \int_{t/2}^t \frac{(t-s)^\alpha}{s^{\alpha+\mu} (t+\xi-s)^2} ds [f]_{C_{\alpha+\mu}^\alpha(X)} \\ &\leq 2^{\alpha+\mu} M_2 \int_0^{+\infty} \sigma^\alpha (\sigma+1)^{-2} d\sigma [f]_{C_{\alpha+\mu}^\alpha(X)}; \\ I_3 &\leq M_1 t^{\alpha+\mu} \xi^{1-\alpha} (t+\xi)^{-1} \|f(t)\| \leq M_1 \|f\|_{B_\mu(X)}. \end{aligned}$$

To conclude the proof, it is sufficient to argue as we did in the proof of Theorem 3.2.17 and of Corollary 3.2.18(i)(ii). \square

3.2.4 Space regularity

We consider several regularity assumptions on f , which are suggested by the behavior of $t \mapsto e^{tA} u_0$ and of $t \mapsto d/dt e^{tA} u_0 = A e^{tA} u_0$ as functions with values in $D_A(\alpha, \infty)$. To begin with, we consider the case where $u_0 \in D_A(\alpha+1, \infty)$, then $A e^{tA} u_0$ is continuous for $t > 0$ and bounded near $t = 0$ with values in $D_A(\alpha, \infty)$. If in addition $u_0 \in D_A(\alpha+1)$, then $t \mapsto A e^{tA} u_0$ is continuous up to $t = 0$ with values in $D_A(\alpha)$. So, first we consider the case where f is bounded with values in $D_A(\alpha, \infty)$, and it has some continuity property, at least for $t > 0$. Then we consider the case of unbounded f .

The case where f is bounded with values in $D_A(\alpha, \infty)$

Theorem 3.2.20. *Let $0 < \alpha < 1$, and let $f \in C((0, T]; X) \cap B((0, T]; D_A(\alpha, \infty))$. Then $v = e^{tA} * f$ has values in $D(A)$, it is differentiable for $t > 0$ with values in X ,*

and it is the classical solution of

$$v'(t) = Av(t) + f(t), \quad 0 < t \leq T, \quad v(0) = 0. \quad (3.2.34)$$

Moreover, v' and Av belong to $C((0, T]; X) \cap B((0, T]; D_A(\alpha, \infty))$, Av belongs to $BUC^\alpha([0, T]; X)$, and there is C such that

$$\|v'\|_{B(D_A(\alpha, \infty))} + \|Av\|_{B(D_A(\alpha, \infty))} + \|Av\|_{BUC^\alpha(X)} \leq C\|f\|_{B(D_A(\alpha, \infty))}. \quad (3.2.35)$$

In addition,

- (i) if $f \in C([0, T]; X) \cap B([0, T]; D_A(\alpha, \infty))$, then v' and Av are continuous with values in X up to $t = 0$, and v is a strict solution of (3.2.34);
- (ii) if $f \in C((0, T]; D_A(\alpha, \infty)) \cap B([0, T]; D_A(\alpha, \infty))$, then $v \in C((0, T]; D_A(\alpha + 1, \infty)) \cap B([0, T]; D_A(\alpha + 1, \infty)) \cap C^1((0, T]; D_A(\alpha, \infty))$.

Proof Let us show that v is a classical solution of (3.2.34), and that (3.2.35) holds. For $0 \leq t \leq T$, $v(t)$ belongs to $D(A)$, and

$$\|Av(t)\| \leq M_{1,\alpha} \int_0^t (t-s)^{\alpha-1} ds \|f\|_{B(D_A(\alpha, \infty))} = \frac{T^\alpha M_{1,\alpha}}{\alpha} \|f\|_{B(D_A(\alpha, \infty))}. \quad (3.2.36)$$

Moreover, for $0 < \xi \leq 1$,

$$\begin{aligned} \|\xi^{1-\alpha} A e^{\xi A} Av(t)\| &= \xi^{1-\alpha} \left\| \int_0^t A^2 e^{(t+\xi-s)A} f(s) ds \right\| \\ &\leq M_{2,\alpha} \xi^{1-\alpha} \int_0^t (t+\xi-s)^{\alpha-2} ds \|f\|_{B(D_A(\alpha, \infty))} \\ &\leq \frac{M_{2,\alpha}}{1-\alpha} \|f\|_{B(D_A(\alpha, \infty))}, \end{aligned} \quad (3.2.37)$$

so that Av is bounded with values in $D_A(\alpha, \infty)$.

Let us show that Av is uniformly Hölder continuous with values in X : for $0 \leq s < t \leq T$, we have

$$\begin{aligned} &\|Av(t) - Av(s)\| \\ &\leq \left\| A \int_0^s (e^{(t-\sigma)A} - e^{(s-\sigma)A}) f(\sigma) d\sigma \right\| + \left\| A \int_s^t e^{(t-\sigma)A} f(\sigma) d\sigma \right\| \\ &\leq \left(M_{2,\alpha} \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{\alpha-2} d\tau + M_{1,\alpha} \int_s^t (t-\sigma)^{\alpha-1} d\sigma \right) \|f\|_{B(D_A(\alpha, \infty))} \\ &\leq \left(\frac{M_{2,\alpha}}{\alpha(1-\alpha)} + \frac{M_{1,\alpha}}{\alpha} \right) (t-s)^\alpha \|f\|_{B(D_A(\alpha, \infty))}, \end{aligned} \quad (3.2.38)$$

so that Av is uniformly α -Hölder continuous in $[0, T]$. Estimate (3.2.35) follows now from (3.2.36), (3.2.37), (3.2.38). Moreover, thanks to Lemma 3.2.6, v is a classical solution of (3.2.34) if $f \in C((0, T]; X)$, and it is a strict solution if $f \in C([0, T]; X)$.

Let us prove statement (ii). Let $f \in C((0, T]; D_A(\alpha, \infty)) \cap B([0, T], D_A(\alpha, \infty))$. We are going to show that Av belongs to $C([a, T]; D_A(\alpha, \infty))$ for every $a \in (0, T)$. For $\varepsilon > 0$, let $\delta \in (0, \varepsilon \wedge a]$ be such that for $t_1, t_2 \in [a, T]$, $|t_1 - t_2| \leq \delta$, we have $\|f(t_1) - f(t_2)\|_{D_A(\alpha, \infty)} \leq \varepsilon$, here $\varepsilon \wedge a := \min\{\varepsilon, a\}$. Then for $a \leq r < t \leq T$, $t - r \leq \delta$, and for $0 < \xi \leq 1$

$$\begin{aligned}
& \left\| \xi^{1-\alpha} A e^{\xi A} (Av(t) - Av(r)) \right\| \\
& \leq \left\| \xi^{1-\alpha} A^2 \int_{r-\delta}^r e^{(\xi+s)A} [f(t-s) - f(r-s)] ds \right\| \\
& \quad + \left\| \xi^{1-\alpha} A^2 \int_0^{r-\delta} e^{(\xi+s)A} [f(t-s) - f(r-s)] ds \right\| \\
& \quad + \left\| \xi^{1-\alpha} A^2 \int_r^t e^{(\xi+s)A} f(t-s) ds \right\| \\
& \leq \frac{M_{2,\alpha}}{1-\alpha} \xi^{1-\alpha} (\xi^{-1+\alpha} - (\xi+r-\delta)^{-1+\alpha}) \varepsilon \\
& \quad + \frac{M_{2,\alpha}}{1-\alpha} \xi^{1-\alpha} ((\xi+r-\delta)^{-1+\alpha} - (\xi+r)^{-1+\alpha}) 2 \|f\|_{B([0,T]; D_A(\alpha, \infty))} \\
& \quad + \frac{M_{2,\alpha}}{1-\alpha} \xi^{1-\alpha} ((\xi+r)^{-1+\alpha} - (\xi+t)^{-1+\alpha}) \|f\|_{B([0,T]; D_A(\alpha, \infty))} \\
& \leq \frac{M_{2,\alpha}}{1-\alpha} \left(\varepsilon + 3 \|f\|_{B([0,T]; D_A(\alpha, \infty))} \frac{\varepsilon^{1-\alpha}}{a^{1-\alpha}} \right).
\end{aligned}$$

Taking the supremum over $\xi \in (0, 1]$, we see that $[Av(t) - Av(r)]_{D_A(\alpha, \infty)}$ goes to 0 as $t - r$ goes to 0. Since Av is continuous with values in X , then $v \in C((0, T]; D_A(\alpha + 1, \infty))$. From the equality $v' = Av + f$ it follows that v' is continuous in $(0, T]$ with values in $D_A(\alpha, \infty)$, and hence that v is continuously differentiable in $(0, T]$ with values in $D_A(\alpha, \infty)$. \square

Corollary 3.2.21. *Let $0 < \alpha < 1$, $u_0 \in X$, $f \in C((0, T]; X) \cap B((0, T]; D_A(\alpha, \infty))$, and let u be the mild solution of (3.2.1). Then $u \in C^1((0, T]; X) \cap C((0, T]; D(A))$, and $u \in B([\varepsilon, T]; D_A(\alpha, \infty))$ for every $\varepsilon \in (0, T)$. Moreover, the following statements hold.*

- (i) *if $u_0 \in \overline{D(A)}$, then u is a classical solution;*
- (ii) *if $u_0 \in D(A)$, $Au_0 \in \overline{D(A)}$, and $f \in C([0, T]; X) \cap B([0, T]; D_A(\alpha, \infty))$, then u is a strict solution;*
- (iii) *if $u_0 \in D_A(\alpha + 1, \infty)$ and $f \in C([0, T]; X) \cap B([0, T]; D_A(\alpha, \infty))$, then u' and Au belong to $C([0, T]; X) \cap B([0, T]; D_A(\alpha, \infty))$, Au belongs to $BUC^\alpha([0, T]; X)$, and there is C such that*

$$\begin{aligned}
& \|u'\|_{B(D_A(\alpha, \infty))} + \|Au\|_{B(D_A(\alpha, \infty))} + \|Au\|_{BUC^\alpha(X)} \\
& \leq C (\|f\|_{B(D_A(\alpha, \infty))} + \|u_0\|_{D_A(\alpha+1, \infty)});
\end{aligned} \tag{3.2.39}$$

- (iv) *if $u_0 \in D_A(\alpha + 1, \infty)$ and $f \in C((0, T]; D_A(\alpha, \infty))$, then $u \in C^1((0, T]; D_A(\alpha, \infty)) \cap C((0, T]; D_A(\alpha + 1, \infty))$.*

Corollary 3.2.22. *Let $0 < \alpha < 1$, and let $f \in C([0, T]; D_A(\alpha))$, $u_0 \in D_A(\alpha + 1)$. Then the solution u of problem (3.2.1) belongs to $C([0, T]; D_A(\alpha + 1)) \cap C^1([0, T]; D_A(\alpha))$, and u' belongs to $buc^\alpha([0, T]; X)$.*

The case where f is unbounded with values in $D_A(\alpha, \infty)$

In the next theorem we show that problem (3.2.1) may have a classical solution even if f is unbounded with values in $D_A(\alpha, \infty)$ near $t = 0$.

Theorem 3.2.23. *Let $0 < \alpha < 1$, $f \in L_1(0, T; X) \cap C((0, T]; X) \cap B([\varepsilon, T]; D_A(\alpha, \infty))$ for every $\varepsilon \in (0, T)$, and let $u_0 \in X$. Then the mild solution u of problem (3.2.1) belongs to $C((0, T]; D(A)) \cap C^1((0, T]; X) \cap B([\varepsilon, T]; D_A(\alpha + 1, \infty))$, for every $\varepsilon \in (0, T)$. If in addition, $u_0 \in D(A)$, then u is a classical solution.*

If f belongs also to $C((0, T]; D_A(\alpha, \infty))$ (respectively, to $C((0, T]; D_A(\alpha))$), then v' and Av belong to $C((0, T]; D_A(\alpha, \infty))$ (respectively, to $C((0, T]; D_A(\alpha))$).

Proof The proof is similar to the proof of Theorem 3.2.16. We split u as $u(t) = e^{tA}u_0 + v(t)$, where $v = e^{tA} * f$. The first addendum is obviously continuous in $(0, T]$ with values in $D(A)$, and it belongs to $C([0, T]; X)$ if and only if $u_0 \in \overline{D(A)}$.

Let us consider the function v . By Proposition 3.2.10, v is continuous in $[0, T]$ with values in X . Moreover, for $0 < \varepsilon \leq t \leq T$,

$$v(t) = e^{(t-\varepsilon)A}v(\varepsilon) + \int_{\varepsilon}^t e^{(t-s)A}f(s)ds = v_1(t) + v_2(t),$$

where v_1 belongs to $C^\infty((\varepsilon, T]; D(A^n))$ for every n , and v_2 belongs to $C([\varepsilon, T]; D(A)) \cap C^1([\varepsilon, T]; X) \cap B([\varepsilon, T]; D_A(\alpha + 1, \infty))$, thanks to Theorem 3.2.20. Since ε is arbitrary, then $u \in C((0, T]; D(A)) \cap C^1((0, T]; X)$.

If in addition $u_0 \in \overline{D(A)}$, then $u \in C([0, T]; X)$, so that it is a classical solution thanks to Lemma 3.2.6.

Let now f be continuous in $(0, T]$ with values in $D_A(\alpha, \infty)$ (respectively, $D_A(\alpha)$). For $0 < \varepsilon \leq t \leq T$, split $v(t) = v_1(t) + v_2(t)$ as above. Then v_1 belongs to $C^\infty((\varepsilon, T]; D(A^n))$ for every $n \in \mathbb{N}$, and v_2 belongs to $C([\varepsilon, T]; D_A(\alpha + 1, \infty))$ (respectively, to $C([\varepsilon, T]; D_A(\alpha + 1))$) thanks to Theorem 3.2.20(ii) (respectively, to Corollary 3.2.22) applied in the interval $[\varepsilon, T]$. Since ε is arbitrary, then v' and Av belong to $C((0, T]; D_A(\alpha, \infty))$ (respectively, to $C((0, T]; D_A(\alpha))$). \square

If we know how $\|f(t)\|_{D_A(\alpha, \infty)}$ blows up as $t \rightarrow 0$, we may give precise information on the behavior of $u(t)$ as $t \rightarrow 0$, getting also optimal regularity results. In the next theorem we consider the case where f belongs to $B_\theta((0, T]; D_A(\alpha, \infty))$, that is $\|t^\theta f(t)\|_{D_A(\alpha, \infty)}$ is bounded, for some $\theta \in (0, 1)$.

Theorem 3.2.24. *Let $0 < \alpha, \theta < 1$. and let $f \in C((0, T]; X) \cap B_\theta((0, T]; D_A(\alpha, \infty))$. Then $v = e^{tA} * f$ has values in $D(A)$, it is differentiable with values in X for $t > 0$, and it is a classical solution of (3.2.34). Moreover, v' and Av belong to $C((0, T]; X) \cap B_\theta((0, T]; D_A(\alpha, \infty))$, Av belongs to $C_\theta^\alpha((0, T]; X)$, and there is C such that*

$$\|v'\|_{B_\theta(D_A(\alpha, \infty))} + \|Av\|_{B_\theta(D_A(\alpha, \infty))} + \|Av\|_{C_\theta^\alpha(X)} \leq C\|f\|_{B_\theta(D_A(\alpha, \infty))}. \quad (3.2.40)$$

In particular, if $\theta = \alpha$ then v is bounded with values in $D(A)$. If $\theta < \alpha$, then Av belongs to $BUC^{\alpha-\theta}([0, T]; X)$, and there is C such that

$$\|Av\|_{BUC^{\alpha-\theta}(X)} \leq C \|f\|_{B_\theta(D_A(\alpha, \infty))}. \quad (3.2.41)$$

If f belongs also to $C((0, T]; D_A(\alpha, \infty))$ (respectively, to $C((0, T]; D_A(\alpha))$), then v' and Av belong to $C((0, T]; D_A(\alpha, \infty))$ (respectively, to $C((0, T]; D_A(\alpha))$).

Proof By Theorem 3.2.23, v is continuous in $(0, T]$ with values in $D(A)$, and continuously differentiable in $(0, T]$ with values in X . Moreover,

$$\begin{aligned} \|Av(t)\| &\leq M_{1,\alpha} \int_0^t (t-s)^{\alpha-1} s^{-\theta} ds \|f\|_{B_\theta(D_A(\alpha, \infty))} \\ &= M_{1,\alpha} t^{\alpha-\theta} \int_0^1 (1-\sigma)^{\alpha-1} \sigma^{-\theta} d\sigma \|f\|_{B_\theta(D_A(\alpha, \infty))} \\ &\leq K t^{\alpha-\theta} \|f\|_{B_\theta(D_A(\alpha, \infty))}. \end{aligned} \quad (3.2.42)$$

Fix now any $r \in (0, T)$. For $t \geq r$, it holds

$$Av(t) = Ae^{(t-r/2)A} v(r/2) + \int_{r/2}^t Ae^{(t-s)A} f(s) ds = Av_1(t) + Av_2(t).$$

Denoting by C_α the norm of the embedding $D_A(\alpha, 1) \subset D_A(\alpha, \infty)$ and using (3.2.5), we get

$$\begin{aligned} \|Av_1(t)\|_{D_A(\alpha, \infty)} &\leq \frac{K_{0,\alpha} C_\alpha}{(t-r/2)^\alpha} \|Av(r/2)\| \\ &\leq \frac{K_{0,\alpha} C_\alpha}{(r/2)^\alpha} K (r/2)^{\alpha-\theta} \|f\|_{B_\theta(D_A(\alpha, \infty))} = \frac{K_{0,\alpha} C_\alpha K}{(r/2)^\theta} \|f\|_{B_\theta(D_A(\alpha, \infty))}. \end{aligned}$$

Moreover, from Theorem 3.2.20 applied in the interval $[r/2, T]$, we get for every $t \in [r/2, T]$,

$$\|Av_2(t)\|_{D_A(\alpha, \infty)} \leq C \sup_{r/2 \leq s \leq T} \|f(s)\|_{D_A(\alpha, \infty)} \leq \frac{C}{(r/2)^\theta} \|f\|_{B_\theta(D_A(\alpha, \infty))}.$$

Summing up and taking $t = r$ we see that

$$\|r^\theta Av(r)\|_{D_A(\alpha, \infty)} \leq (2^\theta K_{0,\alpha} C_\alpha K + 2^\theta C) \|f\|_{B_\theta(D_A(\alpha, \infty))},$$

which implies that $Av \in B_\theta((0, T]; D_A(\alpha, \infty))$.

The proof of the statements about the Hölder continuity of Av is similar to the proof of statements (ii)-(iii) of Proposition 3.2.10. We have seen above that $v(r/2)$ belongs to $D(A)$. Moreover, we have seen in the proof of Corollary 3.2.18 that for every $x \in D(A)$ the function $t \mapsto Ae^{(t-r/2)A} x$ belongs to $C_\alpha^\alpha((r/2, T]; X)$, and its C_α^α -norm is less or equal to $C \|Ax\|$, for some $C > 0$. Using estimate (3.2.42) we get

$$\|Av_1\|_{BUC^\alpha([r, T]; X)} \leq C (r/2)^{-\alpha} \|Av(r/2)\| \leq CK (r/2)^{-\theta} \|f\|_{B_\theta(D_A(\alpha, \infty))}.$$

Due again to Theorem 3.2.20, Av_2 belongs to $BUC^\alpha([r/2, T]; X)$, and that

$$\|Av_2\|_{BUC^\alpha([r/2, T]; X)} \leq C\|f\|_{B([r/2, T]; D_A(\alpha, \infty))} \leq C(r/2)^{-\theta}\|f\|_{B_\theta(D_A(\alpha, \infty))}.$$

Summing up we get

$$\|Av\|_{BUC^\alpha([r, T]; X)} \leq \text{const.} \cdot r^{-\theta}\|f\|_{B_\theta(D_A(\alpha, \infty))},$$

which implies that Av belongs to $C_\theta^\alpha((0, T]; X)$. Estimate (3.2.40) follows now easily.

Let us prove that if $\theta < \alpha$ then $Av \in BUC^{\alpha-\theta}([0, T]; X)$. For $0 \leq r < t \leq T$, we have

$$\begin{aligned} \|Av(t) - Av(r)\| &\leq \left\| \int_0^r \left(\int_{r-s}^{t-s} A^2 e^{\sigma A} d\sigma \right) f(s) ds \right\| + \left\| \int_r^t A e^{(t-s)A} f(s) ds \right\| \\ &\leq \left(M_{2,\alpha} \int_0^r s^{-\theta} \int_{r-s}^{t-s} \frac{1}{\sigma^{2-\alpha}} d\sigma ds + M_{1,\alpha} \int_r^t \frac{1}{(t-s)^{1-\alpha} s^\theta} ds \right) \|f\|_{B_\theta(D_A(\alpha, \infty))}. \end{aligned}$$

Following the proof of statement (iii) of Proposition 3.2.10, we find

$$\begin{aligned} \|Av(t) - Av(r)\| &\leq \frac{1}{\alpha - \theta} \left(M_{2,\alpha} \int_0^1 \frac{d\sigma}{(1-\sigma)^{1-\theta} \sigma^\theta} \right. \\ &\quad \left. + M_{1,\alpha} \int_0^1 \frac{d\sigma}{(1-\sigma)^{1-\alpha} \sigma^\theta} \right) (t-r)^{\alpha-\theta} \|f\|_{B_\theta(D_A(\alpha, \infty))}, \end{aligned}$$

and estimate (3.2.41) follows.

The last statement follows from Theorem 3.2.23. \square

Corollary 3.2.25. *Let $0 < \alpha < 1$, $u_0 \in X$, $f \in C((0, T]; X) \cap B_\theta((0, T]; D_A(\alpha, \infty))$, and let u be the mild solution of problem (3.2.1). The following statements hold.*

(i) *If $u_0 \in \overline{D(A)}$, then u is a classical solution.*

(ii) *If $u_0 \in D_A(\alpha + 1 - \theta, \infty)$, then u' and Au belong to $B_\theta((0, T]; D_A(\alpha, \infty))$, Au belongs to $C_\theta^\alpha((0, T]; X)$, and there is C such that*

$$\begin{aligned} \|u'\|_{B_\theta(D_A(\alpha, \infty))} + \|Au\|_{B_\theta(D_A(\alpha, \infty))} + \|Au\|_{C_\theta^\alpha(X)} \\ \leq C \left(\|f\|_{B_\theta(D_A(\alpha, \infty))} + \|u_0\|_{D_A(\alpha+1-\theta, \infty)} \right). \end{aligned} \quad (3.2.43)$$

Moreover, in the case where $\theta < \alpha$, then $u \in BUC^{\alpha-\theta}([0, T]; D(A))$, and there is C such that

$$\|Au\|_{BUC^{\alpha-\theta}(X)} \leq C \left(\|f\|_{B_\theta(D_A(\alpha, \infty))} + \|u_0\|_{D_A(\alpha+1-\theta, \infty)} \right). \quad (3.2.44)$$

(iii) *If in addition $f \in C((0, T]; D_A(\alpha, \infty))$ (respectively, $f \in C((0, T]; D_A(\alpha))$), then u' , Au belong to $C((0, T]; D_A(\alpha, \infty))$ (respectively, to $C((0, T]; D_A(\alpha))$).*

(iv) In the case $\theta = \alpha$, if $u_0 \in D(A)$, $Au_0 \in \overline{D(A)}$, and $f \in C([0, T]; X) \cap B_\alpha((0, T]; D_A(\alpha, \infty))$, $\lim_{t \rightarrow 0} \|t^\alpha f(t)\|_{D_A(\alpha, \infty)} = 0$, then u is a strict solution and

$$\lim_{t \rightarrow 0} \|t^\alpha Au(t)\|_{D_A(\alpha, \infty)} = \lim_{t \rightarrow 0} \|t^\alpha u'(t)\|_{D_A(\alpha, \infty)} = 0.$$

Theorem 3.2.26. Let $0 < \alpha, \mu < 1$, and let $f \in C((0, T]; X) \cap B_\mu((0, T]; X) \cap B_{\alpha+\mu}((0, T]; D_A(\alpha, \infty))$. Then $v = e^{tA} * f$ is a classical solution of (3.2.34), v' , Av belong to $B_\mu((0, T]; X) \cap B_{\alpha+\mu}((0, T]; D_A(\alpha, \infty))$, Av belongs to $C_{\alpha+\mu}^\alpha((0, T]; X)$, and there is C such that

$$\begin{aligned} & \|v'\|_{B_\mu(X)} + \|v'\|_{B_{\alpha+\mu}(D_A(\alpha, \infty))} + \|Av\|_{B_{\alpha+\mu}(D_A(\alpha, \infty))} \\ & + \|Av\|_{C_{\alpha+\mu}^\alpha(X)} \leq C \left(\|f\|_{B_\mu(X)} + \|f\|_{B_{\alpha+\mu}(D_A(\alpha, \infty))} \right). \end{aligned} \quad (3.2.45)$$

If f belongs also to $C((0, T]; D_A(\alpha, \infty))$ (respectively, to $C((0, T]; D_A(\alpha))$), then v' and Av belong to $C((0, T]; D_A(\alpha, \infty))$ (respectively, to $C((0, T]; D_A(\alpha))$).

Proof By Theorem 3.2.23, v is continuous in $(0, T]$ with values in $D(A)$. Let us prove that Av belongs to $B_\mu((0, T]; X)$. For $0 < t \leq T$, we have

$$\|t^\mu Av(t)\| \leq t^\mu \left\| \int_0^{t/2} Ae^{(t-s)A} f(s) ds \right\| + t^\mu \left\| \int_{t/2}^t Ae^{(t-s)A} f(s) ds \right\|,$$

so that

$$\begin{aligned} \|t^\mu Av(t)\| & \leq M_1 t^\mu \int_0^{t/2} (t-s)^{-1} s^{-\mu} ds \|f\|_{B_\mu(X)} \\ & \quad + 2^\mu M_{1,\alpha} \int_{t/2}^t (t-s)^{\alpha-1} s^{-\alpha} ds \|f\|_{B_{\alpha+\mu}(D_A(\alpha, \infty))} \\ & \leq M_1 \int_0^{1/2} (1-\sigma)^{-1} \sigma^{-\mu} d\sigma \|f\|_{B_\mu(X)} \\ & \quad + 2^\mu M_{1,\alpha} \int_{1/2}^1 (1-\sigma)^{\alpha-1} \sigma^{-\alpha} d\sigma \|f\|_{B_{\alpha+\mu}(D_A(\alpha, \infty))}. \end{aligned}$$

From now on, the proof is similar to the proof of Theorem 3.2.24. \square

Corollary 3.2.27. Let $0 < \alpha, \mu < 1$, $u_0 \in X$, $f \in C((0, T]; X) \cap B_\mu((0, T]; X) \cap B_{\alpha+\mu}((0, T]; D_A(\alpha, \infty))$, and let u be the mild solution of problem (3.2.1). Then

(i) if $u_0 \in \overline{D(A)}$, then u is a classical solution;

(ii) if $u_0 \in D_A(1-\mu, \infty)$, then u' and Au belong to $B_\mu((0, T]; X) \cap B_{\alpha+\mu}((0, T]; D_A(\alpha, \infty))$, Au belongs to $C_{\alpha+\mu}^\alpha((0, T]; X)$, and there is $C > 0$ such that

$$\begin{aligned} & \|u'\|_{B_\mu(X)} + \|u'\|_{B_{\alpha+\mu}(D_A(\alpha, \infty))} + \|Au\|_{B_{\alpha+\mu}(D_A(\alpha, \infty))} + \|Au\|_{C_{\alpha+\mu}^\alpha(X)} \\ & \leq C \left(\|f\|_{B_\mu(X)} + \|f\|_{B_{\alpha+\mu}(D_A(\alpha, \infty))} + \|u_0\|_{D_A(1-\mu, \infty)} \right). \end{aligned} \quad (3.2.46)$$

3.2.5 A further regularity result

Theorem 3.2.28. *Let $0 < \theta, \beta < 1$, with $\theta + \beta \neq 1$. Let*

$$f \in BUC^\theta([0, T]; D_A(\beta, \infty)), \quad u_0 \in D(A), \quad Au_0 + f(0) \in D_A(\theta + \beta, \infty).$$

Then the mild solution u of problem (3.2.1) is strict, u' and Au belong to $BUC^\theta([0, T]; D_A(\beta, \infty))$, $u' \in B([0, T]; D_A(\theta + \beta, \infty))$, $Au \in BUC^{\theta+\beta}([0, T]; X)$, and there exists $C > 0$ such that

$$\begin{aligned} & \|u\|_{BUC^{\theta+1}(D_A(\beta, \infty))} + \|Au\|_{BUC^\theta(D_A(\beta, \infty))} + \|u'\|_{B(D_A(\theta+\beta, \infty))} + \|Au\|_{BUC^{\theta+\beta}(X)} \\ & \leq C \left(\|f\|_{BUC^\theta(D_A(\beta, \infty))} + \|u_0\|_{D(A)} + \|Au_0 + f(0)\|_{D_A(\theta+\beta, \infty)} \right). \end{aligned} \quad (3.2.47)$$

Proof Let us consider problem (3.2.1) as an evolution equation in the space $Y = D_A(\beta, \infty)$. The domain of the part of A in Y is $D(A_Y) = D_A(\beta + 1, \infty)$, and the Reiteration Theorem 2.4.19 yields

$$(Y, D(A_Y))_{\theta, \infty} = D_A(\theta + \beta, \infty).$$

The function f is uniformly Hölder continuous with values in Y , moreover u_0 belongs to $D(A_Y)$ and $Au_0 + f(0)$ belongs to $D_{A_Y}(\theta, \infty) = (Y, D(A_Y))_{\theta, \infty}$. By Theorem 3.2.13(iii), the mild solution in Y is strict and belongs to $BUC^{\theta+1}([0, T]; D_A(\beta, \infty))$, moreover $u' \in B([0, T]; D_A(\theta + \beta, \infty))$.

It remains to show that $Au \in BUC^{\theta+\beta}([0, T]; X)$. Thanks to (3.2.23),

$$Au(t) = \int_0^t Ae^{(t-s)A}(f(s) - f(t))ds + (e^{tA} - 1)f(t) + Ae^{tA}u_0.$$

In the case $\theta + \beta < 1$, $Au(t) - Au(r)$ may be split for $0 \leq r < t \leq T$ as

$$\begin{aligned} Au(t) - Au(r) &= \int_0^r A(e^{(t-s)A} - e^{(r-s)A})(f(s) - f(r))ds \\ &+ \int_r^t Ae^{(t-s)A}(f(s) - f(t))ds + [(e^{(t-r)A} - I)(f(t) - f(r))] \\ &+ [(e^{tA} - e^{rA})(f(r) - f(0))] + [(e^{tA} - e^{rA})(Au_0 + f(0))] \\ &= \sum_{k=1}^5 I_k. \end{aligned}$$

Each addendum I_k may be estimated using the methods of Subsection 3.2.4, getting finally

$$\begin{aligned} \|Au(t) - Au(r)\| &\leq \left(\frac{M_{2,\beta}}{(1 - \beta - \theta)(\beta + \theta)} + \frac{2M_{1,\beta}}{\beta + \theta} + \frac{M_{1,\beta}}{\beta} \right) \\ &\cdot (t - r)^{\beta+\theta} [f]_\theta^{D_A(\beta, \infty)} + \frac{M_{1,\beta+\theta}}{\beta + \theta} (t - r)^{\beta+\theta} \|Au_0 + f(0)\|_{D_A(\theta+\beta, \infty)}. \end{aligned}$$

Therefore, Au is uniformly $(\beta + \theta)$ -Hölder continuous with values in X .

Let us consider the case $\theta + \beta > 1$. To prove that Au belongs to $BUC^{\theta+\beta}([0, T]; X)$, it is sufficient to show that u' belongs to $BUC^{\theta+\beta-1}([0, T]; D(A))$. Indeed, in this case Au' is continuous, so that

$$\left\| \frac{Au(t+h) - Au(t)}{h} - Au'(t) \right\| = \left\| \frac{1}{h} \int_t^{t+h} (Au'(s) - Au'(t)) ds \right\|.$$

Letting $h \rightarrow 0$, we find that Au is differentiable and

$$(Au)' = Au' \in BUC^{\theta+\beta-1}([0, T]; X).$$

For $0 \leq t \leq T$, we have

$$u'(t) = \int_0^t Ae^{(t-s)A}(f(s) - f(t))ds + e^{tA}(f(t) - f(0)) + e^{tA}(Au_0 + f(0)),$$

from which it follows that $u'(t) \in D(A)$, and

$$\|Au'(t)\| \leq \left(\frac{M_{2,\beta}}{\theta + \beta - 1} + M_{1,\beta} \right) T^{\theta+\beta-1} [f]_{\theta}^{D_A(\beta,\infty)} + M_0 \|A(Au_0 + f(0))\|.$$

Moreover, for $0 \leq r < t \leq T$,

$$\begin{aligned} Au'(t) - Au'(r) &= \int_0^r A^2 (e^{(t-s)A} - e^{(r-s)A}) (f(s) - f(r)) ds \\ &\quad + \int_r^t A^2 e^{(t-s)A} (f(s) - f(t)) ds + (Ae^{tA} - Ae^{rA})(f(r) - f(0)) \\ &\quad + Ae^{(t-r)A}(f(t) - f(r)) + A(e^{tA} - e^{rA})(Au_0 + f(0)) \\ &= \sum_{k=1}^5 J_k. \end{aligned}$$

Again, each J_k may be estimated arguing as in Subsection 3.2.4, getting

$$\begin{aligned} \|Au'(t) - Au'(r)\| &\leq \left(\frac{M_{3,\beta}}{(2 - \beta - \theta)(\beta + \theta - 1)} + \frac{2M_{2,\beta}}{\beta + \theta - 1} + M_{1,\beta} \right) \\ &\cdot (t - r)^{\beta+\theta-1} [f]_{\theta}^{D_A(\beta,\infty)} + \frac{M_{1,\beta+\theta-1}}{\beta + \theta - 1} (t - r)^{\beta+\theta-1} \|A(Au_0 + f(0))\|_{D_A(\theta+\beta-1,\infty)}. \end{aligned}$$

Therefore, Au' is uniformly $(\theta + \beta - 1)$ -Hölder continuous, and the statement follows. \square

3.3 Fully nonlinear equation

We now apply the results of the previous section to study fully nonlinear equations. Again we follow Lunardi's [24] approach, but we impose the simplifying assumption that the domain D of the linearization is dense in the basic Banach space X . This simplification is adapted to our application to moving boundaries porous media.

Let D be a Banach space endowed with the norm $\|\cdot\|_D$, continuously embedded and dense in X , and let \mathcal{O} be an open set in D . We consider the initial value problem

$$u'(t) = F(t, u(t)), \quad t > 0; \quad u(0) = u_0, \quad (3.3.1)$$

where $F : [0, T] \times \mathcal{O} \rightarrow X$ is a nonlinear operator, $T \in (0, +\infty)$, and $u_0 \in \mathcal{O}$. The key assumption on problem (3.3.1) is

$$\begin{cases} \mathcal{O} \subset D \text{ is open, } F \in BC^2([0, T] \times \mathcal{O}; X) \text{ and} \\ F_v(t, v) \in \mathcal{H}(D, X) \text{ for all } (t, v) \in [0, T] \times \mathcal{O}. \end{cases} \quad (3.3.2)$$

3.3.1 Local existence, uniqueness and regularity

We now state the main local existence theorem. It is convenient to choose an arbitrary initial time $t_0 \in [0, T)$, and to consider the initial value problem:

$$u'(t) = F(t, u(t)), \quad t_0 < t \leq t_0 + \delta; \quad u(t_0) = x_0, \quad (3.3.3)$$

with $\delta \in (0, T - t_0]$ and $x_0 \in \mathcal{O}$. We shall find a local strict solution, that is a solution $u \in C([t_0, t_0 + \delta]; D) \cap C^1([t_0, t_0 + \delta]; X)$, with δ small, which in addition belongs to the weighted Hölder space $C_\alpha^\alpha((t_0, t_0 + \delta]; D)$.

Theorem 3.3.1. *Let $\alpha \in (0, 1)$. Further, assume that (3.3.2) holds true. Fix $\bar{t} \in [0, T]$ and $\bar{u} \in \mathcal{O}$. Then there are $\delta = \delta(\bar{t}, \bar{u}) > 0$, $r = r(\bar{t}, \bar{u}) > 0$ such that*

- (i) *for every $t_0 \in [\bar{t} - r, \bar{t} + r] \cap [0, T]$, and $x_0 \in \mathcal{O}$ such that $\|x_0 - \bar{u}\|_D \leq r$, there is a strict solution $u \in C([t_0, t_0 + \delta]; D) \cap C^1([t_0, t_0 + \delta]; X)$ of (3.3.3) in $[t_0, t_0 + \delta]$;*
- (ii) *u belongs to $C_\alpha^\alpha((t_0, t_0 + \delta]; D)$, u' belongs to $B_\alpha((t_0, t_0 + \delta]; (X, D)_{\alpha, \infty})$, and in addition*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha [u]_{BUC^\alpha([t_0 + \varepsilon, t_0 + 2\varepsilon]; D)} = 0. \quad (3.3.4)$$

Moreover, u is the unique solution of (3.3.3) belonging to

$$\bigcup_{0 < \beta < 1} C_\beta^\beta((t_0, t_0 + \delta]; D) \cap C([t_0, t_0 + \delta]; D).$$

Proof We look for a solution of (3.3.3) belonging to the metric space

$$Y = \left\{ u \in C_\alpha^\alpha((t_0, t_0 + \delta]; D) \cap C([t_0, t_0 + \delta]; D); \right. \\ \left. u(t_0) = x_0, \|u(\cdot) - \bar{u}\|_{C_\alpha^\alpha((t_0, t_0 + \delta]; D)} \leq \rho \right\}, \quad (3.3.5)$$

endowed with the distance of $C_\alpha^\alpha((t_0, t_0 + \delta]; D)$, where $\delta < T - t_0$ and $\rho < \text{dist}(\bar{u}, \partial\mathcal{O})$ are positive numbers to be chosen later. Clearly, Y is a closed set in $C_\alpha^\alpha((t_0, t_0 + \delta]; D)$.

Set moreover

$$A = F_u(\bar{t}, \bar{u}).$$

Since D endowed with the graph norm of A is a Banach space, the norms $\|\cdot\|_D$ and $\|\cdot\|_{D(A)}$ are equivalent. Then it is easy to see that, due to (3.3.2), for every $u \in Y$ the function $t \mapsto F(t, u(t)) - Au(t)$ belongs to $C^\alpha((t_0, t_0 + \delta]; X) \cap C([t_0, t_0 + \delta]; X)$. By (3.3.2), there exists $M > 0$ such that for $(t, u) \in [0, T] \times \mathcal{O}$,

$$F(t, u), F_t(t, u), F_x(t, u), F_{xt}(t, u) \text{ and } F_{xx}(t, u) \text{ are bounded by } M.$$

Define a nonlinear operator Γ on Y , by $\Gamma(u) = v$, where v is the solution of

$$v'(t) = Av(t) + [F(t, u(t)) - Au(t)], \quad t_0 < t \leq t_0 + \delta; \quad v(t_0) = x_0. \quad (3.3.6)$$

By Corollary 3.2.18 (iii), for every $u \in Y$, $\Gamma(u)$ belongs to $C^\alpha((t_0, t_0 + \delta]; D) \cap C([t_0, t_0 + \delta]; D)$. It is clear that a function $u \in Y$ is a solution of (3.3.3) if and only if it is a fixed point of Γ . We shall show that Γ is a contraction and maps Y into itself, provided $\delta, \rho, |t_0 - \bar{t}|$, and $\|x_0 - \bar{u}\|_D$ are suitably small. Let C be the constant given by Corollary 3.2.18 (ii), and let $\gamma, \gamma_\alpha \geq 1$ be such that

$$\begin{cases} \gamma^{-1}\|y\|_D \leq \|y\| + \|Ay\| \leq \gamma\|y\|_D, & \forall y \in D, \\ \gamma_\alpha^{-1}\|y\|_{(X,D)_{\alpha,\infty}} \leq \|y\|_{D_A(\alpha,\infty)} \leq \gamma_\alpha\|y\|_{(X,D)_{\alpha,\infty}}, & \forall y \in (X,D)_{\alpha,\infty}. \end{cases} \quad (3.3.7)$$

Obviously, C, γ, γ_α depend only on \bar{t} and \bar{u} . For simplicity, if B is any Banach space, we write as usual $C(B), C^\alpha(B)$ for $C([t_0, t_0 + \delta]; B), C^\alpha((t_0, t_0 + \delta]; B)$. For $v_1, v_2 \in Y$, we have

$$\begin{aligned} \|\Gamma(v_1) - \Gamma(v_2)\|_{C^\alpha(D)} &\leq \gamma\|\Gamma(v_1) - \Gamma(v_2)\|_{C^\alpha(D(A))} \\ &\leq \gamma C\|F(\cdot, v_1(\cdot)) - F(\cdot, v_2(\cdot)) - A(v_1(\cdot) - v_2(\cdot))\|_{C^\alpha(X)}. \end{aligned} \quad (3.3.8)$$

For each $t \in [t_0, t_0 + \delta]$, we have

$$\begin{aligned} &\|F(t, v_1(t)) - F(t, v_2(t)) - A(v_1(t) - v_2(t))\| \\ &\leq \left\| \int_0^1 [F_x(t, \sigma v_1(t) + (1 - \sigma)v_2(t)) - A] d\sigma (v_1(t) - v_2(t)) \right\| \\ &\leq \int_0^1 \|F_x(t, \sigma v_1(t) + (1 - \sigma)v_2(t)) - A\|_{\mathcal{L}(D,X)} d\sigma \|v_1(t) - v_2(t)\|_D \\ &\leq (\|F_{xt}\|_\infty |t - \bar{t}| + \|F_{xx}\|_\infty \rho) \|v_1(t) - v_2(t)\|_D \\ &\leq M (|t_0 - \bar{t}| + \delta + \rho) \|v_1(t) - v_2(t)\|_D, \end{aligned} \quad (3.3.9)$$

and for $t_0 + \varepsilon \leq s < t \leq t_0 + \delta$,

$$\begin{aligned} &\|F(t, v_1(t)) - F(t, v_2(t)) - F(s, v_1(s)) + F(s, v_2(s)) \\ &\quad - A(v_1(t) - v_2(t) - v_1(s) + v_2(s))\| \\ &\leq \left\| \int_0^1 [F_x(t, \sigma v_1(t) + (1 - \sigma)v_2(t)) \right. \\ &\quad \left. - F_x(s, \sigma v_1(s) + (1 - \sigma)v_2(s))] d\sigma (v_1(s) - v_2(s)) \right\| \\ &\quad + \left\| \int_0^1 (F_x(t, \sigma v_1(t) + (1 - \sigma)v_2(t)) - A) d\sigma \right. \\ &\quad \left. \cdot (v_1(t) - v_2(t) - v_1(s) + v_2(s)) \right\| \\ &\leq (\|F_{xt}\|_\infty (t - s) + \|F_{xx}\|_\infty \rho \varepsilon^{-\alpha} (t - s)^\alpha) \|v_1 - v_2\|_{C(D)} \\ &\quad + (\|F_{xt}\|_\infty |t - \bar{t}| + \|F_{xx}\|_\infty \rho) \varepsilon^{-\alpha} [v_1 - v_2]_{C^\alpha(D)} (t - s)^\alpha, \end{aligned} \quad (3.3.10)$$

so that

$$\begin{aligned} & [F(\cdot, v_1(\cdot)) - F(\cdot, v_2(\cdot)) - A(v_1(\cdot) - v_2(\cdot))]_{C_\alpha^\alpha(X)} \\ & \leq M(|t_0 - \bar{t}| + \delta + \rho) \|v_1 - v_2\|_{C_\alpha^\alpha(D)}. \end{aligned} \quad (3.3.11)$$

Using now (3.3.8), we find

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{C_\alpha^\alpha(D)} \leq 2\gamma CM (|t_0 - \bar{t}| + \delta + \rho) \|v_1 - v_2\|_{C_\alpha^\alpha(D)}.$$

Therefore, Γ is a 1/2-contraction provided

$$\begin{cases} |t_0 - \bar{t}| + \delta \leq \delta_0 = (8\gamma CM)^{-1}, \\ \rho < \min\{\text{dist}(\bar{u}, \partial\mathcal{O}), (8\gamma CM)^{-1}\}. \end{cases} \quad (3.3.12)$$

Let t_0, δ, ρ satisfy (3.3.12). We now show that Γ maps Y into itself if δ, ρ are sufficiently small, t_0 is sufficiently close to \bar{t} , and x_0 is sufficiently close to \bar{u} . If $v \in Y$, we have, denoting by u_0 the constant function $u_0(t) = x_0, \forall t \in [t_0, t_0 + \delta]$,

$$\begin{aligned} & \|\Gamma(v) - \bar{u}\|_{C_\alpha^\alpha(D)} \leq \|\Gamma(v) - \Gamma(u_0)\|_{C_\alpha^\alpha(D)} + \|\Gamma(u_0) - \bar{u}\|_{C_\alpha^\alpha(D)} \\ & \leq \frac{1}{2}\|v - u_0\|_{C_\alpha^\alpha(D)} + \|\Gamma(u_0) - u_0\|_{C_\alpha^\alpha(D)} + \|x_0 - \bar{u}\|_D \\ & \leq \frac{1}{2}\|v - \bar{u}\|_{C_\alpha^\alpha(D)} + \|\Gamma(u_0) - u_0\|_{C_\alpha^\alpha(D)} + \frac{3}{2}\|x_0 - \bar{u}\|_D \end{aligned} \quad (3.3.13)$$

since $\Gamma(u_0) - u_0$ is the solution w of

$$w'(t) = Aw(t) + F(t, x_0), \quad t_0 < t \leq t_0 + \delta; \quad w(t_0) = 0, \quad (3.3.14)$$

then

$$\begin{aligned} \Gamma(u_0)(t) - u_0 &= \int_{t_0}^t e^{(t-s)A} [F(s, x_0) - F(\bar{t}, \bar{u})] ds \\ &+ \int_{t_0}^t e^{(t-s)A} F(\bar{t}, \bar{u}) ds = I_1 + I_2, \quad t_0 \leq t \leq t_0 + \delta. \end{aligned} \quad (3.3.15)$$

By means of Theorem 3.2.17 and (3.3.7) again, we get

$$\|I_1\|_{C_\alpha^\alpha(D)} \leq \gamma C \|F(\cdot, x_0) - F(\bar{t}, \bar{u})\|_{C_\alpha^\alpha(X)}.$$

For each $t \in [t_0, t_0 + \delta]$, we have

$$\begin{aligned} \|F(t, x_0) - F(\bar{t}, \bar{u})\| &\leq \|F(t, x_0) - F(\bar{t}, x_0)\| + \|F(\bar{t}, x_0) - F(\bar{t}, \bar{u})\| \\ &\leq \|F_t\|_\infty |t - \bar{t}| + \|F_x\|_\infty \|x_0 - \bar{u}\|_D \\ &\leq M(|t_0 - \bar{t}| + \delta + \|x_0 - \bar{u}\|_D), \end{aligned}$$

and for $t_0 + \varepsilon \leq s < t \leq t_0 + \delta$,

$$\varepsilon^\alpha \|F(t, x_0) - F(s, x_0)\| \leq \delta^\alpha \|F_t\|_\infty |t - s| \leq M\delta |t - s|^\alpha,$$

thus

$$\|F(\cdot, x_0) - F(\bar{t}, \bar{u})\|_{C_\alpha^\alpha(X)} \leq M(|t_0 - \bar{t}| + 2\delta + \|x_0 - \bar{u}\|_D).$$

This means

$$\|I_1\|_{C_\alpha^\alpha(D)} \leq \gamma CM(|t_0 - \bar{t}| + 2\delta + \|x_0 - \bar{u}\|_D),$$

so that if

$$|t_0 - \bar{t}| + \delta \leq \delta_1 = \frac{\rho}{12\gamma CM},$$

then

$$\|I_1\|_{C_\alpha^\alpha(D)} \leq \frac{\rho}{6} + \gamma CM \|x_0 - \bar{u}\|_D.$$

Moreover, since $F(\bar{t}, \bar{u}) \in X = \bar{D}$, then $\lim_{\delta \rightarrow 0} \|I_2\|_{C_\alpha^\alpha(D)} = 0$. Indeed, one checks easily that for every $y \in X$ it holds

$$\left\| \int_{t_0}^t e^{(t-s)A} y ds \right\|_{C_\alpha^\alpha(D)} \leq \text{const.} \|y\|,$$

with constant independent of t_0, δ, y , and that for every $y \in D$,

$$\lim_{\delta \rightarrow 0} \left\| \int_{t_0}^t e^{(t-s)A} y ds \right\|_{C_\alpha^\alpha(D)} = 0. \quad (3.3.16)$$

Therefore (3.3.16) holds for every $y \in \bar{D} = X$. It yields $\lim_{\delta \rightarrow 0} \|I_2\|_{C_\alpha^\alpha(D)} = 0$, so that there is $\delta_2 > 0$ (depending on \bar{t} and \bar{u}) such that, if $\delta \leq \delta_2$, then

$$\|I_2\|_{C_\alpha^\alpha(D)} \leq \frac{\rho}{6}.$$

Summing up we find that if

$$|t_0 - \bar{t}| + \delta \leq \min\{\delta_0, \delta_1\} \text{ and } \delta \leq \delta_2,$$

then

$$\|\Gamma(u_0) - u_0\|_{C_\alpha^\alpha(D)} \leq \frac{\rho}{3} + \gamma CM \|x_0 - \bar{u}\|_D.$$

Further, if $\|x_0 - \bar{u}\|_D \leq \frac{\rho}{6\gamma CM + 9}$, using (3.3.13), we get

$$\|\Gamma(v) - \bar{u}\|_{C_\alpha^\alpha(D)} \leq \rho.$$

This means Γ maps Y into itself. Therefore there is a unique fixed point u of Γ in Y , which is a solution of (3.3.3). Statement (i) follows.

Let us prove that statement (ii) holds. The derivatives of the functions in the range of Γ belong to $B_\alpha((t_0, t_0 + \delta]; (X, D)_{\alpha, \infty})$ thanks to Corollary 3.2.18(ii). Since u is a fixed point of Γ , then u' is in $B_\alpha((t_0, t_0 + \delta]; (X, D)_{\alpha, \infty})$.

Concerning (3.3.4), set

$$Y' = \left\{ v \in Y; \lim_{\varepsilon \rightarrow 0} [v]_{C_\alpha^\alpha((t_0, t_0 + \varepsilon]; D)} = 0 \right\}.$$

Y' is closed in $C_\alpha^\alpha((t_0, t_0 + \delta]; D)$, and $\Gamma(Y') \subset Y'$ (the proof is the same used above to show that $\Gamma(Y) \subset Y$). The fixed point u of Γ belongs to Y' , which means that u satisfies (3.3.4).

Let us prove uniqueness. If u, v are solutions of (3.3.3) belonging to $C([t_0, t_0 + \delta]; D) \cap C_\beta^\beta((t_0, t_0 + \delta]; D)$ for some $\beta \in (0, 1)$, set

$$t_1 = \sup \{t \in [t_0, t_0 + \delta]; u|_{[t_0, t]} = v|_{[t_0, t]}\}. \quad (3.3.17)$$

Assume by contradiction that $t_1 < t_0 + \delta$. Then $u(t_1) = v(t_1)$, because both u and v are continuous. Setting $x_1 = u(t_1) = v(t_1)$, both u and v are solution of

$$w'(t) = F(t, w(t)), \quad t_1 < t \leq t_0 + \delta; \quad w(t_1) = x_1, \quad (3.3.18)$$

where $x_1 \in \mathcal{O}$ and $F(t_1, x_1) = u'(t_1)$. Taking $\bar{t} = t_1$, $\bar{u} = x_1$, and replacing α by any $\gamma < \beta$ in point (i), we obtain the existence of $\delta_1 > 0$ such that problem (3.3.18) has a unique strict solution in the set

$$Y_1 = \left\{ u \in C_\gamma^\gamma((t_1, t_1 + \delta_1]; D) \cap C([t_1, t_1 + \delta_1]; D); \right. \\ \left. u(t_1) = x_1, \|u(\cdot) - x_1\|_{C_\gamma^\gamma((t_1, t_1 + \delta_1]; D)} \leq \rho_1 \right\},$$

provided ρ_1 and δ_1 are sufficiently small. Now, both $u|_{[t_1, t_1 + \delta_1]}$ and $v|_{[t_1, t_1 + \delta_1]}$ belong to Y_1 if δ_1 is small: in particular, $\|u(t) - x_1\|_D \leq \rho_1$, $\|v(t) - x_1\|_D \leq \rho_1$ for δ_1 small, because u and v are continuous and they assume the value x_1 at $t = t_1$; moreover for $t_0 < t_1 + \varepsilon \leq s < t \leq t_1 + \delta_1 \leq t_0 + \delta$ and for every $a \in (0, 1)$ it holds

$$\|u(t) - u(s)\|_D = \|u(t) - u(s)\|_D^a \|u(t) - u(s)\|_D^{1-a} \\ \leq \left(\sup_{t_1 \leq s < t \leq t_1 + \delta_1} \|u(t) - u(s)\|_D \right)^a \left([u]_{C_\beta^\beta((t_0, t_0 + \delta]; D)} (t-s)^\beta \varepsilon^{-\beta} \right)^{1-a}.$$

Choosing $a = 1 - \gamma/\beta$, we get

$$[u]_{C_\gamma^\gamma((t_1, t_1 + \delta_1]; D)} \leq \left(\sup_{t_1 \leq s < t \leq t_1 + \delta_1} \|u(t) - u(s)\|_D \right)^a \left([u]_{C_\beta^\beta((t_0, t_0 + \delta]; D)} \right)^{1-a},$$

and, similarly,

$$[v]_{C_\gamma^\gamma((t_1, t_1 + \delta_1]; D)} \leq \left(\sup_{t_1 \leq s < t \leq t_1 + \delta_1} \|v(t) - v(s)\|_D \right)^a \left([v]_{C_\beta^\beta((t_0, t_0 + \delta]; D)} \right)^{1-a}.$$

Since both u and v are continuous with values in D , we get

$$[u]_{C_\gamma^\gamma((t_1, t_1 + \delta_1]; D)} \leq \rho_1, \quad [v]_{C_\gamma^\gamma((t_1, t_1 + \delta_1]; D)} \leq \rho_1,$$

provided δ_1 is small enough.

Therefore, $u|_{[t_1, t_1 + \delta_1]} = v|_{[t_1, t_1 + \delta_1]}$, but this contradicts the definition of t_1 . Hence $u = v$, and statement (ii) follows. \square

Corollary 3.3.2. *Under the assumptions of Theorem 3.3.1, for every $\bar{t} \in [0, T]$ and $\bar{u} \in \mathcal{O}$ there is $K = K(\bar{t}, \bar{u}) > 0$ such that for every $t_0 \in [\bar{t} - r, \bar{t} + r]$, and $x_0, x_1 \in \mathcal{O}$ with $\|x_i - \bar{u}\|_D \leq r$, we have, denoting by u_i the solution of (3.3.3) with initial value x_i , $i = 0, 1$,*

$$\begin{aligned} & \|u_0 - u_1\|_{C_\alpha^\alpha((t_0, t_0 + \delta); D)} + \|u'_0 - u'_1\|_{C_\alpha^\alpha((t_0, t_0 + \delta); X)} \\ & + \sup_{t_0 \leq t \leq t_0 + \delta} (t - t_0)^\alpha \|u'_0(t) - u'_1(t)\|_{(X, D)_{\alpha, \infty}} \leq K \|x_0 - x_1\|_D. \end{aligned}$$

If the initial datum x_0 is “more regular”, in the sense that

$$F(t_0, x_0) \in (X, D)_{\alpha, \infty}, \quad (3.3.19)$$

then the local solution of (3.3.3) is uniformly α -Hölder continuous with values in D up to $t = t_0$, and the corresponding continuous dependence result holds, as the following theorem shows.

Theorem 3.3.3. *Let the assumptions of Theorem 3.3.1 be satisfied, and let in addition (3.3.19) hold. If $u : [t_0, t_0 + \delta] \rightarrow D$ is the solution of (3.3.3) given by Theorem 3.3.1, then*

$$u \in BUC^\alpha([t_0, t_0 + \delta]; D) \cap BUC^{1+\alpha}([t_0, t_0 + \delta]; X), \quad u' \in B([t_0, t_0 + \delta]; (X, D)_{\alpha, \infty}).$$

Moreover, for every $\bar{t} \in [0, T]$, $\bar{u} \in \mathcal{O}$ such that $F(\bar{t}, \bar{u}) \in (X, D)_{\alpha, \infty}$, there are $r_0 = r_0(\bar{t}, \bar{u}) > 0$, $K_0 = K_0(\bar{t}, \bar{u}) > 0$ with the following property: for every $t_0 \in [\bar{t} - r_0, \bar{t} + r_0]$, and $x_0, x_1 \in \mathcal{O}$ such that

$$\|x_i - \bar{u}\|_D \leq r_0, \quad \|F(t_0, x_i) - F(\bar{t}, \bar{u})\|_{(X, D)_{\alpha, \infty}} \leq r_0, \quad i = 0, 1,$$

we have, denoting by u_i , $i = 0, 1$, the solution of (3.3.3) with initial value x_i ,

$$\begin{aligned} & \|u_0 - u_1\|_{BUC^\alpha([t_0, t_0 + \delta]; D)} + \|u'_0 - u'_1\|_{BUC^\alpha([t_0, t_0 + \delta]; X)} + \|u'_0 - u'_1\|_{B([t_0, t_0 + \delta]; (X, D)_{\alpha, \infty})} \\ & \leq K_0 (\|x_0 - x_1\|_D + \|F(t_0, x_0) - F(t_0, x_1)\|_{(X, D)_{\alpha, \infty}}). \end{aligned} \quad (3.3.20)$$

3.3.2 The maximally defined solution

Throughout the subsection, $F : [0, T] \times \mathcal{O} \rightarrow X$ is a nonlinear function satisfying assumption (3.3.2). For each $u_0 \in \mathcal{O}$, Theorem 3.3.1 yields existence and uniqueness of a local solution u of (3.3.1) in the space $C_\alpha^\alpha((0, \delta]; D)$, enjoying property (3.3.4). Since $u \in BUC^\alpha([\delta/2, \delta]; D) \cap BUC^{1+\alpha}([\delta/2, \delta]; X)$, then $u'(\delta) = F(\delta, u(\delta))$ belongs to $(X, D)_{\alpha, \infty}$ (see Proposition 3.1.29), so that, thanks to Theorem 3.3.3, u can be continued to some interval $[\delta, \delta_1]$ with $\delta_1 > \delta$, in such a way the extension belongs to $C_\alpha^\alpha((0, \delta_1]; D)$. So we define $u = u(\cdot; u_0)$ by

$$\left\{ \begin{array}{l} I = I(u_0) = \bigcup \left\{ [0, \delta]; \text{ problem (3.3.1) has a solution } u_\delta \in \right. \\ \left. C([0, \delta]; D) \cap C_\beta^\beta((0, \delta]; D) \text{ for some } \beta \in (0, 1) \right\}, \\ u : I \rightarrow D, \quad u(t) = u_\delta(t) \text{ for } t \in [0, \delta] \subset I. \end{array} \right. \quad (3.3.21)$$

u is well defined thanks to the uniqueness part of Theorem 3.3.1. We shall see in the next proposition that u has no continuous extension with values in D . So, it is the maximally defined solution of problem (3.3.1). We set

$$\tau = \tau(u_0) = \sup I(u_0).$$

Proposition 3.3.4. *If $u(\cdot; u_0)$ is uniformly continuous with values in D , then either*

$$\lim_{t \rightarrow \tau(u_0)} u(t; u_0) \in \partial \mathcal{O}, \quad (3.3.22)$$

or

$$I(u_0) = [0, T]. \quad (3.3.23)$$

A sufficient condition for u be uniformly continuous in I is given by next lemma.

Lemma 3.3.5. *Let the assumptions of Theorem 3.3.1 hold, and let $u : I \rightarrow D$ be the maximally defined solution of problem (3.3.1). If the orbit*

$$\{u(t); 0 \leq t < \tau\}$$

is relatively compact in D , then $u : [0, \tau) \rightarrow D$ is uniformly continuous.

The maximally defined solution depends continuously on the initial value, as the following proposition show.

Proposition 3.3.6. *Let $\bar{u} \in \mathcal{O}$, and fix $\bar{\tau} \in (0, \tau(\bar{u}))$. Then there are $\varepsilon = \varepsilon(\bar{u}, \bar{\tau}) > 0$, $H = H(\bar{u}, \bar{\tau}) > 0$ such that if*

$$u_0 \in \mathcal{O} \quad \text{and} \quad \|u_0 - \bar{u}\|_D \leq \varepsilon,$$

then

$$\tau(u_0) \geq \bar{\tau}$$

and

$$\begin{aligned} & \|u(\cdot; u_0) - u(\cdot; \bar{u})\|_{C_\alpha^\alpha((0, \bar{\tau}]; D)} + \|u_t(\cdot; u_0) - u_t(\cdot; \bar{u})\|_{C_\alpha^\alpha((0, \bar{\tau}]; X)} \\ & + \sup_{0 < t \leq \bar{\tau}} t^\alpha \|u_t(\cdot; u_0) - u_t(\cdot; \bar{u})\|_{(X, D)_{\alpha, \infty}} \leq H \|u_0 - \bar{u}\|_D. \end{aligned}$$

Chapter 4

Stabilization of flows through porous media

In this chapter we study the motion of an incompressible homogeneous Newtonian fluid in a rigid porous medium of infinite extent. The fluid is bounded below by a fixed layer having an external source, and above by a free surface moving under the influence of gravity. The flow is governed by Darcy's law. In [12], J. Escher and G. Simonett studied the case of the fluid bounded below by a fixed impermeable layer, instead of a fixed layer having an external source, i.e. $b \equiv 0$. They proved that the problem (P) with $b \equiv 0$ is locally in time well-posed. However, the question of the stability of the equilibrium solution $(u, f) \equiv (c, c)$ was not addressed in [12]. In this chapter, we shall prove that the equilibrium solution $(u, f) \equiv (c, c)$ is exponentially stable if the term injection rate b satisfies $b(c) = 0$ and $b'(c) > 0$.

4.1 Modelling of flows through porous media

In the following, we describe the physical model we are interested in. We consider a system composed of two fluid phases that simultaneously occupy the entire void space of a porous medium. We shall refer to one fluid (subscript w) as the **wetting** fluid (e.g. water), and to the other one (subscript n) as the **nonwetting** fluid (e.g. air). In addition, subscript s denotes the solid (e.g. rock). Here the wetting fluid is an incompressible homogenous Newtonian fluid.

4.1.1 The mass balance equations

Our starting point is the mass balance equation for the wetting fluid in a rigid, isotropic and linearly elastic porous medium. The mass balance for the fluid phase is expressed

by

$$\begin{aligned} \frac{\partial(nS_w\rho_w)}{\partial t} = & -\nabla \cdot \{nS_w(\rho_w\mathbf{V}_w - \mathbf{D}_w \cdot \nabla\rho_w)\} \\ & - \overline{\rho_w(\mathbf{V}_w - \mathbf{V}_{wn}) \cdot \nu_{wn}^{wn}} \Sigma_{wn} \\ & - \overline{\rho_w(\mathbf{V}_w - \mathbf{V}_{ws}) \cdot \nu_{ws}^{ws}} \Sigma_{ws}, \end{aligned} \quad (4.1.1)$$

(see (2.4.10) in [5]), where n denotes the porosity, ρ_w , S_w and \mathbf{V}_w denote the density, the saturation and the mass weighted velocity of the wetting fluid phase, respectively. The term $\rho_w\mathbf{V}_w$ expresses the wetting fluid's mass flux, while $-\mathbf{D}_w \cdot \nabla\rho_w$ expresses the wetting fluid's dispersive mass flux. The symbols \mathbf{V}_{wn} and \mathbf{V}_{ws} denote the velocity of (wetting) fluid-(nonwetting) fluid interface S_{wn} and (wetting) fluid-solid interface S_{ws} , respectively. Moreover, the term $-\overline{\rho_w(\mathbf{V}_w - \mathbf{V}_{wn}) \cdot \nu_{wn}^{wn}} \Sigma_{wn}$ expresses the influx of the mass across the fluid-fluid interface relative to the possibly moving fluid-fluid interface, and the term $-\overline{\rho_w(\mathbf{V}_w - \mathbf{V}_{ws}) \cdot \nu_{ws}^{ws}} \Sigma_{ws}$ expresses the influx of the mass across the fluid-solid interface relative to the possibly moving fluid-solid interface. Following J. Bear and Y. Bachmat[5], we assume that

A₁ The mass fluxes due to dispersion is much smaller than the mass flux and will therefore be neglected. i.e.

$$|\rho_w\mathbf{V}_w| \gg |\mathbf{D}_w \cdot \nabla\rho_w|.$$

A₂ The fluid-solid interface is a material surface with respect to the wetting fluid's mass, i.e., no mass of the wetting fluid phase crosses S_{ws} .

A₃ The fluid-fluid interface is also a material surface with respect to the wetting fluid's mass, i.e., no mass of the wetting fluid phase crosses S_{wn} .

With these three assumptions, (4.1.1) reduces to

$$\frac{\partial(nS_w\rho_w)}{\partial t} = -\nabla \cdot (nS_w\rho_w\mathbf{V}_w). \quad (4.1.2)$$

Equation (4.1.2) is commonly used as the basic (macroscopic) differential mass balance equation of a phase. So the mass balance for the solid phase can be expressed by an equation similar to (4.1.2), namely

$$\frac{\partial\{(1-n)\rho_s\}}{\partial t} = -\nabla \cdot \{(1-n)\rho_s\mathbf{V}_s\}, \quad (4.1.3)$$

where ρ_s and \mathbf{V}_s are the solid's density and velocity, respectively. Let

$$\mathbf{q}_w = nS_w\mathbf{V}_w \quad \text{and} \quad \mathbf{q}_{wr} = nS_w(\mathbf{V}_w - \mathbf{V}_s)$$

be the mass weighted specific discharge and the relative mass weighted specific discharge (i.e., relative to the solid), respectively.

The fluid's mass balance (4.1.2) can be rewritten in the form

$$\nabla \cdot (\rho_w \mathbf{q}_{wr}) + n S_w \frac{D_s \rho_w}{Dt} + S_w \rho_w \frac{D_s n}{Dt} + n \rho_w \frac{D_s S_w}{Dt} + n S_w \rho_w \nabla \cdot \mathbf{V}_s = 0, \quad (4.1.4)$$

where $D_s(\cdot)/Dt = \partial(\cdot)/\partial t + \mathbf{V}_s \cdot \nabla(\cdot)$ is the *material derivative* with respect to the solid.

Equation (4.1.3) can be rewritten in the form

$$\frac{1}{1-n} \frac{D_s(1-n)}{Dt} + \frac{1}{\rho_s} \frac{D_s \rho_s}{Dt} + \nabla \cdot \mathbf{V}_s = 0. \quad (4.1.5)$$

By eliminating $\nabla \cdot \mathbf{V}_s$ from (4.1.4) and (4.1.5), we obtain

$$\nabla \cdot (\rho_w \mathbf{q}_{wr}) + n S_w \frac{D_s \rho_w}{Dt} + \frac{S_w \rho_w}{1-n} \frac{D_s n}{Dt} + n \rho_w \frac{D_s S_w}{Dt} - \frac{n S_w \rho_w}{\rho_s} \frac{D_s \rho_s}{Dt} = 0. \quad (4.1.6)$$

A_4 The solid phase (at the microscopic level!) preserves its volume. This means

$$\frac{D_s \rho_s}{Dt} = 0.$$

With assumption A_4 , equation (4.1.6) reduces to

$$\nabla \cdot (\rho_w \mathbf{q}_{wr}) + n S_w \frac{D_s \rho_w}{Dt} + \frac{S_w \rho_w}{1-n} \frac{D_s n}{Dt} + n \rho_w \frac{D_s S_w}{Dt} = 0. \quad (4.1.7)$$

A_5 The solid matrix is macroscopically fixed in space, i.e. $\mathbf{V}_s = 0$.

With assumption A_4 and A_5 , equation (4.1.5) reduces to

$$\frac{\partial n}{\partial t} = 0. \quad (4.1.8)$$

With assumption A_5 , we know that

$$\mathbf{q}_w = \mathbf{q}_{wr} \quad (4.1.9)$$

Moreover, equation (4.1.7) also reduces to

$$\nabla \cdot (\rho_w \mathbf{q}_w) + n \frac{\partial(S_w \rho_w)}{\partial t} = 0. \quad (4.1.10)$$

A_6 The density is constant, i.e., $\rho_w = \text{const.}$, and the microscopic boundary of the wetting phase is such that $\frac{\partial(n S_w)}{\partial t} = 0$.

With assumption A_6 and (4.1.8), equation (4.1.10) reduces to

$$\nabla \cdot \mathbf{q}_w = 0 \quad (4.1.11)$$

This conservation law will be of fundamental importance in the following.

4.1.2 The momentum balance equation

We next assume that

A_7 The wetting fluid is microscopically isochoric flow.

A_8 The fluid-solid interface is much larger than the fluid-fluid interface, i.e., $S_{ws} \gg S_{wn}$.

With assumption A_8 , we know

$$\int_{S_{ws}} \sigma_w \cdot \nu_w dS \gg \int_{S_{wn}} \sigma_w \cdot \nu_w dS,$$

where σ_w denotes the stress tensor of the wetting fluid. This means the fluid-fluid momentum transfer is much smaller than the fluid-solid one. Therefore, we may neglect the momentum transferred between the fluid phases that together occupy the void space.

With assumption A_5 , A_7 and A_8 , the momentum balance equation of the wetting fluid can be written in the form

$$\begin{aligned} \rho_w \left\{ \frac{\partial q_{wi}}{\partial t} + \frac{\partial}{\partial x_j} \left(\frac{q_{wi} q_{wj}}{\theta_w} \right) \right\} &= -\theta_w \left(\frac{\partial p_w}{\partial x_j} + \rho_w g \frac{\partial z}{\partial x_j} \right) T_{wji}^* \\ &+ \mu_w \frac{\partial^2 q_{wi}}{\partial x_j \partial x_j} - \mu_w \frac{C_w}{\Delta_w^2} \alpha_{ij}^{(ws)} q_{wj}, \end{aligned} \quad (4.1.12)$$

(see (2.6.48) in [5]), where we use *Einstein's summation convention*. In (4.1.12), $\theta_w (= nS_w)$ and p_w denote the content and the pressure of the wetting fluid, respectively. Moreover, g is *gravity acceleration*, z is the z -coordinate of the position, μ_w is the wetting fluid's *dynamic viscosity*, C_w is a shape factor associated with the ws -surface area, on the w side of this surface. Moreover, $\Delta_w = \theta_w / \Sigma_{ws}$, here Σ_{ws} is the specific area of the fluid-solid interface within per unit volume of the consider domain, \mathbf{T}_w^* and $\alpha^{(ws)}$ are two tensorial properties of the configuration of the ws -surface in saturated, single phase flow.

In (4.1.12), the term on the l.h.s. represents the inertial force acting on the wetting fluid, per unit volume of porous medium.

The first term on the r.h.s. of (4.1.12) represents the resultant force acting on the wetting fluid, due to gravity and to pressure gradient, per unit volume of porous medium.

The second term on the r.h.s. of (4.1.12) represents the force acting on the wetting fluid, due to the viscous resistance to its flow inside the wetting fluid, per unit volume of porous medium.

The last term on the r.h.s. of (4.1.12) expresses the viscous resistance, or viscous drag force exerted by the solid phase on the wetting fluid at their contact surfaces, per unit volume of porous medium.

4.1.3 Darcy's law

We now introduce the fundamental construction relation for the velocity of the wetting fluid.

A_9 The wetting fluid is governed by the Darcy's law (cf. [20]), i.e.,

$$\mathbf{q}_w = -\mathbf{K}\nabla u.$$

Namely, the flow in a given domain is such that the viscous resistance force, due to the momentum transfer at the solid-fluid interface, is much larger than both the inertial force and the viscous resistance to the flow inside the fluid, i.e.

$$\left| \mu_w \frac{C_w}{\Delta_w^2} \alpha_{ij}^{(ws)} q_{wj} \right| \gg \left| \rho_w \left\{ \frac{\partial q_{wi}}{\partial t} + \frac{\partial}{\partial x_j} \left(\frac{q_{wi} q_{wj}}{\theta_w} \right) \right\} \right|,$$

and

$$\left| \mu_w \frac{C_w}{\Delta_w^2} \alpha_{ij}^{(ws)} q_{wj} \right| \gg \left| \mu_w \frac{\partial^2 q_{wi}}{\partial x_j \partial x_j} \right|.$$

Under these conditions, the momentum balance equation (4.1.12) reduces to

$$q_{wj} = -\frac{k_{wjl}}{\mu_w} \left\{ \frac{\partial p_w}{\partial x_l} + \rho_w g \frac{\partial z}{\partial x_l} \right\}, \quad (4.1.13)$$

where

$$k_{wjl} = \frac{\theta_w \Delta_w^2}{C_w} (\alpha_{ji}^{(ws)})^{-1} T_{wil}^* \quad (4.1.14)$$

is called the *effective permeability* for the wetting fluid in a porous medium. Assumption A_6 permits to introduce a so-called *piezometric head* u (see [5] p.175), by setting

$$u(x, y) := \frac{p_w(x, z)}{\rho_w g} + z, \quad (x, z) \in \mathbb{R}^2 \times \mathbb{R}. \quad (4.1.15)$$

Then, (4.1.13) reduces to

$$q_{wj} = -K_{ji} \frac{\partial u}{\partial x_i}, \quad (4.1.16)$$

where the second rank symmetrical tensor

$$K_{ji} = k_{wji} \frac{\rho_w g}{\mu_w}$$

is a coefficient called *hydraulic conductivity*. Note that assumption A_9 is the same as equation (4.1.16). In general \mathbf{K} is a symmetric, uniformly positive definite 3×3 -matrix. However, for homogeneous and isotropic media, we have $\mathbf{K} = k \cdot id_{\mathbb{R}^3}$ for some positive constant k , (see [5] p.189).

Therefore, by means of Darcy's law, the mass balance equation (4.1.11) reduces to

$$\Delta u = 0. \quad (4.1.17)$$

A_{10} *The content of the wetting fluid is independent of position, i.e., $\nabla\theta_w = 0$.*

With assumption A_6 and A_{10} , we know that $\theta_w = \text{const.}$ Moreover, the Darcy's law can be rewritten in the form

$$\mathbf{V}_w = -\frac{k}{nS_w}\nabla u = -\frac{k}{\theta_w}\nabla u. \quad (4.1.18)$$

4.1.4 The boundary conditions

To describe the shape of the moving boundary, we assume that

A_{11} *The capillary pressure $p_c = p_n - p_w$ is continuous, here p_n denotes the pressure of the nonwetting fluid.*

We assume that the air in the unsaturated zone is at atmospheric pressure, normalized to be zero i.e. $p_n = 0$. With assumption A_{11} , we know that p_w is continuous and the shape of the moving fluid-fluid interface Γ_t at time t is then given by the relation

$$(x, z) \in \Gamma_t \text{ if and only if } z = \sup\{y \in \mathbb{R}; p_w(t, x, y) > 0\}. \quad (4.1.19)$$

Finally, we assume that the interface Γ_t is parameterized over \mathbb{R}^2 , i.e., we assume that there is a function f such that

$$(x, z) \in \Gamma_t \text{ if and only if } z = f(t, x).$$

In particular, we find that $\Gamma_t = \Gamma_{f(t)} = \text{graph}(f(t, \cdot))$. Moreover, (4.1.15) and (4.1.19) fuse to

$$u = f \text{ on } \Gamma_f. \quad (4.1.20)$$

Finally, setting $F(t, y) := z - f(t, x)$ for $y = (x, z) \in \mathbb{R}^2 \times \mathbb{R}$ and $t \geq 0$, the interface Γ_f can also be described by the *conservative property* that F is identically equal to zero on Γ_f . Hence, we get

$$\frac{d}{dt}F(t, y) = \partial_t F(t, y) + \nabla F(t, y) \cdot \dot{y} = 0 \text{ on } \Gamma_f. \quad (4.1.21)$$

Using (4.1.18), we may express the velocity \dot{y} by $-\frac{k}{\theta_w}\nabla u$. Consequently, observing $\partial_t F = -\partial_t f$, $\nabla F = (-\nabla_x f, 1)$, and ν is the outer normal of Γ_t ,

$$\nu := \frac{(-\nabla_x f, 1)}{\sqrt{1 + |\nabla_x f|^2}}.$$

(4.1.21) becomes

$$\partial_t f + \frac{k}{\theta_w} \sqrt{1 + |\nabla_x f|^2} \partial_\nu u = 0 \text{ on } \Gamma_f. \quad (4.1.22)$$

A_{12} The porous medium is bounded below by a fixed layer Γ_0 with an external source Γ^m , and this external source depends on the wn -interface.

With assumption A_1 and A_{12} , the general boundary condition for mass on Γ_0 can be expressed by

$$(\rho_w \mathbf{V}_w + \Gamma^m) \cdot \nu = 0, \quad (4.1.23)$$

(see (2.7.6) in [5]), where $\nu = (0, -1)$. Using (4.1.18), equation (4.1.23) can be rewritten in the form

$$\partial_z u = \frac{\theta_w}{k\rho_w} b, \quad (4.1.24)$$

where $b = -\Gamma^m \cdot \nu$ is called *interface-dependent injection rate*. Since Γ^m depends on the wn -interface, the function b also depends on f .

Finally, adding initial condition for the interface f , i.e., $f(0, \cdot) = f_0$, as well as the normalization $\lim_{|(x,z)| \rightarrow \infty} u(t, x, z) = c$, we see that (4.1.17), (4.1.20), (4.1.22), and (4.1.24) fit, up to multiplicative constants, into

$$\begin{aligned} \Delta u &= 0 & \text{in } & \Omega_{f,T}, \\ \partial_z u &= b(f(t, x)) & \text{on } & \Gamma_{0,T}, \\ u &= f & \text{on } & \Gamma_{f,T}, \\ \lim_{|(x,z)| \rightarrow \infty} u(t, x, z) &= c & \text{on } & [0, T), \\ \partial_t f + \sqrt{1 + |\nabla_x f|^2} \partial_\nu u &= 0 & \text{on } & \dot{\Gamma}_{f,T}, \\ f(0, \cdot) &= f_0 & \text{on } & \mathbb{R}^2, \end{aligned} \quad (P)_{f_0, c}$$

where $T > 0$ is fixed, and

$$\begin{aligned} \Omega_{f(t)} &:= \{(x, z) \in \mathbb{R}^2 \times \mathbb{R}; 0 < z < f(t, x)\}, \\ \Gamma_{f(t)} &:= \{(x, z) \in \mathbb{R}^2 \times \mathbb{R}; z = f(t, x)\}, \\ \Omega_{f,T} &:= \{(t, y) \in [0, T) \times \mathbb{R}^3; y \in \Omega_{f(t)}\}, \\ \Gamma_{f,T} &:= \{(t, y) \in [0, T) \times \mathbb{R}^3; y \in \Gamma_{f(t)}\}, \\ \dot{\Gamma}_{f,T} &:= \{(t, y) \in (0, T) \times \mathbb{R}^3; y \in \Gamma_{f(t)}\}, \\ \Gamma_{0,T} &:= [0, T) \times \Gamma_0. \end{aligned}$$

4.2 The full system

In this section, we shall describe the problem we investigate in detail. Now we consider the case of the problem $(P)_{f_0, c}$ in one space dimension (see Figure 1),

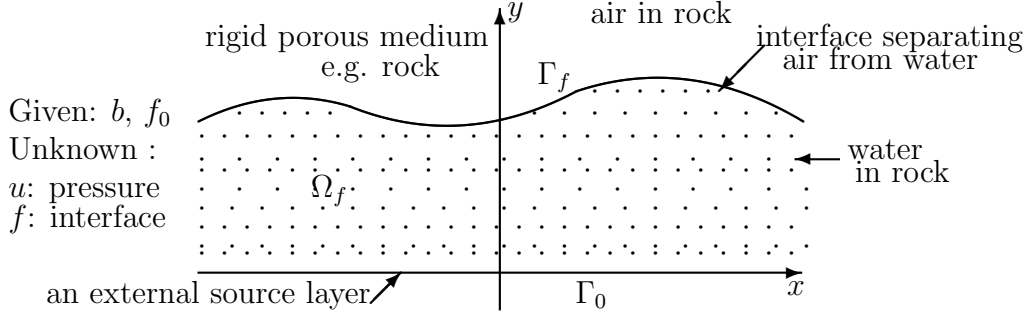


Figure 1

i.e.

$$\Delta u(t, x, y) = 0, \quad t \in J, (x, y) \in \Omega_{f(t)}, \quad (4.2.1)$$

$$\frac{\partial u}{\partial y}(t, x, y)|_{y=0} = b(f(t, x)), \quad t \in J, x \in \mathbb{R}, \quad (4.2.2)$$

$$u(t, x, y)|_{y=f(t, x)} = f(t, x), \quad t \in J, x \in \mathbb{R}, \quad (4.2.3)$$

$$\lim_{|x|+|y| \rightarrow \infty} u(t, x, y) = c, \quad t \in J, (x, y) \in \Omega_{f(t)}, \quad (4.2.4)$$

$$\partial_t f(t, x) + (-f_x, 1) \cdot \nabla u(t, x, y) = 0, \quad t \in \dot{J}, (x, y) \in \Gamma_{f(t)}, \quad (4.2.5)$$

$$f(0, x) = f_0(x), \quad x \in \mathbb{R}, \quad (4.2.6)$$

where $T > 0$ is fixed, and $J = [0, T)$, $\dot{J} = J \setminus \{0\}$. The injection rate b is a given smooth function from \mathbb{R} to \mathbb{R} .

Recall that $\mathfrak{A} = \left\{ g \in BC^2(\mathbb{R}); \inf_{x \in \mathbb{R}} \{c + g(x)\} > 0 \right\}$. Given $s > 0$, let $h^s(\mathbb{R})$ be the closure of $\mathcal{S}(\mathbb{R})$ in $BUC^s(\mathbb{R})$, and

$$h^s := h^s(\mathbb{R}), \quad \text{and} \quad h_{\mathfrak{A}}^s := h^s \cap \mathfrak{A}.$$

Remark 4.2.1. For $s \geq 2$, $h_{\mathfrak{A}}^s$ is open in h^s . Indeed, for $\forall g \in h_{\mathfrak{A}}^s$, let

$$M = \inf_{x \in \mathbb{R}} \{c + g(x)\},$$

then $M > 0$. Thanks to (2.1.1) and the definition of h^s , we have $h^s \hookrightarrow BUC^2(\mathbb{R}) \hookrightarrow BC(\mathbb{R})$. So that there exists $C > 0$ such that

$$\|u\|_{\infty} \leq C \|u\|_{h^s}, \quad \forall u \in h^s.$$

For $\forall h \in h^s$ satisfying $\|h - g\|_{h^s} \leq M/2C$, we get

$$\|h - g\|_{\infty} \leq C \|h - g\|_{h^s} \leq C \frac{M}{2C} = \frac{M}{2}.$$

Then we have

$$c + h(x) = c + g(x) + h(x) - g(x) \geq M - \frac{M}{2} = \frac{M}{2} > 0 \quad \text{for } \forall x \in \mathbb{R}.$$

Thus $h \in \mathfrak{A}$. This means $h \in h_{\mathfrak{A}}^s$, i.e., $h_{\mathfrak{A}}^s$ is open in h^s . \square

4.3 Transformation

4.3.1 The diffeomorphism

To study the moving boundary problem, in general, we usually first transform a moving boundary problem into a fixed boundary problem, then study the corresponding fixed boundary problem. So we now consider a reference domain $\Omega := \mathbb{R} \times (0, 1)$. In this chapter, we always consider $f \in h^{2+\alpha} \cap \tilde{\mathfrak{A}}$, where $\alpha \in (0, 1)$. Then we take $g = f - c \in h_{\tilde{\mathfrak{A}}}^{2+\alpha}$, and define $\varphi_g : \Omega_f \rightarrow \Omega$ by

$$\varphi_g(x', y') := \left(x', 1 - \frac{y'}{c + g(x')} \right), \quad (x', y') \in \Omega_f,$$

and the inverse function $\varphi_g^{-1} : \Omega \rightarrow \Omega_f$ by

$$\varphi_g^{-1}(x, y) := (x, (1 - y)(c + g(x))), \quad (x, y) \in \Omega.$$

Thus we can transform the moving domain Ω_f into the reference domain Ω by φ_g , (see Figure 2).

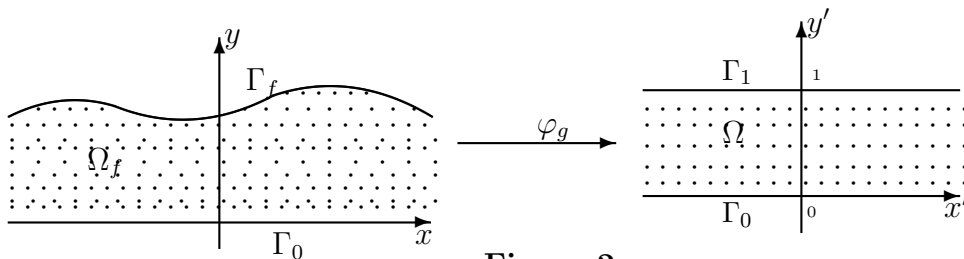


Figure 2

Lemma 4.3.1. *Given $f \in BUC^k(\mathbb{R})$, $k \in \mathbb{N}$ with $\inf_{x \in \mathbb{R}} \{f(x)\} > 0$, and $\phi_f(x, y) := (x, 1 - \frac{y}{f(x)})$. Then we have*

$$\phi_f \in Diff^k(\Omega_f, \Omega).$$

Proof It is obvious that $\phi_f \in C^k(\Omega_f)$ and

$$D\phi_f(x, y) = \begin{pmatrix} 1 & 0 \\ \frac{yf'(x)}{f^2(x)} & \frac{-1}{f(x)} \end{pmatrix}.$$

Hence, by means of the assumption of f , $\det(D\phi_f) = -\frac{1}{f} \neq 0$, i.e. ϕ_f is a local C^k -diffeomorphism. Furthermore, observe that $y \mapsto 1 - \frac{y}{f(x)}$ is strictly decreasing diffeomorphism. Thus this completes the proof. \square

4.3.2 The transformed operators

Let

$$\begin{aligned}\varphi_* u &:= \varphi_*^g u := u \circ \varphi_g^{-1} && \text{for } u \in C(\overline{\Omega_f}), \\ \varphi^* v &:= \varphi_g^* v := v \circ \varphi_g && \text{for } v \in C(\overline{\Omega}),\end{aligned}$$

denote the *push forward* and *pull back* operators, respectively. Then we define the following transformed operators:

$$\begin{aligned}\mathcal{A}(g)v &:= -\varphi_*^g \Delta(\varphi_g^* v), \\ \mathcal{B}_i(g)v &:= \varphi_*^g (\gamma_i(\nabla(\varphi_g^* v)|n_i)) \quad i = 0, 1.\end{aligned}$$

Here $(\cdot|\cdot)$ denotes inner product in \mathbb{R}^2 , $\gamma_i : \Omega \rightarrow \Gamma_i (i = 0, 1)$ are trace operators, $\Gamma_i = \mathbb{R} \times \{i\}$, $n_0 = (-g_x, 1)$ and $n_1 = (0, -1)$. Then problem (4.2.1)–(4.2.6) is transformed to

$$\mathcal{A}(g)v = 0 \quad \text{in } J \times \Omega, \quad (4.3.1)$$

$$v = g \quad \text{on } J \times \Gamma_0, \quad (4.3.2)$$

$$\mathcal{B}_1(g)v = \tilde{b}(g) \quad \text{on } J \times \Gamma_1, \quad (4.3.3)$$

$$\lim_{|z| \rightarrow \infty} v(t, z) = 0 \quad \text{on } J, \quad (4.3.4)$$

$$\partial_t g + \mathcal{B}_0(g)v = 0 \quad \text{on } \dot{J} \times \Gamma_0, \quad (4.3.5)$$

$$g(0, \cdot) = g_0 \quad \text{on } \mathbb{R}, \quad (4.3.6)$$

where $\tilde{b}(g) := -b(c + g)$ and $g_0(x) := f_0(x) - c$.

Proposition 4.3.2. *The problem (4.2.1)–(4.2.6) and the problem (4.3.1)–(4.3.6) are equivalent.*

Proof In fact, if (u, f) is a solution of problem (4.2.1)–(4.2.6), then it is obvious that $(\varphi_* u - c, f - c)$ is a solution of problem (4.3.1)–(4.3.6). Hence it need to check that $(\varphi^* v + c, g + c)$ is a solution of problem (4.2.1)–(4.2.6), if (v, g) is a solution of problem (4.3.1)–(4.3.6). Therefore we only need to check equations (4.2.1), (4.2.3) and (4.2.5). Let $u := \varphi^* v + c$, $f := g + c$ and $(\xi, \eta) = \varphi_g(x, y)$. Clearly, $\xi = x$. Thanks to Lemma 4.3.1, we know $\varphi_g \in Diff^2(\Omega_f, \Omega)$. Furthermore, deduced from (4.3.1), we get

$$\Delta(\varphi^* v + c) = \Delta(\varphi_g^* v) = 0.$$

It satisfies (4.2.1). Since

$$\begin{aligned}\frac{\partial u}{\partial y}(x, y) \Big|_{y=0} &= (\nabla \varphi_g^* v|(0, 1))(x, y) \Big|_{y=0} = -(\nabla \varphi_g^* v|n_1)(x, y) \Big|_{y=0} \\ &= -(\nabla \varphi_g^* v|n_1) \varphi_g^{-1}(\xi, \eta) \Big|_{\eta=1} = -\mathcal{B}_1 v(\xi, \eta) \\ &= -\tilde{b}(g(\xi)) = b(f(\xi)) = b(f(x)).\end{aligned}$$

Similarly, we can check that equation (4.2.5) holds. \square

Lemma 4.3.3. *Given $g \in \mathfrak{A}$, we have*

$$\mathcal{A}(g) = \sum_{j,k=1}^2 -a_{jk}(g)\partial_j\partial_k + a_2(g)\partial_2, \quad \mathcal{B}_i(g) = \sum_{j=1}^2 b_{ji}(g)\gamma_i\partial_j, \quad i = 0, 1,$$

and

$$\sum_{j,k=1}^2 a_{jk}(g)\xi^j\xi^k \geq \underline{\alpha}(g)|\xi|^2 \quad \text{for } \xi \in \mathbb{R}^2,$$

where $\pi(x, y) := 1 - y$ for $(x, y) \in \bar{\Omega}$ and

$$\begin{aligned} a_{11}(g) &:= 1, & a_{12}(g) &:= a_{21}(g) := \frac{\pi g_x}{c+g}, \\ a_{22}(g) &:= \frac{1}{(c+g)^2}(1 + \pi^2 g_x^2), & a_2(g) &:= \frac{\pi}{c+g} \left(\frac{2g_x^2}{c+g} - g_{xx} \right), \\ b_{10} &:= -g_x, & b_{20}(g) &:= -\frac{1}{c+g}(1 + g_x^2), \\ b_{11}(g) &:= 0, & b_{21}(g) &:= \frac{1}{c+g}, \\ \underline{\alpha}(g) &:= \frac{1}{1 + (c+g)^2 + \pi^2 g_x^2}. \end{aligned}$$

Proof Let $\tilde{g}_{jk} := \tilde{g}_{jk}(g) := (\partial_j\varphi_g^{-1}|\partial_k\varphi_g^{-1})$, $1 \leq j, k \leq 2$, denote the components of the metric tensor. Then

$$[\tilde{g}_{jk}] = \begin{pmatrix} 1 + \pi^2 g_x^2 & -\pi(c+g)g_x \\ -\pi(c+g)g_x & (c+g)^2 \end{pmatrix}.$$

Moreover, it is easily verified that $\tilde{g} := \det[\tilde{g}_{jk}] = (c+g)^2$ and that

$$[\tilde{g}^{jk}] := [\tilde{g}_{jk}]^{-1} = \begin{pmatrix} 1 & \frac{\pi g_x}{c+g} \\ \frac{\pi g_x}{c+g} & \frac{1}{(c+g)^2}(1 + \pi^2 g_x^2) \end{pmatrix}.$$

Now, the assertion follows from the well-known formulas

$$\begin{aligned} \mathcal{A}(g)v &= -\varphi_g^* \Delta(\varphi_g^* v) = -\frac{1}{\sqrt{\tilde{g}}} \sum_{j,k=1}^2 \partial_j(\sqrt{\tilde{g}}\tilde{g}^{jk}\partial_k v), \\ \mathcal{B}_i(g)v &= \varphi_g^* (\gamma_i(\nabla(\varphi_g^* v)|n_i)) = \varphi_g^* \left(\gamma_i \left[\sum_{k=1}^2 \partial_j \varphi_g^k \partial_k v \right]_{1 \leq j \leq 2} |n_i \right), \quad i = 0, 1. \end{aligned}$$

Where φ_g^k ($1 \leq k \leq 2$) denote the components of φ_g i.e. $\varphi_g = (\varphi_g^1, \varphi_g^2)$.

To prove the second assertion, we fix $(x, y) \in \bar{\Omega}$ and suppress it in our notation. Set

$$\underline{\alpha} := \underline{\alpha}(g) := [1 + (c+g)^2 + \pi^2 g_x^2]^{-1}, \quad \beta := \beta(g) := 2(c+g),$$

and note that

$$\underline{\alpha}^{-2} - \beta^2 = [1 - (c + g)^2 - \pi^2 g_x^2]^2 + 4\pi^2 g_x^2 \geq 0.$$

Hence we find that $\underline{\alpha}^2 \beta^2 \leq 1$. This implies that

$$0 \leq 1 - \underline{\alpha}^2 \beta^2 \leq \left(1 - \frac{\underline{\alpha}^2 \beta^2}{2}\right)^2.$$

Observe that the smallest eigenvalue of $[\tilde{g}^{j^k}]$ is given by

$$\lambda(g) = \frac{1}{2(c+g)^2} \left\{ 1 + (c+g)^2 + \pi^2 g_x^2 - \sqrt{[1 + (c+g)^2 + \pi^2 g_x^2]^2 - 4(c+g)^2} \right\}.$$

Consequently,

$$\lambda(g) = \frac{2}{\underline{\alpha} \beta^2} \left(1 - \sqrt{1 - \underline{\alpha}^2 \beta^2}\right) \geq \frac{2}{\underline{\alpha} \beta^2} \left[1 - \left(1 - \frac{\underline{\alpha}^2 \beta^2}{2}\right)\right] = \underline{\alpha},$$

which completes the proof of Lemma 4.3.3. \square

In the following, we study problem (4.3.1)–(4.3.6) in the little Hölder spaces.

Lemma 4.3.4. *Let $\alpha \in (0, 1)$ be given. Then*

$$(\mathcal{A}(\cdot), \mathcal{B}_0(\cdot), \mathcal{B}_1(\cdot)) \in C^\infty(h_{\mathfrak{A}}^{2+\alpha}, \mathcal{L}(h^{2+\alpha}(\Omega), h^\alpha(\Omega) \times h^{1+\alpha} \times h^{1+\alpha})),$$

and

$$\begin{aligned} \partial \mathcal{A}(g)[h, v] &= \frac{2\pi}{c+g} \left\{ \left(\frac{g_x h}{c+g} - h_x \right) \partial_1 \partial_2 v + \left(\frac{h}{(c+g)^2} \left(\frac{1}{\pi} + \pi g_x^2 \right) - \frac{\pi}{c+g} g_x h_x \right) \partial_2^2 v \right. \\ &\quad \left. - \left(\frac{2g_x^2 h}{(c+g)^2} - \frac{4g_x h_x + g_{xx} h}{2(c+g)} + \frac{h_{xx}}{2} \right) \partial_2 v \right\}, \\ \partial \mathcal{B}_0(g)[h, v] &= -h_x \gamma_0 \partial_1 v + \frac{1}{c+g} \left(\frac{h + h g_x^2}{c+g} - 2g_x h_x \right) \gamma_0 \partial_2 v, \\ \partial \mathcal{B}_1(g)[h, v] &= -\frac{h}{(c+g)^2} \gamma_1 \partial_2 v, \end{aligned}$$

for $g \in h_{\mathfrak{A}}^{2+\alpha}$, $h \in h^{2+\alpha}$ and $v \in h^{2+\alpha}(\Omega)$.

Proof It is obvious that the first assertion holds. Given $g \in h_{\mathfrak{A}}^{2+\alpha}$. For $h \in h^{2+\alpha}$ and

$v \in h^{2+\alpha}(\Omega)$, we have

$$\begin{aligned}
\partial\mathcal{A}(g)[h, v] &= \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{A}(g + \varepsilon h)v - \mathcal{A}(g)v}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[-\partial_1^2 v - \frac{2\pi(g_x + \varepsilon h_x)}{c + g + \varepsilon h} \partial_1 \partial_2 v - \frac{1 + \pi^2(g_x + \varepsilon h_x)^2}{(c + g + \varepsilon h)^2} \partial_2^2 v \right. \\
&\quad + \frac{\pi}{c + g + \varepsilon h} \left(\frac{2(g_x + \varepsilon h_x)^2}{c + g + \varepsilon h} - (g_{xx} + \varepsilon h_{xx}) \right) \partial_2 v + \partial_1^2 v \\
&\quad \left. + \frac{2\pi g_x}{c + g} \partial_1 \partial_2 v + \frac{1 + \pi^2 g_x^2}{(c + g)^2} \partial_2^2 v - \frac{\pi}{c + g} \left(\frac{2g_x^2}{c + g} - g_{xx} \right) \partial_2 v \right] \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[\frac{2\pi(g_x \varepsilon h - (c + g)\varepsilon h_x)}{(c + g)(c + g + \varepsilon h)} \partial_1 \partial_2 v \right. \\
&\quad + \frac{(1 + \pi^2 g_x^2)\varepsilon h(2(c + g) + \varepsilon h) - (c + g)^2 \pi^2 \varepsilon h_x(2g_x + \varepsilon h_x)}{(c + g)^2 (c + g + \varepsilon h)^2} \partial_2^2 v \\
&\quad + \frac{\pi}{c + g + \varepsilon h} \left(\frac{2((c + g)^2 \varepsilon h_x(2g_x + \varepsilon h_x) - g_x^2 \varepsilon h(2(c + g) + \varepsilon h))}{(c + g)^2 (c + g + \varepsilon h)} \right. \\
&\quad \left. + \frac{g_{xx} \varepsilon h}{c + g} - \varepsilon h_{xx} \right) \partial_2 v \Big] \\
&= \frac{2\pi(g_x h - (c + g)h_x)}{(c + g)^2} \partial_1 \partial_2 v + \frac{2h(1 + \pi^2 g_x^2) - 2\pi^2(c + g)g_x h_x}{(c + g)^3} \partial_2^2 v \\
&\quad + \frac{\pi}{c + g} \left(\frac{4g_x h_x}{c + g} - \frac{4g_x^2 h}{(c + g)^2} + \frac{g_{xx} h}{c + g} - h_{xx} \right) \partial_2 v.
\end{aligned}$$

Similarly, the other assertions hold. \square

4.3.3 Isomorphisms induced by elliptic boundary value problems

In the following, we shall introduce an elliptic regularity theorem.

Theorem 4.3.5. *a) Let $g \in h_{\mathfrak{A}}^{2+\alpha}$, $\lambda \geq 0$ and $\mu > 0$ be given. Then*

$$\begin{aligned}
(\lambda + \mathcal{A}(g), \gamma_0, (c + g)\mathcal{B}_1(g)) &\in \text{Isom} (h^{2+\alpha}(\Omega), h^\alpha(\Omega) \times h^{2+\alpha} \times h^{1+\alpha}), \\
(\lambda + \mathcal{A}(g), \mu\gamma_0 + \mathcal{B}_0(g), \mathcal{B}_1(g)) &\in \text{Isom} (h^{2+\alpha}(\Omega), h^\alpha(\Omega) \times h^{1+\alpha} \times h^{1+\alpha}).
\end{aligned}$$

b) Given $\lambda_0 \geq 0$ and $g \in h_{\mathfrak{A}}^{2+\alpha}$, there exists a positive constant C , depending on $\|g\|_{h^{2+\alpha}}$, λ_0 , α and c such that

$$\|u\|_{h^{2+\alpha}(\Omega)} \leq C \left(\|(\lambda + \mathcal{A}(g))u\|_{h^\alpha(\Omega)} + \|\gamma_0 u\|_{h^{2+\alpha}} + \|(c + g)\mathcal{B}_1(g)u\|_{h^{1+\alpha}} \right) \quad (4.3.7)$$

for all $u \in h^{2+\alpha}(\Omega)$ and $\lambda \in [0, \lambda_0]$.

Proof The proof is based on the classical estimates of Agmon-Douglis-Nirenberg [2] for elliptic equations, see also the Theorem 3.5 in [12]. \square

Remark 4.3.6. Theorem 4.3.5 ensures that the following problem: given $g \in h_{\mathfrak{A}}^{2+\alpha}$ and $\lambda \geq 0$,

$$\begin{cases} \lambda v + \mathcal{A}(g)v = v_1 & \text{in } \Omega, \\ \gamma_0 v = v_2 & \text{on } \Gamma_0, \\ (c + g)\mathcal{B}_1(g)v = v_3 & \text{on } \Gamma_1, \end{cases}$$

for every $(v_1, v_2, v_3) \in h^\alpha(\Omega) \times h^{2+\alpha} \times h^{1+\alpha}$, there exists a unique solution $v \in h^{2+\alpha}(\Omega)$. \square

In particular, given $g \in h_{\mathfrak{A}}^{2+\alpha}$, thanks to Remark 4.3.6, we can define

$$\mathcal{R}(g) := (\mathcal{A}(g), \gamma_0, (c + g)\mathcal{B}_1(g))^{-1}$$

and its restrictions

$$\begin{aligned} \mathcal{S}(g) &:= \mathcal{R}(g)|_{h^\alpha(\Omega) \times \{0\} \times \{0\}}, \\ \mathcal{T}(g) &:= \mathcal{R}(g)|_{\{0\} \times h^{2+\alpha} \times \{0\}}, \\ \mathcal{U}(g) &:= \mathcal{R}(g)|_{\{0\} \times \{0\} \times h^{1+\alpha}}. \end{aligned}$$

Obviously, let $v_1 := \mathcal{T}(g)h_1$, $v_2 := \mathcal{U}(g)h_2$ for $h_1 \in h^{2+\alpha}$, $h_2 \in h^{1+\alpha}$, then we get

$$\begin{aligned} \mathcal{A}(g)v_1 = 0 & \quad \text{in } \Omega, \quad \gamma_0 v_1 = h_1 & \quad \text{on } \Gamma_0, \quad \mathcal{B}_1(g)v_1 = 0 & \quad \text{on } \Gamma_1; \\ \mathcal{A}(g)v_2 = 0 & \quad \text{in } \Omega, \quad \gamma_0 v_2 = 0 & \quad \text{on } \Gamma_0, \quad (c + g)\mathcal{B}_1(g)v_2 = h_2 & \quad \text{on } \Gamma_1. \end{aligned}$$

Lemma 4.3.7. *We have*

$$\mathcal{T}(\cdot) \in C^\infty(h_{\mathfrak{A}}^{2+\alpha}, \mathcal{L}(h^{2+\alpha}, h^{2+\alpha}(\Omega))), \quad \mathcal{U}(\cdot) \in C^\infty(h_{\mathfrak{A}}^{2+\alpha}, \mathcal{L}(h^{1+\alpha}, h^{2+\alpha}(\Omega))),$$

and

$$\partial \mathcal{T}(g)[h, \cdot] = -\mathcal{S}(g)\partial \mathcal{A}(g)[h, \mathcal{T}(g)\cdot], \quad \partial \mathcal{U}(g)[h, \cdot] = -\mathcal{S}(g)\partial \mathcal{A}(g)[h, \mathcal{U}(g)\cdot]$$

for $g \in h_{\mathfrak{A}}^{2+\alpha}$, and $h \in h^{2+\alpha}$.

Proof For simplicity, let $X := h_{\mathfrak{A}}^{2+\alpha}$, $F_i := h^{i+\alpha}(\Omega)$ ($i = 0, 1, 2$), $E_j := h^{j+\alpha}$ ($j = 1, 2$).

a) Thanks to Remark 4.2.1, we know that X is an open subset of E_2 . Moreover, letting

$$A(g) := (\mathcal{A}(g), \gamma_0, (c + g)\mathcal{B}_1(g)), \quad g \in X.$$

By means of Lemma 4.3.4 and Theorem 4.3.5, we also know that

$$A \in C^\infty(X, \text{Isom}(F_2, F_0 \times E_2 \times E_1))$$

with (note that $(c + g)\mathcal{B}_1(g) = \gamma_1 \partial_2$ is independent of g)

$$\partial A(g)h = (\partial \mathcal{A}(g)h, 0, 0) \quad \text{for } h \in E_2.$$

b) Given $A \in \text{Isom}(F_2, F_0 \times E_2 \times E_1)$, define $j(A) := A^{-1}$. Since $\text{Isom}(F_2, F_0 \times E_2 \times E_1)$ is open in $\mathcal{L}(F_2, F_0 \times E_2 \times E_1)$, then

$$j \in C^\infty(\text{Isom}(F_2, F_0 \times E_2 \times E_1), \mathcal{L}(F_0 \times E_2 \times E_1, F_2)),$$

with

$$\partial j(A)\tilde{A} = -A^{-1}\tilde{A}A^{-1}$$

for $A \in \text{Isom}(F_2, F_0 \times E_2 \times E_1)$ and $\tilde{A} \in \mathcal{L}(F_2, F_0 \times E_2 \times E_1)$.

c) Let $R \in \mathcal{L}(F_0 \times E_2 \times E_1, F_2)$ be given, and define $p(R) \in \mathcal{L}(E_2, F_2)$ by

$$p(R)h := R(0, h, 0) \text{ for } h \in E_2.$$

Then $p \in \mathcal{L}(\mathcal{L}(F_0 \times E_2 \times E_1, F_2), \mathcal{L}(E_2, F_2))$ and consequently

$$\partial p(R)S = p(S) \text{ for } R, S \in \mathcal{L}(F_0 \times E_2 \times E_1, F_2).$$

So that $\mathcal{T} = p \circ j \circ A$ and

$$\begin{aligned} \partial \mathcal{T}(g)h &= \partial(p \circ j \circ A(g))h \\ &= \partial p(j(A(g)))\partial j(A(g))\partial A(g)h \\ &= \partial p(j(A(g)))\partial j(A(g))(\partial \mathcal{A}(g)h, 0, 0) \\ &= \partial p(j(A(g)))(-A^{-1}(g)(\partial \mathcal{A}(g)h, 0, 0)A^{-1}(g)) \\ &= p(-A^{-1}(g)(\partial \mathcal{A}(g)h, 0, 0)A^{-1}(g)), \end{aligned}$$

i.e.

$$\begin{aligned} \partial \mathcal{T}(g)[h, v] &= p(-A^{-1}(g)(\partial \mathcal{A}(g)h, 0, 0)A^{-1}(g))v \\ &= -A^{-1}(g)(\partial \mathcal{A}(g)h, 0, 0)A^{-1}(g)(0, v, 0) \\ &= -\mathcal{R}(g)(\partial \mathcal{A}(g)h, 0, 0)\mathcal{T}(g)v \\ &= -\mathcal{R}(g)(\partial \mathcal{A}(g)[h, \mathcal{T}(g)v], 0, 0) \\ &= -\mathcal{S}(g)\partial \mathcal{A}(g)[h, \mathcal{T}(g)v] \end{aligned}$$

for $h, v \in E_2$.

It is similar to $\partial \mathcal{U}(g)h$, thus we can get

$$\partial \mathcal{U}(g)[h, \cdot] = -\mathcal{S}(g)\partial \mathcal{A}(g)[h, \mathcal{U}(g)\cdot].$$

□

4.4 Reduced equation

4.4.1 Equivalent formulation

In this subsection, we shall show that problem (4.3.1)–(4.3.6) can be reduced to an evolution problem.

Given $g \in h_{\mathfrak{A}}^{2+\alpha}$, we define

$$\Phi_{\hat{b}}(g) := \mathcal{B}_0(g) \left[\mathcal{T}(g)g + \mathcal{U}(g)\hat{b}(g) \right].$$

Here $\widehat{b}(g) := (c + g)\widetilde{b}(g) = -(c + g)b(c + g)$. Suppose that $g_0 \in h_{\mathfrak{A}}^{2+\alpha}$. A function $g : J \rightarrow h^{1+\alpha}$ is said to be a classical solution of

$$\dot{g} + \Phi_{\widehat{b}}(g) = 0, \quad g(0) = g_0, \quad (4.4.1)$$

iff $g \in C(J, h_{\mathfrak{A}}^{2+\alpha}) \cap C^1(J, h^{1+\alpha})$ and g satisfies (4.4.1) pointwise.

Proposition 4.4.1. *Let $g_0 \in h_{\mathfrak{A}}^{2+\alpha}$ be given.*

i) Suppose that g is a classical solution of problem (4.4.1) on J . Let $v(t) := \mathcal{T}(g(t))g(t) + \mathcal{U}(g(t))\widehat{b}(g(t))$. Then the pair (v, g) is a classical solution of problem (4.3.1)–(4.3.6) on J , having the additional regularity

$$g \in C(J, h_{\mathfrak{A}}^{2+\alpha}) \cap C^1(J, h^{1+\alpha}), \quad v(t, \cdot) \in h^{2+\alpha}(\Omega), \quad t \in J. \quad (4.4.2)$$

ii) Suppose that (v, g) is a classical solution of problem (4.3.1)–(4.3.6) on J , having the regularity (4.4.2). Then g is a classical solution of problem (4.4.1) on J .

Proof If g is a classical solution of (4.4.1), then let $v = \mathcal{T}(g)g + \mathcal{U}(g)\widehat{b}(g)$, we can check that (v, g) is a classical solution of (4.3.1)–(4.3.6).

If (v, g) is a classical solution of (4.3.1)–(4.3.6), by the uniqueness, then we know that

$$v = \mathcal{T}(g)g + \mathcal{U}(g)\widehat{b}(g).$$

Thus, g is a classical solution of (4.4.1). □

Corollary 4.4.2. $\Phi_{\widehat{b}} \in C^\infty(h_{\mathfrak{A}}^{2+\alpha}, h^{1+\alpha})$ and

$$\begin{aligned} \partial\Phi_{\widehat{b}}(g)h &= \partial\mathcal{B}_0(g)[h, \mathcal{T}(g)g + \mathcal{U}(g)\widehat{b}(g)] + \mathcal{B}_0(g)\mathcal{T}(g)h + \mathcal{B}_0(g)\mathcal{U}(g)\widehat{b}'(g)h \\ &\quad - \mathcal{B}_0(g)\mathcal{S}(g)\partial\mathcal{A}(g)[h, \mathcal{T}(g)g] - \mathcal{B}_0(g)\mathcal{S}(g)\partial\mathcal{A}(g)[h, \mathcal{U}(g)\widehat{b}(g)] \end{aligned}$$

for $g \in h_{\mathfrak{A}}^{2+\alpha}$ and $h \in h^{2+\alpha}$.

Proof By means of Lemma 4.3.4 and Lemma 4.3.7, we know

$$\begin{aligned} \mathcal{B}_0(\cdot) &\in C^\infty(h_{\mathfrak{A}}^{2+\alpha}, \mathcal{L}(h^{2+\alpha}(\Omega), h^{1+\alpha})), \\ \mathcal{T}(\cdot) &\in C^\infty(h_{\mathfrak{A}}^{2+\alpha}, \mathcal{L}(h^{2+\alpha}, h^{2+\alpha}(\Omega))), \\ \mathcal{U}(\cdot) &\in C^\infty(h_{\mathfrak{A}}^{2+\alpha}, \mathcal{L}(h^{1+\alpha}, h^{2+\alpha}(\Omega))). \end{aligned}$$

Then, since \widehat{b} is smooth, for $\forall g \in h_{\mathfrak{A}}^{2+\alpha}$, we get

$$\mathcal{T}(g)g \in h^{2+\alpha}(\Omega), \quad \mathcal{U}(g)\widehat{b}(g) \in h^{2+\alpha}(\Omega) \quad \text{and} \quad \mathcal{B}_0(g) \in \mathcal{L}(h^{2+\alpha}(\Omega), h^{1+\alpha}).$$

Thus, by definition, $\Phi_{\widehat{b}}(g) \in h^{1+\alpha}$, i.e., $\Phi_{\widehat{b}}(g) : h_{\mathfrak{A}}^{2+\alpha} \rightarrow h^{1+\alpha}$. Further, we also get that

$$\Phi_{\widehat{b}} \in C^\infty(h_{\mathfrak{A}}^{2+\alpha}, h^{1+\alpha}).$$

Then $\partial\Phi_{\widehat{b}}(g) \in \mathcal{L}(h^{2+\alpha}, h^{1+\alpha})$. Using Lemma 4.3.4 and Lemma 4.3.7 again, by chain law, we obtain

$$\begin{aligned} \partial\Phi_{\widehat{b}}(g)h &= \partial\mathcal{B}_0(g)[h, \mathcal{T}(g)g + \mathcal{U}(g)\widehat{b}(g)] + \mathcal{B}_0(g)\mathcal{T}(g)h + \mathcal{B}_0(g)\mathcal{U}(g)\widehat{b}'(g)h \\ &\quad - \mathcal{B}_0(g)\mathcal{S}(g)\partial\mathcal{A}(g)[h, \mathcal{T}(g)g] - \mathcal{B}_0(g)\mathcal{S}(g)\partial\mathcal{A}(g)[h, \mathcal{U}(g)\widehat{b}(g)] \end{aligned}$$

for $g \in h_{\mathfrak{A}}^{2+\alpha}$ and $h \in h^{2+\alpha}$. □

4.4.2 The linearization

For simplicity, we always assume that

$$\widehat{b}(0) = 0. \quad (4.4.3)$$

Thanks to Proposition 4.4.1, we only need to study problem (4.4.1). By (4.4.3), we know that $g = 0$ is the stationary solution of (4.4.1). To study problem (4.4.1), using the theory of Section 3.3, we first consider its corresponding linear problem: For a suitable open set $\mathcal{O} \subset h_{\mathfrak{A}}^{2+\alpha}$ such that $g_0 \in \mathcal{O}$,

$$g' + \partial\Phi_{\widehat{b}}(\bar{g})g = G(g, \bar{g}), \quad (4.4.4)$$

where $\bar{g} \in \mathcal{O}$ and $G(g, \bar{g}) = \partial\Phi_{\widehat{b}}(\bar{g})g - \Phi_{\widehat{b}}(g)$.

Since we are concerning about the behavior near $g = 0$, we take $\bar{g} \equiv 0$. This means that we shall study the following problem

$$g' + \partial\Phi_{\widehat{b}}(0)g = G(g); \quad g(0) = g_0, \quad (4.4.5)$$

here $G(g) = \partial\Phi_{\widehat{b}}(0)g - \Phi_{\widehat{b}}(g)$. Before studying (4.4.5), we first study the operator $\partial\Phi_{\widehat{b}}(0)$. By Corollary 4.4.2, we get

$$\partial\Phi_{\widehat{b}}(0)h = \mathcal{B}_0(0)\mathcal{T}(0)h + \mathcal{B}_0(0)\mathcal{U}(0)\widehat{b}'(0)h. \quad (4.4.6)$$

4.4.3 Fourier multipliers

In the following we shall express the operator $\partial\Phi_{\widehat{b}}(0)$ as a Fourier multiplier.

Lemma 4.4.3. *For $h \in h^{2+\alpha}$, we have*

$$\begin{aligned} \mathcal{T}(0)h &= \mathcal{F}^{-1} \left(\frac{\cosh(c|\xi|(1-y))}{\cosh(c|\xi|)} \mathcal{F}h \right), \\ \mathcal{U}(0)h &= \mathcal{F}^{-1} \left(\frac{\sinh(c|\xi|y)}{c|\xi| \cosh(c|\xi|)} \mathcal{F}h \right). \end{aligned}$$

Proof In fact, we consider two problems

$$\mathcal{A}(0)v_1 = 0 \text{ in } \Omega, \quad v_1 = h_1 \text{ on } \Gamma_0, \quad \mathcal{B}_1(0)v_1 = 0 \text{ on } \Gamma_1, \quad (4.4.7)$$

and

$$\mathcal{A}(0)v_2 = 0 \text{ in } \Omega, \quad v_2 = 0 \text{ on } \Gamma_0, \quad c\mathcal{B}_1(0)v_2 = h_2 \text{ on } \Gamma_1. \quad (4.4.8)$$

Here $\mathcal{A}(0) = -\partial_1^2 - \frac{1}{c^2}\partial_2^2$, $\mathcal{B}_1(0) = \frac{1}{c}\gamma_1\partial_2$.

First, we consider $\mathcal{A}(0)v = 0$. By Fourier transformation with respect to x , we have

$$\mathcal{F}v(\xi, y) = C_1 e^{c|\xi|y} + C_2 e^{-c|\xi|y}, \quad C_1, C_2 \text{ are two undetermined constants.}$$

i) For v_1 , we have

$$\begin{aligned} C_1 + C_2 &= \mathcal{F}h_1, \\ C_1|\xi|e^{c|\xi|} - C_2|\xi|e^{-c|\xi|} &= 0. \end{aligned}$$

Then

$$C_1 = \frac{\mathcal{F}h_1}{1 + e^{2c|\xi|}}, \quad C_2 = e^{2c|\xi|} \frac{\mathcal{F}h_1}{1 + e^{2c|\xi|}}.$$

Therefore,

$$\mathcal{F}v_1 = \frac{\cosh(c|\xi|(1-y))}{\cosh(c|\xi|)} \mathcal{F}h_1.$$

Since $v_1 = \mathcal{T}(0)h_1$ is a solution of problem (4.4.7), we have

$$\mathcal{T}(0) = \mathcal{F}^{-1} \frac{\cosh(c|\xi|(1-y))}{\cosh(c|\xi|)} \mathcal{F}.$$

ii) For v_2 , we have

$$\begin{aligned} C_1 + C_2 &= 0, \\ cC_1|\xi|e^{c|\xi|} - cC_2|\xi|e^{-c|\xi|} &= \mathcal{F}h_2. \end{aligned}$$

Then

$$C_1 = \frac{\mathcal{F}h_2}{2c|\xi| \cosh(c|\xi|)}, \quad C_2 = -\frac{\mathcal{F}h_2}{2c|\xi| \cosh(c|\xi|)}.$$

Therefore,

$$\mathcal{F}v_2 = \frac{\sinh(c|\xi|y)}{c|\xi| \cosh(c|\xi|)} \mathcal{F}h_2.$$

Since $v_2 = \mathcal{U}(0)h_2$ is a solution of problem (4.4.8), we have

$$\mathcal{U}(0) = \mathcal{F}^{-1} \frac{\sinh(c|\xi|y)}{c|\xi| \cosh(c|\xi|)} \mathcal{F}.$$

□

Since $B_0(0) = -\frac{1}{c}\gamma_0\partial_2$, by Lemma 4.4.3 and (4.4.6), it is easy to obtain that

$$\partial\Phi_{\hat{b}}(0)h = \mathcal{F}^{-1} \frac{c|\xi| \sinh(c|\xi|) - \hat{b}'(0)}{c \cosh(c|\xi|)} \mathcal{F}h, \quad \forall h \in h^{2+\alpha}. \quad (4.4.9)$$

4.5 Sectorial operator

In the section, we shall investigate the operator $A_{\hat{b}} := \partial\Phi_{\hat{b}}(0) : h^{2+\alpha} \subset h^{1+\alpha} \rightarrow h^{1+\alpha}$, and show $-A_{\hat{b}}$ is a sectorial operator. Thanks to Proposition 3.1.15, we need to know that the spectrum and estimate of $-A_{\hat{b}}$.

4.5.1 Spectral properties

In the subsection, we shall investigate the spectrum of the operator $A_{\widehat{b}}$. To do it, we first shall study the spectrum of $A_{\widehat{b}}$ in $L_2(\mathbb{R})$, then in Sobolev spaces $H^s(\mathbb{R})$, $s > 0$. Finally, we use Sobolev's embedding theorem to get our results.

By Corollary 4.4.2 and (4.4.9), we have

$$A_{\widehat{b}} \in \mathcal{L}(h^{2+\alpha}, h^{1+\alpha}), \quad (4.5.1)$$

and

$$A_{\widehat{b}} = \mathcal{F}^{-1} a_{\widehat{b}}(\xi) \mathcal{F},$$

where

$$a_{\widehat{b}}(\xi) = \frac{c|\xi| \sinh(c|\xi|) - \widehat{b}'(0)}{c \cosh(c|\xi|)} = \frac{c\xi \sinh(c\xi) - \widehat{b}'(0)}{c \cosh(c\xi)}.$$

For simplicity, we assume that $\widehat{b}'(0) < 0$. By the definition of $a_{\widehat{b}}(\xi)$, we know that $a_{\widehat{b}}(\xi)$ is an even function, $a_{\widehat{b}}(\xi) > 0$ for $\xi \in \mathbb{R}$ and $\lim_{\xi \rightarrow +\infty} a_{\widehat{b}}(\xi) = +\infty$. Thus there exists a constant $\omega_0 > 0$, such that $\omega_0 = \min_{\xi \in \mathbb{R}} \{a_{\widehat{b}}(\xi)\}$.

First, we consider an auxiliary problem, i.e. studying $A_{\widehat{b}}$ on $L_2(\mathbb{R})$.

Lemma 4.5.1. $A_{\widehat{b}}$ is self-adjoint on $L_2(\mathbb{R})$, and

$$(A_{\widehat{b}}u, u)_{L_2} \geq \omega_0 \|u\|_2^2, \quad \text{for } u \in L_2(\mathbb{R}).$$

Proof For every $u, v \in L_2(\mathbb{R})$, thanks to Plancherel's theorem, i.e.

$$(u, v)_{L_2} = (\mathcal{F}u, \mathcal{F}v)_{L_2},$$

we have

$$\begin{aligned} (u, A_{\widehat{b}}^*v)_{L_2} &= (A_{\widehat{b}}u, v)_{L_2} = (\mathcal{F}^{-1} a_{\widehat{b}}(\xi) \mathcal{F}u, v)_{L_2} = (a_{\widehat{b}}(\xi) \mathcal{F}u, \mathcal{F}v)_{L_2} \\ &= (\mathcal{F}u, \overline{a_{\widehat{b}}(\xi)} \mathcal{F}v)_{L_2} = (u, \mathcal{F}^{-1} \overline{a_{\widehat{b}}(\xi)} \mathcal{F}v)_{L_2}. \end{aligned}$$

Since $\overline{a_{\widehat{b}}(\xi)} = a_{\widehat{b}}(\xi)$, we can get that

$$A_{\widehat{b}}^* = \mathcal{F}^{-1} \overline{a_{\widehat{b}}(\xi)} \mathcal{F} = A_{\widehat{b}},$$

i.e. $A_{\widehat{b}}$ is self-adjoint.

Moreover, for every $u \in L_2(\mathbb{R})$, we get that

$$\begin{aligned} (A_{\widehat{b}}u, u)_{L_2} &= (a_{\widehat{b}}(\xi) \mathcal{F}u, \mathcal{F}u)_{L_2} = \int_{\mathbb{R}} a_{\widehat{b}}(\xi) |\mathcal{F}u|^2 d\xi \\ &\geq \omega_0 \int_{\mathbb{R}} |\mathcal{F}u|^2 d\xi = \omega_0 \|u\|_2^2. \end{aligned}$$

□

Proposition 4.5.2.

$$\rho(A_{\widehat{b}}) \supseteq \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda < \omega_0\}.$$

Proof For every $\lambda \in \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda < \omega_0\}$, since $a_{\widehat{b}}(\xi) \geq \omega_0$, we get $|\lambda - a_{\widehat{b}}(\xi)| \geq |\lambda - \omega_0|$. By means of Plancherel's theorem, we have

$$\begin{aligned} \|(\lambda - A_{\widehat{b}})u\|_2^2 &= ((\lambda - A_{\widehat{b}})u, (\lambda - A_{\widehat{b}})u)_{L_2} \\ &= (\mathcal{F}^{-1}(\lambda - a_{\widehat{b}}(\xi))\mathcal{F}u, \mathcal{F}^{-1}(\lambda - a_{\widehat{b}}(\xi))\mathcal{F}u)_{L_2} \\ &= ((\lambda - a_{\widehat{b}}(\xi))\mathcal{F}u, (\lambda - a_{\widehat{b}}(\xi))\mathcal{F}u)_{L_2} \\ &= \int_{\mathbb{R}} |\lambda - a_{\widehat{b}}(\xi)|^2 |\mathcal{F}u|^2 d\xi \\ &\geq |\lambda - \omega_0|^2 \|u\|_2^2. \end{aligned} \tag{4.5.2}$$

Hence $\lambda - A_{\widehat{b}}$ is injective. Now we will prove that $\lambda - A_{\widehat{b}}$ is surjective. Suppose $u \in \operatorname{Range}(\lambda - A_{\widehat{b}})^\perp$, then, by Lemma 4.5.1, we have

$$0 = \left| ((\lambda - A_{\widehat{b}})u, u)_{L_2} \right| = \left| \lambda \|u\|_2^2 - (A_{\widehat{b}}u, u)_{L_2} \right| \geq |\lambda - \omega_0| \|u\|_2^2.$$

Thus, $u = 0$, i.e. $\lambda - A_{\widehat{b}}$ is surjective. Hence $(\lambda - A_{\widehat{b}})^{-1}$ exists. From (4.5.2), we obtain that $(\lambda - A_{\widehat{b}})^{-1}$ is bounded. This means that $\lambda \in \rho(A_{\widehat{b}})$. \square

Second, we consider the operator $A_{\widehat{b}}$ on the Sobolev Space $H^s := H^s(\mathbb{R})$ (s is an arbitrary positive real number). The inner product of H^s is expressed by

$$(u, v)_{H^s} = (\Lambda^s \mathcal{F}u, \Lambda^s \mathcal{F}v)_{L_2} \quad \text{for } u, v \in H^s,$$

here $\Lambda^s = (1 + \xi^2)^{s/2}$. Similarly, we can also get that $A_{\widehat{b}}$ is self-adjoint and

$$\rho(A_{\widehat{b}}) \supseteq \{z \in \mathbb{C}; \operatorname{Re}(z) < \omega_0\}.$$

In the following, we will introduce the Sobolev's embedding theorem (cf. [28]).

Theorem 4.5.3 (Sobolev's Embedding Theorem). *If $s > k + \frac{n}{2}$, then*

$$H^s(\mathbb{R}^n) \hookrightarrow BUC^k(\mathbb{R}^n).$$

Lemma 4.5.4. *Let E, F be Banach spaces, and X be a topological space. Further assume that $X \hookrightarrow E \hookrightarrow F$. If X is dense in E , then X and E have the same closure in F , that is, $\overline{X} = \overline{E}$ in F .*

Proof Thank to the assumption, it is clear that $\overline{X} \subset \overline{E}$. Thus it is sufficient to prove $\overline{E} \subset \overline{X}$. For $\forall u \in \overline{E}$ and $\varepsilon > 0$, there exists some $u_0 \in E$ such that

$$\|u - u_0\|_F < \frac{\varepsilon}{2}.$$

Since $E \hookrightarrow F$, there exists a positive constant C such that

$$\|v\|_F \leq C\|v\|_E \quad \text{for } v \in E.$$

Furthermore since X is dense in E , there also exists some $u_1 \in X$ such that

$$\|u_0 - u_1\|_E < \frac{\varepsilon}{2C}.$$

Hence,

$$\|u - u_1\|_F \leq \|u - u_0\|_F + \|u_0 - u_1\|_F < \frac{\varepsilon}{2} + C\|u_0 - u_1\|_E < \varepsilon.$$

This means $u \in \overline{X}$. □

Corollary 4.5.5. *For $s > p > 0$, $h^s \xrightarrow{d} h^p$.*

Proof It is easy to know that

$$\mathcal{S}(\mathbb{R}) \xrightarrow{d} h^s \hookrightarrow BUC^p(\mathbb{R}).$$

Thanks to Lemma 4.5.4, we get

$$\overline{h^s} = \overline{\mathcal{S}(\mathbb{R})} = h^p \text{ in } BUC^p(\mathbb{R}).$$

This means h^s is dense in h^p . □

Corollary 4.5.6. *If $s > k + \frac{3}{2}$, then $H^s \xrightarrow{d} h^{k+\alpha}$.*

Proof By *Sobolev's Embedding Theorem*, we know that $H^s \hookrightarrow BUC^{k+1}(\mathbb{R})$. Moreover it is well-known that $BUC^{k+1}(\mathbb{R}) \hookrightarrow BUC^{k+\alpha}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R}) \xrightarrow{d} H^s$. Then,

$$\mathcal{S}(\mathbb{R}) \xrightarrow{d} H^s \hookrightarrow BUC^{k+\alpha}(\mathbb{R}).$$

By the definition of *little Hölder space*, we know that

$$h^{k+\alpha} = \overline{\mathcal{S}(\mathbb{R})} \quad \text{in } BUC^{k+\alpha}(\mathbb{R}).$$

Thanks to Lemma 4.5.4, we get

$$h^{k+\alpha} = \overline{H^s} \quad \text{in } BUC^{k+\alpha}(\mathbb{R}).$$

This means

$$H^s \xrightarrow{d} h^{k+\alpha}.$$

□

Remark 4.5.7. If $(\lambda - A_{\tilde{\delta}})^{-1} \in \mathcal{L}(BUC^{1+\alpha}(\mathbb{R}), BUC^{2+\alpha}(\mathbb{R}))$ for $Re\lambda < \omega_0$, then $(\lambda - A_{\tilde{\delta}})^{-1} \in \mathcal{L}(h^{1+\alpha}, h^{2+\alpha})$.

Proof By assumption, we know that $(\lambda - A_{\hat{b}})^{-1} \in \mathcal{L}(h^{1+\alpha}, BUC^{2+\alpha}(\mathbb{R}))$. For every $x \in h^{1+\alpha}$, thanks to Corollary 4.5.6, there exists a sequence (x_i) in H^s ($s \geq 4$) such that $x_i \rightarrow x$ in $h^{1+\alpha}$. Thus,

$$(\lambda - A_{\hat{b}})^{-1}x_i \rightarrow (\lambda - A_{\hat{b}})^{-1}x \quad \text{in } BUC^{2+\alpha}(\mathbb{R}).$$

Since $(\lambda - A_{\hat{b}})^{-1} \in \mathcal{L}(H^s)$, we conclude that

$$(\lambda - A_{\hat{b}})^{-1}x \text{ belongs to the closure of } H^s \text{ in } BUC^{2+\alpha}(\mathbb{R}).$$

Again thanks to Corollary 4.5.6, we have $(\lambda - A_{\hat{b}})^{-1}x \in h^{2+\alpha}$. \square

In the following, we will prove that

$$(\lambda - A_{\hat{b}})^{-1} \in \mathcal{L}(h^{1+\alpha}, h^{2+\alpha})$$

for every $\lambda \in \{z \in \mathbb{C}; \operatorname{Re}(z) < \omega_0\}$. Thanks to Remark 4.5.7, we only need to prove that

$$(\lambda - A_{\hat{b}})^{-1} \in \mathcal{L}(BUC^{1+\alpha}(\mathbb{R}), BUC^{2+\alpha}(\mathbb{R}))$$

for every $\lambda \in \{z \in \mathbb{C}; \operatorname{Re}(z) < \omega_0\}$. To do that, we shall use the theory of Fourier multipliers. Recall that,

$$\mathcal{M} := \left(\{a \in L_\infty(\mathbb{R}^n); (1 + |\xi|^2)^{|\alpha|/2} \partial^\alpha a \in L_\infty(\mathbb{R}^n), |\alpha| \leq [n/2] + 1\}, \|\cdot\|_{\mathcal{M}} \right),$$

where

$$\|a\|_{\mathcal{M}} := \max_{|\alpha| \leq [n/2] + 1} \|(1 + |\xi|^2)^{|\alpha|/2} \partial^\alpha a\|_\infty.$$

In particular, we only need the case of $n = 1$.

Before using the theory of Fourier multipliers, we shall introduce a parameter dependent function. Given μ and β , we can define a C^∞ function

$$\varphi_{\mu, \beta}(\xi) := \frac{\cosh \xi}{\mu \cosh \xi - \xi \sinh \xi + \beta}.$$

Then we can calculate that

$$\partial_\xi \varphi_{\mu, \beta}(\xi) = \frac{\sinh \xi \cosh \xi + \xi + \beta \sinh \xi}{(\mu \cosh \xi - \xi \sinh \xi + \beta)^2} \sim O(\xi^{-2}) \quad \text{as } \xi \rightarrow \infty. \quad (4.5.3)$$

Moreover, as $\xi \rightarrow \infty$,

$$d(1 + \xi^2)^{\frac{1}{2}} = (1 + \xi^2)^{-\frac{1}{2}} \xi \sim O(1). \quad (4.5.4)$$

Thus, we can get the following statement:

Lemma 4.5.8. *If $\operatorname{Re} \lambda < \omega_0$, then $(\lambda - A_{\hat{b}})^{-1} \in \mathcal{L}(BUC^{1+\alpha}(\mathbb{R}), BUC^{2+\alpha}(\mathbb{R}))$.*

Proof Now we consider two functions

$$m_1(\xi) = (\lambda - a_{\hat{b}}(\xi))^{-1} \quad \text{and} \quad m_2(\xi) = (1 + \xi^2)^{\frac{1}{2}}(\lambda - a_{\hat{b}}(\xi))^{-1}. \quad (4.5.5)$$

For fixed λ ($Re\lambda < \omega_0$), then we have

$$m_1(\xi) = c\varphi_{c\lambda, \hat{b}'(0)}(c\xi) \quad \text{and} \quad m_2(\xi) = c(1 + \xi^2)^{\frac{1}{2}}\varphi_{c\lambda, \hat{b}'(0)}(c\xi).$$

By means of Leibniz' rule, (4.5.3) and (4.5.4), we know that

$$\partial^k m_1(\xi) \sim O(\xi^{-k-1}) \quad \text{and} \quad \partial^k m_2(\xi) \sim O(\xi^{-k}) \quad \text{as } \xi \rightarrow \infty \text{ for } k = 0, 1,$$

i.e. $\|m_1\|_{\mathcal{M}} < \infty$ and $\|m_2\|_{\mathcal{M}} < \infty$. Clearly,

$$(\lambda - A_{\hat{b}})^{-1} = \mathcal{F}^{-1}m_1\mathcal{F} \quad \text{and} \quad (1 - \Delta)^{\frac{1}{2}}(\lambda - A_{\hat{b}})^{-1} = \mathcal{F}^{-1}m_2\mathcal{F}.$$

Then, thanks to Corollary 2.3.2, we get that

$$(1 - \Delta)^{\frac{1}{2}}(\lambda - A_{\hat{b}})^{-1} \in \mathcal{L}(BUC^{1+\alpha}(\mathbb{R})),$$

and

$$(\lambda - A_{\hat{b}})^{-1} \in \mathcal{L}(BUC^{1+\alpha}(\mathbb{R})).$$

Since $(1 - \Delta)^{\frac{1}{2}} \in Isom(BUC^{2+\alpha}(\mathbb{R}), BUC^{1+\alpha}(\mathbb{R}))$, we can conclude that

$$(\lambda - A_{\hat{b}})^{-1} \in \mathcal{L}(BUC^{1+\alpha}(\mathbb{R}), BUC^{2+\alpha}(\mathbb{R})).$$

□

Thanks to Remark 4.5.7 and Lemma 4.5.8, we get

$$(\lambda - A_{\hat{b}})^{-1} \in \mathcal{L}(h^{1+\alpha}, h^{2+\alpha}) \quad \text{for } Re\lambda < \omega_0.$$

By Lemma 4.5.8, we also know that $\rho(A_{\hat{b}}) \supseteq \{z \in \mathbb{C}; Re(z) < \omega_0\}$ on $h^{1+\alpha}$. For $0 \leq \omega < \omega_0$, we denote that

$$S_\omega = \{z \in \mathbb{C}; Re(z) \leq \omega\}.$$

We have known that

$$(\lambda - A_{\hat{b}})^{-1} \in \mathcal{L}(h^{1+\alpha}) \quad \text{for } \lambda \in S_\omega, \quad (4.5.6)$$

so we obtain that

$$(\lambda - \omega)(\lambda - A_{\hat{b}})^{-1} \in \mathcal{L}(h^{1+\alpha}) \quad \text{for } \lambda \in S_\omega.$$

Now we consider a function:

$$m_\omega(\lambda, \xi) = (\lambda - \omega)(\lambda - a_{\hat{b}}(\xi))^{-1},$$

for $\lambda \in S_\omega$ and $\xi \in \mathbb{R}$. Clearly

$$(\lambda - \omega)(\lambda - A_{\hat{b}})^{-1} = \mathcal{F}^{-1}m_\omega(\lambda, \xi)\mathcal{F}.$$

4.5.2 Estimates of the symbol

Lemma 4.5.9.

$$|m_\omega(\lambda, \xi)| < 1 \text{ for } \lambda \in S_\omega, \xi \in \mathbb{R}. \quad (4.5.7)$$

Proof Since $a_{\widehat{b}}(\xi) \geq \omega_0$, then

$$\operatorname{Re}\lambda - a_{\widehat{b}}(\xi) \leq \operatorname{Re}\lambda - \omega_0 < \operatorname{Re}\lambda - \omega \leq 0 \text{ for } \lambda \in S_\omega.$$

By means of

$$|\lambda - a_{\widehat{b}}(\xi)|^2 = (\operatorname{Im}\lambda)^2 + (\operatorname{Re}\lambda - a_{\widehat{b}}(\xi))^2,$$

we get

$$|\lambda - a_{\widehat{b}}(\xi)| > |\lambda - \omega| \geq 0,$$

i.e. $|m_\omega(\lambda, \xi)| < 1$. □

Theorem 4.5.10. *There exists a constant $C_\omega > 0$, such that*

$$|\lambda - \omega| \cdot \|(\lambda - A_{\widehat{b}})^{-1}\|_{\mathcal{L}(h^{1+\alpha})} \leq C_\omega \text{ for } \forall \lambda \in S_\omega. \quad (4.5.8)$$

Proof By Corollary 2.3.2, we only need to prove that $\|m_\omega(\lambda, \xi)\|_{\mathcal{M}}$ is independent of $\lambda \in S_\omega$. Since $m_\omega(\lambda, \xi)$ is an even function for ξ , we only need to consider the case $\xi \geq 0$. By direct calculation,

$$\partial_\xi m_\omega(\lambda, \xi) = \frac{c^2(\lambda - \omega) \left(\cosh(c\xi) \sinh(c\xi) + c\xi + \widehat{b}'(0) \sinh(c\xi) \right)}{\left(c\lambda \cosh(c\xi) - c\xi \sinh(c\xi) + \widehat{b}'(0) \right)^2} \quad (4.5.9)$$

for $\xi \geq 0$ and $\lambda \in S_\omega$. Recall that

$$\lambda - a_{\widehat{b}}(\xi) = \frac{c\lambda \cosh(c\xi) - c\xi \sinh(c\xi) + \widehat{b}'(0)}{c \cosh(c\xi)} \quad (4.5.10)$$

and

$$|\lambda - a_{\widehat{b}}(\xi)| \geq |\operatorname{Re}\lambda - a_{\widehat{b}}(\xi)| \geq |\omega - \omega_0|. \quad (4.5.11)$$

Thanks to Lemma 4.5.9 and (4.5.9), we have

$$\begin{aligned} Q &:= (1 + \xi^2)^{\frac{1}{2}} |\partial_\xi m_\omega(\lambda, \xi)| \\ &\leq \frac{c(1 + \xi^2)^{\frac{1}{2}} \left| \cosh(c\xi) \sinh(c\xi) + c\xi + \widehat{b}'(0) \sinh(c\xi) \right|}{\cosh(c\xi) \left| c\lambda \cosh(c\xi) - c\xi \sinh(c\xi) + \widehat{b}'(0) \right|}. \end{aligned} \quad (4.5.12)$$

Now we consider Q with two cases.

Case 1: For $\operatorname{Re}\lambda < 0$, recalling that $\widehat{b}'(0) < 0$, we have

$$\begin{aligned} \left| c\lambda \cosh(c\xi) - c\xi \sinh(c\xi) + \widehat{b}'(0) \right| &\geq \left| c\operatorname{Re}\lambda \cosh(c\xi) - c\xi \sinh(c\xi) + \widehat{b}'(0) \right| \\ &\geq c\xi \sinh(c\xi). \end{aligned}$$

Then,

$$Q \leq \frac{(1 + \xi^2)^{\frac{1}{2}} \left| \cosh(c\xi) \sinh(c\xi) + c\xi + \widehat{b}'(0) \sinh(c\xi) \right|}{\xi \cosh(c\xi) \sinh(c\xi)}. \quad (4.5.13)$$

By (4.5.13), there exist $\xi_0 > 0$ and $C > 0$ (independent of λ) such that $Q \leq C$ for $\xi \geq \xi_0$. By (4.5.11), we also have

$$Q \leq \frac{(1 + \xi^2)^{\frac{1}{2}} \left| \cosh(c\xi) \sinh(c\xi) + c\xi + \widehat{b}'(0) \sinh(c\xi) \right|}{\cosh^2(c\xi) |\omega - \omega_0|},$$

then there exists $C_\omega > 0$ such that $Q \leq C_\omega$ for $0 \leq \xi \leq \xi_0$. Thus Q is bounded independently of λ .

Case 2: For $Re\lambda \geq 0$, then $0 \leq Re\lambda \cosh(c\xi) \leq \omega_0 \cosh(c\xi)$. Since

$$\lim_{\xi \rightarrow \infty} \frac{\cosh(c\xi)}{\xi \sinh(c\xi)} = 0,$$

there exists $\xi_1 > 0$ such that $\omega_0 \cosh(c\xi) \leq \frac{\xi \sinh(c\xi)}{2}$ for $\xi > \xi_1$. Then, for $\xi > \xi_1$, we have

$$\begin{aligned} cRe\lambda \cosh(c\xi) - c\xi \sinh(c\xi) + \widehat{b}'(0) &\leq c\omega_0 \cosh(c\xi) - c\xi \sinh(c\xi) + \widehat{b}'(0) \\ &\leq -\frac{c}{2}\xi \sinh(c\xi) \leq 0. \end{aligned}$$

This means $\left| c\lambda \cosh(c\xi) - c\xi \sinh(c\xi) + \widehat{b}'(0) \right| \geq \frac{c}{2}\xi \sinh(c\xi)$ for $\xi > \xi_1$. Thus, similar to case 1, we also obtain that Q is bounded independently of λ .

Therefore, Q is bounded independently of λ . Furthermore, by Lemma 4.5.9, we get that $\|m_\omega(\lambda, \xi)\|_{\mathcal{M}}$ is independent of $\lambda \in S_\omega$. \square

Theorem 4.5.11. $-A_{\widehat{b}}$ is a sectorial operator, and

$$\sup\{Re\lambda; \lambda \in \sigma(-A_{\widehat{b}})\} \leq -\omega_0 < 0. \quad (4.5.14)$$

Moreover, $-A_{\widehat{b}} \in \mathcal{H}(h^{2+\alpha}, h^{1+\alpha})$.

Proof For every $0 < \omega < \omega_0$, we denote $\widetilde{S}_\omega := \{\lambda \in \mathbb{C}; Re\lambda \geq -\omega\}$. Then it is clear that

$$\lambda \in \widetilde{S}_\omega \text{ if and only if } -\lambda \in S_\omega.$$

Then according to (4.5.6), we know $\widetilde{S}_\omega \subset \rho(-A_{\widehat{b}})$ and

$$(\lambda + A_{\widehat{b}})^{-1} = -(-\lambda - A_{\widehat{b}})^{-1} \in \mathcal{L}(h^{1+\alpha}, h^{2+\alpha}) \quad \text{for } \forall \lambda \in \widetilde{S}_\omega. \quad (4.5.15)$$

Thanks to Theorem 4.5.10, there exists a constant $C > 0$ such that

$$|-\lambda - \omega| \cdot \|(-\lambda - A_{\widehat{b}})^{-1}\|_{\mathcal{L}(h^{1+\alpha})} \leq C \quad \text{for } \forall \lambda \in \widetilde{S}_\omega,$$

i.e.

$$|\lambda - (-\omega)| \cdot \|R(\lambda, -A_{\widehat{b}})\|_{\mathcal{L}(h^{1+\alpha})} \leq C \quad \text{for } \forall \lambda \in \widetilde{S}_\omega. \quad (4.5.16)$$

This means $-A_{\widehat{b}}$ is a sectorial operator. Since ω is arbitrary, (4.5.14) holds true.

Further, by means of (4.5.1) and (4.5.15), we know that

$$-\omega + A_{\widehat{b}} \in \mathcal{L}is(h^{2+\alpha}, h^{1+\alpha}).$$

Thus $-A_{\widehat{b}} \in \mathcal{H}(h^{2+\alpha}, h^{1+\alpha})$. □

Remark 4.5.12. The proof of Theorem 4.5.11 shows that the assumption $\widehat{b}'(0) < 0$ is only need to get (4.5.14). In particular, we also have $-A_{\widehat{b}} \in \mathcal{H}(h^{2+\alpha}, h^{1+\alpha})$ in the case $\widehat{b}'(0) \geq 0$. □

4.6 Main results

4.6.1 Stability of the transformed problem

In the next theorem, we shall show the existence of the solution to (4.4.1).

Theorem 4.6.1 (Well-posedness). *Assume that $\widehat{b}(0) = 0$. Then there exists $r > 0$ such that (4.4.1) possesses a classical solution, provided $\|g_0\|_{h^{2+\alpha}} < r$.*

Proof Due to Remark 4.5.12, we know

$$-\partial\Phi_{\widehat{b}}(0) \in \mathcal{H}(h^{2+\alpha}, h^{1+\alpha}).$$

Thanks to Corollary 4.4.2 and Proposition 3.1.18, there exists $r \in (0, c)$ such that

$$-\partial\Phi_{\widehat{b}}(h) \in \mathcal{H}(h^{2+\alpha}, h^{1+\alpha}) \quad \forall \|h\|_{h^{2+\alpha}} < r.$$

Now we take $\mathcal{O} := B(0, r)$ in $h^{2+\alpha}$. Since $r < c$, we can know that $\mathcal{O} \subset h_{\mathfrak{A}}^{2+\alpha}$. By Corollary 4.5.5, we know that $h^{2+\alpha}$ is dense in $h^{1+\alpha}$. Using Corollary 4.4.2 again, we know that (3.3.2) is satisfied. If $\|g_0\|_{h^{2+\alpha}} < r$, then $g_0 \in \mathcal{O}$. Further, thanks to Theorem 3.3.1, we can know that (4.4.1) possesses a classical solution. □

In the following, we study the asymptotic behavior near the equilibrium $g \equiv 0$. Recall that

$$G(g) := A_{\widehat{b}}g - \Phi_{\widehat{b}}(g).$$

Clearly, G is a C^∞ function. By the assumption of \widehat{b} , we can calculate that $G(0) = 0, \partial G(0) = 0$. Notice that

$$\partial_t g = -A_{\widehat{b}}g + G(g).$$

Moreover, we have known that the important condition (4.5.14) is satisfied. Applying the principle of linearized stability([24], Theorem 9.1.2), we obtain that

Theorem 4.6.2 (Exponential stability). *Assume that $\widehat{b}(0) = 0$ and $\widehat{b}'(0) < 0$. For any $\omega \in [0, \omega_0)$, there are positive constants r and C , such that for any $g_0 \in h_{\mathfrak{X}}^{2+\alpha}$ with $\|g_0\|_{h^{2+\alpha}} < r$, we get that the maximal existence interval of the solution g of (4.4.1) is $[0, +\infty)$, and the estimate*

$$\|g(t)\|_{h^{2+\alpha}} + \|g'(t)\|_{h^{1+\alpha}} \leq Ce^{-\omega t} \|g_0\|_{h^{2+\alpha}} \quad \forall t \geq 0, \quad (4.6.1)$$

holds.

Thanks to Proposition 4.4.1, we know that problem (4.3.1)–(4.3.6) has a solution (v, g) . Since $g(t) \in h_{\mathfrak{X}}^{2+\alpha}$ and $b \in C^\infty$, then $\widehat{b}(g(t)) \in C^2$. In the following, we will get the estimate of v .

Theorem 4.6.3. *Assume as Theorem 4.6.2. For any $\omega \in [0, \omega_0)$, there are positive constants r_0 and $C = C(r_0)$, such that for any $g_0 \in h_{\mathfrak{X}}^{2+\alpha}$ with $\|g_0\|_{h^{2+\alpha}} < r_0$, we get the estimate*

$$\|v(t)\|_{h^{2+\alpha}(\Omega)} \leq Ce^{-\omega t} \|g_0\|_{h^{2+\alpha}} \quad \forall t \geq 0, \quad (4.6.2)$$

where $v := \mathcal{T}(g)g + \mathcal{U}(g)\widehat{b}(g)$.

Proof Let r be the constant in the Theorem 4.6.2. Take $r_0 < r$, by Theorem 4.6.2, there exist positive constants $M_1 = M_1(r_0)$ and $M_2 = M_2(r_0)$, such that

$$\|g(t)\|_{h^{2+\alpha}} \leq M_1 \text{ for } \forall t \geq 0 \quad \text{and} \quad \|\widehat{b}\|_{C^2([-M_1, M_1])} \leq M_2.$$

Since $\widehat{b}(0) = 0$, we have

$$|\widehat{b}(g(t))| = |\widehat{b}(g(t)) - \widehat{b}(0)| \leq M_2 \|g(t)\|_\infty. \quad (4.6.3)$$

Again since $(\widehat{b}(g(t, x)))' = \widehat{b}'(g(t, x))g'(t, x)$, then

$$|(\widehat{b}(g(t)))'| \leq M_2 \|g'(t)\|_\infty. \quad (4.6.4)$$

Moreover, for $x \neq y$,

$$\begin{aligned} \frac{|(\widehat{b}(g(t, x)))' - (\widehat{b}(g(t, y)))'|}{|x - y|^\alpha} &\leq \frac{|\widehat{b}'(g(t, x))(g'(t, x) - g'(t, y))|}{|x - y|^\alpha} \\ &\quad + \frac{|(\widehat{b}'(g(t, x)) - \widehat{b}'(g(t, y)))g'(t, y)|}{|x - y|^\alpha} \\ &\leq M_2 (\|g(t)\|_{h^{1+\alpha}} + \|g'(t)\|_\infty) \cdot [g(t)]_\alpha. \end{aligned} \quad (4.6.5)$$

By (4.6.3), (4.6.4) and (4.6.5), there exists a positive constant $C = C(M_1, M_2)$ such that

$$\|\widehat{b}(g(t))\|_{h^{1+\alpha}} \leq C \|g(t)\|_{h^{1+\alpha}}. \quad (4.6.6)$$

By Theorem 4.3.5 b), we know that there exists a positive constant $C_0 = C_0(M_1)$ independent of t , such that

$$\|\mathcal{T}(g(t))\|_{\mathcal{L}(h^{2+\alpha}, h^{2+\alpha}(\Omega))} \leq C_0, \quad \|\mathcal{U}(g(t))\|_{\mathcal{L}(h^{1+\alpha}, h^{2+\alpha}(\Omega))} \leq C_0 \quad \forall t \geq 0.$$

Furthermore, since $h^{2+\alpha} \hookrightarrow h^{1+\alpha}$, by (4.6.1) and (4.6.6), we have

$$\|v(t)\|_{h^{2+\alpha}(\Omega)} \leq C(r_0)(\|g(t)\|_{h^{2+\alpha}} + \|\widehat{b}(g(t))\|_{h^{1+\alpha}}) \leq C(r_0)e^{-\omega t}\|g_0\|_{h^{2+\alpha}} \quad \forall t \geq 0. \quad (4.6.7)$$

□

4.6.2 Stability estimates for the interface and pressure

Thanks to Proposition 4.3.2, we get that problem (4.2.1)–(4.2.6) has a solution (u, f) . In the following, we shall get the estimate of u . To do it, we need some lemmas and remarks.

Remark 4.6.4. Since $\widehat{b}(g) = -(c+g)b(c+g)$, we can calculate that

$$\widehat{b}'(g) = -b(c+g) - (c+g)b'(c+g).$$

Therefore,

$$\left. \begin{array}{l} \widehat{b}(0) = 0 \\ \widehat{b}'(0) < 0 \end{array} \right\} \iff \left\{ \begin{array}{l} b(c) = 0 \\ b'(c) > 0 \end{array} \right. \quad (4.6.8)$$

□

Lemma 4.6.5. *Let E, F and G be Banach spaces, and let X and Y be an open subset of E and F , respectively. Let further $0 < \alpha < 1$. If $h, \tau \in BUC^\alpha(X, Y)$, $\psi \in BUC^\alpha(Y, G)$, and if $\theta \in C^1(X, Y)$ has a bounded derivative, then*

$$\phi := \psi \circ \theta \in BUC^\alpha(X, G), \quad \rho := h \cdot \tau \in BUC^\alpha(X, Y).$$

In particular,

$$[\phi]_\alpha \leq [\psi]_\alpha \|\theta'\|_\infty^\alpha, \quad [\rho]_\alpha \leq [h]_\alpha \|\tau\|_\infty + \|h\|_\infty [\tau]_\alpha. \quad (4.6.9)$$

Proof It is obvious for the function ρ . Therefore, we only need prove the case of function ϕ . By assumptions, we easily know that $\phi \in BC(X, G)$. For $\forall x, y \in X$,

$$\begin{aligned} \|\phi(x) - \phi(y)\|_G &= \|\psi(\theta(x)) - \psi(\theta(y))\|_G \\ &\leq [\psi]_\alpha \|\theta(x) - \theta(y)\|_F^\alpha \\ &\leq [\psi]_\alpha (\|\theta'\|_\infty \|x - y\|_E)^\alpha \\ &\leq [\psi]_\alpha \|\theta'\|_\infty^\alpha \|x - y\|_E^\alpha. \end{aligned}$$

Thus, $\phi \in BUC^\alpha(X, G)$ and $[\phi]_\alpha \leq [\psi]_\alpha \|\theta'\|_\infty^\alpha$. □

Lemma 4.6.6. *If $g \in h_{\mathfrak{A}}^{2+\alpha}$, then $\varphi_g \in C^{2+\alpha}(\Omega_f, \Omega)$ and $\partial_k \varphi_g \in BUC^{1+\alpha}(\Omega_f, \mathbb{R}^2)$ ($k = 1, 2$). Furthermore, $\|\partial_k \varphi_g\|_{BUC^{1+\alpha}(\Omega_f, \mathbb{R}^2)}$ only depends on $\|g\|_{h^{2+\alpha}}$ and constant M , where $M := \inf_{x \in \mathbb{R}} \{c + g(x)\}$.*

Proof Denote the k th component of φ_g by φ_k ($k = 1, 2$), i.e. $\varphi_g = (\varphi_1, \varphi_2)$. In order to prove $\varphi_g \in C^{2+\alpha}$, it is sufficient to prove that $\varphi_1, \varphi_2 \in C^{2+\alpha}$. By the definition of \mathfrak{A} , there exists a positive constant M such that $\inf_{x \in \mathbb{R}} \{c + g(x)\} \geq M$. Moreover, we can calculate that

$$\begin{aligned} \partial \varphi_1(x, y) &= (1, 0), & \partial \varphi_2(x, y) &= \left(\frac{yg'(x)}{(c+g(x))^2}, -\frac{1}{c+g(x)} \right), \\ \partial^2 \varphi_1(x, y) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \partial^2 \varphi_2(x, y) &= \begin{pmatrix} \frac{yg''(x)}{(c+g(x))^2} - \frac{2g'^2(x)y}{(c+g(x))^3} & \frac{g'(x)}{(c+g(x))^2} \\ \frac{g'(x)}{(c+g(x))^2} & 0 \end{pmatrix}. \end{aligned}$$

Moreover,

$$\begin{aligned} \left| \frac{yg'(x)}{(c+g(x))^2} \right| &\leq \frac{\|g'\|_\infty}{M}, & \left| \frac{1}{c+g(x)} \right| &\leq \frac{1}{M}, \\ \left| \frac{yg''(x)}{(c+g(x))^2} - \frac{2g'^2(x)y}{(c+g(x))^3} \right| &\leq \frac{\|g''\|_\infty}{M} + \frac{2\|g'\|_\infty^2}{M^2}, & \left| \frac{g'(x)}{(c+g(x))^2} \right| &\leq \frac{\|g'\|_\infty}{M^2}. \end{aligned}$$

For $0 < \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} < \frac{1}{2}$, we have

$$\begin{aligned} &\left| \frac{y_1 g''(x_1)}{(c+g(x_1))^2} - \frac{y_2 g''(x_2)}{(c+g(x_2))^2} \right| \\ &\leq \frac{1}{M^4} \left| (y_1 - y_2) g''(x_1) (c+g(x_2))^2 + y_2 (g''(x_1) - g''(x_2)) (c+g(x_2))^2 \right. \\ &\quad \left. + y_2 g''(x_2) (2c + g(x_1) + g(x_2)) (g(x_2) - g(x_1)) \right| \\ &\leq \frac{1}{M^4} \left[\|g''\|_\infty (c + \|g\|_\infty)^2 |y_1 - y_2| + \|g\|_{h^{2+\alpha}} (c + \|g\|_\infty)^3 |x_1 - x_2|^\alpha \right. \\ &\quad \left. + 2\|g''\|_\infty (c + \|g\|_\infty)^2 \|g'\|_\infty |x_1 - x_2| \right] \\ &\leq C(|x_1 - x_2|^\alpha + |y_1 - y_2|^\alpha) \\ &\leq 2C \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 \right]^{\frac{\alpha}{2}}, \end{aligned}$$

and

$$\begin{aligned}
& \left| \frac{g'^2(x_1)y_1}{(c+g(x_1))^3} - \frac{g'^2(x_2)y_2}{(c+g(x_2))^3} \right| \\
& \leq \frac{1}{M^6} \left| (y_1 - y_2)g'^2(x_1)(c+g(x_2))^3 + y_2(g'(x_1) - g'(x_2))(g'(x_1) + g'(x_2)) \cdot \right. \\
& \quad (c+g(x_2))^3 + y_2g'^2(x_2)\{(c+g(x_1))^2 + (c+g(x_1))(c+g(x_2)) \\
& \quad \left. + (c+g(x_2))^2\}(g(x_2) - g(x_1)) \right| \\
& \leq \frac{1}{M^6} (\|g'\|_\infty^2(c + \|g\|_\infty)^3|y_1 - y_2| + 2\|g''\|_\infty\|g'\|_\infty(c + \|g\|_\infty)^4|x_1 - x_2| \\
& \quad + 3\|g'\|_\infty^3(c + \|g\|_\infty)^3|x_1 - x_2|) \\
& \leq C(|x_1 - x_2|^\alpha + |y_1 - y_2|^\alpha) \\
& \leq 2C[(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{\alpha}{2}}.
\end{aligned}$$

For $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \geq \frac{1}{2}$, we also have

$$\begin{aligned}
\left| \frac{y_1g''(x_1)}{(c+g(x_1))^2} - \frac{y_2g''(x_2)}{(c+g(x_2))^2} \right| & \leq \frac{2\|g''\|_\infty}{M} \leq \frac{\|g''\|_\infty}{M} \cdot [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{\alpha}{2}}, \\
\left| \frac{g'^2(x_1)y_1}{(c+g(x_1))^3} - \frac{g'^2(x_2)y_2}{(c+g(x_2))^3} \right| & \leq \frac{2\|g'\|_\infty^2}{M^2} \leq \frac{4\|g'\|_\infty^2}{M^2} \cdot [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{\alpha}{2}}.
\end{aligned}$$

Thus, $\varphi_1, \varphi_2 \in C^{2+\alpha}$ and $\partial_i\varphi_1, \partial_i\varphi_2 \in BUC^{1+\alpha}(\Omega_f, \mathbb{R})$ ($i = 1, 2$). Furthermore, $\|\partial_i\varphi_1\|_{BUC^{1+\alpha}(\Omega_f, \mathbb{R})}$, and $\|\partial_i\varphi_2\|_{BUC^{1+\alpha}(\Omega_f, \mathbb{R})}$, only depend on $\|g\|_{h^{2+\alpha}}$ and M . Finally, since $\partial_i\varphi_g = (\partial_i\varphi_1, \partial_i\varphi_2)$, then $\partial_i\varphi_g \in BUC^{1+\alpha}(\Omega_f, \mathbb{R}^2)$, and $\|\partial_i\varphi_g\|_{BUC^{1+\alpha}(\Omega_f, \mathbb{R}^2)}$ only depends on $\|g\|_{h^{2+\alpha}}$ and M . \square

Remark 4.6.7. Similarly, we also can calculate that $\varphi_g^{-1} \in C^{2+\alpha}(\Omega, \Omega_f)$ and $\partial_k\varphi_g^{-1} \in BUC^{1+\alpha}(\Omega, \mathbb{R}^2)$ ($k = 1, 2$). Furthermore, $\|\partial_k\varphi_g^{-1}\|_{BUC^{1+\alpha}(\Omega, \mathbb{R}^2)}$ only depends on $\|g\|_{h^{2+\alpha}}$. Hence, we can obtain that the following assertion. If $\psi := \phi \circ \varphi_g$, then we have

$$\psi \in BUC^{2+\alpha}(\Omega_f) \quad \text{iff} \quad \phi \in BUC^{2+\alpha}(\Omega).$$

Furthermore, there exist $C_1, C_2 > 0$ (only depend on M and $\|g\|_{h^{2+\alpha}}$) such that

$$C_1\|\phi\|_{BUC^{2+\alpha}(\Omega)} \leq \|\psi\|_{BUC^{2+\alpha}(\Omega_f)} \leq C_2\|\phi\|_{BUC^{2+\alpha}(\Omega)}, \quad (4.6.10)$$

where M is defined by Lemma 4.6.6.

Proof " \Leftarrow " Suppose $\phi \in BUC^{2+\alpha}(\Omega)$ and $\varphi_g = (\varphi_1, \varphi_2)$. Thanks to Lemma 4.6.5, we know that $\varphi_g \in C^{2+\alpha}(\Omega_f, \Omega)$. Then we have $\psi \in BC^2(\Omega_f)$. By chain law, we can know that $\|\psi\|_\infty \leq \|\phi\|_\infty$ and for $j, k \in \{1, 2\}$,

$$\begin{aligned}
\partial_k\psi & = \sum_{i=1}^2 \partial_i\phi(\varphi_g) \cdot \partial_k\varphi_i = \sum_{i=1}^2 ((\partial_i\phi) \circ \varphi_g) \cdot \partial_k\varphi_i, \\
\partial_j\partial_k\psi & = \sum_{i,l=1}^2 \partial_l\partial_i\phi(\varphi_g) \cdot \partial_j\varphi_l \cdot \partial_k\varphi_i + \sum_{i=1}^2 \partial_i\phi(\varphi_g) \cdot \partial_j\partial_k\varphi_i \\
& = \sum_{i,l=1}^2 ((\partial_l\partial_i\phi) \circ \varphi_g) \cdot \partial_j\varphi_l \cdot \partial_k\varphi_i + \sum_{i=1}^2 ((\partial_i\phi) \circ \varphi_g) \cdot \partial_j\partial_k\varphi_i.
\end{aligned}$$

Again by Lemma 4.6.6 and Lemma 4.6.5, we can conclude that $\partial_j \partial_k \psi \in BUC^\alpha(\Omega_f)$. This implies $\psi \in BUC^{2+\alpha}(\Omega_f)$. Moreover, there exists a constant $C_2 > 0$ (only depends on M and $\|g\|_{h^{2+\alpha}}$) such that

$$\|\psi\|_{BUC^{2+\alpha}(\Omega_f)} \leq C_2 \|\phi\|_{BUC^{2+\alpha}(\Omega)}.$$

" \implies " By Lemma 4.3.1, $\varphi_g \in Diff^{2+\alpha}(\Omega_f, \Omega)$. Similar to Lemma 4.6.6, we can get the similar properties for φ_g^{-1} . Therefore the statement can be obtained similarly. Moreover, there exists a constant $C > 0$ (only depends on $\|g\|_{h^{2+\alpha}}$) such that

$$\|\phi\|_{BUC^{2+\alpha}(\Omega)} \leq C \|\psi\|_{BUC^{2+\alpha}(\Omega_f)}.$$

Take $C_1 = \frac{1}{C}$, we can get the estimate (4.6.10). □

Corollary 4.6.8. *Given $g \in h_{\mathfrak{A}}^{2+\alpha}$. If $\psi = \phi \circ \varphi_g$, then $\psi \in h^{2+\alpha}(\Omega_f)$ iff $\phi \in h^{2+\alpha}(\Omega)$.*

Proof By Remark 4.6.7, we have known that $\psi \in BUC^{2+\alpha}(\Omega_f)$ iff $\phi \in BUC^{2+\alpha}(\Omega)$. For $\phi \in h^{2+\alpha}(\Omega)$, there exists a sequence $\{\phi_i\} \in \gamma_\Omega \mathcal{S}(\mathbb{R}^2)$ such that

$$\phi_i \rightarrow \phi \quad \text{in} \quad BUC^{2+\alpha}(\Omega) \quad \text{as} \quad i \rightarrow \infty.$$

Since $g \in h_{\mathfrak{A}}^{2+\alpha}$ and Remark 4.2.1, there also exists a sequence $\{g_i\} \in \mathcal{S}(\mathbb{R}) \cap \mathfrak{A}$ such that

$$g_i \rightarrow g \quad \text{in} \quad BUC^{2+\alpha}(\mathbb{R}) \quad \text{as} \quad i \rightarrow \infty.$$

Thanks to Lemma 4.3.1, we know $\varphi_{g_i} \in Diff^\infty(\Omega_f, \Omega)$. Thus $\phi_i \circ \varphi_{g_i} \in \gamma_{\Omega_f} \mathcal{S}(\mathbb{R}^2)$. Moreover,

$$\phi_i \circ \varphi_{g_i} \rightarrow \phi \circ \varphi_g \quad \text{in} \quad BUC^{2+\alpha}(\Omega_f) \quad \text{as} \quad i \rightarrow \infty.$$

This implies $\psi \in h^{2+\alpha}(\Omega_f)$. The converse is similar. □

Theorem 4.6.9. *Assume that $b(c) = 0$ and $b'(c) > 0$. For any $\omega \in [0, \omega_0)$, there exist positive constants r_1 and $C = C(r_1)$, such that for any $f_0 - c \in h_{\mathfrak{A}}^{2+\alpha}$ with $\|f_0 - c\|_{h^{2+\alpha}} < r_1$, the problem (4.2.1)–(4.2.6) has a unique global solution $(u(t, x, y), f(t, x))$ such that $u - c \in C([0, +\infty), h^{2+\alpha}(\Omega_{f(t)}))$, and $f - c \in C([0, +\infty), h_{\mathfrak{A}}^{2+\alpha})$. Furthermore, we get the estimate*

$$\|f(t) - c\|_{h^{2+\alpha}} + \|f'(t)\|_{h^{1+\alpha}} + \|u(t) - c\|_{h^{2+\alpha}(\Omega_{f(t)})} \leq Ce^{-\omega t} \|f_0 - c\|_{h^{2+\alpha}}, \quad \forall t \geq 0. \quad (4.6.11)$$

Proof Let r, r_0 be the constants in the Theorem 4.6.2 and Theorem 4.6.3 respectively, and $g_0 := f_0 - c$. Take $r_1 \leq \min\{r, r_0\}$, by Remark 4.6.4, Theorem 4.6.2 and Theorem 4.6.3, we know that if $f_0 - c \in h_{\mathfrak{A}}^{2+\alpha}$ with $\|f_0 - c\|_{h^{2+\alpha}} < r_1$, then we have

$$g \in C([0, +\infty), h_{\mathfrak{A}}^{2+\alpha}) \cap C^1([0, +\infty), h^{1+\alpha}) \quad \text{and} \quad v \in C([0, +\infty), h^{2+\alpha}(\Omega)).$$

Moreover, there exists a constant $C_0 > 0$ such that

$$\|g(t)\|_{h^{2+\alpha}} + \|g'(t)\|_{h^{1+\alpha}} + \|v\|_{h^{2+\alpha}(\Omega)} \leq C_0 e^{-\omega t} \|f_0 - c\|_{h^{2+\alpha}}, \quad \forall t \geq 0. \quad (4.6.12)$$

Thanks to Proposition 4.4.1 and Proposition 4.3.2, we get that problem (4.2.1)–(4.2.6) exists a unique solution $(u(t, x, y), f(t, x)) = (v(t, \varphi_{g(t,x)}(x, y)) + c, g(t, x) + c)$. Due to (4.6.12), we know $\|g(t)\|_{h^{2+\alpha}} \leq C_0 r_1$ for $\forall t \geq 0$. Let $M(t) := \inf_{x \in \mathbb{R}} \{c + g(t, x)\}$, and we take r_1 small enough such that $r_1 < \frac{c}{2C_0}$, then we can know that

$$M(t) \geq c - \|g(t)\|_{\infty} \geq \frac{c}{2} \quad \text{for } \forall t \geq 0.$$

It is similar to Remark 4.6.7, we can get $u(t) \in BUC^{2+\alpha}(\Omega_{f(t)})$ and there exists $C_1 = C_1(r_1) > 0$ (independent of t) such that

$$\|u(t) - c\|_{BUC^{2+\alpha}(\Omega_{f(t)})} \leq C_1 \|v(t)\|_{h^{2+\alpha}(\Omega)} \quad \text{for } \forall t \geq 0. \quad (4.6.13)$$

By Corollary 4.6.8, $u(t) - c \in h^{2+\alpha}(\Omega_{f(t)})$. Let $C = C_0(C_1 + 1)$, by (4.6.13) and (4.6.12), we get

$$\|f(t) - c\|_{h^{2+\alpha}} + \|f'(t)\|_{h^{1+\alpha}} + \|u(t) - c\|_{h^{2+\alpha}(\Omega_{f(t)})} \leq C e^{-\omega t} \|f_0 - c\|_{h^{2+\alpha}} \quad \text{for } \forall t \geq 0.$$

□

Remark 4.6.10. By Darcy's law $\mathbf{V}_w = -\nabla u$, the y component of the velocity \mathbf{V}_w of the flow on the fixed layer can be pictured as the following (see Figure 3):

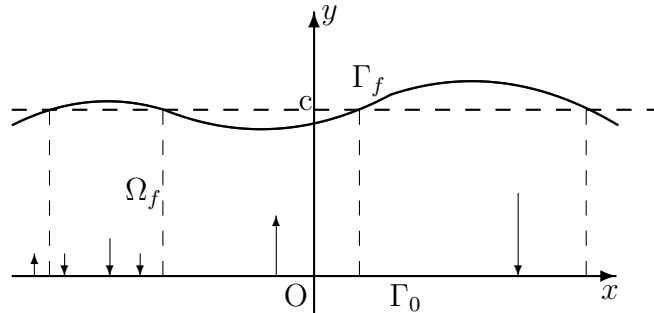


Figure 3

This implies that when the interface function f is larger than the constant c , the external source layer will supply an instant negative injection rate which causes the interface to fall. On the other hand, when the interface function f is less than the constant c , the external source layer will supply an instant positive injection rate which leads the interface to increase. Thus $(u, f) \equiv (c, c)$ is a stationary solution of problem (4.2.1)–(4.2.6). □

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