Additional material to the paper 'On necessity and robustness of dissipativity in economic model predictive control'

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Abstract

This technical report contains additional material to the paper
"On necessity and robustness of dissipativity in economic model predictive control"
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in particular some extensions and proofs. References and labels in this technical report (in particular Equation labels (1)–(26), references [1]–[23], and all theorem numbers etc.) refer to those in that paper.

In this technical report, we need the following additional notation. For a function $F(x) : \mathbb{R}^n \to \mathbb{R}^m$, denote by $\nabla_x F(y)$ the Jacobian matrix of F with respect to x, evaluated at a point y. If F is scalar, then $\nabla_x F(y)$ denotes the gradient of F with respect to x, evaluated at a point y. Furthermore, for a function $G(x) : \mathbb{R}^n \to \mathbb{R}$, $\nabla_x^2 G(y)$ denotes the Hessian of G evaluated at a point y. For $x \in \mathbb{R}^n$ and $\varepsilon \geq 0$, define by $B_\varepsilon(x)$ the ball of radius ε around x, i.e., $B_\varepsilon(x) := \{y \in \mathbb{R}^n : |y - x| \leq \varepsilon\}$. For a set $S \subseteq \mathbb{R}^n$, denote by $\overline{S}$ its closure.

I. Proof of Theorem 4

For simplicity and without loss of generality, in the following we assume again that $\ell(x^*, u^*) = 0$. Similar to the proof of Theorem 3, we want to induce a contradiction by constructing a feasible state/input sequence pair $\hat{x}(\cdot), \hat{u}(\cdot)$ violating (6). By assumption, the linearization of system (2) at the optimal steady-state (x^*, u^*) is controllable, and hence system (2) is locally controllable at x^* in n time steps (see [18,Section 3.7]). This means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each pair of states $y', y'' \in B_\delta(x^*)$, there exists an input/state sequence pair $x'(\cdot), u'(\cdot)$ such that $x'(0) = y'$, $x'(n) = y''$, and $(x'(i), u'(i)) \in B_\varepsilon(x^*, u^*)$ for all $i \in \mathbb{I}_{[0,n-1]}$. Define $\hat{\varepsilon} := \max_{B_\varepsilon(x^*,u^*) \subseteq \mathbb{Z}} \varepsilon$, and denote the corresponding δ by $\hat{\delta}$. Note that $\hat{\varepsilon} > 0$ as $(x^*, u^*) \in \text{int}(\mathbb{Z})$ by assumption.

Now assume for contradiction that the system is uniformly suboptimally operated off steady-state, but it is not dissipative on $\mathbb{Z}^0$. By Theorem 2, this is equivalent to the fact that the available storage is unbounded on $\mathbb{X}^0$, and hence as in the proof of Theorem 3 we conclude that for each $r \in \mathbb{I}_{\geq 0}$, there exist some $y \in \mathbb{X}^0$ and a state/input sequence pair $x_r(\cdot), u_r(\cdot)$ together with a time instant $T_r \in \mathbb{I}_{\geq 0}$ such that $x_r(0) = y$, $(x_r(k), u_r(k)) \in \mathbb{Z}^0$ for all $k \in \mathbb{I}_{\geq 0}$ and (19) is satisfied. Note that due to continuity of ℓ and compactness of $\mathbb{Z}$, it follows that $T_r \to \infty$ as $r \to \infty$. As the system is uniformly suboptimally operated off steady-state, there exists $\bar{t} \in \mathbb{I}_{\geq 1}$ such that for all feasible sequences at least one of the conditions (8a)–(8b) is satisfied with $\delta = \hat{\delta}$. Now define c as

$$c := \max \left\{ n \max_{(x,u) \in B_{\hat{\varepsilon}}(x^*,u^*)} \ell(x,u), -\bar{t} \min_{(x,u) \in \mathbb{Z}} \ell(x,u) \right\} \tag{27}$$

and consider a state/input sequence pair $x_r(\cdot), u_r(\cdot)$ with $r \geq 1 + 3c$. Note that for this sequence, $T_r \geq 3\bar{t} + 1$ as $-r < 3\bar{t} \min_{(x,u) \in \mathbb{Z}} \ell(x,u)$. Hence, due to uniform suboptimal operation off steady-state, we conclude that

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$|x_r(s_1)-x^*| \leq \hat{\delta}$ for some $s_1 \in \mathbb{I}_{[1,\bar{t}]}$. Furthermore, as $\sum_{k=0}^{s_1-1} \ell(x_r(k),u_r(k)) \geq s_1 \min_{(x,u)\in\mathbb{Z}} \ell(x,u) \geq -c$ by definition of c in (27), we have

$$\sum_{k=s_1}^{T_r-1} \ell(x_r(k),u_r(k)) \leq -(1+2c) \tag{28}$$

and $T_r - s_1 \geq 2\bar{t}+1$ as $s_1 \leq \bar{t}$. We can now apply the above argument to the shifted sequence $x_r'(s) = x_r(s+s_1)$ and conclude by uniform suboptimal operation off steady-state that $|x_r'(s_2)-x^*| = |x_r(s_1+s_2)-x^*| \leq \hat{\delta}$ for some $s_2 \in \mathbb{I}_{[1,\bar{t}]}$. Furthermore, $\sum_{k=s_1+s_2}^{T_r-1} \ell(x_r(k),u_r(k)) \leq -(1+c)$ by definition of c in (27), and $T_r - s_1 - s_2 \geq \bar{t}+1$ as $s_2 \leq \bar{t}$. Repeating again the above argument, we conclude that $|x_r(s_1+s_2+s_3)-x^*| \leq \hat{\delta}$ for some $s_3 \in \mathbb{I}_{[1,\bar{t}]}$. We can now distinguish two different cases. Either we have

$$\sum_{k=s_1+s_2+s_3}^{T_r-1} \ell(x_r(k),u_r(k)) \geq -c, \tag{29}$$

or (29) does not hold, in which case the definition of c in (27) implies that $T_r - (s_1+s_2+s_3) > \bar{t}$. In the latter case, we can apply the above argument recursively to obtain time instances s_i, $i \in \mathbb{I}_{\geq 4}$, with $|x_r(s_1+\cdots+s_i)-x^*| \leq \hat{\delta}$ until

$$\sum_{k=s_1+\cdots+s_j}^{T_r-1} \ell(x_r(k),u_r(k)) \geq -c, \tag{30}$$

for some $j \in \mathbb{I}_{\geq 4}$. Note that $j \leq T_r - \bar{t}$, as (30) is fulfilled as soon as $s_1+\cdots+s_j \geq T_r - \bar{t}$ due to the definition of c in (27) and $s_1+\cdots+s_j \geq j$.

Summarizing the above, we have proven that both $|x_r(s_1)-x^*| \leq \hat{\delta}$ and $|x_r(s_1+\cdots+s_j)-x^*| \leq \hat{\delta}$, and

$$\sum_{k=s_1}^{s_1+\cdots+s_j-1} \ell(x_r(k),u_r(k)) \overset{(28),(30)}{\leq} -(1+c). \tag{31}$$

Hence, by local controllability at the optimal steady-state (x^*,u^*), there exists a state/input sequence pair $x'(\cdot), u'(\cdot)$ satisfying $x'(0) = x_r(s_1+\cdots+s_j)$, $x'(n) = x_r(s_1)$, and $(x'(t),u'(t)) \in B_{\hat{\varepsilon}}(x^*,u^*)$ for all $t \in \mathbb{I}_{[0,n]}$. Furthermore, by definition of c in (27) we have

$$\sum_{k=0}^{n-1} \ell(x'(k),u'(k)) \leq c. \tag{32}$$

Now define the following input sequence:

$$\hat{u}\big(k(s_2+\cdots+s_j+n)+i\big) = \begin{cases} u_r(s_1+i) & k \in \mathbb{I}_{\geq 0}, i \in \mathbb{I}_{[0,s_2+\cdots+s_j-1]} \\ u'(i) & k \in \mathbb{I}_{\geq 0}, i \in \mathbb{I}_{[s_2+\cdots+s_j, s_2+\cdots+s_j+n-1]} \end{cases} \tag{33}$$

which results in a cyclic state sequence with $\hat{x}(k(s_2+\cdots+s_j+n)) = x_r(s_1)$ for all $k \in \mathbb{I}_{\geq 0}$. This state/input sequence pair fulfills $(\hat{x}(t),\hat{u}(t)) \in \mathbb{Z}$ for all $t \in \mathbb{I}_{\geq 0}$ by construction, and furthermore we obtain for all $k \in \mathbb{I}_{\geq 0}$:

$$\sum_{i=0}^{s_2+\cdots+s_j+n-1} \ell\Big(\hat{x}\big(k(s_2+\cdots+s_j+n-1)+i\big), \hat{u}\big(k(s_2+\cdots+s_j+n-1)+i\big)\Big) \overset{(31)-(33)}{\leq} -1. \tag{34}$$

But this implies that

$$\liminf_{T\to\infty} \sum_{k=0}^{T-1} \frac{\ell(\hat{x}(k),\hat{u}(k))}{T} \overset{(33)}{=} \frac{1}{s_2+\cdots+s_j+n} \sum_{i=0}^{s_2+\cdots+s_j+n-1} \ell(\hat{x}(i),\hat{u}(i)) \overset{(34)}{\leq} -\frac{1}{s_2+\cdots+s_j+n} < 0$$

contradicting (6), i.e., optimal steady-state operation. Hence we conclude that the system (2) is dissipative on $\mathbb{Z}^0$ with respect to the supply rate $s(x,u) := \ell(x,u) - \ell(x^*,u^*)$. $\square$

II. Proof of Theorems 5 and 6

Before proving Theorems 5 and 6, we first recall some well-known facts from nonlinear programming which will be needed in the following. To this end, consider again the optimization problem (21) and assume that f_0, h and g are twice continuously differentiable. For every feasible point y, denote by $A(y) := \{1 \leq j \leq n_g : g_j(y) = 0\}$ the set of active inequality constraints at y. We say that a feasible point y is *regular* [23], if the gradients $\nabla_y h_i(y)$, $1 \leq i \leq n_h$, and $\nabla_y g_j(y)$, $j \in A(y)$, are linearly independent. We then have the following first-order necessary conditions for optimality, known as the Karush-Kuhn-Tucker (KKT) conditions:

Proposition 1 ([23]): Suppose that y^* is regular and a local minimizer of problem $\mathcal{P}$. Then there exist unique Lagrange multipliers $\mu \in \mathbb{R}^{n_h}$ and $\nu \in \mathbb{R}^{n_g}$ such that the following holds:

$$\nabla_y f_0(y^*) + \sum_{i=1}^{n_h} \mu_i \nabla_y h_i(y^*) + \sum_{j=1}^{n_g} \nu_j \nabla_y g_j(y^*) = 0, \tag{35}$$

$$\nu_j \geq 0, \quad \nu_j g_j(y^*) = 0, \quad 1 \leq j \leq n_g. \tag{36}$$

□

Furthermore, a KKT point (y^*, μ, ν) is said to satisfy the *strong second order sufficiency condition* [20,21] if

$$w^T \Big(\nabla_y^2 f_0(y^*) + \sum_{i=1}^{n_h} \mu_i \nabla_y^2 h_i(y^*) + \sum_{j=1}^{n_g} \nu_j \nabla_y^2 g_j(y^*) \Big) w > 0, \tag{37}$$

for all $w \neq 0$ such that $\nabla_y h_i(y^*)^T w = 0$, $1 \leq i \leq n_h$, and $\nabla_y g_j(y^*)^T w = 0$, for all j such that $j \in A(y^*)$ and $\nu_j > 0$.

Proposition 2 ([23,20]): Suppose that y^* is a feasible point of problem $\mathcal{P}$ which is regular and together with some (μ, ν) satisfies the KKT conditions (35)-(36) as well as the strong second order sufficiency condition (37). Then y^* is a strict local minimizer of problem $\mathcal{P}$. □

Remark: In order for Proposition 2 to hold, it suffices that a slightly weaker condition than the strong second order sufficiency condition holds. Namely, (37) has to hold only for such w which in addition to the above requirements also fulfill $\nabla_y g_j(y^*) w \leq 0$, for all j such that $j \in A(y^*)$ and $\nu_j = 0$ [23]. In this paper, we use the strong second order sufficiency condition as it allows us to apply certain sensitivity results also in the case where strict complementarity (i.e., $\nu_j > 0$ for all $j \in A(y^*)$) does not hold [20,21]. □

A. Proof of Theorem 5

The proof of Theorem 5 consists of two parts. First, the sensitivity analysis in nonlinear programming [20–22] is applied to conclude that for sufficiently small $|\varepsilon|$, there exists a steady-state $(x^*(\varepsilon), u^*(\varepsilon))$ which is continuous in ε and a local minimizer of problem $\mathcal{P}_\ell[\varepsilon]$. We then show that the storage function $\lambda(x;\varepsilon)$ can be modified continuously in ε such that $(x^*(\varepsilon), u^*(\varepsilon))$ is also a local minimizer of problem $\mathcal{P}_\gamma[\varepsilon]$. In the second part, we show that $(x^*(\varepsilon), u^*(\varepsilon))$ is not only a local but also a global minimizer of problems $\mathcal{P}_\ell[\varepsilon]$ and $\mathcal{P}_\gamma[\varepsilon]$, which implies that indeed $(x^*(\varepsilon), u^*(\varepsilon)) \in S_\varepsilon^*$ according to the definition of $\mathcal{P}_\ell[\varepsilon]$ in (23), and that the system (2) is dissipative for all $(x,u) \in \mathbb{Z}_\varepsilon$ with respect to the supply rate $s(x,u;\varepsilon) = \ell(x,u) - \ell(x^*(\varepsilon), u^*(\varepsilon))$ according to the definition of $\mathcal{P}_\gamma[\varepsilon]$ in (25).

Part 1: Let $h_s(x,u) := x - f(x,u)$ and $\Lambda(x,u;\varepsilon) := \lambda(x;\varepsilon) - \lambda(f(x,u);\varepsilon)$. As $(x^*(0), u^*(0))$ is regular and a strict minimizer of both problems $\mathcal{P}_\ell[0]$ and $\mathcal{P}_\gamma[0]$, by Proposition 1 there exist unique Lagrange multipliers $\mu_\ell(0) \in \mathbb{R}^n$, $\nu_\ell(0) \in \mathbb{R}^r$ and $\nu_\gamma(0) \in \mathbb{R}^r$ such that the following is satisfied:

$$\nabla_{(x,u)} \ell(x^*(0), u^*(0)) + \nabla_{(x,u)} h_s(x^*(0), u^*(0))^T \mu_\ell(0) + \nabla_{(x,u)} g(x^*(0), u^*(0); 0)^T \nu_\ell(0) = 0, \tag{38}$$

$$\nabla_{(x,u)} \ell(x^*(0), u^*(0)) + \nabla_{(x,u)} \Lambda(x^*(0), u^*(0); 0) + \nabla_{(x,u)} g(x^*(0), u^*(0); 0)^T \nu_\gamma(0) = 0. \tag{39}$$

Now consider the term $\nabla_{(x,u)} \Lambda(x^*(0), u^*(0); 0)$. We obtain

$$\begin{aligned} \nabla_{(x,u)} \Lambda(x^*(0), u^*(0); 0) &= [I \ 0]^T \nabla_x \lambda(x^*(0); 0) - \nabla_{(x,u)} f(x^*(0), u^*(0))^T \nabla_x \lambda(f(x^*(0), u^*(0)); 0) \\ &= \Big([I \ 0]^T - \nabla_{(x,u)} f(x^*(0), u^*(0))^T \Big) \nabla_x \lambda(x^*(0); 0) = \nabla_{(x,u)} h_s(x^*(0), u^*(0))^T \nabla_x \lambda(x^*(0); 0) \end{aligned} \tag{40}$$

where the first equality follows from the chain rule, the second is due to the fact that $(x^*(0), u^*(0))$ is a steady-state, i.e., $f(x^*(0), u^*(0)) = x^*(0)$, and the third follows from the definition of h_s. Comparing (38) with (39) and using (40) as well as the fact that the Lagrange multipliers are unique, we obtain

$$\nabla_x \lambda(x^*(0); 0) = \mu_\ell(0), \quad \nu_\gamma(0) = \nu_\ell(0). \tag{41}$$

Next, as $(x^*(0), u^*(0))$ is assumed to fulfill the strong second order sufficiency condition (37) for problem $\mathcal{P}_\ell[0]$, from the sensitivity analysis in[1] [20, Theorem 2] (see also [21, Section 4]and [22, Theorem 5.2]) we obtain the result that there exists $0 < \varepsilon_1 \leq \varepsilon_{\max}$ such that for all $|\varepsilon| \leq \varepsilon_1$, the problem $\mathcal{P}_\ell[\varepsilon]$ has a unique local minimizer[2] $(x^*(\varepsilon), u^*(\varepsilon))$ which is regular and continuous in ε as well as the corresponding unique Lagrange multipliers $\mu_\ell(\varepsilon)$ and $\nu_\ell(\varepsilon)$.

Now consider the problem $\mathcal{P}_\gamma[\varepsilon]$ defined in (24)–(25), where the function $\lambda(x; \varepsilon)$ is defined by

$$\lambda(x; \varepsilon) := \lambda(x; 0) + \tilde{\lambda}(\varepsilon)^T x \tag{42}$$

with

$$\tilde{\lambda}(\varepsilon) := \mu_\ell(\varepsilon) - \nabla_x \lambda(x^*(\varepsilon); 0). \tag{43}$$

Note that as $x^*(\varepsilon)$ is continuous in ε, the same holds true for $\nabla_x \lambda(x^*(\varepsilon); 0)$. As furthermore also $\mu_\ell(\varepsilon)$ is continuous in ε, it follows that $\tilde{\lambda}(\varepsilon)$ is continuous in ε with $\tilde{\lambda}(0) = 0$ due to (41) and (43). But this implies that also the function $\lambda(x; \varepsilon)$ and hence also $\gamma(x, u; \varepsilon)$ are continuous in ε. Moreover, from the above and Assumption (i) in Theorem 5 it follows that $\nabla^2_{(x,u)} \gamma(x^*(\varepsilon), u^*(\varepsilon); \varepsilon)$ is continuous in ε.

As next step, we want to show that $(x^*(\varepsilon), u^*(\varepsilon))$ is not only a strict local minimizer of problem $\mathcal{P}_\ell[\varepsilon]$, but also of problem $\mathcal{P}_\gamma[\varepsilon]$. By Proposition 2, this can be concluded if $(x^*(\varepsilon), u^*(\varepsilon))$ satisfies both the KKT conditions (35)–(36) (with some $\nu_\gamma(\varepsilon)$) and the strong second order sufficiency condition (37) for problem $\mathcal{P}_\gamma[\varepsilon]$. We first verify the KKT conditions. Taking $\nu_\gamma(\varepsilon) = \nu_\ell(\varepsilon)$, (36) is immediately satisfied. Furthermore, (35) equals (39) with 0 replaced by ε. Using (40) with 0 replaced by ε as well as (42) and (43), we obtain

$$\begin{aligned}
&\nabla_{(x,u)} \ell(x^*(\varepsilon), u^*(\varepsilon)) + \nabla_{(x,u)} \Lambda(x^*(\varepsilon), u^*(\varepsilon); \varepsilon) + \nabla_{(x,u)} g(x^*(\varepsilon), u^*(\varepsilon); \varepsilon)^T \nu_\gamma(\varepsilon) \\
&\overset{(40)}{=} \nabla_{(x,u)} \ell(x^*(\varepsilon), u^*(\varepsilon)) + \nabla_{(x,u)} h_s(x^*(\varepsilon), u^*(\varepsilon))^T \nabla_x \lambda(x^*(\varepsilon); \varepsilon) + \nabla_{(x,u)} g(x^*(\varepsilon), u^*(\varepsilon); \varepsilon)^T \nu_\gamma(\varepsilon) \\
&\overset{(42),(43)}{=} \nabla_{(x,u)} \ell(x^*(\varepsilon), u^*(\varepsilon)) + \nabla_{(x,u)} h_s(x^*(\varepsilon), u^*(\varepsilon))^T \mu_\ell(\varepsilon) + \nabla_{(x,u)} g(x^*(\varepsilon), u^*(\varepsilon); \varepsilon)^T \nu_\ell(\varepsilon) = 0.
\end{aligned}$$

The last equality follows from the fact that by Proposition 1, equation (38) is satisfied with 0 replaced by ε as $(x^*(\varepsilon), u^*(\varepsilon))$ is a strict local minimizer of problem $\mathcal{P}_\ell[\varepsilon]$. Hence $(x^*(\varepsilon), u^*(\varepsilon))$ satisfies the KKT conditions for problem $\mathcal{P}_\gamma[\varepsilon]$.

Next, we show that $(x^*(\varepsilon), u^*(\varepsilon))$ also satisfies the strong second order sufficiency condition (37) for problem $\mathcal{P}_\gamma[\varepsilon]$, which reads

$$w^T \Big(\nabla^2_{(x,u)} \gamma(x^*(\varepsilon), u^*(\varepsilon); \varepsilon) + \sum_{j=1}^r \nu_{\gamma,j}(\varepsilon) \nabla^2_{(x,u)} g_j(x^*(\varepsilon), u^*(\varepsilon); \varepsilon) \Big) w > 0, \tag{44}$$

for all $w \neq 0$ such that $\nabla_{(x,u)} g_j(x^*(\varepsilon), u^*(\varepsilon); \varepsilon)^T w = 0$, $j \in A(x^*(\varepsilon), u^*(\varepsilon))$ and $\nu_{\gamma,j}(\varepsilon) > 0$. Note that as $\nu_\gamma(\varepsilon) = \nu_\ell(\varepsilon)$ is continuous in ε, for sufficiently small $|\varepsilon|$ it holds that $\nu_{\gamma,j}(\varepsilon) > 0$ for all j such that $\nu_{\gamma,j}(0) > 0$. But then, due to continuity reasons, (44) is satisfied for sufficiently small $|\varepsilon|$ as it is satisfied by assumption for $\varepsilon = 0$. Namely, it this were not the case, then there would exist a sequence $\{(x^*(\varepsilon_k), u^*(\varepsilon_k))\}$ with $\varepsilon_k \to 0$ and a corresponding sequence $\{w_k\}$ with $|w_k| = 1$ such that

$$w_k^T \big(\nabla^2_{(x,u)} \gamma(x^*(\varepsilon_k), u^*(\varepsilon_k); \varepsilon_k) + \sum_{j=1}^r \nu_{\gamma,j}(\varepsilon_k) \nabla^2_{(x,u)} g_j(x^*(\varepsilon_k), u^*(\varepsilon_k); \varepsilon_k) \big) w_k \leq 0$$

[1]In various sensitivity results like [20] it is assumed that the function g is twice continuously differentiable in (x, u, ε). However, differentiability with respect to ε can be relaxed to Assumption (i) in Theorem 5, if only continuity (but not differentiability) of the locally optimal solution and the corresponding Lagrange multipliers with respect to ε shall be established (compare [21] and [22, Section 5]), which is what we need here.

[2]With a slight abuse of notation, we already denote this local minimizer by $(x^*(\varepsilon), u^*(\varepsilon))$, although we earlier reserved this notation for a global minimizer of problem $\mathcal{P}_\ell[\varepsilon]$. We will show later that $(x^*(\varepsilon), u^*(\varepsilon))$ is not only a local but indeed a global minimizer of problem $\mathcal{P}_\ell[\varepsilon]$.

and $\nabla_{(x,u)} g_j(x^*(\varepsilon_k), u^*(\varepsilon_k); \varepsilon_k)^T w_k = 0$ for all $j \in A(x^*(\varepsilon_k), u^*(\varepsilon_k))$ and $\nu_{\gamma,j}(\varepsilon_k) > 0$. As the sequence w_k is bounded, it has a convergent subsequence. Taking the limit over such a subsequence results in a contradiction to (44) with $\varepsilon = 0$, as $\nabla^2_{(x,u)}\gamma(x^*(\varepsilon), u^*(\varepsilon); \varepsilon)$, $\nu_{\gamma,j}(\varepsilon)$, $\nabla_{(x,u)} g_j(x^*(\varepsilon), u^*(\varepsilon); \varepsilon)$ and $\nabla^2_{(x,u)} g_j(x^*(\varepsilon), u^*(\varepsilon); \varepsilon)$ are continuous in ε.

Part 2: Summarizing the above, we have shown that for sufficiently small $|\varepsilon|$, there exists $(x^*(\varepsilon), u^*(\varepsilon))$ which is continuous in ε and is a strict local minimizer of both problems $\mathcal{P}_\ell[\varepsilon]$ and $\mathcal{P}_\gamma[\varepsilon]$. What remains to show is that for sufficiently small $|\varepsilon|$, $(x^*(\varepsilon), u^*(\varepsilon))$ is also a global minimizer of both problems $\mathcal{P}_\ell[\varepsilon]$ and $\mathcal{P}_\gamma[\varepsilon]$. To this end, consider the following. As $(x^*(\varepsilon), u^*(\varepsilon))$ is a strict local minimizer of problem $\mathcal{P}_\gamma[\varepsilon]$ and furthermore g, γ and $(x^*(\varepsilon), u^*(\varepsilon))$ are continuous in ε, there exists $\delta > 0$ such that $(x^*(\varepsilon), u^*(\varepsilon))$ is a strict minimizer of γ on the set $\mathbb{Z}_\varepsilon \cap B_\delta(x^*(0), u^*(0))$. Furthermore, according to Assumption (ii) of Theorem 5, $(x^*(0), u^*(0))$ is a strict global minimizer of problem $\mathcal{P}_\gamma[0]$, i.e., $(x^*(0), u^*(0))$ uniquely minimizes $\gamma(x, u; 0)$ over $\mathbb{Z}_0$. Hence $\gamma(x, u; 0) > 0$ for all $(x, u) \in \overline{\mathbb{Z}_0 \setminus B_\delta(x^*(0), u^*(0))}$. But then, due to continuity of γ in (x, u, ε), it holds that also $\gamma(x, u; \varepsilon) > 0$ for all $(x, u) \in \overline{\mathcal{N}(\mathbb{Z}_0) \setminus B_\delta(x^*(0), u^*(0))}$ for each sufficiently small neighborhood $\mathcal{N}(\mathbb{Z}_0)$ of $\mathbb{Z}_0$ and sufficiently small $|\varepsilon|$. For any such open neighborhood, let $g_{j,\min}(\varepsilon) := \min_{(x,u)\in\mathbb{Z}_{\max}\setminus\mathcal{N}(\mathbb{Z}_0)} g_j(x, u, \varepsilon)$, for all $j \in \mathbb{I}_{[1,r]}$. Note that $g_{j,\min}(\varepsilon)$ is well defined[3] by compactness of $\mathbb{Z}_{\max} \setminus \mathcal{N}(\mathbb{Z}_0)$ and continuity of g. Furthermore $g_{j,\min}(\varepsilon) > 0$ for sufficiently small $|\varepsilon|$ as g is continuous in ε and $g_{j,\min}(0) > 0$. But then, by definition of $\mathbb{Z}_\varepsilon$ in (22) and the fact that $\mathbb{Z}_\varepsilon \subseteq \mathbb{Z}_{\max}$ for all $0 \le |\varepsilon| \le \varepsilon_{\max}$, it follows that for sufficiently small $|\varepsilon|$, $\mathbb{Z}_\varepsilon \subseteq \mathcal{N}(\ mathbb{Z}_0)$ for any open neighborhood $\mathcal{N}(\mathbb{Z}_0)$ of $\mathbb{Z}_0$, and thus $\gamma(x, u; \varepsilon) > 0$ for all $(x, u) \in \overline{\mathbb{Z}_\varepsilon \setminus B_\delta(x^*(0), u^*(0))}$. Together with the above established fact that $(x^*(\varepsilon), u^*(\varepsilon))$ is a strict minimizer of γ on the set $\mathbb{Z}_\varepsilon \cap B_\delta(x^*(0), u^*(0))$ and $\gamma(x^*(\varepsilon), u^*(\varepsilon); \varepsilon) = 0$, this implies that $(x^*(\varepsilon), u^*(\varepsilon))$ is indeed a strict global minimizer of problem $\mathcal{P}_\gamma[\varepsilon]$. But this in particular implies that $(x^*(\varepsilon), u^*(\varepsilon))$ is also a strict global minimizer of problem $\mathcal{P}_\ell[\varepsilon]$ due to the definition of problems $\mathcal{P}_\ell[\varepsilon]$ in (23) and $\mathcal{P}_\gamma[\varepsilon]$ in (25), and the definition of γ (see (24)). But this means that $S^*_\varepsilon = \{(x^*(\varepsilon), u^*(\varepsilon))\}$, i.e., $(x^*(\varepsilon), u^*(\varepsilon))$ is indeed the optimal steady-state.

Thus, we have established that there exists $0 < \bar{\varepsilon} \le \varepsilon_{\max}$ such that for all $|\varepsilon| \le \bar{\varepsilon}$, the system (2) is dissipative for all $(x, u) \in \mathbb{Z}_\varepsilon$ with respect to the supply rate $s(x, u; \varepsilon) = \ell(x, u) - \ell(x^*(\varepsilon), u^*(\varepsilon))$, and the corresponding storage function $\lambda(x; \varepsilon)$ is defined in (42)–(43). This concludes the proof of Theorem 5. □

B. Proof of Theorem 6

As γ is convex by assumption, also γ_{ad} defined in (26) with $\lambda_{ad}(x) := \lambda(x)$ is convex, as the two functions only differ by a constant term. Furthermore, due to the definition of problems $\mathcal{P}_{\ell,ad}$ and $\mathcal{P}_{\gamma_{ad}}$ and the definition of γ_{ad}, each global minimizer of problem $\mathcal{P}_{\gamma_{ad}}$ which is a steady-state is also a global minimizer of problem $\mathcal{P}_{\ell,ad}$. As the KKT conditions are sufficient for optimality in case of a convex optimization problem, for the first statement of Theorem 6 to hold it is sufficient to show that for each feasible steady-state $(y, w) \in S$, there exists a function $g_{ad}(x, u)$ which is convex and continuously differentiable in (x, u), such that the KKT conditions (35)–(36) for Problem $\mathcal{P}_{\gamma_{ad}}$ with $\lambda_{ad}(x) = \lambda(x)$ are satisfied at (y, w). It is easily seen that this is possible by choosing, e.g., g_{ad} as a scalar linear function $g_{ad}(x, u) := [x^T \ u^T]a + b$, where $a \in \mathbb{R}^{n+m}$ and $b \in \mathbb{R}$ are such that (i) $a = -\nabla_{(x,u)}\gamma(y, w)$ if $\nabla_{(x,u)}\gamma(y, w) \neq 0$ ($a \neq 0$ otherwise) and (ii) $g_{ad}(y, w) = 0$. Then, (35)–(36) are satisfied with $\nu_{g_{ad}} = 1$ and $\nu_g = 0$ if $\nabla_{(x,u)}\gamma(y, w) \neq 0$, and $\nu_{g_{ad}} = 0$ and $\nu_g = 0$ otherwise. The second statement of the Theorem directly follows from satisfaction of the KKT conditions and the above consideration that each global minimizer of problem $\mathcal{P}_{\gamma_{ad}}$ which is a steady-state is also a global minimum of problem $\mathcal{P}_{\ell,ad}$. □

III. Extension to general supply rates

In this section, we show how the robustness results of Theorem 5 can be extended to general parameter dependent supply rates and constraint sets. To this end, in the following let $s(x, u; \varepsilon)$ denote some general supply rate which depends on parameters $\varepsilon \in \mathbb{R}^s$ and let the constraint set $\mathbb{Z}_\varepsilon$ be defined as in (22). As before, we assume that there exists some $\varepsilon_{\max} > 0$ and some compact set $\mathbb{Z}_{\max}$ such that for all $0 \le |\varepsilon| \le \varepsilon_{\max}$, the set $\mathbb{Z}_\varepsilon$ is non-empty and $\mathbb{Z}_\varepsilon \subseteq \mathbb{Z}_{\max}$. The function γ is defined analogously to (24), but now with general supply rate s, i.e.,

$$\gamma(x, u; \varepsilon) := s(x, u; \varepsilon) + \lambda(x; \varepsilon) - \lambda(f(x, u); \varepsilon). \tag{45}$$

[3] In case that $\mathbb{Z}_{\max} \setminus \mathcal{N}(\mathbb{Z}_0) = \emptyset$, by convention $g_{j,\min}(\varepsilon) := \infty$.

Again, we want to analyze under what conditions system (2) is robustly dissipative for changing ε, i.e., $\gamma(x, u; \varepsilon) \geq 0$ for all $(x, u) \in \mathbb{Z}_\varepsilon$. In the following, let $(x^*(\varepsilon), u^*(\varepsilon))$ denote a global minimizer of problem $\mathcal{P}_\gamma[\varepsilon]$ as defined in (25). We then have to show that $\gamma(x^*(\varepsilon), u^*(\varepsilon); \varepsilon) \geq 0$ in order to conclude that the system is dissipative. Note that a necessary condition for this to hold is that $s(x, u; \varepsilon) \geq 0$ for all steady-states $(x, u) \in \mathbb{Z}_\varepsilon$, as $\gamma(x, u; \varepsilon) = s(x, u; \varepsilon)$ there, which will be assumed in the following. In order to extend Theorem 5 to the case of general storage functions, we define the optimization problem

$$\mathcal{P}_s[\varepsilon] := \mathcal{P}([x\ u], s, x - f(x, u), g). \tag{46}$$

We are now ready to state the following result:

Theorem: Suppose that the following is satisfied:

(i) The functions f, s and g are twice continuously differentiable in (x, u). Furthermore, s and g as well as their first and second derivatives with respect to (x, u) are continuous in ε.

(ii) For $\varepsilon = 0$, $(x^*(0), u^*(0))$ is the unique global minimizer of problem $\mathcal{P}_\gamma[0]$ satisfying $\gamma(x^*(0), u^*(0); 0) \geq 0$, i.e., system (2) is dissipative for all $(x, u) \in \mathbb{Z}_0$ with respect to the supply rate $s(x, u; 0)$. The corresponding storage function $\lambda(x; 0)$ is twice continuously differentiable in x.

(iii) If $(x^*(0), u^*(0))$ is a steady-state, then $(x^*(0), u^*(0))$ is regular and satisfies the strong second order sufficiency condition (37) for problems $\mathcal{P}_s[0]$ and $\mathcal{P}_\gamma[0]$.

Then there exists $\bar{\varepsilon}$ with $0 < \bar{\varepsilon} \leq \varepsilon_{\max}$ such that for all $|\varepsilon| \leq \bar{\varepsilon}$ the system (2) is dissipative for all $(x, u) \in \mathbb{Z}_\varepsilon$ with respect to the supply rate $s(x, u; \varepsilon)$ and storage function $\lambda(x; \varepsilon)$, where $\lambda(x; \varepsilon)$ is twice continuously differentiable in x and continuous in ε. □

Proof: We distinguish the following three different cases.

Case 1: $\gamma(x^*(0), u^*(0); 0) > 0$. In this case, $\gamma(x, u; 0) > 0$ for all $(x, u) \in \mathbb{Z}_0$, as $(x^*(0), u^*(0))$ is the unique global minimizer of problem $\mathcal{P}_\gamma[0]$. Hence by continuity of γ in (x, u), we also have $\gamma(x, u; 0) > 0$ for all $(x, u) \in \mathcal{N}(\mathbb{Z}_0)$ for each sufficiently small neighborhood $\mathcal{N}(\mathbb{Z}_0)$ of $\mathbb{Z}_0$. As was shown in Part 2 of the proof of Theorem 5, under the given assumptions it holds that for sufficiently small $|\varepsilon|$, $\mathbb{Z}_\varepsilon \subseteq \mathcal{N}(\mathbb{Z}_0)$ for any open neighborhood $\mathcal{N}(\mathbb{Z}_0)$ of $\mathbb{Z}_0$. Therefore, as also s is continuous in ε, it follows that for sufficiently small $|\varepsilon|$, $\gamma(x, u; \varepsilon) > 0$ for all $(x, u) \in \mathbb{Z}_\varepsilon$ with $\lambda(x; \varepsilon) = \lambda(x, 0)$ in the definition of γ in (45).

Case 2: $\gamma(x^*(0), u^*(0); 0) = 0$ and $(x^*(0), u^*(0))$ is not a steady-state. Choose some $\bar{\lambda} \in \mathbb{R}^n$ such that $\bar{\lambda}^T(x^*(0) - f(x^*(0), u^*(0))) > 0$, which is possible as $(x^*(0), u^*(0))$ is not a steady-state. Due to continuity reasons, there exists $\delta > 0$ such that also $\bar{\lambda}^T(x - f(x, u)) > 0$ for all $(x, u) \in B_\delta(x^*(0), u^*(0)) \cap \mathbb{Z}_0$. Moreover, for $(x, u) \in \mathbb{Z}_0 \setminus B_\delta(x^*(0), u^*(0))$, we have $\gamma(x, u; 0) \geq \gamma_{\min}$ for some $\gamma_{\min} > 0$ as $(x^*(0), u^*(0))$ is the unique minimizer of problem $\mathcal{P}_\gamma[0]$, γ is continuous in (x, u) and $\mathbb{Z}_0$ is compact. Now consider the function $\bar{\lambda}(x; 0) = \lambda(x; 0) + \bar{\lambda}^T x$. Choosing $|\bar{\lambda}|$ small enough such that $\bar{\lambda}^T(x - f(x, u)) > -\gamma_{\min}$ for all $(x, u) \in \mathbb{Z}_0$, we obtain that $\bar{\gamma}(x, u; 0)$ defined via (45) with storage function $\lambda(x; 0)$ replaced by $\bar{\lambda}(x; 0)$ satisfies $\bar{\gamma}(x, u; 0) > 0$ for all $(x, u) \in \mathbb{Z}_0$, which allows us to apply Case 1 again.

Case 3: $\gamma(x^*(0), u^*(0)) = 0$ and $(x^*(0), u^*(0))$ is a steady-state. This case can be proven analogously to Theorem 5, but now considering the Problem $\mathcal{P}_s[\varepsilon]$ as defined in (46) instead of $\mathcal{P}_\ell[\varepsilon]$. Namely, one can again apply the sensitivity results of [20–22] to conclude that the minimizer of Problem $\mathcal{P}_s[\varepsilon]$ as well as the corresponding Lagrange multipliers vary continuously in ε for small $|\varepsilon|$. Then, the storage function $\lambda(x; \varepsilon)$ can be defined as in (42)–(43), i.e., $\lambda(x\varepsilon) := \lambda(x; 0) + \tilde{\lambda}(\varepsilon)^T x$ with $\tilde{\lambda}(\varepsilon)$ continuous in ε and $\tilde{\lambda}(0) = 0$, which allows us to show that the minimizers of Problems $\mathcal{P}_\gamma[\varepsilon]$ and $\mathcal{P}_s[\varepsilon]$ coincide. In a second step, this local result can be extended to a global one, as shown in Part 2 of the proof of Theorem 5, and hence dissipativity can be established.

Summarizing the above, we have shown that there exists $\bar{\varepsilon}$ such that for all $|\varepsilon| \leq \bar{\varepsilon}$ the system (2) is dissipative for all $(x, u) \in \mathbb{Z}_\varepsilon$ with respect to the supply rate $s(x, u; \varepsilon)$ and storage function

$$\lambda(x; \varepsilon) = \begin{cases} \lambda(x; 0) & \text{if } \gamma(x^*(0), u^*(0)) > 0, \\ \lambda(x; 0) + \bar{\lambda}^T x & \text{if } \gamma(x^*(0), u^*(0)) = 0 \text{ and } x^*(0) \neq f(x^*(0), u^*(0)), \\ \lambda(x; 0) + \tilde{\lambda}(\varepsilon)^T x & \text{else.} \end{cases} \tag{47}$$

The proof is concluded by noting that $\lambda(x; \varepsilon)$ is twice continuously differentiable in x and continuous in ε as claimed. □

Remark 1: We note that the uniqueness property of $(x^*(0), u^*(0))$ in Assumption (ii) is not needed in Case 1, and also in Case 2 if $\bar{\lambda}$ is such that $\bar{\lambda}^T(x_i^*(0) - f(x_i^*(0), u_i^*(0))) > 0$ for all global minimizers $(x_i^*(0), u_i^*(0))$ of Problem $\mathcal{P}_\gamma[0]$. The same holds true for the differentiability assumptions on f, s, g and λ, which can be relaxed to mere continuity in the above cases. Of course, then also the function $\lambda(x; \varepsilon)$ is only continuous and not twice continuously differentiable in x. □

Remark 2: While the setting of Theorem 5, i.e., the particular storage function used in economic MPC, principally fits into the generalized framework of this section, we point out the following subtle but important difference. Namely, in Theorem 5 it was not assumed a priori that the optimal steady-state $(x^*(\varepsilon), u^*(\varepsilon))$, and hence also the supply rate $s(x, u; \varepsilon)$, are continuous in ε, as was done in the above theorem. Hence in the proof of Theorem 5, we could not directly relate the minimizer of Problem $\mathcal{P}_\gamma[\varepsilon]$ to Problem $\mathcal{P}_s[\varepsilon]$ as in this section, but we applied the sensitivity analysis to Problem $\mathcal{P}_\ell[\varepsilon]$ instead. This allowed us to verify a posteriori that the optimal steady-state $(x^*(\varepsilon), u^*(\varepsilon))$, and hence also the supply rate s, are indeed continuous in ε. □