Essays on Fractional Cointegration and Seasonal Long Memory

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Abstract

This thesis contains five essays on fractional cointegration, seasonal fractional cointegration and seasonal long memory. After an introduction in the first Chapter, Chapter 2 reviews competing tests for fractional cointegration, since no standard approach emerged so far. It provides a synthesis of the literature and a detailed comparative Monte Carlo study to guide empirical researchers in their choice of appropriate methodologies. Special attention is paid to empirically relevant issues such as assumptions about the form of the underlying process and the ability to distinguish between short-run correlation and long-run equilibria. It is found that several approaches are severely oversized in presence of correlated short-run components and that the methods show different performance in terms of power when applied to common-component models instead of triangular systems.

In Chapter 3, the previously analyzed methods are applied in the context of the European government bond market. It is commonly found that the markets for long-term government bonds of EMU countries were highly integrated prior to the subprime mortgage and EMU debt crisis. In contrast to this, it is shown that there were periods of integration and disintegration that coincide with bull- and bear-market periods in the stock market. This finding is based on the interrelation between market integration and fractional cointegration in the context of the common currency area. An econometric argument about the spectral behavior of long-memory time series leads to the conclusion that there is a stronger differentiation with respect to default risks during periods of disintegration, so that the dynamics of the yields implied the possibility of macroeconomic and fiscal divergence between the EMU countries before the crisis periods.

Chapter 4 deals with possible breaks in the persistence structure of a fractional cointegrating relationship. It introduces test procedures for no fractional cointegration that are robust for such a break. They are based on the supremum of the Hassler and Breitung (2006) test statistic for no cointegration over possible breakpoints in the long-run equilibrium that are shown to converge to the supremum of a chi-squared distribution if correctly standardized, and that this convergence is uniform. An empirical application to European benchmark government bonds shows the dissolution of fractional cointegrating relationships with the beginning of the European debt crisis.

The following Chapters 5 and 6 consider another phenomenon in time series, namely seasonality, in particular seasonal long memory. Chapter 5 examines multivariate seasonal data and the concomitant possibility of seasonal fractional cointegration. It proposes two multivariate seasonal long-memory models and derives a seasonal multiple local Whittle

estimator for the seasonal memory parameters and the seasonal cointegrating vector based on Robinson et al. (2008). Finally, in an application to financial high frequency data, seasonal fractional cointegration between realized volatility and trading volume for a daily cycle is found.

Chapter 6 takes a different perspective and deals with univariate seasonal time series. Seasonal behavior often is modeled with dummy variables or deterministic functions disregarding stochastic components. Therefore, a test for seasonal long memory with a known frequency is proposed. Based on this test, it is found that deterministic seasonality is an accurate model for the Dow Jones Industrial Average (DJIA) index but not for the component stocks. These still exhibit significant and persistent periodicity after seasonal de-meaning so that more evolved seasonal long-memory models are required to model their behavior.

Keywords: Fractional Cointegration, Semiparametric Estimating and Testing, Seasonal Long Memory, EMU Government Bonds, Volatility

Zusammenfassung

Diese Arbeit enthält fünf Aufsätze über fraktionale Kointegration, saisonale fraktionale Kointegration und saisonales langes Gedächtnis. Nach einer Einführung im ersten Kapitel behandelt Kapitel 2 konkurrierende Tests für fraktionale Kointegration, da sich bisher kein Standardansatz herausgestellt hat. Es bietet eine Zusammenfassung der Literatur und eine detaillierte Monte Carlo Studie, um Forscher bei der Auswahl geeigneter Methoden zu unterstützen. Besonderes Augenmerk wird auf empirisch relevante Fragestellungen gelegt, wie z.B. Annahmen über die Form des zugrunde liegenden Prozesses und die Fähigkeit der Verfahren, zwischen kurzfristiger Korrelation und langfristigen Gleichgewichten zu unterscheiden. Es wird festgestellt, dass mehrere Ansätze bei korrelierten Kurzzeitkomponenten ihr Signifikanzniveau nicht einhalten und dass die Methoden unterschiedliche Gütemerkmale aufweisen, wenn sie auf "common-component models" anstelle von "triangular systems" angewendet werden.

In Kapitel 3 werden die zuvor analysierten Methoden im Kontext des europäischen Staatsanleihenmarktes angewendet. Es wird allgemein angenommen, dass die Märkte für langfristige Staatsanleihen der EWU-Länder vor der Subprime-Hypotheken- und europäischen Schuldenkrise stark integriert waren. Im Gegensatz dazu zeigt sich, dass es Phasen der Integration und Desintegration gab, die mit Bullen- und Bärenmarktphasen am Aktienmarkt zusammenfallen. Diese Feststellung beruht auf dem Zusammenhang zwischen Marktintegration und fraktionaler Kointegration im Rahmen des gemeinsamen Währungsraums. Ein ökonometrisches Argument über die Spektraldichte von Long-Memory Zeitreihen führt zu dem Schluss, dass es eine stärkere Differenzierung zwischen Anleihen mit unterschiedlichen Ausfallrisiken in Zeiten der Desintegration gibt, sodass die Dynamik der Renditen die Möglichkeit makroökonomischer und fiskalischer Divergenzen zwischen den EWU-Ländern vor den Krisenzeiten implizierte.

Kapitel 4 befasst sich mit möglichen Brüchen in der Persistenzstruktur einer fraktionalen Kointegrationsbeziehung. Es führt Testverfahren für fraktionale Kointegration ein, die für einen solchen Bruch robust sind. Die vorgeschlagenen Tests basieren auf dem Supremum der Hassler and Breitung (2006) Teststatistik für keine Kointegration über mögliche Bruchpunkte im langfristigen Gleichgewicht. Sie konvergieren, korrekt standardisiert, gleichmäßig zum Supremum einer Chi-Quadrat-Verteilung. In einer empirischen Anwendung der Tests auf europäische Staatsanleihen wird die Auflösung von fraktional kointegrierenden Beziehungen mit Beginn der europäischen Schuldenkrise gezeigt.

Die nachfolgenden Kapitel 5 und 6 betrachten ein weiteres Phänomen in Zeitreihen, nämlich die Saisonalität, insbesondere das saisonale lange Gedächtnis. In Kapitel 5 werden multivariate saisonale Daten und die damit verbundene Möglichkeit der saisonalen fraktionalen Kointegration betrachtet und zwei multivariate saisonale long-memory Modelle vorgeschlagen. Zudem wird ein saisonaler multipler local-Whittle Schätzer für die saisonalen Gedächtnisparameter und den saisonalen Kointegrationsvektor basierend auf Robinson et al. (2008) hergeleitet. Schließlich zeigt eine Anwendung auf Hochfrequenzdaten im Finanzbereich saisonale fraktionale Kointegration zwischen realisierter Volatilität und Handelsvolumen für einen täglichen Zyklus.

Kapitel 6 nimmt einen anderen Blickwinkel ein und beschäftigt sich mit univariaten saisonalen Zeitreihen. Das saisonale Verhalten wird oft mit Dummy-Variablen oder deterministischen Funktionen modelliert, ohne Berücksichtigung stochastischer Komponenten. Daher wird ein Test für saisonales langes Gedächtnis an einer bekannten Frequenz vorgeschlagen, mit dem gezeigt wird, dass deterministische Saisonalität ein treffendes Modell für den Dow Jones Industrial Average (DJIA) Index ist, nicht aber für die Komponentenaktien. Diese weisen nach der Entfernung deterministischer Saisonalität immer noch eine signifikante und persistente Periodizität auf, sodass fortgeschrittene saisonale long-memory Modelle erforderlich sind, um ihr Verhalten zu modellieren.

Schlagwörter: Fraktionale Kointegration, Semiparametrisch Schätzen und Testen, Saisonales langes Gedächtnis, EWU Staatsanleihen, Volatilität

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Chapter 1

Introduction

With time series data, it is important to gain an understanding of the dependence structure. In cases where data is more persistent than can be captured with standard linear short-memory processes, an alternative model is required. One popular extension is fractional integration or long memory which is the focus of this thesis. It is characterized by a hyperbolical decay of autocorrelations (in contrast to the exponential decay in short-memory models) and unbounded spectral peaks, and it is common practice to base inference on the latter. Hence, long-memory is mostly analyzed in the frequency domain and exploits the relation $f(\lambda) \sim G|\lambda|^{-2d}$ as $\lambda \to 0$ where $f(\lambda)$ is the (pseudo) spectral density of a time series depending on the frequencies λ, G is a finite constant, and d is the long-memory parameter (e.g. Beran et al. (2013)). This definition focuses only on the long-memory parameter so that semiparametric inference methods are suitable. They have the advantage of disregarding all short-run dynamics and avoiding misspecifications. Popular examples of such semiparametric memory estimates in univariate systems include the (exact) local Whittle or Gaussian semiparametric estimation by Künsch (1986), Robinson (1995a), and Shimotsu and Philips (2005), and the log-periodogram regression or GPH estimation by Geweke and Porter-Hudak (1983).

Examples where data is strongly-dependent includes on the one hand natural science such as meteorology and hydrology (Gil-Alana (2008), Montanari et al. (1997) among many others), and is on the other hand found in economics as well. This includes for example volatility of equity or exchange rates, inflation rates, interest rates, real output, consumption, and income (Diebold and Rudebusch (1989), Cheung (1993), Baillie et al. (1996), Tsay (2000), and Gil-Alana (2006) among many others).

Moving to multivariate data broadens the range of possible phenomena, in particular concerning the interdependence of data. The basic and most obvious concept is correlation, but a more advanced quantity of connection is cointegration. It became popular with the seminal paper of Engle and Granger (1987) and finds many applications with exchange rates (e.g. purchasing power parity, forward premium (Cheung and Lai (1993), Baillie and Bollerslev (1994)), stock market volatility (Christensen and Nielsen (2006), Morana and Beltratti (2008)) and inflation rates (Chen and Hurvich (2003), Nielsen (2010)).

Murray (1994) explains standard cointegration with a vivid example. The paths of a curious puppy and its drunken owner can be described as random walks, and both walk independently of each other but roughly in the direction of their home. The dog is not bound to its owner by a leash at a fixed distance. This part describes the cointegration, i.e. the long-term equilibrium relationship. Occasionally the owner calls for her dog and it barks as an answer. In order not to increase the distance (i.e., the linear combination of their paths) between each other too much, both adjust their paths and close the gap a little. This shows that cointegration is an equilibrium concept describing the co-movement of multivariate time series. If variables are cointegrated, they are driven by the same (non-observable) common stochastic trend; in the dog-example this is captured by the two trying to get home. Formally, the example refers to time series integrated of order one and a linear combination of the series that is integrated of order zero. A more relaxed definition of cointegration does not require integer integration. Hence, the processes only need to be integrated by some fractional order that is identical in all of them, and the linear combination needs to have a lower order of (fractional) integration. This concept is referred to as fractional cointegration and the subject of the Chapters 2 to 4.

Chapter 2 and 3 deal with semiparametric tests for no fractional cointegration and rank estimation procedures. First, Chapter 2 reviews the methods that were introduced during the past 15 years. However, no standard approach prevails so far. The essay addresses this lack of comparison and discusses two issues regarding the performance of the tests depending on the data generating process based on a comprehensive Monte Carlo study. First, how does correlation in the short-run components influence the size of the tests? This is an important question for practical applications since one usually only suspects cointegration in cases when there is obvious correlation in the data. However, we find that some methods tend to mistake correlation for cointegration so that they do not hold the nominal size level. This includes the rank estimation of Robinson and Yajima (2002) and Nielsen and Shimotsu (2007) that is one of the more popular procedures, and Marmol and Velasco (2004) and Hualde and Velasco (2008). Second, how does the model structure influence the power of the tests? There are mainly two possibilities in the literature: In a bivariate context, one variable is considered to be the common stochastic trend and the other is a perturbation of it (triangular model) or both variables are perturbations of the common trend. Furthermore, some tests are restrictive in their assumptions decreasing the range of applicability. It is therefore recommended to carefully choose the tests applied in empirical applications. Overall, we recommend to use the tests of Chen and Hurvich (2006) and Souza et al. (2018), since they are not sensitive to correlated short-term dynamics, are applicable in most scenarios and have good power properties.

Chapter 3 applies the previously analyzed tests to the European government bond market. This is motivated by a definition of market integration that is based on the law of one price which requires equality or, in a less strict sense, an equilibrium of the prices. It is commonly assumed that the introduction of the euro led to complete government bond market integration. According to the employed definition of market integration, this requires the existence of cointegrating relationships. By testing this, we find periods of integration and disintegration that correspond to bull- and bear-market periods on the stock market. Further regression analyses confirm this finding and show that market risk is a relevant driver as well. An economic argument shows that the yield spreads are the cointegrating residuals of a potential cointegrating relationship between the yields, and an econometric argument indicates that default risk dominates liquidity risk in determining the persistence of the yield spreads. This leads to the conclusion that there is a stronger differentiation between the default risks of government bonds during bear-market periods. Furthermore, the partial absence of cointegration indicates, at least during bear markets, the possibility of macroeconomic and fiscal divergence of the EMU countries although the overall low level of the spreads implies a very low probability of this scenario.

The essay in Chapter 4 uses the same EMU government bond data set as the previous Chapter and partially builds on its results. These were that fractional cointegrating relationships do not need to be constant over time which is also in line with the literature on changing persistence. This chapter abandons the semiparametric world and considers parametric tests for cointegration that are robust if not the full sample is cointegrated but potentially only parts of it. It is based on the test of Hassler and Breitung (2006) and combines it with subsample testing procedures introduced by Davidson and Monticini (2010). The asymptotic properties are derived and shown to be standard. In addition, we suggest an estimator that is able to determine the location of the break from a cointegrated subsample towards a non-cointegrated subsample. The application to the EMU data shows that the Dutch and Finish yields might be permanently cointegrated with the German one. In contrast, the other countries are only cointegrated in some part of the considered period and in particular seem to disintegrate during the European debt crisis.

The last two Chapters consider seasonality in addition to long memory and fractional cointegration. Seasonality is relevant, for example, in macroeconomic data like unemployment on a monthly level, but it also becomes more and more important in financial data. This seasonality might be modeled accurately with deterministic structures but there is also evidence of stochastic seasonal components. In this context seasonal long memory comes into play which is found for example in inflation rates (Arteche (2012), Peiris and Asai (2016) among others), unemployment (e.g. Gil-Alana (2007)) and intraday volatility (e.g. Deo et al. (2006)). Technically, seasonal long memory can be defined by an analogous spectral property $f(\lambda \pm \omega) \sim C_\omega \lambda^{-2d}$ as $\lambda \to 0$ that describes an unbounded spectral peak not at the origin but at a specific frequency ω. Again, $f(\lambda + \omega)$ is the (pseudo) spectral density and C_ω is a finite constant. Common models are the seasonally fractionally integrated model (SARFIMA) by Porter-Hudak (1990) and the k-factor Gegenbauer (GARMA) process by Woodward et al. (1998) where the latter became popular in the context of volatility.

Chapter 5 is situated in a bivariate setting, like the previous chapters, and examines seasonal fractional cointegration. The essay defines the concept, proposes two models that can generate multivariate seasonal long memory as well as seasonal fractional cointegration, and examines the relevant spectral properties for estimation. Next, it introduces a semiparametric multiple seasonal local Whittle estimator that estimates the seasonal long memory parameters and the seasonal cointegrating relationship, and it derives the asymptotic properties. By estimating the asymptotic variance, asymptotic confidence intervals and tests become feasible. Both, the estimate and the asymptotic Wald test exhibit good finite sample properties shown by a Monte Carlo study. Finally, seasonal fractional cointegration is not only a possible theoretical framework, but an empirical application to intraday realized volatility and trading volume finds evidence of seasonal fractional cointegration with a daily cycle.

The last essay in Chapter 6 is based on the same data set as the previous Chapter and focuses on seasonal long memory in a univariate context. It addresses the increasing availability of high-frequency data, the thereby induced seasonality on an intraday basis, and the potentially different nature of seasonality, i.e. deterministic seasonal patterns versus stochastic ones. The essay investigates the question whether it is suitable to assume deterministic seasonality in intraday trading volume and realized volatility. Therefore, it proposes a semiparametric test for seasonal long memory, i.e., stochastic seasonality, at a specific frequency having the advantage of not assuming a certain model structure. Simulations show that the test works well in finite samples. The main finding of the empirical application is that deterministic seasonality might be suitable for index data, but single stock data tends to exhibit stochastic seasonal structures in addition to deterministic patterns.

Chapter 2

A Comparison of Semiparametric Tests for Fractional Cointegration

Co-authored with Christian Leschinski and Philipp Sibbertsen.

2.1 Introduction

The concept of cointegration derives its popularity from the fact that it allows to model equilibrium relationships between non-stationary time series. In practice, however, standard cointegration analysis can often not be applied, since the $I(1)/I(0)$ framework is too restrictive. For example, the series of interest may be persistent but not have a unit root, or the deviations from the equilibrium may be more persistent than the $I(0)$ model allows.

Fractional cointegration overcomes these shortcomings, by allowing for non-integer integration orders of the variables in the system and any (possibly non-zero) memory order in the cointegrating residuals as long as it is reduced compared to the original system. Consequently, fractional cointegration promises to facilitate the modeling of a larger number of equilibrium relationships compared to standard cointegration.

This has led to the development of various testing and rank estimation procedures to determine whether fractional cointegration is present in a multivariate time series.

Parametric approaches include Johansen (2008), Łasak (2010), Johansen and Nielsen (2012), Łasak and Velasco (2015), and Johansen and Nielsen (2019), among others, who consider fractional extensions of the cointegrated VAR model of Johansen (1988). Furthermore, Breitung and Hassler (2002) introduce a trace test to determine the cointegrating rank, Avarucci and Velasco (2009) suggest rank estimation in a regression framework, and Hassler and Breitung (2006) develop a time domain residual-based test for fractional cointegration.

Semiparametric approaches, on the other hand, have the advantage that they allow the researcher to focus on the long-run relationship between the series and do not require the specification of short-run dynamics. This literature encompasses the spectral-based rank estimation procedure of Robinson and Yajima (2002) and its extension by Nielsen and Shimotsu (2007), a Hausmann-type test based on the multivariate local Whittle estimator introduced by Robinson (2008), a number of residual-based tests for the null hypothesis of no fractional cointegration developed by Marmol and Velasco (2004), Chen and Hurvich (2006), Hualde and Velasco (2008), and Wang et al. (2015), a variance-ratio test proposed by Nielsen (2010), a test based on a GPH-type estimate of the cointegration strength introduced by Souza et al. (2018) and a rank estimation procedure based on an eigenanalysis of the autocovariance function from Zhang et al. (2019).

Unfortunately, the domain of applicability of most of these procedures is much more restrictive than the definition of fractional cointegration. Some are only applicable in stationary systems — some only in non-stationary systems. Some procedures require the reduction in memory to be more than 1/2 — some only require the memory of the cointegrating residuals to be less than 1/2.

Furthermore, there are different assumptions about the form of the fractionally cointegrated system. Some approaches assume that one of the observed series itself is an observation of the common underlying trend. Other approaches assume an unobserved common underlying trend. We refer to these models as the triangular system and the common-components model. Which of these assumptions is more suitable in practice depends on the specific application. On the one hand, it may be appropriate to think of the risk-free interest rate as an observed common component that is perturbed by risk premia if the yields of risky bonds are realized so that a triangular model can be used. For cointegrated pairs of stocks, on the other hand, it is unclear why the price of one stock should be interpreted as a perturbed version of another stock price so that a common-components model is more appropriate. Finally, even though the development of each of these procedures to determine whether fractional cointegration is present is a major theoretical contribution, relatively little effort has been devoted to analyze how they perform compared to each other.

Here, we try to address these issues by providing a survey of all the rank estimation and testing procedures discussed above. To study the relative performance of the competing approaches, we conduct an extensive Monte Carlo analysis of their size and power properties. It is found that several procedures - namely those of Nielsen and Shimotsu (2007) or Robinson and Yajima (2002), Marmol and Velasco (2004), and Hualde and Velasco (2008) show severe finite sample size distortions in multivariate systems with correlated short-run components. The relative performance in terms of power depends on the form of the system under considerations. For triangular systems and non-stationary common-components models the test of Souza et al. (2018) performs best overall, whereas the test of Chen and Hurvich (2006) is preferable for stationary common-components models.

The rest of the paper is structured as follows. The next Section gives the definition and model of fractional cointegration we adopt and briefly reviews the basic estimation methods required by the tests. Section 2.3 is divided into two subsections describing two types of tests, 2.3.1 containing the tests based on a spectral matrix and 2.3.2 summarizing the tests based on cointegrating residuals, Section 2.4 presents finite sample results, and Section 2.5 concludes.

2.2 Fractional Cointegration — Models and Definitions

A p-dimensional vector-valued time series X_t has long memory if its spectral density fulfills

$$f_X(\lambda) \sim \Lambda_j(d)\, G\, \overline{\Lambda_j(d)}, \quad \text{as } \lambda \to 0^+, \tag{2.1}$$

where G is a real, symmetric and non-negative definite matrix, $\Lambda_j(d) = \text{diag}\left(\lambda^{-d_1} e^{i\pi d_1/2}, ..., \lambda^{-d_p} e^{i\pi d_p/2}\right)$ is a $p \times p$ diagonal matrix, $\overline{\Lambda_j(d)}$ is its complex conjugate transpose and '$\sim$' implies that for each element the ratio of real and imaginary parts on the left- and right-hand side tends to one.

The element in the a-th row and b-th columns of the spectral matrix $f_X(\lambda)$ is denoted by $f_{ab}(\lambda) \sim g_{ab}\lambda^{-2d}$ for $a, b \in \{1, ..., p\}$ where g_{ab} denotes the respective element of G. The periodogram of X_t at the Fourier frequencies is given by

$$I_X(\lambda_j) = w_X(\lambda_j)\overline{w_X(\lambda_j)}, \tag{2.2}$$

with $w_X(\lambda) = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} X_t e^{i\lambda t}$, and $\lambda_j = 2\pi j/T$, for $j = 1, ..., \lfloor T/2 \rfloor$, where $\lfloor \cdot \rfloor$ denotes the greatest integer smaller than the argument.

There is a number of different definitions of fractional cointegration in the literature. The most common one goes back directly to Engle and Granger (1987). According to this definition the p-dimensional vector-valued time series X_t is cointegrated of rank r, if all components of X_t are integrated of order d (denoted by $I(d)$), and there exists a non-singular matrix β so that the r linear combinations $v_t = \beta' X_t$ are $I(d - b_a) = I(d_{v_a})$ with $d > b_a > 0$ for all $a = 1, ..., r$. The matrix β is called the cointegrating matrix and each of its columns is a cointegrating vector. The elements of the vector v_t are the cointegrating residuals. Other definitions are given by Johansen (1995), Flôres Jr and Szafarz (1996), Marinucci and Robinson (2001), and Robinson and Yajima (2002) who also provide a discussion of the implications of the different definitions.

Standard cointegration is a special case of the definition above where $d = 1$ and $d_{v_a} = 0$ for all a. In this setup the system is non-stationary, whereas the cointegrating residuals are stationary. In contrast to that, fractional cointegration allows for a more flexible model so that several cases can be distinguished: weak cointegration ($b < 0.5$), strong cointegration ($b > 0.5$), stationary cointegration ($0 < d_v < d < 0.5$), or non-stationary cointegration ($0.5 < d_v < d$).

In general, (fractional) cointegration is an equilibrium concept where the persistence of the cointegrating residual d_v determines the speed of adjustment towards the cointegration equilibrium $\beta' X_t$, and shocks have no permanent influence on the equilibrium as long as $d_v < 1$ holds.

As an example, consider the fractionally (co-)integrated bivariate model with $X_t = (X_{1t}, X_{2t})'$, where

$$X_{1t} = c_1 + \xi_1 Y_t + \Delta^{-(d-b_1)} u_{1t} \mathbb{1}(t > 0) \tag{2.3}$$

$$X_{2t} = c_2 + \xi_2 Y_t + \Delta^{-(d-b_2)} u_{2t} \mathbb{1}(t > 0) \tag{2.4}$$

$$\text{and} \qquad Y_t = \Delta^{-d} e_t \mathbb{1}(t > 0). \tag{2.5}$$

Here $u_t = (u_{1t}, u_{2t})'$ is a weakly-dependent zero-mean process with constant covariance matrix Ω_u and spectral density matrix $f_u(\lambda)$, e_t (with variance σ_e^2 and spectral density $f_e(\lambda)$) is a univariate weakly-dependent zero-mean process that is allowed to be correlated with u_t, and L denotes the lag-operator so that $LY_t = Y_{t-1}$. The fractional difference operator $\Delta^d = \text{diag}\left\{(1-L)^d, ..., (1-L)^d\right\}$ is defined in terms of the binomial expansion so that $(1-L)^d = \sum_{k=0}^{\infty} \binom{d}{k} (-1)^k L^k$, with $\binom{d}{k} = \frac{d(d-1)(d-2)...(d-(k-1))}{k!}$, and it is of the same dimension as the process that it is applied to. Furthermore, $\mathbb{1}(\cdot)$ denotes the indicator function that takes the value one if its argument is true and is zero, otherwise. Finally, it is assumed that $d \in (0, 1]$ and $d \geq b_1, b_2 \geq 0$.

The truncated processes $\Delta^{-(d-b_a)} u_{at} \mathbb{1}(t > 0)$ are fractionally-integrated processes of type-II, which means they are only asymptotically stationary for $d < 1/2$, but in contrast to type-I processes they are still defined for $d > 1/2$. For a detailed discussion cf. Marinucci and Robinson (1999).

In this bivariate model there can be at most one cointegrating relationship. In this case $r = 1$ and β itself is a cointegrating vector. Obviously, if the linear combination $\beta' X_t = v_t$ has reduced memory, the same is true for every scalar multiple of it. To identify the cointegrating vector, it is therefore customary to apply some kind of normalization such as setting the first element of the vector to unity. In Equations (2.3) to (2.5), fractional cointegration arises if $\xi_1, \xi_2 \neq 0$, and $b_1, b_2 > 0$. In this case the normalized cointegrating vector is $\beta = \left(1, -\frac{\xi_1}{\xi_2}\right)' = \left(1, -\tilde{\beta}\right)'$ and the cointegrating residual v_t is $I(d - b) = I(d_v)$, where $b = \min(b_1, b_2)$.

Note that this model is a common-components model, but it also nests a triangular system. This is obtained as a special case if $\Omega_{u,22} = 0$ so that X_{2t} is a direct (rescaled) observation of the underlying common trend and only X_{1t} is perturbed with a cointegration error so that $b = b_1$.

Standard cointegration in the $I(1)/I(0)$ framework is obtained as a special case if $d = 1$ and $b_1 = b_2 = 1$. It is also possible to have $\xi_1, \xi_2 \neq 0$, so that both X_{1t} and X_{2t} contain the common component Y_t, but they are not cointegrated. This is the case if $b_1 = b_2 = 0$.

2.3 Tests for no Fractional Cointegration

In the following, we provide a comprehensive review of semiparametric tests and estimation procedures that can be used to determine the order of fractional cointegration in a p-dimensional vector-valued time series X_t. According to the definition discussed above, this requires that the components of X_t are integrated of the same order.

In practice, this can either be assumed based on domain specific knowledge, or it can be tested with tests for the equality of memory parameters that allow for cointegration introduced by, for example, Robinson and Yajima (2002), Nielsen and Shimotsu (2007), Hualde (2013), and Wang and Chan (2016). In particular Robinson and Yajima (2002) discuss in detail how to partition a vector-valued time series into subvectors with equal memory parameters. These can then be used for further cointegration analysis.

In the following, it will be assumed that all components of X_t are $I(d)$, which means we abstract from these pre-testing issues to focus on the actual tests for the null of no fractional cointegration. For all tests the hypotheses are defined by

H_0: X_t is not fractionally cointegrated ($d = d_v$),
H_1: X_t is fractionally cointegrated ($d > d_v$).

In contrast to standard $I(1)/I(0)$ cointegration, the memory parameter d is unknown in fractionally cointegrated systems and has to be estimated. Since we are in a setting that potentially entails cointegration, multivariate memory estimation might not be feasible so that the memory parameters are estimated univariately. If not stated otherwise, the estimates involved in the tests are the means of the univariate memory estimates for the components of the system.

The tests presented in this Section apply the most common estimators: the log-periodogram estimator $\widehat{d}_{GPH}$ of Geweke and Porter-Hudak (1983) and Robinson (1995b), the local Whittle estimator $\widehat{d}_{LW}$ of Künsch (1987) and Robinson (1995a), or the exact local Whittle estimator $\widehat{d}_{ELW}$ of Shimotsu and Philips (2005) and Shimotsu (2010). All of these estimators are periodogram based and employ the first m Fourier frequencies. The general requirement is that $m < \lfloor T/2 \rfloor$ tends to infinity more slowly than T, so that $\frac{1}{m} + \frac{m}{T} \to 0$ as $T \to \infty$ and even the largest frequency $2\pi m/T$ is asymptotically local to the zero frequency.

To estimate the cointegrating relationship $\beta' X_t = v_t$ when $r = 1$, the vector is partitioned such that $X_t = (y_t, x_t)$, where y_t is a scalar and x_t is $(p-1) \times 1$. By doing so, the focus is on one possible cointegrating relation $y_t = \tilde{\beta} x_t + v_t$ where $\tilde{\beta}$ is $(p-1)$-dimensional.

As in standard cointegration analysis the vector $\tilde{\beta}$ can be estimated with ordinary least squares (OLS) as long as $d > 1/2$ so that the series remains non-stationary. In stationary long-memory time series, OLS is inconsistent in presence of correlation between the stationary regressors and the innovation term v_t (cf. Robinson (1994)).

Robinson (1994) and Robinson and Marinucci (2001) introduce an alternative estimator of the cointegrating vector that is based on the periodogram local to the zero frequency. In contrast to OLS, this narrow-band frequency domain least squares (NBLS) estimator is consistent under cointegration for all values of d and has a non-normal limiting distribution in the non-stationary region. Christensen and Nielsen (2006) extend the asymptotic results to the stationary region where the estimate follows an asymptotic normal distribution and Nielsen and Frederiksen (2011) provide a correction of the asymptotic bias under weak fractional cointegration.

Estimating the linear cointegrating relationship with NBLS requires calculating the averaged cross-periodogram of x_t with itself and y_t by $I^{av}_{xx}(\lambda_j) = \frac{2\pi}{T}\sum_{j=1}^{m}\omega_x(\lambda_j)\overline{\omega_x(\lambda_j)}$ and $I^{av}_{xy}(\lambda_j) = \frac{2\pi}{T}\sum_{j=1}^{m}\omega_x(\lambda_j)\overline{\omega_y(\lambda_j)}$. The NBLS estimate of $\tilde{\beta}$ is then defined by

$$\widehat{\beta}_m = I^{av}_{xx}(\lambda_j)^{-1} I^{av}_{xy}(\lambda_j). \tag{2.6}$$

The bandwidth m has to fulfill the usual local-to-zero condition as $T \to \infty$. If not specified otherwise, this is the estimator we employ to estimate the cointegrating vector. Other estimators suggested in the literature include estimation based on the eigenvectors of a version of $I^{av}_X(\lambda_j)$ (cf. Chen and Hurvich (2006)) and joint estimation with the memory parameters in multivariate local Whittle approaches such as those of Robinson et al. (2008) and Shimotsu (2012).

The following review is divided into tests based on the spectral density local to the origin (Section 2.3.1) and tests based on estimates of the cointegrating residuals (Section 2.3.2). Of course, this distinction is not clear cut, since some of the residual-based approaches also use the spectral properties of the potential cointegrating residuals and for example the test of Nielsen (2010) is presented as a variance-ratio test. Many different categorizations would be possible. Here, we refer to those approaches as "spectral-based" that rely on the properties of the spectrum of the observed series X_t itself, and those that rely on the spectrum of the cointegrating residual are called "residual-based".

2.3.1 Tests based on the Spectral Matrix

A number of procedures to determine the fractional cointegrating rank of the p-dimensional time series X_t are based on properties of the rescaled spectral matrix local to the zero frequency. This is denoted by G in Equation (2.1) and has reduced rank if and only if X_t is fractionally cointegrated. If fractional cointegration is present, the number of eigenvalues that are equal to zero corresponds to the cointegrating rank r and therefore to the number of cointegrating relationships.

Based on this property Robinson and Yajima (2002) introduce an information criterion to determine the fractional cointegration rank that is extended to non-stationary processes by Nielsen and Shimotsu (2007).

To obtain an estimate $\widehat{G}$ of G, the first step of the procedure consists in applying the univariate exact local Whittle estimator of Shimotsu and Philips (2005) and Shimotsu (2010) to each component of X_t separately, using bandwidth m. In contrast to a multivariate local Whittle estimate that has the inverse of $\widehat{G}$ in its objective function and is therefore not consistent under fractional cointegration, each of the univariate estimates is consistent for the memory order d of X_t. The memory order d is therefore estimated by the pooled estimator $\widehat{d}_{ELW}$ that is the arithmetic mean of the univariate estimates. The estimate of $\widehat{G}(\widehat{d}_{ELW})$ is then defined by

$$\widehat{G}(\widehat{d}_{ELW}) = \frac{1}{m_1} \sum_{j=1}^{m_1} \mathrm{Re}\; I_{\Delta^d}(\lambda_j),$$

where I_{Δ^d} is the periodogram of $\Delta^{\widehat{d}_{ELW}} X_t$. The bandwidths have to fulfill $\frac{m_1}{m} \to 0$ in order to ensure faster convergence of $\widehat{d}_{ELW}$ than of $\widehat{G}(\widehat{d}_{ELW})$.[1] Denote the empirical eigenvalues calculated from $\widehat{G}(\widehat{d}_{ELW})$ and sorted in descending order by $\widehat{\delta}_{a,G}$ for $a = 1, ..., p$. The cointegrating rank can then be estimated using a model selection criterion that is based on the partial sum of the sorted eigenvalues

$$\widehat{r}_{NS} = \arg \min_{k=0,\ldots,p-1} \left(n(T)(p-k) - \sum_{a=1}^{p-k} \widehat{\delta}_{a,G} \right), \tag{2.7}$$

where $n(T)$ is a function which fulfills $n(T) + \frac{1}{\sqrt{m_1}\, n(T)} \to 0$ as $T \to \infty$ so that $n(T)$ goes to zero but more slowly than the estimation error in the eigenvalues that is of order $\mathcal{O}_P\left(m_1^{-1/2}\right)$. Asymptotically, the expression is therefore minimal if only estimates of non-zero eigenvalues are included in the sum.

To deal with situations in which the scales of the components in X_t are different, Nielsen and Shimotsu (2007) suggest to base the procedure on the correlation matrix $\widehat{P}(\widehat{d}_{ELW}) = \widehat{R}(\widehat{d}_{ELW})^{-1/2}\widehat{G}(\widehat{d}_{ELW})\widehat{R}(\widehat{d}_{ELW})^{-1/2}$ instead of $\widehat{G}$, where $\widehat{R}(\widehat{d}_{ELW}) = \mathrm{diag}\,(\widehat{g}_{11}, ..., \widehat{g}_{pp})$ contains the diagonal elements of $\widehat{G}(\widehat{d}_{ELW})$. This is admissible since the rank of $\widehat{P}$ is the same as that of $\widehat{G}$ in the limit. Nielsen and Shimotsu (2007) point out that this approach works better in simulations and also recommend to use the bandwidth $n(T) = m_1^{-0.3}$. The cointegrating rank estimate is consistent for $r \in \{0, ..., p-1\}$. It is applicable for systems of dimension $p \geq 2$, and it does not impose restrictions on d and b.

A similar rank estimation procedure based on the average of finitely many tapered periodogram ordinates local to the origin was also proposed by Chen and Hurvich (2003).

The aforementioned inconsistency of the multivariate local Whittle estimator under fractional cointegration is the basis for a test procedure originally proposed by Marinucci and Robinson (2001). They suggest a Hausman-type test that compares multivariate

[1] We follow the notation of Nielsen and Shimotsu (2007) and use m_1 for the bandwidth in the estimation of $G(d)$ and m for that of d. Note that Robinson and Yajima (2002) chose the opposite notation.

and univariate local Whittle estimates. Under the null hypothesis of no cointegration the multivariate estimator is efficient and both estimators are consistent. Under the alternative of fractional cointegration, on the other hand, the univariate estimator remains consistent, whereas the multivariate one does not.

This idea is formalized by Robinson (2008). The test statistic is based on the objective function of the multivariate local Whittle estimator (cf. Lobato (1999), Shimotsu (2007))

$$S(d) = \log\det \widehat{G}^*(d) - \frac{2pd}{m}\sum_{j=1}^{m}\log\lambda_j \quad \text{with} \quad \widehat{G}^*(d) = \frac{1}{m}\sum_{j=1}^{m} I_X(\lambda_j)\lambda_j^{2d}$$

and its derivative

$$s^*(d) = \operatorname{tr}\left(\widehat{G}^*(d)^{-1}\widehat{H}^*(d)\right) \tag{2.8}$$
$$\text{with} \quad \widehat{H}^*(d) = \frac{1}{m}\sum_{j=1}^{m}\nu_j I_X(\lambda_j)\lambda_j^{2d} \quad \text{and} \quad \nu_j = \log j - \frac{1}{m}\sum_{k=1}^{m}\log k.$$

Similar to the previous procedure, the memory parameter d is estimated by pooling the univariate estimates obtained by applying the local Whittle estimator to each of the component series. The equally weighted average is denoted by $\widehat{d}_{LW}$.

To obtain a test statistic, the derivative $s^*(d)$ from (2.8) is evaluated at this averaged univariate estimate:

$$W^*_{Rob} = \frac{ms^*(\widehat{d}_{LW})^2}{N^2\operatorname{tr}(\widehat{F}^{*2}) - p}; \tag{2.9}$$
$$\widehat{F}^* = \widehat{R}^{*^{-1/2}}\,\widehat{G}^*(\widehat{d}_{LW})\,\widehat{R}^{*^{-1/2}}, \quad \widehat{R}^* = \operatorname{diag}(\widehat{g}^*_{11}, ..., \widehat{g}^*_{pp}),$$

where $\widehat{g}^*_{aa}$, $a = 1, ..., p$, are the diagonal elements of $\widehat{G}^*(\widehat{d}_{LW})$. The scaled derivative $m^{1/2}s^*(\widehat{d}_{LW})$ is asymptotically normal so that the test follows a χ^2_1-distribution if appropriately standardized by the term in the denominator.

The test generates power because $G(d)$ is singular under the alternative of fractional cointegration so that the inverse $\widehat{G}^*(\widehat{d}_{LW})^{-1}$ of the estimate and consequently the trace $s^*\left(\widehat{d}_{LW}\right)$ become large.

This is a score-type test that avoids the calculation of the multivariate local Whittle estimator that can be numerically expansive. Since the efficiency of the multivariate estimate is obtained with a single Newton step from the univariate estimate in direction of the multivariate one, $s^*\left(\widehat{d}_{LW}\right)$ is directly proportionate to the difference between the efficient and the inefficient estimate.

This test allows series of dimensions larger than two, but it is restricted to processes with $d \in (-1/2, 1/2)$ and focuses on the empirically relevant range $d \in (0, 1/2)$. Hence, non-stationary processes are not allowed. An extension based on a trimmed version of the local Whittle estimator is proposed, but the size properties of this test in simulations appear to depend heavily on the sample size.[2]

An alternative way to allow for non-stationary processes would be to base the test on the objective function of the multivariate exact local Whittle estimator (as in Shimotsu (2012), but without allowing for fractional cointegration) and univariate ELW estimates. Since the exact local Whittle estimates have the same asymptotic properties as the local Whittle estimate for $d \in (-1/2, 1/2)$, the test would have the same limiting distribution.

For a bivariate process with known $d \in (0, 1]$, Souza et al. (2018) propose a test based on an estimate of b obtained from the determinant of the trimmed and truncated spectral matrix of the fractionally differenced process via a log-periodogram regression.

Denote the fractionally differenced process by $\Delta^d X_t = (\Delta^d X_{1t}, \Delta^d X_{2t})'$ with spectral density matrix $f_{\Delta^d}(\lambda)$, then the determinant $D_{\Delta^d}(\lambda)$ of $f_{\Delta^d}(\lambda)$ depends on the memory reduction parameter $b \in [0, d]$ and can be approximated by

$$D_{\Delta^d}(\lambda) \sim \tilde{g}|1 - e^{-i\lambda}|^{2b}, \quad \text{as} \quad \lambda \to 0^+, \tag{2.10}$$

where $\tilde{g}$ is a constant and finite scalar.

Under cointegration, $f_{\Delta^d}(\lambda)$ does not have full rank near the origin (like G in (2.1)) so that its determinant $D_{\Delta^d}(\lambda)$ approaches zero as $\lambda \to 0^+$. The memory reduction b can be estimated from the logged version of Equation (2.10) using a log-periodogram type regression,

$$\log D_{\Delta^d}(\lambda) \sim \log \tilde{g} + 2b \log |1 - e^{-i\lambda}| + \log \frac{\tilde{g}^*(\lambda)}{\tilde{g}}, \quad \text{as} \quad \lambda \to 0^+,$$

where $\lim_{\lambda \to 0^+} \tilde{g}^*(\lambda) = \tilde{g}$.

In order to make the estimation of b feasible, the empirical determinant $\widehat{D}_{\Delta^d}(\lambda)$ has to be calculated from an estimate $\widehat{f}_{\Delta^d}(\lambda)$ of the spectral density at the Fourier frequencies with order numbers $j = l, l + (2l - 1), l + 2(2l - 1), ..., m - (2l - 1), m$ with $l + 1 < m < T$. The latter is obtained from the locally averaged periodogram

$$\widehat{f}_{\Delta^d}(\lambda_j) = \frac{1}{2l - 1} \sum_{k=j-(l-1)}^{j+(l-1)} I_{\Delta^d}(\lambda_k),$$

where $I_{\Delta^d}(\lambda_k)$ is the periodogram of $\Delta^d X_t$. At each j the estimate $\widehat{f}_{\Delta^d}(\lambda_j)$ is thus a local average of the periodogram at frequency j and the $l + 1$ frequencies to its left and right and the λ_j are spaced so that the local averages are non-overlapping.

[2]These results are available from the authors upon request.

The resulting estimator for the cointegrating strength b is given by

$$\widehat{b}_{GPH} = \left(\sum_{j=l+1}^{m} \tilde{Z}_j^{*2} \right)^{-1} \sum_{j=l+1}^{m} \tilde{Z}_j^{*} \log \widehat{D}_{\Delta^d}(\lambda_j),$$

where $\tilde{Z}_j^* = Z_j^* - \bar{Z}^*$, $Z_j^* = \log|1 - e^{i\lambda}| = \log(2 - 2\cos(\lambda_j))$, and $\bar{Z}^*$ is the mean of the Z_j^*.

Under the null hypothesis of no fractional cointegration we have $b = 0$. Under this condition, and assuming that l and m fulfill the condition $\frac{l+1}{m} + \frac{m}{T} + \frac{1}{m} + \frac{\log m}{m} \to 0$ as $T \to \infty$, the estimate $\widehat{b}_{GPH}$ is consistent and asymptotic normal with variance $\sigma_b^2 = \frac{1}{m}(\Psi^{(1)}(2l+1) + \Psi^{(1)}(2l))$, where $\Psi^{(1)}(x) = \frac{\delta^2 \log \Gamma(x)}{\delta x^2}$ is the polygamma function of order 1 and $\Gamma(\cdot)$ denotes the gamma function.

The null hypothesis of no fractional cointegration can thus be tested using a simple t-test:

$$W_{SRFB} = \frac{\widehat{b}_{GPH}}{\sigma_b} \xrightarrow{d} N(0,1). \tag{2.11}$$

The method has no restrictions regarding the range of d and b but is only applicable to bivariate processes. For practical purposes, d is usually unknown and has to be estimated, but as shown in our simulation study in Section 2.4 this has no severe implications for the quality of the test. However, a thorough theoretical examination of this aspect would be interesting for further research.

2.3.2 Tests based on Cointegrating Residuals

By the definition of fractional cointegration the memory d_v of the linear combination $v_t = \beta' X_t$ is lower than that of X_t itself. Under the null hypothesis of no fractional cointegration one can still write $v_t = \beta' X_t = y_t - \tilde{\beta} x_t$, since y_t can still depend on the values of the other components of X_t. The difference to the cointegrated case is only that $d_v = d$. It is therefore natural to test for fractional cointegration by testing $d_v = d$ (or $b = 0$) versus $d_v < d$ (or $b > 0$) based on an estimate $\widehat{v}_t$ of the potential cointegrating residual.

Under weak non-stationary fractional cointegration so that $d > d_v > 1/2$, Marmol and Velasco (2004) suggest a Hausman (1978)-type F-test that compares the OLS estimate $\widehat{\beta}^{OLS}$ of the cointegrating vector with an alternative estimate $\widehat{\beta}^{NB}$ with opposite consistency characteristics.

The OLS estimator $\widehat{\beta}^{OLS}$ is consistent for $\tilde{\beta}$ under the alternative (as long as $d > 1/2$) but inconsistent under the null hypothesis. Marmol and Velasco (2004) propose an alternative estimator $\widehat{\beta}^{NB}$ that is consistent for the vector $\tilde{\beta}$ under the null hypothesis but inconsistent under the alternative. The estimator is given by

$$\begin{aligned} \widehat{\beta}^{NB}(\widehat{d}_x, \widehat{d}_v) &= \widehat{G}_{xx}^{MV}(\widehat{d}_x)^{-1}\widehat{g}_{xy}^{MV}(\widehat{d}_v), \\ \text{where} \quad \widehat{G}_{xx}^{MV}(d) &= \frac{2\pi}{m_2}\sum_{j=1}^{m_2}\tilde{\Lambda}_j(d)^{-1}\text{Re}\left\{I_{xx}(\lambda_j)\right\}\tilde{\Lambda}_j^{-1}(d), \\ \widehat{g}_{xy}^{MV}(d) &= \sum_{j=1}^{m_2}\text{Re } I_{xy}(\lambda_j)\lambda_j^{2(d-1)}, \end{aligned}$$

$\tilde{\Lambda}_j(d) = \text{diag}(\lambda_j^{1-d}, ..., \lambda_j^{1-d})$ and where $I_{xx}(\lambda_j)$ and $I_{xy}(\lambda_j)$ are the respective elements of the periodogram $I_{\Delta X \Delta X}(\lambda_j)$ of the differenced process ΔX_t and m_2 is subject to the usual bandwidth conditions.

The estimator is closely related to the narrow band least squares estimator $\widehat{\beta}_m$ from (2.6) but uses a rescaled version of the periodogram. In fact, $\widehat{\beta}^{NB}(0,0)$ would be equivalent to the NBLS estimate based only on the real part of the periodogram.

Inconsistency under the alternative is only obtained through the choice $\widehat{\beta}^{NB}(\widehat{d}_x, \widehat{d}_v)$, where $\widehat{d}_v$ is estimated from the OLS residuals. Since under the alternative $\widehat{v}_t^{OLS}$ is a consistent estimate of the cointegrating residual, $\widehat{d}_v \to d_v < d$, whereas $\widehat{d}_x$ is estimated from the original series and is consistent for d. Under the null hypothesis, on the other hand, $\widehat{\beta}^{OLS}$ is inconsistent so that $\widehat{v}_t^{OLS}$ is just some linear combination of $I(d)$ series, $\widehat{d}_v \to d$, and $\widehat{\beta}^{NB}(\widehat{d}_x, \widehat{d}_v)$ is consistent for $\tilde{\beta}$.

Since the process is non-stationary, the memory is estimated by local Whittle from the differenced process. ALternatively, d could be estimated using a tapered local Whittle estimator, or by the exact or fully extended local Whittle estimator.

The test statistic compares both estimates of $\tilde{\beta}$ where the normalizing variance $\widehat{V}^{MV}$ is estimated from the periodogram of the OLS residuals $\widehat{v}_t^{OLS}$ and that of x_t so that

$$\widehat{V}_{MV} = \left(\sum_{j=-m}^{m} I_{xx}(\lambda_j)\right)^{-1} \sum_{j=-m}^{m} I_{xx}(\lambda_j) I_{\widehat{v}\widehat{v}}(\lambda_j) \left(\sum_{j=-m}^{m} I_{xx}(\lambda_j)\right)^{-1}.$$

This leads to the test statistic

$$W_{MV} = \frac{1}{p-1}\left(\widehat{\beta}^{OLS} - \widehat{\beta}^{NB}\right)' \widehat{V}_{MV}^{-1}\left(\widehat{\beta}^{OLS} - \widehat{\beta}^{NB}\right). \tag{2.12}$$

The choices of m and m_2 are not linked, but both have to satisfy the condition $(m^{d-2} + m^{\gamma-1}\log T)\log^2 T + \frac{m}{T} \to 0$ as $T \to \infty$, with $\gamma > 0$ which is fulfilled if $m \sim T^\eta$, $\eta \in (0,1)$.

The asymptotic distribution is non-standard and depends on the memory parameter d. It is given by

$$W_{MV} \xrightarrow{d} \frac{1}{p-1} \int_0^1 W_y(d;r)W_x(d;r)'drV^{-1} \int_0^1 W_x(d;r)W_y(d;r)dr,$$

$$\text{with} \quad V = \int_0^1 \gamma_R(s) \left\{\gamma_{xx}(s) + \gamma'_{xx}(s) + \gamma_{xx}(1-s) + \gamma'_{xx}(1-s)\right\} ds,$$

$$\gamma_R(s) = \int_0^{1-s} W_y(d;r)W_y(d;r+s)dr,$$

$$\text{and} \quad \gamma_{xx}(s) = \int_0^{1-s} W_x(d;r)W_x(d;r+s)'dr,$$

where $W_y(d;r)$ is a fractional Brownian bridge, and $W_x(d;r)$ is a $p \times 1$ vector of independent fractional Brownian bridges.

Critical values are tabulated in Marmol and Velasco (2004) for dimensions up to $p = 5$ and different forms of detrending that affect the type of the fractional Brownian bridges. The test statistic W_{MV} diverges under the alternative since both $\widehat{\beta}^{NB}$ and $\widehat{V}_{MV}^{-1}$ diverge under fractional cointegration.

Although the consistency of the test is derived assuming $d > d_v > 0.5$, Marmol and Velasco (2004) state that the test remains consistent if the stationarity border is crossed by the cointegrating residuals, i.e. $d > 0.5 > d_v$. Our simulations in Section 2.4 confirm this.

A direct residual based test is proposed by Chen and Hurvich (2006) who estimate the possible cointegrating subspaces using eigenvectors of the averaged periodogram local to the zero frequency. The process X_t is assumed to be stationary after taking $(q-1)$ integer differences which allows $d \in (q-1.5, q-0.5)$. In order to account for possible over-differentiation the complex-valued taper $h_t = 0.5(1-e^{i2\pi t/T})$ of Hurvich and Chen (2000) is applied to the data. The tapered discrete Fourier transform and periodogram of X_t are defined by

$$w_X^{tap}(\lambda_j) = \frac{1}{\sqrt{2\pi \sum_t |h_t^{(q-1)}|^2}} \sum_{t=1}^{T} h_t^{(q-1)} X_t e^{i\lambda_j t},$$

$$I_X^{tap}(\lambda_j) = w_X(\lambda_j)\overline{w_X(\lambda_j)}.$$

Next, define the averaged periodogram matrix of X_t by

$$I_X^{av}(\lambda_j) = \sum_{j=1}^{m_3} \operatorname{Re}\left(I_X^{tap}(\lambda_j)\right),$$

where m_3 is a fixed positive integer fulfilling $m_3 > p+3$. The eigenvalues of $I_X^{av}(\lambda_j)$ sorted in descending order are denoted by $\widehat{\delta}_{a,I_X^{av}}$ and the corresponding eigenvectors are given by $\widehat{\chi}_{a,I_X^{av}}$, for $a = 1, ..., p$. Under the alternative hypothesis, if there are $r > 0$ cointegrating relationships, the matrix consisting of the first r eigenvectors provides a consistent estimate of the cointegrating subspace.

To construct a test for the null hypothesis of no fractional cointegration the potential cointegrating residuals v_t are estimated by multiplying X_t with the eigenvectors $\widehat{\chi}_{a,I_X^{av}}$ so that $\widehat{v}_{at}^{av} = \widehat{\chi}_{a,I_X^{av}}' X_t$, for $a = 1, ..., p$.

The memory of the p residual processes is estimated with the local Whittle estimator using bandwidth m but calculated using shifted Fourier frequencies $\lambda_{\tilde{j}}$ with $\tilde{j} = j + (q - 1)/2$ to account for the tapering of order q. These estimates are denoted by $\widehat{d}_{v_a,\widetilde{LW}}$, and they remain consistent and asymptotic normal.

Since there can be at most $p - 1$ cointegrating relationships in a p-dimensional time series, the first residual corresponding to the largest eigenvalue cannot be a cointegrating residual. Its memory must therefore equal the common memory d of X_t. The last residual $\widehat{v}_{pt}^{av}$ corresponding to the smallest eigenvalue, on the other hand, is most likely to be a cointegrating residual if there is cointegration so that its memory is reduced by b under cointegration.

The test idea of Chen and Hurvich (2006) is therefore to compare the estimated memory orders from the residual series $\widehat{v}_{1t}^{av}$ and $\widehat{v}_{pt}^{av}$. Hence, the test compares the estimated memory parameters $\widehat{d}$ (first residual) and $\widehat{d}_v$ (last residual). Chen and Hurvich (2006) show that

$$\sqrt{m}\left(\widehat{d}_{v_a,\widetilde{LW}} - \widehat{d}_{v_b,\widetilde{LW}}\right) \xrightarrow{d} N\left(0, V_{CH,q}\left(1 - \frac{G_{ab}^2}{G_{aa}G_{bb}}\right)\right)$$
$$\text{with} \quad V_{CH,q} = \frac{1}{2}\,\frac{\Gamma(4q-3)\Gamma^4(q)}{\Gamma^4(2q-1)}.$$

A conservative test statistic is therefore given by

$$W_{CH} = \sqrt{m}\,\frac{\left(\widehat{d}_{v_1,\widetilde{LW}} - \widehat{d}_{v_p,\widetilde{LW}}\right)}{\sqrt{V_{CH,q}}}. \tag{2.13}$$

The tests rejects if W_{CH} is larger than the standard normal quantile $z_{1-\alpha/2}$. It is very versatile, since it does not impose restrictions on the cointegration strengh b and can be applied to stationary as well as non-stationary long memory processes, but it requires a priori knowledge about the location of d in the parameter space to determine the order of differencing.

Hualde and Velasco (2008) propose another testing strategy in a residual-based regression framework. As before, the series X_t is partitioned such that $X_t = (y_t, x_t')'$ and they consider the single-equation regression $y_t = \tilde{\beta} x_t + v_t$.

The test idea is based on the observation that the fractionally differenced residual $\Delta^{d_x} v_t$ is unrelated to the long-run level of x_t under the null hypothesis. This is because

$\Delta^{d_x} v_t$ is $I(0)$ and x_t is $I(d)$. The cross-spectrum of x_t and $\Delta^{d_x} v_t$ should therefore be zero at frequencies local to zero. Possible dependence between the short-run components u_t and e_t in (2.3) would manifest itself in form of a non-zero cross-spectrum at higher frequencies.

The test statistic of Hualde and Velasco (2008) is therefore based on the quantity $\widehat{\tau}_m$ defined as

$$\widehat{\tau}_m = \sum_{j=1}^{m} w_x(-\lambda_j)\zeta(\lambda_j) w_{\Delta^{d_v,d}X}(\lambda_j)$$

where $\Delta^{d_v,d} X_t = \left(\Delta^{\widehat{d_v}} y_t, \Delta^{\widehat{d}} x_t'\right)'$ and $\zeta(\lambda_j) = (1, 0'_{p-1})\, \widehat{f}_X(\lambda_j)^{-1}$. The projection vector $\zeta(\lambda_j)$ estimates the discrete Fourier transform (DFT) of the residual process v_t from $w_{\Delta^{d_v,d}X}(\lambda_j)$ — the DFT of the fractionally differenced process $\Delta^{d_v,d} X_t$. As usual for these semiparametric approaches, it is assumed that $m \leq T/2$ and $m/T \to 0$, as $T \to \infty$.

This leads to the test statistic

$$\begin{aligned} W_{HV} &= \widehat{\tau}_m' \widehat{V}_{HV}^{-1} \widehat{\tau}_m \qquad (2.14)\\ \text{with} \quad \widehat{V}_{HV} &= \sum_{j=0}^{m} a_j \text{ Re } \kappa(\lambda_j) I_{XX}(\lambda_j), \\ \text{and} \quad \kappa(\lambda_j) &= (1, 0'_{p-1})\, \widehat{f}_X(\lambda_j)^{-1} (1, 0'_{p-1})' = \zeta(\lambda_j)\, (1, 0'_{p-1})', \end{aligned}$$

where the weights are defined by $a_j = 1$ if $j \in \{0, T/2\}$ and $a_j = 2$ otherwise. Under the null hypothesis this test statistic follows an asymptotic χ^2_{p-1}-distribution. Under the alternative the test develops power, since d_v is estimated from the NBLS estimate of the cointegrating residuals. Since these have reduced memory under the alternative, the first component of $\Delta^{d_v,d} X_t$ (y_t) is $I(b)$ instead of $I(0)$ and the cross spectrum of the underdifferenced estimate of v_t and x_t in $\widehat{\tau}_m$ becomes non-zero. As before, the memory orders are estimated using consistent estimators that account for the (possible) non-stationarity of the data — for example the exact local Whittle estimator of Shimotsu and Philips (2005).

A modified test with more power in bivariate systems $X_t = (X_{1t}, X_{2t})'$ is calculated with $\tilde{\tau}_m$ instead of $\widehat{\tau}_m$:

$$\tilde{\tau}_m = \sum_{j=0}^{m} a_j \frac{\text{Re}\left(I_{\Delta^{\widehat{d_v}} X_1, X_2}(\lambda_j) - \frac{\tilde{f}_{12}(\lambda_j)}{\tilde{f}_{22}(\lambda_j)} I_{\Delta^{\widehat{d_v}} X_2, X_2}(\lambda_j)\right)}{\widehat{f}_{11}(\lambda_j) - \frac{\widehat{f}_{12}(\lambda_j)\widehat{f}_{21}(\lambda_j)}{\widehat{f}_{22}(\lambda_j)}},$$

$$\text{with} \quad \widehat{f}_{\Delta^{\widehat{d}}}(\lambda_j) = \frac{1}{2m+1} \sum_{k=j-m}^{j+m} I_{\Delta^{\widehat{d}} X}(\lambda_k) \quad \text{and} \quad \tilde{f}_{\Delta^{\widehat{d_v}}}(\lambda_j) = \frac{1}{2m+1} \sum_{k=j-m}^{j+m} I_{\Delta^{\widehat{d_v}} X}(\lambda_k).$$

Here, the respective elements of the spectral matrices are denoted by $\widehat{f}_{ab}(\lambda_j)$ and $\widetilde{f}_{ab}(\lambda_j)$ with $a, b \in \{1, 2\}$. This is the same as $\widehat{\tau}_m$ but with $\widehat{f}_{12}(\lambda_j)$ replaced by $\widetilde{f}_{12}(\lambda_j)$ that is constructed using $\widehat{d}_v$ so that it also diverges under the alternative and constitutes an additional source of power. The asymptotic χ^2_{p-1}-distribution is unaffected by this modification.

It is not necessary to impose any restrictions on the range of d and d_v except for those implied by fractional cointegration, and processes of dimensions higher than two are allowed. The asymptotic χ^2_{p-1} distribution depends only on the dimension of the process. Furthermore, the memory parameters are allowed to differ as long as two components of X_t share the same memory parameter and the vector is sorted so that the component with the highest memory comes first.

Nielsen (2010) introduces a sequential testing approach in order to test the null hypothesis of no fractional cointegration and to determine the cointegrating rank. The method is based on a variance-ratio statistic and imposes the assumption that the process X_t is non-stationary and the potential cointegrating residual process is stationary with $d_v < 0.5 < d$. Denote the demeaned process by $Z_t = X_t - \overline{X_t}$, where $\overline{X_t}$ is the vector of arithmetic means of the component series. The fractionally integrated version of Z_t is denoted by $\widetilde{Z}_t = \Delta^{-\epsilon} Z_t$. Then the variance ratio is given by

$$K_T(\epsilon) = A_T C_T^{-1},$$
$$\text{with } A_T = \sum_{t=1}^{T} Z_t Z_t', \quad \text{and} \quad C_T = \sum_{t=1}^{T} \tilde{Z}_t \tilde{Z}_t.$$

Taking the ratio has the advantage of eliminating the processes' variance from the asymptotic distribution. The eigenvalues of $K_T(\epsilon)$ sorted in ascending order are denoted by $\widehat{\delta}_{a,K}$ with $a = 1, ..., p$.

Similar to the spectral matrix G, the rank of $K_T(\epsilon)$ is reduced to $p-r$ under fractional cointegration. This leads to a non-parametric trace statistic whose structure is similar to the trace statistic of Johansen and Juselius (1990) in the parametric context

$$W_{Niel}(\epsilon) = T^{2\epsilon} \sum_{k=1}^{p-r} \widehat{\delta}_{k,K}, \quad r = 1, ..., p-1, \tag{2.15}$$

where r is the number of cointegrating relations under the null hypothesis. Using (2.15) the cointegrating rank can be determined by a sequence of tests of the null hypothesis H_0: $r = r_0$ vs. H_1: $r > r_0$.

The limiting distribution is given by

$$W_{Niel}(\epsilon) \xrightarrow{d} \text{tr} \left\{ \int_0^1 W_{n-r}(d; s) W_{n-r}(d; s)' ds \left(\int_0^1 \widetilde{W}_{n-r}(d+\epsilon; s) \widetilde{W}_{n-r}(d+\epsilon; s)' ds \right)^{-1} \right\},$$

where $W_{n-r}(d,u) = B_d^{n-r}(u) - \int_0^1 B_d^{n-r}(v)dv$, $\widetilde{W}_{n-r}(d+\epsilon;u) = B_{d+\epsilon}^{n-r}(u) - \int_0^u \frac{(u-v)^{\epsilon-1}}{\Gamma(\epsilon)}dv$ $\int_0^1 B_{d+\epsilon}^{n-r}(v)dv$, B_d^{n-r} is a $n-r$ dimensional vector of mutually independent standard fractional Brownian motions of type II, and the Brownian motions driving the fractional Brownian motions B_d^{n-r} and $B_{d+\epsilon}^{n-r}$ are identical.

This asymptotic distribution is non-standard and depends on the dimension p, the cointegrating rank r, the order of fractional integration ϵ and d. In practice d can be estimated consistently, and the other parameters are known. Critical values for $d = 1$, $\epsilon = 0.1$, and $p-r = 1, 2, ..., 8$ are given by Nielsen (2010), who recommends to use $\epsilon = 0.1$ to integrate the process because it leads to higher power than larger values whereas smaller values improve power slightly but lead to size distortions at the same time. For more details confer Nielsen et al. (2009). Note that choosing a different order of fractional summation changes the limiting distribution which implies that the test performance is free from user-chosen tuning parameters.

To see why this test can be considered to be residual-based, note that

$$\widehat{\delta}_{a,K} = \frac{\widehat{\eta_a}' A_T \widehat{\eta_a}}{\widehat{\eta_a}' C_T \widehat{\eta_a}} = \frac{\sum_{t=1}^T \widehat{v}_t^2}{\sum_{t=1}^T \widetilde{v}_t^2},$$

where $\widehat{\eta}_a$ denotes the eigenvector corresponding to the ath eigenvalue. Since the first r eigenvectors are consistent estimates of the cointegrating space (cf. Theorem 3 in Nielsen (2010)), the first r eigenvalues are thus given by the ratio of the sum of the squared cointegrating residuals and the sum of squares of their ϵ times integrated version $\widetilde{v}_t$.

Here the squares are estimators of the respective process variances and it is assumed that $d > 1/2 > d_v$. Therefore, under the null hypothesis of no fractional cointegration the enumerator grows with rate $\mathcal{O}_P(T^{2d})$ and the more persistent denominator grows with rate $\mathcal{O}_P(T^{2(d+\epsilon)})$, so that the eigenvalue has rate $\mathcal{O}_P(T^{-2\epsilon})$.

Under the alternative of fractional cointegration with $d_v < 1/2$, the process v_t is stationary so that the process variance is finite and the enumerator grows with rate $\mathcal{O}_P(T)$. The denominator that may or may not be stationary due to the integration with ϵ is $\mathcal{O}_P(T^{\max\{1/2, d-b+\epsilon\}})$. Consequently, the eigenvalue is $O_P(T^{\min\{0, 1-2(d-b+\epsilon)\}})$, so that it goes to zero more slowly than under the null hypothesis.

The test is restrictive in that it requires non-stationary processes and, preferably, stationary residual processes, but as shown by his Monte Carlo simulation the test still exhibits power if $d_v > 0.5$ and $b > 0$. Furthermore, it is applicable to multivariate systems and is able to estimate the number of cointegrating relations.

Wang et al. (2015) propose a simple residual-based test in a bivariate setting, where $X_t = (X_{1t}, X_{2t})'$. The test statistic is based on the partial sum of $\Delta^{d_v} Z_{2t}$, which is the demeaned second component series fractionally differenced with the memory order of the potential cointegrating residual v_t. It is given by

$$W_{WWC} = T^{-1/2} \frac{\sum_{t=1}^{T} \Delta^{\widehat{d}_v} Z_{2t}}{\sqrt{2\pi \widehat{f}_{22}(0)}},$$

where f_{22} is the spectral density of either u_{2t} or e_t in (2.3), depending on whether a triangular model or a common-components model is assumed.

Under the null hypothesis $d_v = d$ so that $\Delta^{d_v} Z_{2t}$ is $I(0)$ and the appropriately rescaled sum is asymptotically standard normal. Under the alternative $\Delta^{d_v} Z_{2t}$ is $I(b)$, so that the test statistic diverges with rate $\mathcal{O}_P(T^b)$.

To make this test statistic feasible the spectral density f_{22} can be estimated from the periodogram of the fractionally differenced process $\Delta^{\widehat{d}} Z_{2t}$ following the approach of Hualde (2013):

$$\widehat{f}_{22}(0) = \frac{1}{(2m+1)} \sum_{j=-m}^{m} I_{\Delta^{\widehat{d}} Z_2}(\lambda_j),$$

where $I_{\Delta^{\widehat{d}} Z_2}(\lambda_j)$ is the periodogram of $\Delta^{\widehat{d}} Z_{2t}$.

While Wang et al. (2015) are agnostic about the method that is used for the estimation of the memory parameters d and d_v, they assume that $d > 1/2$ so that the cointegrating vector can be estimated using ordinary least squares. The memory orders can then be estimated from $\widehat{v}_t^{OLS}$ and Z_{2t} using any of the common semiparametric estimates such as ELW with bandwidth m as in $\widehat{f}_{22}$ that fulfills the usual bandwidth conditions.

The method does not impose any restrictions on the fractional cointegrating strength b. As the Monte Carlo simulations below show, the non-stationarity requirement ($d > 1/2$) can be circumvented if the cointegrating residual v_t is based on the NBLS estimate of the cointegrating vector instead of the OLS estimate.

Zhang et al. (2019) propose an alternative estimator of the cointegrating space that is based on the eigenvectors of the non-negative matrix

$$\widehat{M} = \sum_{j=0}^{j_0} \widehat{\Omega}_Z(j) \widehat{\Omega}_Z(j)',$$

where $\widehat{\Omega}_Z(j) = \frac{1}{T} \sum_{t=1}^{T-j} Z_{t+j} Z_t'$ is the autocovariance matrix at lag j and j_0 is a fixed integer. The matrix $\widehat{M}$ is thus the sum of the outer products of the first j_0 autocovariance matrices with themselves. The outer product is used instead of the covariance matrices $\widehat{\Omega}_Z(j)$ to ensure that there is no information cancellation over different lags in $\widehat{M}$. It is assumed that $d > 0.5$ and $d_v < 0.5$.

The eigenvalues of $\widehat{M}$ in descending order are denoted by $\widehat{\delta}_{a,M}$ for $a = 1, ..., p$ and the corresponding eigenvectors are denoted by $\widehat{\chi}_{a,M}$. Similar to the matrix G in (2.1), the first

$p-r$ eigenvalues of M are non-zero, whereas the remaining r are zero. For known r the eigenvectors corresponding to the r smallest eigenvalues provide a consistent estimate of the cointegrating space.

If r is unknown, the p potential cointegrating residuals are estimated using the eigenvectors so that $\widehat{v}_{at}^M = \widehat{\chi}'_{a,M} X_t$. By the same argument as in the procedure of Chen and Hurvich (2006), the residual corresponding to the smallest eigenvalue is most likely a cointegrating residual with reduced memory of $d_v = d - b$ and the residual corresponding to the largest eigenvalue is $I(d)$.

The cointegrating rank can be estimated using a simple criterion based on the summed autocorrelations of the potential cointegrating residuals. Define

$$Q_a(k_0) = \sum_{k=1}^{k_0} \widehat{\rho}_a(k),$$
$$\text{with } \widehat{\rho}_a(k) = \frac{\frac{1}{T-k}\sum_{t=1}^{T-k}(\widehat{v}_{a,t+k}^M - \overline{\widehat{v}_{at}^M})(\widehat{v}_{at} - \overline{\widehat{v}_{at}^M})'}{\frac{1}{T}\sum_{t=1}^{T}(\widehat{v}_{at}^M - \overline{\widehat{v}_{at}^M})^2},$$

where $\overline{\widehat{v}_{at}^M}$ is the mean of $\widehat{v}_{at}^M$. The cointegrating rank estimator counts the instances when the averaged autocorrelation is smaller than a threshold $c_0 \in (0,1)$:

$$\widehat{r}_{ZRY} = \sum_{a=1}^{p} \mathbb{1}\left\{\frac{Q_a(k_0)}{k_0} < c_0\right\}. \tag{2.16}$$

If the residual $\widehat{v}_{at}^M$ is stationary ($d_v < 1/2$), the rescaled sum of autocorrelations $Q_a(k_0)/k_0$ converges to zero asymptotically for $k_0 \to \infty$, since the autocorrelations are asymptotically proportionate to k^{2d_v-1}. Under certain regularity conditions this estimate is consistent. Even though the consistency is only proven for $r \geq 1$ in Theorem 4.2 of Zhang et al. (2019), our simulations below show that it also works well in discriminating between $r = 0$ and $r = 1$.

It should be noted that the authors define $r = p$ if all components of X_t are $I(0)$. This leads to some abuse of notation and r cannot be interpreted as the cointegrating rank in a narrow sense. Based on their simulations Zhang et al. (2019) recommend to use $j_0 = 5$, $k_0 = 20$ and $c_0 = 0.3$. The estimator is easy to implement and applicable to higher dimensional processes. However, the requirement of $d > 0.5$ and $d_v < 0.5$ is restrictive.

2.4 Monte Carlo Study

The asymptotic properties of all tests and rank estimates presented in Section 2.3 are derived by the respective authors, and some of them also present simulations to explore the finite sample results of the test statistics. This however is not the case for all tests and a comprehensive comparative study suited to guide the choice of appropriate methods in practical applications is entirely missing. To close this gap, we conduct an extensive Monte Carlo study. In addition to general results, we are particularly interested in answering two empirically motivated questions.

i.) How does correlation between the underlying short-run components influence the size of the tests? This question is particularly important, since applied researchers will generally want to test for fractional cointegration if two related series seem to be co-moving. Similar trajectories however, can also be generated by persistent processes with highly correlated innovations. Tests for the null hypothesis of no fractional cointegration should therefore be robust to a relatively high degree of correlation between the short-run components of the series.

ii.) Is there a notable difference in the power of the tests depending on whether the data is generated from a triangular model or from a common-components model? Both models are used in the literature to motivate and construct testing procedures, but to our knowledge simulation results are typically based on the triangular representation. In practice, either model could be justified — depending on the application. For example, if one is considered with potential fractional cointegrating relationships between stock prices, it is not clear why one of the stock prices should be seen as a perturbed version of the other one (as it is the case in the triangular model that treats the series in an asymmetric way) so that the common-components model is more suitable. In contrast to that, in the case of the potential parity between implied volatility and the expected average realized volatility over the next month (the so-called implied-realized parity analyzed by Christensen and Prabhala (1998), Christensen and Nielsen (2006), and Nielsen (2007), among others), there is theoretical reason to assume that the implied volatility is a perturbed version of the expected average future realized volatility, since it contains a variance-risk premium (cf. Chernov (2007)). Therefore, a triangular model is more suitable.

We focus on three data generating processes (DGPs) based on the general model from equations (2.3) to (2.5). For simplicity we set $c_1 = c_2 = 0$ and $b = b_1 = b_2$, so that the processes are mean zero and have a common memory reduction parameter.

A simple bivariate model without fractional cointegration is constructed by setting $\xi_1 = \xi_2 = 0$. This model - referred to as DGP1 - is given by

$$X_{1t} = \Delta^{-d} u_{1t} \mathbb{1}\{t > 0\}, \tag{2.17}$$

$$X_{2t} = \Delta^{-d} u_{2t} \mathbb{1}\{t > 0\}, \tag{2.18}$$

where correlation between u_{1t} and u_{2t} is allowed. This is our size-DGP.

For the power simulations, we consider a triangular model and a common-components model. In both cases we set $\xi_1 = \xi_2 = 1$, which implies a cointegrating vector of $\beta = (1, -1)'$.

The triangular model DGP2 is given by

$$X_{1t} = Y_t + \Delta^{-(d-b)} u_{1t} \mathbb{1}\{t > 0\}, \tag{2.19}$$

$$X_{2t} = Y_t, \tag{2.20}$$

and the common-components model DGP3 is defined by

$$X_{1t} = Y_t + \Delta^{-(d-b)} u_{1t} \mathbb{1}\{t > 0\}, \tag{2.21}$$

$$X_{2t} = Y_t + \Delta^{-(d-b)} u_{2t} \mathbb{1}\{t > 0\}. \tag{2.22}$$

In both DGP2 and DGP3 we have $Y_t = \Delta^{-d} e_t \mathbb{1}\{t > 0\}$. The underlying short-run components u_{1t} and u_{2t}, or u_{1t} and e_t — depending on the DGP — have unit variance and correlation ρ.

We consider sample sizes of $T \in \{100, 500, 1000, 2500\}$ and values of $d \in \{0.4, 0.7, 1\}$ in the stationary and non-stationary region. Under fractional cointegration, the memory reduction b is linked to the value of d so that $b \in \{d/3, d\}$. Consequently, there is either a memory reduction to 0 if $b = d$ or a weaker form of cointegration if $b = d/3$. In order to examine the impact of correlation between the short-run components, we consider $\rho \in \{0, 0.45, 0.9, 0.99\}$.

The semiparametric nature of the tests and rank estimates requires several bandwidth choices. The memory estimation with (E)LW estimators involved in all methods is based on the bandwidth m that determines the number of frequencies included in the estimation. We use $m = \lfloor T^{\delta_m} \rfloor$ with $\delta_m = \{0.65, 0.75\}$ to account for sensitivities regarding bandwidth choice. With regard to the other bandwidth choices, we follow the recommendations by the authors: $m_1 = \lfloor T^{\delta_m - 0.1} \rfloor$ and $p(T) = m_1^{-0.3}$ for Nielsen and Shimotsu (2007) or Robinson and Yajima (2002), $l = 1$ for Souza et al. (2018), $m_3 = 25$ for Chen and Hurvich (2006), $c_0 = 0.3$, $j_0 = 5$ and $k_0 = 20$ for Zhang et al. (2019), and for Marmol and Velasco (2004) we set $m = \lfloor T^{2/3} \rfloor$ and $m_2 = \lfloor T^{\delta_m} \rfloor$. All tests are carried out allowing for a non-zero mean.

The results presented are based on 5000 replications and a nominal significance level of $\alpha = 0.05$. Since the tests impose different conditions on d and d_v, we mark the cells in the tables in gray where the methods have well-defined asymptotic properties and are supposed to deliver good results. In some cases the methods give satisfactory results beyond these limitations. For example, we implement the method of Wang et al. (2015) using a NBLS estimate of the cointegrating vector instead of the OLS estimate. This makes the test applicable in stationary time series as well as in non-stationary ones.

Since the limiting distributions of the non-pivotal test statistics of Marmol and Velasco (2004) and Nielsen (2010) depend on d and it is assumed that $d > 1/2$, it is unclear which

	ρ	0			0.45			0.9			0.99		
method	T/d	0.4	0.7	1	0.4	0.7	1	0.4	0.7	1	0.4	0.7	1
NS07*	100	0.000	0.000	0.014	0.130	0.138	0.241	1.000	1.000	0.998	1.000	1.000	1.000
	500	0.000	0.000	0.001	0.000	0.000	0.049	1.000	1.000	0.994	1.000	1.000	1.000
	1000	0.000	0.000	0.000	0.000	0.000	0.021	1.000	1.000	0.989	1.000	1.000	1.000
	2500	0.000	0.000	0.000	0.000	0.000	0.006	1.000	1.000	0.976	1.000	1.000	1.000
CH06	100	0.219	0.119	0.021	0.105	0.073	0.029	0.077	0.040	0.032	0.075	0.033	0.033
	500	0.177	0.058	0.032	0.076	0.031	0.021	0.060	0.026	0.017	0.064	0.027	0.020
	1000	0.136	0.051	0.031	0.064	0.025	0.018	0.045	0.018	0.018	0.046	0.020	0.017
	2500	0.129	0.044	0.023	0.059	0.023	0.015	0.039	0.018	0.012	0.040	0.017	0.012
HV08	100	0.001	0.008	0.015	0.017	0.022	0.024	0.128	0.098	0.058	0.312	0.263	0.130
	500	0.002	0.018	0.020	0.027	0.029	0.022	0.309	0.141	0.028	0.764	0.440	0.098
	1000	0.003	0.023	0.022	0.039	0.029	0.021	0.399	0.128	0.028	0.805	0.501	0.076
	2500	0.003	0.027	0.022	0.060	0.035	0.021	0.494	0.140	0.024	0.835	0.535	0.057
SRFB18	100	0.114	0.117	0.101	0.107	0.111	0.110	0.112	0.121	0.114	0.109	0.115	0.105
	500	0.054	0.054	0.049	0.049	0.052	0.052	0.046	0.055	0.046	0.056	0.052	0.045
	1000	0.041	0.047	0.042	0.043	0.043	0.040	0.044	0.047	0.048	0.045	0.043	0.044
	2500	0.037	0.039	0.029	0.035	0.037	0.037	0.036	0.036	0.034	0.033	0.035	0.035
R08	100	0.174	0.183	0.092	0.050	0.059	0.066	0.036	0.048	0.039	0.042	0.045	0.039
	500	0.233	0.254	0.104	0.049	0.080	0.063	0.052	0.066	0.041	0.049	0.062	0.043
	1000	0.239	0.275	0.090	0.053	0.080	0.064	0.051	0.076	0.046	0.057	0.074	0.042
	2500	0.265	0.304	0.084	0.055	0.094	0.056	0.054	0.084	0.039	0.050	0.084	0.040
WWC15	100	0.080	0.095	0.090	0.079	0.087	0.094	0.081	0.096	0.098	0.078	0.091	0.094
	500	0.069	0.068	0.075	0.068	0.068	0.074	0.065	0.074	0.074	0.072	0.072	0.068
	1000	0.066	0.064	0.067	0.069	0.062	0.066	0.065	0.056	0.067	0.055	0.060	0.068
	2500	0.057	0.059	0.056	0.059	0.052	0.061	0.060	0.057	0.059	0.055	0.049	0.061
ZRY18*	100	0.101	0.644	0.652	0.071	0.597	0.627	0.057	0.560	0.573	0.062	0.565	0.576
	500	0.401	0.058	0.000	0.288	0.043	0.000	0.270	0.036	0.000	0.281	0.030	0.000
	1000	0.548	0.000	0.000	0.410	0.000	0.000	0.408	0.000	0.000	0.401	0.000	0.000
	2500	0.677	0.000	0.000	0.517	0.000	0.000	0.505	0.000	0.000	0.510	0.000	0.000
N10	100		0.035	0.059		0.041	0.061		0.053	0.053		0.058	0.063
	500		0.048	0.055		0.048	0.052		0.064	0.057		0.060	0.055
	1000		0.057	0.056		0.056	0.055		0.070	0.060		0.064	0.055
	2500		0.067	0.054		0.078	0.056		0.078	0.056		0.076	0.053
MV04	100		0.034	0.057		0.036	0.066		0.104	0.101		0.332	0.219
	500		0.041	0.052		0.047	0.056		0.158	0.092		0.496	0.234
	1000		0.039	0.052		0.047	0.055		0.141	0.081		0.490	0.224
	2500		0.036	0.052		0.045	0.056		0.127	0.069		0.442	0.197

Table 2.1: Size (*rank estimation) based on DGP1 with $\delta_m = 0.75$. We abbreviate the methods with the initial letters of the authors' names and the year of publication.

critical values would be used in the stationary region. The respective fields are therefore left blank.

It should be noted that the methods of Nielsen and Shimotsu (2007) (or Robinson and Yajima (2002)) and Zhang et al. (2019) are not tests but rank estimates. Instead of the rejection frequency, we therefore report the ratio of correctly estimated cointegrating ranks. Therefore, the results cannot be interpreted as size or power and in the size table and graphs the estimates should yield 0 instead of 0.05.

Table 2.1 displays size results based on DGP 1 with $\delta_m = 0.75$ and a nominal size level of 5%. The methods that have well defined asymptotic properties across all parameter constellations covered in the table are those of Nielsen and Shimotsu (2007), Chen and Hurvich (2006), Hualde and Velasco (2008), and Souza et al. (2018). It can be observed that all of these methods achieve good size properties for $\rho = 0$, except for the test of Chen and Hurvich (2006), when $d = 0.4$. If ρ increases, however, only the tests of Souza et al. (2018) and Chen and Hurvich (2006) do not over-reject.[3] For low values of d the test of Hualde and Velasco (2008) already becomes oversized for $\rho = 0.45$ and as ρ increases it becomes oversized for higher values of d, too. The rank estimation procedure of Robinson and Yajima (2002) and Nielsen and Shimotsu (2007) is even more affected and estimates a cointegrating rank of one in nearly all cases if $\rho \geq 0.9$.

In addition to the tests of Souza et al. (2018) and Chen and Hurvich (2006), the modified version of the test by Wang et al. (2015) that is based on the NBLS estimator instead of OLS also maintains satisfactory size properties across all values of ρ and d.

The group of procedures that is only applicable to non-stationary systems consists of Marmol and Velasco (2004), Nielsen (2010), and Zhang et al. (2019). It can be observed that the procedure of Marmol and Velasco (2004) behaves similar to that of Hualde and Velasco (2008) in the sense that it is very liberal for higher values of ρ and lower values of d. Zhang et al. (2019) estimate the cointegrating rank based on the mean autocorrelation of the residuals under the assumption that the original processes are non-stationary and the residual process is stationary. For non-stationary series and larger sample sizes the procedure correctly estimates the cointegrating rank to be zero — independently of the degree of correlation. The variance-ratio statistic of Nielsen (2010) turns out to be slightly liberal for $d = 0.7$ in larger samples, but this effect is independent of the degree of correlation.

Finally, the test of Robinson (2008) is only applicable for stationary systems. Here, it can be observed that the test does not hold its size for $\rho = 0$. This is because the Hausman-testing principle requires one of the estimates of the memory parameter to be more efficient than the other one, but the multivariate estimate is not more efficient in absence of correlation. For other values of ρ, however, the test has good size properties. Interestingly, the test also has good size properties if $d = 1$, even though it assumes stationarity. The intermediate value of $d = 0.7$, on the other hand, leads to a moderately oversized test.

[3] The test of Chen and Hurvich (2006) is conservative by construction as discussed in the previous section.

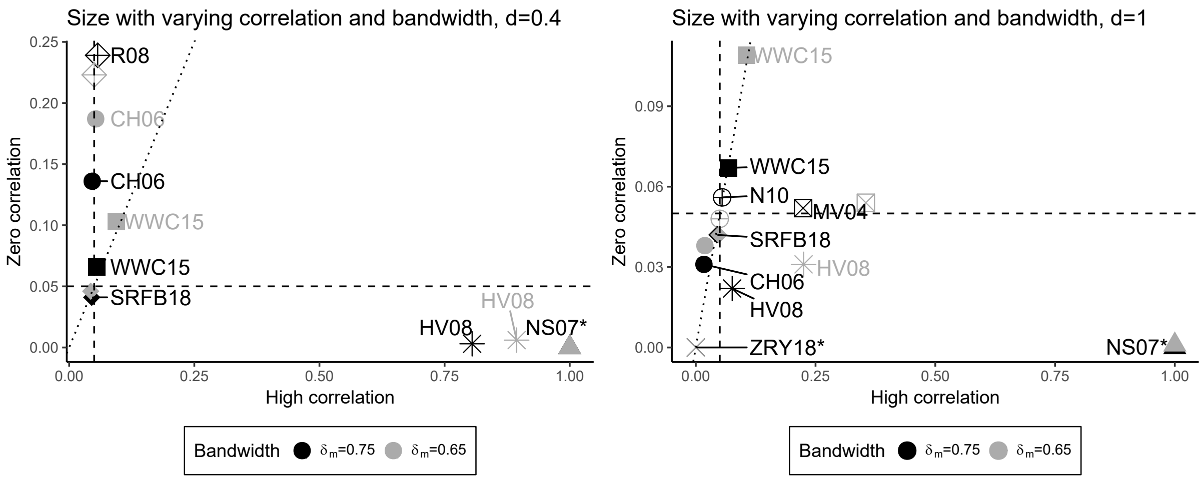

Figure 2.1: Size (*rank estimation) based on DGP1 depending on correlation $\rho \in \{0, 0.99\}$ and bandwidth $\delta_m \in \{0.65, 0.75\}$ with $T = 1000$.

Figure 2.1 analyzes the interaction between the degree of correlation ρ and the choice of the bandwidth δ_m. It shows the size of the tests in scatterplots where the results with high correlation ($\rho = 0.99$) are plotted against results with no correlation ($\rho = 0$). On the left-hand side only tests that allow for stationary processes (with $d = 0.4$) and on the right-hand side (where $d = 1$) the non-stationarity-robust tests, i.e. all except that of Robinson (2008) are displayed. The dashed lines mark the nominal size level of 0.05, so that ideally all points would lie on the intersection between these two lines. The dotted line is the bisector implying that methods above the bisector do better with correlation and methods below the bisector do better without. Black symbols give results with a bandwidth parameter of $\delta_m = 0.75$ and gray symbols with $\delta_m = 0.65$.

Overall, it can be observed that the bandwidth choice has a limited effect on the performance of the procedures. A notable exception is the test by Wang et al. (2015), where the size improves considerably as the bandwidth is increased. In general, correlation in the underlying short-run component is mistaken for cointegration more often in stationary systems than in non-stationary ones.

As a robustness check for the finite sample analysis of the size properties conducted here, we consider two alternative size DGPs in Tables A.1 and A.2 in the appendix A. These results are obtained from DPG2 and DGP3 where we set $b = 0$, so that the time series are not cointegrated. In this case, the processes share a common trend Y_t, but the linear combination $v_t = X_{1t} - X_{2t}$ does not have reduced memory so that the definition of cointegration is not fulfilled. The results show that the tests that are already heavily affected by correlation show even more distorted size results if there is a common component in the DGP. The general picture, however, remains unaffected.

method	ρ		0			0.45			0.9			0.99	
	T/d	0.4	0.7	1	0.4	0.7	1	0.4	0.7	1	0.4	0.7	1
	d_v	0	0	0	0	0	0	0	0	0	0	0	0
NS07*	100	0.989	0.998	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	2500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
CH06	100	0.556	0.349	0.337	0.317	0.291	0.110	0.046	0.205	0.020	0.007	0.179	0.010
	500	0.998	0.443	1.000	0.991	0.179	1.000	0.999	0.081	1.000	1.000	0.071	1.000
	1000	1.000	0.811	1.000	1.000	0.518	1.000	1.000	0.277	1.000	1.000	0.241	1.000
	2500	1.000	0.998	1.000	1.000	0.972	1.000	1.000	0.869	1.000	1.000	0.833	1.000
HV08	100	0.788	0.989	1.000	0.882	0.998	1.000	0.958	1.000	1.000	0.959	1.000	1.000
	500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	2500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
SRFB18	100	0.720	0.976	0.999	0.709	0.982	0.999	0.642	0.928	0.993	0.259	0.586	0.914
	500	0.993	1.000	1.000	0.994	1.000	1.000	0.983	1.000	1.000	0.794	0.961	0.993
	1000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.949	0.994	0.999
	2500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000
R08	100	0.044	0.245	0.572	0.009	0.023	0.200	0.049	0.060	0.065	0.671	0.463	0.147
	500	0.870	1.000	1.000	0.368	0.869	0.998	0.202	0.239	0.454	1.000	0.957	0.599
	1000	0.997	1.000	1.000	0.799	0.999	1.000	0.347	0.343	0.573	1.000	0.991	0.709
	2500	1.000	1.000	1.000	0.999	1.000	1.000	0.576	0.486	0.707	1.000	1.000	0.804
WWC15	100	0.634	0.896	0.966	0.618	0.887	0.970	0.426	0.855	0.965	0.307	0.832	0.959
	500	0.829	0.967	0.993	0.807	0.968	0.994	0.694	0.961	0.993	0.630	0.956	0.995
	1000	0.867	0.982	0.998	0.850	0.979	0.998	0.763	0.977	0.997	0.695	0.978	0.997
	2500	0.917	0.992	0.998	0.896	0.988	0.999	0.817	0.990	0.999	0.787	0.990	0.999
ZRY18*	100	0.020	0.403	0.797	0.007	0.339	0.780	0.006	0.289	0.770	0.008	0.289	0.764
	500	0.082	0.983	1.000	0.032	0.974	1.000	0.016	0.953	1.000	0.010	0.954	1.000
	1000	0.119	1.000	1.000	0.042	1.000	1.000	0.011	0.998	1.000	0.011	0.998	1.000
	2500	0.200	1.000	1.000	0.049	1.000	1.000	0.013	1.000	1.000	0.007	1.000	1.000
N10	100		0.431	0.978		0.356	0.965		0.270	0.950		0.262	0.954
	500		0.996	1.000		0.988	1.000		0.981	1.000		0.974	1.000
	1000		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000
	2500		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000
MV04	100		0.481	0.974		0.748	0.994		0.854	0.998		0.866	0.999
	500		0.985	1.000		0.993	1.000		0.997	1.000		0.997	1.000
	1000		0.997	1.000		0.999	1.000		1.000	1.000		1.000	1.000
	2500		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000

Table 2.2: Power (*rank estimation) with $b = d$ and $\delta_m = 0.75$ for the triangular model (DGP2).

Overall, in terms of size for bivariate systems and taking the range of admissible parameter values into account, we find that the test of Souza et al. (2018) has the best performance, followed by those of Chen and Hurvich (2006) and Wang et al. (2015).

To analyze the power of the procedures, we focus on the triangular representation in DGP2 with $b = d$ so that the memory reduces to zero in the cointegrating relation. Again δ_m is set to 0.75. The results are shown in Table 2.2. In the following, we focus on the results for parameter constellations for which the tests have reasonable size properties. It should be noted that the procedure of Marmol and Velasco (2004) is not theoretically justified for any of the parameter constellations considered, since it assumes that the series are non-stationary, and $d_v > 1/2$.

It can be seen that the rank estimate of Nielsen and Shimotsu (2007) correctly identifies the presence of fractional cointegration even in relatively small samples. Since the estimate works well under the null hypothesis if ρ is low, it clearly outperforms its competitors in this situation. The power of the test of Hualde and Velasco (2008) is also high, but it suffers from similar size issues in case of strongly correlated short-run components.

Among the tests that are more widely applicable the approach of Souza et al. (2018) generates higher power than that of Wang et al. (2015) (except for $\rho = 0.99$), which in turn outperforms the approach of Chen and Hurvich (2006). Furthermore, it can be seen that the test of Souza et al. (2018) also outperforms more restrictive approaches such as those of Robinson (2008) and Nielsen (2010). For the test of Chen and Hurvich (2006) we can observe that the power is lower for $d = 0.7$ than for other values of d. Furthermore, the power becomes non-monotonic in T in some cases. This effect is likely to be caused by the fact that the order of differentiation required may be estimated incorrectly for intermediate values of d. The approach of Zhang et al. (2019) performs similar to that of Nielsen (2010).

With regard to the test of Robinson (2008), it is noteworthy that the power is considerably lower for $\rho = 0.9$ than it is for $\rho = 0.45$ or $\rho = 0.99$. Further simulation results on this V-shaped dependence pattern between the power of the test and ρ (not reported here) show that the test has no power if $\rho = 0.8$ and its power is very low in a neighborhood of this point. The size of this neighborhood shrinks to zero as the sample size increases.

The test of Marmol and Velasco (2004) develops good power for non-stationary values of d, even though its theoretical properties are not derived under this alternative.

Overall, we find that the rank estimation of Nielsen and Shimotsu (2007) performs best in identifying the correct order of fractional cointegration if the correlation between the series is low. Among the more broadly applicable methods the test of Souza et al. (2018) clearly performs best in terms of size and power.

The same experiment is repeated for a weakly cointegrated scenario where we set $b = d/3$. In this case the test of Marmol and Velasco (2004) becomes applicable for $d = 1$ and those of Nielsen (2010) and Zhang et al. (2019) are no longer applicable for $d = 1$. Table A.3 in the appendix A shows the results. It can be seen that the general ordering of the tests in terms of power remains the same in the weakly cointegrated case.

method	ρ	0			0.45			0.9			0.99		
	T/d	0.4	0.7	1	0.4	0.7	1	0.4	0.7	1	0.4	0.7	1
	d_v	0	0	0	0	0	0	0	0	0	0	0	0
NS07*	100	0.624	0.866	0.968	0.993	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	500	0.658	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1000	0.688	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	2500	0.803	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
CH06	100	0.646	0.912	0.696	0.599	0.896	0.628	0.530	0.872	0.547	0.532	0.873	0.524
	500	0.995	0.756	1.000	0.991	0.661	1.000	0.985	0.611	0.999	0.981	0.592	1.000
	1000	1.000	0.783	1.000	1.000	0.592	1.000	1.000	0.437	1.000	1.000	0.423	1.000
	2500	1.000	0.988	1.000	1.000	0.950	1.000	1.000	0.871	1.000	1.000	0.847	1.000
HV08	100	0.172	0.711	0.986	0.387	0.875	0.994	0.725	0.977	1.000	0.887	0.995	1.000
	500	0.883	1.000	1.000	0.986	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000
	1000	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	2500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
SRFB18	100	0.366	0.775	0.979	0.417	0.837	0.986	0.467	0.868	0.990	0.474	0.874	0.990
	500	0.800	1.000	1.000	0.856	1.000	1.000	0.895	1.000	1.000	0.904	1.000	1.000
	1000	0.964	1.000	1.000	0.983	1.000	1.000	0.992	1.000	1.000	0.991	1.000	1.000
	2500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
R08	100	0.109	0.533	0.860	0.054	0.368	0.770	0.018	0.183	0.601	0.013	0.158	0.573
	500	0.950	1.000	1.000	0.904	1.000	1.000	0.837	1.000	1.000	0.816	1.000	1.000
	1000	0.999	1.000	1.000	0.999	1.000	1.000	0.996	1.000	1.000	0.996	1.000	1.000
	2500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
WWC15	100	0.391	0.769	0.933	0.477	0.798	0.939	0.500	0.835	0.939	0.509	0.833	0.942
	500	0.634	0.928	0.989	0.688	0.942	0.988	0.733	0.943	0.989	0.742	0.945	0.990
	1000	0.721	0.959	0.992	0.761	0.968	0.994	0.806	0.966	0.990	0.802	0.970	0.993
	2500	0.807	0.979	0.997	0.826	0.982	0.997	0.856	0.983	0.999	0.873	0.982	0.998
ZRY18*	100	0.015	0.363	0.785	0.010	0.347	0.770	0.005	0.308	0.754	0.006	0.300	0.756
	500	0.045	0.978	1.000	0.029	0.966	1.000	0.016	0.958	1.000	0.014	0.955	1.000
	1000	0.072	1.000	1.000	0.035	0.999	1.000	0.017	0.999	1.000	0.014	0.998	1.000
	2500	0.091	1.000	1.000	0.042	1.000	1.000	0.018	1.000	1.000	0.015	1.000	1.000
N10	100		0.151	0.879		0.175	0.887		0.202	0.881		0.201	0.874
	500		0.962	1.000		0.965	1.000		0.966	1.000		0.967	1.000
	1000		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000
	2500		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000
MV04	100		0.162	0.439		0.141	0.611		0.297	0.919		0.548	0.975
	500		0.217	0.999		0.553	1.000		0.911	1.000		0.953	1.000
	1000		0.491	1.000		0.872	1.000		0.992	1.000		0.993	1.000
	2500		0.952	1.000		0.996	1.000		1.000	1.000		1.000	1.000

Table 2.3: Power (*rank estimation) with $b = d$ and $\delta_m = 0.75$ with the common-component model (DGP3).

Both tables so far are generated based on the triangular model (DGP2), but we are also interested in the performance based on the common-components model (DGP3). Those results are displayed in Table 2.3. It can be seen that there is a number of striking

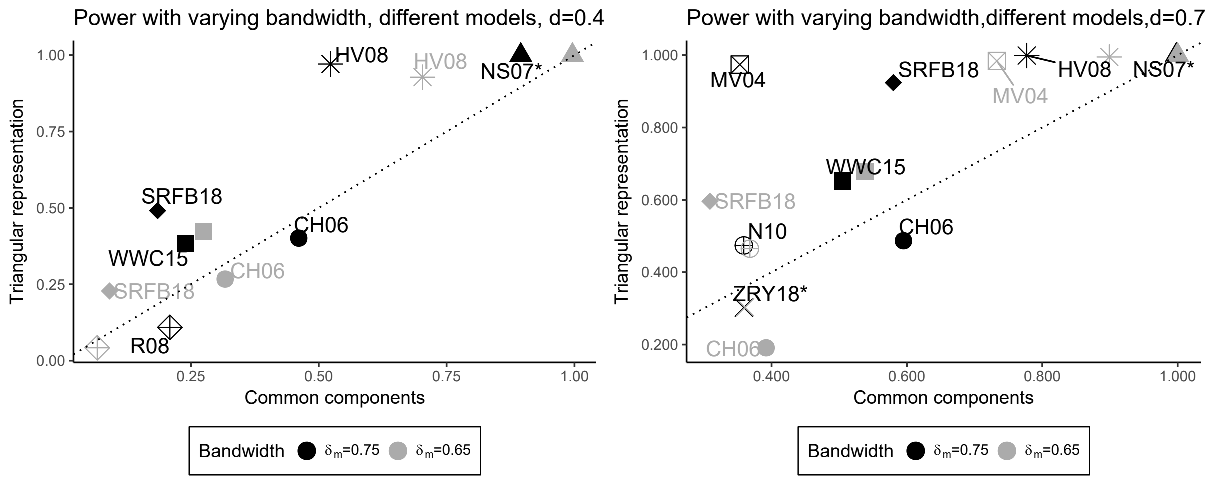

Figure 2.2: Power (*rank estimation) depending on model specification (DGP2 or DGP3) and bandwidth $\delta_m \in \{0.65, 0.75\}$ with $T = 1000$, $\rho = 0.45$, and $b = d/3$.

differences in the relative performance of the tests. For low values of d, the rank estimation procedure of Nielsen and Shimotsu (2007)/Robinson and Yajima (2002) loses precision. At the same time, the test of Chen and Hurvich (2006) becomes more powerful, so that overall the two procedures become comparable in terms of their ability to identify the correct rank. Unfortunately, the non-monotonicity of the test of Chen and Hurvich (2006) for intermediate values of d becomes even more apparent.

The test of Souza et al. (2018) still performs relatively well - especially for larger values of d. The same holds true for that of Wang et al. (2015) which reaches a relatively high power in smaller samples but approaches 1 only slowly.

With respect to the other tests, it can be seen that the variance ratio approach of Nielsen (2010) needs larger samples to develop power, but the test of Hualde and Velasco (2008) has very good power properties — also for low values of ρ where it maintains its size. The procedure of Zhang et al. (2019) turns out to perform better than that of Nielsen (2010) in very small samples.

As for the triangular model, the same analysis is repeated for a weakly cointegrated common-components model where $b = d/3$. The results are shown in Table A.4 in the appendix A. As before, the general ordering remains the same. However, it can be seen that the non-monotonicity of the Chen and Hurvich (2006) test for $d = 0.7$ disappears. This means the appearance of the effect depends on the cointegrating strength. In addition to that, the rank estimation procedure of Nielsen and Shimotsu (2007) completely loses its ability to identify the cointegrating relationship for low values of ρ.

To analyze the effect of the bandwidth choice on the power of the procedures, we conduct a similar analysis to that for the size in Figure 2.2. As before, black symbols represent results with $\delta_m = 0.75$ and gray symbols represent $\delta_m = 0.65$. The values of d and b are selected so that the power of the procedures tends to be low and changes in their behavior are easier to identify. While an increase of the bandwidth leads to a considerable power gain for the tests of Chen and Hurvich (2006), Robinson (2008), and Souza et al. (2018), the approaches of Marmol and Velasco (2004), Hualde and Velasco (2008) and Nielsen and Shimotsu (2007) have higher power with a smaller bandwidth — at least in the common-components model. This, however, might be due to the larger size distortions visible in Figure 2.1. The performance of the other approaches of Nielsen (2010) and Wang et al. (2015) is relatively independent of the bandwidth choice. For the test of Nielsen (2010) this is explained by the fact that the bandwidth only influences the estimate of d that determines the correct set of critical values. The test statistic itself does not depend on the bandwidth.

To explore the behavior of the tests in higher dimensional systems, we conduct another set of simulations in a triangular system of dimension $p = 3$ to determine the size, the power and the precision of estimates of the cointegrating rank r. The results are shown in Tables A.5, A.6 and A.7 in the appendix A. In general, it can be seen that general patterns observed for $p = 2$ are magnified. For example, the test of Chen and Hurvich (2006) becomes oversized in stationary systems also if the series are correlated. Noteworthy is that the test of Robinson (2008) shows very good size properties across all sample sizes, but requires larger sample sizes to accurately estimate the rank if $r > 0$, and it does overestimate the rank if $r = 1$ and $\rho = 0.99$. Furthermore, in larger samples of $T = 1000$ or more the rank estimation procedure of Zhang et al. (2019) shows a very good performance for all values of r.

Overall, in the bivariate setup we find that the methods of Robinson and Yajima (2002) and Hualde and Velasco (2008) have the highest power but they have size issues in case of strongly correlated short-run components. The test of Souza et al. (2018) tends to have the best power among the methods that have satisfactory size properties across all scenarios. In stationary systems with common components the test of Chen and Hurvich (2006) also has good power properties. For $p = 3$, where the test of Souza et al. (2018) is no longer applicable and that of Chen and Hurvich (2006) becomes liberal in stationary (triangular) systems, especially the rank estimation procedure of Zhang et al. (2019) for non-stationary systems can be recommended due to its robustness.

2.5 Conclusion

This review is written with the objective to provide guidance for the selection of methods in practical applications. We judge the methods based on i.) the range of values of d and b that are allowed, ii.) the ability to distinguish correctly between common trends and correlated innovations, and iii.) the performance across different DGPs — namely triangular systems as well as common-components models.

Based on our Monte Carlo studies, we find that some of the proposed approaches have weaknesses in their finite sample behavior in some empirically relevant scenarios — especially in presence of correlated short-run components. This concerns mostly the methods of Nielsen and Shimotsu (2007) (or Robinson and Yajima (2002)), Marmol and Velasco (2004), and Hualde and Velasco (2008). With regard to iii.), we find that the size properties of the tests in the triangular case and the common-components model is generally comparable. For the power of the tests, however, there are important differences between the two cases. In particular, the test of Chen and Hurvich (2006) has much better power for stationary systems under the common components specification, whereas the methods of Robinson and Yajima (2002) and Hualde and Velasco (2008) become worse in their ability to detect fractional cointegration.

Overall, we conclude that the test of Souza et al. (2018) for bivariate systems has the best properties, both theoretically and empirically, and is a good choice for the applied econometrician. It allows for the whole empirically relevant range of d and b, it is robust to correlation, and it provides comparable performance in both — triangular systems and common-components models.

Although the methods of Robinson (2008), Nielsen (2010), and Zhang et al. (2019) turn out to be robust to short-run correlation and are appealing due to their simplicity, they impose practically relevant restrictions on the permissible range of d and b, and they are outperformed by their competitors in terms of power in bivariate systems.

In higher dimensional systems, however, the test of Souza et al. (2018) is no longer applicable and that of Chen and Hurvich (2006) turns out to be liberal in finite samples from stationary processes. Here the test of Robinson (2008) can be recommended for stationary processes and the rank estimation procedure of Zhang et al. (2019) should be preferred for non-stationary systems if the cointegrating residuals can be expected to be stationary.

A Appendix

method	model T/d	triangular 0.4	0.7	1	common component 0.4	0.7	1	noise 0.4	0.7	1
NS07*	100	1.000	0.999	0.985	0.924	0.920	0.849	0.133	0.133	0.240
	500	1.000	1.000	0.963	0.712	0.684	0.644	0.000	0.000	0.049
	1000	1.000	1.000	0.937	0.335	0.345	0.448	0.000	0.000	0.027
	2500	1.000	1.000	0.883	0.009	0.013	0.241	0.000	0.000	0.008
CH06	100	0.076	0.049	0.035	0.089	0.048	0.028	0.093	0.087	0.027
	500	0.057	0.022	0.018	0.058	0.026	0.020	0.081	0.036	0.025
	1000	0.049	0.021	0.016	0.054	0.020	0.017	0.068	0.029	0.016
	2500	0.040	0.017	0.016	0.043	0.020	0.013	0.050	0.025	0.020
HV08	100	0.137	0.113	0.099	0.088	0.091	0.079	0.030	0.052	0.059
	500	0.251	0.121	0.072	0.130	0.078	0.062	0.041	0.048	0.053
	1000	0.334	0.133	0.065	0.186	0.075	0.062	0.051	0.054	0.060
	2500	0.413	0.121	0.057	0.238	0.077	0.057	0.069	0.052	0.054
SRFB18	100	0.108	0.113	0.143	0.115	0.114	0.140	0.112	0.119	0.146
	500	0.050	0.054	0.067	0.054	0.057	0.072	0.053	0.051	0.072
	1000	0.039	0.047	0.055	0.041	0.046	0.065	0.045	0.046	0.059
	2500	0.030	0.038	0.052	0.034	0.033	0.049	0.034	0.041	0.051
R08	100	0.044	0.045	0.040	0.038	0.050	0.039	0.053	0.059	0.075
	500	0.050	0.066	0.038	0.045	0.069	0.047	0.053	0.078	0.053
	1000	0.050	0.070	0.040	0.055	0.072	0.046	0.054	0.080	0.056
	2500	0.048	0.082	0.043	0.052	0.090	0.048	0.056	0.091	0.056
WWC15	100	0.077	0.090	0.095	0.083	0.093	0.095	0.082	0.092	0.099
	500	0.070	0.078	0.075	0.065	0.073	0.077	0.064	0.071	0.079
	1000	0.059	0.064	0.073	0.056	0.068	0.064	0.066	0.064	0.069
	2500	0.056	0.059	0.057	0.060	0.054	0.061	0.053	0.061	0.060
ZRY18*	100	0.061	0.565	0.577	0.067	0.564	0.594	0.068	0.591	0.615
	500	0.295	0.029	0.000	0.281	0.032	0.000	0.281	0.039	0.000
	1000	0.413	0.000	0.000	0.407	0.000	0.000	0.401	0.000	0.000
	2500	0.511	0.000	0.000	0.503	0.000	0.000	0.526	0.000	0.000
N10	100		0.055	0.063		0.046	0.059		0.044	0.057
	500		0.050	0.054		0.056	0.058		0.046	0.057
	1000		0.062	0.056		0.063	0.057		0.056	0.056
	2500		0.073	0.054		0.071	0.063		0.068	0.052
MV04	100		0.081	0.082		0.053	0.068		0.036	0.067
	500		0.132	0.080		0.084	0.067		0.051	0.062
	1000		0.117	0.067		0.073	0.061		0.040	0.057
	2500		0.093	0.062		0.068	0.052		0.044	0.049

Table A.1: Size (*rank estimation) with DGP2 ($b = 0$), DGP3 ($b = 0$), DGP1, $\rho = 0.45$, and $\delta_m = 0.75$.

method	model T/d	triangular 0.4	0.7	1	common components 0.4	0.7	1	noise 0.4	0.7	1
NS07*	100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997
	500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.994
	1000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.990
	2500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.976
CH06	100	0.075	0.036	0.032	0.078	0.035	0.028	0.078	0.041	0.035
	500	0.052	0.023	0.018	0.057	0.022	0.019	0.051	0.027	0.021
	1000	0.046	0.021	0.016	0.056	0.024	0.018	0.053	0.019	0.013
	2500	0.040	0.018	0.012	0.037	0.019	0.013	0.045	0.017	0.012
HV08	100	0.318	0.249	0.164	0.235	0.183	0.122	0.165	0.135	0.106
	500	0.630	0.336	0.110	0.491	0.226	0.091	0.349	0.156	0.076
	1000	0.693	0.352	0.098	0.584	0.251	0.074	0.418	0.164	0.063
	2500	0.753	0.385	0.077	0.655	0.257	0.066	0.517	0.169	0.059
SRFB18	100	0.103	0.123	0.131	0.109	0.122	0.132	0.114	0.119	0.138
	500	0.057	0.053	0.078	0.052	0.057	0.070	0.052	0.052	0.069
	1000	0.043	0.047	0.057	0.043	0.046	0.064	0.043	0.040	0.058
	2500	0.036	0.036	0.049	0.035	0.044	0.051	0.035	0.039	0.052
R08	100	0.044	0.043	0.035	0.045	0.045	0.040	0.044	0.045	0.040
	500	0.048	0.066	0.039	0.052	0.061	0.044	0.049	0.068	0.035
	1000	0.052	0.072	0.041	0.052	0.074	0.035	0.054	0.072	0.043
	2500	0.054	0.084	0.041	0.047	0.090	0.042	0.049	0.082	0.043
WWC15	100	0.080	0.099	0.095	0.080	0.094	0.098	0.080	0.090	0.095
	500	0.065	0.071	0.076	0.069	0.073	0.070	0.071	0.074	0.078
	1000	0.064	0.061	0.067	0.064	0.067	0.074	0.062	0.063	0.067
	2500	0.060	0.059	0.065	0.055	0.053	0.069	0.055	0.059	0.057
ZRY18*	100	0.062	0.549	0.566	0.064	0.557	0.577	0.066	0.548	0.565
	500	0.282	0.031	0.000	0.277	0.026	0.000	0.285	0.035	0.000
	1000	0.400	0.000	0.000	0.418	0.000	0.000	0.400	0.000	0.000
	2500	0.529	0.000	0.000	0.513	0.000	0.000	0.512	0.000	0.000
N10	100		0.056	0.063		0.049	0.059		0.055	0.069
	500		0.060	0.059		0.050	0.060		0.053	0.064
	1000		0.064	0.056		0.061	0.057		0.059	0.052
	2500		0.071	0.050		0.080	0.048		0.069	0.052
MV04	100		0.222	0.153		0.159	0.126		0.103	0.100
	500		0.344	0.150		0.232	0.117		0.157	0.092
	1000		0.321	0.148		0.231	0.107		0.145	0.078
	2500		0.298	0.123		0.197	0.096		0.130	0.070

Table A.2: Size (*rank estimation) with DGP2 ($b = 0$), DGP3 ($b = 0$), DGP1, $\rho = 0.9$, and $\delta_m = 0.75$.

method	ρ	0			0.45			0.9			0.99		
	T/d	0.4	0.7	1	0.4	0.7	1	0.4	0.7	1	0.4	0.7	1
	d_v	0.267	0.467	0.667	0.267	0.467	0.667	0.267	0.467	0.667	0.267	0.467	0.667
NS07*	100	0.946	0.973	0.991	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	500	0.921	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1000	0.848	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	2500	0.585	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
CH06	100	0.163	0.091	0.179	0.132	0.048	0.075	0.076	0.023	0.010	0.009	0.030	0.003
	500	0.347	0.353	0.868	0.258	0.178	0.660	0.345	0.057	0.404	0.771	0.025	0.284
	1000	0.524	0.718	0.990	0.401	0.487	0.947	0.645	0.302	0.924	0.995	0.292	0.953
	2500	0.815	0.983	1.000	0.728	0.925	1.000	0.961	0.918	1.000	1.000	0.932	1.000
HV08	100	0.192	0.354	0.433	0.342	0.585	0.679	0.649	0.944	0.962	0.863	0.988	0.991
	500	0.691	0.883	0.926	0.872	0.987	0.990	0.976	1.000	1.000	0.992	1.000	1.000
	1000	0.888	0.979	0.992	0.971	0.999	1.000	0.997	1.000	1.000	0.999	1.000	1.000
	2500	0.991	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
SRFB18	100	0.246	0.432	0.600	0.250	0.431	0.605	0.236	0.377	0.621	0.162	0.208	0.605
	500	0.352	0.748	0.952	0.352	0.766	0.957	0.349	0.715	0.967	0.247	0.449	0.942
	1000	0.506	0.926	0.999	0.491	0.924	0.997	0.488	0.901	0.997	0.364	0.668	0.991
	2500	0.774	0.997	1.000	0.784	0.999	1.000	0.766	0.996	1.000	0.666	0.909	1.000
R08	100	0.018	0.017	0.029	0.017	0.012	0.012	0.027	0.025	0.017	0.252	0.354	0.141
	500	0.095	0.391	0.765	0.046	0.157	0.479	0.029	0.055	0.224	0.880	0.948	0.448
	1000	0.244	0.746	0.980	0.109	0.384	0.810	0.030	0.091	0.383	0.990	0.995	0.600
	2500	0.596	0.985	1.000	0.337	0.803	0.990	0.034	0.180	0.560	1.000	1.000	0.748
WWC15	100	0.225	0.401	0.557	0.244	0.419	0.565	0.194	0.420	0.640	0.081	0.421	0.644
	500	0.350	0.579	0.750	0.326	0.595	0.759	0.328	0.663	0.828	0.279	0.737	0.863
	1000	0.380	0.647	0.810	0.383	0.652	0.810	0.375	0.755	0.872	0.363	0.808	0.898
	2500	0.458	0.712	0.847	0.461	0.737	0.865	0.461	0.825	0.909	0.456	0.873	0.925
ZRY18*	100	0.026	0.442	0.738	0.020	0.395	0.720	0.016	0.379	0.723	0.017	0.363	0.736
	500	0.097	0.626	0.053	0.069	0.626	0.052	0.053	0.623	0.052	0.051	0.606	0.046
	1000	0.123	0.348	0.001	0.093	0.359	0.000	0.066	0.357	0.001	0.065	0.340	0.000
	2500	0.189	0.062	0.000	0.134	0.057	0.000	0.087	0.063	0.000	0.072	0.058	0.000
N10	100		0.115	0.215		0.122	0.220	0.013	0.134	0.302		0.172	0.404
	500		0.303	0.355		0.297	0.372		0.438	0.531		0.598	0.680
	1000		0.468	0.414		0.474	0.412		0.678	0.614		0.855	0.732
	2500		0.706	0.463		0.715	0.469		0.906	0.685		0.983	0.806
MV04	100		0.139	0.368		0.328	0.580		0.664	0.865		0.789	0.948
	500		0.644	0.861		0.882	0.963		0.990	1.000		0.996	1.000
	1000		0.861	0.968		0.974	0.996		0.998	1.000		1.000	1.000
	2500		0.984	0.999		0.999	1.000		1.000	1.000		1.000	1.000

Table A.3: Power (*rank estimation) under with $b = d/3$ and $\delta_m = 0.75$ for the triangular model (DGP2).

method	ρ	0			0.45			0.9			0.99		
	T/d	0.4	0.7	1	0.4	0.7	1	0.4	0.7	1	0.4	0.7	1
	d_v	0.267	0.467	0.667	0.267	0.467	0.667	0.267	0.467	0.667	0.267	0.467	0.667
NS07*	100	0.347	0.490	0.707	0.963	0.984	0.994	1.000	1.000	1.000	1.000	1.000	1.000
	500	0.017	0.147	0.661	0.951	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1000	0.000	0.029	0.636	0.895	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	2500	0.000	0.000	0.599	0.664	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000
CH06	100	0.221	0.251	0.298	0.181	0.212	0.288	0.171	0.153	0.262	0.154	0.150	0.259
	500	0.389	0.481	0.903	0.317	0.345	0.875	0.288	0.268	0.837	0.267	0.245	0.822
	1000	0.539	0.715	0.994	0.461	0.595	0.987	0.393	0.454	0.978	0.394	0.443	0.980
	2500	0.805	0.981	1.000	0.723	0.956	1.000	0.649	0.905	1.000	0.632	0.896	1.000
HV08	100	0.042	0.080	0.151	0.101	0.170	0.239	0.317	0.418	0.511	0.548	0.661	0.757
	500	0.130	0.237	0.479	0.353	0.533	0.698	0.745	0.892	0.946	0.926	0.976	0.993
	1000	0.215	0.431	0.731	0.523	0.777	0.895	0.856	0.981	0.995	0.960	0.996	1.000
	2500	0.423	0.796	0.965	0.786	0.975	0.995	0.963	1.000	1.000	0.991	1.000	1.000
SRFB18	100	0.167	0.204	0.324	0.149	0.241	0.356	0.171	0.256	0.395	0.170	0.248	0.402
	500	0.134	0.326	0.663	0.151	0.387	0.721	0.170	0.456	0.774	0.170	0.454	0.782
	1000	0.159	0.489	0.889	0.185	0.580	0.919	0.218	0.635	0.943	0.225	0.649	0.952
	2500	0.262	0.805	0.996	0.322	0.875	1.000	0.385	0.916	1.000	0.402	0.930	1.000
R08	100	0.021	0.035	0.060	0.018	0.017	0.032	0.016	0.010	0.013	0.016	0.008	0.013
	500	0.132	0.504	0.845	0.081	0.392	0.777	0.061	0.305	0.700	0.050	0.279	0.685
	1000	0.294	0.848	0.988	0.209	0.762	0.980	0.156	0.659	0.966	0.138	0.643	0.964
	2500	0.668	0.995	1.000	0.545	0.988	1.000	0.447	0.972	1.000	0.420	0.973	1.000
WWC15	100	0.142	0.255	0.401	0.146	0.271	0.426	0.168	0.272	0.425	0.171	0.275	0.432
	500	0.187	0.407	0.619	0.211	0.430	0.632	0.225	0.458	0.632	0.226	0.431	0.635
	1000	0.209	0.479	0.708	0.240	0.505	0.712	0.261	0.514	0.697	0.253	0.513	0.703
	2500	0.249	0.577	0.778	0.282	0.580	0.778	0.293	0.595	0.784	0.317	0.606	0.785
ZRY18*	100	1.973	1.562	1.030	1.979	1.590	1.053	1.979	1.602	1.073	1.985	1.621	1.068
	500	1.909	0.656	0.046	1.931	0.651	0.042	1.951	0.647	0.050	1.948	0.663	0.046
	1000	1.878	0.356	0.001	1.907	0.353	0.001	1.933	0.352	0.000	1.942	0.341	0.001
	2500	1.806	0.062	0.000	1.881	0.054	0.000	1.923	0.064	0.000	1.924	0.065	0.000
N10	100		0.060	0.161		0.077	0.177		0.096	0.179		0.096	0.183
	500		0.197	0.297		0.204	0.285		0.230	0.304		0.241	0.301
	1000		0.341	0.341		0.359	0.345		0.369	0.352		0.374	0.343
	2500		0.586	0.392		0.605	0.396		0.602	0.405		0.610	0.401
MV04	100		0.043	0.129		0.058	0.196		0.234	0.452		0.559	0.751
	500		0.080	0.300		0.200	0.529		0.707	0.901		0.901	0.986
	1000		0.100	0.474		0.353	0.751		0.865	0.979		0.963	0.999
	2500		0.195	0.811		0.665	0.960		0.979	1.000		0.996	1.000

Table A.4: Power (*rank estimation) with $b = d/3$ and $\delta_m = 0.75$ with the common-component model (DGP 3).

method	ρ / T/d	0: 0.4	0: 0.7	0: 1	0.45: 0.4	0.45: 0.7	0.45: 1	0.9: 0.4	0.9: 0.7	0.9: 1	0.99: 0.4	0.99: 0.7	0.99: 1
NS07*	100	0.002	0.003	0.059	0.375	0.385	0.498	1.000	1.000	0.993	1.000	1.000	1.000
	500	0.000	0.000	0.001	0.000	0.001	0.107	1.000	1.000	0.980	1.000	1.000	1.000
	1000	0.000	0.000	0.000	0.000	0.000	0.057	1.000	1.000	0.967	1.000	1.000	1.000
	2500	0.000	0.000	0.000	0.000	0.000	0.022	1.000	1.000	0.928	1.000	1.000	1.000
CH06	100	0.477	0.238	0.016	0.214	0.139	0.020	0.183	0.066	0.023	0.170	0.058	0.024
	500	0.472	0.094	0.050	0.181	0.049	0.031	0.147	0.034	0.031	0.143	0.035	0.030
	1000	0.404	0.090	0.052	0.164	0.041	0.033	0.126	0.033	0.028	0.122	0.033	0.022
	2500	0.300	0.081	0.044	0.128	0.035	0.025	0.098	0.029	0.021	0.095	0.031	0.019
HV08	100	0.005	0.046	0.104	0.040	0.096	0.154	0.488	0.549	0.484	0.883	0.889	0.853
	500	0.002	0.034	0.079	0.043	0.077	0.103	0.562	0.508	0.356	0.891	0.900	0.799
	1000	0.002	0.037	0.075	0.061	0.068	0.096	0.596	0.501	0.299	0.901	0.882	0.756
	2500	0.003	0.030	0.066	0.065	0.060	0.080	0.614	0.481	0.250	0.906	0.885	0.727
R08	100	0.114	0.109	0.052	0.036	0.047	0.030	0.034	0.038	0.035	0.035	0.040	0.033
	500	0.192	0.190	0.060	0.050	0.062	0.038	0.047	0.060	0.039	0.046	0.064	0.041
	1000	0.213	0.224	0.049	0.056	0.064	0.043	0.049	0.065	0.036	0.049	0.072	0.039
	2500	0.232	0.259	0.050	0.049	0.083	0.040	0.050	0.083	0.039	0.053	0.088	0.039
ZRY18*	100	0.003	0.161	0.528	0.002	0.157	0.528	0.006	0.156	0.512	0.012	0.165	0.385
	500	0.061	0.209	0.002	0.067	0.155	0.001	0.053	0.129	0.001	0.072	0.091	0.001
	1000	0.134	0.004	0.000	0.151	0.002	0.000	0.149	0.001	0.000	0.141	0.001	0.000
	2500	0.356	0.000	0.000	0.372	0.000	0.000	0.373	0.000	0.000	0.325	0.000	0.000
MV04	100		0.036	0.054		0.042	0.065		0.126	0.124		0.382	0.272
	500		0.030	0.053		0.047	0.055		0.152	0.089		0.467	0.247
	1000		0.036	0.049		0.050	0.054		0.129	0.078		0.461	0.211
	2500		0.035	0.053		0.042	0.055		0.122	0.075		0.428	0.188
N10	100		0.033	0.048		0.038	0.058		0.053	0.061		0.058	0.065
	500		0.048	0.059		0.049	0.049		0.062	0.059		0.063	0.052
	1000		0.056	0.060		0.063	0.055		0.068	0.054		0.073	0.061
	2500		0.075	0.053		0.075	0.059		0.089	0.058		0.083	0.053

Table A.5: Size (*rank estimation) based on DGP1 with $p = 3$ and $\delta_m = 0.75$.

method	ρ T/d	0 0.4	0 0.7	0 1	0.45 0.4	0.45 0.7	0.45 1	0.9 0.4	0.9 0.7	0.9 1	0.99 0.4	0.99 0.7	0.99 1
							***r* = 1**						
CH06	100	0.650	0.407	0.249	0.408	0.358	0.082	0.083	0.182	0.009	0.027	0.005	0.004
	500	0.999	0.491	1.000	0.997	0.259	1.000	0.999	0.171	1.000	0.254	0.059	0.561
	1000	1.000	0.853	1.000	1.000	0.616	1.000	1.000	0.525	1.000	0.815	0.323	0.991
	2500	1.000	0.998	1.000	1.000	0.987	1.000	1.000	0.969	1.000	0.996	0.957	1.000
HV08	100	0.691	0.985	1.000	0.721	0.979	1.000	0.695	0.980	1.000	0.616	0.982	1.000
	500	0.999	1.000	1.000	0.999	1.000	1.000	0.995	1.000	1.000	0.994	1.000	1.000
	1000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	2500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
R08	100	0.039	0.133	0.212	0.010	0.014	0.039	0.047	0.065	0.038	0.285	0.188	0.073
	500	0.686	0.960	0.989	0.123	0.324	0.766	0.192	0.230	0.159	0.971	0.762	0.279
	1000	0.955	0.998	1.000	0.325	0.712	0.981	0.308	0.325	0.286	0.999	0.912	0.415
	2500	0.999	1.000	1.000	0.767	0.984	1.000	0.565	0.499	0.443	1.000	0.988	0.597
MV04	100		0.454	0.977		0.686	0.991		0.762	0.994		0.771	0.992
	500		0.995	1.000		1.000	1.000		1.000	1.000		1.000	1.000
	1000		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000
	2500		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000
N10	100		0.214	0.832		0.149	0.785		0.110	0.712		0.109	0.716
	500		0.911	1.000		0.833	1.000		0.756	1.000		0.726	1.000
	1000		0.999	1.000		0.996	1.000		0.994	1.000		0.994	1.000
	2500		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000
							***r* = 2**						
CH06	100	0.565	0.900	0.508	0.437	0.891	0.482	0.402	0.832	0.545	0.401	0.829	0.559
	500	0.994	0.753	1.000	0.969	0.842	0.997	0.926	0.936	0.985	0.927	0.952	0.982
	1000	1.000	0.762	1.000	0.999	0.651	1.000	0.997	0.809	1.000	0.996	0.828	1.000
	2500	1.000	0.983	1.000	1.000	0.642	1.000	1.000	0.449	1.000	1.000	0.464	1.000
HV08	100	0.604	0.941	0.997	0.729	0.946	0.997	0.924	0.987	0.999	0.982	0.999	1.000
	500	0.997	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	2500	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
R08	100	0.052	0.345	0.780	0.006	0.028	0.290	0.009	0.002	0.086	0.132	0.024	0.067
	500	0.934	1.000	1.000	0.422	0.964	1.000	0.037	0.009	0.620	0.865	0.342	0.508
	1000	0.999	1.000	1.000	0.871	1.000	1.000	0.059	0.015	0.793	0.989	0.550	0.636
	2500	1.000	1.000	1.000	1.000	1.000	1.000	0.110	0.038	0.988	1.000	0.806	0.757
MV04	100		0.366	0.879		0.465	0.934		0.841	0.993		0.994	1.000
	500		0.763	0.999		0.813	0.999		0.982	1.000		1.000	1.000
	1000		0.847	1.000		0.879	1.000		0.998	1.000		1.000	1.000
	2500		0.915	1.000		0.936	1.000		1.000	1.000		1.000	1.000
N10	100		0.190	0.860		0.066	0.690		0.037	0.527		0.035	0.504
	500		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000
	1000		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000
	2500		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000

Table A.6: Power with $p = 3$, $b = d$, and $\delta_m = 0.75$ for the triangular model.

method	ρ T/d	0 0.4	0 0.7	0 1	0.45 0.4	0.45 0.7	0.45 1	0.9 0.4	0.9 0.7	0.9 1	0.99 0.4	0.99 0.7	0.99 1
						$r = 1$							
NS07	100	0.991	0.997	0.998	0.958	0.966	0.903	0.000	0.000	0.009	0.000	0.000	0.000
	500	1.000	1.000	1.000	1.000	1.000	0.989	0.000	0.000	0.027	0.000	0.000	0.000
	1000	1.000	1.000	1.000	1.000	1.000	0.995	0.000	0.000	0.044	0.000	0.000	0.000
	2500	1.000	1.000	1.000	1.000	1.000	0.999	0.000	0.000	0.077	0.000	0.000	0.000
CH06	100	0.597	0.330	0.247	0.390	0.321	0.089	0.098	0.180	0.036	0.039	0.280	0.032
	500	0.948	0.468	0.981	0.984	0.249	0.990	0.910	0.125	0.743	0.748	0.128	0.439
	1000	0.951	0.841	0.982	0.989	0.612	0.987	0.888	0.423	0.835	0.186	0.174	0.016
	2500	0.969	0.985	0.987	0.996	0.982	0.994	0.879	0.860	0.852	0.005	0.018	0.003
R08	100	0.190	0.247	0.263	0.041	0.085	0.113	0.054	0.098	0.074	0.317	0.183	0.079
	500	0.524	0.568	0.839	0.187	0.615	0.883	0.205	0.279	0.370	0.039	0.070	0.186
	1000	0.527	0.522	0.844	0.567	0.837	0.903	0.305	0.338	0.415	0.002	0.019	0.212
	2500	0.446	0.463	0.861	0.879	0.832	0.902	0.425	0.409	0.460	0.000	0.001	0.207
ZRY18	100	0.001	0.053	0.308	0.000	0.053	0.299	0.008	0.049	0.285	0.003	0.054	0.263
	500	0.008	0.939	1.000	0.003	0.937	1.000	0.014	0.751	1.000	0.001	0.943	0.858
	1000	0.028	0.999	1.000	0.007	0.999	1.000	0.010	0.925	1.000	0.007	0.999	0.933
	2500	0.091	1.000	1.000	0.023	1.000	1.000	0.012	0.994	1.000	0.013	0.996	0.999
N10	100		0.201	0.810		0.143	0.767		0.107	0.700		0.105	0.706
	500		0.900	0.973		0.827	0.979		0.754	0.983		0.723	0.983
	1000		0.982	0.974		0.988	0.972		0.990	0.975		0.991	0.974
	2500		0.970	0.962		0.982	0.967		0.991	0.972		0.991	0.974
						$r = 2$							
NS07	100	0.761	0.918	0.981	0.994	0.996	0.993	1.000	1.000	1.000	1.000	1.000	1.000
	500	0.822	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1000	0.867	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	2500	0.943	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
CH06	100	0.266	0.834	0.321	0.058	0.607	0.375	0.002	0.290	0.479	0.000	0.226	0.494
	500	0.978	0.745	1.000	0.888	0.838	0.997	0.882	0.932	0.985	0.918	0.949	0.982
	1000	1.000	0.757	1.000	0.998	0.645	1.000	0.997	0.805	1.000	0.996	0.823	1.000
	2500	1.000	0.983	1.000	1.000	0.639	1.000	1.000	0.443	1.000	1.000	0.457	1.000
R08	100	0.002	0.011	0.088	0.001	0.000	0.006	0.021	0.077	0.058	0.618	0.565	0.213
	500	0.438	0.995	1.000	0.019	0.204	0.854	0.173	0.619	0.362	1.000	0.993	0.527
	1000	0.940	1.000	1.000	0.188	0.845	0.999	0.324	0.858	0.536	1.000	1.000	0.656
	2500	1.000	1.000	1.000	0.831	1.000	1.000	0.657	0.988	0.660	1.000	1.000	0.763
ZRY18	100	0.012	0.342	0.769	0.003	0.233	0.746	0.007	0.169	0.705	0.020	0.153	0.701
	500	0.034	0.973	1.000	0.003	0.918	1.000	0.002	0.826	1.000	0.023	0.824	1.000
	1000	0.045	0.999	1.000	0.003	0.994	1.000	0.001	0.980	1.000	0.007	0.975	1.000
	2500	0.061	1.000	1.000	0.000	1.000	1.000	0.000	1.000	1.000	0.000	1.000	1.000
N10	100		0.048	0.575		0.010	0.314		0.005	0.180		0.005	0.164
	500		0.806	1.000		0.611	1.000		0.490	0.999		0.475	0.999
	1000		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000
	2500		1.000	1.000		1.000	1.000		1.000	1.000		1.000	1.000

Table A.7: Fraction of correct rank estimation with $p = 3$, $b = d$, and $\delta_m = 0.75$ for the triangular model.

Chapter 3

Fractional Cointegration and EMU Government Bond Market Integration

Co-authored with Christian Leschinski and Philipp Sibbertsen.

3.1 Introduction

We show that even though the yields on long-term government bonds of the major EMU countries were largely co-moving prior to the crisis, the degree of market integration exhibited considerable variation over time. This time variation is related to the stock market sentiment. During bear-market periods, there was no equilibrium mechanism between the yields that would have ensured the subsistence of a stable relationship.

In contrast to our findings, it is nearly universally accepted in the literature on the integration of EMU bond markets that the introduction of the Euro led to essentially complete integration of EMU bond markets that ended with the advent of the subprime mortgage crisis. This was found empirically by contributions such as Ehrmann et al. (2011), Baele et al. (2004), Pozzi and Wolswijk (2012), Christiansen (2014), and Ehrmann and Fratzscher (2017) and is also implicitly assumed by studies on the determinants of yield spreads between government bonds in the euro area, such as Beber et al. (2008), who treat the yield spreads as stationary variables.

The difference between these studies and ours is rooted in the fact that we take a very different perspective from previous contributions to the literature. Instead of focusing on the shock transmission among the spreads or the relative importance of global and local factors, we test for the existence of an equilibrium among the interest rates themselves. Our study adopts a definition of market integration that is widely used in other areas such as the analysis of commodity markets. This definition is directly based on the law of one price and closely connected to the existence of a (fractional) cointegrating relationship. Using it enables us to draw conclusions about market equilibria by applying a wide set of modern methods for the analysis of fractionally cointegrated systems.

Utilizing this direct correspondence between economic theories and statistical concepts allows us to make several major contributions. First, we establish that the EMU bond markets were integrated during bull markets but disintegrated in bear markets. This is achieved directly by testing for pairwise fractional cointegration among the yields and in-directly by considering the persistence of the yield spreads. The yield spreads between the countries are the cointegrating residuals obtained by imposing the cointegrating vector $(1, -1)'$ on the yields. The persistence of the spreads is therefore directly related to the existence of an equilibrium relationship among the yields. Further insights into the dynamics of integration and disintegration in the EMU bond markets are therefore obtained from a rolling window analysis of the memory of the spreads.

The second contribution is to provide insights into the sources of the time-varying persistence in the spreads. To this end, the estimated degree of persistence is regressed on a set of variables that proxy for market sentiment, risk, and risk aversion. The analysis not only confirms the relationship between integration and bull and bear markets, but also shows that the degree of market integration is driven by market risk.

Finally, the third contribution is to provide insights into the possible economic origins of the observed time variation in market integration. Here, we make use of the fact that the yields are the sum of the risk-free rate, the default risk premium, and the liquidity risk premium of the respective country. Due to the special situation in the EMU where (due to the common currency area) the risk-free rate is the same for all countries and Germany is typically assumed to be risk-free, the spreads relative to Germany are solely determined by the default risk premium and the liquidity risk premium. Standard results on the properties of linear combinations of long-memory time series from Chambers (1998) then give rise to two possible mechanisms that can generate the observed time variation in the persistence of the spreads. The first one is that markets expect economic and fiscal divergence within the EMU area in bear markets, whereas they are optimistic about convergence within the euro area in bull markets. The second possible explanation is that markets always assume that divergence is a possibility, but the default risk premium exhibits so little variation in good times that the persistence of the spreads is dominated by the liquidity premium. In contrast to that, in bad times, when risk and risk aversion are high, the persistence of the spreads is dominated by the default risk premium, due to its increased variability.

Both of these arguments lead to the conclusion that (at least in crisis times) the pricing of EMU government bonds implied the possibility of macroeconomic and fiscal divergence between the EMU countries, long prior to the EMU debt crisis. Also, differences between the core and periphery countries are already visible during previous bear-market periods.

The rest of the paper is structured as follows. Section 3.2 provides a discussion of market integration and a discussion of fractional integration and cointegration. Subsequently, Section 3.3 describes the data set and discusses the definition of bull and bear markets. Section 3.4 contains the empirical analysis including formal tests for market integration separately for bull and bear markets, rolling window estimates of the persistence of the spreads, and an analysis of the drivers of the degree of market integration. Finally, Section 3.5 concludes.

3.2 Market Integration, Fractional Integration, and Fractional Cointegration

In international finance, measures for market integration are typically based on factor models for the returns. The most widely adopted approaches in recent years are those of Bekaert and Harvey (1995) and Pukthuanthong and Roll (2009). Bekaert and Harvey (1995) consider two markets to be integrated if their movement is completely determined by global factors, whereas local factors (that are specific to individual countries) are not priced. Similarly, Pukthuanthong and Roll (2009) consider the explanatory power of a multifactor model as a measure for market integration. While both of these measures are intuitive for asset returns, they lack a rigorous foundation in economic theory and they are not readily applicable to bond yields that are typically found to have unit roots.

Here, we therefore consider a different definition that is commonly used for the analysis of commodity markets. According to this definition markets for different goods that are close substitutes, or markets for the same good that are spatially separated are considered to be (economically) integrated with each other if the law of one price (LOP) applies. In the strict sense, the LOP requires that there is a correction mechanism (such as arbitrage) in place that enforces the stability of an equilibrium relationship, and that the form of this equilibrium is such that prices in both markets are exactly the same. The weaker definition of partial market integration only requires the existence of a stable equilibrium relationship but not exact equality between the prices.

For non-stationary prices, this definition is often tied to the concept of cointegration (cf. Ravallion (1986), Ardeni (1989)), since cointegration implies the existence of an equilibrium relationship between unit root processes. In the classical $I(0)/I(1)$ framework, deviations from this equilibrium have to be weakly persistent in the sense that they are stationary and have short memory. This, however, is an unnecessary restriction, since an equilibrium relationship only requires deviations from the mean to be transitory in the sense that they are mean reverting.

We therefore allow for fractional cointegration when testing for (partial) market integration and consider a bivariate system of the form

$$X_{1t} = c_1 + \xi_1 Y_t + \Delta^{-(d-b_1)} u_{1t} \mathbb{1}_{\{t>0\}} \tag{3.1}$$

$$X_{2t} = c_2 + \xi_2 Y_t + \Delta^{-(d-b_2)} u_{2t} \mathbb{1}_{\{t>0\}} \tag{3.2}$$

$$Y_t = \Delta^{-d} e_t \mathbb{1}_{\{t>0\}}, \tag{3.3}$$

where the coefficients c_1, c_2, ξ_1, and ξ_2 are finite, $0 \leq b_1, b_2 \leq d$, L is the lag-operator, the fractional differences $\Delta^d Y_t = (1-L)^d Y_t$ are defined in terms of generalized binomial coefficients such that

$$(1-L)^d = \sum_{k=0}^{\infty} \binom{d}{k} (-1)^k L^k = \sum_{k=0}^{\infty} \pi_k L^k,$$
$$\text{with} \quad \binom{d}{k} = \frac{d(d-1)(d-2)\ldots(d-(k-1))}{k!},$$

and e_t and $u_t = (u_{1t}, u_{2t})'$ are martingale difference sequences. The memory of both X_{1t} and X_{2t} is determined by Y_t so that they are integrated of the same order d, denoted by $X_t \sim I(d)$, where the memory parameter is restricted to $d \in (0,1]$ and $X_t = (X_{1t}, X_{2t})'$. Since it is assumed that $u_{1t} = u_{2t} = e_t = 0$ for all $t \leq 0$, the processes under consideration are fractionally integrated of type-II. For a detailed discussion of type-I and type-II processes confer Marinucci and Robinson (1999). The spectral density of X_t can be approximated by

$$f_X(\lambda) \sim \Lambda_j(d)\, G\, \overline{\Lambda_j(d)}, \quad \text{as } \lambda \to 0^+, \tag{3.4}$$

where G is a real, symmetric, finite, and positive definite matrix, $\Lambda_j(d) = \text{diag}\left(\lambda^{-d} e^{i\pi d/2}, \lambda^{-d} e^{i\pi d/2}\right)$ is a 2×2 diagonal matrix and $\overline{\Lambda_j(d)}$ is its complex conjugate transpose. The periodogram of a process X_t is defined through the discrete Fourier transform $w_X(\lambda_j) = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{T} X_t e^{i\lambda_j t}$ as $I_X(\lambda_j) = w_X(\lambda_j)\overline{w_X(\lambda_j)}$, with Fourier frequencies $\lambda_j = 2\pi j/T$ for $j = 1, ..., \lfloor T/2 \rfloor$, where the operator $\lfloor \cdot \rfloor$ returns the integer part of its argument.

The two series X_{1t} and X_{2t} are said to be fractionally cointegrated, if there exists a linear combination

$$\beta' X_t = v_t,$$

so that the cointegrating residuals v_t are fractionally integrated of order $I(d-b)$ for some $0 < b \leq d$. Obviously, for the model in equations (3.1) to (3.3), this is the case for every multiple of the vector $\left(1, -\frac{\xi_1}{\xi_2}\right)'$ and $b = \min(b_1, b_2)$.

Here, we conclude that markets for EMU government bonds that could be considered as close substitutes are (partially) economically integrated if the yields are fractionally cointegrated with each other. From the definition above, this is the case if there exists an equilibrium relationship between the yields (X_{1t} and X_{2t}) so that the persistence of deviations from the equilibrium denoted by v_t is reduced compared to that of the individual series.[4]

[4] A similar approach that uses fractional cointegration to test for market integration was recently adopted by García-Enríquez et al. (2014).

In the following, we will test this hypothesis in two different ways. First, we apply a number of tests for the null hypothesis of no fractional cointegration among the yields of long-term EMU government bonds. The methods used are semiparametric and do not impose any assumptions on the short-run behavior of the series, apart from mild regularity conditions. This approach has the advantage that we can avoid spurious findings that might arise due to misspecifications. Research on semiparametric tests for fractional cointegration has been an active field in recent years and there is a variety of competing approaches.

The first group of tests is based on the fact that the rank of the matrix G in (3.4) is reduced for fractionally cointegrated systems. This property is used by the rank estimation criterion of Nielsen and Shimotsu (2007) that extends the approach of Robinson and Yajima (2002) to nonstationary processes, the spectral regression approach of Souza et al. (2018), and the Hausman-type test of Robinson (2008). Robinson and Yajima (2002) and Nielsen and Shimotsu (2007) use the singularity of the G matrix in case of cointegration to propose an information criterion that is based on the eigenvalues of an estimate $\hat{G}$.

Souza et al. (2018) use the fractionally differenced process $\Delta^d X_t$ and the fact that the determinant $D_{\Delta^d}(\lambda)$ of $f_{\Delta^d X}(\lambda)$ is of the form $D_{\Delta^d}(\lambda) \sim \tilde{G}|1 - e^{-i\lambda}|^{2b}$, where $\tilde{G}$ is a scalar constant and $0 < \tilde{G} < \infty$. An estimate of b can therefore be obtained via a log-periodogram regression and the hypothesis that $b = 0$ can be tested based on the resulting estimate.

The test of Robinson (2008) is based on the fact that univariate estimates of d for the component series X_{1t} and X_{2t} are consistent both in the absence and in the presence of fractional cointegration. In contrast to that, the objective function of multivariate local Whittle estimates for the memory in X_t depends on the inverse of G, so that the estimator is inconsistent under fractional cointegration. On the other hand, the estimator is more efficient in absence of fractional cointegration, due to its multivariate nature. This provides the basis for a Hausman-type test.

A second group of tests is residual-based, since the cointegrating residuals v_t have reduced memory of order $d - b$ instead of d if a fractional cointegrating relationship exists. Chen and Hurvich (2006) and Wang et al. (2015) provide tests that rely on this property.

The test of Wang et al. (2015) is based on the sum over the fractionally differenced process $\Delta^{\hat{d}_v} X_{2t}$, where $\hat{d}_v$ is an estimate of the memory from the cointegrating residuals obtained using a consistent estimator for the cointegrating vector β such as the narrow-band least squares estimator of Robinson (1994), Robinson and Marinucci (2003), and Christensen and Nielsen (2006), among others. In contrast to that, the test of Chen and Hurvich (2006) is directly based on $\hat{d}_v$, but the cointegrating space is estimated by the eigenvectors of the averaged and tapered periodogram matrix local to the origin.

A third group of tests proposed by Marmol and Velasco (2004) and Hualde and Velasco (2008) relies on the behavior of pairs of estimators for the cointegrating vector β. These pairs include one estimator that is only consistent under the null hypothesis of no fractional cointegration and one estimator that is only consistent under fractional cointegration. While the test of Marmol and Velasco (2004) has a non-standard distribution, the test of Hualde and Velasco (2008) utilizes the GLS estimates of Robinson and Hualde (2003) and has a chi-square distribution.

Finally, Nielsen (2010) suggests a variance ratio test. The test statistic is based on the sum of the eigenvalues of the variance-covariance matrix of the series multiplied with the inverse of the variance-covariance matrix of the fractionally differenced series. This is because the eigenvalues associated with eigenvectors that are in a cointegrating direction are $O_P(1)$, whereas the eigenvalues corresponding to eigenvectors in non-cointegrating directions are $o_P(1)$, for $d - b < 1/2$.

If a cointegrating relationship is found with one of these procedures, the degree of (market) integration corresponds to b — the strength of the relationship. This is because b determines the speed of adjustment towards the equilibrium. The higher b, the stronger the degree of integration and the faster is the adjustment after shocks that cause deviations from the equilibrium. In the cases of Nielsen and Shimotsu (2007) and Robinson (2008), where the methods themselves do not produce an estimate of the cointegrating strength, we estimate it by the difference between the memory of the yields and the memory of the spread. This is because the spreads are the cointegrating residuals obtained by imposing the cointegrating vector $(1, -1)'$, as discussed in detail below.

Using domain specific knowledge about the behavior of the yields in the common currency area also allows us to adopt a second approach and test for cointegration based on simple estimations of the memory parameters in the yield spreads. We denote the interest rate yield on bonds of country i in period t by y_{it} for $i = 1, ..., N$ and $t = 1, ..., T$. The spreads s_{it} are usually formed relative to the yield of the German bonds

$$s_{it} = y_{it} - y_t^{GER}. \tag{3.5}$$

It is commonly assumed that the interest rates of country i can be decomposed into

$$y_{it} = r_t^f + \delta_{it} + l_{it}, \tag{3.6}$$

where r_t^f is the risk-free interest rate, and δ_{it} and l_{it} are the risk premiums for the default risk and liquidity risk of country i. The risk-free rate is the same across countries due to the common currency area. If Germany — the benchmark country — is assumed to have no default risk and no liquidity risk, so that $y_t^{GER} = r_t^f$, it follows that

$$s_{it} = \delta_{it} + l_{it}. \tag{3.7}$$

Therefore, the spreads are the risk premiums associated with the liquidity and default risk of the respective country. If Germany is not assumed to be risk-free, δ_{it} and l_{it} are interpreted as risk premium differentials between the respective country and Germany. However, if the risk of Germany and its variation are low compared to that of the respective country, the behavior of the differentials will still be dominated by the risk premiums of the country. We therefore maintain the assumption that Germany is risk-free to simplify the verbal description of the results.

The risk-free interest rate r_t^f in (3.6) is driven by expected macroeconomic factors such as GDP-growth, inflation rates, and interest rates, and it is widely found to be $I(1)$ (cf. for example Stock and Watson (1988), Mishkin (1992), Chen and Hurvich (2003) and Nielsen (2010))[5]. That means y_{it} and y_t^{GER} can only be cointegrated if r_t^f is removed from the linear combination $\beta'(y_{it}, y_t^{GER})'$, as it is the case in the spreads in (3.7). Forming the spreads according to (3.5) therefore means to impose the cointegrating vector $\beta = (1, -1)'$ on the yields, which is the only possible cointegrating direction according to the theoretical arguments outlined above. The spreads are therefore the cointegrating residuals. Since in this case the cointegrating residuals are not affected by estimation error, we can apply a simple test for the null hypothesis that the memory $d(s_{it})$ of the spread s_{it} of country i at time t is equal to one to test for the null hypothesis of no fractional cointegration among the yields. Formally, we test

$$\begin{aligned} & H_0: \quad d(s_{it}) = 1 \\ \text{versus} \quad & H_1: \quad d(s_{it}) < 1, \end{aligned}$$

for all i and t. If this hypothesis can be rejected, this is statistical evidence for market integration.

To gain a deeper economic understanding of the mechanisms driving market integration and disintegration, reconsider the decomposition of the spreads in equation (3.7). Since the spreads are the cointegrating residuals between the yields, their persistence determines whether there is an equilibrium or not. According to equation (3.7), the spreads consist of two components — the liquidity risk premium l_{it} and the default risk premium δ_{it}. Since credit default swap data is not available for most of the time period before the

[5] Since it is implausible from an economic perspective that interest rates should become very large or very negative, they are often treated as being $I(0)$. Since this is not supported by the finite sample behavior of the yield series, imposing such an assumption will provide an imprecise asymptotic approximation and likely invalid statistical inference.

subprime mortgage crisis, we cannot use this information to disentangle the default and liquidity risk premiums as for example in Longstaff et al. (2005).

We can, however, draw some conclusions based on properties of long-memory processes. Denote the memory of the default risk premium for country i at time t by $d(\delta_{it})$ and let $d(l_{it})$ denote the memory of the liquidity risk premium. To see how the persistence of the aggregate s_{it} relates to the components δ_{it} and l_{it}, the properties of linear combinations of long-memory time series have to be considered. With constant unconditional mean and variance of the component series, it was shown by Chambers (1998) that the memory of a linear combination of long-memory processes is determined by the most persistent series in the combination. For two long-memory series a_t and b_t with memory parameters d_a and d_b this means that $c_t = a_t + b_t$ has long memory of order $d_c = \max\{d_a, d_b\}$. The memory of the spreads s_{it} is therefore either $d(\delta_{it})$, or $d(l_{it})$, according to which is larger.

The reasoning behind this result of Chambers (1998) is as follows. If a_t and b_t are mutually independent, the spectral density of c_t local to the origin is given by

$$f_c(\lambda) \sim G_a|\lambda|^{-2d_a} + G_b|\lambda|^{-2d_b},$$

as $\lambda \to 0$. Here, G_a and G_b denote the long-run variance of the short-memory components in the respective series. Obviously, both of the components on the right-hand side generate poles and the smaller one is dominated by the larger one.[6]

These results are based on the assumption that G_a and G_b are fixed, finite, and positive. In practice, however, there could arise situations in which one of the components is very small compared to the other one. A more fitting theoretical framework for such a situation would be to assume that $G_a/G_b \to 0$, as $T \to \infty$. In this case, the ratio of the long-run variances of the short-memory components depends on the sample size and goes to zero. More formally, let $c_t = a_t + b_t$, with $d_a > d_b$ and $G_a(T)/G_b(T) = o\left(T^{-2(d_a-d_b)}\right)$, then $d_c = d_b$, asymptotically. This implies that in practice the estimated degree of persistence in the spreads s_{it} will be a convex combination of $d(l_{it})$ and $d(\delta_{it})$ that depends on the relative scale of the variation of the two risk premiums.

Most importantly, if the persistence of the spreads is high and that of the liquidity premium is low, than the behavior of the default premium δ_{it} has to be the main driver of the spreads.

3.3 Data and Definition of Bull and Bear Markets

Our analysis is based on the daily interest rates on 10-year maturity benchmark government bonds of eleven EMU countries. As is customary in the literature, we refer to

[6] If a_t and b_t are dependent, there is also an interaction term in $f_c(\lambda)$, but the mechanism remains the same.

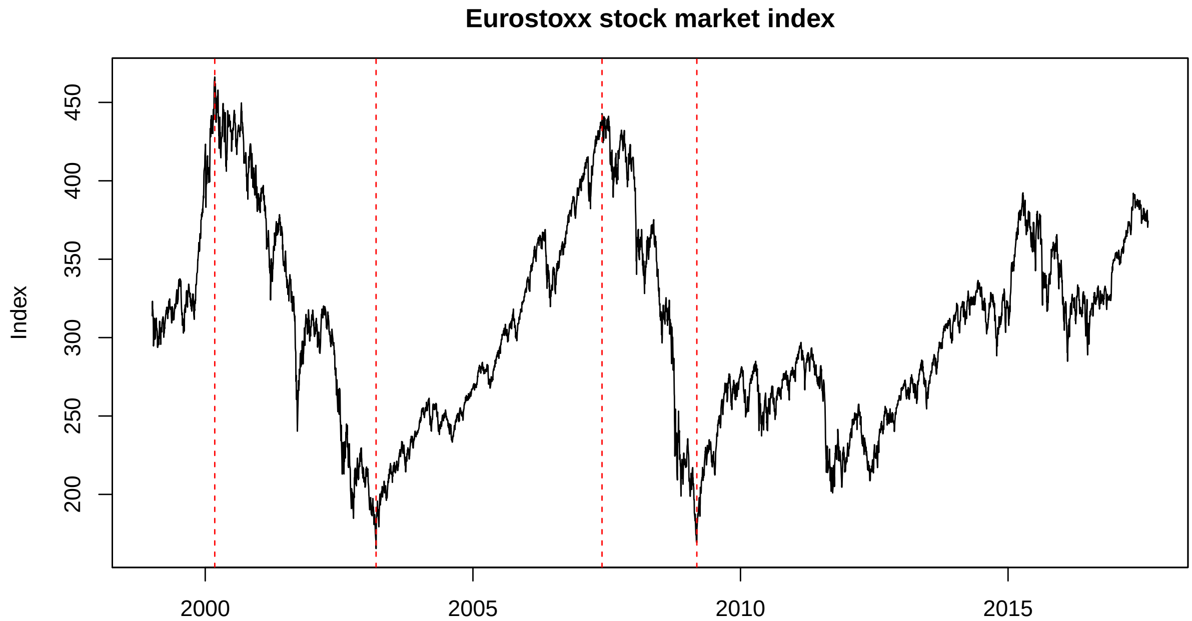

Figure 3.1: Development of the Eurostoxx stock market index and timing of bull and bear markets.

	Begin	Index	End
Bull 1	01/01/1999	313.92	03/05/2000
Bear 1	03/06/2000	466.24	03/11/2003
Bull 2	03/12/2003	165.43	05/31/2007
Bear 2	06/01/2007	442.87	03/08/2009
Crisis	03/09/2009	169.38	08/08/2017

Table 3.1: Definition of bull- and bear-market periods.

Spain, Italy, Portugal, Ireland, and Greece as the periphery countries. Belgium, Austria, Finland, the Netherlands, and France are called the core countries. The data set contains daily (bid) yields on benchmark bonds for these ten countries and for Germany as well as a range of explanatory variables. All series are obtained from Thomson Reuters Eikon and observed between January 1, 1999 and August 8, 2017.

As discussed in the introduction, one of the main objectives of this paper is to show that the degree of EMU bond market integration differs between bull and bear markets. To do so, we need to define which periods are regarded as bull markets and which ones are regarded as bear markets. Since there is no universally accepted definition of bull and bear markets, we simply rely on a visual inspection of the trajectory of the Eurostoxx index. Every bull-market period begins with a local minimum and every bear-market period begins with a local maximum. The timing of these local extrema is indicated by vertical dashed lines in Figure 3.1, and the exact definitions along with the index values at the starting date of the respective series is given in Table 3.1. The first two periods

	ES	**IT**	**PT**	**IE**	**GR**	**BE**	**AT**	**FI**	**NL**	**FR**	**GER**	(s.e.)
Bull 1	1.00	1.02	0.96	1.01	1.00	1.03	1.02	0.94	0.99	0.98	1.01	(0.07)
Bear 1	0.95	0.95	0.94	0.97	0.95	0.95	0.96	0.97	0.95	0.94	0.97	(0.05)
Bull 2	1.05	1.07	1.04	1.07	1.06	1.04	1.04	1.06	1.05	1.05	1.06	(0.04)
Bear 2	0.99	0.91	0.93	1.01	0.91	0.94	0.88	0.95	0.97	0.97	1.01	(0.06)
Crisis	0.89	0.92	0.97	1.02	0.95	0.95	1.00	0.97	0.99	0.99	0.95	(0.03)
Full sample	0.99	1.03	0.96	1.00	0.93	0.93	0.99	1.05	0.98	0.96	1.00	(0.02)

Table 3.2: Memory estimates of the yields for different subperiods. In the Bull 2 period the standard error of the estimate for Ireland is 0.05. The exact definition of the market phases can be found in Table 3.1.

are determined by the Dot-com bubble and the subsequent crash starting on March 6, 2000. The recovery and boom thereafter lasted from March 12, 2003, until May 31, 2007, when the subprime mortgage crisis began. This bear market lasted until March 8, 2009. In the recovery after that, it could be argued that there were several shorter bull- and bear-market periods. However, it can be expected that the mechanisms driving the pricing of EMU government bonds changed permanently with the onset of the EMU debt crisis in October 2009, when the Greek government revised its deficit figures. This is also confirmed empirically by previous studies such as Pozzi and Wolswijk (2012), Christiansen (2014), and Ehrmann and Fratzscher (2017). We therefore focus on the previous bull and bear markets and refer to the post-2009 period as the crisis period.

Estimates of the memory parameters of the yields in each subsample are given in Table 3.2. Here and hereafter, all memory parameters are estimated using the exact local Whittle estimator of Shimotsu (2010) and a bandwidth of $m = \lfloor T^{0.7} \rfloor$. The estimator is a direct extension of that suggested in Shimotsu and Philips (2005), but allowing for non-zero means. It is given by

$$\hat{d}_{ELW} = \arg\min_{-1<d<3.5} \left\{ \log \hat{G}_m(d) - d \left(\frac{2}{m} \sum_{j=1}^{m} \log \lambda_j \right) \right\},$$

where $\lambda_j = 2\pi j/T$, $\hat{G}_m(d) = m^{-1} \sum_{j=1}^{m} I_{\Delta^d x}(\lambda_j)$, and $I_{\Delta^d x}(\lambda)$ denotes the periodogram of the fractionally differenced process $(1-L)^d(X_t - X_1)$. Under mild regularity conditions Shimotsu (2010) show that

$$\sqrt{m}\left(\hat{d}_{ELW} - d\right) \xrightarrow{d} N(0, 1/4).$$

As can be seen in Table 3.2, the estimated memory parameters are statistically indistinguishable from one, so that it is reasonable to assume that the interest rates follow a stochastic trend. This is also supported by formal tests.

3.4 Empirical Analysis

Using the definition of bull and bear markets from the previous section, we now analyze the dynamics of integration and disintegration in EMU government bond markets using several approaches. First, we test for fractional cointegration among the yields, separately for bull and bear markets. Second, to determine the robustness of our findings to the definition of the subperiods, we use the second approach and test in a rolling window whether the order of integration in the spreads is equal to one, so that we do not impose any restrictions on the timing of periods of integration and disintegration. Finally, we conduct a regression analysis to gain further insights into the forces driving these results.

3.4.1 Testing for Market Integration among the Yields

As discussed in Section 3.2, integration in the market for EMU government bonds requires the yields to be pairwise fractionally cointegrated. Since the German government bonds are considered to be the most liquid and essentially risk free, it is customary to use Germany as the base country and to analyze the pairwise relationship of each country with Germany. We therefore adopt this approach and start our analysis by applying tests for the null hypothesis of no fractional cointegration on these pairs in each of the subsamples. The results of this exercise are given in Table 3.3. Empty fields indicate the absence of a significant fractional cointegrating relationship at the 5%-level. Non-empty fields give an estimate of b — the strength of the cointegrating relationship. Larger values of b indicate a stronger equilibrium relationship.

The tests from Section 3.2 are abbreviated by the authors' names and the year of publication. Since the methods employed are based on very different properties of fractionally cointegrated systems, it is not surprising that there is some variation in the findings. However, overall the results show that the majority of interest rates were indeed cointegrated with the German rate during the bull-market periods but not during the bear-market periods. A notable exception is Greece in the first bull market, since it only joined the EMU in 2001, which is during our first bear-market period. When comparing the bull-market periods and bear-market periods, it is immediately noticeable that the tests reject the null hypothesis less often during the bear markets than during the bull markets. Evidence for the existence of an equilibrium relationship during the bear-market periods is mainly found for the core countries. Furthermore, when comparing the strength of the cointegrating relationships that persist during bull and bear markets, we can observe that the strength declines in bear-market periods.

If we consider Finland, for example, deviations from the equilibrium have a memory of approximately $1 - b_8 = 0.25$ in the first bull market. This increases to nearly 0.65 in

		ES	**IT**	**PT**	**IE**	**GR**	**BE**	**AT**	**FI**	**NL**	**FR**
Bull 1	NS07	0.60	0.38	0.41	0.45		0.58	0.54	0.75	0.64	0.62
	SRF16	0.47	0.29	0.45	0.45		0.41	0.59	0.86	0.59	0.71
	MV04	0.60	0.38	0.46	0.53		0.60	0.70	0.75	0.67	0.69
	WWC15	0.60	0.38	0.47	0.53	0.07	0.60	0.70	0.75	0.67	0.69
	CH06	0.58	0.38	0.47	0.50		0.60	0.68	0.74	0.67	0.69
	R08	0.60	0.38	0.41	0.45		0.58	0.54	0.75	0.64	0.62
	HV08	0.60	0.38	0.47	0.53		0.60	0.70	0.75	0.67	0.69
	N10			0.46	0.52		0.54	0.70	0.62	0.65	0.69
Bear 1	NS07	0.16	0.07	0.11	0.14	0.09	0.16	0.15	0.35	0.22	0.34
	SRF16	0.32		0.29	0.29			0.30	0.43	0.29	0.39
	MV04				0.15				0.35	0.17	0.31
	WWC15	0.11			0.15		0.10		0.35	0.17	0.31
	CH06	0.14						0.14	0.35	0.22	0.35
	R08								0.35		0.34
	HV08				0.15				0.35	0.17	0.31
	N10										
Bull 2	NS07	0.49	0.14	0.37	0.46	0.39	0.22	0.46	0.44	0.46	0.32
	SRF16	0.45		0.28	0.43	0.34	0.19	0.36	0.18	0.29	0.28
	MV04	0.49		0.37	0.47	0.39	0.21	0.45	0.48	0.47	0.32
	WWC15	0.49		0.37	0.47	0.39	0.21	0.45	0.48	0.47	0.32
	CH06	0.48	0.14	0.37	0.47	0.39	0.21	0.43	0.46	0.46	0.32
	R08	0.49		0.37	0.46	0.39		0.46	0.44	0.46	0.32
	HV08	0.49		0.37	0.47	0.39	0.21	0.45	0.48	0.47	0.32
	N10	0.49			0.46			0.46	0.44	0.46	0.32
Bear 2	NS07	0.05	0.06	-0.02			-0.04	-0.02	0.15	0.08	0.12
	SRF16							0.26	0.34		0.28
	MV04		0.14				0.09	0.13	0.20	0.15	0.22
	WWC15		0.14					0.13	0.20	0.15	0.22
	CH06		0.16						0.23	0.16	0.23
	R08										0.12
	HV08							0.13	0.20	0.15	0.22
	N10										
Crisis	NS07							0.10	0.10	0.17	0.09
	SRF16				-0.15						
	MV04										
	WWC15										
	CH06								0.11	0.19	
	R08	0.04					0.07		0.10	0.17	
	HV08									0.19	
	N10										

Table 3.3: Strength of the fractional cointegration relationship between the yields of bonds of the respective country and Germany. The exact definition of the market phases can be found in Table 3.1.

the first bear market, before dropping to 0.5 in the second bull market, and rising again to about 0.85 in the second bear market.

When we consider the results for the EMU crisis period, we find that there is no evidence for the existence of an equilibrium relationship for the periphery countries anymore.

Among the core countries some weak evidence is found, but mostly for the Netherlands and Finland. The overwhelming majority of the tests are unable to detect any evidence for market integration during this period.

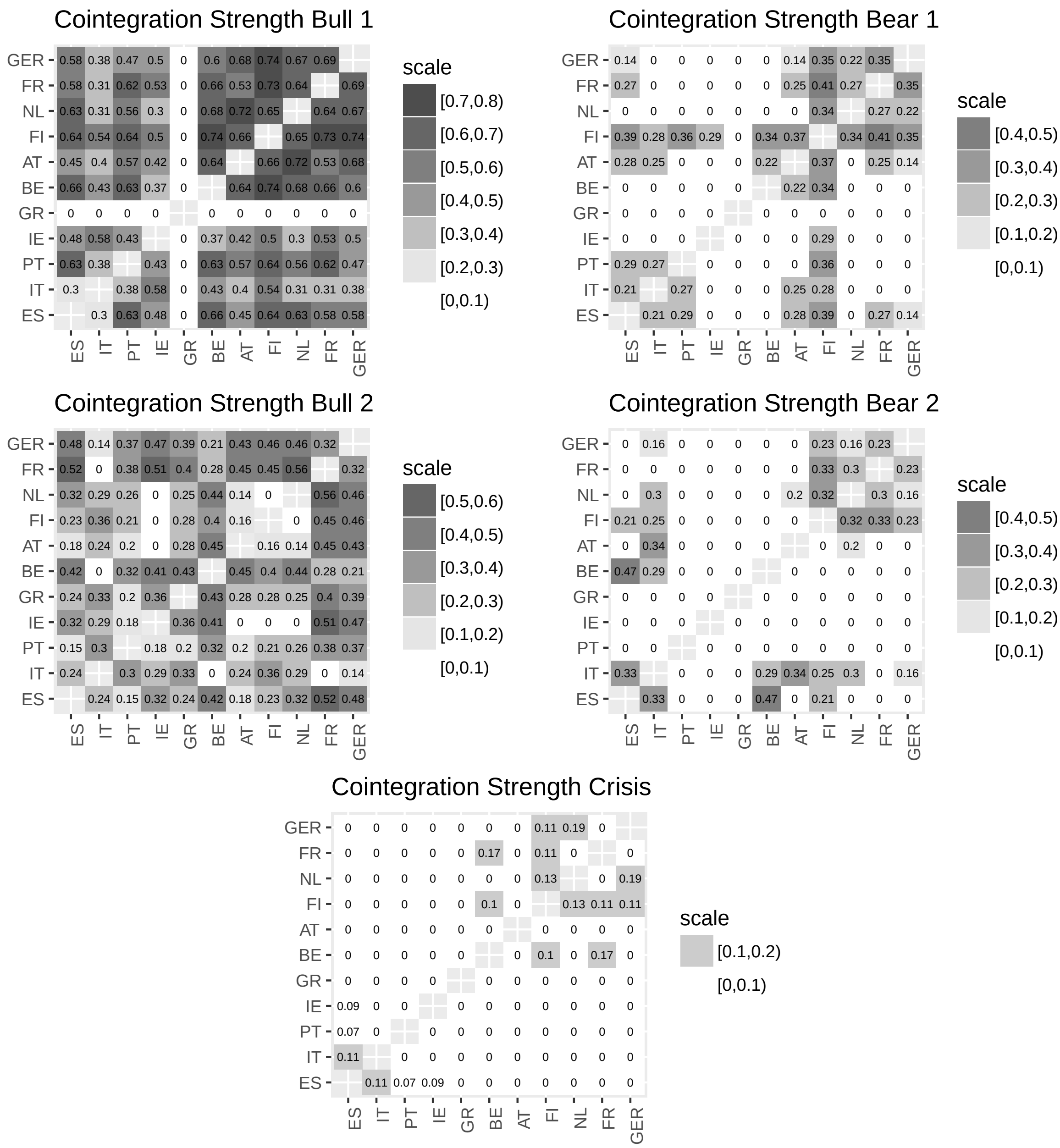

Figure 3.2: Heatmaps for the strength of all pairwise cointegration relationships. The test for the existence of a cointegrating relationships and the estimation of their strength is carried out for different subperiods using the method of Chen and Hurvich (2006). The exact definition of the market phases can be found in Table 3.1.

To gain further insights into the dynamics of market integration between all possible country pairs, we repeat the same analysis using the method of Chen and Hurvich (2006). The results are presented in heatmaps in Figure 3.2. Here, a darker color indicates a strong equilibrium relationship. Clearly, there is much more evidence for pairwise market integration between the countries during the bull-market periods, which are shown on the left-hand side, than during the bear-market periods depicted on the right-hand side.

We observe that, during the bull markets, there is a larger number of cointegrating relationships among the core countries than there is among the periphery countries. During the first bear market, Finland is a notable exception, since it appears to be fractionally cointegrated with all of the core countries and with all of the periphery countries, except for Greece. In the second bear market Italy is an exception, since it is in equilibrium with the majority of core countries. We can also observe that there is a tendency of Portugal, Italy, and Spain to remain in equilibrium with each other during the bear markets. Finally, we observe a clear distinction between periphery countries and core countries during the crisis period. Here, the core countries tend to remain (weakly) integrated with each other, whereas the periphery countries disintegrate completely.

Taken together, we find that there are periods of integration and periods of disintegration associated with bull and bear markets. We can observe that there is stronger market integration between the core countries than between the core and the periphery during bear markets. Finally, we observe a disintegration of the yields for all countries during the crisis. Considering the behavior of the Eurostoxx, the EMU crisis could be regarded as a bull-market period, which usually is a period of integration. The cyclical relationship with periods of integration and disintegration therefore breaks down with the advent of the EMU debt crisis.

An obvious extension of this analysis would be to model the system as a whole and to determine the number of common trends driving it. However, this is econometrically challenging. Methods to determine the cointegrating rank tend to become more unstable as the dimension of the system increases, when the cointegrating strength decreases, and when the correlation of the short memory components increases. Since we are dealing with a system of 11 strongly correlated series that appears to be weakly cointegrated, such an analysis is unlikely to produce reliable results.

3.4.2 Testing for Market Integration among the Yield Spreads

As discussed in Section 3.2, a second approach to test for fractional cointegration is to consider the persistence of the spreads directly. Figure 3.3 shows the spreads for the bull- and bear-market subperiods. Visually, the spreads appear to be less persistent during bull markets than during bear markets. This is also confirmed by the memory estimates in Table 3.4. These findings clearly support those from the previous section. Furthermore, it can be seen that there is little evidence for market integration if the whole sample period is considered.

However, in this context we no longer need to impose specific time periods that are defined to be bull or bear markets. We can therefore gain further insights into the dynamics

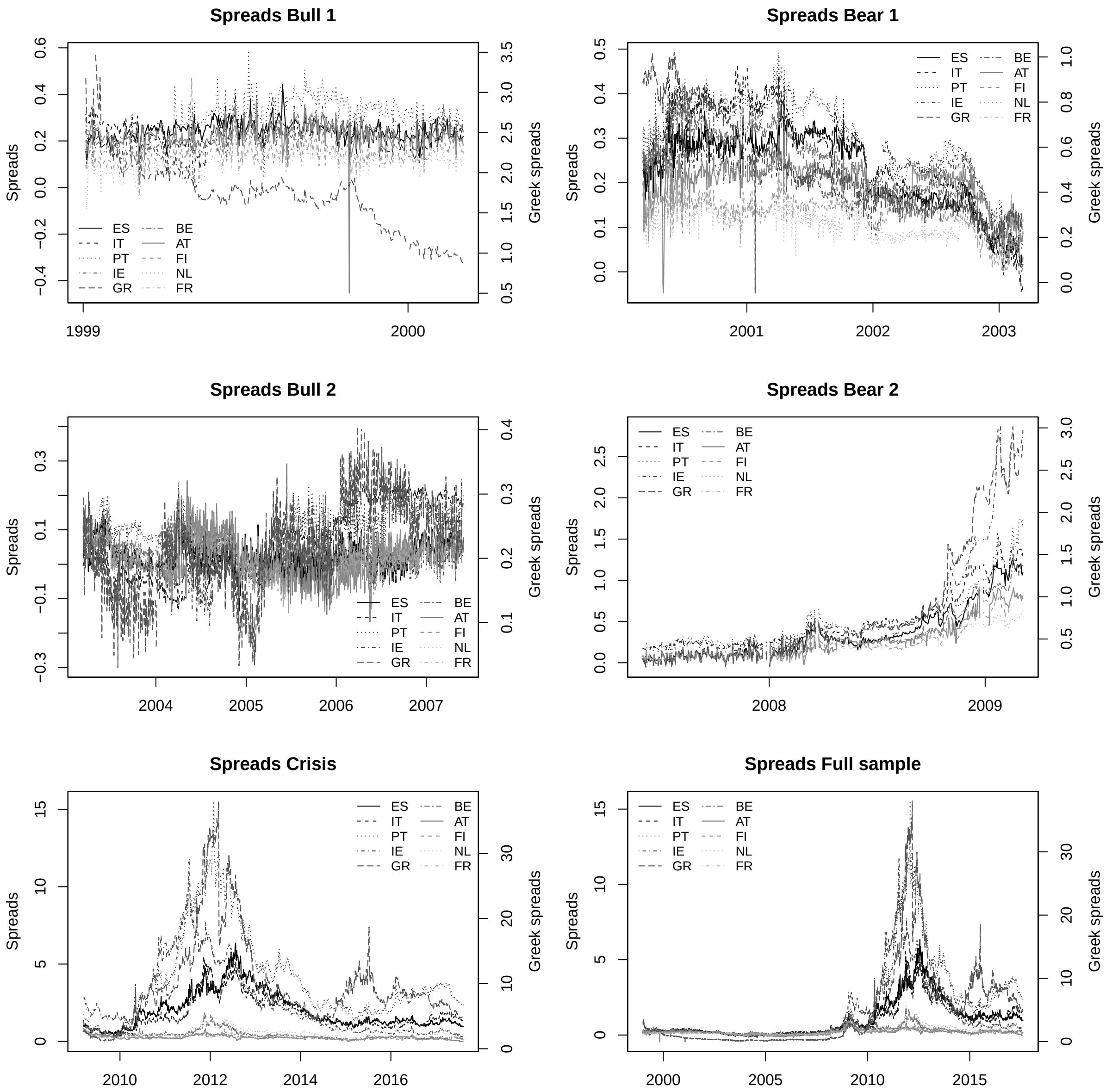

Figure 3.3: Interest rate yield spreads s_{it} relative to Germany. The exact definition of the market phases can be found in Table 3.1.

of economic integration and disintegration among the interest rates in the euro area, by adopting a semiparametric approach and testing for $d(s_{it}) = 1$ in a rolling window. The window size is set to 250 observations, which corresponds to one year, and provides a good trade-off between bias and sampling variation of the estimate.

The results of this exercise are shown in Figure 3.4 for the core countries and in Figure 3.5 for the periphery countries. Each point represents the estimated memory parameter $\hat{d}(s_{it})$ from the window that ends on this date. The horizontal dashed lines represent a 95% confidence band centered around $d(s_{it}) = 1$, based on $1.96/\left(2\sqrt{\sum_{j=1}^{m} \nu_j^2}\right)$, where $\nu_j = \log \lambda_j - m^{-1} \sum_{j=1}^{m} \log \lambda_j$ and $\lambda_j = 2\pi j/250$. This is the typical finite sample correction for the variance of the estimator that is based on its Hessian (cf. Hurvich and Beltrao (1994),

	ES	IT	PT	IE	GR	BE	AT	FI	NL	FR	(s.e.)
Bull 1	0.42	0.63	0.58	0.59	0.89	0.42	0.42	0.24	0.30	0.26	(0.07)
Bear 1	0.81	0.90	0.84	0.83	0.71	0.80	0.82	0.62	0.75	0.61	(0.05)
Bull 2	0.56	0.94	0.68	0.61	0.68	0.86	0.59	0.62	0.59	0.76	(0.04)
Bear 2	0.90	0.83	0.96	1.05	0.99	0.95	0.92	0.78	0.86	0.81	(0.06)
Crisis	0.87	0.90	0.95	0.96	0.95	0.88	0.92	0.87	0.84	0.89	(0.03)
Full sample	0.92	0.99	0.93	0.95	0.93	0.81	0.86	1.00	0.79	0.94	(0.02)

Table 3.4: Memory estimates of the spreads s_{it} relative to Germany for different subperiods. In the Bull 2 period the standard error of the estimate for Ireland is 0.05 and in the full sample the standard error of the estimate for Greece is 0.03. The exact definition of the market phases can be found in Table 3.1.

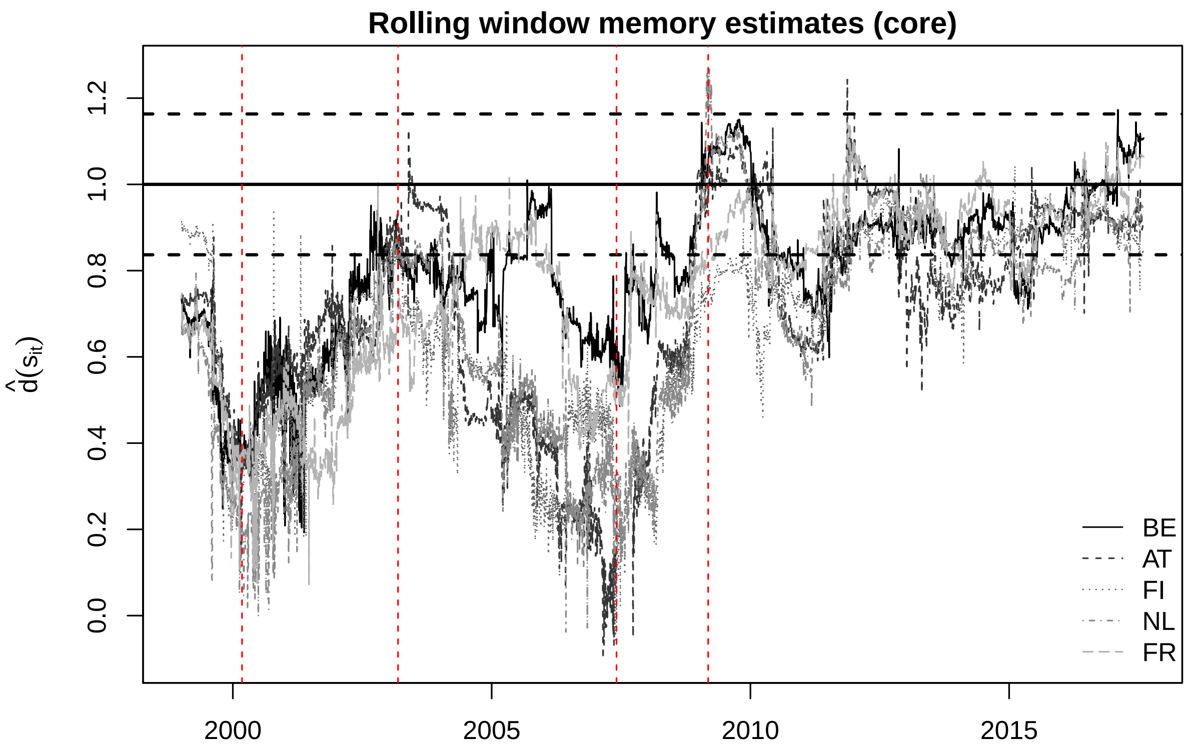

Figure 3.4: Rolling window estimates of the memory $d(s_{it})$ in the spreads of the core countries.

Lemma 1). It is well known that these tests remain liberal even despite the correction. We therefore might reject the hypothesis of no fractional cointegration too often. As before, the vertical dashed lines mark the start and endpoints of the bull- and bear-market periods defined as before.

Considering the results for the core countries in Figure 3.4, we can make several observations. When we move from a bull-market period to a bear-market period, the estimated memory parameter increases as new observations enter the estimation window. Conversely, when we enter a bull market after a bear market, the new observations entering

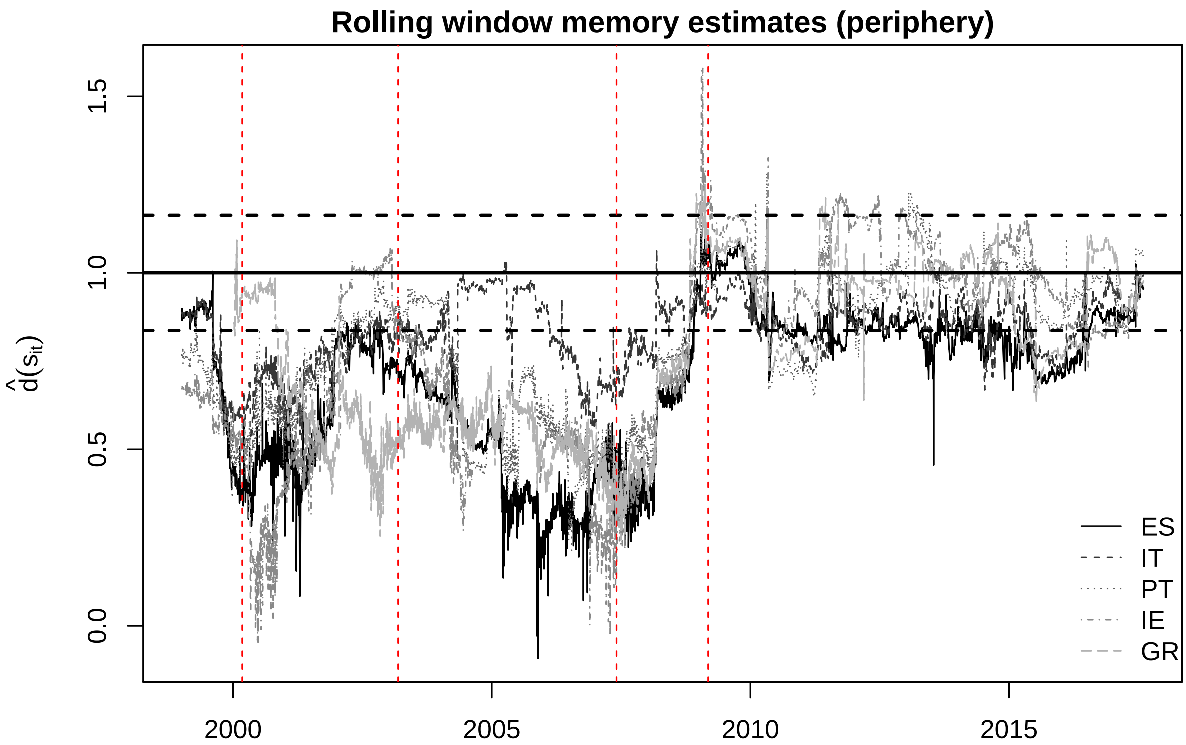

Figure 3.5: Rolling window estimates of the memory $d(s_{it})$ in the spreads of the periphery countries.

the estimation window tend to decrease the estimated memory parameter. A similar pattern can be observed for the periphery countries in Figure 3.5, although they are a bit less homogeneous.

Around the end of the first bear market in 2003, there is an extended period during which the estimated memory parameters indicate the absence of a fractional cointegrating relationship and thus no evidence for market integration.

In both groups there are some deviations from the general pattern. Among the core countries the persistence of the Belgian and French spreads keeps increasing in the initial phase of the second bull market. Similarly, the persistence of the Greek and Italian spreads remains high in the same period. Finally, Ireland shows a somewhat different behavior during the first bull and bear market.

After the second bear market — with the advent of the EMU debt crisis — the relationship breaks down. The estimates of the $d(s_{it})$ are close to 1, and well within the confidence bands, indicating that there is no equilibrium relationship. A notable exception is a short dip in the level of the persistence after April, 2010 when the European Financial Stability Facility (EFSF) was first established. Here, the estimated memory parameters are close to the lower confidence band. However, this period ended quickly thereafter, which implies that the EFSF as a policy measure was not sufficient to effectively calm the market and re-establish an equilibrium.

Overall, the results are clearly in line with those in the previous section that show that there are periods of integration and periods of disintegration that are related to bull markets and bear markets.

3.4.3 Drivers of Market Integration and Disintegration

To gain further insights into the determinants of EMU bond market integration, we conduct a regression analysis of the sources of variation in the estimated memory parameters from Section 3.4.2. The main objective of this analysis is to determine whether the observed time variation in the persistence of the spreads can be explained by factors such as market risk or risk aversion that might also drive bull and bear markets.

Typical measures for market risk or "uncertainty" include realized and implied volatility. Let there be N intraday returns r_{it} observed at trading day t, then the realized variance is given by

$$RV_t = \sum_{i=1}^{N} r_{it}^2,$$

which provides a consistent estimate of the quadratic variation of the respective asset as $N \to \infty$. We therefore consider the realized volatility of the Eurostoxx index as a measure of current market risk. The implied volatility measured by the VIX and its European equivalent, the VSTOXX, is a forward-looking measure that extracts the expected average volatility over the next 22 trading days from a panel of option prices, assuming that market participants are risk-neutral. As discussed in detail in Chernov (2007), the VSTOXX can therefore be decomposed into the expected average volatility over the next month and a risk premium according to

$$VSTOXX_t = E_t[RV_{t+22}^{(22)}] + VP_t,$$

where $RV_t^{(H)} = H^{-1} \sum_{h=0}^{H-1} RV_{t-h}$. Under the assumption of rational expectations, we can obtain an ex post estimate of VP_t via

$$VP_t = VSTOXX_t - RV_{t+22}^{(22)}.$$

It is typically found that the VSTOXX has explanatory power for the flight-to-quality effect (cf. for example Connolly et al. (2005)). Due to the persistence of RV_t and the relationships discussed above, it is unclear whether this explanatory power is due to the current level of market risk RV_t, the expected change in the average market risk over the next month

$$\Delta RV_t = RV_{t+22}^{(22)} - RV_t,$$

or the variance premium VP_t.

The variance premium VP_t has received a lot of attention in the recent literature, since it is related to the degree of risk aversion. Bollerslev et al. (2009) and Bollerslev et al. (2013), for example, show theoretically and empirically that it has some explanatory power for future stock returns, and Bekaert and Hoerova (2014) show that it improves forecasts of future realized volatility.

Instead of including the VSTOXX itself, we therefore consider RV_t, ΔRV_t, and VP_t separately so that it is possible to distinguish the effect of current market risk from that of (expected) future risk and that of changes in risk pricing.

To formally test the hypothesis that the existence and strength of equilibrium relationships between the bonds of the respective country and Germany are driven by bull- and bear-market periods, we include the bull-market indicator ($\mathbb{1}_{bull,t}$) that corresponds to the market phases defined in Section 3.3. Due to the special interest in this variable, we include interaction terms between the bull-market indicator and all market-uncertainty measures.

As additional control variables, the daily return of the Eurostoxx (r_t), the spread between BBB-rated US corporate bonds and AAA-rated US government bonds (BBB_t) as a measure for global risk aversion, and the 3-month Euribor rate ($Euribor_t$) are included. This is motivated by the finding of Ang and Longstaff (2013), who show that financial variables have higher explanatory power than macroeconomic variables at a daily frequency.

For a better approximation by the normal distribution, we consider the log of RV_t and VP_t. Furthermore, due to the different levels of persistence among these variables, the regressors RV_t, ΔRV_t, VP_t, BBB_t and $Euribor_t$ are fractionally differenced to achieve balanced regressions. Finally, the regressors are standardized to have zero mean and unit variance to facilitate the interpretation of the regression coefficients. This leads to the regression equation

$$\begin{aligned}\hat{d}_{t+125}(s_{it}) = {} & \beta_0 + \beta_1\, \mathbb{1}_{bull,t} + \beta_2\, RV_t + \beta_3\, \Delta RV_t + \beta_4\, VP_t + \beta_5\, r_t + \beta_6 BBB_t \\ & + \beta_7\, Euribor_t + \beta_8\, \mathbb{1}_{bull,t} \times RV_t + \beta_9\, \mathbb{1}_{bull,t} \times \Delta RV_t + \beta_{10}\, \mathbb{1}_{bull,t} \times VP_t + \upsilon_t,\end{aligned} \tag{3.8}$$

where υ_t is the innovation term. To achieve the best possible estimation of the respective memory parameters, the dependent variable at time t is the rolling window estimate from period $t + 125$ so that the day of interest is in the middle of the estimation window. We observed in the previous sections that the relationship between market sentiment and persistence of the spreads breaks down in the EMU crisis period. Here, the spreads remain persistent despite the bullish environment due to investors' concerns about sovereign default risks. Our estimation period is therefore restricted to the period up to March 8, 2009 — the end of the second bear market.

An econometric complication lies in the fact that our dependent variable itself is estimated in a rolling window of 250 observations, which induces a long autocorrelation structure. This issue is similar to the problems incurred in long-horizon regressions that test stock return predictability for overlapping time periods. However, in our case the plausible dependence structure is more general than that of asset returns. The dependent variable is not directly observable, and the overlap concerns also past variables. Typical approaches to address this problem, such as those of Hansen and Hodrick (1980) and its extensions by Richardson and Smith (1991) and Hodrick (1992), can therefore not be applied in our setup.

Another common approach is to use HAC estimators with a long lag structure, as for example in Bekaert and Hoerova (2014). This is also the approach we follow here. To account for the autocorrelation caused by the rolling window estimation of the dependent variable, we use a Newey-West estimator with 500 lags. Since this number of lags is relatively large in proportion to the sample size, we cannot resort to standard asymptotics when conducting hypothesis tests. Instead we use so-called fixed-b asymptotics introduced by Kiefer and Vogelsang (2005). Denote the standard HAC estimator based on the first B autocovariances by $\widehat{V}_{HAC}$. Standard asymptotic theory is based on the assumption that $B/T \to 0$, as $T \to \infty$, so that $\widehat{V}_{HAC}$ is consistent for the true variance V. In contrast to this, Kiefer and Vogelsang (2005) assume that $B/T \to b_{HAC}$, where $b_{HAC} \in (0, 1]$ is a fixed non-zero constant. In this case $\widehat{V}_{HAC}$ is no longer consistent, but converges to the true variance V multiplied by a functional of a Brownian bridge process $Q(k, b_{HAC})$. The corresponding t-statistic t_{FB} has a non-standard limiting distribution that depends on both the kernel k used by the HAC estimator and b_{HAC}. Here, $t_{FB} \Rightarrow \frac{W(1)}{\sqrt{Q(k,b_{HAC})}}$, where $W(r)$ is a standard Brownian motion on $r \in [0, 1]$ and for the *Bartlett* kernel $Q(k, b_{HAC})$ is given by

$$Q(k, b_{HAC}) = \frac{2}{b_{HAC}} \left(\int_0^1 \widetilde{W}(r)^2 \mathrm{d}r - \int_0^{1-b_{HAC}} \widetilde{W}(r + b_{HAC}) \widetilde{W}(r) \mathrm{d}r \right),$$

with $\widetilde{W}(r) = W(r) - rW(1)$ denoting a standard Brownian bridge. This approach typically provides better size control in persistent time series and can be particularly useful in our setup, where the number of lags employed by the Newey-West estimator is very large.

The results of this exercise are shown in Table 3.5. It can be seen that the estimated memory parameters are indeed significantly lower in bull markets. The estimated coefficients of the bull-market indicator are negative for all countries but France and significant in 6 out of 10 cases. The reduction in memory in these cases ranges from -0.16 for Finland to -0.32 for Ireland. We also find a significant impact of current risk and future risk changes for the core countries, as well as a number of significant interaction terms

	ES	IT	PT	IE	GR	BE	AT	FI	NL	FR
const	0.68 **	0.80 **	0.80 **	0.77 **	0.67 **	0.74 **	0.73 **	0.60 **	0.63 **	0.60 **
$\mathbb{1}_{bull,t}$	−0.24 **	−0.00	−0.23 **	−0.32 **	−0.10	−0.05	−0.29 **	−0.16 **	−0.20 *	0.06
RV_t	0.17	0.14 *	0.12	0.30	0.18	0.25 *	0.19 **	0.13	0.27	0.22 **
ΔRV_t	0.18	0.14 *	0.12	0.29	0.18	0.25 *	0.19 **	0.13	0.27	0.22 **
VP_t	−0.01	−0.01 **	−0.00	−0.01	−0.01	−0.01 **	−0.00	−0.01 *	−0.01	−0.01 **
r_t	−0.01	−0.01	−0.01	−0.01	−0.01	−0.01 *	−0.01 *	−0.01 *	−0.02 **	−0.01 *
BBB_t	−0.00	−0.00	−0.00	−0.00	0.00	−0.00	−0.00	−0.00 *	−0.00	−0.00
$Euribor_t$	−0.01	−0.00	−0.01	−0.00	−0.00	−0.01	−0.01	−0.01 *	−0.01	−0.00
$\mathbb{1}_{bull,t} \times RV_t$	−0.46 **	−0.18	−0.20	−0.50 *	−0.33 *	−0.07	−0.40 **	−0.31 **	−0.32 *	−0.19
$\mathbb{1}_{bull,t} \times \Delta RV_t$	−0.46 **	−0.18	−0.19	−0.50 *	−0.33 *	−0.07	−0.39 **	−0.31 **	−0.31 *	−0.18
$\mathbb{1}_{bull,t} \times VP_t$	−0.01	0.00	−0.01 *	−0.00	0.01 *	0.00	−0.02 *	−0.00	−0.01	0.00
$R^2_{adj.}$	0.36	0.02	0.38	0.36	0.08	0.08	0.37	0.22	0.21	0.03
b_{HAC}	0.20	0.20	0.20	0.23	0.21	0.20	0.20	0.20	0.20	0.20
$crit_{0.975}$	2.55	2.55	2.55	2.66	2.59	2.55	2.58	2.55	2.55	2.55
$crit_{0.95}$	2.08	2.08	2.08	2.16	2.11	2.08	2.10	2.08	2.08	2.08

Table 3.5: Dependence between bond market integration and ex post determined stock market sentiment. The estimation is carried out for the period 01/01/1999–03/08/2009. The symbols * and ** indicate significance at the 10% level and 5% level, respectively.

between the bull-market dummies and the risk variables. In bear markets a one standard deviation increase in the fractionally differenced realized volatility leads to an increase of the memory parameter of about 0.2, whereas the effect is offset or even reversed in bull markets where the interaction terms come into effect. The variance risk premium does not generally have a significant effect, and where it does, the size of the effect is not economically meaningful. With regard to the quality of the models, the $R^2_{adj.}$ is about 0.35 for Spain, Portugal, Ireland, and Austria and it is around 0.2 for the Netherlands and Finland. For Belgium and Greece the explanatory power is lower and the model fails to explain the time variation in the persistence of the spreads of France and Italy.

The bull- and bear-market periods defined in Section 3.3 are *ex post*, since they require the knowledge of subsequent highs and lows of the index. This information is not available to market participants in real time. Instead, they can consider a *nowcast* of the probability of being in a bull market that is based on past returns. Furthermore, even though the results in Table 3.5 clearly indicate that the market periods specified in Section 3.3 are meaningful for the degree of integration, the regression analysis conducted here does not require long uninterrupted bull- and bear-market periods.

We therefore consider an alternative specification of the bull- and bear-market model, where the state of the Eurostoxx index is determined endogenously, and the bull- and bear-market periods are allowed to be short-lived. This is achieved by using a Markov-switching mean and variance model, where

$$r_t = \mu_{s_t} + \sigma_{s_t}\eta_t, \tag{3.9}$$

	ES	IT	PT	IE	GR	BE	AT	FI	NL	FR
$const.$	0.78 ∗∗	0.86 ∗∗	0.87 ∗∗	0.83 ∗∗	0.79 ∗∗	0.81 ∗∗	0.87 ∗∗	0.72 ∗∗	0.74 ∗∗	0.63 ∗∗
$s_{t\|t}$	−0.28 ∗∗	−0.03	−0.25 ∗∗	−0.26 ∗∗	−0.17	−0.08	−0.37 ∗∗	−0.25 ∗∗	−0.28 ∗∗	0.03
RV_t	0.07	0.02	0.09	0.19	0.08	0.14	0.03	−0.01	0.13	0.09
ΔRV_t	0.07	0.02	0.10	0.19	0.07	0.14	0.04	−0.00	0.14	0.09
VP_t	−0.01	−0.00	−0.01	−0.00	−0.01 ∗∗	0.00	−0.01	−0.00	0.00	0.00
r_t	−0.02 ∗∗	−0.01	−0.01 ∗∗	−0.02 ∗∗	−0.01	−0.01	−0.01 ∗	−0.01 ∗	−0.01 ∗	−0.01
BBB_t	−0.00	−0.00	−0.00	−0.01	0.00	−0.00	−0.00	−0.00 ∗∗	−0.00	−0.00
$Euribor_t$	−0.01 ∗	−0.00	−0.00	−0.01	−0.01	−0.00	−0.00	−0.01	−0.01	−0.01
$s_{t\|t} \times RV_t$	−0.25 ∗∗	0.07	−0.16 ∗	−0.18	−0.26	0.10	−0.18	−0.13	−0.19	0.09
$s_{t\|t} \times \Delta RV_t$	−0.26 ∗∗	0.07	−0.16 ∗	−0.19	−0.26	0.10	−0.18	−0.14	−0.19	0.09
$s_{t\|t} \times VP_t$	−0.00	−0.01	0.00	−0.01	0.01 ∗	−0.01	0.00	−0.01	−0.02	−0.01
$R^2_{adj.}$	0.28	0.03	0.28	0.18	0.13	0.10	0.35	0.25	0.24	0.01
b_{HAC}	0.20	0.20	0.20	0.23	0.21	0.20	0.20	0.20	0.20	0.20
$crit_{0.975}$	2.55	2.55	2.55	2.66	2.59	2.55	2.58	2.55	2.55	2.55
$crit_{0.95}$	2.08	2.08	2.08	2.16	2.11	2.08	2.10	2.08	2.08	2.08

Table 3.6: Dependence between bond market integration and a nowcast of the stock market sentiment. The estimation is carried out for the period 01/01/1999–03/08/2009. The symbols ∗ and ∗∗ indicate significance at the 10% level and 5% level, respectively.

with $\eta_t \overset{iid}{\sim} (0,1)$. Here $s_t \in \{1,2\}$ is a Markov chain with transition probabilities p_{12} and p_{21}. For identification purposes, we assume $\mu_1 > \mu_2$ and call regime one the "bull-market regime". Let $s_{t|t} = P(s_t = 1|r_t, r_{t-1}, ...)$ denote the probability of a bull market at time t conditional on the observations up to time t estimated on the basis of the Markov-switching model. We will refer to $s_{t|t}$ as the market sentiment.

When the model is applied to the Eurostoxx returns, we can observe that the bull-market regime is associated with a positive mean $\hat{\mu}_1 = 0.0008$ and a low standard deviation $\hat{\sigma}_1 = 0.0089$, whereas the bear market regime has a negative mean of $\hat{\mu}_2 = -0.0014$ and a larger standard deviation. The probability to stay in the bull-market regime is estimated to be 0.9884, whereas the probability of staying in the bear-market regime is 0.9746. Therefore both regimes are persistent, but the average bear market is shorter than the average bull market.

The filtered state probabilities $s_{t|t}$ for a bull market are shown in Figure B.2 in the appendix, with the previous dating of bull and bear markets indicated by vertical dashed lines. It can clearly be seen that the bull-market probability seems to be higher during those periods that were previously classified as bull markets. However, in the nowcast there is much more uncertainty about the market environment.

The regression results for $\mathbb{1}_{bull,t}$ replaced with $s_{t|t}$ are shown in Table 3.6. When comparing the results with those in Table 3.5, we find that RV_t and ΔRV_t are no longer significant. This indicates that the bull-market state probability carries all necessary information about the volatility. The reduction of the memory in bull markets compared to bear markets appears to be even higher, but the overall fit of the model is reduced when compared to the specification with the bull-market dummy $\mathbb{1}_{bull,t}$ in Table 3.6.

Since the Markov-switching model gives a non-constant bull-market probability for the EMU-crisis period as well, we can analyze the change of the relationship in the bull market during the EMU crisis. As can be seen in Table B.2 in the appendix, the relationship between the bull-market probability and the persistence of the spreads breaks down completely and the model loses its explanatory power.

Since the regression problem with rolling window estimates of the dependent variable is non-standard, we conduct a further robustness check, where we estimate the memory parameters $d(s_{it})$ separately for each quarter. Similarly, we form quarterly means of the explanatory variables before taking fractional differences of all persistent regressors and standardizing. The results of this exercise are given in Tables B.3 and B.4 in the appendix. The estimated coefficients for the bull-market dummy as well as the model fit are comparable in their magnitude, even though fewer of the estimated coefficients are statistically significant. This can be attributed to the lower number of observations. All evidence for a positive effect of increased risk disappears. Similarly, when using nowcasts of the bull-market probability instead of the bull-market dummy, the effect of the bull-market probability is estimated to be even higher in magnitude and statistically significant in most cases. Again, the evidence for a positive effect of the risk and future risk variable disappears.

Overall, however, we find that the time variation in the estimated memory parameters is well explained by a bull-market indicator and the evolution of current and future risk. This finding holds true for ex post defined bull and bear markets as well as an endogenously determined nowcast of the bull-market probability.

As discussed in Section 3.2, the persistence of the spreads may be driven by that of the default risk premium or that of the liquidity risk premium. Unfortunately, since there were no credit default swaps during the period of interest, we cannot draw any direct conclusions about the memory of the default risk premium. We can, however, consider the bid-ask spreads of the benchmark bonds (ba_{it}) as a proxy for liquidity. Estimates of their memory parameters are provided in Table 3.7, along with estimates of the memory in the yield spreads for the same period. It can be observed that the level of persistence in the bid-ask spreads is much lower than that in the yield spreads. From the theoretical results on the memory of linear combinations discussed above, the persistence of the spreads and thus the periods of integration and disintegration therefore could not have been caused by changes in the persistence of the liquidity risk premium. This would require the persistence of the bid-ask spreads to be as high as that of the spreads. Instead, it has to be caused by changes of the persistence or relative variability of the default risk premium. Further support of this argument is provided by rolling window estimates of the memory in the bid-ask spreads in Figures B.3 and B.4 in the appendix. Here, the estimated memory parameters for the core countries are mostly in the lower stationary region — with the exception of a brief period during the EMU debt crisis, where they reach values around

	ES	**IT**	**PT**	**IE**	**GR**	**BE**	**AT**	**FI**	**NL**	**FR**	(s.e.)
$\hat{d}(s_{it})$	0.90	0.88	0.94	1.03	0.95	0.94	0.90	0.84	0.85	0.90	(0.04)
$\hat{d}(ba_{it})$	0.27	0.29	0.06	0.55	0.24	0.09	0.41	0.24	0.13	0.26	(0.04)

Table 3.7: Memory estimates for the yield spreads s_{it} and the bid-ask spreads ba_{it}. The estimation is carried out for the period from 12/01/2001–03/08/2009. The standard error of the estimate for the bid-ask spread of Ireland is 0.05.

0.6. Similarly, the bid-ask spreads of the majority of periphery countries show low and stable persistence prior to the EMU crisis and higher levels afterwards.

Based on these results, it seems reasonable to assume that $d(\delta_{it}) \geq d(l_{it})$ for all $i = 1, ..., N$ and $t = 1, ..., T$. Hence, the theoretical arguments discussed above give rise to two mechanisms that generate the observed time variation in the memory of the spreads that tends to be one in bear markets but much lower in bull markets: (i) breaks in $d(\delta_{it})$ from $d(\delta_{it}) < 1$ to $d(\delta_{it}) = 1$ and vice versa, or (ii) $d(\delta_{it}) = 1$, for all t, but the relative scale of variations in δ_{it} compared to l_{it} differs for bull and bear markets.

Since the default risk is driven by the macroeconomic and fiscal conditions in the respective country, mean reverting default risk premiums imply the existence of a stable equilibrium relationship between the countries' default risk and the default risk of the benchmark country (Germany). In contrast to that, integrated default risk premiums imply the possibility of divergence between the respective country and Germany, since the variance of integrated series grows linearly with time.

The conclusion in situation (i) would therefore be that market participants considered the possibility of economic and fiscal divergence within the EMU area in bear markets, whereas they expected economic convergence within the currency area in bull markets. In situation (ii), market participants would permanently anticipate the possibility of economic and fiscal divergence between the EMU countries, but the level and variability of the default risk premium is so low during bull markets that the memory properties are dominated by those of the less persistent liquidity risk premium. Conversely, during bear markets risk and risk aversion are high so that the variability of the default risk premium increases relative to that of the liquidity risk premium and the persistence of the spreads is dominated by that of the default risk premium.

These findings provide clear support for the assertion that the persistence of the spreads can be attributed to time variation in either the persistence of the default risk premium or its variability. Both of these arguments ((i) and (ii)) lead to the conclusion that (at least in crisis times) the pricing of EMU government bonds implied the possibility of macroeconomic and fiscal divergence between the EMU countries.

3.5 Conclusion

The analysis in this paper is based on the application of a wide array of modern methods for the analysis of fractionally cointegrated time series, coupled with a careful consideration of the interrelations between the dynamics driving long-term interest rates and spreads, the persistence of these series, and the implications of the relationships for the existence or non-existence of equilibria in the EMU government bond market.

Contrary to previous results in the literature, we find that EMU government bond markets are not continually integrated prior to the EMU debt crisis. Even though the level of the spreads was very small compared to that of the yields, we establish that there were periods during which the spreads became unit root processes so that there was no correction mechanism that would drive the yields back to their equilibrium relationship. This is a critical component of the law of one price, which was therefore not fulfilled. These periods of disintegration tended to coincide with bear-market periods, whereas EMU bond markets tended to be economically integrated if stock markets were bullish. Furthermore, the integration among the core countries used to be more intense than that among the periphery countries and especially the degree of integration between the core and the periphery countries was already low in periods prior to the EMU debt crisis.

Altogether, these results imply that investors do not only shift their portfolios from (comparatively) risky stocks to safer bonds in bear markets as described by flight-to-quality effects, there is also a stronger differentiation between sovereign default risks during these periods. As discussed in the previous section, the nature of this differentiation between the default risks of the different countries implies that at least in bear markets investors did consider the possibility of macroeconomic and fiscal divergence between the EMU countries, even though the low magnitude of the spreads shows that this was considered very unlikely.

B Appendix

		ES	**IT**	**PT**	**IE**	**GR**	**BE**	**AT**	**FI**	**NL**	**GER**
Bull 1	NS07	0.56	0.31	0.64	0.54		0.65	0.50	0.64	0.65	0.64
	SRF16	0.50		0.54	0.49		0.42	0.56	0.81	0.61	0.65
	MV04	0.63	0.30	0.67	0.59		0.66	0.52	0.75	0.67	0.71
	WWC15	0.63	0.31	0.67	0.59	0.09	0.66	0.52	0.75	0.67	0.71
	R08	0.56	0.31	0.64	0.54		0.65	0.50	0.64	0.65	0.64
	CH06	0.59	0.31	0.67	0.54		0.67	0.52	0.74	0.66	0.71
	HV08	0.63	0.31	0.67	0.59		0.66	0.52		0.67	0.71
	N10			0.66	0.58			0.52	0.55	0.63	0.71
Bear 1	NS07	0.22	0.12	0.12	0.10	0.10	0.20	0.17	0.38	0.25	0.33
	SRF16	0.32		0.26			0.22	0.30	0.37	0.26	0.31
	MV04	0.25	0.12	0.13	0.14		0.22	0.23	0.40	0.27	0.28
	WWC15	0.25	0.12	0.13	0.14	0.14	0.22	0.23	0.40	0.27	0.29
	R08	0.22					0.20	0.17	0.38	0.25	0.33
	CH06	0.26						0.24	0.39	0.25	0.33
	HV08	0.25					0.22	0.23	0.40		
	N10									0.26	
Bull 2	NS07	0.52	0.08	0.38	0.49	0.41	0.29	0.46	0.43	0.55	0.32
	SRF16	0.53		0.36	0.43	0.30	0.22	0.44	0.27	0.38	0.25
	MV04	0.52		0.38	0.50	0.40	0.28	0.45	0.46	0.57	0.32
	WWC15	0.52		0.37	0.50	0.40	0.28	0.45	0.46	0.57	0.32
	R08	0.52		0.38	0.49	0.41	0.29	0.46	0.43	0.55	0.32
	CH06	0.52		0.38	0.50	0.41	0.28	0.45	0.46	0.56	0.32
	HV08	0.52				0.40		0.45	0.46	0.57	
	N10	0.52			0.49		0.29	0.46	0.43	0.55	0.32
Bear 2	NS07	0.15	0.14	0.06	-0.04		-0.02	0.10	0.34	0.29	0.15
	SRF16							0.33	0.33	0.34	0.25
	MV04		0.17				0.07	0.18	0.33	0.31	
	WWC15		0.17					0.18	0.33	0.31	0.26
	R08					-0.04			0.34	0.29	0.15
	CH06								0.35	0.32	0.26
	HV08										0.26
	N10										
Crisis	NS07						0.18	0.09	0.12	0.07	0.11
	SRF16						0.14				
	MV04						0.17				
	WWC15										
	R08	0.08				0.02	0.18				
	CH06						0.18				
	HV08										
	N10										

Table B.1: Strength of the fractional cointegration relationship between the yields of bonds of the respective country and France. Empty fields indicate the absence of a significant fractional cointegrating relationship at the 5%-level. Non-empty fields give an estimate of b — the strength of the cointegrating relationship. Larger values of b indicate a stronger equilibrium relationship. The exact definition of the market phases can be found in Table 3.1.

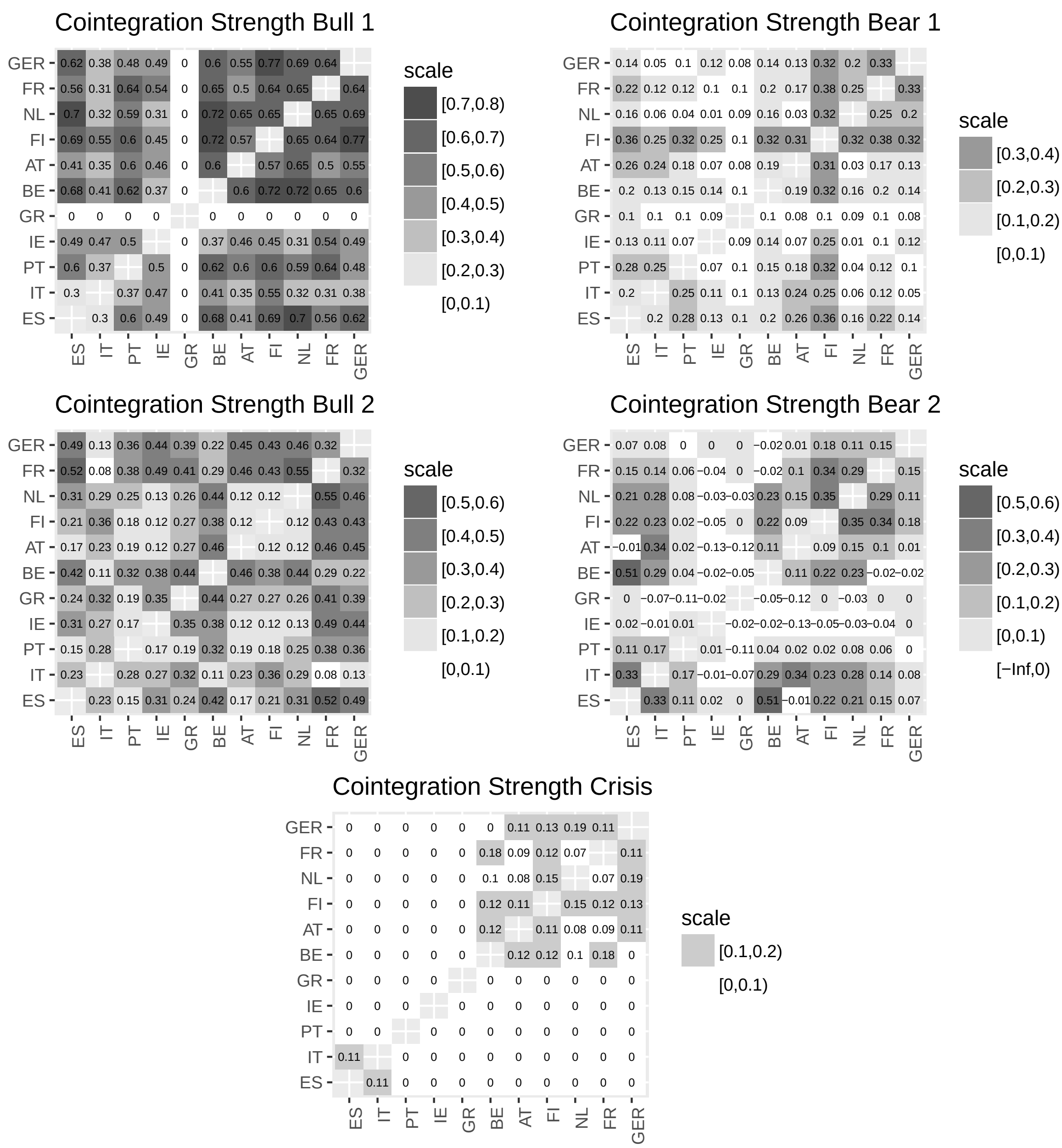

Figure B.1: Heatmaps for the strength of all pairwise cointegration relationships. The test for the existence of a cointegrating relationship and the estimation of their strength are carried out for different subperiods using the method of Nielsen and Shimotsu (2007). Dark fields indicate a strong equilibrium relationship between the countries. The exact definition of the market phases can be found in Table 3.1.

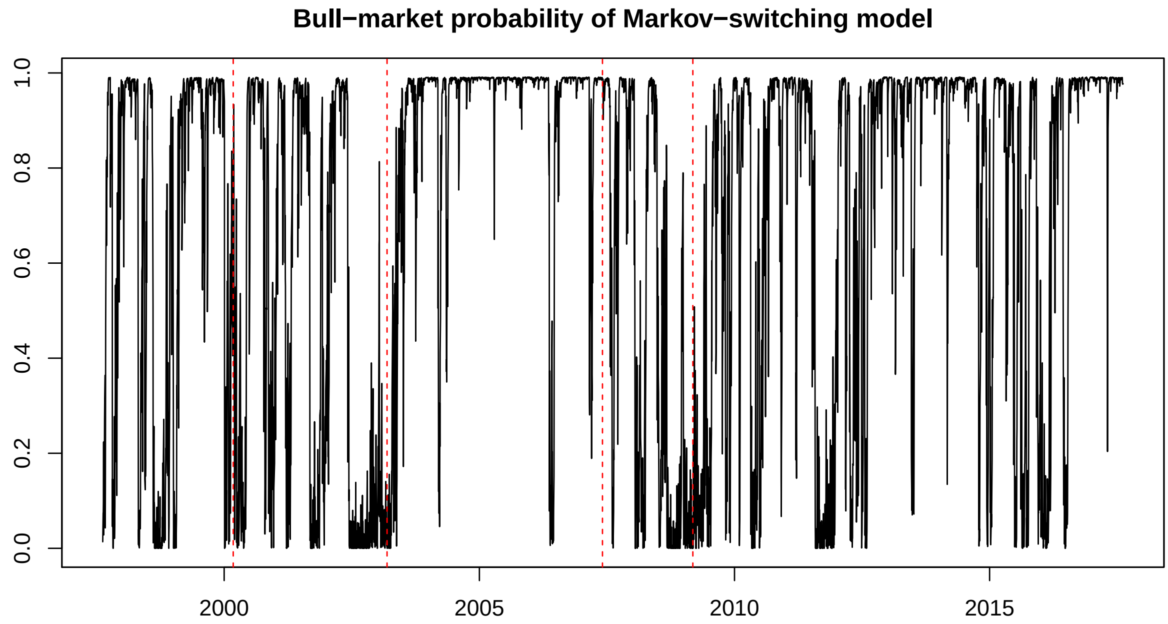

Figure B.2: Nowcast of the market sentiment. The graph shows the filtered probabilities $s_{t|t}$ for a bull-market regime in the Markov-switching mean-variance model (3.9) estimated for the Eurostoxx index. Vertical dashed lines indicate the beginning of a new bull- or bear-market period according to the previous definition as listed in Table 3.1.

	ES	**IT**	**PT**	**IE**	**GR**	**BE**	**AT**	**FI**	**NL**	**FR**
$const.$	0.88 **	0.89 **	0.95 **	1.01 **	0.94 **	0.95 **	0.93 **	0.87 **	0.87 **	0.92 **
$s_{t\|t}$	−0.01	−0.00	0.01	−0.02	0.01	−0.02	−0.07 **	−0.01	−0.01	0.01
RV_t	−0.11	0.00	−0.02	−0.07	−0.00	−0.05	0.01	0.16 **	−0.09	0.12 **
ΔRV_t	−0.10	0.00	−0.01	−0.05	0.01	−0.05	0.02	0.16 **	−0.08	0.12 **
VP_t	0.00	−0.00	−0.00	0.00	−0.00	0.00	−0.00	−0.00 *	0.00	−0.00
r_t	−0.01	−0.00	−0.00 *	−0.00	−0.01	−0.01 *	−0.00	0.00	−0.00 *	−0.00 *
BBB_t	−0.01 **	−0.01 **	−0.01 **	0.00	−0.01	−0.01 *	−0.01 **	0.00	−0.01 **	−0.01 *
$Euribor_t$	−0.00	−0.00	−0.01 **	0.00	−0.00	−0.01	−0.02	−0.02 **	−0.02 *	−0.01
$s_{t\|t} \times RV_t$	0.05	−0.06	−0.00	0.06	−0.09	0.05	−0.01	−0.09	0.09	−0.16
$s_{t\|t} \times \Delta RV_t$	0.04	−0.06	−0.01	0.04	−0.10	0.04	−0.03	−0.09	0.08	−0.16 *
$s_{t\|t} \times VP_t$	−0.01	0.00	0.00	−0.00	0.01	0.00	0.01 *	0.01	0.00	0.01
$R^2_{adj.}$	0.09	0.01	0.04	0.01	0.02	0.07	0.10	0.08	0.09	0.03
b_{HAC}	0.25	0.25	0.25	0.25	0.25	0.25	0.26	0.25	0.25	0.25
$\text{crit}_{0.975}$	2.72	2.72	2.72	2.73	2.73	2.72	2.74	2.73	2.72	2.72
$\text{crit}_{0.95}$	2.20	2.21	2.20	2.21	2.22	2.20	2.22	2.21	2.20	2.20

Table B.2: Dependence between bond market integration and a nowcast of the stock market sentiment during the crisis periods. The estimation is carried out for the period 03/09/2009–08/08/2017. The symbols $*$ and $**$ indicate significance at the 10% level and 5% level, respectively.

	ES	IT	PT	IE	GR	BE	AT	FI	NL	FR
$const.$	0.54 ∗∗	0.81 ∗∗	0.66 ∗∗	0.59 ∗∗	0.62 ∗∗	0.70 ∗∗	0.57 ∗∗	0.50 ∗∗	0.50 ∗∗	0.52 ∗∗
$\mathbb{1}_{bull,t}$	−0.18	−0.03	−0.25 ∗∗	−0.27∗	−0.17	−0.03	−0.26∗	−0.21 ∗∗	−0.18	0.13
RV_t	−0.05	−0.01	−0.01	−0.13	−0.02	−0.12	−0.07	−0.11	−0.05	0.10
ΔRV_t	0.00	−0.01	0.05	0.00	−0.11	−0.06	−0.08	−0.19∗	−0.06	0.14
VP_t	−0.09	−0.13	−0.02	−0.13	−0.19	−0.09	−0.06	−0.13	−0.06	−0.06
r_t	−0.13∗	−0.05	−0.15∗	−0.14∗	0.06	−0.06	−0.06	−0.02	−0.08	−0.16 ∗∗
BBB_t	0.14∗	0.12∗	0.07	0.15∗	0.17∗	0.13∗	0.11	0.10	0.09	0.06
$Euribor_t$	−0.09	−0.11 ∗∗	−0.01	−0.06	−0.13∗	−0.14 ∗∗	−0.06	−0.03	−0.08	−0.06
$\mathbb{1}_{bull,t} \times RV_t$	0.04	0.42 ∗∗	−0.05	−0.11	−0.20	0.60 ∗∗	0.10	0.19	0.15	0.13
$\mathbb{1}_{bull,t} \times \Delta RV_t$	−0.07	0.09	−0.07	−0.09	−0.24	0.21	0.06	0.16	0.11	−0.05
$\mathbb{1}_{bull,t} \times VP_t$	−0.16	−0.05	−0.21	0.08	0.10	−0.06	−0.24	0.09	0.05	−0.08
$R^2_{adj.}$	0.31	0.15	0.26	0.29	0.21	0.25	0.18	0.20	0.11	0.04

Table B.3: Dependence between bond market integration and ex post determined stock market sentiment using quarterly estimates. The estimation is carried out for the period 01/1999–01/2009. The symbols ∗ and ∗∗ indicate significance at the 10% level and 5% level, respectively.

	ES	IT	PT	IE	GR	BE	AT	FI	NL	FR
$const.$	0.67 ∗∗	0.76 ∗∗	0.88 ∗∗	0.69 ∗∗	0.67 ∗∗	0.63 ∗∗	0.73 ∗∗	0.56 ∗∗	0.62 ∗∗	0.54 ∗∗
$s_{t\|t}$	−0.50 ∗∗	0.01	−0.79 ∗∗	−0.56 ∗∗	−0.25	0.04	−0.61 ∗∗	−0.37 ∗∗	−0.53 ∗∗	0.12
RV_t	0.03	−0.08	−0.0001	−0.09	−0.05	−0.17	−0.01	−0.10	−0.02	0.07
ΔRV_t	0.11	−0.05	0.12	0.13	−0.01	−0.06	0.09	−0.05	0.06	0.19
VP_t	−0.06	−0.11	−0.002	0.05	−0.07	−0.06	−0.09	−0.03	0.08	0.03
r_t	−0.12∗	−0.09	−0.09∗	−0.13∗	0.04	−0.12∗	−0.07	−0.06	−0.07	−0.15 ∗∗
BBB_t	0.10	0.12	0.06	0.05	0.12	0.12	0.12	0.04	0.01	0.02
$Euribor_t$	−0.07	−0.11∗	−0.05	−0.09	−0.15 ∗∗	−0.13 ∗∗	−0.07	−0.04	−0.10	−0.06
$s_{t\|t} \times RV_t$	−0.10	0.71 ∗∗	0.21	0.16	0.01	0.84 ∗∗	0.04	0.36	0.33	0.31
$s_{t\|t} \times \Delta RV_t$	−0.18	0.25	0.10	−0.09	−0.39	0.27	−0.15	0.06	0.14	−0.16
$s_{t\|t} \times VP_t$	−0.14	−0.29	−0.20	−0.43	−0.11	−0.33	−0.004	−0.18	−0.32	−0.43
$R^2_{adj.}$	0.49	0.08	0.63	0.43	0.23	0.13	0.42	0.21	0.27	0.08

Table B.4: Dependence between bond market integration and a nowcast of the stock market sentiment using quarterly estimates. The estimation is carried out for the period 01/1999–01/2009. The symbols ∗ and ∗∗ indicate significance at the 10% level and 5% level, respectively.

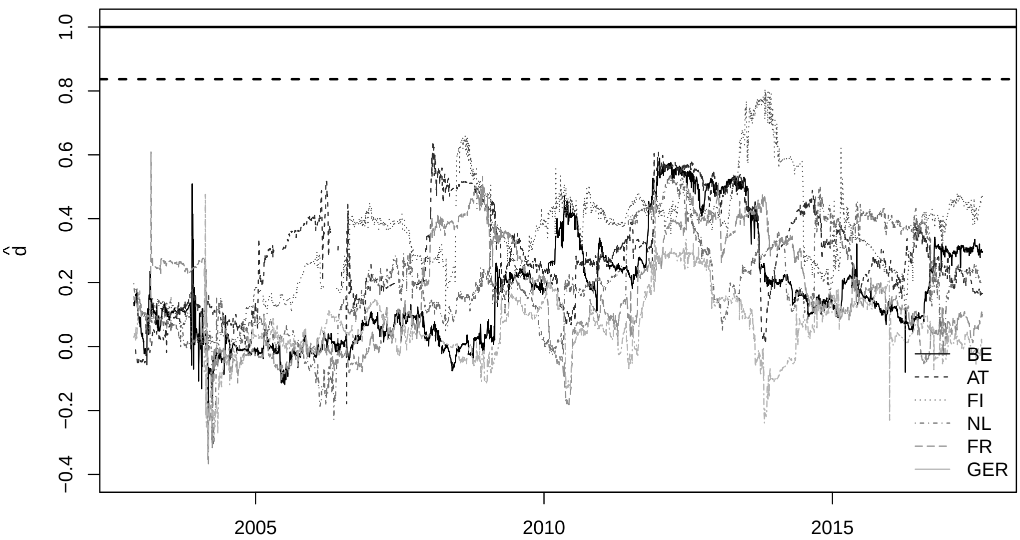

Figure B.3: Memory estimates for the bid-ask spreads of the core countries. In analogy to Figures 3.4 and 3.5, the estimates of d are obtained using the exact local Whittle estimator of Shimotsu and Philips (2005) with a bandwidth of $m = \lfloor T \rfloor^{0.7}$ in a rolling window of 250 observations. Every value represents the estimated memory parameter from the sample ending on the respective day. Vertical dashed lines indicate the timing of bull and bear markets as defined in Table 3.1 and horizontal dashed lines mark pointwise critical values for a test of $H_0: \ d(ba_{it}) = 1$.

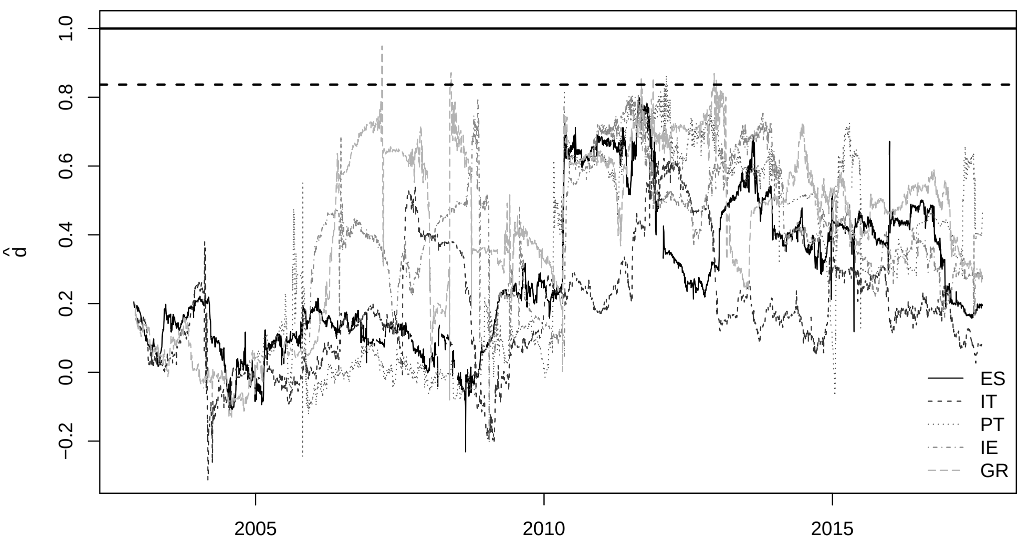

Figure B.4: Memory estimates for the bid-ask spreads of the periphery countries. In analogy to Figures 3.4 and 3.5, the estimates of d are obtained using the exact local Whittle estimator of Shimotsu and Philips (2005) with a bandwidth of $m = \lfloor T \rfloor^{0.7}$ in a rolling window of 250 observations. Every value represents the estimated memory parameter from the sample ending on the respective day. Vertical dashed lines indicate the timing of bull and bear markets as defined in Table 3.1 and horizontal dashed lines mark pointwise critical values for a test of $H_0: \ d(ba_{it}) = 1$.

Chapter 4

Testing for Breaks in the Cointegrating Relationship: On the Stability of Government Bond Markets' Equilibrium

Co-authored with Paulo M. M. Rodrigues and Philipp Sibbertsen.

4.1 Introduction

Since the seminal works of Engle and Granger (1987) and Johansen (1988) cointegration testing has become an important topic of research, both theoretically as well as empirically. The equilibrium relationship between economic and financial variables postulated by many economic theories is typically assumed to be constant over time, i.e., cointegrating relationships do not change. However, this assumption may be too restrictive.

A constant long-run equilibrium may be questionable in light of the growing empirical evidence that economic and financial time series may display persistence changes over time (see, inter alia, Kim (2000), Kim et al. (2002), Busetti and Taylor (2004), and Harvey et al. (2006), for tests when the order of integration is integer; and Giraitis and Leipus (1994), Beran and Terrin (1996), Beran and Terrin (1999), Sibbertsen and Kruse (2009), Hassler and Scheithauer (2011), Hassler and Meller (2014), and Martins and Rodrigues (2014), for tests when the order of integration is some real number). Hence, it is natural to expect that changes in the persistence of economic and financial time series may also originate changes in the long-run equilibrium. This has been substantiated in recent years by a vast literature documenting changes in the historical behaviour of economic and financial variables; see among others, McConnell and Perez-Quiros (2000), Herrera and Pesavento (2005), Cecchetti et al. (2006), Kang et al. (2009) and Halunga et al. (2009).

The impact of structural breaks in the deterministic kernels on cointegration has been widely analysed (see e.g. Hansen (1992), Quintos and Phillips (1993), Hao (1996), Andrews et al. (1996), Bai and Perron (1998), Kuo (1998), Inoue (1999), Johansen et al. (2000), and Lütkepohl et al. (2003), but less attention has been given to the impact of changes in the actual long-run equilibrium (see Martins and Rodrigues (2018)). The focus of this paper is to propose new tests capable of detecting changes in fractional cointegration relationships. We introduce procedures designed to detect changes in the long-run equilibrium between macroeconomic or financial variables based on rolling, recursive forward and recursive reverse estimation of the Hassler and Breitung (2006) test, in the spirit of the approaches proposed by e.g. Davidson and Monticini (2010). Asymptotic results are derived and the performance of the new tests evaluated in an in-depth Monte Carlo exercise. In particular, special attention is devoted to the case of unknown orders of integration of the variables involved due to its empirical relevance. Furthermore, we apply the new test statistics to the government bond market of the European Monetary Union (EMU) finding evidence of segmented fractional cointegration with breaks at the beginning of the European debt crisis.

This paper is organized as follows. Section 2 presents the model specification and assumptions; Section 3 introduces the tests for no cointegration under persistence breaks, a break point estimator, and corresponding asymptotic theory; Section 4 discusses the results of an in-depth Monte Carlo analysis on the finite sample properties of the new tests; Section 5 illustrates the application of the new procedures to the EMU government bond market; Section 6 concludes the paper and finally, an appendix collects all the proofs.

4.2 Model Specification and Assumptions

Consider an m-dimensional process $\mathbf{x_t}$ integrated of order d, $I(d)$, and let y_t be an one-dimensional $I(d)$ process as well. The processes $\mathbf{x_t}$ and y_t are said to be fractionally cointegrated if, considering the regression,

$$y_t = \mathbf{x}_t' \, \beta + u_t, \qquad t = 1, \ldots, T, \tag{4.1}$$

u_t is integrated of order $I(d-b)$ with $b > 0$.

In what follows the focus is on testing the null hypothesis of no fractional cointegration, $H_0 : b = 0$. The usual alternative in this setting is to have fractional cointegration over the whole range of observations, $H_1 : b > 0$. However, we are interested in testing for segmented fractional cointegration. This means that the fractional cointegration relationship may hold only in subsamples of the period under analysis. Therefore, our alternative hypothesis is $H_1 : b_t > 0,$ for $t = \lfloor \lambda_1 T \rfloor + 1, \ldots, \lfloor \lambda_2 T \rfloor$ and $b_t = 0$ elsewhere, with $0 \leq \lambda_1 < \lambda_2 \leq 1$.

The test statistics that will be proposed are based on the approach of Hassler and Breitung (2006), who provide a regression-based test for the null of no fractional cointegration on the residuals, $\hat{u}_t$, of a model as in (1). Before presenting the relevant test statistics let us make the following assumptions:

Assumption 1: Let y_t and $\mathbf{x_t}$ be fractionally integrated of orders d_1 and d_2, respectively with $y_t = 0$ and $\mathbf{x}_t = 0$ for $t \leq 0$.

Assumption 2: The vector $\mathbf{v}_t' := (v_{1,t}, \mathbf{v}_{2,t}') = (\Delta_+^{d_1} y_t,\ \Delta_+^{d_2} \mathbf{x}_t')$, is a stationary vector autoregressive process of order p of the form

$$\mathbf{v}_t = A_1 \mathbf{v}_{t-1} + \cdots + A_p \mathbf{v}_{t-p} + \varepsilon_t \tag{4.2}$$

where $\Delta_+^{d_1} y_t := (1-L)^{d_1} y_t I(t>0)$, $\Delta_+^{d_2} \mathbf{x}_t := (1-L)^{d_2} \mathbf{x}_t I(t>0)$, $I(\cdot)$ is the indicator function, L denotes the usual backshift or lag operator and the error process ε_t is assumed to be independent and identically distributed (iid) with mean zero and covariance matrix,

$$\mathbf{\Sigma} := \begin{pmatrix} \sigma_{11}^2 & \sigma_{21}' \\ \sigma_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}.$$

4.3 Testing for no Cointegration under Persistence Breaks

As in Hassler and Breitung (2006) the cointegrating vector β is not identified under the null hypothesis of no cointegration. Thus, considering that $d_1 = d_2 = d$, we define the following regression model,

$$\Delta_+^d y_t = \Delta_+^d \mathbf{x}_t' \beta + e_t, \qquad \beta := \mathbf{\Sigma}_{22}^{-1} \sigma_{21} \tag{4.3}$$

where $e_t := v_{1,t} - \mathbf{v}_{2,t}' \mathbf{\Sigma}_{22}^{-1} \sigma_{21}$.

The LM test for no cointegration is then applied to the OLS residuals, $\hat{e}_t$, obtained from (4.3), i.e.,

$$\Delta_+^d y_t = \Delta_+^d \mathbf{x}_t' \hat{\beta} + \hat{e}_t$$

where

$$\hat{e}_t := e_t - \sum_{t=1}^{T} \mathbf{v}_{2,t}' e_t \left(\sum_{t=1}^{T} \mathbf{v}_{2,t} \mathbf{v}_{2,t}' \right)^{-1} \mathbf{v}_{2,t}.$$

Specifically, to implement the tests proposed by Hassler and Breitung (2006) and Demetrescu et al. (2008), which is the approach followed in this paper, a regression framework is considered, viz.,

$$\hat{e}_t = \phi \hat{e}_{t-1}^* + \sum_{i=1}^{p} \gamma_i \hat{e}_{t-i} + a_t, \qquad t = 1, \ldots, T, \tag{4.4}$$

where $\hat{e}_{t-1}^* := \sum_{j=1}^{t-1} j^{-1} \hat{e}_{t-j}$ and a_t is a martingale difference sequence. Equation (4.4) is used to test the null $H_0 : \phi = 0$ $(b = 0)$ against the alternative $H_1 : \phi < 0$ $(b > 0)$.

Remark 3.1: Under local alternatives of the form $\mathrm{H}_1 : b = c/\sqrt{T}$ with a fixed $c > 0$, it can be shown that $\phi = -c/\sqrt{T} + O\left(T^{-1}\right)$ and that $\{a_t\}$ is a fractionally integrated noise component. As a result, the heterogeneous behavior of ϕ and the different stochastic properties of a_t provide a sound statistical basis to identify the order of fractional integration of $\{\hat{e}_t\}$. Despite the apparent theoretical simplicity of this framework, the fact that $\hat{e}^*_{t-1}$ converges in mean square sense to $e^{**}_{t-1} := \sum_{j=1}^{\infty} j^{-1} e_{t-j,d}$ under the null hypothesis and Assumption 1, with $\left\{e^{**}_{t-1}\right\}$ being a stationary linear process with non-absolutely summable coefficients, is a source of major technical difficulties for the asymptotic analysis in this context; see e.g. Hassler et al. (2009). □

Remark 3.2: Demetrescu et al. (2008) and Hassler et al. (2009) derive the asymptotic theory of the fractional integration tests under least-squares (LS) estimation of the set of parameters $\kappa := (\phi, \gamma_1, ..., \gamma_p)'$ of a regression as in (4.4), and show that these are $\sqrt{T}$-consistency and asymptotic normal under fairly general conditions. As a result, in a conventional setting as in (4.4) $\mathrm{H}_0 : \phi = 0$ can be tested by means of a standard t-ratio, or some measurable transformation such as its squares. If our assumptions are strengthened such that $a_t \sim iid\mathcal{N}\left(0, \sigma^2\right)$, the specific harmonic weighting upon which $\left\{e^*_{t-1}\right\}$ is constructed in (4.4) also ensures efficient testing. □

In this paper we concentrate on the case of iid errors, e_t, ($p = 0$ in (4.4)) although it is also possible to allow for serial correlation in the innovations. Following Demetrescu et al. (2008) this can be accommodated through parametric augmentation as in (4.4) allowing for $p > 0$.

4.3.1 The Test Statistics

As we are interested in testing for no fractional cointegration against the alternative of segmental fractional cointegration, we apply the Hassler and Breitung (2006) test on a subinterval defined by the truncation points λ_1 and λ_2 with $0 \leq \lambda_1 < \lambda_2 \leq 1$. Thus, for λ_1 and λ_2 fixed we consider the statistic,

$$t(\hat{e}(\lambda_1, \lambda_2)) = \frac{\sqrt{\lfloor \lambda_2 T \rfloor - \lfloor \lambda_1 T \rfloor} \sum_{t=\lfloor \lambda_1 T \rfloor + 1}^{\lfloor \lambda_2 T \rfloor} \hat{e}_t(\lambda_1, \lambda_2) \hat{e}^*_{t-1}(\lambda_1, \lambda_2)}{\sqrt{\sum_{t=\lfloor \lambda_1 T \rfloor + 1}^{\lfloor \lambda_2 T \rfloor} \hat{e}^{*2}_{t-1}(\lambda_1, \lambda_2)} \sqrt{\frac{1}{T-1} \sum_{t=\lfloor \lambda_1 T \rfloor + 1}^{\lfloor \lambda_2 T \rfloor} \hat{e}^2_t(\lambda_1, \lambda_2)}} \tag{4.5}$$

where $\hat{e}_t(\lambda_1, \lambda_2)$ are the subsample based residuals and $\hat{e}^*_{t-1}(\lambda_1, \lambda_2)$ the corresponding harmonic weighted residuals as defined in (4.4).

However, since the breakpoints, λ_1 and λ_2, are usually unknown we adopt the split sample testing approach proposed by Davidson and Monticini (2010), and define the following sets on which the tests will be performed:

$$\Lambda_S = \left\{\left\{0, \frac{1}{2}\right\}, \left\{\frac{1}{2}, 1\right\}\right\} \tag{4.6}$$

$$\Lambda_{0f} = \{\{0, s\} : s \in [\lambda_0, 1]\} \tag{4.7}$$

$$\Lambda_{0b} = \{\{s, 1\} : s \in [0, 1 - \lambda_0]\} \tag{4.8}$$

$$\Lambda_{0R} = \{\{s, s + \lambda_0\} : s \in [0, 1 - \lambda_0]\} \tag{4.9}$$

where Λ_S represents a simple split sample with just two elements; Λ_{0f} and Λ_{0b} denote forward- and backward-running incremental samples, respectively of minimum length $\lfloor \lambda_0 T \rfloor$ and maximum length T; Λ_{0R} defines a rolling sample of fixed length $\lfloor \lambda_0 T \rfloor$, and finally $\lambda_0 \in (0, 1)$ is fixed and needs to be chosen by the practitioner. Davidson and Monticini (2010) consider two additional sets, namely $\Lambda_S^* = \Lambda_S \cup \{0, 1\}$ and $\Lambda_{0R}^* = \Lambda_{0R} \cup \{0, 1\}$.

Therefore, considering the sets in (6) to (9), our proposed test procedures against breaks in the fractional cointegration relation are the split sample tests,

$$\mathcal{T}_S := \max_{\{\lambda_1, \lambda_2\} \in \Lambda_S} t^2(\hat{e}(\lambda_1, \lambda_2)); \tag{4.10}$$

$$\mathcal{T}_S^* := \max_{\{\lambda_1, \lambda_2\} \in \Lambda_S^*} t^2(\hat{e}(\lambda_1, \lambda_2)); \tag{4.11}$$

the incremental (recursive) tests

$$\mathcal{T}_{I_f}(\lambda) := \max_{\lambda_0 \leq \lambda \leq 1} t^2(\hat{e}(0, \lambda)); \tag{4.12}$$

$$\mathcal{T}_{I_b}(\lambda) := \max_{0 \leq \lambda \leq 1 - \lambda_0} t^2(\hat{e}(\lambda, 1)); \tag{4.13}$$

the rolling sample test

$$\mathcal{T}_R(\lambda) := \max_{0 \leq \lambda \leq 1 - \lambda_0} t^2(\hat{e}(\lambda, \lambda + \lambda_0)); \tag{4.14}$$

$$\mathcal{T}_R^*(\lambda) := \max_{\{\lambda_1, \lambda_2\} \in \Lambda_{0R}^*} t^2(\hat{e}(\lambda_1, \lambda_2)). \tag{4.15}$$

We can state these statistics in general form as,

$$\mathcal{T}_K(\lambda_1, \lambda_2) := \max_{\lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2} t^2(\hat{e}(\lambda_1, \lambda_2)), \; K = S, S^*, I_f, I_b, R, R^\star. \tag{4.16}$$

4.3.2 Asymptotic Results

To characterize the asymptotic behavior of the test statistics in (4.10) - (4.15), consider first Theorem 1 provided next, which states the asymptotic normality of the test statistic in (4.5) and which is the main building block of the test statistics $\mathcal{T}_K(\lambda_1, \lambda_2)$, $K = S, S^*, I_f, I_b, R, R^\star$.

Theorem 4.3.1. Assuming that the data is generated from (4.1) and that Assumptions 1 and 2 hold, it follows under the null hypothesis of no fractional cointegration that, as $T \to \infty$,

$$t(\hat{e}(\lambda_1, \lambda_2)) \Rightarrow N(0, 1), \tag{4.17}$$

where $\Rightarrow$ denotes weak convergence.

Hence, based on the result of Theorem 1 we can now state the limit results for the test statistics introduced in (4.10) - (4.15).

Theorem 4.3.2. Assuming that the data is generated from (4.1) and that Assumptions 1 and 2 hold, under the null hypothesis of no fractional cointegration it follows, as $T \to \infty$, that

$$\mathcal{T}_K(\lambda_1, \lambda_2) \Rightarrow \sup_{\lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2} \chi_1^2, \;\; K = S, S^*, I_f, I_b, R, R^*. \tag{4.18}$$

As a next step we provide an estimator of the break point τ under the alternative. The estimator basically consists of minimizing the sum of squared residuals of a regression as in (4.3). Thus, our break point estimator is

$$\hat{\tau} = \underset{\tau \in \Delta}{arg\ inf}\ [\tau T]^{-2\hat{d}} \sum_{t=1}^{[\tau T]} \hat{e}_t^2(\tau) \tag{4.19}$$

where, $\Delta := (\delta; (1 - \delta))$ and $0 < \delta < 0.5$ is an interval eliminating the first and last observations to have enough observations at hand for the break point estimation. For this statistic, the following consistency result can be stated:

Theorem 4.3.3. Assuming that the break is from the cointegrated subsample to the non-cointegrated subsample and that Assumptions 1 and 2 hold, as $T \to \infty$, than

$$\hat{\tau} \to \tau_0. \tag{4.20}$$

where τ_0 denotes the true break fraction.

Remark 3.3: If the break is from the non-cointegrated to the cointegrated sample then the reversed sum of squared residuals, from T to $\lfloor \tau T \rfloor$, can be used to consistently estimate the break fraction τ_0. □

4.4 Monte Carlo Study

In this Section, we analyze the finite-sample properties of the residual-based tests for segmented fractional cointegration introduced above by means of Monte Carlo simulation. The data generation process (DGP) considered for the empirical size and power analysis is

$$y_t = x_t + e_t, \qquad t = 1, ..., T \tag{4.21}$$

$$x_t = x_{t-1} + v_t, \tag{4.22}$$

$$(1-L)^{(1-b_t)} e_t = a_t, \tag{4.23}$$

where

$$\begin{pmatrix} v_t \\ a_t \end{pmatrix} \sim iidN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$$

For $\rho = 0$, x_t is strictly exogenous whereas for $\rho \neq 0$, x_t is correlated with e_t (i.e. endogenous).

For implementation of the tests we compute the OLS residuals,

$$\hat{e}_t = y_t - \hat{\alpha} - \hat{\beta} x_t, \tag{4.24}$$

run the test regression in (4) on these residuals ($\hat{e}_t$) and compute the different test statistics introduced in the previous section, i.e., $\mathcal{T}_S^*$, $\mathcal{T}_{I_f}(\lambda_0)$, $\mathcal{T}_{I_b}(\lambda_0)$, and $\mathcal{T}_R(\lambda_0)$, as well as the full sample test proposed by Hassler and Breitung (2006), which we denote as $\mathcal{T}_{HB}$. All results reported are for a 5% significance level and are based on 5000 Monte Carlo replications. We present results for sample sizes $T = \{250, 500\}$.

For benchmarking purposes, we consider the test statistics computed either for iid innovations as in Breitung and Hassler (2002) or using Eicker-White's correction against heteroskedasticity as in Demetrescu et al. (2008).

To compute the critical values for the tests we generate data from

$$y_t = x_t + e_t, \qquad t = 1, ..., T \tag{4.25}$$

$$(1-L)^{d_1} x_t = v_t, \tag{4.26}$$

$$(1-L)^{d_1} e_t = a_t, \tag{4.27}$$

with $d_1 = \{0.5, 0.6, ..., 1\}$ and computed the critical values as the average of the critical values obtained for each d_1 considered at a specific significance level (see Table 4.1).

	$\mathcal{T}_S^*$	$\mathcal{T}_{I_f}(\lambda_0)$	$\mathcal{T}_{I_b}(\lambda_0)$	$\mathcal{T}_R(\lambda_0)$
		$T = 250$		
1%	9.438	7.722	7.699	7.172
5%	5.960	4.458	4.471	4.112
10%	4.470	3.130	3.133	2.867
		$T = 500$		
1%	8.888	7.387	7.405	6.862
5%	5.737	4.293	4.296	3.955
10%	4.381	3.000	3.006	2.767

Table 4.1: Critical values for subsample tests. For implementation of the tests we considered $\lambda_0 = 0.5$ and all results are based on 5000 Monte Carlo replications.

4.4.1 Empirical Rejection Frequencies

For the analysis of the finite sample rejection frequencies under the null and alternative hypothesis, we consider three experiments:

Experiment 1: Constant cointegration relation over the whole sample.

Experiment 2: Spurious regime in the first part of the sample and a fractional cointegrated regime in the second part, i.e.,

$$\begin{cases} b_t = 0 & for \quad t = 1, ..., \lfloor \lambda T \rfloor \\ b_t > 0 & for \quad t = \lfloor \lambda T \rfloor + 1, ..., T \end{cases} . \tag{4.28}$$

Experiment 3: Fractional cointegrated regime in the first part of the sample and a spurious regime in the second part of the sample, i.e.,

$$\begin{cases} b_t > 0 & for \quad t = 1, ..., \lfloor \lambda T \rfloor \\ b_t = 0 & for \quad t = \lfloor \lambda T \rfloor + 1, ..., T \end{cases} \tag{4.29}$$

with $\lambda \in \{0.3, 0.5, 0.7\}$ in both experiments 2 and 3.

In the case of Experiment 1, data is generated from (4.21) - (4.23), where y_t and x_t are both I(1) variables and $b_t = b = \{0, 0.05, 0.10, ..., 0.50\}$ which allows us to look at the empirical rejection frequencies under the null hypothesis (empirical size, $b = 0$) as well as under the alternative (finite sample power, $b_t > 0$). The first observation we can make from the upper panel of Table 4.2 is that for $T = 250$, with the exception of $\mathcal{T}_{HB}$ (which displays an empirical size of 8.4%), all other tests have acceptable finite sample size (ranging between 5.2% and 6.1%). As the sample size increases to $T = 500$ all tests improve in size (for $\mathcal{T}_{HB}$ the empirical rejection frequency under the null hypothesis reduces to 6.4% whereas for the other subsample tests it ranges between 4.5% and 4.9%).

$\rho = 0$										
	$T = 250$					$T = 500$				
b	$\mathcal{T}_S^*$	$\mathcal{T}_{I_f}(\lambda_0)$	$\mathcal{T}_{I_b}(\lambda_0)$	$\mathcal{T}_R(\lambda_0)$	$\mathcal{T}_{HB}$	$\mathcal{T}_S^*$	$\mathcal{T}_{I_f}(\lambda_0)$	$\mathcal{T}_{I_b}(\lambda_0)$	$\mathcal{T}_R(\lambda_0)$	$\mathcal{T}_{HB}$
0.00	0.061	0.061	0.058	0.052	0.084	0.049	0.046	0.045	0.048	0.064
0.05	0.153	0.182	0.181	0.115	0.225	0.208	0.251	0.255	0.158	0.306
0.10	0.384	0.444	0.448	0.258	0.514	0.624	0.694	0.692	0.434	0.752
0.15	0.690	0.756	0.757	0.485	0.811	0.932	0.959	0.959	0.758	0.969
0.20	0.899	0.938	0.939	0.697	0.960	0.995	0.998	0.998	0.943	0.999
0.25	0.987	0.995	0.995	0.879	0.996	0.999	1	1	0.995	1
0.30	1	1	1	0.957	1	1	1	1	0.999	1
0.35	1	1	1	0.992	1	1	1	1	1	1
0.40	1	1	1	0.998	1	1	1	1	1	1
0.45	1	1	1	1	1	1	1	1	1	1
0.50	1	1	1	1	1	1	1	1	1	1

$\rho = 0.8$										
	$T = 250$					$T = 500$				
b	$\mathcal{T}_S^*$	$\mathcal{T}_{I_f}(\lambda_0)$	$\mathcal{T}_{I_b}(\lambda_0)$	$\mathcal{T}_R(\lambda_0)$	$\mathcal{T}_{HB}$	$\mathcal{T}_S^*$	$\mathcal{T}_{I_f}(\lambda_0)$	$\mathcal{T}_{I_b}(\lambda_0)$	$\mathcal{T}_R(\lambda_0)$	$\mathcal{T}_{HB}$
0.00	0.048	0.053	0.051	0.039	0.080	0.051	0.055	0.055	0.047	0.066
0.05	0.159	0.199	0.205	0.094	0.263	0.314	0.374	0.365	0.149	0.403
0.10	0.454	0.546	0.548	0.217	0.625	0.796	0.852	0.850	0.426	0.862
0.15	0.798	0.871	0.868	0.418	0.907	0.983	0.988	0.989	0.747	0.989
0.20	0.964	0.980	0.979	0.630	0.987	0.999	0.999	0.999	0.938	0.999
0.25	0.996	0.999	0.998	0.789	0.999	1	1	1	0.988	1
0.30	1	1	1	0.915	1	1	1	1	0.998	1
0.35	1	1	1	0.967	1	1	1	1	0.999	1
0.40	1	1	1	0.986	1	1	1	1	1	1
0.45	1	1	1	0.994	1	1	1	1	1	1
0.50	1	1	1	0.999	1	1	1	1	1	1

Table 4.2: Rejection frequencies of tests in Experiment 1 with $\lambda_0 = 0.5$.

Also in terms of power an improvement is observed. In the lower panel with endogenous x_t, we observe lower empirical sizes for $T = 250$ compared to the exogenous case and slightly higher sizes for $T = 500$. The power is always better than with exogenous x_t. Overall, all tests are relatively robust to endogeneity. Note, that of the set of sequential tests proposed, the best performing in both cases are the recursive tests, $\mathcal{T}_{I_f}(\lambda_0)$ and $\mathcal{T}_{I_b}(\lambda_0)$, although, as expected, $\mathcal{T}_{HB}$ displays in the case of Experiment 1 the overall best performance.

In the case of Experiment 2, the sample is divided into two sub-periods where in the first sub-period there is no cointegration ($b = 0$) and in the second the variables are cointegrated ($b > 0$). We allow the change into the cointegrated regime to be early in the sample ($\lambda = 0.3$), in the middle of the sample ($\lambda = 0.5$) and late in the sample ($\lambda = 0.7$). We consider a similar exercise in Experiment 3 except that the first sub-period corresponds to cointegration ($b > 0$) and the second to a spurious regression ($b = 0$). From Table 4.3

	T = 250					T = 500				
b	$\mathcal{T}_S^*$	$\mathcal{T}_{I_f}(\lambda_0)$	$\mathcal{T}_{I_b}(\lambda_0)$	$\mathcal{T}_R(\lambda_0)$	$\mathcal{T}_{HB}$	$\mathcal{T}_S^*$	$\mathcal{T}_{I_f}(\lambda_0)$	$\mathcal{T}_{I_b}(\lambda_0)$	$\mathcal{T}_R(\lambda_0)$	$\mathcal{T}_{HB}$
$\lambda = 0.3$										
0.00	0.055	0.058	0.061	0.054	0.076	0.058	0.055	0.056	0.056	0.065
0.05	0.079	0.077	0.079	0.051	0.104	0.082	0.083	0.083	0.057	0.096
0.10	0.101	0.100	0.103	0.050	0.128	0.133	0.125	0.136	0.052	0.141
0.15	0.134	0.129	0.144	0.051	0.166	0.189	0.173	0.182	0.053	0.191
0.20	0.161	0.151	0.167	0.054	0.189	0.254	0.222	0.238	0.051	0.243
0.25	0.202	0.178	0.194	0.053	0.221	0.311	0.265	0.272	0.052	0.293
0.30	0.237	0.210	0.230	0.050	0.257	0.375	0.325	0.339	0.050	0.351
0.35	0.281	0.247	0.262	0.051	0.298	0.453	0.393	0.410	0.052	0.420
0.40	0.310	0.275	0.293	0.056	0.324	0.499	0.424	0.437	0.052	0.454
0.45	0.353	0.307	0.313	0.049	0.359	0.537	0.467	0.473	0.058	0.493
0.50	0.397	0.341	0.353	0.055	0.393	0.594	0.514	0.527	0.052	0.543
$\lambda = 0.5$										
0.00	0.051	0.060	0.060	0.048	0.078	0.054	0.059	0.057	0.054	0.069
0.05	0.092	0.094	0.099	0.063	0.126	0.114	0.115	0.118	0.090	0.132
0.10	0.159	0.147	0.155	0.091	0.182	0.279	0.224	0.239	0.164	0.248
0.15	0.269	0.207	0.227	0.142	0.260	0.474	0.331	0.345	0.228	0.361
0.20	0.411	0.285	0.298	0.204	0.336	0.658	0.426	0.436	0.283	0.454
0.25	0.530	0.345	0.358	0.240	0.400	0.775	0.532	0.531	0.344	0.560
0.30	0.640	0.409	0.418	0.267	0.463	0.832	0.593	0.594	0.373	0.619
0.35	0.727	0.462	0.473	0.297	0.518	0.871	0.657	0.654	0.404	0.676
0.40	0.770	0.515	0.518	0.325	0.565	0.894	0.707	0.700	0.425	0.727
0.45	0.811	0.565	0.566	0.328	0.618	0.906	0.739	0.732	0.432	0.757
0.50	0.832	0.611	0.612	0.348	0.653	0.924	0.766	0.766	0.452	0.783
$\lambda = 0.7$										
0.00	0.060	0.062	0.058	0.053	0.085	0.056	0.056	0.058	0.054	0.066
0.05	0.114	0.128	0.133	0.072	0.166	0.154	0.167	0.178	0.080	0.188
0.10	0.207	0.216	0.232	0.090	0.266	0.342	0.360	0.364	0.119	0.386
0.15	0.346	0.344	0.347	0.105	0.398	0.583	0.546	0.538	0.137	0.572
0.20	0.509	0.465	0.467	0.125	0.518	0.739	0.657	0.646	0.181	0.679
0.25	0.625	0.534	0.542	0.145	0.592	0.847	0.741	0.724	0.210	0.757
0.30	0.726	0.619	0.613	0.159	0.660	0.887	0.780	0.766	0.239	0.796
0.35	0.798	0.669	0.664	0.177	0.714	0.912	0.818	0.810	0.279	0.831
0.40	0.837	0.710	0.700	0.197	0.738	0.927	0.837	0.825	0.311	0.850
0.45	0.871	0.742	0.735	0.227	0.774	0.933	0.843	0.832	0.336	0.854
0.50	0.884	0.761	0.751	0.237	0.789	0.946	0.868	0.854	0.369	0.878

Table 4.3: Rejection frequencies of tests in Experiment 2 with $\lambda_0 = 0.5$.

we observe first that the overall best performing test of the sequential tests introduced is $\mathcal{T}_S^*$ followed by $\mathcal{T}_{I_f}(\lambda_0)$. The overall test $\mathcal{T}_{HB}$, although slightly oversized, also displays interesting power performance. The good behavior of $\mathcal{T}_S^*$ is clearly observable in the larger sample ($T = 500$) where it stands out particularly for $\lambda = 0.5$ and $\lambda = 0.7$. For $\lambda = 0.3$ the difference of $\mathcal{T}_S^*$ with regards to $\mathcal{T}_{HB}$ is not as marked.

Table 4.4 reports results for the case where there is cointegration in the first sub-period and in the second sub-period the results are spurious. In this case the rolling approach

$\mathcal{T}_R(\lambda_0)$ displays interesting behavior, particularly for $b_t > 0.15$ and $T = 250$ and for $b_t > 0.1$ when $T = 500$. The $\mathcal{T}_S^*$ statistic also displays good power performance.[7]

	$T = 250$					$T = 500$				
b	$\mathcal{T}_S^*$	$\mathcal{T}_{I_f}(\lambda_0)$	$\mathcal{T}_{I_b}(\lambda_0)$	$\mathcal{T}_R(\lambda_0)$	$\mathcal{T}_{HB}$	$\mathcal{T}_S^*$	$\mathcal{T}_{I_f}(\lambda_0)$	$\mathcal{T}_{I_b}(\lambda_0)$	$\mathcal{T}_R(\lambda_0)$	$\mathcal{T}_{HB}$
$\lambda = 0.3$										
0.00	0.061	0.061	0.058	0.052	0.084	0.058	0.055	0.054	0.050	0.076
0.05	0.111	0.137	0.128	0.115	0.171	0.152	0.169	0.165	0.162	0.208
0.10	0.257	0.273	0.270	0.258	0.326	0.399	0.401	0.394	0.416	0.462
0.15	0.438	0.432	0.426	0.479	0.504	0.709	0.661	0.655	0.753	0.717
0.20	0.653	0.617	0.603	0.682	0.677	0.915	0.832	0.832	0.934	0.868
0.25	0.830	0.741	0.735	0.863	0.788	0.980	0.910	0.908	0.988	0.929
0.30	0.927	0.828	0.819	0.940	0.864	0.995	0.939	0.937	0.998	0.954
0.35	0.962	0.873	0.865	0.978	0.899	0.998	0.958	0.954	0.998	0.966
0.40	0.986	0.908	0.902	0.993	0.933	1.000	0.974	0.973	1.000	0.979
0.45	0.995	0.926	0.920	0.997	0.944	0.999	0.982	0.980	1.000	0.986
0.50	0.997	0.948	0.943	0.998	0.961	0.999	0.981	0.979	1.000	0.986
$\lambda = 0.5$										
0.00	0.058	0.059	0.061	0.057	0.081	0.049	0.048	0.050	0.051	0.069
0.05	0.097	0.095	0.093	0.230	0.123	0.115	0.112	0.114	0.360	0.152
0.10	0.193	0.169	0.163	0.509	0.222	0.311	0.237	0.229	0.686	0.288
0.15	0.350	0.250	0.243	0.726	0.305	0.591	0.365	0.363	0.879	0.425
0.20	0.529	0.344	0.334	0.845	0.406	0.823	0.495	0.494	0.962	0.556
0.25	0.702	0.430	0.413	0.926	0.494	0.934	0.602	0.593	0.987	0.651
0.30	0.828	0.516	0.504	0.965	0.574	0.970	0.678	0.675	0.997	0.724
0.35	0.888	0.560	0.551	0.983	0.623	0.980	0.752	0.746	0.996	0.789
0.40	0.937	0.633	0.623	0.991	0.684	0.989	0.780	0.773	0.998	0.820
0.45	0.953	0.673	0.664	0.994	0.721	0.989	0.813	0.817	0.998	0.845
0.50	0.967	0.711	0.703	0.996	0.756	0.991	0.849	0.848	0.999	0.877
$\lambda = 0.7$										
0.00	0.058	0.057	0.055	0.057	0.080	0.051	0.052	0.050	0.054	0.076
0.05	0.071	0.079	0.072	0.079	0.104	0.077	0.085	0.077	0.095	0.113
0.010	0.108	0.107	0.104	0.123	0.139	0.120	0.123	0.117	0.155	0.154
0.15	0.136	0.135	0.129	0.158	0.172	0.181	0.165	0.154	0.223	0.206
0.20	0.163	0.161	0.155	0.197	0.202	0.241	0.222	0.208	0.292	0.269
0.25	0.205	0.191	0.183	0.238	0.245	0.285	0.262	0.250	0.340	0.314
0.30	0.230	0.217	0.212	0.268	0.272	0.351	0.310	0.296	0.411	0.357
0.35	0.263	0.249	0.241	0.306	0.306	0.402	0.359	0.353	0.456	0.418
0.40	0.291	0.274	0.265	0.341	0.328	0.436	0.398	0.388	0.485	0.462
0.45	0.347	0.321	0.314	0.386	0.376	0.496	0.447	0.444	0.543	0.504
0.50	0.368	0.341	0.332	0.413	0.401	0.527	0.484	0.482	0.566	0.543

Table 4.4: Rejection frequencies of tests in Experiment 3 with $\lambda_0 = 0.5$.

We also apply the break point estimator to data from Experiment 3 and residuals from a regression without constant in order to detect a break from cointegration to no

[7] We have also performed simulations with EW corrected statistics, however since the results are qualitatively similar to those reported in Tables 4.2 - 4.4 we have decided not to include them in the paper for the sake of space. These can however be obtained from the authors.

cointegration. Table 4.5 shows the estimated break fraction for different choices of δ. This choice does not have any influence on the results. Therefore for practical purposes, a small δ is recommended in order to keep a large part of the data in the analysis. With small b, there is a tendency to locate the break in the middle of the sample, but the results improve as the cointegrating strength b increases and for the largest b the accuracy is good. Hence, with strong cointegrating relations, the break point estimator delivers reliable results. If there is permanent cointegration, the break is estimated at the end of the admissible window. If the data is generated from Experiment 2, the regression residuals are reversed before applying the break point estimator. The results remain the same and are available upon request.

δ		0.05			0.1			0.15	
$b\backslash\lambda$	0.3	0.5	0.7	0.3	0.5	0.7	0.3	0.5	0.7
0.10	0.564	0.604	0.688	0.559	0.598	0.676	0.550	0.589	0.659
0.15	0.503	0.558	0.667	0.509	0.560	0.666	0.514	0.559	0.665
0.20	0.461	0.526	0.661	0.458	0.526	0.658	0.472	0.524	0.660
0.25	0.424	0.499	0.655	0.437	0.501	0.657	0.436	0.503	0.658
0.30	0.410	0.483	0.654	0.412	0.488	0.656	0.414	0.494	0.659
0.35	0.389	0.470	0.653	0.397	0.473	0.656	0.404	0.478	0.656
0.40	0.373	0.458	0.655	0.381	0.461	0.655	0.392	0.470	0.656
0.45	0.365	0.446	0.648	0.374	0.457	0.651	0.387	0.463	0.653
0.50	0.358	0.448	0.647	0.375	0.453	0.648	0.380	0.458	0.653
no break		0.938			0.890			0.842	

Table 4.5: Break point estimates with $T = 1000$ and 5000 Monte Carlo replications.

4.5 Empirical Application

In this Section, we apply the tests introduced in Section 4.3 to benchmark government bonds of countries that are part of the European Monetary Union (EMU). The analysis is based on daily observations between 01.01.1999 and 08.08.2017 (about 4,800 observations per country) of 10-year-to-maturity benchmark government bonds of eleven EMU countries (Spain, Italy, Portugal, Ireland, Greece, Belgium, Austria, Finland, the Netherlands, France and Germany). The data is obtained from Thomson Reuters Eikon.

According to Leschinski et al. (2018), market integration requires the existence of a (fractional) cointegrating relationship among the goods of the market under consideration. Regarding the European bond market, it is generally accepted that the market is integrated after the introduction of the Euro and prior to the EMU debt crisis or at least up to the subprime mortgage crisis (Baele et al. (2004), Ehrmann et al. (2011), Pozzi and Wolswijk (2012), Christiansen (2014), and Ehrmann and Fratzscher (2017), among others) so that we would expect fractional cointegration during this period. This conclusion

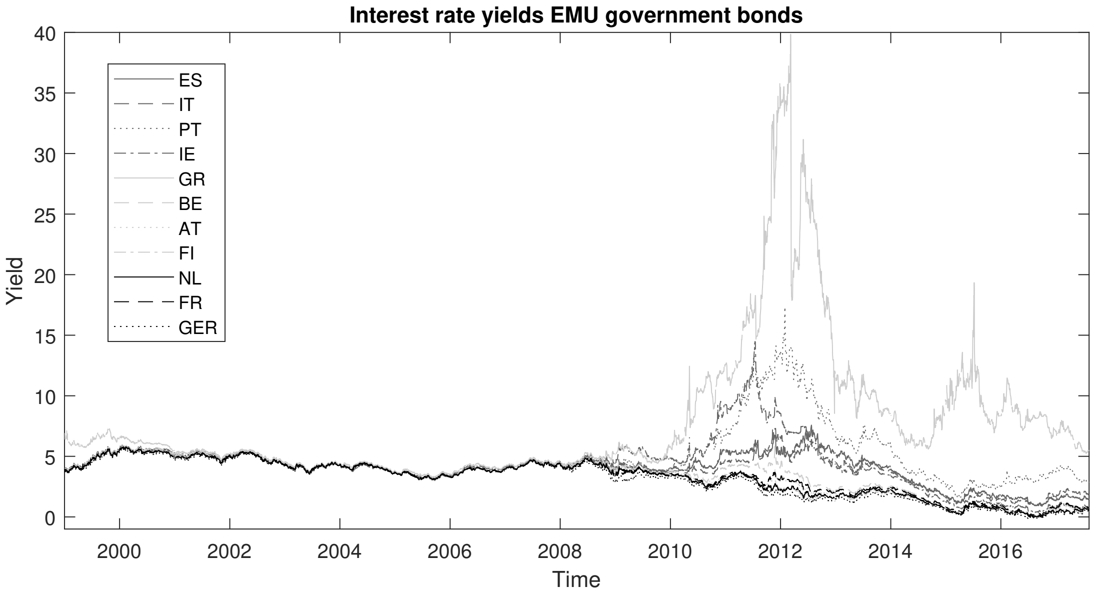

Figure 4.1: Yields of EMU government bonds.

is supported by Figure 4.1 that shows how the bond yields co-move in the beginning. When the crisis began in 2008-2010, they drift apart so that no market integration and no cointegration is assumed any longer. Therefore, it is likely that testing for no cointegration over the full sample does not allow us to reject the null hypothesis. However, with the new tests introduced in this paper we expect to be able to detect cointegration with breaks in the cointegrating relationship in the sense that under the alternative we have fractional cointegration in a certain subsample and no cointegration elsewhere.

	ES	IT	PT	IE	GR	BE	AT	FI	NL	FR	GER
ADF	0.93	0.93	0.93	0.92	0.93	0.93	0.93	0.93	0.93	0.93	0.93
KPSS	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01

Table 4.6: p-values of ADF- and KPSS-tests.

The order of integration of our data is unknown so that we apply unit root and stationarity tests (Table 4.6). The ADF-test, augmented based on the Schwert's rule and including a drift, cannot reject the unit root and the KPSS-test rejects stationarity for all countries leading to the conclusion that $d_i = 1$ for all countries' yields. This might be implausible from an economic perspective. However, the finite sample behavior suggests a unit root which is consistent with results available in the literature on fractional cointegration, confer for example Chen and Hurvich (2003) and Nielsen (2010). The cointegrating regressions are carried out in a bivariate setting where the yield of country i, y_{it}, is regressed on the German yield, $y_{GER,t}$:

$$y_{it} = \beta_0 + \beta_1 y_{GER,t} + e_t, \text{ for } i = 1, ..., 10. \tag{4.30}$$

The residuals obtained from the regressions in (4.30) are used for testing in the split, incremental and rolling sample versions of the test where λ_0 is set to 0.2 and 0.5, respectively. The Hassler-Breitung test is applied to the full sample. In order to account for autocorrelation, we augment the lagged regression (4.4) using the Schwert's rule as suggested in Demetrescu et al. (2008), and we use Eicker-White (EW) heteroscedasticity-robust standard errors as it is more suitable in our empirical setting. The results are given in Table 4.7 and bold numbers indicate rejection at the 5% significance level.

	$\mathcal{T}_{HB}$	$\mathcal{T}_S^*$	$\mathcal{T}_{I_f}(0.2)$	$\mathcal{T}_{I_b}(0.2)$	$\mathcal{T}_R(0.2)$	$\mathcal{T}_{I_f}(0.5)$	$\mathcal{T}_{I_b}(0.5)$	$\mathcal{T}_R(0.5)$
ES	0.05	0.05	1.85	0.70	**8.02**	1.85	0.04	1.66
IT	0.31	0.31	2.51	1.88	**15.31**	2.51	0.79	1.87
PT	2.46	2.46	2.55	3.26	**14.87**	2.55	2.66	2.14
IE	0.04	0.04	**4.90**	4.09	**28.71**	**4.90**	0.09	2.65
GR	0.29	0.55	3.18	2.21	**5.65**	3.18	2.21	2.70
BE	0.45	1.67	**8.30**	2.20	**15.68**	**8.30**	0.66	**6.52**
AT	2.91	4.20	**11.45**	**8.77**	**37.06**	**4.57**	4.22	**6.38**
FI	3.43	**28.03**	**33.84**	**5.98**	**24.43**	**29.30**	**5.00**	**28.92**
NL	**11.42**	**11.42**	**19.34**	**11.15**	**23.19**	**11.60**	**11.15**	**11.36**
FR	2.91	2.91	**11.99**	**5.53**	**11.92**	**11.99**	**5.45**	**9.18**

Table 4.7: Values of test statistic with $\lambda_0 = 0.2$ and $\lambda_0 = 0.5$ with EW heteroscedasticity-robust standard errors, and parametric augmentation to correct for autocorrelation (Schwert's rule).

The Hassler-Breitung test does not reject the null of no cointegration on the full sample for all countries except for the Dutch yield, and the split sample test finds cointegration between the German and Dutch and the German and Finnish yields. The incremental tests with $\lambda_0 = 0.2$ reject the null hypothesis for Austria, Finland, the Netherlands and France in the backward-rolling window and additionally for Ireland and Belgium in the forward-rolling window. Thus, segmented cointegration is found for countries that were less affected by the financial crisis and no cointegration for those more strongly affected. The rolling sample tests rejects the null of no cointegration for all regression pairs. Overall, the results meet the expectation that the European yields are not cointegrated over the whole period. With the new tests for segmented cointegration, we find that the European yields were cointegrated in at least part of the sample.

Davidson and Monticini (2010) recommend the use of $\lambda_0 = 0.5$ because a break must occur in either the first half of the sample or the second. Nonetheless, choosing $\lambda_0 = 0.2$ leads neither to disadvantages nor to advantages which was also confirmed in the Monte Carlo exercise. With $\lambda_0 = 0.5$, the results for the incremental tests are very similar to those with $\lambda_0 = 0.2$, but we get less rejections with the rolling sample test. This could imply that a shorter period than 50% of the sample is fractionally cointegrated or, at least, that the evidence for segmented fractional cointegration for the countries that were most affected by the financial crisis is ambiguous.

The finding of segmented cointegration for the Netherlands does not contradict the rejection of the Hassler-Breitung test as it also has power, albeit less, in the presence of segmented cointegration. The other way round, the tests for segmented cointegration also have power if the cointegrating relation is permanent as they include the full sample as well.

ES 05.05.2010	IT 24.05.2010	PT 27.04.2010	IE 28.04.2010 15.08.2014*	GR 22.04.2010
BE 21.11.2008	AT 14.12.2001	FI 06.12.2002	NL 21.10.2002	FR 21.11.2008

Table 4.8: Break date estimates with $\delta = 0.05$.

In order to gain a deeper understanding of the dynamics, we estimate the break date with the break point estimator proposed in (4.19) based on the regression residuals (without constant). We set $\delta = 0.05$ and impose a minimum length of $\lfloor 0.1T \rfloor$ between the sequentially estimated breaks. The results are given in Table 4.8. The breaks for Spain, Italy, Portugal, Ireland and Greece are estimated in April and May of 2010, hence shortly after the start of the European debt crisis. For France and Belgium we obtain the exact same date in November 2008, i.e. two years earlier than for the previous countries. For Austria, Finland and the Netherlands the breaks are located at the end of 2001 and 2002. We also look at reversed residuals in order to identify potential breaks from no cointegration to cointegration that are indicated by an asterisk. There is one found for Ireland implying that the Irish yield is cointegrated with the German one until 2010, then the cointegrating relationship temporarily dissolves and reemerges in 2014.

If we consider the sample starting 1999 up to the first break, there is still evidence of unit roots in the data and we find the breaks given in Table 4.9. As they are also 'forward'-breaks implying the dissolution of cointegration, they contradict the first found break dates. In the sample between the break date estimates, we do not find 'backward'-breaks that would justify the first break, except for Italy. For Italy, it implies a short period of no cointegration between 2002 and 2004. For the other countries, the 'backward'-break might be too small to be detected or there is a smooth transition. Therefore, it is not clear for Spain, Portugal, Ireland and Greece at which point exactly the relationship with Germany dissolves. The test results in Table 4.7 suggest a short period of cointegration because the rolling test rejects with $\lambda_0 = 0.2$ but not with $\lambda_0 = 0.5$ for these countries.

Strictly speaking, the direction of the estimated break dates for Finland and the Netherlands in 2000 and 2002 imply no cointegration for most of the sample. This contradicts the findings of the tests in Table 4.7 that state rather strong evidence of cointegration, in particular for the Dutch yield. Therefore, we conclude that they are permanently cointegrated.

ES	IT	PT	IE	GR
04.03.2002	16.10.2002	10.12.2001	30.09.2008	30.10.2008
BE	AT	FI	NL	FR
—	05.01.2001	08.05.2000*	11.02.2000*	13.12.2007

Table 4.9: Break dates with $\delta = 0.05$ before the first break in Table 4.8.

Considering the sample from the first break date until 2017, we estimate the break dates in Table 4.10. Those are 'backward'-breaks implying the emergence of a fractional cointegrating relation. They are located in 2012 and 2013 for most of the countries. For Austria, there is another 'forward'-break in 2008, but after that we also find a 'backward'-break on 05.09.2012.

ES	IT	PT	IE	GR
24.05.2013*	02.05.2013*	12.12.2012*	11.12.2014*	12.10.2012*
BE	AT	FI	NL	FR
11.12.2012*	30.10.2008	—	—	05.09.2012*

Table 4.10: Break dates with $\delta = 0.05$ after the first break in Table 4.8.

In Table C.1 in the appendix, all found break dates from sequential estimation are collected, and in all subsamples the data still exhibits unit roots. The table contains further break dates for some countries in 2000 and in 2016 that imply no cointegration at the edges of the sample. However, the dates are very close to the edges, and the Monte Carlo simulation showed estimates very close to the margins in the case of permanent cointegration. Therefore, the validity of the breaks in the small subsamples close to the edges is doubtful and we rather suspect continuous cointegration in the border-subsamples.

All in all, based on the co-movements in Figure 4.1 and the rejections in Table 4.7, we conclude that the yields of the countries were fractionally cointegrated with that of Germany after the introduction of the euro until the European debt crisis. The break point estimates point to the dissolution of fractional cointegrating relationships and market integration at the beginning of the European debt crisis in 2010 although the breaks might have occurred earlier for Spain, Italy, Portugal and Ireland. In 2012/2013 the cointegrating relationships are reestablished. For Finland and the Netherlands the results indicate permanent cointegration.

4.6 Conclusion

In this paper, we present tests for the null of no fractional cointegration against the alternative of segmented fractional cointegration. To do this we develop new tests based on the procedure of Hassler and Breitung (2006) combined with ideas from Davidson and

Monticini (2010). We introduce split sample, forward- and backward-running incremental sample and rolling sample tests for segmented cointegration. We show that the limit distribution of all of these statistics converge to the supremum of a chi-squared distribution. Furthermore, a break point estimator based on minimizing the sum of squared residuals is also proposed.

An in-depth Monte Carlo analysis shows the satisfying size and power properties of our tests in various situations. However, it turns out that the split sample test performs best in terms of power when the break occurs from the spurious to the fractionally cointegrated regime wherever the breakpoint is. On the other hand, if the break is from the fractionally cointegrated regime to the spurious regime, the rolling window test has the best power properties for all possible breakpoints. Therefore, we recommend application of both the split sample and the rolling window tests.

As segmented fractional cointegration is a very likely empirical situation we investigate daily EMU government bonds between January 1999 and August 2017. We find constant fractional cointegration for the Dutch and Finish government bond yields with Germany. For the other countries, namely Spain, Italy, Portugal, Greece, Ireland, Belgium, and France we find segmented fractional cointegration with a period of no fractional cointegration during the European debt or financial crisis.

C Appendix

Before we prove the Theorems define

$$\mathbf{e}^{'}(\lambda_1,\lambda_2) := (e_{\lfloor \lambda_1 T\rfloor+2},\dots,e_{\lfloor \lambda_2 T\rfloor})$$

and

$$\mathbf{e}^{*'}(\lambda_1,\lambda_2) := (e^*_{\lfloor \lambda_1 T\rfloor+1},\dots,e^*_{\lfloor \lambda_2 T\rfloor}).$$

Proof of Theorem 1:
From Lemma A in Hassler and Breitung (2006) we have directly:

$$\begin{aligned}
\frac{1}{\lfloor \lambda_2 T\rfloor - \lfloor \lambda_1 T\rfloor}\mathbf{e}^{'}(\lambda_1,\lambda_2)\mathbf{e}(\lambda_1,\lambda_2) &\overset{P}{\to} \sigma^2 \\
\frac{1}{(\lfloor \lambda_2 T\rfloor - \lfloor \lambda_1 T\rfloor)^{1/2}}\mathbf{e}^{'}(\lambda_1,\lambda_2)\mathbf{e}^*(\lambda_1,\lambda_2) &\Rightarrow N\left(0;\sigma^4\frac{\pi^2}{6}\right) \\
\frac{1}{\lfloor \lambda_2 T\rfloor - \lfloor \lambda_1 T\rfloor}\mathbf{e}^{*'}(\lambda_1,\lambda_2)\mathbf{e}^*(\lambda_1,\lambda_2) &\overset{P}{\to} \sigma^2\frac{\pi^2}{6}.
\end{aligned} \tag{31}$$

The rest of the proof follows exactly the lines of the proof of proposition 3 in Hassler and Breitung (2006) with the only difference that we localize their arguments to the interval $t = \lfloor \lambda_1 T\rfloor + 1,\dots,\lfloor \lambda_2 T\rfloor$. For ease of readability we recall their arguments here.
Defining $\hat{e}_t(\lambda_1,\lambda_2) = e_t(\lambda_1,\lambda_2) - \mathbf{e}^{'}(\lambda_1,\lambda_2)\mathbf{V}_2(\lambda_1,\lambda_2)(\mathbf{V}_2^{'}(\lambda_1,\lambda_2)\mathbf{V}_2(\lambda_1,\lambda_2))^{-1}v_{2,t}(\lambda_1,\lambda_2)$ and $\hat{e}^*_{t-1}(\lambda_1,\lambda_2) = e^*_{t-1}(\lambda_1,\lambda_2) - \mathbf{e}^{'}(\lambda_1,\lambda_2)\mathbf{V}_2(\lambda_1,\lambda_2)(\mathbf{V}_2^{'}(\lambda_1,\lambda_2)\mathbf{V}_2(\lambda_1,\lambda_2))^{-1}v^*_{2,t-1}(\lambda_1,\lambda_2)$ we have

$$\begin{aligned}
\hat{\mathbf{e}}'(\lambda_1,\lambda_2)\hat{\mathbf{e}}(\lambda_1,\lambda_2) &= \mathbf{e}'(\lambda_1,\lambda_2)\mathbf{e}(\lambda_1,\lambda_2) - r_T'\mathbf{V}_2'(\lambda_1,\lambda_2)\mathbf{e}(\lambda_1,\lambda_2), \\
\hat{\mathbf{e}}^{*\prime}(\lambda_1,\lambda_2)\hat{\mathbf{e}}^*(\lambda_1,\lambda_2) &= \mathbf{e}^{*\prime}(\lambda_1,\lambda_2)\mathbf{e}^*(\lambda_1,\lambda_2) - 2r_T'\mathbf{V}_2^{*\prime}(\lambda_1,\lambda_2)\mathbf{e}^*(\lambda_1,\lambda_2) \\
&\quad + r_T'\mathbf{V}_2^{*\prime}(\lambda_1,\lambda_2)\mathbf{V}_2^*(\lambda_1,\lambda_2)r_T, \\
\hat{\mathbf{e}}^{*\prime}(\lambda_1,\lambda_2)\hat{\mathbf{e}}(\lambda_1,\lambda_2) &= \mathbf{e}^{*\prime}(\lambda_1,\lambda_2)\mathbf{e}(\lambda_1,\lambda_2) - r_T'\mathbf{V}_2^{*\prime}(\lambda_1,\lambda_2)\mathbf{e}(\lambda_1,\lambda_2) \\
&\quad - r_T'\mathbf{V}_2'(\lambda_1,\lambda_2)\mathbf{e}^*(\lambda_1,\lambda_2) + r_T'\mathbf{V}_2^{*\prime}(\lambda_1,\lambda_2)\mathbf{V}_2(\lambda_1,\lambda_2)r_T,
\end{aligned}$$

with $r_T = (\mathbf{V}_2'(\lambda_1,\lambda_2)\mathbf{V}_2(\lambda_1,\lambda_2))^{-1}\mathbf{V}_2'(\lambda_1,\lambda_2)\mathbf{e}(\lambda_1,\lambda_2)$, $\mathbf{V}_2 = \left(\mathbf{V}_{2,2}',\dots,\mathbf{V}_{2,T}'\right)$. By Assumption 2 and the iid assumption for v_t it holds

$$\begin{aligned}
\mathbf{V}_2'(\lambda_1,\lambda_2)\mathbf{e}(\lambda_1,\lambda_2) &= O_P(T^{1/2}), \\
r_T &= O_P(T^{-1/2}), \\
\mathbf{V}_2^{*\prime}\mathbf{e}^* &= O_P(T),
\end{aligned}$$

and

$$\frac{1}{\lfloor\lambda_2 T\rfloor - \lfloor\lambda_1 T\rfloor}\mathbf{V}_2^{*\prime}(\lambda_1,\lambda_2)\mathbf{e}(\lambda_1,\lambda_2) \rightarrow 0,$$
$$\frac{1}{\lfloor\lambda_2 T\rfloor - \lfloor\lambda_1 T\rfloor}\mathbf{V}_2^{\prime}(\lambda_1,\lambda_2)\mathbf{e}^*(\lambda_1,\lambda_2) \rightarrow 0.$$

From (31) we now have:

$$\frac{1}{\lfloor\lambda_2 T\rfloor - \lfloor\lambda_1 T\rfloor}\hat{\mathbf{e}}^{\prime}(\lambda_1,\lambda_2)\hat{\mathbf{e}}(\lambda_1,\lambda_2)$$
$$= \frac{1}{\lfloor\lambda_2 T\rfloor - \lfloor\lambda_1 T\rfloor}\mathbf{e}^{\prime}(\lambda_1,\lambda_2)\mathbf{e}(\lambda_1,\lambda_2) + o_P(1) \xrightarrow{P} \sigma^2$$
$$\frac{1}{(\lfloor\lambda_2 T\rfloor - \lfloor\lambda_1 T\rfloor)^{1/2}}\hat{\mathbf{e}}^{\prime}(\lambda_1,\lambda_2)\hat{\mathbf{e}}^*(\lambda_1,\lambda_2)$$
$$= \frac{1}{(\lfloor\lambda_2 T\rfloor - \lfloor\lambda_1 T\rfloor)^{1/2}}\mathbf{e}^{\prime}(\lambda_1,\lambda_2)\mathbf{e}^*(\lambda_1,\lambda_2) + o_P(1) \Rightarrow N\left(0;\sigma^4\frac{\pi^2}{6}\right)$$
$$\frac{1}{\lfloor\lambda_2 T\rfloor - \lfloor\lambda_1 T\rfloor}\hat{\mathbf{e}}^{*\prime}(\lambda_1,\lambda_2)\hat{\mathbf{e}}^*(\lambda_1,\lambda_2)$$
$$= \frac{1}{\lfloor\lambda_2 T\rfloor - \lfloor\lambda_1 T\rfloor}\mathbf{e}^{*\prime}(\lambda_1,\lambda_2)\mathbf{e}^*(\lambda_1,\lambda_2) + o_P(1) \xrightarrow{P} \sigma^2\frac{\pi^2}{6}$$

which proves the theorem. □

Proof of Theorem 2:

The proof follows directly from the results in Theorem 1 and the arguments in Davidson and Monticini (2010). □

Proof of Theorem 3:

Assume that the break is from cointegration to non-cointegration. This is before the break the residuals are of integration order $d-b$ whereas they are of order d after the break. Denote by $\hat{d}$ the estimated integration order based on the whole sample. Then we have $d-b \leq \hat{d} \leq d$.

We thus have

$$\lfloor\tau T\rfloor^{-2\hat{d}}\sum_{t=1}^{\lfloor\tau T\rfloor}\hat{e}_t^2(\tau) = O_P(T^{(d-b)-\hat{d}})1_{[\tau\leq\tau_0]} + \infty 1_{[\tau>\tau_0]}$$

which proves the theorem. □

	ES	IT	PT	IE	GR	BE	AT	FI	NL	FR
1999										
2000	*23.03.2000*	*31.01.2000*	*09.05.2000*		*03.02.2000*		*25.01.2000*	*08.05.2000*	*11.02.2000*	
2001		*10.04.2001*	**10.12.2001**				**14.12.2001**	**19.04.2001**		
2002	**04.03.2002**	**16.10.2002**						**06.12.2002**	**21.10.2002**	
2003				**04.07.2003**						
2004		*30.04.2004*								
2005										
2006										
2007										**13.12.2007**
2008	**05.09.2008**	**05.09.2008**	**15.09.2008**	**30.09.2008**	**30.10.2008**	**21.11.2008**	**30.10.2008**			**21.11.2008**
2009										
2010	**05.05.2010**	**24.05.2010**	**27.04.2010**	**28.04.2010**	**22.04.2010**					
2011										
2012			*12.12.2012*	**02.08.2012**	*12.10.2012*	*11.12.2012*	*05.09.2012*			*05.09.2012*
2013	*24.05.2013*	*02.05.2013*								
2014				*15.08.2014*	**29.12.2014**					
2015							**24.06.2015**			
2016	**29.01.2016**	**02.02.2016**	**29.01.2016**	**08.02.2016**		**08.02.2016**				**06.01.2016**
2017										

Table C.1: All break dates in pairwise cointegrating regressions with the German yield. **Bold dates** indicate 'forward'-breaks and *italic dates* indicate 'backward'-breaks.

Chapter 5

Seasonal Fractional Cointegration

Co-authored with Philipp Sibbertsen.

5.1 Introduction

It is well known that high-frequency time series like intraday stock returns, realized volatilities or trading volumes often exhibit long memory as well as periodic behavior. Univariately, especially the volatility series have been modelled with seasonal long-memory processes such as k-factor Gegenbauer processes (Gray et al. (1989), Woodward et al. (1998)) or with seasonally fractionally integrated processes (Porter-Hudak (1990), Ray (1993)). In this paper, we examine these concepts in a multivariate context.

Univariately, the above-mentioned models can be estimated with seasonal versions of popular long memory estimators, i.e., log-periodogram regression or Gaussian semiparametric estimation. Arteche and Robinson (2000) propose both types of estimators in asymmetric seasonal long memory models that require trimming of periodogram ordinates. Further literature includes Hassler (1994) and Reisen et al. (2006) who both suggest alternative forms of log-periodogram regression. Arteche and Robinson (2000) also introduce a test for asymmetry that compares the memory estimates before and after the seasonal spectral pole, and Arteche (2002) proposes a test for equality of seasonal memory parameters in k-factor GARMA models.

This shows that seasonal long memory has mostly been analyzed univariately, so far. Exceptions are for example Gil-Alana (2005) who analyzes consumption and income finding potential evidence of seasonal fractional cointegration, and Gil-Alana (2010) who applies a seasonal fractional multivariate model to GDP and unemployment without allowing cointegration. Standard seasonal cointegration was introduced quickly after the seminal paper of Engle and Granger (1987) by Hylleberg et al. (1990) in the context of seasonal unit roots and the $I(1)/I(0)$-framework. We extend this to non-integer orders of integration and introduce multivariate seasonal long memory models that are able to capture seasonal fractional cointegration. Furthermore, we propose a seasonal version of the multiple local Whittle estimator by Robinson et al. (2008) that is able to estimate seasonal memory parameters and the seasonal cointegrating relation at the same time.

Since this estimator is semiparametric and in the frequency domain, we focus on spectral properties of our models as well.

The existing literature also includes some empirical applications of seasonal long memory. For example, Arteche (2007) and Arteche (2012) analyze Spanish inflation volatility. They find no general evidence of asymmetry but of seasonal long memory in absolute and squared inflation values in addition to standard long memory which is also found by Arteche and Robinson (2000) for British inflation. In this paper, we take a different perspective and apply our estimator to financial data. Realized volatilities and trading volume exhibit standard long memory and, on an intraday basis, seasonality as well. This seasonality is usually well-described by seasonal long memory, and we find evidence in favor of seasonal fractional cointegration in daily and half-daily cycles.

The rest of the paper is organized as follows: Section 5.2 briefly reviews the univariate SARFIMA and GARMA models and provides bivariate extensions including their spectral properties; Section 5.3 deals with the definition of seasonal fractional cointegration and how it can be captured in the models from the previous Section 5.2. Section 5.4 introduces the seasonal multiple local Whittle estimator and corresponding asymptotic theory. The following Section 5.5 analyzes the finite sample properties of the estimator and asymptotic Wald tests, Section 5.6 applies the method to trading volume and realized volatility of Dow Jones component stocks, and Section 5.7 concludes the paper.

5.2 Seasonal Long Memory Models

Porter-Hudak (1990), Ray (1993), Ooms (1995) introduce the seasonally fractionally integrated SARFIMA model which is constructed as

$$Y_t = (1 - L^S)^{-d}\varepsilon_t,$$

where ε_t is a short-memory process with continuous, bounded, and positive spectral density $f_\varepsilon(\lambda)$, L is the lag-operator, $\left(1 - L^S\right)^d = \sum_{h=0}^{\infty}\binom{d}{h}\left(-L^S\right)^h$ is the seasonal fractional differencing operator with seasonal long memory parameter $d \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, and S is the seasonal periodicity. It determines the number of seasonal frequencies $\omega_s = \frac{2\pi s}{S}$ with $s = 1, ..., \lfloor S/2 \rfloor$ where $\lfloor\cdot\rfloor$ denotes the largest integer smaller than the argument. By construction, a process generated from this model has the same memory parameter d at all seasonal frequencies. The corresponding spectral density is given by

$$f_Y(\lambda) = f_\varepsilon(\lambda)\Big(2\big(1 - \cos(S\lambda)\big)\Big)^{-d} = f_\varepsilon(\lambda)\left(2\sin\left(\frac{S\lambda}{2}\right)\right)^{-2d} \tag{5.1}$$

with a local approximation $f_Y(\lambda \pm \omega_s) \sim C_1\lambda^{-2d_{\omega_s}}$ for $\lambda \to 0$ with $0 < C_1 < \infty$. Further properties of SARFIMA models are summarized comprehensibly for example by Bisognin and Lopes (2009). Here and in the following, we focus on seasonal frequencies and neglect standard long memory at the zero frequency. However, it can easily be included in the following models by multiplying the noise process with $(1-L)^{-d_0}$.

The k-factor Gegenbauer ARMA (GARMA) model from Gray et al. (1989), Giraitis and Leipus (1995), and Woodward et al. (1998) is given by

$$X_t = \prod_{l=1}^{k}(1 - 2\cos\omega_l L + L^2)^{-d_l}\varepsilon_t,$$

where $(1-2u_l L + L^2)^{-d_l}$ with $u_l = \cos\omega_l$ is the Gegenbauer polynomial

$$\left(1 - 2u_l L + L^2\right)^{-d_l} = \sum_{h=0}^{\infty} C_h^{(d_l)}(u_l)L^h$$

$$\text{with } C_h^{(d_l)}(u_l) = \sum_{k=0}^{\lfloor h/2 \rfloor} \frac{(-1)^k (2u_l)^{h-2k}\Gamma(d_l - k + h)}{k!(h-2k)!\Gamma(d_l)},$$

where $\Gamma(\cdot)$ denotes the gamma function and $\omega_l \in [0,\pi]$, $l = 1, ..., k$ are seasonal frequencies. This implies k seasonal frequencies ω_l each having an individual memory parameter d_l so that there are peaks of different magnitude at the frequencies ω_l in the spectrum. The Gegenbauer polynomial requires $d_l \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ for $0 < \omega_l < \pi$ and $d_l \in \left(-\frac{1}{4}, \frac{1}{4}\right)$ for $\omega_l \in \{0, \pi\}$ for stationarity and invertibility of the model. The spectral density of the GARMA model is given by

$$f_X(\lambda) = f_\varepsilon(\lambda) \prod_{l=1}^{k} |2(\cos\lambda - \cos\omega_l)|^{-2d_l} \tag{5.2}$$

with a local approximation $f_X(\lambda \pm \omega_l) \sim C_2\lambda^{-2d_l}$ as $\lambda \to 0$, where $0 < C_2 < \infty$. If $k > 1$, C_2 includes the interaction of the considered frequency ω_l with the remaining $k-1$ seasonal frequencies (for details confer Hassler (1994) and Giraitis and Leipus (1995)).

The local spectral approximation of both models can be rewritten to

$$f(\lambda \pm \omega) \sim C\lambda^{-2d_\omega} \quad \text{as} \quad \lambda \to 0$$

according to Arteche and Robinson (2000) where $0 < C < \infty$.

We introduce bivariate extensions of the above-mentioned models. Here, the focus is on bivariate models that benefit from the advantage of having no issue of identification so that they are most popular in empirical fractional cointegration analyses.

The basic model structure is $Z_t = \Delta(d)\,\epsilon_t$ or $Z_t = \Phi(d)\,\epsilon_t$ where $\Delta(d)$ and $\Phi(d)$ are bivariate diagonal matrices containing the seasonal fractionally integration and the Gegenbauer filter respectively

$$\Delta(d) = \text{diag}\left((1-L^S)^{-d_1},\, (1-L^S)^{-d_2}\right), \tag{5.3}$$

$$\Phi(d) = \text{diag}\left(\prod_{l=1}^{k}(1-\cos\omega_l L + L^2)^{-d_{1l}},\, \prod_{l=1}^{k}(1-\cos\omega_l L + L^2)^{-d_{2l}}\right), \tag{5.4}$$

and ϵ_t is defined as a bivariate short-memory process with continuous, bounded, and positive definite spectral density matrix $f_\epsilon(\lambda)$ for the rest of the paper. For ease of notation, we assume that both GARMA series share the same seasonal frequencies so that we have the same k in both processes. The corresponding spectral density matrices are given by

$$f_Z(\lambda) = \Xi(d) f_\epsilon(\lambda) \Xi^*(d) \quad \text{or} \quad f_Z(\lambda) = \Psi(d) f_\epsilon(\lambda)\Psi^*(d)$$

with

$$\Xi(d) = \text{diag}\left((1-e^{-iS\lambda})^{-d_1}, (1-e^{-iS\lambda})^{-d_2}\right),$$

$$\Psi(d) = \text{diag}\left(\prod_{l=1}^{k}(1-2\cos\omega_l e^{-i\lambda}+e^{-2i\lambda})^{-d_{1l}}, \prod_{l=1}^{k}(1-2\cos\omega_l e^{-i\lambda}+e^{-2i\lambda})^{-d_{2l}}\right)$$

for the bivariate SARFIMA based on (5.3), and the bivariate GARMA based on (5.4). In both cases * indicates complex conjugation and transposition. Auto-spectra, $f_{Z_{11}}(\lambda)$, $f_{Z_{22}}(\lambda)$, collapse to the real-valued univariate versions in (5.1) and (5.2). In contrast, the off-diagonal elements are complex-valued and yield a phase shift γ if the memory parameters of the series are different. In the SARFIMA setting the upper cross spectral density is given by

$$f_{Z_{12}}(\lambda) = f_{\epsilon_{12}}(\lambda)\left(2\sin\left(\frac{S\lambda}{2}\right)\right)^{-(d_1+d_2)} e^{-i\pi\frac{(d_2-d_1)}{2}}$$

where the phase shift is analogous to the standard fractionally integrated model $\gamma = \pi\frac{(d_2-d_1)}{2}$ (confer for example Kechagias and Pipiras (2015)). In the GARMA setting, the cross spectral density is given by

$$f_{Z_{12}}(\lambda) = f_{\epsilon_{12}}(\lambda)\prod_{l=1}^{k}|2(\cos\lambda - \cos\omega_l)|^{-(d_{1l}+d_{2l})}e^{-i\pi(d_{2l}-d_{1l})}$$

where the phase shift is $\gamma_l = \pi(d_{2l}-d_{1l})$. The phase shift tied to the Gegenbauer filter is thus twice the phase shift tied to the fractional integration filter. The difference between the model-specific phase shifts is intuitive because, at frequencies $\omega \in \{0,\pi\}$, the Gegenbauer filter is a squared version of the fractional integration filter, namely $(1-L)^2$ for

$\omega_l = 0$ and $(1+L)^2$ for $\omega_l = \pi$. This can also be seen in the univariate spectral densities of the models, since for $\omega = 0$ the GARMA spectral density collapses basically to the squared SARFIMA equation. This square cancels the $\frac{1}{2}$ in the phase shift.

For the purpose of estimating long memory semiparametrically, only the spectral density matrix local to the frequency of interest ω is important. Under the assumption of symmetric poles, we approximate the spectral density matrix of both models by

$$f_Z(\omega \pm \lambda) \sim \Lambda(d_\omega) C_\omega \Lambda^*(d_\omega),$$
$$\Lambda(d_\omega) = \text{diag}\left(\lambda^{-d_{1\omega}} e^{-id_{1\omega}\pi/2}, \lambda^{-d_{2\omega}} e^{-id_{2\omega}\pi/2}\right) \quad \text{as} \quad \lambda \to 0, \tag{5.5}$$

where C_ω is a real, symmetric, finite, and positive-definite matrix, $\omega \in (0,\pi)$, and $d_{1\omega}, d_{2\omega} \in (-1/2, 1/2)$.[8]

5.3 Seasonal Fractional Cointegration

In line with the standard definition of cointegration and earlier definitions of seasonal cointegration (cf. Engle et al. (1989), Hylleberg et al. (1990), Arteche (1998)) we define seasonal fractional cointegration in the following way:

A p-dimensional long-memory vector time series Z_t where each component is associated with the long-memory parameter d_ω at frequency ω is said to be seasonally fractionally cointegrated if there exists a vector $\beta_\omega \neq 0$ such that $v_t = \beta'_\omega Z_t$ has long-memory parameter $d_\omega - b_\omega$ at frequency ω, $b_\omega > 0$.

Strictly speaking, the term "cointegrated" is based on the concept of integration so that the Gegenbauer filter would not fall into this category. A more general term could be "common cyclical long memory". However, we here stick to the generally-used term "cointegration".

This definition is quite general as it allows for cointegration at specific frequencies ω without assuming cointegration at other frequencies. Furthermore, the cointegrating relations β_ω and the memory reductions b_ω are frequency-specific. However, as usual with cointegration, we require the same memory parameter d_ω in both time series and we exclude the possibility of cointegration across frequencies. Cointegration across frequencies would imply a long-term relationship between a daily cycle and a weekly cycle, for example. Even if this is an interesting aspect, we consider its importance to be rather low. Moreover, it might be hard to identify and estimate such a relationship.

[8] With this approximation $\Lambda(d_\omega)$, we ignore the correct phase specification for GARMA models which leads to a slightly higher approximation error. However, for practical purposes this error is negligible, as shown in our simulations in Section 5.5. Furthermore, it is natural not to assume a particular data generating process in semiparametric estimators.

We consider the bivariate system

$$BZ_t = v_t \quad \text{with} \quad B = \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix}, \tag{5.6}$$

where we observe Z_t, and v_t is the unobservable underlying process. By construction, all cointegrating relations are subject to the same cointegrating vector $(1, -\beta)$, and the first element in v_t can be referred to as the cointegrating residual. First, let us consider bivariate SARFIMA time series with

$$v_t = \begin{pmatrix} (1 - L^S)^{-(d-b)} \epsilon_{1t} \\ (1 - L^S)^{-d} \epsilon_{2t} \end{pmatrix}.$$

If $\beta \neq 0$ and $b > 0$, Z_t is cointegrated and both observed time series have the same memory parameter d at all seasonal frequencies $\omega_s = \frac{2\pi s}{S}$. Hence, the model is cointegrated at all seasonal frequencies ω_s with the same strength b. By allowing for a more complex multiplicative model structure, like $(1 - L^{S_1})^{-d_1}(1 - L^{S_2})^{-d_2}$, it is possible to have different memory parameters at different frequencies. This feature is already contained in the GARMA model. Assume

$$v_t = \begin{pmatrix} \prod_{l=1}^{k} (1 - \cos \omega_l L + L^2)^{-(d_l - b_l)} \epsilon_{1t} \\ \prod_{l=1}^{k} (1 - \cos \omega_l L + L^2)^{-d_l} \epsilon_{2t} \end{pmatrix}.$$

Here, each frequency ω_l is tied to a specific memory parameter so that the cointegrating strength b_l can differ across the frequencies. By setting $b_l = 0$, the series are not cointegrated at that frequency without the need of changing β. Further it is not necessary, that all ω_l are identical, only the ones that are cointegrated must be. This structure allows for a lot of flexibility as each frequency and the behavior at that frequency can be addressed individually.

Note that we focus on bivariate data because higher dimensional cointegration models are always confronted with problems of identification, and ambiguous decisions and results. In addition, bivariate models cover the usual empirical applications in the context of cointegration, like for example income-consumption or volatility-trading volume relations.

5.4 A Seasonal Multiple Local Whittle Estimator

In order to estimate the seasonal memory parameters and cointegrating relation in (5.6), we combine the semiparametric frequency-domain approach of Robinson et al. (2008), Shimotsu (2012), Arteche (1998), and Arteche and Robinson (2000) assuming a fixed phase shift $\gamma = \pi\frac{(d_{2\omega} - d_{1\omega})}{2}$. We define the parameter vector $\theta_\omega = (d_{1\omega}, d_{2\omega}, \beta_\omega)$, and the spectral density of v_t, $f_v(\omega \pm \lambda) \sim \Lambda(d_\omega) C_\omega \Lambda^*(d_\omega)$ as $\lambda \to 0$ with $\Lambda(d_\omega)$ as in (5.5).

Note that the memory parameters $d_{1\omega}$, $d_{2\omega}$ are not associated with the observed data Z_t but with the underlying noise processes v_t. For cointegration we need $d_{1\omega} < d_{2\omega}$ and $\beta_\omega \neq 0$. In this case Z_{1t} and Z_{2t} both share the memory parameter $d_{2\omega}$, and the unobserved cointegration residual v_{1t} has the reduced memory parameter $d_{1\omega}$.

The local Whittle log-likelihood function after the spectral pole is given by

$$Q(\theta_\omega, C_\omega) = \frac{1}{m}\sum_{j=1}^{m}\left\{\log\det\left(\Lambda_j(d_\omega)\, C_\omega\, \Lambda_j^*(d_\omega)\right) + \operatorname{tr}\left((\Lambda_j(d_\omega)\, C_\omega\, \Lambda_j^*(d_\omega))^{-1}\, B\, I_v(\omega+\lambda_j)\, B'\right)\right\}.$$

Here, $\Lambda_j(d_\omega) = \operatorname{diag}\left(\lambda_j^{-d_{1\omega}} e^{-i\pi d_{1\omega}/2}, \lambda_j^{-d_{2\omega}} e^{-i\pi d_{2\omega}/2}\right)$ with Fourier frequencies $\lambda_j = 2\pi j/T$ for $j = 1, ..., \lfloor T/2 \rfloor$. The periodogram matrix is defined by $I_v(\omega \pm \lambda_j) = w_v(\omega \pm \lambda_j) w_v^*(\omega \pm \lambda_j)$ through the discrete Fourier transform $w_v = \frac{1}{\sqrt{2\pi T}}\sum_{t=1}^{T} v_t e^{it(\omega \pm \lambda_j)t}$. We obtain the estimate

$$\hat{C}_\omega(\theta_\omega) = \operatorname{Re}\left\{\frac{1}{m}\sum_{j=1}^{m}\Lambda_j(d_\omega)^{-1}\, B\, I_v(\omega+\lambda_j)\, B'\, \Lambda_j^*(d_\omega)^{-1}\right\},$$

and the objective function to minimize

$$R(\theta_\omega) = \log\det\left(\hat{C}_\omega(\theta_\omega)\right) - 2(d_{1\omega} + d_{2\omega})\frac{1}{m}\sum_{j=1}^{m}\log\lambda_j$$

with $i \in \{1, 2\}$. The estimator is defined as

$$\hat{\theta}_\omega = \arg\min_{\theta_\omega} R(\theta_\omega). \tag{5.7}$$

In order to show consistency and asymptotic normality we need to introduce the following assumptions.

Assumption 5.4.1. For $\alpha \in (0, 2]$:

$$f(\omega \pm \lambda) = \Lambda(d_\omega) C_\omega \Lambda^*(d_\omega)\left(1 + \mathcal{O}(\lambda^\alpha)\right)$$

and $0 < d_{1\omega}, d_{2\omega} < 0.5$.

Assumption 5.4.2. In a neighborhood $(-\delta, 0) \cup (0, \delta)$ of ω, $f_{aa}(\lambda)$ is differentiable and

$$\left|\frac{d}{d\lambda} f_{aa}(\omega \pm \lambda)\right| = \mathcal{O}(\lambda^{-1-2d_a}), \qquad \lambda \to 0^+$$

for $a = 1, 2$.

Assumption 5.4.3. $z_t = Ez_0 + \sum_{j=0}^{\infty} A_j \varepsilon_{t-j}$ where ε_t is a martingale difference sequence with $E\|\varepsilon_t\| < \infty$, $E[\varepsilon_t \varepsilon_t^{'} | F_{t-1}] = R$ where the diagonal elements of R are equal to 1, F_{t-1} is the σ-field generated by ε_s, $s \leq t-1$ and ε_t and $\varepsilon_t \varepsilon_t^{'} - R$ are uniformly integrable.

Assumption 5.4.4. $\theta_\omega \in \Theta$ for a compact set $\Theta \in R^3$ such that $\Theta = \Theta_\beta \times \Theta_d$, with Θ_β and Θ_d chosen as follows. Take $\Theta_d = \{d_\omega : -\eta_1 \leq d_{1\omega} \leq d_{2\omega} - \eta_2 \leq \frac{1}{2} - \eta_2 - \eta_3\}$, where η_i are arbitrarily small positive numbers satisfying $0 < \eta_1 < \min(\eta_2, \eta_3)$ and $\eta_2 + \eta_3 < \frac{1}{2}$. This includes the short-memory case $d_{1\omega} = 0$ and allows for some $d_{1\omega} < 0$. Θ_β can be taken to be an arbitrarily large interval including $\{0\}$.

Assumption 5.4.5.

$$\frac{1}{m} + \frac{m}{T} \to 0, \qquad T \to \infty.$$

Theorem 5.4.1 (Consistency). Under Assumptions 5.4.1 to 5.4.5 and with $\nu = d_{2\omega} - d_{1\omega}$

$$\hat{d}_\omega \xrightarrow{P} d_\omega, \quad \hat{\beta}_\omega = \beta_\omega + o_P\left(\left(\frac{m}{T}\right)^\nu\right), \qquad T \to \infty.$$

Proof. The proof follows very closely the lines of Theorem 3 in Robinson (2008) where we only need to replace $I_v(\lambda_j)$ by $I_v(\omega + \lambda_j)$. □

Note that the convergence rate of the cointegrating relation $\hat{\beta}_\omega$ depends on the difference of the memory parameters. To prove asymptotic normality we need the following set of assumptions:

Assumption 5.4.6. Let $A(\lambda) = (A_1(\omega + \lambda), A_2(\omega + \lambda))'$ where $A_a(\lambda) = \sum_{j=0}^{\infty} d_{aj} e^{ij(\lambda)} = (A_a^1(\lambda), A_a^2(\lambda))$. Assume

$$\frac{d}{d\lambda} A_a^k(\omega \pm \lambda) = \mathcal{O}\left(\frac{A_a^k(\omega \pm \lambda)}{\lambda}\right), \qquad \lambda \to 0^+$$

for $a, k = 1, 2$.

Assumption 5.4.7. Assumption 5.4.3 holds and $E[\varepsilon_a(t)\varepsilon_b(t)\varepsilon_c(t) \| F_{t-1}] = \mu_{abc}$ with $\|\mu_{abc}\| < \infty$. For the fourth moment we have $E[\varepsilon_a(t)\varepsilon_b(t)\varepsilon_c(t)\varepsilon_d(t) \| F_{t-1}] = 3 + \kappa_{abcd}$ with $\|\kappa_{abcd}\| < \infty$ for $a, b, c, d = 1, 2$.

Assumption 5.4.8. θ_ω is an interior point of Θ.

Assumption 5.4.9. For any $c < \infty$ and $b \in (0, 2]$

$$\frac{(\log T)^c}{m} + \frac{m^{1+2b}(\log m)^2}{T^{2b}} \to 0, \qquad T \to \infty.$$

Theorem 5.4.2 (Normality). Under Assumption 5.4.1 and 5.4.6 to 5.4.9

$$m^{1/2} \Delta_T (\hat{\theta}_\omega - \theta_\omega) \xrightarrow{d_\omega} N(0, \Sigma_\omega^{-1}), \qquad T \to \infty$$

with $\Delta_T = \text{diag}(\lambda_m^{-\nu}, 1, 1)$ and where the elements of Σ_ω are defined by

$$\Sigma_{11} = 2\mu \left((1-2\nu)^{-1} - (1-\nu)^{-2} \cos^2(\gamma) \right) C_{22}/C_{11},$$
$$\Sigma_{22} = \Sigma_{33} = 4 + \left(\pi^2/4 - 1 \right) 2\mu\rho^2$$
$$\Sigma_{23} = \Sigma_{32} = -\left(\pi^2/4 - 1 \right) 2\mu\rho^2,$$
$$\Sigma_{12} = \Sigma_{21} = -2\mu\nu(1-\nu)^{-2} \cos(\gamma) C_{12}/C_{11} + \pi\mu(1-\nu)^{-1} \sin(\gamma) C_{12}/C_{11},$$
$$\Sigma_{13} = \Sigma_{31} = -\Sigma_{12}$$

with $\mu = (1-\rho^2)^{-1}$, $\rho = C_{12}/(C_{11}C_{22})^{1/2}$ where C_{aa} are the respective elements of C_ω, and $\gamma = \nu\pi/2$.

Proof. The proof follows very closely the lines of the proof of Theorem 4 in Robinson et al. (2008) and Theorem 2 in Shimotsu (2012) only that the periodogram $I_v(\lambda_j)$ needs to be replaced by $I_v(\omega + \lambda_j)$ wherever it appears and we fix $\gamma = \nu\pi/2$ as in Shimotsu (2012). □

5.5 Monte Carlo Study

In order to analyze the finite sample performance of the estimator, we consider the basic DGP as in (5.6)

$$BZ_t = v_t \quad \text{with} \quad B = \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix},$$

and distinguish SARFIMA time series with $S = 13$

$$v_t = \begin{pmatrix} (1-L)^{-(d-b)}(1-L^{13})^{-(d-b)}\epsilon_{1t} \\ (1-L)^{-d}(1-L^{13})^{-d}\epsilon_{2t} \end{pmatrix}, \tag{5.8}$$

and GARMA time series with $k_1 = k_2 = 7$ and $\omega_l \in \left\{0, \frac{2\pi}{13}, \frac{4\pi}{13}, \frac{6\pi}{13}, \frac{8\pi}{13}, \frac{10\pi}{13}, \frac{12\pi}{13}\right\}$

$$v_t = \begin{pmatrix} \prod_{l=1}^{7}(1 - \cos\omega_l L + L^2)^{-(d_l-b_l)}\epsilon_{1t} \\ \prod_{l=1}^{7}(1 - \cos\omega_l L + L^2)^{-d_l}\epsilon_{2t} \end{pmatrix}. \tag{5.9}$$

In both cases $\epsilon_t = (\epsilon_{1t}, \epsilon_{2t})'$ is bivariate and uncorrelated white noise. By setting $S = 13$, we obtain the same seasonal frequencies in the SARFIMA model as in the GARMA model, and we also reproduce the situation from the empirical application in the next Section. As an example, we state the estimation results $\hat{\theta} = (\hat{d}_1, \hat{d}_2, \hat{\beta})$ for the first seasonal frequency, i.e. $\omega = \omega_2 = \frac{2\pi}{13} \approx 0.48$, and we drop the index ω_2 for ease of readability.

The following parameter grids are regarded: cointegration and noncointegration by $\beta \in \{0, 1\}$, stationary long memory parameters $d, d_l \in \{0.2, 0.4\}$, and $b, b_l \in \{0.5d, d\}$. Here, $b = d$ implies a memory reduction to zero so that the cointegrating residual has short memory, and $b = 0.5d$ is a reduction to 0.2 (0.1 respectively), i.e., a weaker cointegrating relation where the residual still exhibits long memory. In the GARMA model, all memory parameters are identical, i.e., $d_l = d$ and $b_l = b \; \forall l$. Note that these definitions translate as $d_1 = d - b$ and $d_2 = d$ into the estimation. Furthermore, we set $T \in \{500, 1000, 2000\}$ and $m = \lfloor 1 + T^{\delta_m} \rfloor$ with $\delta_m \in \{0.55, 0.6, 0.65\}$. The bandwidth choice is crucial for semiparametric estimation of seasonal long memory. If the bandwidth is too large, the estimation comprises periodogram ordinates that belong to the neighboring seasonal peak so that the results are biased. Here, we take a look at the first seasonal peak at frequency $\omega \approx 0.48$. This frequency belongs to the $j = \frac{\omega T}{2\pi}$-th periodogram ordinate and is also equal to the spacing between the poles. With $\delta_m = 0.55$ we do not have any interference for our considered numbers of observations, but with $\delta_m = 0.6$ there is a collision if $T = 500$, and with $\delta_m = 0.65$ additionally if $T = 1000$.

The optimization intervals are set to $\Theta_\beta = [-5, 5]$ and $\Theta_d = \{d : -0.001 \leq d_1 \leq d_2 -0.002 \leq 0.496\}$. Hence, we include short-memory and require a minimum difference of 0.002 between the memory estimates.

Table 5.1 gives the bias and RMSE of $\hat{\theta} = (\hat{d}_1, \hat{d}_2, \hat{\beta})$ in (5.7) when d_2 is set to 0.4. The upper panel is based on SARFIMA time series from (5.8) and the lower on GARMA time series from (5.9). In both cases, as explained in the previous Sections, the phase shift is implicitly set to $\gamma = \pi \frac{d_2 - d_1}{2}$ by the definition of $\Lambda_j(d)$.

The bias of the memory parameters is rather small and always negative, i.e., the true memory is underestimated. Furthermore, it is interesting that the bias of $\hat{d}_2$ is always larger than that of $\hat{d}_1$. If $d_1 = 0$, the bias of $\hat{d}_1$ turns positive and becomes larger (around 0.06). Although the optimization interval includes zero, the interval border (-0.001) is too close to the true value 0 yielding a positive bias. To obtain more precise estimates if $d_1 = 0$, the optimization interval has to be altered. For example, $\eta_1 = 0.1$ reduces the bias by half although it is still positive. However, this does not fulfill Assumption 5.4.4 anymore because increasing η_1 requires increasing η_2, η_3 as well so that the upper limit of the interval shrinks and the minimum distance between d_1 and d_2 grows. As we want to focus on long memory, we prioritize reliable results in the long memory case and accept the bias in case of short-memory cointegrating errors. In contrast to univariate local Whittle estimation, both estimates are insensitive to the existence or nonexistence of cointegration because of the simultaneous estimation of β. The bias of β is smaller than that of the memory parameters and also negative in most cases. Further, it is smaller if $\beta = 0$ than if $\beta = 1$. Table D.2 in the appendix contains the analogous results with $d_2 = 0.2$. It shows that $d_1 = 0.1$ leads to positive bias as well because 0.1 is still close to the optimization interval border. The remaining results are similar to Table 5.1.

	δ_m	0.55						0.6						0.65					
	d_1	0.2			0			0.2			0			0.2			0		
β	T	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$
SARFIMA																			
									Bias										
0	500	-0.033	-0.090	-0.007	0.084	-0.094	0.006	-0.106	-0.304	0.011	0.074	-0.304	0.004	-0.107	-0.286	0.001	0.059	-0.288	-0.006
	1000	-0.015	-0.030	0.012	0.067	-0.031	-0.002	-0.047	-0.084	0.021	0.054	-0.088	0.000	-0.128	-0.315	0.009	0.045	-0.315	0.008
	2000	-0.005	-0.006	-0.010	0.053	-0.007	-0.001	-0.015	-0.022	-0.001	0.042	-0.023	-0.002	-0.062	-0.117	0.004	0.034	-0.118	0.000
1	500	-0.032	-0.092	-0.015	0.083	-0.096	-0.009	-0.105	-0.303	-0.061	0.073	-0.303	-0.026	-0.106	-0.287	-0.042	0.058	-0.288	-0.025
	1000	-0.015	-0.031	-0.020	0.065	-0.035	-0.002	-0.045	-0.082	-0.013	0.055	-0.085	0.002	-0.128	-0.317	-0.071	0.045	-0.314	-0.051
	2000	-0.004	-0.007	-0.018	0.052	-0.008	-0.002	-0.015	-0.021	-0.004	0.043	-0.024	-0.001	-0.062	-0.116	-0.001	0.035	-0.118	0.000
									RMSE										
0	500	0.100	0.136	0.842	0.106	0.137	0.559	0.124	0.312	1.154	0.092	0.311	0.973	0.122	0.295	1.112	0.074	0.296	0.984
	1000	0.087	0.083	0.630	0.084	0.082	0.222	0.083	0.111	0.609	0.068	0.114	0.236	0.136	0.320	1.106	0.056	0.320	0.882
	2000	0.071	0.062	0.438	0.066	0.062	0.110	0.058	0.059	0.313	0.053	0.059	0.094	0.077	0.126	0.419	0.043	0.127	0.110
1	500	0.102	0.136	0.831	0.105	0.139	0.581	0.123	0.311	1.158	0.092	0.311	1.001	0.121	0.296	1.106	0.073	0.296	0.938
	1000	0.087	0.084	0.632	0.082	0.086	0.224	0.082	0.108	0.573	0.069	0.112	0.231	0.136	0.321	1.135	0.056	0.319	0.946
	2000	0.070	0.062	0.439	0.065	0.064	0.124	0.060	0.059	0.338	0.054	0.061	0.096	0.077	0.126	0.387	0.044	0.127	0.109
GARMA																			
									Bias										
0	500	-0.033	-0.098	-0.013	0.085	-0.099	0.002	-0.108	-0.293	0.003	0.073	-0.293	0.010	-0.109	-0.273	-0.001	0.059	-0.271	0.014
	1000	-0.014	-0.031	0.005	0.067	-0.033	-0.003	-0.047	-0.082	-0.009	0.054	-0.086	0.002	-0.129	-0.304	-0.029	0.045	-0.304	0.013
	2000	-0.001	-0.005	0.010	0.053	-0.007	-0.002	-0.015	-0.022	0.000	0.043	-0.022	0.003	-0.062	-0.115	0.004	0.036	-0.116	-0.001
1	500	-0.031	-0.099	-0.036	0.084	-0.100	-0.023	-0.108	-0.295	-0.040	0.072	-0.291	-0.043	-0.108	-0.270	-0.049	0.059	-0.272	-0.004
	1000	-0.014	-0.032	-0.012	0.066	-0.033	-0.003	-0.047	-0.083	-0.011	0.055	-0.086	-0.003	-0.130	-0.306	-0.028	0.045	-0.305	-0.019
	2000	-0.002	-0.005	-0.010	0.053	-0.007	-0.001	-0.015	-0.021	-0.003	0.042	-0.021	0.001	-0.061	-0.115	-0.005	0.035	-0.116	-0.002
									RMSE										
0	500	0.101	0.142	0.812	0.107	0.142	0.581	0.126	0.302	1.136	0.091	0.301	0.953	0.124	0.283	1.062	0.073	0.280	0.933
	1000	0.085	0.084	0.592	0.084	0.086	0.249	0.083	0.109	0.626	0.068	0.113	0.244	0.137	0.309	1.085	0.057	0.309	0.844
	2000	0.071	0.061	0.442	0.067	0.063	0.111	0.059	0.060	0.344	0.053	0.060	0.099	0.077	0.124	0.405	0.045	0.125	0.109
1	500	0.101	0.143	0.804	0.106	0.143	0.590	0.125	0.303	1.090	0.090	0.299	0.957	0.123	0.280	1.028	0.073	0.281	0.870
	1000	0.085	0.085	0.630	0.083	0.085	0.230	0.082	0.110	0.599	0.068	0.113	0.236	0.137	0.311	1.080	0.056	0.310	0.868
	2000	0.070	0.062	0.457	0.066	0.062	0.109	0.059	0.058	0.332	0.053	0.058	0.093	0.076	0.124	0.404	0.043	0.126	0.108

Table 5.1: Bias and RMSE of parameter estimates $\hat{\theta}$ when $d_2 = 0.4$. The upper panel shows results for SARFIMA time series and the lower panel shows results for GARMA time series. In both cases the phase shift is set to $\gamma = \pi \frac{d_2 - d_1}{2}$.

The RMSE's for $\hat{d}_1$ and $\hat{d}_2$ in Table 5.1 are similar and smaller than 0.1 in large samples. For $\hat{\beta}$ it is much larger (in small samples around 1), but it decreases to below 0.5 in large samples. This implies that the variability of $\hat{\beta}$ is larger than that of the memory estimates and it also indicates a slight instability of the estimate, but the stronger the cointegrating relation, the more stable the results become. For example, if the cointegrating residual is short-memory ($\hat{d}_1 = 0$) in the largest sample, the RMSE decreases to the same level as for the memory estimates (around 0.1).

These observations hold for both DGPs, SARFIMA and GARMA. This is particularly interesting for the GARMA-case where the time series are estimated with an incorrectly-specified spectral density. However, the results are very similar to those based on SARFIMA time series. Table D.3 (and D.4) in the appendix contains the results when γ is fixed to $\pi(d_2 - d_1)$. The memory results are unaffected by this choice. For $\hat{\beta}$ the results change slightly, but it is hard to tell in which direction because of the instability of the estimation. Therefore, for practical purposes in finite samples, the specification of the phase parameter seems not too important.

As expected from the discussion of bandwidth, all estimates are biased if the bandwidth choice leads to inclusion of the neighboring peak. While $\hat{\beta}$ has a slightly higher bias (and doubled RMSE), it is particularly influencing the memory estimates that become zero if too many periodogram ordinates that already belong to the next seasonal peak are included. Therefore, we recommend to choose the bandwidth carefully and to resort to rather small bandwidths.

	δ_m	0.55						0.65					
	d_1	0.5 d_2			0			0.5 d_2			0		
β	T	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$
$d_2 = 0.2$													
0	500	0.279	0.560	0.044	0.134	0.567	0.045	0.207	0.332	0.025	0.113	0.320	0.019
	1000	0.344	0.779	0.043	0.106	0.765	0.046	0.186	0.290	0.021	0.101	0.291	0.015
	2000	0.420	0.905	0.049	0.082	0.898	0.058	0.423	0.916	0.036	0.073	0.921	0.041
1	500	0.287	0.551	0.407	0.128	0.550	0.458	0.209	0.321	0.310	0.105	0.325	0.312
	1000	0.342	0.762	0.495	0.102	0.757	0.619	0.177	0.302	0.297	0.099	0.293	0.295
	2000	0.415	0.905	0.587	0.076	0.902	0.789	0.449	0.916	0.566	0.075	0.912	0.783
$d_2 = 0.4$													
0	500	0.464	0.907	0.082	0.114	0.906	0.091	0.321	0.458	0.064	0.109	0.425	0.018
	1000	0.670	0.993	0.094	0.086	0.996	0.129	0.299	0.414	0.056	0.090	0.387	0.012
	2000	0.854	1.000	0.091	0.069	1.000	0.143	0.882	1.000	0.056	0.055	1.000	0.046
1	500	0.468	0.904	0.617	0.111	0.900	0.792	0.325	0.437	0.463	0.101	0.420	0.472
	1000	0.671	0.995	0.761	0.077	0.995	0.968	0.307	0.402	0.442	0.098	0.390	0.482
	2000	0.851	1.000	0.873	0.068	1.000	0.997	0.879	1.000	0.888	0.055	1.000	0.998

Table 5.2: Parameter significance tests with the SARFIMA-model and $\gamma = \pi\frac{d_2 - d_1}{2}$. The significance level is set to $\alpha = 0.05$.

By estimating the asymptotic variance Σ, we are able to construct asymptotic confidence intervals for the parameter estimates and test hypotheses. We can address, for example, the question whether cointegration exists by testing $H_0 : \beta = 0$ vs. $H_1 : \beta \neq 0$. The second condition for cointegration, $d_1 < d_2$, is fulfilled by construction of the optimization interval. We are also able to test whether the cointegrating error is short-memory by formulating the hypotheses $H_0 : d_1 = 0$ vs. $H_1 : d_1 \neq 0$. Table 5.2 gives such Wald test rejection frequencies at the 5%-significance level for SARFIMA time series. The upper panel shows results for $d_2 = 0.2$, and the lower panel for $d_2 = 0.4$. Gray-colored cells contain size results and non-colored cells are power results.

First, we consider the results for $\hat{\beta}$. If $d_2 = 0.2$, the empirical size is close to the nominal level, but if $d_2 = 0.4$, the test becomes liberal. This is associated with lower power when $d_2 = 0.2$ than when $d_2 = 0.4$. In contrast, the power is higher if the cointegrating error is short-memory which can be explained by less biased β estimation in this case.

Testing short-memory cointegrating errors is rather liberal. The main reason is probably due to biased estimation if $d_1 = 0$ and this bias impedes the test performance. However, the test approaches the nominal significance level with growing sample size. Of course, the larger d is, the higher the power. With $d_1 = 0.1$, power is around 0.4 but doubles to 0.8 to 0.9 if $d = 0.2$. For $d = 0.4$, the power properties are very good. Again, as in the estimation setting, the performance of testing d is not affected by the existence or nonexistence of cointegration. Overall, the test properties are good if the sample is large enough. This is not surprising because we employ an estimate of the asymptotic variance.

Table D.1 in the appendix displays analogous results based on GARMA time series which are basically the same as in Table 5.2.

5.6 Application to Trading Volume and Realized Volatility

In this Section, we apply the estimation method from Section 5.4 to intraday trading volume and realized volatility of the component stocks of the Dow Jones Industrial Average (DJIA) index obtained from the Thomson Reuters Tick Database as five-minute data for a time period starting January 2014 and ending December 2015. We use aggregated half-hourly logarithmic data in the regular trading hours 9:30 to 16:00, i.e., 13 observations per day, so that the seasonal periodicity is set to $S = 13$.

For this type of data, trading volume and return volatility, there is much literature on the contemporaneous relation and high positive correlation, cf. for example Chen et al. (2001) and Mougoué and Aggarwal (2011) among many others. A possible theoretical explanation for this observation is the mixture-of-distribution hypothesis (MDH) that states that price changes and trading volume are driven by the same latent information arrival process (Clark (1973), Tauchen and Pitts (1983), Andersen (1996), Bollerslev et al. (2018)) in the sense that (unexpected) good (or bad) news increase (or decrease) the price through higher trading activity. This could imply cointegration in the context of a common components model, i.e., one underlying stochastic trend, the information arrival. Bollerslev and Jubinski (1999) address this potential long-run dynamic relationship and find evidence in favor of this hypothesis.

Our setting is a triangular model so that we must decide how to normalize the data. From theoretical considerations it is not clear whether RV_t should be the dependent or driving variable in the cointegrating relation, and the literature pursues differing orders as well.

For example, Bollerslev et al. (2018) use the differences-of-opinion model[9] of Kandel and Pearson (1995) and reformulate it to the equilibrium equation Volume $= |\beta_0 + \beta_1 \cdot$ Price Change$|$. The absolute price change is equal to the absolute return that in turn is an estimate of the volatility. Hence, this equilibrium equation implies cointegration between trading volume and volatility which we approximate through realized volatility as it is the natural estimate on an intraday basis. Our cointegration model from (5.6) can capture this by setting $Z_{1t} = Vol_t$ and $Z_{2t} = RV_t$ so that

$$\begin{aligned} Vol_t &= v_{1t} + \beta \cdot RV_t \\ RV_t &= v_{2t}. \end{aligned}$$

In contrast, Bollerslev and Jubinski (1999) use $(RV_t, Vol_t)'$ in a log-periodogram regression analysis as well as Fleming and Kirby (2011) in multivariate linear regression. This suggests the order $Z_{1t} = RV_t$ and $Z_{2t} = Vol_t$ so that our setting is

$$\begin{aligned} RV_t &= v_{1t} + \beta \cdot Vol_t \\ Vol_t &= v_{2t}. \end{aligned}$$

Furthermore, Mougoué and Aggarwal (2011), as well as Chen et al. (2001), find bidirectional Granger causality which can be interpreted as a hint for cointegrating relations. This again leaves room for different interpretations of the normalization of Z_t.

So far, the relationship between realized volatility and trading volume is most often analysed with daily data or even coarser sampling frequencies (for an exception confer Bollerslev et al. (2018)). Here, we deal with intraday data that exhibits seasonality. Therefore, we want to examine whether the MDH-implied cointegrating relation holds at seasonal frequencies. As it is not clear, how the time series should be ordered we consider both possibilities.

[9]The difference-of-opinion model describes how new information is processed in the different ways by the traders, since they disagree on the expected payoff and interpret, for examples, announcements in different manners.

The Monte Carlo results suggest to use a rather small bandwidth, i.e., $\delta_m = 0.55$. However, it is always important to consider the number of observations at hand. Here, all time series have more than 6,500 observations each so that employing $\delta_m = 0.55$ means that 125 periodogram ordinates are included when there are about 500 periodogram ordinates between the poles. Therefore, $\delta_m = 0.65$ (300 ordinates) is a valid choice as well that excludes neighbouring peaks and includes almost no contradictory information.

We only state the results for the first two seasonal frequencies that correspond to a daily ($\omega_2 = 0.48$) and half-daily ($\omega_2 = 0.96$) cycle. Fractional cointegration at larger seasonal frequencies corresponding to shorter cycles implies an equilibrium, for example, at a-sixth-daily (equal to about one hour) cycle. This is a very short time horizon so that it is not surprising that we find very few significant results at the large seasonal frequencies.

Table 5.3 contains the estimation results with bandwidth $\delta_m = 0.55$ for the seasonal frequencies $\omega_2 = 0.48$ (daily cycle) and $\omega_3 = 0.96$ (half-daily cycle), and the normalization $Z_t = (RV_t, Vol_t)'$. The small numbers below the estimates represent the deviation in asymptotic 95% confidence intervals. If zero is not included, we can reject the hypothesis $H_0 : \beta = 0$, and in order to make identification easier we mark the cells in gray when the hypothesis is rejected at the 5% significance level.

The (common) memory parameters $\hat{d}_2$ at the daily frequency are between 0.3 and the upper bound, except for Apple where it is much lower (around 0.07). The memory parameters of the potential cointegrating relation $\hat{d}_1$ are lower and lie between 0.15 and 0.4 implying a memory reduction for example close to the lower bound (0.003 for Verizon), around 0.17 for JPMorgan Chase and 0.25 for Travelers Companies Inc. The cointegrating relation $\hat{\beta}$ is significant in all cases and always positive except for the index. Most values are around 1.4 to 2.4, only some are larger (3 to 5).

At the half-daily frequency, memory parameters are lower ($\hat{d}_2$: 0.15 to 0.3) except for Apple where it is slightly larger than at the daily frequency. The memory parameters $\hat{d}_1$ are lower (0.05 to 0.25) than at the daily frequency as well, and in all cases the memory reduction is smaller. For example, the relation $\hat{d}_1 < \hat{d}_2$ for Travelers Companies Inc and United Technologies is probably only enforced by the optimization interval and the true memory parameters might not obey this constraint. Consequently, we do not get significant β-estimates. As the difference of the memory parameters influences the estimation of β, the standard error becomes very large if the difference is too small. The remaining $\hat{d}_1$ range between 0.003 for Microsoft (where we cannot reject a short-memory cointegrating error) and 0.17 for Exxon Mobil. The β estimates are in a similar range as at the daily frequency.

Overall, we find evidence in favor of seasonal fractional cointegration. Hence, the equilibrium (long-run) relation postulated by the MDH holds in seasonal structures and implies a daily, and partially also half-daily, equilibrium towards that trading volume and realized volatility are driven.

	$\omega_2 = 0.48$			$\omega_3 = 0.96$				$\omega_2 = 0.48$			$\omega_3 = 0.96$		
	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$		$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$
.DJI	0.415	0.496	0.106	0.193	0.331	-0.044	**KO**	0.231	0.496	1.390	0.162	0.250	1.367
	0.084		1.591	0.085		0.819		0.084		0.356	0.085		1.602
AAPL.O	0.034	0.069	3.546	0.043	0.109	2.261	**MCD**	0.286	0.496	1.800	0.157	0.227	2.303
	0.068		2.677	0.081		1.831		0.078		0.460	0.074		1.999
AXP	0.257	0.496	1.835	0.179	0.247	0.639	**MMM**	0.228	0.496	1.833	0.126	0.224	1.708
	0.079		0.436	0.088		2.594		0.081		0.398	0.080		1.427
BA	0.291	0.335	4.749	0.188	0.260	3.478	**MRK**	0.235	0.496	1.593	0.208	0.245	3.694
	0.064		2.257	0.069		1.804		0.081		0.411	0.070		4.236
CAT	0.317	0.496	1.837	0.190	0.229	3.325	**MSFT.O**	0.167	0.292	2.042	0.044	0.211	1.585
	0.077		0.487	0.070		3.704		0.073		0.729	0.076		0.518
CSCO.O	0.337	0.438	2.297	0.189	0.241	4.975	**NKE**	0.340	0.381	5.000	0.244	0.296	4.486
	0.068		0.869	0.065		2.367		0.064		2.336	0.067		2.751
CVX	0.383	0.441	3.756	0.219	0.279	4.165	**PFE**	0.251	0.301	3.076	0.105	0.144	4.847
	0.065		1.803	0.068		2.535		0.066		1.831	0.066		3.323
DD	0.238	0.382	2.106	0.124	0.161	2.988	**PG**	0.274	0.355	3.197	0.142	0.188	5.000
	0.072		0.672	0.070		3.767		0.066		1.254	0.066		3.034
DIS	0.272	0.372	3.004	0.119	0.164	5.000	**TRV**	0.249	0.496	1.898	0.190	0.192	0.495
	0.068		0.968	0.065		2.774		0.079		0.467	0.088		83.501
GE	0.305	0.421	2.017	0.161	0.229	1.962	**UNH**	0.289	0.496	1.671	0.132	0.185	4.638
	0.071		0.758	0.073		1.561		0.081		0.540	0.067		2.987
GS	0.282	0.496	1.763	0.149	0.196	2.750	**UTX**	0.268	0.496	2.103	0.214	0.216	0.661
	0.081		0.459	0.077		3.490		0.078		0.486	0.088		76.631
HD	0.292	0.496	1.732	0.232	0.297	3.149	**V**	0.232	0.496	1.742	0.185	0.238	3.592
	0.078		0.485	0.070		2.095		0.078		0.350	0.069		2.865
IBM	0.292	0.468	1.726	0.083	0.174	1.673	**VZ**	0.289	0.322	4.935	0.164	0.218	3.113
	0.077		0.567	0.079		1.381		0.064		3.281	0.071		2.679
INTC.O	0.281	0.324	4.423	0.087	0.255	1.429	**WMT**	0.294	0.336	4.774	0.112	0.187	3.124
	0.064		2.326	0.082		0.724		0.064		2.504	0.072		1.722
JNJ	0.317	0.457	2.418	0.167	0.230	3.790	**XOM**	0.346	0.496	1.547	0.132	0.306	1.691
	0.071		0.790	0.068		2.328		0.080		0.641	0.079		0.622
JPM	0.299	0.473	1.910	0.106	0.155	2.482							
	0.073		0.559	0.071		2.541							

Table 5.3: Estimation results for $Z_t = (RV_t, Vol_t)'$ with bandwidth $\delta_m = 0.55$ and $\gamma = \pi\frac{d_2 - d_1}{2}$. The small numbers below the estimates represent the deviation in asymptotic 95% confidence intervals.

Table D.5 in the appendix contains the results for $\delta_m = 0.65$, i.e. when including more periodogram ordinates. In this case, the memory estimates become smaller which is already indicated by the Monte Carlo results where we observe a larger negative bias with larger bandwidths. We further get slightly less significant $\hat{\beta}$-results and in general also smaller values. One exception is the index where we now obtain significant results in contrast to the results with $\delta_m = 0.55$ and, surprisingly, $\hat{\beta}$ is negative whereas we have positive estimates in single stock data. This underlines the potentially different nature of index-data in comparison to individual stock data.

As discussed above, the normalization of the data is not clear in advance and we consider $Z_t = (Vol_t, RV_t)'$ as well. These results are given in Table D.6 in the appendix. We observe similar memory parameter estimates but almost no significant $\hat{\beta}$. This emphasizes the importance of how data is sorted in cointegration analyses.

Furthermore, the nature of seasonality is not always clear and stochastic and deterministic models compete. Here, we consider a purely stochastic model, but it is possible to include deterministic components, for example by seasonally demeaning the data beforehand. By doing so, we still obtain significant long memory estimates but there are hardly any significant non-zero β estimates.

5.7 Conclusion

This paper deals with multivariate extensions of seasonal long memory. We formulate two bivariate models that capture fractional seasonality as well as seasonal fractional cointegration and provide a suitable estimator for the seasonal memory parameters and the seasonal cointegrating relation based on the multiple local Whittle estimator by Robinson et al. (2008).

Our Monte Carlo analysis shows the reliability of the estimation results and a satisfactory performance in asymptotic Wald tests. The semiparametric nature of the method has the advantage of disregarding short-run dynamics and avoiding misspecification. However, the bandwidth has to be chosen carefully in order to avoid interference of neighboring seasonal poles. Therefore, we recommend to examine the location of seasonal poles (based on theoretical considerations or empirical estimation methods) and to chose a rather small bandwidth.

The empirical application shows the importance of how data is organized in the context of cointegration. Depending on the specification, we find evidence in favor of seasonal fractional cointegration between realized volatility and trading volume of all Dow Jones component stocks. This is a phenomenon postulated by the MDH in the long-run that appears to hold in daily cycles as well.

Our estimator disregards the possibility of asymmetric spectral poles that might arise at seasonal peaks, but they might be accommodated by trimming as in Arteche (2002). In his case the asymptotic theory has to be reexamined which we leave for future research.

D Appendix

	δ_m	0.55						0.65					
	d_1	$0.5\ d_2$			0			$0.5\ d_2$			0		
β	T	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$
$d_2 = 0.2$													
0	500	0.300	0.545	0.044	0.124	0.560	0.045	0.196	0.322	0.024	0.115	0.313	0.020
	1000	0.348	0.761	0.046	0.100	0.765	0.049	0.190	0.294	0.019	0.104	0.303	0.016
	2000	0.426	0.905	0.048	0.087	0.910	0.051	0.421	0.923	0.031	0.070	0.919	0.040
1	500	0.279	0.552	0.409	0.132	0.551	0.442	0.197	0.349	0.315	0.118	0.320	0.305
	1000	0.351	0.765	0.507	0.102	0.762	0.621	0.185	0.307	0.279	0.102	0.297	0.309
	2000	0.423	0.909	0.590	0.073	0.900	0.794	0.437	0.920	0.569	0.061	0.915	0.782
$d_2 = 0.4$													
0	500	0.470	0.894	0.082	0.116	0.901	0.096	0.308	0.525	0.059	0.107	0.521	0.027
	1000	0.677	0.992	0.093	0.083	0.994	0.125	0.296	0.491	0.049	0.093	0.466	0.011
	2000	0.860	1.000	0.082	0.073	0.999	0.137	0.880	1.000	0.056	0.064	1.000	0.044
1	500	0.478	0.890	0.616	0.113	0.893	0.792	0.310	0.546	0.486	0.106	0.514	0.536
	1000	0.677	0.993	0.761	0.076	0.994	0.971	0.290	0.480	0.462	0.094	0.464	0.519
	2000	0.865	1.000	0.877	0.069	1.000	0.997	0.887	1.000	0.885	0.054	1.000	0.998

Table D.1: Parameter significance tests with the GARMA-model and $\gamma = \pi\frac{d_2-d_1}{2}$. The significance level is set to $\alpha = 0.05$.

	δ_m	0.55						0.6						0.65					
	d_1	0.1			0			0.1			0			0.1			0		
β	T	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$
SARFIMA																			
		Bias																	
0	500	0.022	-0.019	0.025	0.085	-0.016	-0.012	-0.017	-0.108	-0.013	0.074	-0.108	-0.012	-0.024	-0.105	-0.015	0.059	-0.105	-0.011
	1000	0.013	-0.002	0.004	0.066	-0.001	0.019	-0.007	-0.038	0.001	0.053	-0.040	0.016	-0.042	-0.129	0.023	0.045	-0.127	-0.010
	2000	0.006	0.006	0.002	0.052	0.001	0.017	-0.003	-0.010	-0.021	0.042	-0.012	-0.004	-0.027	-0.058	0.000	0.034	-0.058	0.000
1	500	0.021	-0.021	-0.047	0.084	-0.018	-0.057	-0.017	-0.108	-0.120	0.074	-0.107	-0.079	-0.024	-0.107	-0.046	0.059	-0.104	-0.049
	1000	0.012	-0.003	-0.035	0.065	-0.004	-0.029	-0.009	-0.039	-0.047	0.054	-0.037	0.003	-0.043	-0.128	-0.039	0.046	-0.127	-0.074
	2000	0.005	0.004	-0.028	0.052	0.002	-0.009	-0.006	-0.012	-0.017	0.043	-0.012	-0.008	-0.025	-0.059	-0.004	0.035	-0.059	-0.007
		RMSE																	
0	500	0.085	0.097	1.089	0.108	0.093	1.003	0.063	0.125	1.215	0.092	0.125	1.207	0.058	0.120	1.240	0.074	0.120	1.169
	1000	0.070	0.082	0.979	0.083	0.082	0.819	0.058	0.078	0.973	0.066	0.077	0.802	0.057	0.137	1.274	0.057	0.135	1.157
	2000	0.060	0.070	0.831	0.065	0.068	0.554	0.049	0.058	0.829	0.053	0.057	0.441	0.048	0.074	0.893	0.043	0.073	0.550
1	500	0.086	0.096	1.038	0.107	0.095	1.023	0.063	0.125	1.237	0.093	0.124	1.166	0.057	0.121	1.247	0.073	0.119	1.158
	1000	0.070	0.085	0.962	0.083	0.081	0.778	0.058	0.079	0.966	0.067	0.076	0.781	0.058	0.135	1.213	0.057	0.134	1.162
	2000	0.059	0.069	0.852	0.064	0.068	0.527	0.051	0.058	0.761	0.054	0.057	0.469	0.047	0.075	0.888	0.044	0.074	0.583
GARMA																			
		Bias																	
0	500	0.022	-0.020	-0.003	0.083	-0.018	0.017	-0.016	-0.110	-0.026	0.073	-0.107	-0.009	-0.026	-0.106	0.012	0.060	-0.106	0.005
	1000	0.014	-0.005	-0.004	0.066	-0.003	-0.005	-0.009	-0.039	-0.033	0.052	-0.038	-0.002	-0.042	-0.128	0.001	0.046	-0.126	0.001
	2000	0.006	0.005	0.002	0.053	0.005	0.003	-0.004	-0.009	-0.009	0.043	-0.010	0.012	-0.027	-0.058	-0.003	0.034	-0.057	0.018
1	500	0.019	-0.019	-0.065	0.085	-0.021	-0.047	-0.018	-0.108	-0.092	0.073	-0.109	-0.067	-0.025	-0.103	-0.039	0.060	-0.105	-0.058
	1000	0.013	-0.002	-0.053	0.066	-0.002	-0.035	-0.008	-0.039	-0.038	0.052	-0.037	-0.020	-0.042	-0.126	-0.060	0.046	-0.127	-0.062
	2000	0.007	0.005	-0.028	0.052	0.005	-0.023	-0.005	-0.008	-0.018	0.043	-0.009	0.004	-0.026	-0.058	-0.027	0.034	-0.059	-0.012
		RMSE																	
0	500	0.087	0.097	1.076	0.105	0.095	1.034	0.062	0.126	1.224	0.092	0.123	1.174	0.058	0.121	1.226	0.074	0.120	1.204
	1000	0.072	0.083	0.982	0.082	0.081	0.771	0.057	0.078	1.003	0.066	0.077	0.797	0.058	0.136	1.227	0.057	0.134	1.143
	2000	0.060	0.069	0.841	0.066	0.068	0.562	0.049	0.057	0.799	0.053	0.058	0.472	0.049	0.074	0.846	0.043	0.073	0.532
1	500	0.084	0.098	1.032	0.108	0.094	1.002	0.061	0.125	1.288	0.091	0.126	1.204	0.057	0.119	1.205	0.075	0.119	1.175
	1000	0.072	0.084	0.956	0.083	0.083	0.793	0.056	0.080	1.009	0.066	0.075	0.791	0.057	0.134	1.244	0.058	0.134	1.165
	2000	0.060	0.069	0.808	0.065	0.070	0.590	0.049	0.056	0.770	0.054	0.056	0.450	0.047	0.074	0.870	0.042	0.074	0.529

Table D.2: Bias and RMSE of parameter estimates when $d_2 = 0.2$. The upper panel shows results for the SARFIMA-model and the lower panel shows results for the GARMA-model. In both cases the phase shift is set to $\gamma = \pi \frac{d_2 - d_1}{2}$.

δ_m		0.55						0.6						0.65					
d_1		0.2			0			0.2			0			0.2			0		
β	T	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$
SARFIMA																			
									Bias										
0	500	-0.030	-0.090	0.000	0.086	-0.093	0.001	-0.103	-0.303	-0.026	0.076	-0.304	0.005	-0.104	-0.286	-0.009	0.061	-0.288	-0.002
	1000	-0.013	-0.030	0.010	0.069	-0.030	0.003	-0.045	-0.084	0.004	0.055	-0.088	-0.003	-0.126	-0.314	0.013	0.046	-0.315	0.008
	2000	-0.004	-0.006	0.009	0.053	-0.007	-0.001	-0.014	-0.022	-0.005	0.043	-0.022	-0.001	-0.062	-0.117	0.005	0.035	-0.117	0.000
1	500	-0.028	-0.093	-0.037	0.085	-0.096	-0.033	-0.101	-0.302	-0.106	0.076	-0.303	-0.095	-0.103	-0.287	-0.102	0.060	-0.288	-0.054
	1000	-0.013	-0.032	-0.027	0.066	-0.034	-0.002	-0.044	-0.082	-0.021	0.056	-0.085	-0.002	-0.126	-0.316	-0.041	0.046	-0.314	-0.045
	2000	-0.003	-0.007	-0.010	0.053	-0.008	-0.003	-0.014	-0.021	-0.004	0.044	-0.024	0.000	-0.061	-0.116	-0.010	0.036	-0.118	0.000
									RMSE										
0	500	0.100	0.136	0.961	0.108	0.137	0.629	0.122	0.311	1.556	0.095	0.312	1.361	0.120	0.294	1.477	0.076	0.296	1.184
	1000	0.087	0.083	0.608	0.085	0.082	0.187	0.082	0.111	0.626	0.069	0.114	0.185	0.134	0.319	1.308	0.057	0.320	1.044
	2000	0.071	0.062	0.402	0.067	0.063	0.071	0.058	0.059	0.279	0.053	0.059	0.064	0.077	0.126	0.396	0.044	0.127	0.071
1	500	0.101	0.137	1.156	0.108	0.140	0.765	0.121	0.310	1.559	0.094	0.311	1.393	0.118	0.296	1.508	0.074	0.296	1.228
	1000	0.087	0.085	0.798	0.083	0.086	0.249	0.081	0.109	0.708	0.070	0.112	0.228	0.134	0.321	1.333	0.057	0.319	1.043
	2000	0.070	0.062	0.495	0.066	0.064	0.111	0.059	0.059	0.309	0.055	0.061	0.064	0.077	0.126	0.399	0.044	0.127	0.070
GARMA																			
									Bias										
0	500	-0.030	-0.098	0.027	0.087	-0.100	0.010	-0.105	-0.293	0.004	0.076	-0.293	-0.004	-0.106	-0.273	-0.004	0.061	-0.271	0.006
	1000	-0.012	-0.031	0.007	0.068	-0.033	-0.002	-0.045	-0.082	0.006	0.055	-0.086	-0.001	-0.127	-0.304	0.007	0.047	-0.304	0.010
	2000	0.000	-0.005	0.002	0.054	-0.006	0.000	-0.014	-0.022	0.005	0.043	-0.022	0.002	-0.061	-0.115	0.007	0.036	-0.116	0.000
1	500	-0.028	-0.099	-0.079	0.086	-0.100	-0.023	-0.104	-0.294	-0.088	0.074	-0.291	-0.100	-0.105	-0.270	-0.061	0.061	-0.272	-0.034
	1000	-0.012	-0.033	-0.033	0.067	-0.033	0.002	-0.045	-0.083	-0.020	0.056	-0.086	-0.005	-0.127	-0.305	-0.062	0.046	-0.305	-0.025
	2000	-0.001	-0.005	-0.015	0.054	-0.007	0.000	-0.014	-0.021	-0.005	0.043	-0.021	0.001	-0.060	-0.115	0.002	0.035	-0.116	-0.001
									RMSE										
0	500	0.100	0.142	1.064	0.109	0.142	0.692	0.122	0.302	1.498	0.094	0.302	1.280	0.121	0.282	1.432	0.075	0.281	1.161
	1000	0.085	0.084	0.708	0.085	0.086	0.262	0.082	0.110	0.696	0.069	0.113	0.188	0.135	0.309	1.284	0.058	0.309	0.946
	2000	0.070	0.061	0.458	0.068	0.063	0.101	0.059	0.060	0.334	0.054	0.060	0.065	0.077	0.124	0.391	0.045	0.125	0.099
1	500	0.101	0.143	1.148	0.109	0.144	0.831	0.122	0.303	1.540	0.092	0.299	1.341	0.120	0.280	1.358	0.075	0.281	1.090
	1000	0.085	0.085	0.879	0.084	0.085	0.280	0.082	0.110	0.685	0.069	0.113	0.221	0.135	0.311	1.275	0.057	0.310	0.973
	2000	0.071	0.062	0.512	0.067	0.062	0.090	0.059	0.058	0.313	0.054	0.058	0.062	0.075	0.124	0.333	0.044	0.126	0.070

Table D.3: Bias and RMSE of parameter estimates when $d_2 = 0.4$. The upper panel shows results for the SARFIMA-model and the lower panel shows results for the GARMA-model. In both cases the phase shift is set to $\gamma = \pi(d_2 - d_1)$.

	δ_m	0.55						0.6						0.65					
	d_1	0.1			0			0.1			0			0.1			0		
β	T	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$
SARFIMA																			
									Bias										
	500	0.024	-0.019	-0.005	0.087	-0.017	-0.022	-0.015	-0.108	-0.013	0.076	-0.108	-0.010	-0.023	-0.105	-0.004	0.061	-0.105	-0.003
0	1000	0.014	-0.002	-0.003	0.067	-0.002	0.018	-0.005	-0.038	-0.037	0.054	-0.040	0.023	-0.041	-0.129	0.011	0.047	-0.127	0.007
	2000	0.007	0.005	0.012	0.053	0.001	-0.002	-0.002	-0.010	-0.008	0.043	-0.012	0.004	-0.026	-0.059	0.007	0.035	-0.058	0.009
	500	0.024	-0.022	-0.090	0.086	-0.019	-0.079	-0.015	-0.108	-0.085	0.076	-0.107	-0.120	-0.023	-0.107	-0.101	0.060	-0.104	-0.082
1	1000	0.014	-0.003	-0.054	0.067	-0.004	-0.069	-0.007	-0.040	-0.050	0.055	-0.038	-0.008	-0.042	-0.128	-0.076	0.047	-0.127	-0.046
	2000	0.007	0.004	-0.042	0.053	0.002	-0.007	-0.004	-0.012	-0.038	0.044	-0.012	-0.012	-0.024	-0.059	-0.041	0.036	-0.059	-0.012
									RMSE										
	500	0.086	0.097	1.347	0.109	0.094	1.202	0.062	0.124	1.428	0.094	0.125	1.468	0.057	0.120	1.349	0.075	0.120	1.298
0	1000	0.070	0.082	1.146	0.084	0.082	0.888	0.057	0.078	1.066	0.067	0.077	0.807	0.057	0.136	1.291	0.057	0.135	1.221
	2000	0.060	0.070	0.891	0.066	0.069	0.538	0.049	0.058	0.854	0.054	0.057	0.396	0.047	0.074	0.839	0.044	0.073	0.480
	500	0.087	0.096	1.403	0.109	0.095	1.333	0.062	0.125	1.420	0.094	0.124	1.502	0.056	0.121	1.446	0.074	0.119	1.301
1	1000	0.071	0.084	1.197	0.085	0.082	1.043	0.057	0.079	1.110	0.068	0.076	0.874	0.057	0.135	1.280	0.057	0.134	1.245
	2000	0.059	0.069	0.990	0.065	0.068	0.583	0.051	0.058	0.841	0.055	0.057	0.445	0.047	0.075	0.884	0.044	0.074	0.547
GARMA																			
									Bias										
	500	0.024	-0.020	-0.012	0.085	-0.018	0.039	-0.014	-0.110	-0.012	0.075	-0.107	0.000	-0.024	-0.106	-0.025	0.061	-0.106	-0.035
0	1000	0.016	-0.006	0.017	0.067	-0.003	-0.014	-0.007	-0.039	-0.008	0.053	-0.038	0.025	-0.041	-0.128	-0.014	0.047	-0.126	-0.006
	2000	0.008	0.005	0.023	0.054	0.005	0.009	-0.003	-0.009	-0.011	0.043	-0.010	0.001	-0.027	-0.058	-0.006	0.035	-0.057	-0.004
	500	0.022	-0.020	-0.068	0.088	-0.022	-0.100	-0.015	-0.108	-0.129	0.075	-0.109	-0.118	-0.023	-0.103	-0.071	0.061	-0.105	-0.094
1	1000	0.016	-0.002	-0.083	0.068	-0.003	-0.070	-0.006	-0.040	-0.045	0.054	-0.037	-0.014	-0.041	-0.126	-0.072	0.047	-0.127	-0.052
	2000	0.008	0.004	-0.034	0.053	0.005	-0.012	-0.004	-0.008	-0.024	0.044	-0.009	0.003	-0.025	-0.058	-0.033	0.035	-0.059	-0.013
									RMSE										
	500	0.087	0.097	1.338	0.107	0.095	1.242	0.061	0.126	1.481	0.094	0.123	1.387	0.056	0.121	1.427	0.076	0.120	1.351
0	1000	0.072	0.083	1.128	0.084	0.081	0.869	0.057	0.078	1.060	0.067	0.077	0.869	0.057	0.135	1.254	0.058	0.134	1.192
	2000	0.060	0.069	0.957	0.067	0.068	0.521	0.049	0.057	0.802	0.054	0.058	0.376	0.048	0.074	0.811	0.044	0.073	0.483
	500	0.085	0.097	1.356	0.110	0.094	1.366	0.060	0.125	1.591	0.092	0.126	1.565	0.056	0.119	1.407	0.076	0.120	1.354
1	1000	0.073	0.084	1.176	0.085	0.083	0.989	0.056	0.080	1.165	0.067	0.075	0.894	0.057	0.134	1.282	0.059	0.134	1.189
	2000	0.060	0.069	1.001	0.066	0.070	0.601	0.049	0.056	0.807	0.055	0.056	0.447	0.047	0.074	0.830	0.043	0.074	0.470

Table D.4: Bias and RMSE of parameter estimates when $d_2 = 0.2$. The upper panel shows results for the SARFIMA-model and the lower panel shows results for the GARMA-model. In both cases the phase shift is set to $\gamma = \pi(d_2 - d_1)$.

	$\omega_2 = 0.48$			$\omega_3 = 0.96$				$\omega_2 = 0.48$			$\omega_3 = 0.96$		
	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$		$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$
.DJI	0.321	0.393	-1.526	0.110	0.235	-0.905	**KO**	0.179	0.424	1.224	0.122	0.174	0.869
	0.054		1.242	0.056		0.630		0.055		0.282	0.056		1.682
AAPL.O	0.076	0.078	0.601	0.042	0.111	0.841	**MCD**	0.230	0.415	1.354	0.119	0.188	1.253
	0.056		24.534	0.056		1.112		0.053		0.409	0.053		1.334
AXP	0.201	0.419	1.384	0.107	0.133	1.999	**MMM**	0.184	0.379	1.874	0.087	0.148	1.337
	0.055		0.366	0.053		4.323		0.052		0.411	0.054		1.503
BA	0.233	0.287	2.899	0.111	0.195	1.702	**MRK**	0.215	0.403	1.696	0.126	0.169	2.959
	0.044		1.372	0.052		0.984		0.051		0.429	0.047		2.428
CAT	0.233	0.421	1.570	0.139	0.172	3.412	**MSFT.O**	0.097	0.236	1.429	0.033	0.187	1.072
	0.051		0.360	0.045		2.828		0.052		0.471	0.054		0.420
CSCO.O	0.262	0.353	1.743	0.153	0.188	3.351	**NKE**	0.257	0.319	2.436	0.175	0.247	1.763
	0.047		0.704	0.044		2.382		0.045		1.137	0.051		1.259
CVX	0.279	0.337	2.646	0.164	0.221	2.633	**PFE**	0.192	0.247	1.923	0.063	0.121	2.350
	0.044		1.247	0.048		1.717		0.045		1.182	0.049		1.440
DD	0.200	0.326	1.822	0.095	0.125	1.071	**PG**	0.230	0.295	2.172	0.089	0.147	2.199
	0.048		0.578	0.053		2.791		0.045		1.153	0.049		1.574
DIS	0.220	0.333	1.834	0.118	0.156	2.715	**TRV**	0.232	0.394	1.723	0.146	0.148	0.498
	0.049		0.630	0.046		2.234		0.053		0.527	0.056		50.403
GE	0.203	0.327	1.640	0.097	0.177	0.341	**UNH**	0.255	0.396	1.623	0.125	0.153	3.245
	0.048		0.508	0.056		0.793		0.052		0.56	0.046		3.701
GS	0.209	0.401	1.476	0.123	0.168	1.534	**UTX**	0.208	0.402	1.819	0.135	0.137	0.492
	0.054		0.363	0.055		2.142		0.052		0.413	0.056		49.891
HD	0.210	0.413	1.309	0.175	0.216	3.354	**V**	0.197	0.395	1.508	0.133	0.175	2.235
	0.054		0.358	0.045		2.207		0.052		0.363	0.049		2.318
IBM	0.228	0.344	1.716	0.075	0.132	0.706	**VZ**	0.187	0.264	2.259	0.096	0.167	1.248
	0.050		0.637	0.056		1.383		0.046		0.996	0.054		1.243
INTC.O	0.252	0.289	1.684	0.112	0.181	1.462	**WMT**	0.256	0.303	3.025	0.053	0.161	1.496
	0.047		1.840	0.053		1.191		0.043		1.526	0.055		0.778
JNJ	0.278	0.360	2.596	0.131	0.199	1.800	**XOM**	0.236	0.385	1.408	0.122	0.230	1.131
	0.045		1.002	0.050		1.386		0.053		0.452	0.055		0.684
JPM	0.240	0.355	1.799	0.067	0.137	0.781							
	0.049		0.622	0.054		1.067							

Table D.5: Estimation results for $Z_t = (RV_t, Vol_t)'$ with bandwidth $\delta_m = 0.65$ and $\gamma = \pi \frac{d_2 - d_1}{2}$. The small numbers below the estimates represent the deviation in asymptotic 95% confidence intervals.

	$\omega_2 = 0.48$			$\omega_3 = 0.96$				$\omega_2 = 0.48$			$\omega_3 = 0.96$		
	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$		$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$	$\hat{d}_1$	$\hat{d}_2$	$\hat{\beta}$
.DJI	0.481	0.483	0.797	0.232	0.249	-4.253	**KO**	0.471	0.480	4.045	0.216	0.218	0.500
	0.068		19.073	0.062		2.380		0.063		4.774	0.079		16.373
AAPL.O	-0.001	0.117	0.300	0.076	0.078	0.543	**MCD**	0.466	0.468	0.553	0.196	0.203	-3.788
	0.086		0.269	0.081		15.019		0.082		19.625	0.063		5.767
AXP	0.469	0.471	0.449	0.217	0.221	2.177	**MMM**	0.470	0.474	3.711	0.179	0.183	-3.524
	0.083		16.730	0.063		8.528		0.062		10.846	0.063		9.256
BA	0.338	0.401	0.224	0.234	0.251	-1.354	**MRK**	0.485	0.487	0.527	0.231	0.243	-2.547
	0.086		0.835	0.066		2.650		0.079		17.332	0.063		3.759
CAT	0.440	0.496	0.374	0.208	0.220	-2.636	**MSFT.O**	0.226	0.302	0.252	0.157	0.163	-1.444
	0.087		0.672	0.063		4.151		0.088		0.587	0.067		9.400
CSCO.O	0.445	0.461	2.167	0.236	0.268	-0.700	**NKE**	0.392	0.463	0.727	0.296	0.305	-3.290
	0.066		4.267	0.074		1.530		0.078		0.830	0.063		4.923
CVX	0.465	0.477	2.606	0.263	0.277	-2.029	**PFE**	0.306	0.314	2.820	0.100	0.152	-0.282
	0.064		4.909	0.064		3.143		0.065		8.432	0.076		0.722
DD	0.375	0.377	0.498	0.129	0.146	-2.872	**PG**	0.389	0.391	0.519	0.156	0.235	-0.093
	0.083		22.885	0.063		2.890		0.080		26.79	0.086		0.464
DIS	0.342	0.428	-0.016	0.132	0.199	-0.388	**TRV**	0.471	0.473	-0.364	0.165	0.216	-0.215
	0.086		0.519	0.081		0.662		0.075		16.999	0.081		0.542
GE	0.414	0.458	1.064	0.211	0.213	-0.280	**UNH**	0.490	0.496	3.251	0.158	0.168	-4.294
	0.074		1.359	0.084		24.616		0.063		7.150	0.062		4.264
GS	0.476	0.478	0.550	0.182	0.184	0.509	**UTX**	0.494	0.496	0.526	0.162	0.261	-0.014
	0.082		16.492	0.078		15.838		0.080		16.109	0.086		0.272
HD	0.491	0.493	0.555	0.276	0.285	-3.381	**V**	0.491	0.493	0.547	0.226	0.235	-3.054
	0.081		19.694	0.063		5.486		0.081		18.387	0.063		5.071
IBM	0.431	0.433	0.491	0.147	0.149	0.517	**VZ**	0.304	0.383	0.417	0.192	0.224	0.437
	0.083		21.038	0.080		20.065		0.084		0.625	0.081		1.129
INTC.O	0.351	0.360	2.649	0.196	0.198	0.508	**WMT**	0.364	0.373	3.257	0.161	0.171	-2.372
	0.065		7.669	0.079		18.610		0.064		7.002	0.063		4.373
JNJ	0.464	0.466	0.526	0.213	0.224	-2.989	**XOM**	0.440	0.496	1.362	0.255	0.257	0.509
	0.079		22.510	0.063		4.164		0.069		0.834	0.082		19.511
JPM	0.466	0.470	2.101	0.150	0.155	2.007							
	0.065		13.141	0.065		11.884							

Table D.6: Estimation results for $Z_t = (Vol_t, RV_t)'$ with bandwidth $\delta_m = 0.55$ and $\gamma = \pi \frac{d_2 - d_1}{2}$. The small numbers below the estimates represent the deviation in asymptotic 95% confidence intervals.

Chapter 6

Seasonal Long Memory in Intraday Volatility and Trading Volume of Dow Jones Stocks

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