Additional material to the paper "Nonlinear moving horizon estimation in the presence of bounded disturbances"

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Abstract

This technical report contains additional material to the paper
"Nonlinear moving horizon estimation in the presence of bounded disturbances" by M. A. Müller, in Automatica, 2017, 79, 306–314, DOI: 10.1016/j.automatica.2017.01.033,
in particular the proofs of Lemma 7 and Theorem 17. References and labels in this technical report (in particular Equation labels (1)–(33), references [1]–[20], and all theorem numbers etc.) refer to those in that paper.

I. Proof of Lemma 7

In the proof of Lemma 7, we will make use of the following properties, which hold for all $\alpha \in \mathcal{K}$, all $\beta \in \mathcal{KL}$, and all $a_i \in \mathbb{R}_{\geq 0}$ with $i \in \mathbb{I}_{[1,n]}$ (for a proof, see, e.g., [15, Appendix A]):

$$\alpha(a_1 + \cdots + a_n) \leq \alpha(na_1) + \cdots + \alpha(na_n) \tag{34}$$

$$\beta(a_1 + \cdots + a_n, t) \leq \beta(na_1, t) + \cdots + \beta(na_n, t) \tag{35}$$

Now consider a moving horizon estimator with some arbitrary (but fixed) estimation horizon $N \in \mathbb{I}_{\geq 1}$. Since $\sum_{i=t-N}^{t-1} \ell(\omega(i|t), \nu(i|t)) \geq \max_{i\in\mathbb{I}_{[t-N,t-1]}} \ell(\omega(i|t), \nu(i|t))$, it follows from Assumptions 4–5 that for each $t \in \mathbb{I}_{\geq N}$, the optimal value function $J_N^0(t) := J_N(\hat{x}(t-N|t), \hat{\boldsymbol{w}}(t))$ of problem (2)–(3) is lower bounded for all $i \in \mathbb{I}_{[t-N,t-1]}$ by[1]

$$J_N^0(t) \geq \; \delta_1 \underline{\gamma}_p(|\hat{x}(t-N|t) - \hat{x}(t-N)|) + (\delta + \delta_2)(\underline{\gamma}_w(|\hat{w}(i|t)|) + \underline{\gamma}_v(|\hat{v}(i|t)|)). \tag{36}$$

Furthermore, since we have $\sum_{i=t-N}^{t-1} \ell(\omega(i|t), \nu(i|t)) \leq N \max_{i\in\mathbb{I}_{[t-N,t-1]}} \ell(\omega(i|t), \nu(i|t))$, again by Assumptions 4 and 5 and due to optimality we conclude that for each $t \in \mathbb{I}_{\geq N}$, $J_N^0(t)$ is upper bounded by

$$\begin{aligned} J_N^0(t) &\leq J_N(x(t-N), \{w(t-N), \dots, w(t-1)\}) \\ &\leq \delta_1 \overline{\gamma}_p(|x(t-N) - \hat{x}(t-N)|) + (\delta + N\delta_2)(\overline{\gamma}_w(\|\boldsymbol{w}\|_{[t-N,t-1]}) + \overline{\gamma}_v(\|\boldsymbol{v}\|_{[t-N,t-1]})) \\ &\leq \delta_1 \overline{\gamma}_p(|x(t-N) - \hat{x}(t-N)|) + (\delta + N\delta_2)(\overline{\gamma}_w(\|\boldsymbol{w}\|) + \overline{\gamma}_v(\|\boldsymbol{v}\|)). \end{aligned} \tag{37}$$

Combining (36) with (37), we obtain that for all $t \in \mathbb{I}_{\geq N}$ and all $i \in \mathbb{I}_{[t-N,t-1]}$

$$\begin{aligned} |\hat{w}(i|t)| &\overset{(36)}{\leq} \underline{\gamma}_w^{-1}(J_N^0(t)/(\delta + \delta_2)) \\ &\overset{(37)}{\leq} \underline{\gamma}_w^{-1}\Big(\big(\delta_1 \overline{\gamma}_p(|x(t-N) - \hat{x}(t-N)|) + (\delta + N\delta_2)(\overline{\gamma}_w(\|\boldsymbol{w}\|) + \overline{\gamma}_v(\|\boldsymbol{v}\|))\big)/(\delta + \delta_2)\Big) \\ &\overset{(34)}{\leq} \underline{\gamma}_w^{-1}\Big(\big(3\delta_1 \overline{\gamma}_p(|x(t-N) - \hat{x}(t-N)|)\big)/(\delta + \delta_2)\Big) \\ &\quad + \underline{\gamma}_w^{-1}\left(\frac{3(\delta + N\delta_2)\overline{\gamma}_w(\|\boldsymbol{w}\|)}{\delta + \delta_2}\right) + \underline{\gamma}_w^{-1}\left(\frac{3(\delta + N\delta_2)\overline{\gamma}_v(\|\boldsymbol{v}\|)}{\delta + \delta_2}\right). \end{aligned} \tag{38}$$

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[1] To see that this is true for $t = N$, note that $\hat{x}(0) = \bar{x}_0$.

An analogous upper bound can be obtained for $|\hat{v}(i|t)|$, where $\underline{\gamma}_w^{-1}$ in all three terms on the right hand side of (38) is replaced by $\underline{\gamma}_v^{-1}$. Finally, again from (36) and (37), we obtain that for all $t \in \mathbb{I}_{\geq N}$

$$
\begin{aligned}
&|\hat{x}(t-N|t) - \hat{x}(t-N)| \overset{(36)}{\leq} \underline{\gamma}_p^{-1}(J_N^0(t)/\delta_1) \\
&\overset{(37)}{\leq} \underline{\gamma}_p^{-1}\big(\overline{\gamma}_p(|x(t-N) - \hat{x}(t-N)|) + (\delta + N\delta_2)(\overline{\gamma}_w(\|\boldsymbol{w}\|) + \overline{\gamma}_v(\|\boldsymbol{v}\|))/\delta_1\big) \\
&\overset{(34)}{\leq} \underline{\gamma}_p^{-1}(3\overline{\gamma}_p(|x(t-N) - \hat{x}(t-N)|)) + \underline{\gamma}_p^{-1}(3(\delta + N\delta_2)\overline{\gamma}_w(\|\boldsymbol{w}\|)/\delta_1) + \underline{\gamma}_p^{-1}(3(\delta + N\delta_2)\overline{\gamma}_v(\|\boldsymbol{v}\|)/\delta_1) \qquad (39)
\end{aligned}
$$

Next, consider some time $t \in \mathbb{I}_{\geq N}$. We now apply the i-IOSS property (5) with $x_1 = x(t-N)$, $x_2 = \hat{x}(t-N|t)$, $\boldsymbol{w}_1 = \{w(t-N), \dots, w(t-1)\}$, $\boldsymbol{w}_2 = \{\hat{w}(t-N|t), \dots, \hat{w}(t-1|t)\}$, and $\tau = N$. Since $x(t) = x(N; x_1, \boldsymbol{w}_1)$, $\hat{x}(t) = \hat{x}(t|t) = x(N; x_2, \boldsymbol{w}_2)$, and $h(x(i)) = y(i) - v(i)$ as well as $h(\hat{x}(i|t)) = y(i) - \hat{v}(i|t)$ for all $i \in \mathbb{I}_{[t-N,t-1]}$, from (5) we obtain

$$
\begin{aligned}
|x(t) - \hat{x}(t)| &\leq \beta(|x_1 - x_2|, N) + \gamma_1(\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{[0,N-1]}) + \gamma_2(\|h_{\boldsymbol{w}_1}(\boldsymbol{x}) - h_{\boldsymbol{w}_2}(\boldsymbol{x})\|_{[0,N-1]}) \\
&= \beta(|x(t-N) - \hat{x}(t-N|t)|, N) + \gamma_1(\sup_{i \in \mathbb{I}_{[t-N,t-1]}} |w(i) - \hat{w}(i|t)|) + \gamma_2(\sup_{i \in \mathbb{I}_{[t-N,t-1]}} |v(i) - \hat{v}(i|t)|). \qquad (40)
\end{aligned}
$$

The three terms on the right hand side of (40) can be upper bounded as follows. For the first term, we obtain

$$
\begin{aligned}
&\beta(|x(t-N) - \hat{x}(t-N|t)|, N) \\
&\leq \beta\big(|x(t-N) - \hat{x}(t-N)| + |\hat{x}(t-N|t) - \hat{x}(t-N)|, N\big) \\
&\overset{(35)}{\leq} \beta(2|x(t-N) - \hat{x}(t-N)|, N) + \beta(2|\hat{x}(t-N|t) - \hat{x}(t-N)|, N) \\
&\overset{(39),(35)}{\leq} \beta(2|x(t-N) - \hat{x}(t-N)|, N) + \beta(6\underline{\gamma}_p^{-1}(3\overline{\gamma}_p(|x(t-N) - \hat{x}(t-N)|)), N) \\
&\quad + \beta(6\underline{\gamma}_p^{-1}(3(\delta + N\delta_2)\overline{\gamma}_w(\|\boldsymbol{w}\|)/\delta_1), N) + \beta(6\underline{\gamma}_p^{-1}(3(\delta + N\delta_2)\overline{\gamma}_v(\|\boldsymbol{v}\|)/\delta_1), N) \qquad (41) \\
&\overset{(6),(10)}{\leq} c_\beta 2^p |x(t-N) - \hat{x}(t-N)|^p \Psi(N) + c_\beta 6^p (3\overline{c}_p/\underline{c}_p)^{p/a} |x(t-N) - \hat{x}(t-N)|^p \Psi(N) \\
&\quad + c_\beta 6^p (3/\underline{c}_p)^{p/a} \overline{\gamma}_w(\|\boldsymbol{w}\|)^{p/a} ((\delta + N\delta_2)/\delta_1)^{p/a} \Psi(N) + c_\beta 6^p ((3/\underline{c}_p)^{p/a} \overline{\gamma}_v(\|\boldsymbol{v}\|)^{p/a} ((\delta + N\delta_2)/\delta_1)^{p/a} \Psi(N) \qquad (42)
\end{aligned}
$$

For the second term on the right hand side of (40), we obtain

$$
\begin{aligned}
&\gamma_1(\sup_{i \in \mathbb{I}_{[t-N,t-1]}} |w(i) - \hat{w}(i|t)|) \leq \gamma_1(\|\boldsymbol{w}\| + \sup_{i \in \mathbb{I}_{[t-N,t-1]}} |\hat{w}(i|t)|) \\
&\overset{(38),(34)}{\leq} \gamma_1\left(4\underline{\gamma}_w^{-1}\left(\frac{3\delta_1\overline{\gamma}_p(|x(t-N) - \hat{x}(t-N)|)}{\delta + \delta_2}\right)\right) \\
&\quad + \gamma_1(4\underline{\gamma}_w^{-1}(3(\delta + N\delta_2)\overline{\gamma}_v(\|\boldsymbol{v}\|)/(\delta + \delta_2))) + \gamma_1(4\|\boldsymbol{w}\|) + \gamma_1(4\underline{\gamma}_w^{-1}(3(\delta + N\delta_2)\overline{\gamma}_w(\|\boldsymbol{w}\|)/(\delta + \delta_2))) \\
&\overset{(10),(8)}{\leq} c_1(3\overline{c}_p)^{\alpha_1}(\delta_1/(\delta + \delta_2))^{\alpha_1}|x(t-N) - \hat{x}(t-N)|^{a\alpha_1} \\
&\quad + c_1(3(\delta + N\delta_2)\overline{\gamma}_v(\|\boldsymbol{v}\|)/(\delta + \delta_2))^{\alpha_1} + \gamma_1(4\|\boldsymbol{w}\|) + c_1(3(\delta + N\delta_2)\overline{\gamma}_w(\|\boldsymbol{w}\|)/(\delta + \delta_2))^{\alpha_1} \qquad (43)
\end{aligned}
$$

Analogously, for the third term on the right hand side of (40), we obtain

$$
\begin{aligned}
&\gamma_2(\sup_{i \in \mathbb{I}_{[t-N,t-1]}} |v(i) - \hat{v}(i|t)|) \\
&\leq \gamma_2\left(4\underline{\gamma}_v^{-1}\left(\frac{3\delta_1\overline{\gamma}_p(|x(t-N) - \hat{x}(t-N)|)}{\delta + \delta_2}\right)\right) \\
&\quad + \gamma_2(4\underline{\gamma}_v^{-1}(3(\delta + N\delta_2)\overline{\gamma}_w(\|\boldsymbol{w}\|)/(\delta + \delta_2))) + \gamma_2(4\|\boldsymbol{v}\|) + \gamma_2(4\underline{\gamma}_v^{-1}(3(\delta + N\delta_2)\overline{\gamma}_v(\|\boldsymbol{v}\|)/(\delta + \delta_2))) \\
&\overset{(10),(8)}{\leq} c_2(3\overline{c}_p)^{\alpha_2}(\delta_1/(\delta + \delta_2))^{\alpha_2}|x(t-N) - \hat{x}(t-N)|^{a\alpha_2} \\
&\quad + c_2(3(\delta + N\delta_2)\overline{\gamma}_w(\|\boldsymbol{w}\|)/(\delta + \delta_2))^{\alpha_2} + \gamma_2(4\|\boldsymbol{v}\|) + c_2(3(\delta + N\delta_2)\overline{\gamma}_v(\|\boldsymbol{v}\|)/(\delta + \delta_2))^{\alpha_2} \qquad (44)
\end{aligned}
$$

Inserting (42)–(44) into (40) results in (12) with $\hat{\beta}$, $\hat{\varphi}_w$, and $\hat{\varphi}_v$ as defined in (13)–(15), which completes the proof of Lemma 7. □

II. Proof of Theorem 17

The proof of Theorem 17 proceeds similarly to the one of Theorem 14. Applying again Lemma 7 with $\Psi(s)$, δ_1, δ_2, and δ as in the theorem statement and exploiting the fact that $\kappa^{-p/a}q \leq 1$, it follows that (16) holds for all $t \in \mathbb{I}_{\geq N}$ with φ_w and φ_v given by (29)–(30) and $\overline{\beta}$ defined by

$$\begin{aligned} \overline{\beta}(r,N) &:= c_\beta(2^p + 6^p(3\overline{c}_p/\underline{c}_p)^{p/a})r^p q^N \\ &+ c_1(3\overline{c}_p)^{\alpha_1} r^{a\alpha_1} N^{\alpha_1}\kappa^{\alpha_1 N} + c_2(3\overline{c}_p)^{\alpha_2} r^{a\alpha_2} N^{\alpha_2}\kappa^{\alpha_2 N}. \end{aligned} \tag{45}$$

for all $r \geq 0$ and all $N \in \mathbb{I}_{\geq 1}$. Since $\kappa \leq 1/e$, it follows that both $N^{\alpha_1}\kappa^{\alpha_1 N}$ and $N^{\alpha_2}\kappa^{\alpha_2 N}$ are decreasing in N for $N \in \mathbb{I}_{\geq 1}$. Hence also $\overline{\beta}(r,N)$ is decreasing in N for $N \in \mathbb{I}_{\geq 1}$ and fixed $r > 0$. But this means that for $N = 0$, we can again extend $\overline{\beta}$ arbitrarily such that $\overline{\beta} \in \mathcal{KL}$ and $\overline{\beta}(r,0) \geq r$ for all $r \geq 0$.

Now fix $\mu > 0$ and let $r_{\max}(N) := \max\{(1/2)(\overline{\beta}(e_{\max},0) + \varphi_w(w_{\max}) + \varphi_v(v_{\max})), (1+\mu)(\varphi_w(w_{\max}) + \varphi_v(v_{\max}))\}$. As in the proof of Theorem 14, we have $r_{\max}(N) = \mathcal{O}(N^\alpha)$ with $\alpha = \max\{\alpha_1, \alpha_2\}$. But then, since $\lim_{N\to\infty} N^{\varepsilon_1}\varepsilon_2^N = 0$ for all $\varepsilon_1 \geq 0$ and all $0 \leq \varepsilon_2 < 1$, it follows that for each $\hat{\alpha}$ satisfying $\max\{q, \kappa^{\alpha_1}, \kappa^{\alpha_2}\} < \hat{\alpha} < 1$, there exists $N_0 \in \mathbb{I}_{\geq 1}$ such that for all $N \in \mathbb{I}_{\geq N_0}$ the following three conditions are satisfied:

$$\begin{aligned} 3c_\beta(2^p + 6^p(3\overline{c}_p/\underline{c}_p)^{p/a})2^p r_{\max}(N)^{p-1} q^N &\leq \hat{\alpha}^N \\ 3c_1(3\overline{c}_p)^{\alpha_1} 2^{a\alpha_1} r_{\max}(N)^{a\alpha_1 - 1} N^{\alpha_1}\kappa^{\alpha_1 N} &\leq \hat{\alpha}^N \\ 3c_2(3\overline{c}_p)^{\alpha_2} 2^{a\alpha_2} r_{\max}(N)^{a\alpha_2 - 1} N^{\alpha_2}\kappa^{\alpha_2 N} &\leq \hat{\alpha}^N \end{aligned} \tag{46}$$

Then, for all $N \in \mathbb{I}_{\geq N_0}$ and for all $0 \leq r \leq r_{\max}(N)$, it follows that $\overline{\beta}(2r,N) \leq r\hat{\alpha}^N \leq r\hat{\alpha}^{N-N_0}$. From here, we can proceed as in the proof of Theorem 9, replacing $(N/N_0)^{-\hat{\alpha}}$ by $\hat{\alpha}^{N-N_0}$ at the respective places. □