Additional material to the paper

“Nonlinear moving horizon estimation in the presence of bounded disturbances”

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Abstract

This technical report contains additional material to the paper

"Nonlinear moving horizon estimation in the presence of bounded disturbances"
in particular the proofs of Lemma 7 and Theorem 17. References and labels in this technical report (in particular
Equation labels (1)–(33), references [1]–[20], and all theorem numbers etc.) refer to those in that paper.

I. PROOF OF LEMMA 7

In the proof of Lemma 7, we will make use of the following properties, which hold for all \( \alpha \in \mathbb{K} \), all \( \beta \in \mathbb{K}\mathbb{L} \),
and all \( \alpha_i \in \mathbb{R}_{\geq 0} \) with \( i \in [1,n] \) (for a proof, see, e.g., [15, Appendix A]):

\[
\alpha(a_1 + \cdots + a_n) \leq \alpha(na_1) + \cdots + \alpha(na_n) \tag{34}
\]

\[
\beta(a_1 + \cdots + a_n, t) \leq \beta(na_1, t) + \cdots + \beta(na_n, t) \tag{35}
\]

Now consider a moving horizon estimator with some arbitrary (but fixed) estimation horizon \( N \in \mathbb{I}_{\geq 1} \). Since
\( \sum_{i=t-N}^{t-1} \ell(\omega(i|t), \nu(i|t)) \geq \max_{i \in [t-N,t-1]} \ell(\omega(i|t), \nu(i|t)) \), it follows from Assumptions 4–5 that for each \( t \in \mathbb{I}_{\geq N} \),
the optimal value function \( J_N^0(t) := J_N(\hat{x}(t-N|t), w(t)) \) of problem (2)–(3) is lower bounded for all \( i \in [t-N,t-1] \)
by\(^1\)

\[
J_N^0(t) \geq \delta_1 \gamma_w(\| \hat{x}(t-N|t) - \hat{x}(t-N) \|) + (\delta + \delta_2)(\gamma_v(\| \hat{w}(i|t) \|) + \gamma_v(\| \hat{v}(i|t) \|)). \tag{36}
\]

Furthermore, since we have \( \sum_{i=t-N}^{t-1} \ell(\omega(i|t), \nu(i|t)) \leq N \max_{i \in [t-N,t-1]} \ell(\omega(i|t), \nu(i|t)) \), again by Assumptions 4 and 5 and due to optimality we conclude that for each \( t \in \mathbb{I}_{\geq N} \), \( J_N^0(t) \) is upper bounded by

\[
J_N^0(t) \leq J_N(x(t-N), \{w(t-N), \ldots, w(t-1)\})
\leq \delta_1 \gamma_p(\| x(t-N) - \hat{x}(t-N) \|) + (\delta + N\delta_2)(\gamma_w(\| \hat{w}(i|t) \|) + \gamma_v(\| \hat{v}(i) \|))
\leq \delta_1 \gamma_p(\| x(t-N) - \hat{x}(t-N) \|) + (\delta + N\delta_2)(\gamma_w(\| \hat{w} \|) + \gamma_v(\| \hat{v} \|)). \tag{37}
\]

Combining (36) with (37), we obtain that for all \( t \in \mathbb{I}_{\geq N} \) and all \( i \in [t-N,t-1] \)

\[
|\hat{w}(i|t)| \leq \gamma_w^{-1}(J_N^0(t)/\delta) \tag{36}
\]

\[
\leq \gamma_w^{-1}\left(\frac{\delta_1 \gamma_p(\| x(t-N) - \hat{x}(t-N) \|) + (\delta + N\delta_2)(\gamma_w(\| \hat{w} \|) + \gamma_v(\| \hat{v} \|))}{\delta + \delta_2}\right) \tag{37}
\]

\[
\leq \gamma_w^{-1}\left(\frac{3(\delta + N\delta_2)\gamma_w(\| \hat{w} \|)}{\delta + \delta_2}\right)
\]

\[
\leq \gamma_w^{-1}\left(\frac{3(\delta + N\delta_2)\gamma_v(\| \hat{v} \|)}{\delta + \delta_2}\right). \tag{38}
\]

\(^1\)To see that this is true for \( t = N \), note that \( \hat{x}(0) = \bar{x}_0 \).

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An analogous upper bound can be obtained for $|\hat{v}(i|t)|$, where $\gamma_w^{-1}$ in all three terms on the right hand side of (38) is replaced by $\gamma_w^{-1}$. Finally, again from (36) and (37), we obtain that for all $t \in \mathbb{I}_{\geq N}$

$$
|\hat{x}(t - N|t) - \hat{x}(t - N)| \leq \gamma_p^{-1}(J_N^0(t)/\delta_1)
$$

$$(36)$$

$$
\leq \gamma_p^{-1}(\tau_p(|x(t - N) - \hat{x}(t - N)|) + (\delta + N\delta_2)(\tau_w(||w||) + \tau_v(||v||))/\delta_1)
$$

$$(37)$$

$$
\leq \gamma_p^{-1}(3\tau_p(|x(t - N) - \hat{x}(t - N)|)) + \gamma_p^{-1}(3(\delta + N\delta_2)\tau_w(||w||)/\delta_1) + \gamma_p^{-1}(3(\delta + N\delta_2)\tau_v(||v||)/\delta_1)
$$

$$(34)$$

Next, consider some time $t \in \mathbb{I}_{\geq N}$. We now apply the i-IOSS property (5) with $x_1 = x(t - N|t)$, $x_2 = \hat{x}(t - N|t)$, $w_1 = \{(w(t - N), \ldots, w(t - 1))\}$, $w_2 = \{\hat{w}(t - N|t), \ldots, \hat{w}(t - 1|t)\}$, and $r = N$. Since $x(t) = x(N; x_1, w_1)$, $\hat{x}(t) = \hat{x}(t|t) = x(N; x_2, w_2)$, and $h(x(i)) = y(i) - v(i)$ as well as $h(\hat{x}(i|t)) = y(i) - \hat{v}(i|t)$ for all $i \in \mathbb{I}_{[t - N, t - 1]}$, from (5) we obtain

$$
|x(t) - \hat{x}(t)| \leq \beta(|x_1 - x_2|, N) + \gamma_1(||w_1 - w_2||_{[0, N-1])} + \gamma_2(||h_{w_1}(x) - h_{w_2}(x)||_{[0, N-1)})
$$

$$
= \beta(|x(t - N) - \hat{x}(t - N|t)|, N) + \gamma_1(\sup_{i \in \mathbb{I}_{[t - N, t - 1]}}|w(i) - \hat{w}(i|t)|) + \gamma_2(\sup_{i \in \mathbb{I}_{[t - N, t - 1]}}|v(i) - \hat{v}(i|t)|).
$$

$$(40)$$

The three terms on the right hand side of (40) can be upper bounded as follows. For the first term, we obtain

$$
\beta(|x(t - N) - \hat{x}(t - N|t)|, N)
$$

$$
\leq \beta(|x(t - N) - \hat{x}(t - N)| + |\hat{x}(t - N|t) - \hat{x}(t - N)|, N)
$$

$$
(35)
$$

$$
\leq \beta(2|x(t - N) - \hat{x}(t - N)|, N) + \beta(2|\hat{x}(t - N|t) - \hat{x}(t - N)|, N)
$$

$$
\beta(2|x(t - N) - \hat{x}(t - N)|, N) + \beta(2|\hat{x}(t - N|t) - \hat{x}(t - N)|, N)
$$

$$
(39),(35)
$$

$$
\leq \beta(2|x(t - N) - \hat{x}(t - N)|, N) + \beta(6\gamma_p^{-1}(3\tau_p(|x(t - N) - \hat{x}(t - N)|)), N)
$$

$$
+ \beta(6\gamma_p^{-1}(3(\delta + N\delta_2)\tau_w(||w||)/\delta_1), N) + \beta(6\gamma_p^{-1}(3(\delta + N\delta_2)\tau_v(||v||)/\delta_1), N)
$$

$$
(41)
$$

$$
\leq c_\beta 2\beta|x(t - N) - \hat{x}(t - N)|^p\Psi(N) + c_\beta 6\beta^p(3\tau_p/\tau_p)^{p/\alpha}|x(t - N) - \hat{x}(t - N)|^p\Psi(N)
$$

$$
+ c_\beta 6\beta^p(3\tau_p/\tau_p)^{p/\alpha}\tau_w(||w||)^{p/\alpha}((\delta + N\delta_2)/\delta_1)^{p/\alpha}\Psi(N) + c_\beta 6\beta^p(3\tau_p/\tau_p)^{p/\alpha}\tau_v(||v||)^{p/\alpha}((\delta + N\delta_2)/\delta_1)^{p/\alpha}\Psi(N)
$$

$$
(42)
$$

The second term on the right hand side of (40), we obtain

$$
\gamma_1(\sup_{i \in \mathbb{I}_{[t - N, t - 1]}}|w(i) - \hat{w}(i|t)|) \leq \gamma_1(||w|| + \sup_{i \in \mathbb{I}_{[t - N, t - 1]}}|\hat{w}(i|t)|)
$$

$$
(38),(34)
$$

$$
\leq \gamma_1\left(\frac{4\gamma_w^{-1}}{2w}\left(3\delta_1\tau_p(|x(t - N) - \hat{x}(t - N)|)/\delta + \delta_2\right)\right)
$$

$$
+ \gamma_1(4\gamma_w^{-1}(3(\delta + N\delta_2)\tau_v(||v||)/(\delta + \delta_2))) + \gamma_1(4||w||) + \gamma_1(4\gamma_w^{-1}(3(\delta + N\delta_2)\tau_w(||w||)/(\delta + \delta_2))
$$

$$
(10),(8)
$$

$$
\leq c_1(3\tau_p)^{\alpha_2}(\delta_1/(\delta + \delta_2))^{\alpha_2}|x(t - N) - \hat{x}(t - N)|^{\alpha_2}
$$

$$
+ c_1(3(\delta + N\delta_2)\tau_v(||v||)/(\delta + \delta_2))^{\alpha_2} + \gamma_1(4||w||) + c_1(3(\delta + N\delta_2)\tau_w(||w||)/(\delta + \delta_2))^{\alpha_2}
$$

$$
(43)
$$

Analogously, for the third term on the right hand side of (40), we obtain

$$
\gamma_2(\sup_{i \in \mathbb{I}_{[t - N, t - 1]}}|v(i) - \hat{v}(i|t)|)
$$

$$
\leq \gamma_2\left(\frac{4\gamma_w^{-1}}{2w}\left(3\delta_1\tau_p(|x(t - N) - \hat{x}(t - N)|)/\delta + \delta_2\right)\right)
$$

$$
+ \gamma_2(4\gamma_w^{-1}(3(\delta + N\delta_2)\tau_w(||w||)/(\delta + \delta_2))) + \gamma_2(4||v||) + \gamma_2(4\gamma_w^{-1}(3(\delta + N\delta_2)\tau_v(||v||)/(\delta + \delta_2))
$$

$$
(10),(8)
$$

$$
\leq c_2(3\tau_p)^{\alpha_2}(\delta_1/(\delta + \delta_2))^{\alpha_2}|x(t - N) - \hat{x}(t - N)|^{\alpha_2}
$$

$$
+ c_2(3(\delta + N\delta_2)\tau_w(||w||)/(\delta + \delta_2))^{\alpha_2} + \gamma_2(4||v||) + c_2(3(\delta + N\delta_2)\tau_v(||v||)/(\delta + \delta_2))^{\alpha_2}
$$

$$
(44)
$$

Inserting (42)–(44) into (40) results in (12) with $\hat{\beta}$, $\hat{\varphi}_w$, and $\hat{\varphi}_v$ as defined in (13)–(15), which completes the proof of Lemma 7. \qed
II. PROOF OF THEOREM 17

The proof of Theorem 17 proceeds similarly to the one of Theorem 14. Applying again Lemma 7 with \( \Psi(s) \), \( \delta_1 \), \( \delta_2 \), and \( \delta \) as in the theorem statement and exploiting the fact that \( \kappa^{-p/\alpha}q \leq 1 \), it follows that (16) holds for all \( t \in \mathbb{I} \geq N \) with \( \varphi_w \) and \( \varphi_v \) given by (29)–(30) and \( \overline{\beta} \) defined by

\[
\overline{\beta}(r, N) := c_\beta(2^p + 6^p(3\overline{\varepsilon}_p/c_{\beta})^{p/\alpha})r^p q^N \\
+ c_1(3\overline{\varepsilon}_p)^{\alpha_1} r^{\alpha_1} N^{\alpha_1} \kappa^{\alpha_1} N + c_2(3\overline{\varepsilon}_p)^{\alpha_2} r^{\alpha_2} N^{\alpha_2} \kappa^{\alpha_2} N. \tag{45}
\]

for all \( r \geq 0 \) and all \( N \in \mathbb{I} \geq 1 \). Since \( \kappa \leq 1/e \), it follows that both \( N^{\alpha_1} \kappa^{\alpha_1} N \) and \( N^{\alpha_2} \kappa^{\alpha_2} N \) are decreasing in \( N \) for \( N \in \mathbb{I} \geq 1 \). Hence also \( \overline{\beta}(r, N) \) is decreasing in \( N \) for \( N \in \mathbb{I} \geq 1 \) and fixed \( r > 0 \). But this means that for \( N = 0 \), we can again extend \( \overline{\beta} \) arbitrarily such that \( \overline{\beta} \in \mathcal{K} \mathcal{L} \) and \( \overline{\beta}(r, 0) \geq r \) for all \( r \geq 0 \).

Now fix \( \mu > 0 \) and let \( r_{\text{max}}(N) := \max\{(1/2)(\overline{\beta}(e_{\max}, 0) + \varphi_v(w_{\max}) + \varphi_v(v_{\max})), (1 + \mu)(\varphi_w(w_{\max}) + \varphi_v(v_{\max}))\} \). As in the proof of Theorem 14, we have \( r_{\text{max}}(N) = O(N^\alpha) \) with \( \alpha = \max\{\alpha_1, \alpha_2\} \). But then, since \( \lim_{N \to \infty} N^{\varepsilon_1} e_N^N = 0 \) for all \( \varepsilon_1 \geq 0 \) and all \( 0 \leq \varepsilon_2 < 1 \), it follows that for each \( \hat{\alpha} \) satisfying \( \max\{q, \kappa^{\alpha_1}, \kappa^{\alpha_2}\} < \hat{\alpha} < 1 \), there exists \( N_0 \in \mathbb{I} \geq 1 \) such that for all \( N \in \mathbb{I} \geq N_0 \), the following three conditions are satisfied:

\[
3c_\beta(2^p + 6^p(3\overline{\varepsilon}_p/c_{\beta})^{p/\alpha})2^p r_{\text{max}}(N)^{p-1} q^N \leq \hat{\alpha}^N \\
3c_1(3\overline{\varepsilon}_p)^{\alpha_1} 2^{\alpha_1} r_{\text{max}}(N)^{\alpha_1-1} N^{\alpha_1} \kappa^{\alpha_1} N \leq \hat{\alpha}^N \\
3c_2(3\overline{\varepsilon}_p)^{\alpha_2} 2^{\alpha_2} r_{\text{max}}(N)^{\alpha_2-1} N^{\alpha_2} \kappa^{\alpha_2} N \leq \hat{\alpha}^N. \tag{46}
\]

Then, for all \( N \in \mathbb{I} \geq N_0 \) and for all \( 0 \leq r \leq r_{\text{max}}(N) \), it follows that \( \overline{\beta}(2r, N) \leq r^{\hat{\alpha} N} \leq r^{\hat{\alpha} N - N_0} \). From here, we can proceed as in the proof of Theorem 9, replacing \( (N/N_0)^{-\hat{\alpha}} \) by \( \hat{\alpha}^{N-N_0} \) at the respective places. \( \square \)