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Every finite system of $\mathbf{T_1}$ uniformities comes from a single distance structure

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Abstract. Using the general notion of distance function introduced in an earlier paper, a construction of the finest distance structure which induces a given quasi-uniformity is given. Moreover, when the usual defining condition $x\,U_\varepsilon\,y :\Leftrightarrow d(y,x) \leqslant \varepsilon$ of the basic entourages is generalized to $nd(y,x) \leqslant n\varepsilon$ (for a fixed positive integer n), it turns out that if the value-monoid of the distance function is commutative, one gets a countably infinite family of quasi-uniformities on the underlying set. It is then shown that at least every finite system and every descending sequence of T_1 quasi-uniformities which fulfil a weak symmetry condition is included in such a family. This is only possible since, in contrast to real metric spaces, the distance function need not be symmetric.

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1. Introduction

Since Fréchet's invention of real metric spaces in [2], many generalizations of this concept have been studied in the literature. Much research has been done on *generalized metric spaces,* in which the distance functions are replaced by certain set systems (cf. [9]). On the contrary, many authors independently suggested more general types of *distance functions*, the references [8], [12], [7], [5], [6], [11], [10], and [1] are only a small selection. In [3] and [4], a common framework for most if not all of these general concepts of distance functions has been developed to a certain extent.

In this paper, the induction of quasi-uniformities on a *distance space* $(X, d, \underline{M}, P)$ will be studied. In such a structure, $d : X \times X \to \underline{M}$ is a general *distance function* on X, that is, it fulfils the zero-distance condition $d(x, x) = 0$ and the triangle inequality $d(x, y) + d(y, z) \geqslant d(x, z)$, and takes its values in a quasi-ordered monoid (q. o. m.) $\underline{M} = (M, +, 0, \leqslant)$. The set $P \subseteq M$ must be a *set of positives* (or *idempotent zero-filter*) for $\underline{M}$, that is, a filter of $(M, \leqslant)$ with infimum 0 such that, for every $\varepsilon \in P$, there is $\delta \in P$ with $2\delta \leqslant \varepsilon$. The triple $(d, \underline{M}, P)$ is called a *distance structure* on X. For examples and categorical aspects of distance functions on various mathematical objects, see [3, 4].

Using Kelley's metrization lemma, one can easily show that every quasi-uniformity is induced by a suitable multi-quasi-pseudo-metric, that is, a "quasi-pseudo-metric" taking values in a real vector space instead of the non-negative reals. There is no doubt that this fact must have been noticed early. In this article however, we will see that also every finite *family* of T_1 uniformities (and many families of T_1 quasi-uniformities) on a fixed set X comes from a single distance structure. In Theorem 8, this is proved by constructing the finest such structure. This construction is a combinatorially more complex variant of the construction of a finest distance structure for a given quasi-uniformity, which is given in Theorem 2. In contrast to multi-quasi-pseudo-metric spaces, the "topological" information in the resulting spaces will be mostly contained in the set of positives P rather than in the distance function d itself. For example, each T_1 quasi-uniformity on some set X can be induced using one and the same distance function.

2. Preliminaries

In generalization of the usual definition of entourages in a metric space, let

$$U_n(\varepsilon) := \big\{(x, y) \in X \times X : nd(y, x) \leqslant n\varepsilon\big\}$$

for every $\varepsilon \in P$ and every positive integer n. As P is a filter, the set $\mathcal{E}_n := \{U_n(\varepsilon) : \varepsilon \in P\}$ is a base for a filter $\mathcal{U}_n$ of reflexive relations on X for each n. Moreover, when $\underline{M}$ is commutative,

$$nd(y, x) \leqslant \delta \geqslant nd(z, y) \text{ implies } nd(z, x) \leqslant n\big(d(z, y) + d(y, x)\big) \leqslant 2\delta,$$

so that, for every $\varepsilon \in P$, there is $\delta \in P$ with $U_n(\delta)^2 \subseteq U_n(\varepsilon)$, that is, $\mathcal{U}_n$ is a quasi-uniformity.

Of course, there are certain relationships between the $\mathcal{U}_n$, and in many cases most of them coincide. Obviously,

$$n = n_1 + \cdots + n_k \text{ implies} U_{n_1}(\varepsilon) \cap \cdots \cap U_{n_k}(\varepsilon) \subseteq U_n(\varepsilon).$$

Also, $nd(x, y) \leqslant nmd(x, y) + (m-1)nd(y, x)$, so that

$$(2m-1)n\delta \leqslant n\varepsilon \text{ implies} U_m(\delta) \cap U_n^{-1}(\delta) \subseteq U_n(\varepsilon).$$

For a *positive* d (that is, when $d(x, y) \geqslant 0$ for all x, y),

$$n \leqslant m \text{ and } m\delta \leqslant n\varepsilon \text{ imply } U_m(\delta) \subseteq U_n(\varepsilon). \quad (\dagger)$$

On the other hand, a *symmetric* d (that is, one with $d(x, y) = d(y, x)$) fulfils $2d(x, y) = d(x, y) + d(y, x) \geqslant d(x, x) = 0$, so that here the implication (†) holds at least when $m - n$ is even. This proves the following

Lemma 2.1.

(a) $n = n_1 + \cdots + n_k$ *implies* $\mathcal{U}_n \subseteq \mathcal{U}_{n_1} \vee \cdots \vee \mathcal{U}_{n_k}$*, in particular, the map* $n \mapsto \mathcal{U}_n$ *is antitone with respect to divisibility.*
(b) *For all* n, m, $\mathcal{U}_n \subseteq \mathcal{U}_n^{-1} \vee \mathcal{U}_m$.
(c) *For a positive* d*, all* $\mathcal{U}_n$ *coincide.*
(d) *For a symmetric* d *and all* $k \geqslant 1$, $\mathcal{U}_{2k} = \mathcal{U}_2 \subseteq \mathcal{U}_1 = \mathcal{U}_{2k-1}$.

Note that there are indeed natural distance functions which are neither positive nor symmetric, the most important being perhaps the distance $x^{-1}y$ on groups, introduced by Menger [8]:

Example 2.2. Let $G := [0, 2\pi)$ be the additive group of real numbers modulo 2π, $\underline{M} := (\mathcal{P}(G), +, \{0\}, \subseteq)$ the power set of G ordered by set inclusion and with the usual element-wise addition, $P := \{(-\delta, \delta) : \delta \in (0, 2\pi]\}$. Then $d(x, y) := \{y - x\}$ defines a *skew-symmetric* distance function (that is, one with $d(x, y) + d(y, x) = 0$), and $\mathcal{U}_1$ is the usual “Euclidean” uniformity on G, while $\mathcal{U}_n$ is this uniformity “modulo $\frac{2\pi}{n}$” since

$$x \, U_n(-\delta, \delta) \, y \iff x - y \in \bigcup_{k \in n} (-\delta + \tfrac{2k\pi}{n}, \tfrac{2k\pi}{n} + \delta).$$

Likewise, for $X := \mathbb{C} \setminus \{0\}$, $\underline{M}' := \underline{M} \otimes [0, \infty)$, $P' := P \times (0, \infty)$, and $d'(x, y) := \big(d'(\arg x, \arg y), \big||y| - |x|\big|\big)$, the uniformity $\mathcal{U}_n$ of $(d', \underline{M}', P')$ induces the Euclidean topology “modulo multiplication with nth roots of unity”.

3. Finest distance functions

Like for other topological structures on a set X, we might compare two distance functions d, d' resp. distance structures $\underline{D} = (d, \underline{M}, P)$ and $\underline{D}' = (d', \underline{M}', P')$ on X with respect to their *fineness*. If the implication

$$\begin{aligned} & d(x_1, y_1) + \cdots + d(x_n, y_n) \leqslant d(z_1, w_1) + \cdots + d(z_m, w_m) \\ \Longrightarrow \quad & d'(x_1, y_1) + \cdots + d'(x_n, y_n) \leqslant d'(z_1, w_1) + \cdots + d'(z_m, w_m) \end{aligned}$$

holds for all $x_i, y_i, z_i, w_i \in X$, we say that d is *finer* than d'. If, additionally, for all $\varepsilon' \in P'$, there is $\varepsilon \in P$ such that

$$d(x_1, y_1) + \cdots + d(x_n, y_n) \leqslant \varepsilon \Longrightarrow d'(x_1, y_1) + \cdots + d'(x_n, y_n) \leqslant \varepsilon'$$

for all $x_i, y_i \in X$, we say that $\underline{D}$ is finer than $\underline{D}'$.

For a convenient notation, let me introduce the free monoid F of all words in X that have even length and define

$$\begin{aligned} d(x_1 y_1 \cdots x_n y_n) \quad &:= \quad d(x_1, y_1) + \cdots + d(x_n, y_n), \\ s \, R_d \, t \quad &:\Leftrightarrow \quad d(s) \leqslant d(t) \quad (s, t \in F). \end{aligned}$$

By definition, $(F, \circ, 0, R_d)$ is a q. o. m., where $\circ$ is concatenation and 0 is the empty word. Given any quasi-order R on F which is compatible to $\circ$ (that is, whenever $(F, \circ, 0, R)$ is a q. o. m.), the following construction leads to a distance function d_R if and only if

$$xx \, R \, 0 \, R \, xx \text{ and } xz \, R \, xyyz \quad \text{for all } x, y, z \in X. \quad (\star)$$

Let $(M_R, \subseteq) := \theta(F, R)$ be the *lower set completion* of (F, R), that is, the system of all lower sets $RA := \{s : s\,R\,t \text{ for some } t \in A\}$ of (F, R) with set inclusion as partial order. Define an associative operation $+_R$ on M_R and its neutral element 0_R by

$$RA +_R RB := R\{s \circ t : s \in A \text{ and } t \in B\} \quad \text{for all } A, B \subseteq F$$

and $0_R := R\{0\}$. Then let

$$d_R : \begin{cases} X \times X & \to \quad \underline{M_R} = (M_R, +_R, 0_R, \subseteq) \\ (x, y) & \mapsto \quad R\{xy\}. \end{cases}$$

It was shown in [3] that d_{R_d} is equivalent to d, which motivates calling R_d the *generating quasi-order of* d. Moreover, when $R_\perp$ is the smallest quasi-order on F which fulfils ($\star$) and is compatible with $\circ$ then $d_\perp := d_{R_\perp}$ is a finest distance function on X. In this relation, the step from $s \in F$ to an upper neighbour w. r. t. $R_\perp$ consists of inserting a pair yy at an arbitrary position in s or removing a pair yy after an even number of letters in s, while the step to a lower neighbour is made by removing a pair yy at an arbitrary position or inserting a pair yy after an even number of letters.

4. Induction of a single quasi-uniformity

We are now ready for the first main result of this paper:

Theorem 4.1. *Every quasi-uniformity $\mathcal{V}$ admits a finest distance structure $(d_{\mathcal{V}}, \underline{M}_{\mathcal{V}}, P_{\mathcal{V}})$ for which $\mathcal{V} = \mathcal{U}_1$.*

Proof. Let $\mathcal{V}$ be some quasi-uniformity on X and $V_0 := \bigcap \mathcal{V}$. We will see that the essential information about $\mathcal{V}$ is contained in the set of positives $P_{\mathcal{V}}$ which we must construct, while the generating quasi-order $R_{d_{\mathcal{V}}}$ is fully determined by the very weak condition that $xy\, R_{d_{\mathcal{V}}}\, zz$ must hold for any triple $x, y, z \in X$ which fulfils $y\, V_0\, x$ (otherwise $d_{\mathcal{V}}(x, y) \not\leq \varepsilon$ for some $\varepsilon \in P_{\mathcal{V}}$, in contradiction to $V_0 \subseteq \mathcal{U}_1(\varepsilon)$). Therefore, let R be the smallest quasi-order on F that is compatible with $\circ$ and fulfils

$$x'y'\, R\, 0\, R\, xx \text{ and } xz\, R\, xyyz \quad \text{for all } x, y, z, x', y' \in X \text{ with } y'\, V_0\, x'. \quad (\star')$$

If we find a suitable s. o. p. P such that (d_R, P) induces $\mathcal{V}$ then R must obviously be the smallest relation (and thus d_R a finest distance function) with this property.

Now observe that each of the resulting entourages $U_1(\varepsilon)$ has to include some entourage $V_1 \in \mathcal{V}$, hence every $\varepsilon \in P$ must include some set $\{xy \in F : y\, V_1\, x\}$ with $V_1 \in \mathcal{V}$. Since $0_R = R\{xx\}$ is a neutral element, ε must even include the set

$$\{xy \in F : y\, V_0 V_1 V_0\, x\} \subseteq 0_R +_R \{xy \in F : y\, V_1\, x\} +_R 0_R.$$

The same must be true for any $\delta \in P$ which fulfils $\delta +_R \delta \subseteq \varepsilon$, so that ε must also include a set $\{xyx'y' \in F : y\, V_0 V_2 V_0\, x, y'\, V_0 V_2 V_0\, x'\} \subseteq \delta +_R \delta$ for some $V_2 \in \mathcal{V}$. This process of replacing some ε by some 2δ can be continued, and in order to describe it formally, let us define W to be the smallest set of tuples of positive integers that contains the 1-tuple (1) and fulfils

$$(n_1, \dots, n_{i-1}, n_i + 1, n_i + 1, n_{i+1}, \dots, n_k) \in W$$

whenever $(n_1, \dots, n_k) \in W$ and $1 \leqslant i \leqslant k$. One can think of the elements of W as coding exactly those terms of the form '$\varepsilon_{n_1} + \cdots + \varepsilon_{n_k}$' that can be obtained when we start with the term 'ε_1' and then successively replace an arbitrary summand 'ε_n' by the term '$\varepsilon_{n+1} + \varepsilon_{n+1}$'. Accordingly, one shows by induction that for each element ε_1 of a set of positives P there is a sequence $\varepsilon_2, \varepsilon_3, \dots$ in P such that

$$(n_1, \dots, n_k) \in W \text{ implies } \varepsilon_{n_1} + \cdots + \varepsilon_{n_k} \leqslant \varepsilon_1.$$

In our situation, this observation implies that for each $\varepsilon \in P$ there must be a sequence $\mathcal{S} = (V_1, V_2, \dots)$ in $\mathcal{V}$ with the property that ε includes the set $A_\mathcal{S}$ of all words $v_1 w_1 \cdots v_k w_k \in F$ for which there is some $(n_1, \dots, n_k) \in W$ such that $w_i\, V_0 V_{n_i} V_0\, v_i$ for $i = 1, \dots, k$. In particular, $\varepsilon_\mathcal{S} := RA_\mathcal{S} \subseteq R\varepsilon = \varepsilon$. It turns out that this is the only restraint on the set of positives $P_\mathcal{V}$. More precisely, we will see that the system

$$B := \{\varepsilon_\mathcal{S} : \mathcal{S} \text{ is a sequence in } \mathcal{V}\}$$

of lower sets of (F, R) is a base for a set of positives of $(M_R, +_R, 0_R, \subseteq)$, and that the distance structure (d_R, P) induces the quasi-uniformity $\mathcal{V}$. It is then clear that P is the largest set of positives with this property, so that $(d_\mathcal{V}, P_\mathcal{V}) := (d_R, P)$ is a finest distance structure inducing $\mathcal{V}$.

Since $\mathcal{V}$ is a filter and the map $\mathcal{S} \mapsto \varepsilon_\mathcal{S}$ is isotone in every component of $\mathcal{S}$, B is a filter-base. In order to show that P is a s. o. p., we first observe that $(n_1, \dots, n_k), (m_1, \dots, m_l) \in W$ implies

$$(n_1 + 1, \dots, n_k + 1, m_1 + 1, \dots, m_l + 1) \in W.$$

Indeed, after increasing each index by one, the replacements that produce $(n_1, \dots, n_k)$ and $(m_1, \dots, m_l)$ from the tuple (1) can be combined to a sequence of replacements that produce $(n_1 + 1, \dots, n_k + 1, m_1 + 1, \dots, m_l + 1)$ from the tuple $(2, 2)$.

Hence also $v_1 w_1 \cdots v_k w_k, v'_1 w'_1 \cdots v'_l w'_l \in \varepsilon_{(V_2, V_3, V_4, \dots)}$ implies

$$v_1 w_1 \cdots v_k w_k v'_1 w'_1 \cdots v'_l w'_l \in \varepsilon_{(V_1, V_2, V_3, \dots)}$$

for each sequence $(V_1, V_2, \dots)$ in $\mathcal{V}$. Secondly, we must prove that $\bigcap B = 0_R$, which is the harder part. Let $s = x_1 z_1 \cdots x_m z_m \in \bigcap B$ and $V_1 \in \mathcal{V}$. I will show that $z_j\, V_0 V_1 V_0\, x_j$ holds for all $j = 1, \dots, m$. Choose a sequence $\mathcal{S} = (V_1, V_2, \dots)$ in $\mathcal{V}$ such that $V_{i+1} V_0 V_{i+1} \subseteq V_i$ for all $i \geqslant 1$ (such a sequence always exists in a quasi-uniformity). Note that $(n_1, \dots, n_k) \in W$ then implies $V_0 V_{n_1} V_0 V_{n_2} V_0 \cdots V_0 V_{n_k} V_0 \subseteq V_0 V_1 V_0$. Now $s \in RA_\mathcal{S}$, that is, there exists a word $v_1 w_1 \cdots v_k w_k$ and a k-tuple $(n_1, \dots, n_k) \in W$ such that $w_i\, V_0 V_{n_i} V_0\, v_i$ for $i = 1, \dots, k$ and $s\, R\, v_1 w_1 \cdots v_k w_k$. The latter means that, starting with $v_1 w_1 \cdots v_k w_k$, one gets $x_1 z_1 \cdots x_m z_m$ in finitely many steps in each of which some pair of letters is inserted or removed corresponding to the condition $(\star')$. Now take the k-tuple

$$\psi := (w_1\, V_0 V_{n_1} V_0\, v_1, \dots, w_k\, V_0 V_{n_k} V_0\, v_k)$$

of formulae (which express true propositions about the word $v_1 w_1 \cdots v_k w_k$) and modify it, analogously to those finitely many steps, in the following way: (i) if (because of $xz\, R\, xyyz$) a pair yy is being removed after an odd number of letters, replace the two consecutive formulae $\dots V_0\, y, y\, V_0 \cdots$ in ψ by one formula $\dots V_0 \cdots$ (that is, erase the symbols '$y, y\, V_0$'); (ii) if (because of $0\, R\, xx$) a pair xx is being removed after an even number of letters, remove the corresponding formula $x \dots x$ from ψ; (iii) if (because of $x'y'\, R\, 0$) a pair $x'y'$ is inserted, insert the formula $y'\, V_0\, x'$ at the respective position in ψ. By definition of R, all these modifications preserve the truth of all formulae in the tuple, and each formula in the resulting tuple $(\psi_1, \dots, \psi_k)$ expresses a true proposition of the form

$$\psi_j = z_j\, V_0 V_{n_a} V_0 V_{n_{a+1}} V_0 \dots V_0 V_{n_b} V_0\, x_j$$

with $1 \leqslant a, b \leqslant k$. Since all V_{n_i} are reflexive, ψ_j thus implies

$$z_j\, V_0 V_{n_1} V_0 V_{n_2} V_0 \dots V_0 V_{n_k} V_0\, x_j,$$

hence $z_j\, V_0 V_1 V_0\, x_j$. Because V_1 was chosen arbitrarily, we conclude that $z_j\, V_0\, x_j$ for all j, and therefore $x_1 z_1 \cdots x_m z_m\, R\, 0$.

Finally, we have to show that (d_R, P) induces the quasi-uniformity $\mathcal{V}$. For $V \in \mathcal{V}$, choose $V_1 \in \mathcal{V}$ such that $V_0 V_1 V_0 \subseteq V$, then choose a sequence $\mathcal{S}$ as in the preceding paragraph. There we have shown that, in particular,

$$d_R(x, z) \subseteq RA_{\mathcal{S}} \text{ implies } (z, x) \in V_0 V_1 V_0 \subseteq V.$$

On the other hand, for each $\varepsilon \in P$ there is some sequence $\mathcal{S} = (V_1, \dots)$ in $\mathcal{V}$ such that $\varepsilon_{\mathcal{S}} \subseteq \varepsilon$, and

$$(z, x) \in V_1 \subseteq V_0 V_1 V_0 \text{ implies} d_R(x, z) \subseteq \varepsilon_{\mathcal{S}} \subseteq \varepsilon.$$

□

A somewhat astonishing consequence of this construction is that *one* distance function is compatible to *all* T_1 quasi-uniformities on X:

Corollary 4.2. *The distance function $d_\perp$ is the finest distance function d on X such that for each T_1 quasi-uniformity $\mathcal{V}$ on X there is a s. o. p. P such that $(d_\perp, P)$ induces $\mathcal{V}$ (namely $P = P_{\mathcal{V}}$).*

5. Induction of systems of quasi-uniformities

I will now extend this result to certain systems of quasi-uniformities and show that, in particular, every finite system and every descending sequence of T_1 uniformities is part of some system $(\mathcal{U}_n)_{n\in\omega}$.

Some additional notation: Intervals of integers will be designated by $[a, b]$. A pair of letters $xy \in F$ is a *syllable* of a word $s \in F$ if and only if it occurs in s after an even number of letters. Let $\tilde{s} \in F$ be the word s after deletion of all syllables of the form xx $(x \in X)$. The length of $\tilde{s}$ in letters is designated by

$\ell(s)$, and s_a is the ath letter of $\tilde{s}$ for any position $a \in [1, \ell(s)]$. The subword of $\tilde{s}$ from position a to b is $s_{a,b}$. Moreover, let $\lambda(x, s)$ and $\sigma(xy, s)$ denote the number of occurrences of the letter x resp. the syllable xy in $\tilde{s}$. Finally, $(xy)^r = xy \cdots xy$ is a word consisting of r equal syllables.

The next constructions mainly rely on four lemmata. For the moment, let us fix some words $s, t \in F$ with $s\, R_\perp\, t$, where

$$\tilde{t} = (v_1 w_1)^{r_1} \cdots (v_\varrho w_\varrho)^{r_\varrho}, \quad v_i \neq w_i, \quad \text{and all } r_i \text{ are even.}$$

Then $\tilde{s}$ can be derived from $\tilde{t}$ by a finite number of successive deletions of pairs of identical letters which are neighbours at the time of deletion. A guiding example: for $s = yy\,xy\,zz\,xy\,uz\,uz\,R_\perp\,xy\,xy\,zz\,zu\,uz\,uz\,xx\,uz = t$, the deletion steps could be this: in $\tilde{t} = xy\,xy\,zu\,uz\,uz\,uz$, first delete uu, giving $xy\,xy\,zz\,uz\,uz$, then delete zz, giving $xy\,xy\,uz\,uz = \tilde{s}$.

We now also fix such a sequence of deletions and let $D \subseteq [1, \ell(t)]$ be the set of positions in $\tilde{t}$ whose corresponding letters are deleted in one of these steps (in the example: $D = [5, 8]$). For $a \in D$, let $\pi(a) \in [1, \ell(t)]$ be that position in $\tilde{t}$ such that t_a and $t_{\pi(a)}$ build a deleted pair (in the example: $\pi(5) = 8$ and $\pi(6) = 7$). Finally, we write $a \curvearrowright b$ if and only if a and $b-1$ are even numbers in D such that $a < \pi(a) = b - 1$ (in the example: $6 \curvearrowright 8$). Note that because t_c and $t_{\pi(c)}$ must first become neighbours before they can be deleted, $a \curvearrowright \cdots \curvearrowright b$ implies that (i) $[a, b-1] \subseteq D$, (ii) $\pi(c) \in [a, b-1]$ for all $c \in [a, b-1]$, and thus (iii) $\lambda(x, t_{a,b-1})$ is even for all $x \in X$.

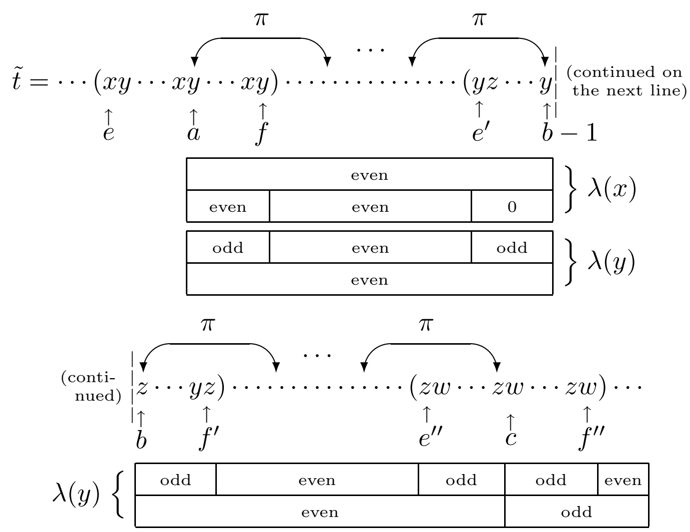
Lemma 5.1. *Assume $a \curvearrowright \cdots \curvearrowright b \curvearrowright \cdots \curvearrowright c$, $t_a = t_{b-1}$, and $t_b = t_{c-1}$. Then*

(a) *$t_{a-1} = t_b$ or $t_{b-1} = t_c$.*
(b) *If $t_{a-1} \neq t_b$ then $\lambda(t_a, t_{c,\ell(t)})$ is odd.*
(c) *If $t_{b-1} \neq t_c$ then $\lambda(t_b, t_{1,a-1})$ is odd.*

Proof. Let $e, f, e', f', e'', f'' \in [1, \ell(t)]$ with $e < a \leqslant f < e' < b \leqslant f' < e'' < c \leqslant f''$ such that $t_{e,f}$, $t_{e',f'}$, and $t_{e'',f''}$ are three of the defining subwords $(v_i w_i)^{r_i}$ of $\tilde{t}$. Moreover, let $x := t_{a-1}$, $y := t_a = t_{b-1}$, $z := t_b = t_{c-1}$, and $w := t_c$, and assume $x \neq z$. The situation and the parity arguments that will follow are sketched in Figure 1.

Because of $x \neq z$, we have $\lambda(x, t_{e',b-1}) = 0$. Moreover, $\lambda(x, t_{f+1,e'-1})$ is even (since all r_i are even), and $\lambda(x, t_{a,b-1})$ is even because of (iii), so that also $\lambda(x, t_{a,f})$ is even and $\lambda(y, t_{a,f})$ is odd (since $|[a, f]|$ is odd). As before, $\lambda(y, t_{f+1,e'-1})$ and $\lambda(y, t_{a,b-1})$ are even, thus $\lambda(y, t_{e',b-1})$ is odd. Because all r_i are even, $\lambda(y, t_{b,f'})$ is also odd. Again, $\lambda(y, t_{f'+1,e''-1})$ and $\lambda(y, t_{b,c-1})$ are even, hence $\lambda(y, t_{e'',c-1})$ is odd. In particular, $y \in \{z, w\}$, that is, $y = w$ (as yz is a syllable of $\tilde{t}$), and $\lambda(y, t_{c,f''})$ is also odd. Finally, $\lambda(y, t_{c,\ell(t)})$ is odd because $\lambda(y, t_{f'',\ell(t)})$ is even. This proves (a) and (b), whereas (c) is strictly analogous to (b). $\square$

Figure 1. Situation in Lemma 5.1



Lemma 5.2.

(a) *Assume that* $a_0 \curvearrowright b_0 \curvearrowright a_1 \curvearrowright b_1 \cdots a_k \curvearrowright b_k \curvearrowright c$ *with* $t_{a_0} = \cdots = t_{a_k} = y$, *and* $t_{b_0} = \cdots = t_{b_k} = z$. *Then* $t_{a_0-1} = z$ *or* $y = t_c$.

(b) *Assume that* $a \curvearrowright \cdots \curvearrowright b$ *with* $t_a = t_{b-1}$, *and* $t_{a-1} \neq t_b$. *Then both* $\lambda(t_a, t_{1,a-1})$ *and* $\lambda(t_a, t_{b,\ell(t)})$ *are odd.*

Proof. (a) Define e'', f'' as above. Similarly, for each $i \in [0, k]$, find positions $e_i, f_i, e'_i, f'_i \in [1, \ell(t)]$ with $e_i < a_i \leqslant f_i < e'_i < b_i \leqslant f'_i$ such that t_{e_i,f_i} and $t_{e'_i,f'_i}$ are two of the defining subwords of $\tilde{t}$. Assuming $t_{a_0-1} = x \neq z$, one proves that $\lambda(y, t_{b_0,f'_0})$ is odd exactly as before. Since, for $i \in [1, k]$, all of $\lambda(y, t_{b_{i-1},a_i-1})$, $\lambda(y, t_{a_i,b_i-1})$, $\lambda(y, t_{f'_{i-1}+1,e_i-1})$, $\lambda(y, t_{e_i,f_i})$, $\lambda(y, t_{f_i+1,e'_i-1})$, and $\lambda(y, t_{e'_i,f'_i})$ are even, and since also $\lambda(y, t_{b_k,c-1})$ and $\lambda(y, t_{f'_k+1,e''-1})$ are even, we conclude that $\lambda(y, t_{e'',c-1})$ is odd, hence $y = t_c$.

(b) Again as in the previous lemma, one proves that, for $y := t_a$, the number $\lambda(y, t_{b,f'})$ is odd, so that the first claim follows because $\lambda(y, t_{f',\ell(t)})$ is even. The second claim is just the dual. □

Lemma 5.3. *Assume that* $s_{e-1}s_e = xz$ *is the syllable of* $\tilde{s}$ *that remains after all the deletions in a subword* $t_{a-1,b}$ *of* $\tilde{t}$, *with* $a < b$, $t_{a-1} = x$, *and* $t_b = z$. *Then there is* $y \in X$ *such that* $\lambda(y, s) > 0$, $\sigma(xy, t_{a-1,b}) > 0$, *and* $\sigma(yz, t_{a-1,b}) > 0$.

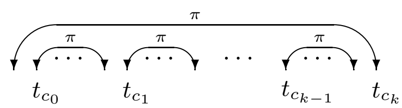
Proof. Although t_a and t_{b-1} may be different, we find $k \geqslant 2$, $c_1, \ldots, c_k \in [1, \ell(t)]$, and $y_0, y_1, \ldots, y_k \in X$ such that

$$a = c_1 \curvearrowright \cdots \curvearrowright c_2 \curvearrowright \cdots \curvearrowright c_3 \cdots c_{k-1} \curvearrowright \cdots \curvearrowright c_k \leqslant b,$$

$t_{c_i} = t_{c_{i+1}-1} = y_i$ for $i \in [1, k-1]$, $y_0 = x$, $y_k = z$, and $y_i \neq y_j$ for $i \neq j$ (Start with $a =: c'_1 \curvearrowright c'_2 \curvearrowright \cdots \curvearrowright c'_l := b$ and $y'_i := t_{c'_i}$. As long as there are indices $j > i > 1$ with $y'_i = y'_j$, remove all the indices $i+1, \ldots, j$, so that finally all remaining y'_i are different. Since $y'_1 = t_a \neq z = y'_l$, at least $k \geq 2$ of the original indices are not removed, including the index 1, and the corresponding c'_i build the required positions $c_1, \ldots, c_k$).

Then $k = 2$ since otherwise Lemma 5.1 (a) would imply that either $y_0 = y_2$ or $y_1 = y_3$. With y_1 for y and c_2 for b, Lemma 5.2 (b) implies that $\lambda(y, t_{1,a-1})$ is odd. Now, also $\lambda(y, s_{1,e-1})$ is odd, because $c \in [1, a-1] \cap D$ implies $\pi(c) \in [1, a-1]$ (since the letter x at position $a-1$ is not deleted). In particular, $\lambda(y, s_{1,e-1}) > 0$. □

Lemma 5.4. *Assume that* $k \geqslant 2$, $c_0 \curvearrowright c_1 \cdots c_{k-1} \curvearrowright c_k$, $c_k \in D$, *and* $\pi(c_k) = c_0 - 1$, *representing a number of deletions of the form*



Let $t' := t_{c_0-1} t_{c_0} t_{c_1-1} t_{c_1} \cdots t_{c_k-1} t_{c_k}$ *be the word consisting only of the "boundary letters", and* $i \in [0, k]$. *Then* $\sigma(t_{c_i-1} t_{c_i}, t') = \sigma(t_{c_i} t_{c_i-1}, t')$.

Proof. Put $c_{-1} := c_k$. Obviously, $t_{c_{i-1}} = t_{c_i-1}$ for all $i \in [1, k]$, and $t_{c_k} = t_{c_0-1}$. If also $t_{c_{i-1}-1} = t_{c_i}$ for all $i \in [0, k]$ then k must be odd (since $t_{c_k} \neq t_{c_0}$), and $\sigma(t_{c_i-1} t_{c_i}, t') = \sigma(t_{c_i} t_{c_i-1}, t') = k/2$. Otherwise, there are $r \geqslant 1$ positions $i(1) < \cdots < i(r)$ in $[0, k]$ with $t_{c_{i(j)-1}-1} \neq t_{c_{i(j)}}$. Then $i(j+1) - i(j)$ is even for all j (otherwise, put $a_0 := c_{i(j)-1}$, $b_0 := c_{i(j)}, \ldots$, $c := c_{i(j+1)-1}$ and apply Lemma 5.2 (a)). In case that all $i(j)$ are even, we have

$$t_{c_{k-1}} \neq t_{c_k} = t_{c_0-1} = t_{c_1} = t_{c_i}$$

for all odd i, so that k must be odd. On the other hand, if all $i(j)$ are odd, we have

$$t_{c_k} = t_{c_0} - 1 \neq t_{c_0} = t_{c_i}$$

for all even i, so that again k must be odd. This shows that t' is of one of the following two forms:

$$\begin{aligned} t' &= \quad (yxxy)^{m_0} (yz_1z_1y)^{m_1} \cdots (yz_{r-1}z_{r-1}y)^{m_{r-1}} (yxxy)^{m_r} \\ \text{or} \quad t' &= xy(yxxy)^{m_0} (yz_1z_1y)^{m_1} \cdots (yz_{r-1}z_{r-1}y)^{m_{r-1}} (yxxy)^{m_r} yx, \end{aligned}$$

from which the claim follows immediately. □

Now we are ready for the construction. Let p_i be the ith odd prime number, and $S(A) := \{a_1 + \cdots + a_k : k \geqslant 1,\ a_i \in A\}$ for any set A of integers. In the next theorem, we need the following sets of even numbers: for any positive integer u, let $q_{uj} = \frac{2}{p_j} \prod_{i=1}^{u} p_i$ for all $j \in [1, u]$, $Q_u := \{q_{u1}, \ldots, q_{uu}\}$, and $Q_{uj} := Q_u \setminus \{q_{uj}\}$. It is easy to see that then, for each $j \in [1, u]$ and $k \in S(Q_{uj})$, $k - q_{uj} \notin S(Q_{uj})$ (since p_j divides k but not q_{uj}).

Theorem 5.5.

(a) *Let $\mathcal{V}_1, \dots, \mathcal{V}_u$ be a finite system of T_1 quasi-uniformities such that, for all $i, j \in [1, u]$, $\mathcal{V}_j \subseteq \mathcal{V}_j^{-1} \vee \mathcal{V}_i$. Then there is a finest s. o. p. P such that, for $j \in [1, u]$, $\mathcal{V}_j = \mathcal{U}_{q_{uj}}$.*

(b) *Let $\mathcal{V}_1 \supseteq \mathcal{V}_2 \dots$ be a descending sequence of T_1 quasi-uniformities such that, for all j and all $U \in \mathcal{V}_j$, there are $V_1 \in \mathcal{V}_1, V_2 \in \mathcal{V}_2, \dots$ with $V_j^{-1} \cap \bigcup_{i \neq j} V_i \subseteq U$. Then there is a finest s. o. p. P such that $\mathcal{V}_j = \mathcal{U}_{2^j}$ for all j.*

Proof. For part (a), let $I := [1, u]$, while for part (b), let I be the set of natural numbers. In both cases, P is defined quite analogously to the proof of Theorem 4.1: its filter-base is now the system

$$B := \{\varepsilon_{\mathcal{S}} : \mathcal{S} \text{ is a sequence in } \mathcal{V}\}$$

of lower sets $\varepsilon_{\mathcal{S}} = R_\perp A_{\mathcal{S}}$ of $R_\perp$, where $\mathcal{V} := \prod_i \mathcal{V}_i$, and the definition of $A_{\mathcal{S}}$ changes to this: for

$$\mathcal{S} = ((V_{11}, V_{12}, \dots), (V_{21}, V_{22}, \dots), \dots),$$

$A_{\mathcal{S}}$ is now the set of all words $(v_1 w_1)^{r_1} (v_2 w_2)^{r_2} \cdots (v_\varrho w_\varrho)^{r_\varrho} \in F$ for which there is some $(n_1, \dots, n_\varrho) \in W$ and some tuple of indices $(i_1, \dots, i_\varrho)$ such that, for all $a \in [1, \rho]$, $w_a \, V_{n_a i_a} \, v_a$ and either $r_a = q_{u i_a}$ (for the proof of (a)) or $r_a = 2^{i_a}$ (for the proof of (b)).

As before, P turns out to be a s. o. p., where the only major change is the proof of $\bigcap B = 0_R$: Let $s \in \bigcap B$, $\sigma(xz, s) > 0$, and $V = (V_{11}, V_{12}, \dots) \in \mathcal{V}$. Choose $\mathcal{S}$ so that $V_{k+1,i} V_{k+1,i} \subseteq V_{ki}$ for all $i \in I$ and all k, and some $t \in A_{\mathcal{S}}$ with $s \, R_\perp \, t$. Assume that $\tilde{t} = (v_1 w_1)^{r_1} (v_2 w_2)^{r_2} \cdots (v_\rho w_\rho)^{r_\rho}$. If $\sigma(xz, t) > 0$, put $y_V := x$, otherwise choose some $y_V \in X$ with $\lambda(y_V, s) > 0$, $\sigma(x y_V, t) > 0$, and $\sigma(y_V z, t) > 0$, according to Lemma 5.3. Since $\ell(s)$ is finite and $\mathcal{V}$ is filtered, there is some y such that, for all $V \in \mathcal{V}$, there is $V' \in \mathcal{V}$ with $V' \leqslant V$ and $y_{V'} = y$, where $\leqslant$ denotes component-wise set inclusion. Consequently, $x \, U_V \, y \, U_V \, z$ for all $V \in \mathcal{V}$, where $U_V = \bigcup_i V_{1i}$. This implies that $x, y \in \bigcap \mathcal{V}_i$ and $x, y \in \bigcap \mathcal{V}_{i'}$ for some $i, i' \in I$, hence $x = y = z$. Since this is a contradiction to $x \neq z$, we have shown that $\tilde{s}$ is the empty word, that is, $s \in 0_R$.

Finally, let us show that $\mathcal{V}_j = \mathcal{U}_{q_{uj}}$ resp. $\mathcal{V}_j = \mathcal{U}_{2^j}$ for each $j \in I$. Fix some $j \in I$ and let $V_{0j} \in \mathcal{V}_j$. Because of the premises, the following choices can now be made. For part (a), choose for all $i \in I \setminus \{j\}$ some $V_{0i} \in \mathcal{V}_j$ and $V_{1i} \in \mathcal{V}_i$ such that $(V_{0i})^{-1} \cap V_{1i} \subseteq V_{0j}$. Then choose $V_{1j} \in \mathcal{V}_j$ such that $V_{1j} \subseteq V_{0i}$ for all of the finitely many $i \in I \setminus \{j\}$. For part (b), choose instead some $(V_{11}, V_{12}, \dots) \in \mathcal{V}$ with $V_{1h} = V_{1j} \subseteq V_{0j}$ for all $h \leqslant j$ and $(V_{1j})^{-1} \cap \bigcup_{i \neq j} V_{1i} \subseteq V_{0j}$.

After that, choose the remaining components of a sequence

$$\mathcal{S} = \big((V_{11}, V_{12}, \dots), (V_{21}, V_{22}, \dots), \dots\big)$$

in $\mathcal{V}$ so that $V_{k+1,i} V_{k+1,i} \subseteq V_{ki}$ for all $i \in I$ and all k, and assume that $rd_{R_\perp}(x, y) \leqslant \varepsilon_{\mathcal{S}}$, that is, $s := (xz)^r \, R_\perp \, t \in A_{\mathcal{S}}$ with (a) $r = q_{uj}$ resp. (b) $r = 2^j$. We have to show that $z \, V_{0j} \, x$.

By definition of $A_{\mathcal{S}}$, we have $\tilde{t} = (v_1 w_1)^{r_1}(v_2 w_2)^{r_2} \cdots (v_\rho w_\rho)^{r_\rho}$, and there is some corresponding tuple $(i_1, \ldots, i_\rho)$. Since the only letters in $\tilde{s}$ are x and z, there are exactly r occurrences of the syllable xz in $\tilde{t}$ which are not deleted (because otherwise Lemma 5.3 would imply the existence of a third letter y in $\tilde{s}$). All other occurrences of xz in $\tilde{t}$ are deleted as part of some set of deletions of the form represented in Lemma 5.4, that is, there are $c_0, \ldots, c_k$ with properties as in Lemma 5.4 and with $t_{c_i-1} t_{c_i} = xz$ for some $i \in [0, k]$. Then the lemma implies that $\sigma(xz, t) = r + \sigma(zx, t) =: k$.

For (a): If $(v_a w_a)^{r_a} = (xz)^{q_{uj}}$ for some $a \in [1, \rho]$, then $i_a = j$ and

$$(z, x) \in V_{n_a, i_a} \subseteq V_{1j} \subseteq V_{0j}.$$

Otherwise, we know that $k \in S(Q_{uj})$, that is, $\sigma(zx, t) = k - q_{uj} \in S(Q_u) \setminus S(Q_{uj})$, so that $(v_a w_a)^{r_a} = (zx)^{q_{uj}}$ and $i_a = j$ for some $a \in [1, \rho]$. Also, $(v_b w_b)^{r_b} = (xz)^{q_{ui}}$ and $i_b = i$ for some $b \in [1, \rho]$ and some $i \in I \setminus \{j\}$, so that $(z, x) \in (V_{1j})^{-1} \cap V_{1i} \subseteq V_{0j}$.

For (b) instead: If $(v_a w_a)^{r_a} = (xz)^{2^i}$ for some $a \in [1, \rho]$ and $i \leqslant j$, then $i_a = i$ and $(z, x) \in V_{n_a, i_a} \subseteq V_{1i} \subseteq V_{0j}$. Otherwise, k is a multiple of 2^{j+1} so that $\sigma(zx, t) = k - 2^j$ is not such a multiple. Therefore, $(v_a w_a)^{r_a} = (zx)^{2^{i_a}}$ and $i_a \leqslant j$ for some $a \in [1, \rho]$. Also, $(v_b w_b)^{r_b} = (xz)^{2^{i_b}}$ and $i_b \neq j$ for some $b \in [1, \rho]$, so that again $(z, x) \in (V_{1i_a})^{-1} \cap V_{1i_b} \subseteq V_{0j}$. □

Unfortunately, this proof highly depends on the fact that $\underline{M}_{R_\perp}$ is not commutative, so that the conjecture that there is also a suitable distance structure with a commutative value monoid is yet unproved.

The most familiar example for a descending sequence of uniformities is perhaps the following. Let $X := C_b[0, 1]$ be the (infinite-dimensional) vector space of bounded, continuous, and real-valued functions on the unit interval $[0, 1]$, and, for positive integers p, let $\mathcal{V}_p$ be the uniformity on X induced by the usual p-norm.

For a second example, take u different primes $p_1, \ldots, p_u$ and let $\mathcal{V}_i$ be the p_i-adic uniformity on the rationals. As these are transitive uniformities with countable bases, we may use a slightly simpler construction. More precisely, a base for $\mathcal{V}_i$ is the set of equivalence relations $U_{i,m} := \{(x, y) : p_i^m$ divides $\nu(|x - y|)\}$, where m is a positive integer, and $\nu(z/n) := z$ whenever z, n have no common divisor (that is, $\nu(q)$ is the nominator of q). Therefore, it suffices to use only those $\varepsilon_{\mathcal{S}}$ where all tuples in $\mathcal{S}$ are equal, that is, $V_{h+1,i} = V_{hi}$ for all i, h. In this case, the resulting s. o. p. P has a countable base $B = \{\varepsilon_m : m$ a positive integer$\}$, where

$$\varepsilon_m := \bigcup_{n=0}^{\infty} \left(n \cdot \bigcup_{\substack{j \in [1,u], \\ (x,y) \in U_{j,m}}} q_{uj} d_\perp(x, y) \right).$$

As a concluding remark, I note that with similar methods, one can show that, for each pair of comparable T_1 uniformities $\mathcal{V}_2 \subseteq \mathcal{V}_1$, there is some *symmetric* distance structure (d, P) such that $\mathcal{U}_i = \mathcal{V}_i$, which gives a complete characterization of the symmetric T_1 case.

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