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Superfield formulation of nonlinear $N = 4$ supermultiplets

S. Bellucci,[1] S. Krivonos,[2] O. Lechtenfeld,[3,*] and A. Shcherbakov[1,2]
[1]INFN-Laboratori Nazionali di Frascati, Via E. Fermi 40, 00044 Frascati, Italy
[2]Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia
[3]Theory Division, Physics Department, CERN, 1211 Geneva 23, Switzerland
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We propose a unified superfield formulation of $N = 4$ off-shell supermultiplets in one spacetime dimension using the standard $N = 4$ superspace. The main idea of our approach is a gluing together of two linear supermultiplets along their fermions. The functions defining such a gluing obey a system of equations. Each solution of this system provides a new supermultiplet, linear or nonlinear, modulo equivalence transformations. In such a way we reproduce all known linear and nonlinear $N = 4$, $d = 1$ supermultiplets and propose some new ones. Particularly interesting is an explicit construction of nonlinear $N = 4$ hypermultiplets.

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I. INTRODUCTION

The main ingredients for the construction of one-dimensional systems with extended $N \geq 4$ supersymmetry are irreducible supermultiplets. Given a set of those, preferably formulated in superspace, one may immediately write the corresponding sigma-model type actions and the general potential terms. In this respect, the almost complete classification of *linear* off-shell representations for one-dimensional supersymmetry [1–3] seems to suffice for constructing any mechanics model with extended supersymmetry. However, a detailed analysis of the corresponding actions reveals a common restriction—the bosonic parts of all actions describe only conformally flat manifolds. Moreover, the prepotentials describing the most general interaction are constrained to obey flat Laplace equations in superspace. These are signals that something essential is missing. The above-mentioned classification of linear representations admits only one possibility: there must exist additional *nonlinear* representations.

The possibility of nonlinear off-shell $N = 4$, $d = 1$ supermultiplets was first noted in [4]. Subsequently, in [5] the first two examples of such nonlinear supermultiplets were explicitly described. One of these examples was reduced from a four-dimensional cousin while the other one was completely new. The next step was taken in [6], with the reduction of the $N = 2$, $d = 4$ hypermultiplet to an off-shell $N = 4$, $d = 1$ supermultiplet. These new nonlinear supermultiplets with four physical bosonic and four fermionic components were explicitly constructed [7], and their formulation in harmonic superspace was proposed [8]. In parallel, the component description of several new nonlinear $N = 8$ supermultiplets was found [9].

Although by now the list of nonlinear $N = 4$ supermultiplets has gotten a bit lengthy, no attempt has as yet been made for their classification. The main obstacle here is the variety of methods by which these supermultiplets have been constructed: Some have been found within the geometric approach based on a nonlinear realization of the $N = 4$ superconformal group [5], others were built by applying the so-called dualization procedure [7]. In further cases, the harmonic superspace constraints just mimic their $N = 4$, $d = 2$ counterparts [4,8]. Moreover, part of these nonlinear supermultiplets have been formulated in terms of components, part in the standard $N = 4$, $d = 1$ superspace, while the rest in harmonic superspace. Clearly, for a classification it is desirable to have a unified description. Yet, such a framework has to be flexible enough not to exclude nonlinear supermultiplets which have yet to be discovered.

The main goal of the present paper is to provide such a unified approach towards nonlinear supermultiplets with $N = 4$ supersymmetry in one spacetime dimension. The key idea is to construct a nonlinear supermultiplet by entangling a pair of linear $N = 4$ supermultiplets. Let us illustrate the main steps of our construction.

For the sake of clarity we momentarily suppress all indices but one. A linear $N = 4$ supermultiplet consists of n physical fields ϕ, 4 fermionic ones ψ and $4 - n$ auxiliary ones A. Taking two such supermultiplets Φ_1 and Φ_2 with n_1 and n_2 physical bosons ϕ_1 and ϕ_2, respectively, we have twice as many fermions ψ_1 and ψ_2 as is required by $N = 4$ supersymmetry. This is not a problem in principle, but to get the minimal representation we must reduce this amount by somehow identifying the fermions of both supermultiplets. Denoting by D and $\bar{D}$ the covariant spinor derivatives, so that $D\Phi_{1,2} = \psi_{1,2} + \ldots$, the most general identification of the two sets of four spinors reads

$$D\Phi_1 = fD\Phi_2 + g\bar{D}\Phi_2 + hD\bar{\Phi}_2 + k\bar{D}\bar{\Phi}_2 \quad \text{with functions } f, g, h, k \text{ of } \Phi_1 \text{ and } \Phi_2. \tag{1.1}$$

As a consequence, the resulting *nonlinear* representation contains only four independent fermions rather than eight. Because of supersymmetry, some of the higher compo-

*On leave of absence from: Leibniz Universität Hannover, Germany.

nents of $\Phi_{1,2}$ will be expressed through lower components. The total number of physical components of the combined representation is just $n_1 + n_2$, leaving $4 - (n_1 + n_2)$ auxiliary fields in total. Since all numbers must be nonnegative, the possibilities are restricted by the inequality $n_1 + n_2 \leq 4$. It turns out that a vanishing n_1 or n_2 will just reproduce the partner supermultiplet, and so the nontrivial list of cases is

$$(n_1, n_2) = (1, 1), (2, 1), (3, 1), (2, 2). \tag{1.2}$$

Clearly enough, the functions f, g, h, and k cannot be completely arbitrary, because the irreducible $N = 4$ superfields $\Phi_{1,2}$ obey some constraints. These constraints imply a system of equations which these functions must satisfy. Each solution to this system gives rise to some irreducible supermultiplet (linear or nonlinear). Of course, some of these solutions may be equivalent to others via some superfield redefinition. Modulo this freedom, one may expect to find some of the known supermultiplets among the solutions. However, it is unexpected—and very satisfying—to see that actually *all known* linear and nonlinear supermultiplets may be constructed in this fashion. Moreover, the set of solutions is large enough to leave room for yet undiscovered nonlinear supermultiplets. The next four sections are devoted to the derivation of these results in the cases (1.2). A final section comments upon the implications of the results for the classification problem and touches upon a number of related issues.

II. 2 = 1 + 1

Let us start with the simplest example. Our goal is to construct the irreducible $N = 4$, $d = 1$ supermultiplet with the two physical bosons starting from two irreducible $N = 4$ supermultiplets containing one physical boson each. In terms of the $N = 4$ superfields the $N = 4$ supermultiplet with one physical boson is completely defined by a scalar superfield obeying the constraints [10]. We need two such supermultiplets, so we introduce two scalar $N = 4$ superfields u and v satisfying

$$D^{(i}\bar{D}^{j)}u = 0, \qquad D^{(i}\bar{D}^{j)}v = 0. \tag{2.1}$$

Here, D and $\bar{D}$ are spinor covariant derivatives obeying the standard super Poincaré algebra

$$\{D^i, \bar{D}_j\} = 2i\delta^i_j\partial_t, \tag{2.2}$$

and the brackets () as usual mean symmetrization over the indices enclosed.

Each of our superfields u, v contains among the components one physical scalar, four physical fermions, and three auxiliary bosons. Clearly, to get the irreducible supermultiplet one has reduce the number of the physical fermions to four. The simplest way to do this is to identify the spinors in both supermultiplets. The most general identifications is achieved in the following way:

$$D^i u = f_1 D^i v + f_2 \bar{D}^i v, \qquad \bar{D}_i u = \bar{f}_1 \bar{D}_i v - \bar{f}_2 D_i v, \tag{2.3}$$

with the arbitrary functions f_1, f_2 depending on both superfields u and v.

The superfield u obeys (2.1), therefore the right-hand side (r.h.s.) in (2.3) should be also antisymmetric over $su(2)$ indices upon action of D^j and $\bar{D}^j$ on them. This leads to the constraints on the functions $f_{1,2}$

$$\begin{aligned} &f_2\frac{\partial f_1}{\partial u} = f_1\frac{\partial f_2}{\partial u} + \frac{\partial f_2}{\partial v}, \qquad -\bar{f}_2\frac{\partial f_2}{\partial u} = \bar{f}_1\frac{\partial f_1}{\partial u} + \frac{\partial f_1}{\partial v}, \\ &\bar{f}_2\frac{\partial \bar{f}_1}{\partial u} = \bar{f}_1\frac{\partial \bar{f}_2}{\partial u} + \frac{\partial \bar{f}_2}{\partial v}, \qquad -f_2\frac{\partial \bar{f}_2}{\partial u} = f_1\frac{\partial \bar{f}_1}{\partial u} + \frac{\partial \bar{f}_1}{\partial v}. \end{aligned} \tag{2.4}$$

Let us note that Eqs. (2.3), being satisfied, reduce also the number of the auxiliary components to two in both supermultiplets expressing some of the auxiliary components through time derivatives of the physical bosons and identifying the remaining ones in both supermultiplets. Thus any solution of the system (2.4) provides us with the $N = 4$ supermultiplet with two physical bosons.

Before going on to solve Eqs. (2.4), one should note that we are free to choose the basic superfields in a different way. Indeed, one may write, for example, the basic constraint on the general superfunction $G(u, v)$ depending in an arbitrary way on u and v

$$\begin{cases} D^i G = f_1 D^i v + f_2 \bar{D}^i v, \\ \bar{D}_i G = \bar{f}_1 \bar{D}_i v - \bar{f}_2 D_i v, \end{cases} \Rightarrow \begin{cases} D^i u = \frac{f_1 - \frac{\partial G}{\partial v}}{\frac{\partial G}{\partial u}} D^i v + \frac{f_2}{\frac{\partial G}{\partial u}} \bar{D}^i v, \\ \bar{D}^i u = \frac{\bar{f}_1 - \frac{\partial G}{\partial v}}{\frac{\partial G}{\partial u}} D^i v - \frac{\bar{f}_2}{\frac{\partial G}{\partial u}} D_i v. \end{cases} \tag{2.5}$$

Clearly, one may use this gauge freedom to completely remove the real part of the function f_1. Thus, from now we impose the following condition:

$$f_1 = if, \qquad \bar{f}_1 = -if, \tag{2.6}$$

where $f(u, v)$ is a real function.

Now we are ready to find the general solution of Eqs. (2.4). First of all, one may easily show that from Eqs. (2.4) and (2.6) it follows that

$$\frac{\partial}{\partial u}(f^2 + f_2\bar{f}_2) = 0, \qquad \frac{\partial}{\partial v}(f^2 + f_2\bar{f}_2) = 0. \tag{2.7}$$

Therefore, $f^2 + f_2\bar{f}_2 = \text{const}$ and we are free to fix this constant

$$f^2 + f_2\bar{f}_2 = 1. \tag{2.8}$$

Now, it is rather convenient to solve Eq. (2.8) as

$$f = \frac{h\bar{h} - 1}{h\bar{h} + 1}, \qquad f_2 = \frac{2ih}{h\bar{h} + 1}, \qquad \bar{f}_2 = -\frac{2i\bar{h}}{h\bar{h} + 1}, \tag{2.9}$$

where h, $\bar{h}$ are two arbitrary functions. Substituting (2.9) in (2.4) we will get the following equations:

$$\bar{h}(ih_u - h_v) = 0, \qquad h(i\bar{h}_u + \bar{h}_v) = 0 \tag{2.10}$$

which have the evident solution

$$h = h(u+iv), \qquad \bar{h} = \bar{h}(u-iv). \tag{2.11}$$

Thus, our basic constraints (2.4) read

$$\begin{aligned} D^i u &= i\frac{h\bar{h}-1}{h\bar{h}+1}D^i v + \frac{2ih}{h\bar{h}+1}\bar{D}^i v, \\ \bar{D}^i u &= -i\frac{h\bar{h}-1}{h\bar{h}+1}\bar{D}^i v + \frac{2i\bar{h}}{h\bar{h}+1}D_i v. \end{aligned} \tag{2.12}$$

The last step is to rewrite the system (2.12) as

$$\begin{aligned} D^i(u+iv) &= h(u+iv)\bar{D}^i(u+iv), \\ \bar{D}^i(u-iv) &= -\bar{h}(u+iv)D^i(u-iv). \end{aligned} \tag{2.13}$$

So, one may construct the $N=4$ supermultiplet with the two physical bosons from the two supermultiplets with one physical bosons by imposing on them the constraints (2.13).

It is quite easy to recognize which supermultiplets we constructed. If the functions $h=\bar{h}=0$, the constraints (2.13) describe the standard $N=4$ chiral supermultiplet. If the function $h=\text{const}$, then we deal with the twisted chiral supermultiplet. Finally, if $h\neq \text{const}$ one may multiply the equations in (2.13) by h' and $\bar{h}'$, respectively, to get

$$D^i Z = Z\bar{D}^i Z, \qquad \bar{D}_i\bar{Z} = -\bar{Z}D_i\bar{Z}, \tag{2.14}$$

where

$$Z \equiv h(u+iv), \qquad \bar{Z} \equiv \bar{h}(u-iv). \tag{2.15}$$

The constraints (2.14) defined the nonlinear chiral supermultiplet [5].

Thus, we were able to construct all known $N=4$, $d=1$ supermultiplets with the two physical bosons among the components. Moreover, no other solutions exist within our approach. This is in full agreement with the claim of the paper [11] that all possible two-dimensional supermultiplets include chiral and nonlinear chiral supermultiplets only.

III. 3 = 2 + 1

In this section we will construct the $N=4$ supermultiplets with three physical bosons starting from two supermultiplets with one and two physical bosons, respectively. To describe the $N=4$ supermultiplet with one physical boson we will use the same real $N=4$ superfield u as in the previous section, subject to the constraints (2.1). In addition, the chiral $N=4$ superfield λ, $\bar{\lambda}$

$$D^i\lambda = 0, \qquad \bar{D}^i\bar{\lambda} = 0, \tag{3.1}$$

contains just two physical boson components. Now we have to identify the fermionic components in both supermultiplets as

$$D^i u = f_1 D^i\bar{\lambda} + f_2\bar{D}^i\lambda, \qquad \bar{D}^i u = \bar{f}_1\bar{D}^i\lambda - \bar{f}_2 D^i\bar{\lambda}, \tag{3.2}$$

where $f_{1,2}(u,\lambda,\bar{\lambda})$ are arbitrary functions depending on all our superfields $(u,\lambda,\bar{\lambda})$.

As well as in the previous case, the consistency of (3.2) imposes the restrictions on the functions $f_{1,2}$

$$f_2\frac{\partial f_1}{\partial u} = f_1\frac{\partial f_2}{\partial u} + \frac{\partial f_2}{\partial\bar{\lambda}}, \tag{3.3a}$$

$$\bar{f}_2\frac{\partial \bar{f}_1}{\partial u} = \bar{f}_1\frac{\partial \bar{f}_2}{\partial u} + \frac{\partial \bar{f}_2}{\partial\lambda}, \tag{3.3b}$$

$$-\bar{f}_2\frac{\partial f_2}{\partial u} = \bar{f}_1\frac{\partial f_1}{\partial u} + \frac{\partial f_1}{\partial\lambda}, \tag{3.4a}$$

$$-f_2\frac{\partial \bar{f}_2}{\partial u} = f_1\frac{\partial \bar{f}_1}{\partial u} + \frac{\partial \bar{f}_1}{\partial\bar{\lambda}}. \tag{3.4b}$$

So, any solution of the systems (3.3) and (3.4) describes the irreducible $N=4$ supermultiplet with three physical bosons, modulo possible redefinitions of the superfields. To partially fix this freedom, let us note that we may write the same Eqs. (3.3) and (3.4) on the arbitrary real superfunction $G(u,\lambda,\bar{\lambda})$ instead of u. This will result in the same Eqs. (3.3) and (3.4) for the superfield u but with the modified functions $\tilde{f}_{1,2}$

$$\begin{aligned} \tilde{f}_1 &= \frac{1}{G_u}(f_1 - G_{\bar{\lambda}}), \qquad \tilde{\bar{f}}_1 = \frac{1}{G_u}(\bar{f}_1 - G_\lambda), \\ \tilde{f}_2 &= \frac{1}{G_u}f_2, \qquad \tilde{\bar{f}}_2 = \frac{1}{G_u}\bar{f}_2. \end{aligned} \tag{3.5}$$

Using this freedom we cannot fully remove the real part of the function f_1 as in the previous section. Instead, one may partially restrict f_1 imposing the following condition:

$$\frac{\partial}{\partial\lambda}f_1 + \frac{\partial}{\partial\bar{\lambda}}\bar{f}_1 = 0. \tag{3.6}$$

Before solving the systems (3.3), (3.4), and (3.6) it is useful to demonstrate how the known $N=4$ supermultiplets with three physical bosons appear among the solutions of these equations.

A. Linear tensor supermultiplet

The linear tensor supermultiplet [12] is defined in terms of the $su(2)$ triplets of the bosonic superfields $V^{(ij)}$ subject to the following constraints:

$$\nabla^{(i}V^{jk)} = 0, \qquad \bar{\nabla}^{(i}V^{jk)} = 0, \tag{3.7}$$

where ∇^i, $\bar{\nabla}^i$ is the set of $N=4$ covariant derivatives with the standard superalgebra

$$\{\nabla^i, \bar\nabla_j\} = 2i\delta^i_j\partial_t. \tag{3.8}$$

Redefining the superfields and the covariant derivatives as

$$V^{ii} = \lambda, \quad V^{22} = \bar\lambda, \quad V^{12} = iu, \quad D^1 = \nabla^1, \quad D^2 = -\bar\nabla_2, \quad \bar D_1 = \bar\nabla_1, \quad \bar D_2 = -\nabla^2, \tag{3.9}$$

one may rewrite the basic constraints (3.7) as

$$D^i\lambda = 0, \qquad \bar D_i\bar\lambda = 0, \qquad D^i u = \frac{i}{2}\bar D^i\lambda, \qquad \bar D^i u = \frac{i}{2}D^i\bar\lambda. \tag{3.10}$$

Clearly, the constraints (3.10) coincide with (3.2) if we choose

$$f_1 = \bar f_1 = 0, \qquad f_2 = \frac{i}{2}, \qquad \bar f_2 = -\frac{i}{2}. \tag{3.11}$$

It is trivial to check that (3.11) is a particular solution of the system (3.3), (3.4), and (3.6).

B. Nonlinear tensor supermultiplet

The nonlinear tensor supermultiplet is defined in terms of the three bosonic $N = 4$ superfields $(u, \lambda, \bar\lambda)$ obeying the constraints [4,5]

$$D^i\lambda = 0, \qquad \bar D^i\bar\lambda = 0, \qquad D^i(e^{-iu}\bar\lambda) = -i\bar D^i u, \qquad \bar D^i(e^{iu}\lambda) = -iD^i u. \tag{3.12}$$

Rewriting the second line in the system (3.12) as

$$D^i u = \frac{i}{1+\lambda\bar\lambda}[-\lambda D^i\bar\lambda + e^{iu}\bar D^i\lambda], \qquad \bar D^i u = \frac{i}{1+\lambda\bar\lambda}[\bar\lambda\bar D^i\lambda + e^{-iu}D^i\bar\lambda], \tag{3.13}$$

one may again find the full agreement with (3.2) upon identification

$$f_1 = -i\frac{\lambda}{1+\lambda\bar\lambda}, \qquad f_2 = i\frac{e^{iu}}{1+\lambda\bar\lambda}. \tag{3.14}$$

The expressions (3.14), like to the previous case, provide the particular solution of Eqs. (3.3), (3.4), and (3.6).

C. General solution

Thus, all known supermultiplets with the three physical bosons are present among the solutions of our system (3.3), (3.4), and (3.6). To understand whether there are other solutions describing new $N = 4$ supermultiplets one has to solve Eqs. (3.3), (3.4), and (3.6).

First of all, let us note that the superfields λ, $\bar\lambda$ and the covariant derivatives are charged with respect to $U(1)$ rotations

$$D^i \to e^{i\alpha}D^i, \qquad \bar D^i \to e^{-i\alpha}\bar D^i, \qquad \lambda \to e^{2i\alpha}\lambda, \qquad \bar\lambda \to e^{-2i\alpha}\bar\lambda, \tag{3.15}$$

while the superfield u is chargeless. To keep this $U(1)$ invariance manifest, let us suppose that our functions $f_{1,2}$ are restricted as

$$f_1 = \lambda\tilde f_1(u,z), \qquad f_2 = f_2(u,z), \qquad z \equiv \lambda\bar\lambda. \tag{3.16}$$

With these conditions, Eq. (3.6) reads

$$\frac{\partial}{\partial z}[z(\tilde f_1 + \bar{\tilde f}_1)] = 0. \tag{3.17}$$

Thus, the real part of the function $\tilde f_1$ is completely fixed to be

$$\tilde f_1 + \bar{\tilde f}_1 = \frac{F(u)}{z} \equiv \frac{F(u)}{\lambda\bar\lambda}, \tag{3.18}$$

where $F(u)$ is an arbitrary function depending on the superfield u alone. Substituting (3.18) into our basic constraints (3.2) one may easily check that one can always redefine the superfield u to cancel this part in the constraints. So, from now on, we will impose the further restriction on the functions $\tilde f_1$

$$\tilde f_1 = if(u,z), \qquad \bar{\tilde f}_1 = -if(u,z). \tag{3.19}$$

Thus, Eq. (3.6) is satisfied, while the systems (3.3) and (3.4) read

$$if_2\frac{\partial f}{\partial u} = if\frac{\partial f_2}{\partial u} + \frac{\partial f_2}{\partial z}, \tag{3.20a}$$

$$-i\bar f_2\frac{\partial f}{\partial u} = -if\frac{\partial \bar f_2}{\partial u} + \frac{\partial \bar f_2}{\partial z}, \tag{3.20b}$$

$$zf\frac{\partial f}{\partial u} + i\frac{\partial}{\partial z}(zf) = -\bar f_2\frac{\partial f_2}{\partial u}, \tag{3.21a}$$

$$zf\frac{\partial f}{\partial u} - i\frac{\partial}{\partial z}(zf) = -f_2\frac{\partial \bar f_2}{\partial u}. \tag{3.21b}$$

Summing (3.21a) and (3.21b) we will get the equation

$$\frac{\partial}{\partial u}[zf^2 + f_2\bar f_2] = 0, \tag{3.22}$$

while the difference of these equations produces

$$2i\frac{\partial}{\partial z}(zf) = f_2\frac{\partial \bar f_2}{\partial u} - \bar f_2\frac{\partial f_2}{\partial u}. \tag{3.23}$$

If we further sum Eq. (3.20a) multiplied by $\bar f_2$ with Eq. (3.20b) multiplied by f_2 we will obtain the equation

$$\frac{\partial}{\partial z}(f_2\bar f_2) = if\left[f_2\frac{\partial \bar f_2}{\partial u} - \bar f_2\frac{\partial f_2}{\partial u}\right]. \tag{3.24}$$

Now, combining (3.23) and (3.24) we will have

$$\frac{\partial}{\partial z}[zf^2 + f_2\bar f_2] + f^2 = 0. \tag{3.25}$$

In virtue of (3.22) we immediately conclude from (3.25) that

$$\frac{\partial}{\partial u} f = 0, \tag{3.26}$$

and therefore

$$\frac{\partial}{\partial u}(f_2 \bar{f}_2) = 0 \Rightarrow f_2 = h(z) e^{i\Psi(z,u)}, \quad \bar{f}_2 = \bar{h}(z) e^{-i\Psi(z,u)}. \tag{3.27}$$

Moreover, plugging (3.27) in Eq. (3.23) one may find that $\frac{\partial}{\partial u}\Psi(z, u)$ does not depend on u, and therefore $\Psi(z, u) = \alpha u$, $\alpha = \text{const}$, modulo redefinitions of h, $\bar{h}$. Finally, it follows from (3.21) that

$$h'\bar{h} - h\bar{h}' = 0 \Rightarrow h = \beta e^{\Phi(z)}, \qquad \bar{h} = \bar{\beta} e^{\Phi(z)}, \quad \beta = \text{const}. \tag{3.28}$$

Putting all these together, we have the following semi-solution of our basic system (3.3), (3.4), and (3.6)

$$f_1 = i\lambda f(z), \qquad \bar{f}_1 = -i\bar{\lambda} f(z), \quad f_2 = \beta e^{i\alpha u + \Phi(z)}, \qquad \bar{f}_2 = \bar{\beta} e^{-i\alpha u + \Phi(z)}, \tag{3.29}$$

where two real functions $f(z)$ and $\Phi(z)$ are still restricted to obey

$$\frac{d}{dz}(zf) = -\alpha\beta\bar{\beta} e^{2\Phi}, \qquad \frac{d}{dz}\Phi = \alpha f. \tag{3.30}$$

If $\alpha = 0$ then the solution of (3.30) is trivial and describes the linear tensor supermultiplet. Alternatively, with $\alpha \neq 0$ one may always rescale the superfield u to fix $\alpha = 1$.

The general solution of the system (3.30) with $\alpha = 1$ reads

$$f = \frac{-1 + c_1(-1 + \frac{2c_2}{z^{c_1} + c_2})}{2z}, \qquad e^{\Phi} = \frac{c_1 \sqrt{c_2} z^{(1/2)(c_1 - 1)}}{\sqrt{\beta\bar{\beta}}(z^{c_1} + c_2)}, \tag{3.31}$$

where c_1, c_2 are arbitrary real constants.

D. Action

To understand better what systems can be described by the new nonlinear supermultiplet let us construct the action. The general sigma-model type action may be easily constructed as the integral over $N = 4$ superspace

$$S_1 = \int dt d^2\theta d^2\bar{\theta} L(u, \lambda, \bar{\lambda}). \tag{3.32}$$

Here, $L(u, \lambda, \bar{\lambda})$ is an arbitrary real function.

Before going to the component action and to possible potential terms one has to understand the structure of the auxiliary bosonic components in our supermultiplet. From the beginning we have three auxiliary components in the superfield u and two components in the superfield λ, $\bar{\lambda}$:

$$A = D^i D_i u|, \qquad C = [D^i, \bar{D}_i]u|, \qquad \bar{A} = \bar{D}^i \bar{D}_i u|, \quad B = \bar{D}^i \bar{D}_i \lambda|, \qquad \bar{B} = D^i D_i \bar{\lambda}|, \tag{3.33}$$

where | means limit $\theta = \bar{\theta} = 0$. Let us concentrate on pure bosonic equations discarding all the fermionic terms. Thus, from our basic constraints (3.2) it immediately follows the relation between auxiliary components and time derivatives from physical bosons:

$$\begin{aligned} A &= f_1 B + 4if_2\dot{\lambda}, \\ \bar{A} &= \bar{f}_1 \bar{B} + 4i\bar{f}_2\dot{\bar{\lambda}}, \\ C &= 4i(\bar{f}_1\dot{\lambda} - f_1\dot{\bar{\lambda}}) + f_2 B - \bar{f}_2\bar{B}, \\ 4i\dot{u} &= 4i(\bar{f}_1\dot{\lambda} + f_1\dot{\bar{\lambda}}) - f_2 B - \bar{f}_2\bar{B}. \end{aligned} \tag{3.34}$$

First of all, we conclude that the function f_2 cannot be equal to zero, because otherwise from (3.34) it follows the relation between time derivatives of the physical bosonic components, and therefore we get the on-shell multiplet. With $f_2 \neq 0$ our constraints leave only one auxiliary component in the superfields u, λ, $\bar{\lambda}$ as it should be.

Now one may construct the general potential term for our supermultiplet. To do this, one should notice that from the constraints (3.2) it follows that all the spinor derivatives with respect to θ_2, $\bar{\theta}^2$ may be expressed as θ_1- and $\bar{\theta}^1$-derivatives:

$$\begin{aligned} D^2\bar{\lambda} &= \frac{\bar{f}_1 \bar{D}_1\lambda - \bar{D}_1 u}{\bar{f}_2}, \\ \bar{D}_2\lambda &= \frac{f_1 D^1\bar{\lambda} - D^1 u}{f_2}, \\ D^2 u &= \frac{(f_1\bar{f}_1 + f_2\bar{f}_2)\bar{D}_1\lambda - f_1\bar{D}_1 u}{\bar{f}_2}, \\ \bar{D}_2 u &= \frac{(f_1\bar{f}_1 + f_2\bar{f}_2)D^1\bar{\lambda} - \bar{f}_1 D^1 u}{f_2}. \end{aligned} \tag{3.35}$$

Thus, all the components are sitting in the $N = 2$ superfields $(\tilde{u}, \tilde{\lambda}, \tilde{\bar{\lambda}})$

$$\tilde{u} = u|_{\theta_2 = \bar{\theta}^2 = 0}, \qquad \tilde{\lambda} = \lambda|_{\theta_2 = \bar{\theta}^2 = 0}, \qquad \tilde{\bar{\lambda}} = \bar{\lambda}|_{\theta_2 = \bar{\theta}^2 = 0}. \tag{3.36}$$

Therefore, the most general potential term can be written as

$$S_2 = m \int dt d\theta_1 d\bar{\theta}^1 F(\tilde{u}, \tilde{\lambda}, \tilde{\bar{\lambda}}). \tag{3.37}$$

By construction, the potential term (3.37) is invariant under $N = 2$ supersymmetry realized on the $(\theta_1, \bar{\theta}^1)$. To be invariant under the other implicit $N = 2$ supersymmetry, the prepotential F has to obey the following equation:

$$(f_1\bar{f}_1 + f_2\bar{f}_2)F_{\tilde{u}\tilde{u}} + F_{\tilde{\lambda}\tilde{\bar{\lambda}}} + \bar{f}_1 F_{\tilde{u}\tilde{\bar{\lambda}}} + f_1 F_{\tilde{u}\tilde{\lambda}} = 0. \tag{3.38}$$

So, the most general action for our nonlinear supermultiplet reads

$$S = S_1 + S_2 = \int dt d^2\theta d^2\bar\theta L(u,\lambda,\bar\lambda) + m\int dt d\theta_1 d\bar\theta^1 F(\tilde u,\tilde\lambda,\tilde{\bar\lambda}), \tag{3.39}$$

where the prepotential F is defined as the solution of Eq. (3.38).

Finally, let us present the bosonic sector of the action (3.39)

$$S = \int dt\Big[g\Big(f_2\bar f_2\dot\lambda\dot{\bar\lambda} + \frac14(\bar f_1\dot\lambda + f_1\dot{\bar\lambda} - \dot u)^2\Big) + im[(f_1F_u + F_{\bar\lambda})\dot{\bar\lambda} - (\bar f_1 F_u + F_\lambda)\dot\lambda] - m^2\frac{F_u^2}{g}\Big], \tag{3.40}$$

where

$$g = \frac{16}{f_2\bar f_2}((f_1\bar f_1 + f_2\bar f_2)L_{uu} + \bar f_1 L_{u\bar\lambda} + f_1 L_{u\lambda} + L_{\lambda\bar\lambda}). \tag{3.41}$$

We checked that with the $g=1$ the sigma-model part of the action (3.40) describes a conformally flat (the Weyl tensor is vanishing here) constant positive curvature three-dimensional manifold. Of course, to make any final conclusion about this model one has to fully analyze all fermionic terms. We postpone this analysis for the future.

IV. 4 = 3 + 1

The first way to construct an off-shell nonlinear $N=4$ supermultiplet with four physical bosons is to start with two $N=4$ supermultiplets containing three and one physical bosons, respectively, and then identify the fermionic degrees of freedom in both supermultiplets.

The $N=4$ supermultiplet with three physical bosons is well known [12]. It is called linear tensor supermultiplet and may be described by a real $N=4$ superfield v^{ij}

$$v^{ij} = v^{ji}, \qquad (v^{ij})^\dagger = v_{ij}, \qquad i,j = 1,2,$$

subject to the constraints

$$D^{(i}v^{jk)} = \bar D^{(i}v^{jk)} = 0. \tag{4.1}$$

The constraints (4.1) leave in the tensor supermultiplet just three physical bosons v^{ij}, four fermions ξ^i, $\bar\xi_i$, and one auxiliary boson A

$$v^{ij} = v^{ij}|, \qquad \xi^i = \frac13 D^j v^i_j|, \qquad \bar\xi^i = -\frac13\bar D_j v^{ij}|, \qquad A = \frac{i}{6}D^i\bar D^j v_{ij}|, \tag{4.2}$$

where, as usual, the symbol | means restriction to $\theta = \bar\theta = 0$.

The second supermultiplet with one physical boson we need is an "old tensor" supermultiplet [10]. This supermultiplet may be described by a real superfield u subject to the following constraints:

$$D^iD_iu = \bar D^i\bar D_iu = 0. \tag{4.3}$$

It comprises one physical boson u, once again four fermions ψ^i, $\bar\psi_i$, and a triplet of auxiliary components $A^{(ij)}$

$$u = u|, \qquad \psi^i = D^iu|, \qquad \bar\psi_i = \bar D_iu|, \qquad A^{(ij)} = \frac{i}{2}[\bar D^{(i}, D^{j)}]u|. \tag{4.4}$$

One should stress that the constraints (4.3) describe just the same multiplet with one physical boson we used in the previous sections. The twisted form of the constraints we are using now is preferable for the following reasons. Within our approach we will identify the fermions in both the supermultiplets v^{ij} and u. Clearly, this identification will reduce the number of the auxiliary components in both the supermultiplets A (4.2) and A^{ij} (4.4) to zero, by expressing all these components in terms of four physical components v^{ij} and u. To be manifestly invariant under the $SU(2)$ symmetry realized on the doublet indices (i,j) the three auxiliary components in the superfield u have to form a vector with respect to $SU(2)$. In this case they may be expressed as time derivatives of v^{ij} (plus fermionic terms with the same $SU(2)$ structure). Just this structure of the auxiliary components is provided by the constraints (4.4).

Now we will identify the fermions in both supermultiplets by imposing the following constraints:

$$D^iu = \tfrac13 fD_jv^{ij} - \tfrac13 a^{ij}D^kv_{kj}, \qquad \bar D^iu = \tfrac13\bar f\bar D_jv^{ij} - \tfrac13\bar a^{ij}\bar D^kv_{kj}, \tag{4.5}$$

where the functions $f(u,u)$, $a^{ij}(u,v)$, and their conjugated ones depend on both supermultiplets. In order to have Eqs. (4.5) consistent with (4.1) and (4.3), these functions have to be real and obey the following equations:

$$2f\frac{\partial f}{\partial u} + a_{ij}\frac{\partial a^{ij}}{\partial u} - 2\frac{\partial a^{ij}}{\partial v^{ij}} = 0, \tag{4.6}$$

$$f\frac{\partial a_{ij}}{\partial u} - a_{ij}\frac{\partial f}{\partial u} + 2\frac{\partial f}{\partial v^{ij}} - \frac12\Big(a_{ik}\frac{\partial a^k_j}{\partial u} + a_{jk}\frac{\partial a^k_i}{\partial u}\Big) + \Big(\frac{\partial a^k_i}{\partial v^{kj}} + \frac{\partial a^k_j}{\partial v^{ki}}\Big) = 0. \tag{4.7}$$

As we already explained before, in virtue of (4.6) and (4.7) the auxiliary components of both supermultiplets are expressed in terms of four physical bosons u, v^{ij}

$$A = \frac{1}{f}\Big(\dot u + \frac{1}{2}a_{ij}\dot v^{ij} + \text{fermions}\Big),$$
$$A^{ij} = f\dot v^{ij} + \frac{1}{2}(a^{ik}\dot v_k^j + a^{jk}\dot v_k^i) + a^{ij}A + \text{fermions}. \qquad (4.8)$$

Thus, we indeed have a nonlinear supermultiplet with four physical bosonic and four fermionic degrees of freedom.

Concerning the action, in $N = 4$ superspace it may be easily constructed in the standard way as

$$S = \int d^4\theta dt L(u, v), \qquad (4.9)$$

where Lagrangian L is an arbitrary real function on superfields u and v^{ij}. Passing to the components one may easily find the bosonic part of the action (4.9)

$$S_{\text{bos}} = \int dt G\Big[\frac{1}{2}\dot v^{ij}\dot v_{ij} + \frac{1}{f^2}\Big(\dot u + \frac{1}{2}a_{ij}\dot v^{ij}\Big)^2\Big], \qquad (4.10)$$

with the metric G

$$G \equiv (2f^2 + a_{ij}a^{ij})\frac{\partial^2 L}{\partial u^2} - 4a^{ij}\frac{\partial^2 L}{\partial u \partial v^{ij}} + 4\frac{\partial^2 L}{\partial v^{ij}\partial v_{ij}}. \qquad (4.11)$$

Thus we conclude that the two $N = 4$ supermultiplets (4.1) and (4.3) span a new nonlinear $N = 4$ supermultiplet with four physical bosonic and four fermionic degrees of freedom if they are related as in (4.5) with the functions f and a^{ij} obeying to (4.6) and (4.7).

It is a rather complicated task to find the general solution of the system (4.6) and (4.7). Therefore it is desirable to provide some clarifying examples of systems which could be described with a new nonlinear supermultiplet. Here we present two of the simplest examples.

A. Hypermultiplet

It is evident that the simplest solution of the system (4.6) and (4.7) is given by

$$f = 1, \qquad a^{ij} = 0. \qquad (4.12)$$

In this case the resulting $N = 4$ supermultiplet is the well-known linear hypermultiplet [1,4,5,13] and the bosonic part of the action reads

$$S = 2\int dt\Big(\frac{\partial^2 L}{\partial u^2} + 2\frac{\partial^2 L}{\partial v^{ij}\partial v_{ij}}\Big)\Big[\frac{1}{2}\dot v^{ij}\dot v_{ij} + \dot u^2\Big]. \qquad (4.13)$$

Clearly, the action (4.13) describes conformally flat four-dimensional bosonic manifolds.

B. Nonlinear hypermultiplet and hyper-Kähler sigma model

A more involved example corresponds to the case where both functions f and a^{ij} depend only on tensor supermultiplet v^{ij}. In this case Eqs. (4.6) and (4.7) are simplified to be

$$\frac{\partial a^{ij}}{\partial v^{ij}} = 0, \qquad 2\frac{\partial f}{\partial v^{ij}} - \Big(\frac{\partial a_i^k}{\partial v^{kj}} + \frac{\partial a_j^k}{\partial v^{ki}}\Big) = 0. \qquad (4.14)$$

As a consequence of (4.14) the function f has to be a harmonic one

$$\frac{\partial^2}{\partial v^{ij}\partial v_{ij}} f = 0. \qquad (4.15)$$

If we additionally choose the metric G (4.11) as

$$G = f$$

then the bosonic part of the action acquires the form

$$S = \int dt\Big[\frac{f}{2}\dot v^{ij}\dot v_{ij} + \frac{1}{f}\Big(\dot u + \frac{1}{2}a_{ij}\dot v^{ij}\Big)^2\Big]. \qquad (4.16)$$

In the action (4.16) one may immediately recognize the Gibbons-Hawking ansatz for the four-dimensional hyper-Kähler (HK) sigma-model action with translational (or triholomorphic) isometry [14], provided Eqs. (4.14) are satisfied. Thus, the $N = 4$ supersymmetric sigma models with HK geometry in the bosonic sector may be naturally described within the constructed nonlinear supermultiplet.

Let us notice that the $N = 4$ supersymmetric system with the bosonic action (4.16) has been first constructed in [7] in components. Until now the superfield formulation of the corresponding nonlinear hypermultiplet has been known only in the harmonic superspace [6,8]. The constraints (4.5) together with Eqs. (4.14) provide the superfield description of the nonlinear hypermultiplet in the standard $N = 4$ superspace.

V. 4 = 2 + 2

Another possibility to construct a nonlinear supermultiplet with four physical bosons is to start with two chiral $N = 4$ supermultiplets both containing two physical bosons and then again identify the fermions in both supermultiplets.

Let us introduce two $N = 4$ chiral supermultiplets x and y subject to ordinary constraints

$$D^i x = \bar D^i \bar x = 0, \qquad D^i y = \bar D^i \bar y = 0. \qquad (5.1)$$

The most general variant of identification of the fermions in both supermultiplets reads

$$D^i \bar x = f D^i \bar y + g \bar D^i y, \qquad \bar D_i x = \bar f \bar D^i y - \bar g D_i \bar y, \qquad (5.2)$$

where the arbitrary functions f, g depend on all superfields x, $\bar{x}$, y, and $\bar{y}$.

The self-consistency of the constraints (5.2) imposes the following restriction on the functions f, g:

$$g\frac{\partial f}{\partial \bar{x}} = f\frac{\partial g}{\partial \bar{x}} + \frac{\partial g}{\partial \bar{y}}, \qquad \bar{g}\frac{\partial g}{\partial x} = -\bar{f}\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}, \tag{5.3}$$

and their conjugated. It also follows from (5.2) that the auxiliary components of the superfields x, y

$$\begin{aligned} A &= -\frac{i}{4}\bar{D}^2 x|, \qquad \bar{A} = -\frac{i}{4}D^2\bar{x}|, \\ B &= -\frac{i}{4}\bar{D}^2 y|, \qquad \bar{B} = -\frac{i}{4}D^2\bar{y}| \end{aligned} \tag{5.4}$$

are expressed in terms of physical bosons and fermions as

$$\begin{aligned} A &= \frac{(f\bar{f} + g\bar{g})\dot{\bar{y}} - \bar{f}\dot{\bar{x}}}{g} + \text{fermions}, \\ B &= \frac{f\dot{\bar{y}} - \dot{\bar{x}}}{g} + \text{fermions}. \end{aligned} \tag{5.5}$$

Now we may construct the most general sigma-model action for this supermultiplet

$$S = -\frac{1}{16}\int d^4\theta dt \mathcal{L}(y, \bar{y}, x, \bar{x}). \tag{5.6}$$

After passing to components, the bosonic part of the action (5.6) reads

$$S_B = \int dt G[\dot{x}\dot{\bar{x}} + (f\bar{f} + g\bar{g})\dot{y}\dot{\bar{y}} - f\dot{x}\dot{\bar{y}} - \bar{f}\dot{\bar{x}}\dot{y}], \tag{5.7}$$

where

$$G = \frac{1}{g\bar{g}}[L_{y\bar{y}} + \bar{f}L_{x\bar{y}} + fL_{\bar{x}y} + f\bar{f}L_{x\bar{x}}]. \tag{5.8}$$

The full analysis of this system is out of the scope of the present paper and will be done elsewhere.

VI. CONCLUSION

In this paper we proposed a unified framework for a description of all linear and nonlinear one-dimensional supermultiplets with $N = 4$ supersymmetry, based on ‘‘gluing’’ a pair of linear supermultiplets along their fermions. The functions defining such a gluing obey a system of equations, each solution of which yields an irreducible supermultiplet, linear or nonlinear. A given supermultiplet may appear in several equivalent ways which are related by superfield redefinitions. It is amazing that *all known* $N = 4$ supermultiplets appear in this way, as we showed explicitly.

Furthermore, by iterating this method, all known $N = 4$ supermultiplets may be constructed just from the linear supermultiplet, which features a single physical boson. Gluing this fundamental ingredient to any other $N = 4$ supermultiplet increases n by one, and so any case is eventually reached starting from several copies of the linear supermultiplet. In this respect, this supermultiplet plays a role analogous to the one of the ‘‘root’’ supermultiplet [2,8,15,16], which contains four physical bosons (for $N = 4$ supersymmetry). However, the root supermultiplet is not unique. There is an infinite number of supermultiplets with four bosonic and four fermionic components, while the supermultiplet with one physical boson is unique. Therefore, we believe that our approach is more general.

Why did we ignore the $n = 0$ supermultiplets, which do exist in $d = 1$ supersymmetry? The answer is that gluing such a supermultiplet to an arbitrary one just expresses the components of the combined multiplet through those of the arbitrary supermultiplet only. In other words, we merely generate a superfield redefinition on the arbitrary supermultiplet.

Clearly enough, before discussing the classification issue we should comment on the completeness of the proposed scheme. We expect that the general solutions to the equations in Secs. III, IV, and V will provide us with novel supermultiplets. Therefore, one must first strive to solve in general the systems of differential equations presented here. Second, one has to analyze the equivalence relations among all solutions and characterize the equivalence classes, which are in one-to-one correspondence with the different supermultiplets. Third, not all nonlinear $N = 4$ supermultiplets may be reached directly by gluing two linear multiplets, so one should investigate the gluing process with previously found nonlinear multiplets, as seems natural in an iteration, and also the simultaneous gluing of more than two multiplets. The associativity of iterated gluing is another issue of interest. The idea we utilized in this paper may be easily applied to these cases without any modification. It is only that the ensuing equations are more involved, and the task of solving them is deferred to future work.

Hence, we only stand at the beginning of a classification program. Curiously, our framework offers much more information than we looked for. Indeed, all known supermultiplets correspond merely to the simplest solutions of our equation systems, e.g. solutions with a frozen dependence on one coordinate. Hence, still open is the most intriguing question: To which supermultiplets correspond the general solutions?

Finally, we note that the more complicated problem of constructing $N = 8$ supermultiplets in one spacetime dimension may also be attacked by adapting our framework. In this case one has a larger number of possibilities for gluing together different supermultiplets, but the main needed ingredient—the supermultiplet with one physical boson—is well known. Moreover, the simplest case of joining two such supermultiplets shall give birth to a new nonlinear $N = 8$ supermultiplet with two physical bosons. We intend to report these results elsewhere.

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[1] S. J. Gates, Jr., L. Rana, Maryland University Report No. UMDPP 93-24, 1994; Phys. Lett. B **352**, 50 (1995); **369**, 262 (1996).

[2] M. Faux and S. J. Gates, Jr., Phys. Rev. D **71**, 065002 (2005); S. Bellucci, S. J. Gates, Jr., and E. Orazi, Lect. Notes Phys. **698**, 1 (2006); C. F. Doran, M. G. Faux, S. J. Gates, Jr., T. Hubsch, K. M. Iga, and G. D. Landweber, arXiv:hep-th/0605269; Int. J. Mod. Phys. A **22**, 869 (2007); arXiv:math-ph/0603012; S. Bellucci and S. Krivonos, Lect. Notes Phys. **698**, 49 (2006);

[3] A. Pashnev and F. Toppan, J. Math. Phys. (N.Y.) **42**, 5257 (2001); Z. Kuznetsova, M. Rojas, and F. Toppan, J. High Energy Phys. 03 (2006) 098; F. Toppan, in Proceedings of the Fifth International Conference on Mathematical Methods in Physics IC2006, 2006, CBPF, Rio de Janeiro, Brazil (unpublished); in Proceedings of the 22nd Max Born Symposium in honour of the 70th birthday of Prof. J. Lukierski, Wroclaw, 2006 (unpublished); Z. Kuznetsova and F. Toppan, arXiv:hep-th/0701225.

[4] E. Ivanov and O. Lechtenfeld, J. High Energy Phys. 09 (2003) 073.

[5] E. Ivanov, S. Krivonos, and O. Lechtenfeld, Classical Quantum Gravity **21** 1031 (2004); arXiv:hep-th/0310299.

[6] F. Delduc and S. Krivonos (unpublished).

[7] S. Krivonos and A. Shcherbakov, Phys. Lett. B **637** 119 (2006).

[8] F. Delduc and E. Ivanov, Nucl. Phys. **B753**, 211 (2006).

[9] S. Bellucci, E. Ivanov, S. Krivonos, and O. Lechtenfeld, Nucl. Phys. **B699**, 226 (2004); S. Bellucci, S. Krivonos, and A. Shcherbakov, Phys. Rev. D **73** 085014 (2006); S. Bellucci, S. Krivonos, and A. Marrani, Phys. Rev. D **74** 045005 (2006); E. Ivanov, Phys. Lett. B **639** 579 (2006); S. Bellucci, S. Krivonos, and A. Sutulin, Phys. Rev. D **76**, 065017 (2007)

[10] E. Ivanov, S. Krivonos, and V. Leviant, J. Phys. A **22**, 4201 (1989).

[11] F. Delduc and E. Ivanov, Nucl. Phys. **B787**, 176 (2007).

[12] M. de Crombrugghe and V. Rittenberg, Ann. Phys. (N.Y.) **151**, 99 (1983); E. A. Ivanov and A. V. Smilga, Phys. Lett. B **257**, 79 (1991); V. P. Berezovoj and A. I. Pashnev, Classical Quantum Gravity **8**, 2141 (1991); A. Maloney, M. Spradlin, and A. Strominger, J. High Energy Phys. 04 (2002) 003.

[13] S. Hellerman and J. Polchinski, arXiv:hep-th/9908202.

[14] S. Hawking, Phys. Lett. A **60** 81 (1977); G. Gibbons and S. Hawking, Commun. Math. Phys. **66**, 291 (1979); K. Tod and R. Ward, Proc. R. Soc. Lond. A **368**, 411 (1979).

[15] S. Bellucci, S. Krivonos, A. Marrani, and E. Orazi, Phys. Rev. D **73**, 025011 (2006).

[16] F. Delduc and E. Ivanov, Nucl. Phys. **B770**, 179 (2007).