

Harmonic $\mathcal{N} = 2$ mechanicsAnton Galajinsky^{1,*} and Olaf Lechtenfeld^{2,†}¹Laboratory of Mathematical Physics, Tomsk Polytechnic University, 634050 Tomsk, Lenin Avenue 30, Russian Federation²Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstrasse 2, D-30167 Hannover, Germany

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$\mathcal{N} = 2$ superconformal many-body quantum mechanics in arbitrary dimensions is governed by a single scalar prepotential which determines the bosonic potential and the boson-fermion couplings. We present a special class of such models, for which the bosonic potential is absent. They are classified by homogeneous harmonic functions subject to physical symmetry requirements, such as translation, rotation and permutation invariance. The central charge is naturally quantized. We provide some examples for systems of identical particles in any dimension.

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I. INTRODUCTION

$\mathcal{N} = 4$ superconformal many-body quantum mechanics in one dimension is governed by two scalar prepotentials U and F which obey a coupled set of partial differential equations. While U may vanish, F always takes nonzero values. Recent studies in [1–4] (for related developments see [5–10]) revealed an interesting link between $\mathcal{N} = 4$ quantum mechanics and the Witten, Dijkgraaf, Verlinde, and Verlinde (WDVV) equation [11,12] which plays an important role in $d = 2$ topological field theory [11,12] and $\mathcal{N} = 2$ supersymmetric Yang-Mills theory [13]. Because the WDVV equation underlies a potential deformation of a Fröbenius algebra [14], it relates $\mathcal{N} = 4$ mechanics with Fröbenius manifolds. All $\mathcal{N} = 4$ models with a nontrivial U constructed so far are based on the root systems of simple Lie algebras or Coxeter reflection groups.

A peculiar feature of $\mathcal{N} = 4$ mechanics concerns the center-of-mass coordinate. Although it decouples from the relative particle motion, its nonzero F prepotential generates an inverse-square potential for the center-of-mass motion, thus breaking translation invariance. If this is unwanted, one must give up $\mathcal{N} = 4$ and soften the model to an $\mathcal{N} = 2$ system, which is ruled by the prepotential U alone [15]. Our interest in $\mathcal{N} = 2$ mechanics is also motivated by the desire to go beyond $d = 1$ and to construct new exactly solvable many-body models in higher dimensions and to explore novel correlations (see e.g. [16] and references therein). It is natural to expect that $d > 1$, $\mathcal{N} = 2$ superconformal many-body models will provide new insight into the nonrelativistic version of the AdS/CFT correspondence which has currently sparked substantial interest.

A minimal extension of the Galilei algebra by the dilatation and special conformal generators is known in the literature as the Schrödinger algebra. A conformal exten-

sion obtained by contracting the relativistic conformal $so(d + 1, 2)$ algebra gives an even larger algebra which goes under the name of conformal Galilei algebra (for a recent discussion and further references see e.g. [17]). Because the conformal Galilei algebra requires vanishing mass, the Schrödinger algebra has a better prospect for quantum mechanical applications. Since the translations are part of the Schrödinger algebra, $\mathcal{N} = 2$ interacting many-body quantum mechanics is likely to be the maximal superextension feasible in higher dimensions.

The purpose of this paper is to reconsider the construction of $\mathcal{N} = 2$ n -particle quantum mechanics in d dimensions and to exhibit a new special class of models determined by a single harmonic function. These (n, d) models are characterized by the absence of bosonic interactions, yet retain (quantum) boson-fermion couplings. They are classified by the homogeneous harmonic functions on \mathbb{R}^{nd} subject to physical symmetry requirements (Euclidean and permutation invariance) and quantize the central charge of the $\mathcal{N} = 2$ algebra.

In Sec. II we recall the conventional framework for formulating $\mathcal{N} = 2$ many-body models in one dimension and explore the hitherto unexploited possibility of purely boson-fermion couplings. We show how the Laplace equation arises, explain the central charge quantization and discover solutions related to Lie-algebra root systems.

In Sec. III the analysis is extended beyond one dimension. It is shown that the role of the Laplace equation persists in higher dimensions, but the prepotential is further constrained by Euclidean invariance in \mathbb{R}^d , as part of the $\mathcal{N} = 2$ Schrödinger supersymmetry. We finally present a one-parameter family of (n, d) models as well as a particular $(n, n - 1)$ system, both being invariant under particle permutations. Conclusions follow.

II. SPECIAL $\mathcal{N} = 2$ MECHANICS

The conventional representation of the $d = 1$, $\mathcal{N} = 2$ superconformal algebra on the phase space of n identical particles (with unit mass) is provided by a single prepo-

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tential $U(x_1, \dots, x_n)$ which gives rise to the operators [6]¹

$$\begin{aligned} H &= \frac{1}{2}p_i p_i + \frac{1}{2}\partial_i U(x)\partial_i U(x) - \partial_i \partial_j U(x)\langle \psi_i \bar{\psi}_j \rangle, \\ J &= \langle \psi_i \bar{\psi}_i \rangle, \\ D &= tH - \frac{1}{4}(x_i p_i + p_i x_i), \quad K = -t^2 H + 2tD + \frac{1}{2}x_i x_i, \\ Q &= \psi_i(p_i + i\partial_i U(x)), \quad \bar{Q} = \bar{\psi}_i(p_i - i\partial_i U(x)), \\ S &= x_i \psi_i - tQ, \quad \bar{S} = x_i \bar{\psi}_i - t\bar{Q}, \end{aligned} \quad (1)$$

where the symbol $\langle \dots \rangle$ stands for symmetric (or Weyl) ordering of the fermions. The operators H , D and K generate time translations, dilatations and special conformal transformations, respectively, while Q and \bar{Q} are supersymmetry generators, and S and \bar{S} generate superconformal transformations. The $U(1)$ R -symmetry transformation generated by J affects only the fermions. Note that the prepotential $U(x)$ is defined up to an additive constant.

The operators (1) obey the (anti)commutation relations of the $d = 1$, $\mathcal{N} = 2$ superconformal algebra with central charge C (Hermitian conjugates are omitted)

$$\begin{aligned} [H, D] &= iH, & [K, D] &= -iK, & [Q, D] &= \frac{i}{2}Q, \\ [S, D] &= -\frac{i}{2}S, & [Q, J] &= -Q, & [S, J] &= -S, \\ [H, K] &= 2iD, & [Q, K] &= -iS, & [S, H] &= iQ, \\ \{Q, \bar{Q}\} &= 2H, & \{S, \bar{S}\} &= 2K, \\ \{Q, \bar{S}\} &= -2D - iJ - iC, \end{aligned} \quad (2)$$

provided the prepotential satisfies the linear partial differential equation

$$x_i \partial_i U(x) = -C. \quad (3)$$

The general solution to (3) reads

$$U(x) = -C \ln|x_1| + \Lambda\left(\frac{x_i}{x_j}\right), \quad (4)$$

where $\Lambda\left(\frac{x_i}{x_j}\right)$ is a function of the coordinate ratios $\frac{x_i}{x_j}$ for $i < j$.

In order to extract a class of reasonable models from the infinity of $\mathcal{N} = 2$ systems encoded in the general solution (4), one can impose additional restrictions like permutation symmetry, translation invariance, etc. Another option is to start with a specific bosonic theory,

$$H_B = \frac{1}{2}p_i p_i + V(x) \quad \text{with} \quad (x_i \partial_i + 2)V(x) = 0, \quad (5)$$

and then solve the Hamilton-Jacobi equation

¹We work in the standard coordinate representation, $p_i = -i\frac{\partial}{\partial x_i}$, $[x_i, p_j] = i\delta_{ij}$, and put $\hbar = 1$. The fermionic operators are mutually conjugate via $(\psi_i)^\dagger = \bar{\psi}_i$ and obey the anticommutation relations $\{\psi_i, \psi_j\} = 0$, $\{\bar{\psi}_i, \bar{\psi}_j\} = 0$, $\{\psi_i, \bar{\psi}_j\} = \delta_{ij}$. The t -dependent pieces in the generators are kept explicit so as to have a direct link to the classical theory. Throughout the paper summation over repeated indices is understood.

$$\partial_i U(x)\partial_i U(x) = 2V(x) \quad (6)$$

for the given potential $-V$ and zero energy. Each solution U yields an $\mathcal{N} = 2$ superconformal extension of the original model (5). In particular, in this way one can treat quantum integrable many-body models related to simple Lie algebras, the prominent example being the $\mathcal{N} = 2$ Calogero model [15] (see also [6]).

Among the many possible bosonic starting points, there exist special bosonic potentials V which can be absorbed into a reordering of the fermions. Since a deviation from Weyl ordering produces a term proportional to $\partial_i \partial_j U$ in H , this property translates to the condition

$$\partial_i U(x)\partial_i U(x) + \kappa \partial_i \partial_i U(x) = 0 \quad (7)$$

for some real parameter κ of order \hbar . Note that this forces U to be of order \hbar as well, so that these models are classically free. The value of κ quantifies the deviation from Weyl ordering and takes unit (\hbar) value for normal ordering. If (7) can be solved, then the Hamiltonian may be brought to the form

$$H = \frac{1}{2}p_i p_i - \partial_i \partial_j U(x):\psi_i \bar{\psi}_j:\kappa \quad (8)$$

for a suitable fermionic ordering prescription, so that the interaction contains only boson-fermion couplings. We now describe a class of solutions to (7) with quantized central charge.

The conditions (3) and (7) simplify under the substitution

$$U(x) = \kappa \ln G(x) \quad \text{to} \quad \left(x_i \partial_i + \frac{C}{\kappa}\right)G(x) = 0$$

$$\text{and} \quad \partial_i \partial_i G(x) = 0, \quad (9)$$

so that $G(x)$ is a harmonic homogeneous function of degree $\ell := -\frac{C}{\kappa}$ in \mathbb{R}^n . Such functions are single-valued only for $\ell \in \mathbb{Z}$ and regular at the origin $x_1 = x_2 = \dots = x_n = 0$ for $\ell \geq 0$. These conditions quantize the central charge in units of κ ,

$$C = -\ell\kappa \quad \text{with} \quad \ell = 0, 1, 2, \dots, \quad (10)$$

and restrict the prepotential to²

$$G_\ell(x) = (x_1^2 + \dots + x_n^2)^{\ell/2} Y_\ell \quad (\text{angles}), \quad (11)$$

where Y_ℓ is a linear combination of S^{n-1} spherical harmonics for spin ℓ . Note that linear combinations of G_ℓ are forbidden by the homogeneity condition (9).

Each value of ℓ and choice of Y_ℓ produces a special $\mathcal{N} = 2$ many-body quantum system. The demand for permutation invariance or translation invariance puts restrictions on Y_ℓ , which can be solved. For illustration, we consider a solution related to the positive roots $\{\alpha\}$ of a simple Lie algebra,

²If singular behavior is admitted at coincidence loci $x_i = x_j$, more general solutions appear.

$$G(x) = \prod_{\alpha} (\alpha x). \quad (12)$$

In this case ℓ equals the number of positive roots. That (12) solves Laplace's equation is verified with the use of the same root identities which were previously applied in [4] for solving the WDVV equation (see Sec. 6 in [4] for more details). Permutation and translation invariance is achieved for the A_n root systems, $\{(\alpha x)\} = \{x_i - x_j | 1 \leq i < j \leq n + 1\}$. The interaction potential for these models reads

$$V_{\text{int}} = \sum_{\alpha} \frac{(\alpha \psi)(\alpha \bar{\psi})}{(\alpha x)^2}. \quad (13)$$

III. SPECIAL $\mathcal{N} = 2$ MODELS IN ARBITRARY DIMENSION

We proceed to the construction of $\mathcal{N} = 2$ models in dimensions d greater than 1. At the algebraic level, extra dimensions come with additional generators corresponding to spatial translations P^α , spatial rotations $M^{\alpha\beta}$, Galilei boosts K^α and super Galilei transformations L^α and \bar{L}^α with $\alpha, \beta = 1, \dots, d$. It is assumed that L^α and \bar{L}^α are Hermitian conjugates of each other. The set of generators $\{H, D, K, P^\alpha, K^\alpha, M^{\alpha\beta}\}$ spans a subalgebra known as the Schrödinger algebra. In what follows, Greek letters are reserved for spatial indices while Latin indices label identical particles of unit mass.

Apart from the structure relations (2), which persist in higher dimensions, the nonvanishing (anti)commutation relations of the $\mathcal{N} = 2$ Schrödinger superalgebra include (Hermitian conjugates are omitted)

$$\begin{aligned} [H, K^\alpha] &= -iP^\alpha, & [D, K^\alpha] &= \frac{i}{2}K^\alpha, & [K, P^\alpha] &= iK^\alpha, & [D, P^\alpha] &= -\frac{i}{2}P^\alpha, \\ [M^{\alpha\beta}, L^\gamma] &= i(\delta^{\alpha\gamma}L^\beta - \delta^{\beta\gamma}L^\alpha), & [Q, K^\alpha] &= -iL^\alpha, & [K^\alpha, P^\beta] &= i\delta^{\alpha\beta}M, \\ [M^{\alpha\beta}, P^\gamma] &= i(\delta^{\alpha\gamma}P^\beta - \delta^{\beta\gamma}P^\alpha), & \{Q, \bar{L}^\alpha\} &= P^\alpha, & \{L^\alpha, \bar{L}^\beta\} &= \delta^{\alpha\beta}Z, \\ [M^{\alpha\beta}, K^\gamma] &= i(\delta^{\alpha\gamma}K^\beta - \delta^{\beta\gamma}K^\alpha), & [S, P^\alpha] &= iL^\alpha, & \{S, \bar{L}^\alpha\} &= K^\alpha, \\ [M^{\alpha\beta}, M^{\gamma\delta}] &= i(\delta^{\alpha\gamma}M^{\beta\delta} + \delta^{\beta\delta}M^{\alpha\gamma} - \delta^{\beta\gamma}M^{\alpha\delta} - \delta^{\alpha\delta}M^{\beta\gamma}), & [J, L^\alpha] &= L^\alpha, \end{aligned} \quad (14)$$

where M and Z are the central charges.

In order to build a quantum mechanical representation of this algebra, one introduces bosonic operators x_i^α and p_i^α and fermionic operators ψ_i^α and $\bar{\psi}_i^\alpha$, which obey the (anti)commutation relations

$$[x_i^\alpha, p_j^\beta] = i\delta^{\alpha\beta}\delta_{ij} \quad \text{and} \quad \{\psi_i^\alpha, \bar{\psi}_j^\beta\} = \delta^{\alpha\beta}\delta_{ij}. \quad (15)$$

The fermionic operators are related by Hermitian conjugation, i.e. $(\psi_i^\alpha)^\dagger = \bar{\psi}_i^\alpha$. A representation of the superalgebra (14) can then be constructed in terms of a single prepotential $U(x)$ by analogy with the one-dimensional case,

$$\begin{aligned} Q &= \psi_i^\alpha(p_i^\alpha + i\partial_{\alpha i}U(x)), & \bar{Q} &= \bar{\psi}_i^\alpha(p_i^\alpha - i\partial_{\alpha i}U(x)), & L^\alpha &= \sum_i \psi_i^\alpha, & \bar{L}^\alpha &= \sum_i \bar{\psi}_i^\alpha, \\ S &= x_i^\alpha \psi_i^\alpha - tQ, & \bar{S} &= x_i^\alpha \bar{\psi}_i^\alpha - t\bar{Q}, & J &= \langle \psi_i^\alpha \bar{\psi}_i^\alpha \rangle, & P^\alpha &= \sum_i p_i^\alpha, \\ M^{\alpha\beta} &= (x_i^\alpha p_i^\beta - x_i^\beta p_i^\alpha) - i\langle \psi_i^\alpha \bar{\psi}_i^\beta - \psi_i^\beta \bar{\psi}_i^\alpha \rangle, & K^\alpha &= \sum_i x_i^\alpha - tP^\alpha, \\ C &= -t^2H + 2tD + \frac{1}{2}x_i^\alpha x_i^\alpha, & D &= tH - \frac{1}{4}(x_i^\alpha p_i^\alpha + p_i^\alpha x_i^\alpha), \\ H &= \frac{1}{2}p_i^\alpha p_i^\alpha + \frac{1}{2}\partial_{\alpha i}U(x)\partial_{\alpha i}U(x) - \partial_{\alpha i}\partial_{\beta j}U(x)\langle \psi_i^\alpha \bar{\psi}_j^\beta \rangle, \end{aligned} \quad (16)$$

where we abbreviated $\partial_{\alpha i} = \frac{\partial}{\partial x_i^\alpha}$. This representation fixes the values of the two central charges to $Z = M = n$.³ The commutation relations of the $\mathcal{N} = 2$ Schrödinger superalgebra (14) constrain the prepotential to obey a set of partial differential equations,

$$\begin{aligned} (x_i^\alpha \partial_{\beta i} - x_i^\beta \partial_{\alpha i})U(x) &= 0, & \sum_i \partial_{\alpha i}U(x) &= 0, \\ x_i^\alpha \partial_{\alpha i}U(x) &= -C. \end{aligned} \quad (17)$$

The first two restrictions in (17) come from rotation and translation invariance, while the last one is responsible for conformal symmetry.

³For particles of mass m one has $Z = M = nm$.

Like in one dimension, we would like to absorb the bosonic potential $V = \frac{1}{2} \partial_{\alpha i} U \partial_{\alpha i} U$ into a reordering of the fermions. The condition for this option generalizes our principal equation (7) to

$$\partial_{\alpha i} U(x) \partial_{\alpha i} U(x) + \kappa \partial_{\alpha i} \partial_{\alpha i} U(x) = 0. \quad (18)$$

Introducing $G(x)$ as in (9) one gets

$$\left(x_i^\alpha \partial_{\alpha i} + \frac{C}{\kappa} \right) G(x) = 0 \quad \text{and} \quad \partial_{\alpha i} \partial_{\alpha i} G(x) = 0 \quad (19)$$

besides translation and rotation invariance for $G(x)$.

Formally, the n -particle model in d dimensions is just a special nd -particle model in one dimension. However, the physical symmetry requirement is different: We want the potential to be invariant under permutations of the n particle labels only, and not under permutations of all nd labels. The translation and rotation invariance, on the other hand, is more restrictive in d dimensions, but this may be dealt with by passing to a set of $SO(d)$ invariants built from relative coordinates. We shall see that for $d > 1$ it is possible to construct physically acceptable (n, d) models for identical particles.

Like in one dimension, we consider prepotentials G which are regular at the origin $x_i^\alpha = 0$. We do not know how to write down the most general rotation and translation-invariant harmonic function, but let us present two classes of examples. In order to take into account translation and rotation invariance, we switch to the relative coordinates $r_{ij}^\alpha = x_i^\alpha - x_j^\alpha$ and form $SO(d)$ scalars (r_{ij}, r_{kl}) and $\epsilon_{\alpha_1 \dots \alpha_d} r_{i_1 j_1}^{\alpha_1} \dots r_{i_d j_d}^{\alpha_d}$ from them. Here, $(,)$ and $\epsilon_{\alpha_1 \dots \alpha_d}$ denote the Euclidean scalar product and the Levi-Civita tensor, respectively, in \mathbb{R}^d . It should be kept in mind that these building blocks are not independent; e.g. the triangle rule $r_{ij}^\alpha + r_{jk}^\alpha + r_{ki}^\alpha = 0$ implies that

$$(r_{ij}, r_{jk}) + (r_{jk}, r_{ki}) + (r_{ki}, r_{ij}) = -\frac{1}{2} [(r_{ij}, r_{ij}) + (r_{jk}, r_{jk}) + (r_{ki}, r_{ki})] \quad (\text{no sum}). \quad (20)$$

Our first example is a homogeneous and permutation invariant polynomial of fourth order (thus $C = -4\kappa$),

$$G(x) = \alpha \sum_{i < j, k} (r_{ik}, r_{kj})^2 + \beta \sum_{i < j, k} (r_{ik}, r_{ik})(r_{kj}, r_{kj}) + \gamma \sum_{i < j} (r_{ij}, r_{ij})^2 \quad (21)$$

with free parameters α , β and γ . Computing

$$\begin{aligned} \partial_{\alpha m} \partial_{\alpha m} (r_{ik}, r_{kj})^2 &= 4(r_{ik}, r_{ik}) + 4(r_{kj}, r_{kj}) \\ &\quad - 4(d+1)(r_{ik}, r_{kj}), \\ \partial_{\alpha m} \partial_{\alpha m} (r_{ik}, r_{ik})(r_{kj}, r_{kj}) &= 4d(r_{ik}, r_{ik}) + 4d(r_{kj}, r_{kj}) \\ &\quad - 8(r_{ik}, r_{kj}), \\ \partial_{\alpha m} \partial_{\alpha m} (r_{ij}, r_{ij})^2 &= 8(d+2)(r_{ij}, r_{ij}) \end{aligned} \quad (22)$$

with sums over m only, and employing the identity

$$\sum_{i < j, k} (r_{ik}, r_{kj}) = \frac{2-n}{2} \sum_{i < j} (r_{ij}, r_{ij}) \quad (23)$$

following from (20), one arrives at

$$\partial_{\alpha m} \partial_{\alpha m} G(x) = \delta \sum_{i < j} (r_{ij}, r_{ij}), \quad (24)$$

with δ being a linear expression in α , β and γ . Therefore, solving (19) amounts to putting $\delta = 0$, which is

$$(n-2)(d+5)\alpha + (n-2)(4d+2)\beta + (4d+8)\gamma = 0. \quad (25)$$

Since the scale of $G(x)$ is irrelevant, this linear relation leaves a one-parameter family (21) of (n, d) prepotentials, for $d > 1$ and $n \geq 2$. The formulas also work for $d = 1$, but produce $G(x) \equiv 0$.

Viewing the three particle labels i , j , and k in (21) as the vertices of a triangle, this prepotential appears to be constructed in terms of triangle areas and edge lengths. This suggests to construct other prepotentials in terms of generalized volumes. The simplest such situation, specific to $d = n - 1$ dimensions, provides our second example,

$$G(x) = \epsilon_{\alpha_1 \dots \alpha_{n-1}} r_{12}^{\alpha_1} r_{13}^{\alpha_2} \dots r_{1n}^{\alpha_{n-1}}. \quad (26)$$

This homogeneous polynomial of degree $n - 1$ measures the volume of the simplex spanned by the $n = d + 1$ particle locations and is naturally permutation invariant (up to an irrelevant sign). It trivially solves the Laplace equation since each vector x_i^α occurs at most linearly in (26). Hence, this example describes a valid $(n, n - 1)$ particle model.

IV. CONCLUSIONS

We have constructed new interacting $\mathcal{N} = 2$ many-body quantum mechanics of a special kind: The bosonic potential is absent, but interaction takes place through boson-fermion couplings alone. These couplings are governed by a prepotential $G = e^{U/\kappa}$ which only has to be harmonic and homogeneous. The central charge (in the $\mathcal{N} = 2$ superconformal algebra) is given by the degree of G and therefore naturally quantized. By changing the fermionic ordering prescription, one may generate also a particular bosonic potential which is purely quantum.

In $d = 1$, the admissible prepotentials include models built from the positive roots of simple Lie algebras. The A_n root systems yield translation-invariant models of identical particles. In dimensions $d > 1$, we provided a general framework with $\mathcal{N} = 2$ Schrödinger supersymmetry and gave two example models, one for generic (n, d) with a free parameter and another one for $d = n - 1$.

Finally, let us discuss possible further developments of this work. The quantization of the central charge may be

weakened by letting the particle coordinates parametrize a cone rather than \mathbb{R}^n . The freedom of a deficit angle around the singularity allows for more general harmonic functions and therefore other $\mathcal{N} = 2$ models. In the higher-dimensional situation, our examples were not the most general ones. A physical classification needs an understanding of all homogeneous harmonic functions on \mathbb{R}^{nd} invariant under the $n!$ permutations of the particle labels and under the rigid translations and rotations of \mathbb{R}^d . It

would be interesting to learn how the root-system solutions fit into such a scheme.

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