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$\mathcal{N} = 4$ supersymmetry and the Belavin-Polyakov-Shvarts-Tyupkin instanton

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In this paper we construct the Lagrangian and Hamiltonian formulations of $\mathcal{N} = 4$ supersymmetric systems describing the motion of an isospin particle on a conformally flat four-manifold with $SO(4)$ isometry carrying the non-Abelian field of a Belavin-Polyakov-Shvarts-Tyupkin instanton. The conformal factor can be specified to yield various particular systems, such as superconformally invariant mechanics as well as a particle on the four-sphere, the pseudosphere, or on $\mathbb{R} \times \mathbb{S}^3$. The isospin degrees of freedom arise as bosonic components of an additional fermionic $\mathcal{N} = 4$ supermultiplet, whose other components are rendered auxiliary by a nonlocal redefinition. Our on-shell component action coincides with the one recently proposed in [arXiv:0912.3289].

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I. INTRODUCTION

In the past decade a lot of attention was paid to the higher-dimensional quantum Hall effect. In many respects, the four-dimensional Hall effect, as formulated by Zhang and Hu [1], was a breakthrough result. At the heart of their approach was the second Hopf map, and most subsequent developments utilized and extended this idea. Among the results, we mention an eight-dimensional variant of the Hall effect [2], an extension of the quantum Hall systems to $\mathbb{CP}$ manifolds [3] and hyperbolic versions of the quantum Hall effect [4].

Another activity concerns the extensions of the quantum Hall effect to supersymmetric theories. From a formal point of view, such an extension requires supersymmetric mechanics of an isospin-carrying particle moving in the background of magnetic monopoles. By now it is well known [5–10] that to invent monopole-type interactions in Lagrangian mechanics one has to involve "isospin" variables with a specific kinetic energy of first order in the time derivatives. In supersymmetric systems these isospin variables become part of some supermultiplet, whose spinor components are auxiliary. The first realization of this idea was proposed in [5], where isospin bosonic and auxiliary fermionic degrees of freedom constitute an auxiliary gauge supermultiplet. More recently, [11] has demonstrated that the minimal coupling to an external non-Abelian self-dual background works perfectly in the case of four-dimensional $\mathcal{N} = 4$ supersymmetric mechanics, and in [12] the Lagrangian formulation has been given by the use of harmonic superspace.

Here, we present an alternative approach, which utilizes ordinary superspace together with a nonlocal component redefinition. This procedure was developed in [7] and has been applied to three-dimensional supersymmetric mechanics in [10]. The main idea of this approach is the replacement of physical fermions by auxiliary ones, which we describe as follows. Suppose we have at hand a $(4, 4, 0)$ fermionic supermultiplet Ψ^α with four physical fermions $\{\psi^\alpha, \bar\psi_\alpha\}$ and four auxiliary bosons $\{v^i, \bar v_i\}$, subject to the standard $d = 1$, $\mathcal{N} = 4$ Poincaré supersymmetry transformations

$$\begin{aligned} \delta\psi^1 &= -\bar\epsilon^i \bar v_i, \\ \delta\psi^2 &= \epsilon_i \bar v^i, \\ \delta v^i &= -2i\epsilon^i \dot{\bar\psi}^1 + 2i\bar\epsilon^i \dot{\bar\psi}^2, \\ \delta\bar v_i &= -2i\epsilon_i \dot\psi^1 + 2i\bar\epsilon_i \dot\psi^2. \end{aligned} \tag{1.1}$$

If we make the formal replacement

$$\dot\psi^\alpha \to \chi^\alpha \quad \text{and} \quad \dot{\bar\psi}_\alpha \to \bar\chi_\alpha, \tag{1.2}$$

we get a new supermultiplet $\mathcal{V}^i$ of $(0, 4, 4)$-type with components $\{v^i, \bar v_i, \chi^\alpha, \bar\chi_\alpha\}$, which transform as

$$\begin{aligned} &\delta\chi^1 = -\bar\epsilon^i \dot{\bar v}_i, \qquad \delta\chi^2 = \epsilon_i \dot{\bar v}^i, \\ &\delta v^i = -2i\epsilon^i \bar\chi^1 + 2i\bar\epsilon^i \bar\chi^2, \qquad \delta\bar v_i = -2i\epsilon_i \bar\chi^1 + 2i\bar\epsilon_i \chi^2. \end{aligned} \tag{1.3}$$

The goal is to construct an $\mathcal{N} = 4$ supersymmetric Lagrangian for the components of $\mathcal{V}^i$. To this end, we couple Ψ^α to some "matter" multiplet Q in a standard superspace action $S[\Psi^\alpha, Q]$. If we make sure that the fermionic components ψ^α are cyclic in this action, i.e. they appear only via their derivatives $\dot\psi^\alpha$, then we may perform the replacement (1.2) on the component level and obtain a supersymmetric and local action for the fields appearing in $\mathcal{V}^i$ and Q. To be sure, such a replacement alters the dynamics: In terms of Ψ^α, it amounts to putting to zero the momentum canonically conjugate to ψ^α. However, we are not interested in the dynamics of $S[\Psi^\alpha, Q]$ but in the physics of the new action governing the dynamics of the components of $\mathcal{V}^i$ and Q.

In the present paper, we choose for the matter Q a one-dimensional hypermultiplet Q^{ia} [13–17] and couple it minimally to Ψ^α,

$$S[\Psi^\alpha, Q^{ia}] = \int d^4\theta dt [F(Q) + Y(Q)\Psi^\alpha \bar{\Psi}_\alpha]. \quad (1.4)$$

The condition of cyclicity of ψ^α in this action restricts the function $Y(Q)$ to be harmonic,

$$\frac{\partial^2}{\partial Q^{ia}\partial Q_{ia}} Y = 0. \quad (1.5)$$

This generalizes to four dimensions the cases of one- and three-dimensional $\mathcal{N}=4$ supersymmetric mechanics with isospin variables considered in [7,10]. The simple action (1.4) will lead to a minimal coupling to the instanton if we choose the $SO(4)$ invariant solution of the condition (1.5) as

$$Y = \rho + \frac{2}{Q^{ia}Q_{ia}} \quad \text{for } Q^{ia} \neq 0. \quad (1.6)$$

Later on, the constant $|\rho|$ becomes the size of the instanton.[1] To have an $SO(4)$ invariant system, we also restrict the arbitrary function $F(Q)$ to depend only on the $SO(4)$ invariant combination

$$X = 2/(Q^{ia}Q_{ia}) \quad (1.7)$$

of the hypermultiplet fields Q^{ia}. Thus, we arrive at the same action as proposed in [7],

$$S[\Psi^\alpha, X] = \int d^4\theta dt [F(X) + (X+\rho)\Psi^\alpha \bar{\Psi}_\alpha]. \quad (1.8)$$

The matter has component content $X = \{x, A^{(ij)}, \eta^i, \bar{\eta}_i\}$, which transforms as

$$\delta x = -i\epsilon_i \eta^i - i\bar{\epsilon}^i \bar{\eta}_i, \qquad \delta\eta^i = -\bar{\epsilon}^i \dot{x} - i\bar{\epsilon}^j A^i_j,$$
$$\delta\bar{\eta}_i = -\epsilon_i \dot{x} + i\epsilon_j A^j_i, \qquad \delta A_{ij} = -\epsilon_{(i}\dot{\eta}_{j)} + \bar{\epsilon}_{(i}\dot{\bar{\eta}}_{j)}. \quad (1.9)$$

We stress that it is the composite structure (1.7) of the superfield X which causes our particle to interact with the instanton. If instead we treat X as an independent $\mathcal{N}=4$ superfield, the isospin degrees of freedom will decouple and the resulting system will describe a particle in the field of a Dirac monopole [8,9]. On the other hand, employing the composite-field concept in three dimensions produces a coupling to the Wu-Yang monopole [10].

At this point we perform the integration over the θ's and then apply our replacement recipe (1.2). A straightforward computation yields the $\mathcal{N}=4$ supersymmetric off-shell component action [7]

$$\begin{aligned} S = \int dt \Big[& \frac{1}{8}G\dot{x}^2 - \frac{1}{16}GA^{ij}A_{ij} + \frac{i}{8}G(\dot{\eta}^i\bar{\eta}_i - \eta^i\dot{\bar{\eta}}_i) \\ & + \frac{1}{8}G'\eta^i\bar{\eta}^j A_{ij} - \frac{1}{32}G''\eta^i\eta_i\bar{\eta}_j\bar{\eta}^j \\ & - (x+\rho)(\chi^1\bar{\chi}^2 - \chi^2\bar{\chi}^1) + \frac{i}{4}(x+\rho)(\dot{v}_i\bar{v}^i - v_i\dot{\bar{v}}^i) \\ & + \frac{1}{4}A_{ij}v^i\bar{v}^j + \frac{1}{2}\eta_i(\bar{v}^i\bar{\chi}^2 + v^i\chi^2) \\ & + \frac{1}{2}\bar{\eta}^i(v_i\chi^1 + \bar{v}_i\bar{\chi}^1)\Big], \end{aligned} \quad (1.10)$$

which describes the interaction of eight bosons $\{x, A^{(ij)}, v^i, \bar{v}_i\}$ and eight fermions $\{\eta^i, \bar{\eta}_i, \chi^\alpha, \bar{\chi}_\alpha\}$ living on the one-dimensional worldline of a particle. Here, $G = F''(x)$ is an arbitrary function depending on x only, ρ is a free parameter, and all indices run over 1 and 2. This action is our starting point.

In the following section we perform several changes of variables and eliminate auxiliary ones, in order to bring out explicitly the instanton coupling. In Sec. III we present the supercharges and the Hamiltonian, as well as the four-dimensional translation and rotation generators. The configuration-space metric of our system is $SO(4)$-invariant and conformally flat, and thus depends only on the single ''radial'' function $G(x)$. In Sec. IV we specialize this metric to obtain a few interesting examples, such as a particle on the sphere $\mathbb{S}^4$ interacting with a Belavin-Polyakov-Shvarts-Tyupkin (BPST) instanton located in its center. Finally, in the Conclusion we shortly discuss the bosonic $SO(5)$ symmetry which naturally appears in the latter case and which is explicitly broken by fermionic terms.

II. THE INSTANTON COUPLING

In our model the four-dimensional nature of the theory is encoded in the composite structure of the superfield X (1.7). The net effect of such a representation is summarized in the composite structure of the ''auxiliary'' components A^{ij}, which are now expressed via the components of Q^{ia} as

$$A_{ij} = ix(\dot{q}^a_i q_{ja} + \dot{q}^a_j q_{ia}) - \frac{1}{x}(\eta_i\bar{\eta}_j + \eta_j\bar{\eta}_i). \quad (2.1)$$

Here, we have used a polar representation of the bosonic Q^{ia} components,

$$Q^{ia}Q_{ia}| = \frac{2}{X|} =: \frac{2}{x} \quad \text{and} \quad Q^{ia}| =: \frac{q^{ia}}{\sqrt{x}} \Rightarrow q^{ia}q_{ia} = 2. \quad (2.2)$$

We substitute the expression (2.1) for A_{ij} into the component action (1.10) and eliminate the auxiliary fermions χ^α and $\bar{\chi}_\alpha$ by their equations of motion, obtaining

[1] In four dimensions this constant plays an essential role, in contrast to the three-dimensional case [10].

$$S=\int dt\bigg\{\frac{1}{8}G\bigg[\dot{x}^2+\frac{x^2}{2}\omega^{ij}\omega_{ij}+i(\dot{\eta}^i\bar{\eta}_i-\eta^i\dot{\bar{\eta}}_i)\bigg]-\frac{i}{8}(2G+xG')\omega_{ij}\eta^i\bar{\eta}^j-\frac{x^2G''+6xG'+6G}{32x^2}\eta^2\bar{\eta}^2+\frac{i}{4}(x+\rho)(\dot{v}_i\bar{v}^i-v_i\dot{\bar{v}}^i)-\frac{i}{4}x\omega_{ij}v^i\bar{v}^j-\frac{\rho}{4x(x+\rho)}v^i\bar{v}^j(\eta_i\bar{\eta}_j+\eta_j\bar{\eta}_i)\bigg\}, \tag{2.3}$$

where

$$\omega_{ij}=\dot{q}_i^a q_{ja}+\dot{q}_j^a q_{ia}. \tag{2.4}$$

The action (2.3) describes four physical bosons $\{x, q^{ia}{:}q^{ia}q_{ia}=2\}$, four physical fermions $\{\eta^i,\bar{\eta}_i\}$, and four isospin variables $\{v^i,\bar{v}_i\}$.

The variables we used until now were rather useful for discussing $\mathcal{N}=4$ supersymmetry properties. However, for clarifying the interactions disguised in (2.3) it is preferable to change variables. We do this in two steps.

First, in order to simplify the kinetic terms for all variables, we rescale them to

$$Y^{ia}=\sqrt{\frac{x}{2}}q^{ia}\Rightarrow Y^{ia}Y_{ia}=x,\qquad u^i=\sqrt{Y^2+\rho}\,v^i,\qquad \xi^i=\frac{\sqrt{G}}{2}\eta^i. \tag{2.5}$$

In addition, we introduce the isospin and fermionic spin currents as useful bilinears:

$$I^A=\frac{i}{2}(\sigma^A)_i^j u^i\bar{u}_j,\qquad \Sigma^A=-i(\sigma^A)_i^j\xi^i\bar{\xi}_j,\quad \text{with}\quad A=1,2,3, \tag{2.6}$$

where the σ^A matrices are normalized as $[\sigma^A,\sigma^B]=2i\epsilon^{ABC}\sigma^C$. In terms of these variables the action (2.3) reads

$$S=\int dt\bigg\{\frac{1}{2}GY^2\dot{Y}^{ia}\dot{Y}_{ia}+\frac{i}{2}(\dot{\xi}^i\bar{\xi}_i-\xi^i\dot{\bar{\xi}}_i)-\frac{i}{4}(\dot{u}^i\bar{u}_i-u^i\dot{\bar{u}}_i)-\frac{1}{2Y^2G}(2G+Y^2G')\Omega^A\Sigma^A+\frac{1}{2(Y^2+\rho)}\Omega^AI^A-\frac{Y^4G''+6Y^2G'+6G}{3Y^4G^2}\Sigma^A\Sigma^A+\frac{2\rho}{GY^2(Y^2+\rho)^2}I^A\Sigma^A\bigg\}, \tag{2.7}$$

where we introduced

$$\Omega^A=(\dot{Y}^{ja}Y_{ia}+Y^{ja}\dot{Y}_{ia})(\sigma^A)_j^i. \tag{2.8}$$

Second, we pass to four-dimensional vector coordinates y^μ via

$$Y^{ia}=\frac{1}{\sqrt{2}}\varepsilon^{ik}y^\mu(\sigma^\mu)_k^a\Rightarrow Y^{ia}Y_{ia}=y^\mu y^\mu, \tag{2.9}$$

where the four sigma-matrices are defined as $\sigma^\mu=(i\sigma^A,\mathbf{1})$ with $A=1,2,3$ and $\mu=1,2,3,4$. It is easy to check that the ingredient Ω^A in (2.7) acquires the nice form

$$\Omega^A=2i\eta^A_{\mu\nu}y^\mu\dot{y}^\nu \tag{2.10}$$

involving the self-dual t'Hooft symbol

$$\eta^A_{\mu\nu}=\delta^A_\mu\delta_{\nu4}-\delta^A_\nu\delta_{\mu4}+\epsilon^A{}_{\mu\nu4}\Rightarrow(\delta_{\mu\rho}\delta_{\nu\sigma}-\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma})\eta^A_{\rho\sigma}=0. \tag{2.11}$$

Combining everything, we get the final form of the action,

$$S=\int dt\bigg[\frac{g}{2}\dot{y}^\mu\dot{y}^\mu+\frac{i}{2}(\dot{\xi}^i\bar{\xi}_i-\xi^i\dot{\bar{\xi}}_i)-\frac{i}{4}(\dot{u}^i\bar{u}_i-u^i\dot{\bar{u}}_i)-\frac{i}{y^2g}(g+y^2g')\eta^A_{\mu\nu}y^\mu\dot{y}^\nu\Sigma^A+\frac{i}{y^2+\rho}\eta^A_{\mu\nu}y^\mu\dot{y}^\nu I^A+\frac{2\rho}{(y^2+\rho)^2g}I^A\Sigma^A-\frac{1}{3y^2g^2}(2g+4y^2g'+y^4g'')\Sigma^A\Sigma^A\bigg], \tag{2.12}$$

where the metric function g is defined as

$$g(y^2)=y^2G(y^2). \tag{2.13}$$

The action (2.12) is our main result. It describes $\mathcal{N}=4$ supersymmetric four-dimensional isospin particles moving in the field of a BPST instanton. Indeed, from (2.12) one sees that the bosonic part of the vector potential reads

$$\mathcal{A}_\mu=-\frac{i}{y^2+\rho}\eta^A_{\mu\nu}y^\nu I^A\Rightarrow\mathcal{F}_{\mu\nu}\equiv\partial_\mu\mathcal{A}_\nu-\partial_\nu\mathcal{A}_\mu+[\mathcal{A}_\mu,\mathcal{A}_\nu]=\frac{2i\rho\eta^A_{\mu\nu}I^A}{(y^2+\rho)^2}, \tag{2.14}$$

which is of the familiar instanton form if we may view I^A, as defined in (2.6), as proper isospin matrices.[2]

The on-shell component action (2.12) coincides (modulo some redefinitions) with the one constructed recently within the harmonic superspace approach in [12]. Surely, the most general case of the action (1.4) with an arbitrary prepotential $F(Q)$ and a more general harmonic function $Y(Q)$ could be easily considered.

To close this section, let us comment on the appearance of the t'Hooft symbol in our construction. The definition of Ω^A in (2.8) makes use of the $su(2)$ algebra generated by the

[2]For a solution to the Yang-Mills equations we must have $[I^A,I^B]=2i\epsilon^{ABC}I^C$.

σ^A, which gets embedded into the self-dual part of the $so(4)$ symmetry group via (2.9). If instead we embed into the anti–self-dual part, by replacing σ^μ with $\bar\sigma^\mu = (-i\sigma^A, \mathbf{1})$, we shall arrive at

$$\Omega^A = 2i\bar\eta^A_{\mu\nu}y^\mu\dot y^\nu \quad \text{with} \quad \bar\eta^A_{\mu\nu} = -\delta^A_\mu\delta_{\nu4} + \delta^A_\nu\delta_{\mu4} + \epsilon^A{}_{\mu\nu4}, \tag{2.15}$$

and the vector potential becomes

$$\mathcal{A}_\mu = -\frac{i}{y^2+\rho}\bar\eta^A_{\mu\nu}y^\nu I^A \Rightarrow \Big(\delta_{\mu\rho}\delta_{\nu\sigma} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Big)\mathcal{F}_{\rho\sigma} = 0, \tag{2.16}$$

producing the BPST anti-instanton.

III. HAMILTONIAN AND SUPERCHARGES

In order to find the classical Hamiltonian, we follow the standard procedure for quantizing a system with bosonic and fermionic degrees of freedom. From the action (2.12) we define the momenta $(P_\mu, \pi_i, \bar\pi^i, p_i, \bar p^i)$ conjugated to $(y^\mu, \xi^i, \bar\xi_i, u^i, \bar u_i)$ as

$$P_\mu = g\dot y_\mu - i\eta^A_{\mu\nu}y^\nu\Big(\frac{1}{y^2+\rho}I^A - \frac{g+y^2g'}{y^2g}\Sigma^A\Big),$$
$$\pi_i = \frac{i}{2}\bar\xi_i, \qquad \bar\pi^i = \frac{i}{2}\xi^i, \qquad p_i = -\frac{i}{4}\bar u_i, \qquad \bar p^i = \frac{i}{4}u^i, \tag{3.1}$$

and introduce Dirac brackets for the canonical variables,

$$\{y^\mu, P_\nu\} = \delta^\mu_\nu, \qquad \{\xi^i, \bar\xi_j\} = i\delta^i_j, \qquad \{u^i, \bar u_j\} = 2i\delta^i_j. \tag{3.2}$$

As usual, the canonical momenta P^μ differ by the vector-potential shift from the kinematical momenta

$$\hat P_\mu := g\dot y_\mu = P_\mu + i\eta^A_{\mu\nu}y^\nu\Big(\frac{1}{y^2+\rho}I^A - \frac{g+y^2g'}{y^2g}\Sigma^A\Big), \tag{3.3}$$

whose Dirac brackets contain the instanton field strength,

$$\{\hat P_\mu, \hat P_\nu\} = \frac{2i\rho}{(y^2+\rho)^2}\eta^A_{\mu\nu}I^A - 2i\frac{2g^2+3y^2gg'+y^4(g')^2}{y^2g^2}\eta^A_{\mu\nu}\Sigma^A - 2i\frac{2g^2+2y^4(g')^2+2y^2gg'-y^4gg''}{y^4g^2} \times (y_\mu\eta^A_{\nu\rho} - y_\nu\eta^A_{\mu\rho})y^\rho\Sigma^A. \tag{3.4}$$

One may check that the supercharges

$$\begin{aligned}
Q^i &= \frac{i}{\sqrt{y^2}g}(\delta^i_j\delta_{\mu\nu} - i\eta^A_{\mu\nu}(\sigma^A)^i_j)y_\mu P_\nu\bar\xi^j - \frac{i\sqrt{y^2}}{(y^2+\rho)\sqrt{g}}(\sigma^A)^i_j\bar\xi^jI^A - i\frac{g-y^2g'}{3\sqrt{y^2}g\sqrt{g}}(\sigma^A)^i_j\bar\xi^j\Sigma^A\\
&= \frac{i}{\sqrt{y^2}g}(\delta^i_j\delta_{\mu\nu} - i\eta^A_{\mu\nu}(\sigma^A)^i_j)y_\mu\hat P_\nu\bar\xi^j - \frac{2i}{3}\Big(\frac{2g+y^2g'}{\sqrt{y^2}g\sqrt{g}}\Big)(\sigma^A)^i_j\bar\xi^j\Sigma^A,\\
\bar Q_i &= \frac{i}{\sqrt{y^2}g}(\delta^j_i\delta_{\mu\nu} + i\eta^A_{\mu\nu}(\sigma^A)^j_i)y_\mu P_\nu\xi_j + \frac{i\sqrt{y^2}}{(y^2+\rho)\sqrt{g}}(\sigma^A)^j_i\xi_jI^A + i\frac{g-y^2g'}{3\sqrt{y^2}g\sqrt{g}}(\sigma^A)^j_i\xi_j\Sigma^A\\
&= \frac{i}{\sqrt{y^2}g}(\delta^j_i\delta_{\mu\nu} + i\eta^A_{\mu\nu}(\sigma^A)^j_i)y_\mu\hat P_\nu\xi_j + \frac{2i}{3}\Big(\frac{2g+y^2g'}{\sqrt{y^2}g\sqrt{g}}\Big)(\sigma^A)^j_i\xi_j\Sigma^A,
\end{aligned} \tag{3.5}$$

and the Hamiltonian

$$H = \frac{1}{2g}\hat P^\mu\hat P^\mu - \frac{2\rho}{(y^2+\rho)^2g}I^A\Sigma^A + \frac{1}{3y^2g^2}(2g+4y^2g'+y^4g'')\Sigma^A\Sigma^A \tag{3.6}$$

form the standard $\mathcal{N}=4$, $d=1$ Poincarè superalgebra

$$\{Q^i, \bar Q_j\} = 2i\delta^i_jH, \qquad [Q^i, H] = [\bar Q_j, H] = 0. \tag{3.7}$$

The spin variables u^i, $\bar u_i$ enter the Hamiltonian only through the isospin currents I^A, which commute with everything, excluding themselves:

$$\{I^A, I^B\} = 2i\epsilon^{ABC}I^C \quad \text{and} \quad \{I^AI^A, H\} = 0, \tag{3.8}$$

just forming an $SU(2)$ algebra with respect to the brackets (3.2). Thus, I^A may be interpreted as classical isospin matrices at fixed isospin I. Analogously, also the fermions appear in the Hamiltonian only through the combination Σ^A which likewise obeys

$$\{\Sigma^A, \Sigma^B\} = 2i\epsilon^{ABC}\Sigma^C \quad \text{and} \quad \{\Sigma^A\Sigma^A, H\} = 0, \tag{3.9}$$

thus providing a description for the fermionic spin degrees of freedom. Clearly, the $SO(4)$ invariance of our system is realized in a standard way through the generators

$$M_{\mu\nu} = y_\mu P_\nu - y_\nu P_\mu - \frac{i}{2}\eta^A_{\mu\nu}(I^A + \Sigma^A) \Rightarrow \{M_{\mu\nu}, H\} = 0. \tag{3.10}$$

Thus, we conclude that the Hamiltonian (3.6) indeed describes the motion of an $\mathcal{N} = 4$ supersymmetric isospin particle in a BPST instanton background.

The supercharges (3.5) and Hamiltonian (3.6) have the same structure as those ones presented in [11]. Therefore, our component action (1.10) provides a formulation alternative to the one of [12].

IV. CASES OF SPECIAL INTEREST

So far our consideration was general, and the function $g(y^2)$ in (2.12) was arbitrary. Let us now specify it to produce some cases of particular interest.

A. Superconformal invariant models

Invariance under the most general $\mathcal{N} = 4$ superconformal group $D(2,1;\alpha)$ in one dimension [18] is achieved for the choice [19][3]

$$g(y^2) = (y^2)^{-1-(1/\alpha)} \quad \text{for } \alpha \neq 0. \tag{4.1}$$

In addition, superconformal invariance demands fixing our parameter to $\rho = 0$, so the instanton must have zero size. Of special interest are the subcases $\alpha = -1$ and $\alpha = 1$ corresponding to the $SU(1,1|2)$ and $OSp(4|2)$ superconformal groups, for which the metric is flat (with $z^\mu = y^{-2}y^\mu$ in the second case):

$$\begin{aligned} S_{\alpha=-1} = \int dt \Big[& \frac{1}{2}\dot{y}^\mu \dot{y}^\mu + \frac{i}{2}(\dot{\xi}^i \bar{\xi}_i - \xi^i \dot{\bar{\xi}}_i) \\ & - \frac{i}{4}(\dot{u}^i \bar{u}_i - u^i \dot{\bar{u}}_i) + \frac{i}{y^2}\eta^A_{\mu\nu} y^\mu \dot{y}^\nu (I^A - \Sigma^A) \\ & - \frac{2}{3y^2}\Sigma^A\Sigma^A \Big], \\ S_{\alpha=1} = \int dt \Big[& \frac{1}{2}\dot{z}^\mu \dot{z}^\mu + \frac{i}{2}(\dot{\xi}^i \bar{\xi}_i - \xi^i \dot{\bar{\xi}}_i) \\ & - \frac{i}{4}(\dot{u}^i \bar{u}_i - u^i \dot{\bar{u}}_i) + \frac{i}{z^2}\eta^A_{\mu\nu} z^\mu \dot{z}^\nu (I^A + \Sigma^A) \Big]. \end{aligned} \tag{4.2}$$

As expected, the four-fermion term disappeared in the $OSp(4|2)$ invariant case.

B. $\mathbb{R} \times \mathbb{S}^3$ case

In the limit of $\alpha \to \infty$ we obtain another special case,

$$g(y^2) = (y^2)^{-1}. \tag{4.3}$$

With this choice, the kinetic term for the y^μ variables in the action (2.12) acquires the form

$$\frac{1}{2y^2}\dot{y}^\mu \dot{y}_\mu = \frac{1}{2}\left(\frac{\dot{\tilde{y}}^2}{\tilde{y}^2} + \dot{\hat{y}}^\mu \dot{\hat{y}}^\mu\right) \quad \text{with} \quad y^\mu = \tilde{y}\hat{y}^\mu, \qquad \hat{y}^\mu \hat{y}^\mu = 1, \tag{4.4}$$

and thus we meet an $\mathbb{R} \times \mathbb{S}^3$ geometry in the bosonic sector. The full action then reads

$$\begin{aligned} S_{\alpha\to\infty} = \int dt \Big[& \frac{1}{2y^2}\dot{y}^\mu \dot{y}^\mu + \frac{i}{2}(\dot{\xi}^i \bar{\xi}_i - \xi^i \dot{\bar{\xi}}_i) \\ & - \frac{i}{4}(\dot{u}^i \bar{u}_i - u^i \dot{\bar{u}}_i) + \frac{i}{y^2+\rho}\eta^A_{\mu\nu} y^\mu \dot{y}^\nu I^A \\ & + \frac{2\rho y^2}{(y^2+\rho)^2} I^A \Sigma^A \Big]. \end{aligned} \tag{4.5}$$

C. Sphere $\mathbb{S}^4$ and pseudosphere cases

To describe the sphere $\mathbb{S}^4$ or the pseudosphere one has to choose

$$g = \frac{1}{(\rho + y^2)^2}, \tag{4.6}$$

with $\rho > 0$ for the sphere or $\rho < 0$ for the pseudosphere. The corresponding action becomes

$$\begin{aligned} S_{\mathbb{S}^4} = \int dt \Big[& \frac{1}{2(\rho+y^2)^2}\dot{y}^\mu \dot{y}^\mu + \frac{i}{2}(\dot{\xi}^i \bar{\xi}_i - \xi^i \dot{\bar{\xi}}_i) \\ & - \frac{i}{4}(\dot{u}^i \bar{u}_i - u^i \dot{\bar{u}}_i) + \frac{i}{y^2+\rho}\eta^A_{\mu\nu} y^\mu \dot{y}^\nu I^A \\ & - i\frac{\rho - y^2}{y^2(\rho + y^2)}\eta^A_{\mu\nu} y^\mu \dot{y}^\nu \Sigma^A + 2\rho I^A \Sigma^A \\ & + \frac{2\rho(2y^2 - \rho)}{3y^2}\Sigma^A\Sigma^A \Big]. \end{aligned} \tag{4.7}$$

D. Very simple system

Rather than specializing to a simple bosonic manifold like we did so far, one might try to simplify the fermionic sector instead. Here, the maximal simplification occurs for

$$g = \frac{1}{y^2(y^2+\rho)}. \tag{4.8}$$

With this choice, the system possesses an additional conserved current,

$$W = I^A \Sigma^A. \tag{4.9}$$

In addition, the kinematical momenta simplify to

$$\hat{P}_\mu = P_\mu + i\eta^A_{\mu\nu} y^\nu \frac{1}{y^2+\rho}(I^A + \Sigma^A), \tag{4.10}$$

and, therefore,

[3] $\mathcal{N} = 4$ $D(2,1;\alpha)$ superconformal mechanics without isospin degrees of freedom have been constructed in [20].

$$\{\hat{P}_\mu, \hat{P}_\nu\} = \frac{2i\rho}{(y^2+\rho)^2}\,\eta^A_{\mu\nu}(I^A + \Sigma^A) = \mathcal{F}_{\mu\nu}|_{I\to I+\Sigma}. \tag{4.11}$$

Finally, the "very simple" Hamiltonian reads

$$H = \frac{y^2(y^2+\rho)}{2}\hat{P}_\mu\hat{P}_\mu - \frac{2\rho y^2}{y^2+\rho}\left(I^A\Sigma^A + \frac{1}{3}\Sigma^A\Sigma^A\right). \tag{4.12}$$

V. CONCLUSION

In this paper we have constructed the Lagrangian and Hamiltonian formulations of $\mathcal{N}=4$ supersymmetric systems describing the motion of isospin particles on a conformally flat four-manifold with a BPST $SU(2)$ instanton. Because of $SO(4)$ rotation invariance around the instanton location, the conformal factor depends on the radial variable only. It was further specified to capture some particularly interesting systems, including superconformally invariant mechanics, a particle living on the sphere, the pseudosphere, or on $\mathbb{R}\times\mathbb{S}^3$. The isospin variables entered the action as the bosonic components of an auxiliary $\mathcal{N}=4$ supermultiplet. Its other components were auxiliary and became expressed on-shell through the physical fermions. It is obvious how to generalize the action (1.4) to an arbitrary prepotential $F(Q)$ and a general harmonic function $Y(Q)$.

Starting from the off-shell component action (1.10) [7] we derived the action (2.12), which coincides with the one proposed recently in [12]. It is a matter of taste to prefer one approach over another. What makes us enthusiastic about the present construction is the extreme simplicity of the precursor action (1.4) in ordinary superspace. Of course, one must revert to the component level for applying the nonlocal replacement recipe (1.2).

The sphere case, treated in Sec. IV C, is of enhanced interest due to its possible relation with the four-dimensional Hall effect. Indeed, our system provides a new $\mathcal{N}=4$ supersymmetric extension of the model considered by Zhang and Hu [1], with which it coincides in the bosonic sector. It is tempting to investigate the spectrum of our system and to analyze the role played by the supersymmetry of (4.7).

Unfortunately, not everything looks nice in our system. First of all, the implicit $SO(5)$ symmetry of the four-dimensional Hall effect, which played a crucial role in the computation of spectra in [1], is explicitly broken by $\mathcal{N}=4$ supersymmetry. Indeed, to be the same constant at all points of the four-sphere, the right-hand side of (3.4) can only depend on y^2. This necessary condition results in an equation for the configuration-space metric g,

$$2g^2 + 2y^4(g')^2 + 2y^2gg' - y^4gg'' = 0 \Rightarrow g = \frac{c_1}{y^2(c_2y^2+\rho)}. \tag{5.1}$$

All the cases we considered in Sec. IV belong to this class of metric, except for the sphere and pseudosphere. Thus, the supersymmetry has to be responsible for removing the high degeneracy of the eigenstates presented in the ordinary Hall effect.

A second unpleasant feature of our system is the absence of a confining potential. To accommodate such a potential, auxiliary bosonic degrees of freedom are needed, which requires adding extra supermultiplets to our present scheme. Such multiplets will bring in new physical fermions, and we have not yet a recipe for how to deal with those.

Geometrically, the absence of $SO(5)$ invariance in our systems originates from treating the four-dimensional coordinates y_μ as $SO(5)/SO(4)$ coset-space coordinates, and so a part of the $SO(5)$ symmetry is nonlinearly realized via

$$\delta y_\mu = \frac{\rho - y^2}{2}a_\mu + (a_\nu y^\nu)y_\mu. \tag{5.2}$$

While the four-sphere possesses such an invariance, the constraints defining the $\mathcal{N}=4$ hypermultiplet do not. Instead, as directly follows from [19], the physical bosons of the hypermultiplet parametrize $\mathbb{R}\times\mathbb{S}^3$, a space which cannot carry $SO(5)$ invariance. A possible solution could be to replace the hypermultiplet by some nonlinear supermultiplet, whose bosonic components should parametrize the four-sphere. The most natural candidate for this role is the nonlinear $(4, 8, 4)$ supermultiplet [21,22], which would extend the number of fermions to eight. Nevertheless, we do not expect the corresponding action to enjoy $\mathcal{N}=8$ supersymmetry [6], due to rather strong restrictions on the bosonic metric imposed by the four extra supersymmetries. We are planning to consider these possibilities in more detail elsewhere.

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