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Quantum Suppression of Irregularity in the Spectral Properties of the Kicked Rotator

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The statistical properties of the quasienergy spectrum are used to measure the influence of quantum effects on the quantum kicked rotator which displays chaotic behavior in the classical limit. A transition from orthogonal-ensemble statistics in the semiclassical limit ($\hbar \rightarrow 0$) to Poisson statistics in the quantum regime is observed. In view of previously obtained results for this system the dependence on the irrationality of $\hbar$ is discussed.

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The statistical properties of the spectra of quantum systems have been found to be a significant measure for the degree of integrability of the same system in its classical limit. Using semiclassical arguments, Berry and Tabor[1] have shown that integrable systems have an uncorrelated spectrum showing a Poisson distribution for the energy separations of adjacent levels. On the other hand, the theory of random matrices has been applied successfully to the analysis of the spectral properties of classically chaotic systems.[2,3] In the intermediate regime, i.e., for systems that undergo a transition from nearly integrable to completely chaotic behavior in their classical limit, a transition between these extremal cases has been found as a function of the classical degree of stochasticity.[4,5] Thus the statistical properties of the spectrum may be used as a measure for the degree of regularity in the quantum system.

In this Letter we apply this approach to the quantum kicked rotator, introduced by Casati *et al.*[6] as the basic model to study "quantum chaos." We will present results on how the semiclassical behavior is affected by quantum effects of varying strength,[7] whereas previous work mostly studied the influence of an increasing degree of classical stochasticity in the semiclassical limit. The statistical properties of the spectrum of the kicked rotator have been discussed before by several authors.[8,9] According to the work of Feingold and co-workers,[9] the spacing distribution for driven Hamiltonian systems should be always Poissonian. We will relate our results to this work below.

The kicked rotator is a quantized version of the standard map.[10] The Hamiltonian of the classical system reads

$$H=\frac{p^2}{2I}-\frac{kI}{T}\cos\phi\sum_n \delta(t-nT). \tag{1}$$

Below we use the dimensionless units, i.e., we assume $I=1$, $T=2\pi$. The Hamiltonian equations can be integrated over one period of the driving force to give the classical map (y_n denotes the variable y *after* the nth kick):

$$\phi_{n+1}=\phi_n+2\pi p_n, \tag{2a}$$

$$p_{n+1}=p_n+(k/2\pi)\sin\phi_{n+1}. \tag{2b}$$

The kick strength k controls the stochasticity of the standard map. For $k\rightarrow 0$, the system is nearly integrable; for $k\approx 1$ the last Kolmogorov-Arnol'd-Moser torus disappears and diffusion in the p direction becomes possible.

Note that besides the rotational symmetry $\phi'=\phi+2\pi$ and parity conservation ($p'=-p$, $\phi'=-\phi$), the discrete translation $p'=p+1$ does not change the classical dynamics which defines a (classical) scale of momentum. This periodicity is a consequence of the special time dependence of the driving field. The corresponding canonical transformation $(\phi,p)\rightarrow(\theta,I)$ which leaves the

Hamiltonian (1) invariant can be given in terms of a generating function[11] which reads

$$F_n(\phi,I,t)=(I+n)(\phi-nt)+\tfrac{1}{2}n^2t \tag{3}$$

for the standard map. For any time dependence with bounded spectrum, this translational symmetry in p will not appear.

By application of Floquet's theorem the quantum dynamics can be described by the discrete time-evolution operator

$$U=U(2\pi,0)=T\exp\left[-\frac{i}{\hbar}\int_0^{2\pi}d\tau H(\tau)\right]$$
$$=e^{-(i/\hbar)\pi p^2}e^{i(k/2\pi\hbar)\cos\phi}, \tag{4}$$

which can be given explicitly in p representation[6]:

$$U_{mm'}=e^{-i\pi\hbar m^2}i^{m-m'}J_{m-m'}(k/2\pi\hbar), \tag{5}$$

where $p|m\rangle=\hbar m|m\rangle$ and J_n are Bessel functions of the first kind. As a result of the discrete set of eigenvalues of the operator p, a second momentum scale arises in the quantum system. Depending on whether the classical and the quantum momentum scales are commensurate or not, i.e., $\hbar$ [namely the ratio $\hbar/(2\pi I/T)$ in the original units] is rational or irrational, the propagator will have a continuous (band) or a discrete spectrum. Since we want to calculate the quasispectrum of U numerically, we will confine ourselves to the case of rational $\hbar=M/N$. For NM even one can reduce U to an $N\times N$ matrix[12]

$$U_{mm'}(a)=\frac{1}{N}\exp\left[-i\pi\hbar m^2-i(m-m')\frac{a}{N}\right]\sum_{j=1}^{N}\exp\left[-i(m-m')\frac{2\pi j}{N}+i\frac{k}{2\pi\hbar}\cos\left(\frac{2\pi j+a}{N}\right)\right], \tag{6}$$

where $a\in[0,2\pi]$ is the Bloch number related to the discrete translational invariance $p'=p+M$ of the quantum system. Comparison with the classical system shows that M is the number of first-order resonances within the period of the quantum system. For irrational $\hbar$, the classical periodicity of the system has no quantum analog.

The statistical properties of the spectrum of a quantum system in the semiclassical limit depend on the integrability of the corresponding classical model. A widely used criterion for the characterization of a given spectrum is the distribution of nearest-neighbor spacings. Berry and Tabor[1] showed that for a classically integrable system this distribution is Poissonian. For classically chaotic systems it is possible to map the quantum-mechanical eigenvalue problem to a complicated mechanical system whose integrals of motion can be associated with the spectral distribution function of the quantum system. This procedure gives agreement with certain random matrix ensembles: the Gaussian orthogonal ensemble (GOE) for autonomous systems[2] and the circular orthogonal ensemble (COE) for periodically driven systems.[3] For these ensembles the spacing distribution is given by the Wigner surmise

$$p(x)=\tfrac{1}{2}\pi x e^{-(\pi/4)x^2}, \tag{7}$$

to a very good approximation (the spacing x is measured in units of the mean spacing).

For systems which show a transition from regular to chaotic behavior in the classical limit, several interpolation formulas for the spacing distribution have been introduced (see, e.g., Ref. 5). In this work the one given by Berry and Robnik[13] (quoted as BR in the following) will be used:

$$p(x,q)=\{(1-q)^2\operatorname{erfc}[(\sqrt{\pi}/2)qx]+[2q(1-q)+(\pi/2)q^3x]e^{-(\pi/2)q^2x^2}\}e^{-(1-q)x}. \tag{8}$$

This gives (7) for $q=1$ and the Poisson distribution for $q=0$. Another characteristic statistical quantity is the rigidity Δ_3 which is a measure for correlations in the spectrum:

$$\Delta_3(L)=\frac{1}{L}\min_{A,B}\int_x^{x+L}[n(\epsilon)-A\epsilon-B]^2d\epsilon, \tag{9}$$

where $n(\epsilon)$ is the number of states with energy below ϵ. Measuring L in units of the mean level spacing one finds

$$\Delta_3(L)=L/15 \quad \text{for Poisson statistics,} \tag{10a}$$

$$\Delta_3(L)\simeq 1/\pi^2\ln L-0.007 \quad \text{for OE statistics } (L\geq 15). \tag{10b}$$

As before in the case of the level-spacing distribution a transition between the two limiting cases depending on the parameter q can be observed (see, e.g., Ref. 5).

To study the level statistics of the kicked-rotator system, one has to take into account all symmetries of the system. A well-defined result can be obtained in the subspace defined by specific choices of the "good" quantum numbers parity and Bloch number a only. We analyzed the spectra of $U(a=0)$ in the even-parity subspace numerically for different values of $\hbar$ and kick strength $k=10.0$. For this value of k the classical phase space is covered by a single chaotic trajectory to more than 99%. Thus, from semiclassical arguments one expects the quasispectra to obey OE statistics.

In Fig. 1 we present the results for the level-spacing distribution $p(x)$ and the Δ_3 statistics for two different values of $\hbar$. For $\hbar=1/1944$, the semiclassical prediction

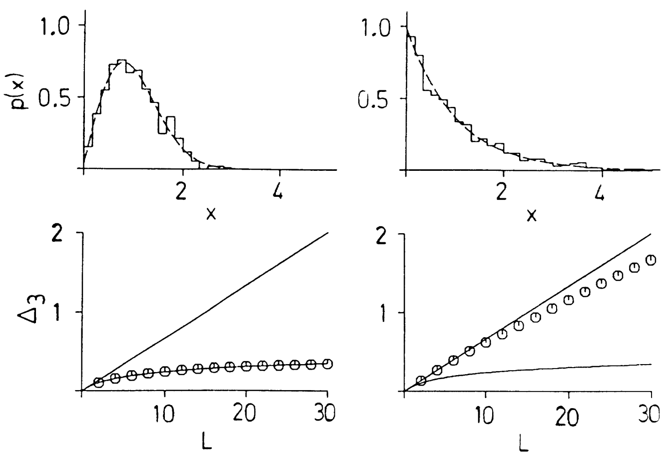

FIG. 1. Level-spacing distribution and Δ_3 statistics of the calculated quasienergy spectra of the quantum kicked rotator for $k=10$. (a) $\hbar=1/1944$ and (b) $\hbar=625/1944$. Histograms in the level-spacing diagrams are numerical data, dashed lines are best fits by Eq. (8). Solid lines in the Δ_3 diagrams are Poisson and OE results (10).

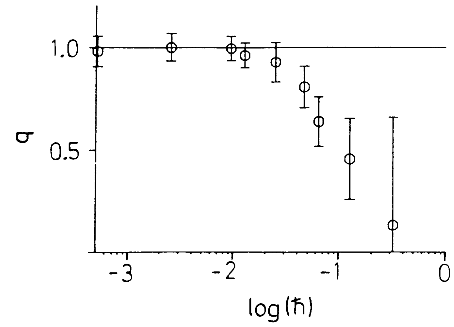

FIG. 2. BR parameter q from numerical data as a function of $\hbar$. The large errors for small values of q are a consequence of the functional dependence of the BR distribution (8) on q.

of OE-like behavior holds, while for $\hbar=625/1944\simeq\frac{1}{3}$ we find a Poisson distribution of the quasienergies. One may use the value of the BR parameter q obtained from a least-squares fit of the numerical results by the BR level-spacing distribution function (8) as a measure for the statistical properties of the quasispectra. Figure 2 shows the results of this procedure obtained for q as a function of $\hbar$ [We diagonalized $U(a=0)$ in the even-parity subspace for $\hbar=n/1944$ which gives 973 levels for each point.] In the semiclassical limit $\hbar\ll 1$, we find OE properties of the quasispectrum while for $\hbar\sim 0.1$ a transition towards Poissonian distribution is observed. [Dimensionless values of $\hbar$ have a physical meaning in the specific system for which they have been defined only. To compare our results with different systems one should note that $\hbar^{-1}$ is the number of (unperturbed) quantum levels per first-order resonance of the classical system.] These results are in good agreement with the ones obtained earlier for a kicked spin system[5] and the general observation that quantum effects tend to suppress the effects of classical stochasticity.[7,14]

It was argued[9] that for irrational $\hbar$ the statistical properties of the spectra should always be Poisson, since the eigenvalue problem for U could be mapped on a quasirandom Anderson model[15] and an exact proof exists[16] that the energy-level separation in an infinite-dimensional Anderson model obeys Poisson statistics. However, the numerical confirmation of this conjecture suffers from the fact that there is no possibility to diagonalize the exact propagation operator in the infinite-dimensional Hilbert space of this system. A truncation of the propagation operator U as given in Eq. (5) is possible by use of the fact that the Bessel functions $J_n(k/2\pi\hbar)$ become negligible for $n\geq k/\pi\hbar$ which gives a band matrix for U. In previous numerical work on the quasispectra of the kicked rotator, this procedure was used for irrational $\hbar\approx 1$ where Poissonian distributions of the eigenvalues have been found. This result was interpreted as a confirmation of the equivalence of driven Hamiltonian quantum systems with the Anderson model. From our results it is clear that this has to be expected for values of the quantum parameter $\hbar\approx 1$ independent of the classical properties of the system. The crucial question is whether this conclusion can also be drawn if one approaches the semiclassical limit. Our results indicate that a transition to Wigner statistics occurs at sufficiently small values of $\hbar$.

To investigate this question further, we have calculated the quasispectrum for $\hbar\ll 1$ (where the truncation procedure is not applicable) by using a rational $M/N\simeq\hbar$ with M,N being relatively prime. For large N the width of the quasienergy bands decreases exponentially and the dependence on the Bloch number a may be neglected.[15]

To prove that for $\hbar\to 0$ the spectral properties again change to OE-like behavior, we investigated the spectra of $U(a)$ for rational approximations to the most irrational value of $\hbar$ near $\hbar_0\propto 1/100$ which may be obtained by adding 1 to the continued-fraction (CF) expansion[11] of $\hbar_0=1/N$:

$$\hbar=\cfrac{1}{N+\cfrac{1}{1+\cfrac{1}{1+\dots}}}\equiv[N,1,1,\dots]. \qquad (11)$$

(The dependence of the time evolution of the quantum kicked rotator on the irrationality of $\hbar$ as measured by an increasing length of the CF was investigated recently by Casati *et al.*[17]) Table I shows the dependence of the BR parameter q on the length of this expansion where for technical reasons different N in the range of 103 to 106 were chosen. A dependence on the "irrationality" of

TABLE I. Dependence of the BR parameter q on the irrationality of $\hbar$ as measured by the length of the CF expansion of $\hbar$.

$\hbar$	CF ($\hbar$)	q	Δq
3/320	[106,1,1,1]	1.000	± 0.204
5/528	[105,1,1,1,1]	0.989	± 0.133
13/1360	[104,1,1,1,1,1,1]	0.995	± 0.059
21/2176	[103,1,1,1,1,1,1,1]	1.000	± 0.085

$\hbar$ has not been found in the range of parameters we were able to treat numerically (up to $\hbar = 21/2176$ which requires the diagonalization of a unitary 1089×1089 matrix). However, any rational value M/N of $\hbar$ corresponds to a pseudorandom lattice with N sites instead of an infinite system and we are not able to show that Wigner statistics persists in the limit $N \rightarrow \infty$ with $\hbar = M/N$ fixed. Therefore the results of the present paper are consistent with those of Ref. 9 if a transition to Poisson statistics appears in this limit.

From the results presented above we conclude that quantum effects can drastically change the spectral properties of systems which are classically chaotic. The influence of the irregular character of the corresponding classical system which is well established in the semiclassical regime disappears. The observed transition of the spectral characteristics as a function of $\hbar$ is analogous to the one found in semiclassical systems ($\hbar \rightarrow 0$) when the parameter that controls the classical stochasticity—k in the standard map—is varied. Both effects can be seen as different aspects of the same phenomenon, namely, the degree of irregularity in a quantum system. It is proportional to the classical one in the semiclassical regime while it vanishes for $\hbar \rightarrow 1$, i.e., classical and quantum irregularity become independent. If measured by the BR parameter q it should be possible to describe this quantum irregularity by a universal scaling function $q(k,\hbar)$. However, a theory which describes this scaling has to go beyond semiclassical arguments and is not available so far.

In the investigation of the spectral characteristics in their dependence on the number-theoretical properties of $\hbar$ we have not been able to verify the conjecture that there should be a transition towards Poisson statistics when one approaches irrational values of $\hbar$ (Table I). It can be argued that this transition occurs for continued-fraction expansion of $\hbar$ even longer than the ones considered in this work. However, one should remember that the difference in the quantum dynamics between rational and irrational values of $\hbar$ is a consequence of the classical periodicity of the standard map in p which was generated by choice of a driving force with unbounded spectrum. Thus, a more realistic system will not show this phenomenon.

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