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Wolf Barth (1942–2016)

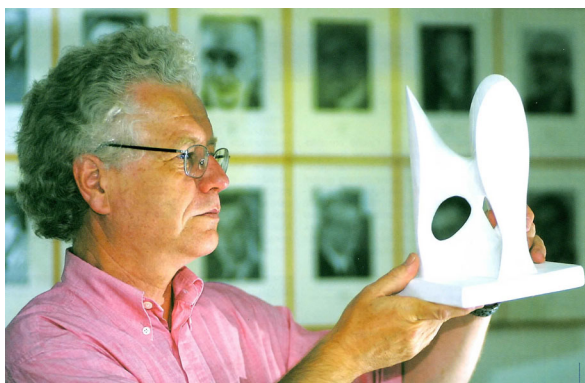**Thomas Bauer[1] · Klaus Hulek[2] · Sławomir Rams[3] · Alessandra Sarti[4] · Tomasz Szemberg[5]**

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[1] Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Straße, Lahnberge, 35032 Marburg, Germany

[2] Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

[3] Institute of Mathematics, Jagiellonian University, Łojasiewicza 6, 30348 Cracow, Poland

[4] Laboratoire de Mathématiques et Applications UMR 7348 du CNRS, Université de Poitiers, Téléport 2 Boulevard Marie et Pierre Curie, BP 30179, 86962 FUTUROSCOPE CHASSENEUIL Cedex, France

[5] Department of Mathematics, Pedagogical University of Cracow, Podchorazych 2, 30-084 Cracow, Poland

1 Life

1.1 Youth

Wolf Paul Barth was born on 20th October 1942 in Wernigerode, a small town in the Harz mountains. This is where his family, originally living in Nuremberg, had sought refuge from the bombing raids. In 1943 the family moved to Simmelsdorf, which is close to Nuremberg, but was less vulnerable to attacks. In 1945 Barth's brother Hannes was born. The family, the father was a town employee, then moved back to Nuremberg and Wolf Barth's home was close to the famous Nuremberg castle—his playgrounds were the ruins of Nuremberg.

In 1961 Wolf Barth successfully completed the Hans Sachs Gymnasium in Nuremberg. He was an excellent student (except in sports), and he then chose to study mathematics and physics at the nearby Universität Erlangen, officially called Friedrich-Alexander Universität Erlangen-Nürnberg.

1.2 Studies and Early Academic Career

As it was the standard at the time, Wolf Barth enrolled for the *Staatsexamen*, the German high school teacher's examination. At this time, Reinhold Remmert was a professor of mathematics in Erlangen and Wolf Barth soon felt attracted to this area of mathematics. So, when Remmert moved to Göttingen in 1963, Barth went with him. Göttingen had at that time started to recover to some extent after its losses in the Nazi period and Hans Grauert was one of the leading figures who attracted many talented mathematicians. And indeed, it was Grauert, who became Barth's second academic teacher and a figure who influenced his mathematical thinking greatly. Wolf Barth not only completed his studies in Göttingen, but in May 1967 he also obtained his PhD with a thesis on *Einige Eigenschaften analytischer Mengen in kompakten komplexen Mannigfaltigkeiten* (Some properties of analytic sets in compact complex manifolds). In 1967 Remmert moved again, this time to take up a chair at Westfälische Wilhelms-Universität Münster where Heinrich Behnke had been a professor, and whose successor Remmert became. Münster, the home of the Behnke school, was at that time one of the international centres of the school of several complex variables. Again, Barth followed Remmert and worked for the next two years in Münster. These were lively times at German universities (the '68 revolt) and Barth became interested in and attracted to the ideas of the 1968 generation.

Wolf Barth went to spend the academic year 1969/70 as a visiting lecturer at MIT in Cambridge, USA, a period where he encountered many new faces and ideas. In 1971, a year after returning from the USA, he obtained his Habilitation in Münster, where he also became a professor. In 1972 Wolf Barth took up a chair at Rijksuniversiteit Leiden in the Netherlands. At the time he was the youngest full professor in Leiden. It was here that he started his close collaboration with Antonius Van de Ven—an encounter which would greatly influence his future mathematical interests and career, as we will outline below. Barth married his wife Regina in 1972.

1.3 The Erlangen Period

In 1976 Wolf Barth was offered a professorship at Universität Erlangen, his original Alma Mater. Erlangen is of course well known to mathematicians, not least through the Klein programme, made famous by the inaugural lecture of Felix Klein, who was a professor there from 1872 to 1875. Another famous professor had been Max Noether, one of the 19th century masters of algebraic geometry—the field of mathematics which had become Wolf Barth's own area of research. Also, his daughter Emmy Noether had been a student in Erlangen where she wrote her thesis in invariant theory under the guidance of Paul Gordan.

This call also reflected the international reputation which Barth had gained in such a short period on the strength of his research; he was offered the very chair which had been established for Max Noether and which had been held by such prominent mathematicians as Heinrich Tietze, Johannes Radon, Wolfgang Krull and Georg Nöbeling. This in itself was a great honour. But it was also the chance to move back to Franconia, his home region, which made this offer irresistible. Wolf Barth truly loved his Franconia and in fact he stayed there for the rest of his life. In 1977 his son Matthias was born and in 1981 his daughter Ursula followed.

In Erlangen, Barth took on multiple duties which came with the position of *Lehrstuhlinhaber* (full professor). It should be emphasized that teaching was not simply a duty for him, he took it very seriously and was very conscientious—we will come back to this aspect of his mathematical life later. As one of the full professors Barth was also constantly involved in running the Institute of Mathematics (which later became part of the Department of Mathematics). In particular, he was Dean of Naturwissenschaftliche Fakultät I from 1981 to 1983.

Barth's arrival in Erlangen created a very lively research atmosphere. In fact, there were two active research seminars related to algebraic geometry in Erlangen at the time: the seminar run by Wolf Barth and the seminar organized by Wulf-Dieter Geyer and Herbert Lange. Often one of the seminars was devoted to lectures on ongoing or recently completed research with many guests and visitors as speakers, whereas for the other seminar a subject was chosen to be presented in much detail by the participants of the seminar in turn. In the early 1980s preparations were made to apply to Deutsche Forschungsgemeinschaft (DFG) for a major research project in complex algebraic geometry. Several research groups collaborated to submit a proposal for a Priority Programme (Schwerpunktprogramm), including the newly established algebraic geometry group in Bayreuth, where Michael Schneider had taken up a chair, as well as the group led by Otto Forster in München. The application was successful and from 1985 onwards the DFG Schwerpunktprogramm *Komplexe Mannigfaltigkeiten* became an important factor of the German research activities in algebraic geometry. Wolf Barth became the coordinator of the programme and oversaw two successful applications for an extension of the programme. This was a very active and fruitful period for the researchers involved and it brought many visitors, both short and long term, to the participating groups in Germany. In particular, the extra funding allowed Wolf Barth to invite students from various countries (Mexico, Israel, Italy, Poland to name a few) to work under his guidance—we will discuss some results of his efforts to support development of research groups abroad in the last section of this account.

For many years Barth was a regular organizer of Oberwolfach meetings. The meeting *Mehrere komplexe Veränderliche*, originally organized by Grauert, Remmert and Stein, is one of the oldest series of Oberwolfach conferences. In 1982 Wolf Barth was asked to replace Karl Stein as one of the organizers. In this capacity, together with Grauert and Remmert, he was in charge of the biannual meetings until 1994, when the organization of the series was taken over by Demailly, Hulek and Peternell. Apart from the regular meetings in this series, he often organized Oberwolfach workshops on more specialized topics, typically with Van de Ven as a co-organizer. In spite of these numerous obligations, Barth also served the mathematical community as an editor. Most importantly, he and Wolf von Wahl were joint Editors-in-Chief of Mathematische Zeitschrift from 1984 to 1990.

Wolf Barth retired on 1st April 2011. Quite tellingly, the lecture he gave during the farewell conference "Groups and Algebraic Geometry", organized by the Emmy-Noether Zentrum to honour his achievements, was called "99 Semester Mathematik". Even after Barth's retirement, his former students sometimes obtained mails with his comments on their recent papers, and his teaching manuscripts were further available from the webpage of Universität Erlangen and were widely used by students. Wolf Barth died on December 30, 2016 in Nuremberg.

2 Wolf Barth—Research

2.1 Barth-Lefschetz Theorems

Wolf Barth's field of research was complex algebraic geometry and his original approach was strongly influenced by the German school of several complex variables, also known as the Behnke school, named after its founder Heinrich Behnke, and later led by Karl Stein, Hans Grauert and Reinhold Remmert. Barth became first famous through his work on the topology of subvarieties of projective space $\mathbb{P}^N$. The starting point of this work is the celebrated Lefschetz hyperplane theorem, which compares the topology of a projective manifold to that of a hyperplane section:

Theorem 2.1 (Lefschetz hyperplane theorem) *Let $X \subset \mathbb{P}^N$ be a subvariety of dimension k and let $Y = X \cap H$ be a hyperplane section such that $U = X \setminus Y$ is smooth. Then the restriction map $H^k(X) \to H^k(Y)$ in (singular) cohomology is an isomorphism for $k < N - 1$ and injective for $k = N - 1$.*

Lefschetz proved this result using the by now famous technique of Lefschetz pencils. Another approach was later developed by Andreotti and Fraenkel using Morse theory. In his paper [6, p. 952] Barth generalized the Lefschetz theorem in the following way.

Theorem 2.2 (Barth) *Let $X, Z \subset \mathbb{P}^N$ be manifolds of dimension n and m respectively, such that $2n \geq N + s$ and $n + m \geq N + r$. Then $H^k(X) \to H^k(X \cap Z)$ is an isomorphism for $k \leq \min\{r - 1, s\}$.*

Barth's proof is analytic. It is based on his earlier work on extending meromorphic functions [5] and uses q-convexity, sheaf cohomology and the Leray spectral sequence. Barth and Larsen [18] further extended the work of Lefschetz from cohomology to homotopy groups. Again, their approach uses analytic tools, such as distance functions in projective space and pseudo-concavity of tubular neighborhoods. In particular, they proved [18, Theorem I]:

Theorem 2.3 (Barth-Larsen) *Let $X \subset \mathbb{P}^N$ be smooth of dimension n with $2n \geq N + 1$. Then X is simply connected, i.e. $\pi_1(X) = 0$.*

This was later strengthened further by Larsen [52]. Indeed, the concept of Barth-Lefschetz theorems became a well known term in the literature. The work of Barth and Larsen also influenced Fulton and Hansen [40] when they proved their famous connectedness theorem. This circle of ideas was further taken up in the work of Badescu, Sommese and Debarre leading, among other things, to Barth-Lefschetz theorems in products of projective spaces and, more generally, homogeneous varieties.

2.2 Vector Bundles

After moving to Leiden, Wolf Barth started collaborating with Van de Ven. It was at this time that Hartshorne [44, p. 1017] formulated his well-known conjecture on complete intersections, which, in its simple form, can be stated as

Conjecture 2.4 (Hartshorne) *Let $X \subset \mathbb{P}^N$ be a smooth manifold of codimension 2. If $N \geq 7$ then X is a complete intersection $X = S_1 \cap S_2$ of two hypersurfaces.*

This conjecture was motivated, on the one hand, by a number of examples in small dimension and, on the other hand, by the fact that projective submanifolds in this range are subject to a number of strong topological constraints. Indeed, by the Lefschetz theorem, the Picard group, i.e. the group of line bundles on X, is a free abelian group of rank one, more precisely, the restriction $\mathrm{Pic}(\mathbb{P}^N) \to \mathrm{Pic}(X)$ is an isomorphism. Further, by the Barth-Larsen Theorem 2.3, X is simply connected. This conjecture of Hartshorne's can be rephrased as a statement about vector bundles on projective space. The connection is via the *Serre construction*, which establishes a relationship between locally complete intersections of codimension 2 and rank 2 vector bundles, generalizing the well-known correspondence between divisors and line bundles. Applied to projective space $\mathbb{P}^N$ of dimension $N \geq 3$ this says the following: Assume that $X \subset \mathbb{P}^N$ is a codimension 2 manifold which is *sub-canonical*. Then there exists a rank 2 vector bundle E on $\mathbb{P}^N$ and a section $s \in H^0(\mathbb{P}^N, E)$ such that $X = \{s = 0\}$. Here sub-canonical means that the canonical line bundle $K_X = \det T_X$, the determinant of the cotangent bundle, is the restriction of a line bundle on $\mathbb{P}^N$, i.e. it is of the form $K_X = \mathcal{O}_X(k) = \mathcal{O}_{\mathbb{P}^N}(k)|_X$ where $\mathcal{O}_{\mathbb{P}^N}(k) = \mathcal{O}_{\mathbb{P}^N}(1)^{\otimes k}$ and $\mathcal{O}_{\mathbb{P}^N}(1)$ is the hyperplane bundle on $\mathbb{P}^N$. Since the canonical bundle can, by the adjunction formula, be written as $K_X = \det N_{X/\mathbb{P}^N} \otimes K_{\mathbb{P}^N}$, where $N_{X/\mathbb{P}^N}$ is the normal bundle of X in $\mathbb{P}^N$, Serre's construction can also be rephrased in more geometric terms as follows: if the determinant of the normal bundle of a codimension 2 submanifold X of $\mathbb{P}^N$ can

be extended to the surrounding projective space, then so can the normal bundle itself. Now it follows from the fact that the restriction map defines an isomorphism on the Picard groups, that every codimension 2 manifold in $\mathbb{P}^N$ is sub-canonical, provided $N \geq 5$. In particular, the Serre construction can be applied in this case and we obtain that $X = \{s = 0\}$ for some section s of a rank 2 bundle E on $\mathbb{P}^N$. Using that the vector bundle E is uniquely determined, one can argue that X is a complete intersection if and only if E is a decomposable rank 2 bundle, i.e. a sum of two line bundles. Hence the Hartshorne conjecture can be restated in terms of vector bundles as

Conjecture 2.5 *If $N \geq 7$, then every rank 2 bundle E on $\mathbb{P}^N$ decomposes, i.e. is the sum of two line bundles.*

It should be noted that there are also no known indecomposable rank 2 bundles (apart from the Tango bundle on $\mathbb{P}^5$ in characteristic 2) for $N = 5, 6$. Barth and Van de Ven [29, Theorem I] proved the asymptotic version of this conjecture. To describe this, we fix a sequence of linear embeddings $i_N : \mathbb{P}^N \to \mathbb{P}^{N+1}$. We say that a vector bundle E on $\mathbb{P}^N$ extends to $\mathbb{P}^{N+1}$ if there exists a vector bundle E' on $\mathbb{P}^{N+1}$ such that $E = i_N^*(E')$. Similarly, we say that a submanifold $X \subset \mathbb{P}^N$ extends to $\mathbb{P}^{N+1}$ (as a submanifold) if there exists a submanifold $X' \subset \mathbb{P}^{N+1}$ with $i_N(X) = i_N(\mathbb{P}^N) \cap X'$.

Theorem 2.6 (Babylonian Tower Theorem, Barth–Van de Ven) *A rank 2 vector bundle E on $\mathbb{P}^N$ which extends to $\mathbb{P}^M$ for all $M \geq N$, splits into the sum of two line bundles.*

By the Serre construction this implies the

Corollary 2.7 *A smooth codimension 2 submanifold $X \subset \mathbb{P}^N$ which extends, as a submanifold, to $\mathbb{P}^M$, for all $M \geq N$, is a complete intersection of two hypersurfaces.*

The paper by Barth and Van de Ven [29] contains many techniques which later became standard tools in the study of vector bundles, such as the idea to investigate vector bundles on $\mathbb{P}^N$ by studying their restriction to lines $L \subset \mathbb{P}^N$. In their article, Barth and Van de Ven also rediscover the Serre construction, as did Horrocks, Hartshorne and Grauert and Mülich on different occasions.

One area where Wolf Barth's influence was, and is, extraordinary is the classification of vector bundles. Classifying vector bundles means, as is typically the case with classification problems in algebraic geometry, that one has to construct a *moduli space*, whose geometric properties one aims to understand. To be able to construct moduli spaces it is necessary to restrict to *stable* objects. As suitable notions of stability one can use *Mumford-Takemoto stability* or, alternatively, *Gieseker stability*. A vector bundle F of rank r on $\mathbb{P}^N$ is called *Mumford-Takemoto semi-stable* if for every non-trivial coherent subsheaf $E \subset F$ the inequality

$$c_1(E)/\mathrm{rank}(E) \leq c_1(F)/\mathrm{rank}(F)$$

holds. Here $c_1 = c_1(E) \in H^2(\mathbb{P}^N, \mathbb{Z}) = \mathbb{Z}$ denotes the first Chern class (which one can interpret as an integer) of a coherent sheaf E. The bundle F is called (properly)

stable if strict inequality always holds. Stable vector bundles are subject to topological constraints: Schwarzenberger [66, Theorem 7] showed that the Chern classes c_i, $i = 1, 2$, of a stable rank 2 bundle F on $\mathbb{P}^2$ satisfy the inequality $c_1^2 < 4c_2$. In [7, Corollary 2] Barth proved that this inequality also holds for stable bundles on any $\mathbb{P}^n$, $n \geq 2$. In this paper Barth also demonstrated the power of the, afterward universally used, method of studying a vector bundle on $\mathbb{P}^N$ by its restriction to lines. If F is a rank 2 vector bundle then one can, after twisting with a suitable line bundle, assume that its first Chern class $c_1(F) \in \{0, -1\}$. If $c_1(F) = 0$ then, by the Grauert-Mülich theorem [43, §6, Satz 2], later extended to higher rank by Spindler, a stable rank 2 bundle F splits as the sum of two trivial line bundles on a general line L, namely $F|_L = \mathcal{O}_L \oplus \mathcal{O}_L$. The lines where this is not the case, i.e. where $F|_L = \mathcal{O}_L(k) \oplus \mathcal{O}_L(-k)$ for some $k > 0$ define a divisor, and hence a curve $C = C(F)$ in the dual projective plane. These lines were called *jumping lines* by Barth and $C(F)$ was called the *curve of jumping lines*. Studying the curves of jumping lines and related objects became a central theme of the theory of vector bundles in the years after Barth's paper.

Constructing moduli spaces and understanding their properties is a crucial question in any classification problem in algebraic geometry. The fundamental theory for moduli spaces of vector bundles on higher-dimensional varieties was developed by Maruyama [54, 55] and Gieseker [41] (for surfaces). It was Barth in [8] who gave a beautiful concrete construction of the moduli spaces $M_{0,c_2}^2(\mathbb{P}^2)$ of rank 2 bundles on $\mathbb{P}^2$ with even first Chern class. The main geometric result is that every stable rank 2 vector bundle F on $\mathbb{P}^2$ with even first Chern class, which we can assume to be $c_1(F) = 0$, can be reconstructed from its curve of jumping lines $C(F)$, which is a plane curve of degree $n = c_2(F)$, together with an ineffective theta characteristic θ, i.e. a root of the canonical bundle without sections, on $C(F)$. The other important contribution of Barth in this paper is, that he made, for the first time, systematic use of monads in the study of moduli of vector bundles. The concept of monad was introduced by Horrocks [46], who considered monads as the elementary building blocks for constructing vector bundles. A *monad* is simply a complex

$$A \xrightarrow{f} B \xrightarrow{g} C$$

where A, B and C are vector bundles, f is a monomorphism of vector bundles, g is an epimorphism such that $g \circ f = 0$. The cohomology of this complex

$$F = \ker(g)/\mathrm{im}(f)$$

is then a vector bundle. The advantage of monads is that one can often take A, B and C to be very simple bundles and in this way reduce a moduli problem to a linear algebra question. For stable rank 2 vector bundles with $c_1(F) = 0$ and $c_2(F) = n$, Barth showed that every such bundle can be realized as the cohomology of a monad of the form

$$n\mathcal{O}_{\mathbb{P}^2} \xrightarrow{a} n\Omega_{\mathbb{P}^2}^1(1) \xrightarrow{c} (n-2)\mathcal{O}_{\mathbb{P}^2}(1)$$

where $\Omega_{\mathbb{P}^2}^1$ is the cotangent bundle on $\mathbb{P}^2$. Moreover, for a given sheaf E, the sheaf $E(k) = E \otimes \mathcal{O}_{\mathbb{P}^2}(k)$ denotes the twist of E by $\mathcal{O}_{\mathbb{P}^2}(k)$, and nE is the n-fold direct sum

of E. The crucial ingredient here is the construction of the map a (which in turn determines the map c). It is given by the middle part $H^1(\mathbb{P}^2, F(-2)) \otimes H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \to H^1(\mathbb{P}^2, F(-1))$ of the cohomology module $\bigoplus_k H^1(\mathbb{P}^2, F(k))$ over the homogeneous coordinate ring $\bigoplus_k H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k))$. Via canonical identifications one can then view a as a *net of quadrics* in $\mathbb{P}^2$ and this net of quadrics determines the pair $(C(F), \theta)$ and vice versa. In particular, $C(F)$ is the discriminant of the net of quadrics and the theta-characteristic encodes how the net can be reconstructed from its discriminant.

Having thus reduced the classification problem of stable rank 2 bundles with even first Chern class to a linear algebra problem, Barth was able to show in [8, p. 83]:

Theorem 2.8 (Barth) *The moduli spaces $M_{0,n}^2(\mathbb{P}^2)$ of stable rank 2 bundles with first Chern class $c_1(F) = 0$ and second Chern class $c_2(F) = n$ are irreducible rational manifolds of dimension $4n - 3$.*

It is worth mentioning that at that time mathematicians still thought it conceivable that all moduli spaces in algebraic geometry are (uni-)rational. It was only at the beginning of the 1980's that Freitag, Tai and Mumford proved that the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g is of general type for $g \geq 7$. This was the first time that such a phenomenon was observed. As it turned out, Barth's proof gave unirationality rather than rationality of $M_{0,n}^2(\mathbb{P}^2)$ and the proof of rationality was finally completed by Maruyama [56]. Barth and Hulek then studied monads more systematically in their paper [19]. At the same time, Beilinson developed his very general approach, leading to the Beilinson spectral sequence [33], thereby linking the classification of vector bundles to derived categories. The geometry of moduli spaces on projective spaces was further pursued and advanced by many authors including Drezet-Le Potier [35], Ellingsrud [36], Forster-Hirschowitz-Schneider [39], Hirschowitz [45], Hulek [48], Le Potier [53], Strømme [67], and many others, cf. also the book by Okonek, Schneider and Spindler [63]. It is now known that all moduli spaces $M_{c_1,c_2}^r(\mathbb{P}^2)$ of stable vector bundles on the projective plane are irreducible and rational. This is, however no longer true on higher-dimensional projective spaces, starting with $\mathbb{P}^3$ [19, Sect. 8].

The late 1970's was also a mathematically very exciting period in other respects. It was at this time that Atiyah and others developed new links between mathematical physics, notably quantum field theory, on the one side, and algebraic and differential geometry on the other side. The first prime example of this is *non-abelian gauge theory* and its connections with vector bundles. Yang-Mills fields can be described as anti-selfdual connections on $SU(2)$-bundles. Imposing asymptotic conditions on these connections at infinity, one obtains self-dual connections on $SU(2)$-bundles on the 4-sphere S^4. At this point Penrose's *twistor theory* comes into play. The twistor space of the 4-sphere S^4 is the complex projective 3-space and the twistor fibration becomes $p : \mathbb{P}^3 \to S^4$. Another interpretation of this map is that $\mathbb{P}^3 = \mathbb{P}(\mathbb{C}^4)$ is the complex projective space and that $S^4 = \mathbb{P}(\mathbb{H}^2)$ is the quaternionic projective line. Identifying $\mathbb{C}^4 \cong \mathbb{H}^2$, the twistor map

$$p : \mathbb{P}^3 = \mathbb{P}(\mathbb{C}^4) \to S^4 = \mathbb{P}(\mathbb{H}^2)$$

is then given by associating to a complex line in $\mathbb{C}^4$ the quaternionic line containing it. Left-multiplication by j moreover gives $\mathbb{P}^3$ a real structure $\sigma : \mathbb{P}^3 \to \mathbb{P}^3$ with $\sigma^2 = -\text{id}$. The fibres of the twistor map $p : \mathbb{P}^3 \to S^4$ are projective lines which are invariant under σ, these are the so-called *real lines*. Now, given a pair (F, d), consisting of an $SU(2)$-bundle F, together with a connection d, one can, by the Atiyah-Ward correspondence [4], consider the pullback $p^*(F)$. Its associated vector bundle E is a $\mathbb{C}^2$-bundle. The connection d defines an almost complex structure on E which, since d is assumed to be anti-selfdual, turns out to be integrable. In other words, E is a holomorphic (and by GAGA thus algebraic) rank 2 vector bundle on $\mathbb{P}^3$. Moreover, E carries a real structure and, in particular, E is trivial on all real lines. It is then not hard to see that E is stable. Thus one can identify self-dual $SU(2)$-connections on S^4 with certain stable algebraic rank 2 vector bundles on $\mathbb{P}^3$ with a real structure, the so-called *instanton bundles*. Under this construction, the instanton number of the connection becomes the second Chern class of E. These vector bundles satisfy the additional property that the cohomology groups $H^1(\mathbb{P}^3, E(-2))$ vanish. This is a translation of the fact that certain differential equations admit only trivial solutions. Now every stable rank 2-bundle E on $\mathbb{P}^3$ with this additional property (these are the so-called *mathematical instanton bundles*) are given by a result of Barth and Hulek [19, Sect. 7] by a monad of the form

$$n\mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{a} (2n+2)\mathcal{O}_{\mathbb{P}^2} \xrightarrow{{}^t a \circ J} n\mathcal{O}_{\mathbb{P}^2}(1)$$

where J is the standard symplectic form on the vector space $\mathbb{C}^{2n+2}$. The map a is then nothing but an $n \times (2n+2)$-matrix with linear entries in the homogeneous coordinates of the projective plane. This can be viewed as a triple of matrices, and is often also called a *Kronecker module*. The famous Atiyah-Drinfeld-Hitchin-Manin (ADHM) correspondence [3] now says that all instanton bundles are given by monads of the above form satisfying a reality condition.

We have already discussed that, conjecturally, all rank 2 bundles on $\mathbb{P}^N$ split for $N \geq 6$. Whereas there are plenty of indecomposable rank 2 vector bundles on $\mathbb{P}^2$ and $\mathbb{P}^3$, only very few examples are known for higher-dimensional projective space. Essentially the only known example in characteristic 0 is the famous *Horrocks-Mumford bundle* $F = F_{\text{HM}}$ on $\mathbb{P}^4$. This is a stable rank 2 vector bundle on $\mathbb{P}^4$ with Chern classes $c_1(F) = 5$ and $c_2(F) = 10$. The Horrocks-Mumford bundle is a beautiful mathematical object which is distinguished by its symmetry group N_5 of order 15.000. The group N_5 is the semi-direct product of the Heisenberg group of level 5, a group of order 125, and the binary icosahedral group $SL(2, \mathbb{Z}/5\mathbb{Z})$. This bundle was constructed by Horrocks and Mumford in [47] by means of a monad, but it can also be obtained via the Serre construction from an abelian surface A embedded as a surface of degree 10 into $\mathbb{P}^4$. Barth, together with Hulek and Moore, studied the many beautiful aspects of the geometry of this vector bundle in detail. The paper [20] contains a complete classification of all *Horrocks-Mumford surfaces*, i.e. all surfaces obtained as zero-sets $A_s = \{s = 0\} \subset \mathbb{P}^4$ of sections $0 \neq s \in H^0(\mathbb{P}^4, F)$. These surfaces are $(1, 5)$-polarized abelian surfaces and their degenerations and this establishes a connection with (modular) compactifications of moduli spaces of abelian surfaces.

2.3 Algebraic Surfaces

Barth's interest in algebraic surfaces originated from his time at Leiden and became the major focus of his work since the 1980's. Three main lines of research can be identified in his work in this area:

- fundamental work in the theory of algebraic surfaces [2, 21–24]
- work on abelian surfaces and Kummer surfaces [10–12, 25, 30, 31] motivated originally by their connection with the Horrocks Mumford bundle [9, 20]
- surfaces with many symmetries and with special geometry: see Sect. 2.4 [13, 15, 16, 26–28, 31].

Algebraic Surfaces—The Classification   One of the central tasks of any mathematical theory is the classification of its objects. A systematic study of algebraic surfaces had been initialized by Max Noether, who as Wolf Barth, was an Ordinarius at Universität Erlangen. The classification of complex projective surfaces was accomplished by Federigo Enriques. His works culminated in the book *Le superficie algebriche* [38] published in 1949, three years after his death. The classification was extended to non-algebraic compact surfaces by Kunihiko Kodaira in the late 60's. One of the key points in the approach of Kodaira is the study of elliptic fibrations of compact surfaces. Barth, together with Gerhard Angermüller, presented in [2] a complete classification of singular elliptic fibres on Enriques surfaces. As is typical for Barth's research, this paper contains a considerable list of explicit examples of Enriques surfaces and elliptic fibrations, in which fibres from the classification appear.

In the paper [21] with Chris Peters, Barth continued his study of Enriques surfaces turning attention to their automorphisms. The main result of this article was quite surprising: the authors proved that the automorphism group of a generic (in the sense of moduli) Enriques surface is large, in particular infinite, whereas it can be small, in particular finite, for special Enriques surfaces. This is quite non-intuitive, as usually, for example for curves or surfaces of general type, the picture is opposite: large automorphism groups are attached to special, hence rare, varieties. The article, based on the global Torelli theorem for projective K3 surfaces, implies, as a byproduct, a result which Barth surely considered *amusing*. It says that a generic Enriques surface has

- 527 elliptic fibrations
- 67 456 realisations as a double plane in $\mathbb{P}^4$
- 5 396 480 realisations as a sextic surface in $\mathbb{P}^3$ passing doubly through the edges of a tetrahedron and
- 25 903 104 ways to be written as a surface of degree 10 in $\mathbb{P}^5$.

Of course, such sample results, amusing as they might be, are only the tip of the iceberg of Barth's main contribution in the 80's, namely his book *Compact Complex Surfaces* [22] co-authored with Chris Peters and Antonius Van de Ven. This book is the first instance where the aforementioned Enriques-Kodaira classification appears in full details in print in one place. The book contains numerous, up to date at that time, results on surfaces of general type and on K3 and Enriques surfaces, including the global Torelli theorem and the theory of periods. The authors' approach to the

classification theory, based on Iitaka's $C_{2,1}$-conjecture, incidentally reproved in the book, was new and original. The book went to print in 1984. Around the same time Shigefumi Mori introduced a completely new way of classifying algebraic varieties of higher dimensions, known nowadays as the Minimal Model Programme. Although aimed at higher-dimensional birational geometry, the minimal program also opens up a new view on surface classification. This is reflected in the second edition of the book, co-authored by Hulek, which appeared in 2004 [23]. It is considerably enlarged and reflects the developments of two decades, further including Reider's results and the ensuing improvement of the treatment of pluricanonical maps, as well as the theory of Donaldson and Seiberg-Witten invariants. The book serves still as a standard text for compact complex surfaces.

At the Roots of Algebraic Geometry—Equations Defining Algebraic Varieties
Barth's study of abelian varieties was strongly influenced by the series of articles by David Mumford [58–60] on equations defining abelian varieties. Mumford's approach is highly abstract. In fact, his first article contains only one equation (expressed in homogeneous coordinates): that of an elliptic curve in a Hesse pencil. There is no doubt that Barth easily handled and created complex abstract objects and arguments in algebraic geometry. In particular in the later stages of his career, however, his research was guided by the desire to be as explicit as possible. He was intrigued by the question how abstract algebraic varieties can be explicitly described in terms of equations involving homogeneous coordinates. The fascination by specific equations was growing over the years, culminating in the beautiful symmetric constructions described in Sect. 2.4.

Working still on the Horrocks-Mumford bundle, Barth suggested in [9] a way to obtain new rank 2 stable algebraic vector bundles on $\mathbb{P}^4$. He showed that for a generic point $P \in \mathbb{P}^4$ the jumping lines (see Sect. 2.2) of F_{HM} passing through P generate a cone with vertex at P over a smooth curve, which is the contact curve of two Kummer surfaces. Barth suggested to reverse the process and to construct vector bundles starting with suitable contact curves. The idea was pursued, among others, by Decker, Narasimhan and Schreyer, but despite these efforts it remains in the legacy of the not yet completed projects envisioned by Wolf Barth.

Adler and van Moerbeke studied in [1] algebraically integrable geodesic flows on $SO(4)$ and related them to affine parts of abelian surfaces in $\mathbb{C}^6$ defined as complete intersections of 4 quadrics. These affine parts come from abelian surfaces embedded in $\mathbb{P}^7$ by complete linear systems of type $(2, 4)$. Inspired by these results, Barth studied in [10] abelian surfaces with $(1, 2)$ polarization and gave a complete description of these surfaces and a description of their moduli space. In a paper dedicated to Friedrich Hirzebruch [11], Barth studied the question whether there exist other abelian surfaces, besides those discovered by Adler and van Moerbeke, which might be related to integrable Hamiltonian systems. He studied explicit equations of such potential abelian surfaces and their symmetries imposed by Heisenberg groups and arrived at the conclusion that among abelian surfaces with general moduli no such surfaces exist.

Wolf Barth continued his studies of abelian surfaces defined by quadratic equations in [12]. There he considered principally polarized abelian surfaces A embedded

in $\mathbb{P}^8$ by the third power of the theta divisor. By a result of Kempf [50] it was known that the homogeneous ideal of such surfaces is generated by forms of degree 2 and 3. Barth showed that quadrics suffice to generate the *ideal sheaf* if and only if the polarized abelian surface A is not a product of two elliptic curves. This paper is yet another example of Barth's interest in concrete equations. He provides explicit equations of quadrics cutting out $A \subset \mathbb{P}^8$.

If A is an abelian surface, then its quotient $X = A/(-1)_A$ by the (-1)-involution is by definition a *Kummer surface*. Its desingularization $\widetilde{X}$, the *smooth Kummer surface* of A, is then a $K3$ surface which contains 16 skew smooth rational curves (corresponding to the half-periods of A). It is a result of Nikulin [62] that, conversely, every Kähler K3 surface with 16 skew smooth rational curves arises in this way as a smooth Kummer surface. Barth was interested in finding such surfaces with many skew rational curves in *three-space*—always with an eye towards concrete realisations of surfaces. In joint work with Nieto he rediscovered in [25] the smooth Kummer surfaces found by Traynard [68]. These contain 16 skew lines and they are associated with abelian surfaces of type $(1, 3)$. It was his idea that one should be able to generalize the construction in order to obtain also smooth Kummer surfaces in $\mathbb{P}^3$ with 16 skew *conics* (instead of lines). This was in fact possible and the surfaces constructed in this way turn out to be very interesting as they contain a surprisingly high number of conics in total [31]. It was shown later [32] that it is even possible to obtain 16 skew smooth rational curves of *any* given degree.

Visualizations of Algebraic Surfaces—From the First Steps to the Imaginary Exhibition In addition to his theoretical work, Barth had a great interest in visualizations of algebraic surfaces. He had a fascination for the models of algebraic surfaces that were produced in the 19th century (made in plaster, sometimes in wood), and he loved them both for the geometric insight they could provide as well as from a purely esthetical point of view. His contribution (with Horst Knörrer) to the volume *Mathematical Models* [24] shows his ample knowledge and his appreciation of these kinds of visualizations.

With the availability of ever more powerful computers, it was his idea that given the polynomial equation $f(x, y, z) = 0$ of an algebraic surface in three-space, it should be possible to generate convincing images of the (real) surface (i.e. of the zero set of f in $\mathbb{R}^3$). The crucial breakthrough in this direction was achieved in the diploma thesis by Stephan Endrass (1992), supervised by Barth. In this thesis, the program `surf` was developed (on a Cadmus workstation), which was able to produce impressive images of algebraic surfaces from their equations. When Barth had constructed his now famous sextic surface [13] (see Sect. 2.4), it was a great moment for him to actually *see* the surface after he had found it by theoretical means. The program `surf` was extended and transferred to other systems during the 90s in Barth's group in Erlangen. Since 2000, further development has been done by the algebraic geometry group in Mainz, and the program is still in constant use nowadays via the GUI program `surfer` (https://imaginary.org/de/program/surfer), which relies on the `surf` kernel for its computations. So we can enjoy the impressive images of algebraic surfaces in the Imaginary project (https://imaginary.org) thanks to Barth's early interest in visualizations. Of course, Barth's sextic surface deserves its prominent place in the exhibition.
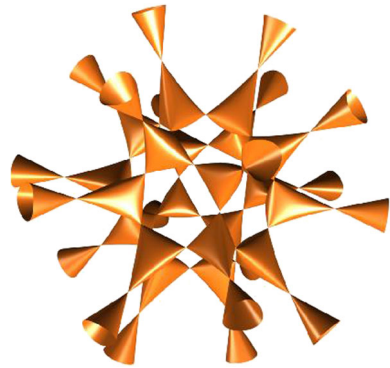
2.4 Symmetries

In the last 20 years of his research activity Wolf Barth got very much interested in the use of symmetries in algebraic geometry. He was fascinated by the beauty of the surfaces that one can produce using symmetries. In particular, he investigated surfaces with many rational curves (resp. singularities obtained by contracting configurations of such curves) and constructed numerous examples.

Recall that an A_1 (resp. an A_2) point of a surface is a singularity locally given by the equation

$$x \cdot y - z^2 = 0$$

(resp. $x \cdot y - z^3 = 0$). Such a point can be locally obtained as the quotient of the two-dimensional complex unit ball by an appropriate $\mathbb{Z}/2\mathbb{Z}$ (resp. $\mathbb{Z}/3\mathbb{Z}$) action. An A_1 (resp. A_2) singularity is also called a *node* (resp. a *cusp*).

A foundational work of W. Barth on this subject is the paper of 1996, [13], where he discovered what is now called *Barth's sextic* (see Fig. 1). It is a surface of degree 6 in three dimensional complex projective space with 65 nodes.

In this paper he considers the symmetry group I of the icosahedron in euclidean three dimensional space $\mathbb{R}^3$. It is well-known that this is isomorphic to the alternating group A_5 and it acts on the ring of coordinates $\mathbb{R}[x, y, z]$. The ring of invariant polynomials is generated by polynomials of degree 2, 6 and 10 respectively. The polynomial of degree 2 can be written as $x^2 + y^2 + z^2$ and the other two invariants of degree 6 and 10 can be written in the following forms (they were already known to Goursat [42]):

$$Q(x, y, z) = \left(\tau^2 x^2 - y^2\right)\left(\tau^2 y^2 - z^2\right)\left(\tau^2 z^2 - x^2\right)$$
$$R(x, y, z) = \left(x^2 - \tau^4 y^2\right)\left(y^2 - \tau^4 z^2\right)\left(z^2 - \tau^4 x^2\right)(x + y + z)(x + y - z)$$
$$\times (x - y + z)(x - y - z)$$

where $\tau = (1 + \sqrt{5})/2$ is the golden ratio. Combining these two equations with the equation of the three dimensional sphere $S : x^2 + y^2 + z^2 - 1 = 0$, Barth obtains two families of symmetric surfaces invariant under the action of I. Barth's sextic belongs

Fig. 2 Barth's decic surface
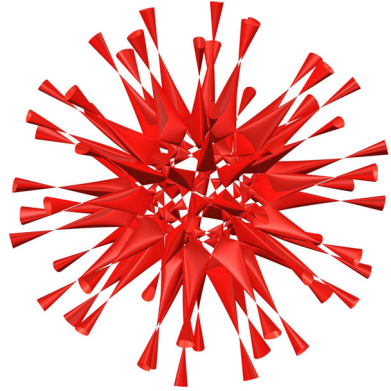
to the family of surfaces with equation in affine coordinates

$$f_\alpha : \ Q(x, y, z) - \alpha \big(x^2 + y^2 + z^2 - 1\big)^2 = 0.$$

For generic choice of the complex parameter α the surface has 45 nodes that belong to special lines of the icosahedron, the *mid lines*, as Barth called them, that connect two opposite mid points of edges. Then Barth imposes geometric conditions in a clever way to get an extra orbit of 20 nodes for the action of I. He finally obtains that by taking the parameter $\alpha = (2\tau + 1)/4$ the surface has in total 65 nodes. A very similar construction was then used by Barth to combine the degree ten polynomial $R(x, y, z)$ and the equation of the sphere S to obtain *Barth's decic* (Fig. 2) that has 345 nodes. As in the previous case, the singular points are located on special families of lines and planes related to the geometric properties of the icosahedron (the 10 planes through the origin, parallel to the twenty faces of the icosahedron).

In the paper [57] of 1984, Miyaoka gave a bound for the maximum number of nodes that a surface of given degree in complex projective three-space can have. The bound is 66 for a surface of degree 6 and 360 for a surface of degree 10. The two examples produced by Barth are the best known examples so far in these two degrees. In particular the discovery of Barth's sextic was quite surprising, since it was thought that no surface of degree 6 surface with 65 nodes could exist: fifteen years before, Catanese and Ceresa [34] had erroneously claimed that 64 was the maximum number of ordinary double points for a sextic surface. Soon after Barth's discovery, Jaffe and Ruberman [49] showed, using coding theory, that in fact 65 is the maximum possible number of nodes for surfaces of degree 6. The problem of the maximum number of nodes in degree six was thus solved!

Barth's sextic remains until now an exceptional and beautiful example of how symmetries are a powerful tool in attacking problems in algebraic geometry.

The lower bound of 345 given by Barth's decic also remains to our knowledge the best so far. It is still not known whether Miyaoka's bound of 360 nodes is possible or not for a surface of degree 10 in complex three-dimensional projective space. The idea of Barth of considering surfaces with many symmetries to attack the problem of finding surfaces with many nodes, paved the way to the discovery of more *world record surfaces*, as Barth termed them, and greatly influenced several mathematicians

working on the subject, in particular Barth's PhD students: S. Endrass and A. Sarti. They found respectively a surface of degree 8 with 168 nodes in 1997, [37], and a surface of degree 12 with 600 nodes in 2001, [65]. These numbers of nodes approach the bounds of Miyaoka which are respectively 174 and 645 and are so far the best known examples. Both the surfaces are very symmetric, they have respectively the symmetries of some extension of order two of the dihedral group D_8 and of the bipolyhedral icosahedral group of order 7200.

The ideas of Barth affected the works of several other mathematicians such as D. van Straten and his students in Mainz, as well as S. Cynk in Crakow. His ideas also influenced the study of other difficult and classical problems in algebraic geometry such as the study of the maximum number of lines on projective surfaces. Around 2001 Barth found a smooth quintic surface with 75 lines that was later described in the paper [69] (the author, as he said, followed a suggestion by W. Barth).

In a paper of 2003 W. Barth and A. Sarti [28] studied the relation of the symmetric surfaces of the PhD thesis of A. Sarti, which had been conducted under the supervision of W. Barth, and K3 surfaces. In particular, they show that the minimal resolution of the quotient of the surface with 600 nodes by the bipolyhedral icosahedral group is a K3 surface with maximal Picard number, namely 20. Wolf Barth had always had a special interest in the beauty of the geometry of K3 surfaces and in his paper [16] he studied divisible sets of rational curves on K3 surfaces. More precisely, let $L_1, \ldots, L_k$, $k \geq 1$, be smooth disjoint rational curves on a K3 surface, Nikulin in [61] showed that if this set is *2-divisible*, i.e. the sum $L_1 + \cdots + L_k$ is equivalent to two times a divisor in the Nèron-Severi group, then $k = 8$ or $k = 16$ and 16 disjoint rational curves are always an even set. This is not the case for eight disjoint rational curves. In [16] Barth characterizes even sets of 8 rational curves on the most geometric projective models of K3 surfaces, such as double covers of $\mathbb{P}^2$ ramified along a sextic curve, quartics in $\mathbb{P}^3$ and double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified along a curve of bi-degree $(4, 4)$. The results provide a geometric expression of the lattice-theoretical results of Nikulin and they deeply affected the work of several other mathematicians, such as B. van Geemen and his students in Milan, working in particular on symplectic involutions on K3 surfaces (which are strictly related to the existence of such even sets).

Wolf Barth also investigated sets of A_2 singularities on surfaces. In general, techniques of enumerative geometry are not sensitive enough to give precise statements on the number of A-D-E points on degree d surfaces in $\mathbb{P}^3$. In the case of quintics this problem was circumvented by Beauville, who associated to a surface with μ nodes a linear code in $\mathbb{F}_2^\mu$. Each word of the code in question is given by a so-called *even set of nodes*. By definition, a set of nodes $P_1, \ldots, P_k$ on a surface X is even if and only if the local $\mathbb{Z}/2\mathbb{Z}$-quotient structure around the k nodes can be defined globally, i.e. when there exists a double cover Y of X branched only at the points $P_1, \ldots, P_k$ with Y smooth along the branch locus. Once the code is defined, one can use coding theory to arrive at bounds on the number of singularities and constraints on their configuration. In the papers [14, 15] Barth generalized the above notion to surfaces with A_2 singularities. In this case one arrives at a ternary code. Barth used this generalization to show that a cuspidal quartic surface can carry at most eight singularities and every K3-surface with nine A_2 points arises as a 3:1 quotient of a

complex torus. This result generalizes the classical and well known construction of Kummer surfaces. Furthermore, he classified all tori occurring in the construction and gave explicit examples of quartics with eight A_2 points. Finally, the papers [26, 27], joint with S. Rams, contain sharp effective bounds on the minimal weights of ternary codes given by low-degree surfaces and the computation of codes for certain surfaces.

It should be pointed out that Barth's work on divisible sets of singularities played an important role in the project of classifying fundamental groups of open Enriques and K3 surfaces which has recently been carried out by J.H. Keum and D.Q. Zhang, whereas his published and unpublished examples appear in various contexts in algebraic geometry (e.g. Barth's quintics surface with 75 lines turns out to be the smooth quintic with the highest Picard number known so far—see [64]).

3 Wolf Barth—The Teacher

Over the years, Wolf Barth developed an ever growing interest in teaching mathematics, which became apparent in a number of activities that are non-routine among top research mathematicians: for instance, when he found that the mathematics education regularly provided for elementary and middle school teachers was not up to the standards that he aimed at, rather than theorizing about this fact, or putting blame on others, he volunteered to restructure the courses in question and to teach them himself for a number of years to come—an instance of the hands-on approach he employed in such situations.

As far as standard courses such as Analysis, Linear Algebra, Abstract Algebra and Complex Analysis are concerned, he developed his unique way of attacking the subjects, enriched by written manuscripts for the students which were not just compilations of the known textbook approaches. One such example is his *Lineare Algebra* text, which later became a Springer book coauthored by Peter Knaber [51].

Also, it was his intention to convey to students his fascination for classical topics in geometry. As an example, he developed and taught a course on *circles* [17]. In his motivation one can see parallels with his activities in the environment protection movement: Barth knew that classical geometry lost its place at universities because it was considered by the majority of academia mathematicians as too elementary. Barth himself considered numerous beautiful theorems related to circles as members of a rare species which deserved protection from vanishing from mankind's intellectual legacy. His attitude to projective geometry was very similar.

Over the years Barth had been continuously involved in the training of the next generation of algebraic geometers. His first student, Wilfred Hulsbergen, obtained his PhD in 1976, the last student of Barth, Thomas Werner, graduated in 2012. Four of the authors of this obituary were Barth's students and the remaining one, Rams, obtained his habilitation with Barth in Erlangen.

The last of the named authors came from Cracow to Erlangen, exactly on the day of German reunification, as an exchange student. He encountered Barth and became his graduate student. It was the time of rapid political changes and growing hopes throughout Europe. All of a sudden, Eastern Europe became part of the Free

Prof. Dr. Wolf Barth and his students at the conference *Groups and Algebraic Geometry*, Erlangen, April 2011. (From left to right: T. Szemberg, Th. Bauer, A. Sarti, W. Barth, K. Hulek, S. Rams)

World. Direct scientific exchange of ideas and people was possible and it was generously supported by various organizations including DFG. Barth successfully applied for considerable grants supporting the library of the Institute of Mathematics of the Jagiellonian University and for four years he coordinated a staff exchange program with Cracow. He visited Cracow himself twice. His visits and the visits of Polish young mathematicians, including Sławomir Cynk, Zbigniew Jelonek and Piotr Tworzewski resulted in transplanting modern algebraic geometry to Cracow, which nowadays is, next to Warsaw, the strongest centre of algebraic geometry in Poland. This development would have been hardly possible without Barth's support and engagement.

Barth also had important connections to Italy. At the end of the 1990's, Graziano Gentili, who at that time was the Head of the *Scuola Matematica Interuniversitaria* invited Wolf Barth to teach a course in algebraic geometry at the Perugia summer school. This is a very famous Italian summer school aiming to prepare young Italian and foreign students to work on a PhD thesis. Barth gave the algebraic geometry course both in 1997 and in 2000, and his lectures were highly appreciated. The fourth named author met Barth at the school in 1997 and then started a PhD thesis in Erlangen with him. Over the years Barth became a good friend of Gentili, whom he invited several times to Erlangen.

* * *

Wolf Barth had a lively personality. He was very much interested in politics, history and contemporary events. He rejected any form of xenophobia and he was actively involved in the protection of the environment. He loved every kind of beauty (flowers, art, music, and all geometric shapes). Material things were not important to him. He loved his family, his children and he was loved in return. He was a brilliant

mathematician, passionate for algebraic geometry where he obtained fundamental results and wrote a foundational book. His famous sextic surface, the Barth surface, is one of the icons of algebraic surfaces. To all who came in contact with him he communicated his great passion and enthusiasm for mathematics.

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