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Noncommutative multi-instantons on $\mathbb{R}^{2n} \times S^2$

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Abstract

Generalizing self-duality on $\mathbb{R}^2 \times S^2$ to higher dimensions, we consider the Donaldson–Uhlenbeck–Yau equations on $\mathbb{R}^{2n} \times S^2$ and their noncommutative deformation for the gauge group U(2). Imposing SO(3) invariance (up to gauge transformations) reduces these equations to vortex-type equations for an Abelian gauge field and a complex scalar on $\mathbb{R}^{2n}_\theta$. For a special S^2-radius R depending on the noncommutativity θ we find explicit solutions in terms of shift operators. These vortex-like configurations on $\mathbb{R}^{2n}_\theta$ determine SO(3)-invariant multi-instantons on $\mathbb{R}^{2n}_\theta \times S^2_R$ for $R = R(\theta)$. The latter may be interpreted as sub-branes of codimension $2n$ inside a coincident pair of noncommutative Dp-branes with an S^2 factor of suitable size.

1. Introduction

Noncommutative deformation is a well established framework for stretching the limits of conventional (classical and quantum) field theories [1,2]. On the nonperturbative side, all celebrated classical field configurations have been generalized to the noncommutative realm. Of particular interest thereof are BPS configurations, which are subject to first-order nonlinear equations. The latter descend from the 4d Yang–Mills (YM) self-duality equations and have given rise to instantons [3], monopoles [4] and vortices [5], among others. Their noncommutative counterparts were introduced in [6,7] and [8], respectively, and have been studied intensely for the past five years (see [9] for a recent review).

String/M theory embeds these efforts in a higher-dimensional context, and so it is important to formulate BPS-type equations in more than four dimensions. In fact, noncommutative instantons in higher dimensions and their brane interpretations have recently been considered in [10–12]. Yet already 20 years ago, generalized self-duality equations for YM fields in more than four dimensions were proposed [13,14] and their solutions investigated, e.g., in [14,15]. For $\mathrm{U}(k)$ gauge theory on a Kähler manifold these equations specialize to the Donaldson–Uhlenbeck–Yau (DUY) equations [16,17]. They are the natural analogues of the 4d self-duality equations.

In this Letter we generalize the DUY equations to the noncommutative spaces $\mathbb{R}^{2n}_\theta \times S^2$ and construct explicit $\mathrm{U}(2)$ multi-instanton solutions even though these equations are not integrable. The key lies in a clever ansatz for

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the gauge potential, due to Taubes [5], which we generalize to higher dimensions and to the noncommutative setting. This SO(3)-invariant ansatz reduces the U(2) DUY equations to vortex-type equations on $\mathbb{R}^{2n}_\theta$. For $n=1$ the latter are the standard vortex equations on $\mathbb{R}^2_\theta$, while for $n=2$ they are intimately related to the Seiberg–Witten monopole equations on $\mathbb{R}^4_\theta$ [18].

2. Donaldson–Uhlenbeck–Yau equations on $\mathbb{R}^{2n}_\theta \times S^2$

2.1. *Manifold* $\mathbb{R}^{2n}_\theta \times S^2$

We consider the manifold $\mathbb{R}^{2n} \times S^2$ with the Riemannian metric

$$\mathrm{d}s^2 = \sum_{\mu,\nu=1}^{2n} \delta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + R^2\big(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\big) = \sum_{i,j=1}^{2n+2} g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{2.1}$$

where $x^1,\dots,x^\mu,\dots,x^{2n}$ are coordinates on $\mathbb{R}^{2n}$ while $x^{2n+1}=\vartheta$ and $x^{2n+2}=\varphi$ parametrize the standard two-sphere S^2 with constant radius R, i.e., $0\leqslant\varphi\leqslant 2\pi$ and $0\leqslant\vartheta\leqslant\pi$. The volume two-form on S^2 reads

$$\sqrt{\det(g_{ij})}\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi =: \omega_{\vartheta\varphi}\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi = \omega \quad\Longrightarrow\quad \omega_{\vartheta\varphi} = -\omega_{\varphi\vartheta} = R^2\sin\vartheta. \tag{2.2}$$

The manifold $\mathbb{R}^{2n}\times S^2$ is Kähler, with local complex coordinates $z^1,\dots,z^n, y$ where

$$z^a = x^{2a-1} - \mathrm{i}x^{2a}, \qquad \bar z^{\bar a} = x^{2a-1} + \mathrm{i}x^{2a} \quad \text{with } a=1,\dots,n \tag{2.3}$$

and

$$y = \frac{R\,\sin\vartheta}{(1+\cos\vartheta)}\exp(-\mathrm{i}\varphi), \qquad \bar y = \frac{R\,\sin\vartheta}{(1+\cos\vartheta)}\exp(\mathrm{i}\varphi), \tag{2.4}$$

so that $1+\cos\vartheta = \frac{2R^2}{R^2+y\bar y}$. In these coordinates, the metric takes the form[1]

$$\mathrm{d}s^2 = \delta_{a\bar b}\,\mathrm{d}z^a\,\mathrm{d}\bar z^{\bar b} + \frac{4\,R^4}{(R^2+y\bar y)^2}\,\mathrm{d}y\,\mathrm{d}\bar y \tag{2.5}$$

with $\delta_{a\bar a} = \delta^{a\bar a} = 1$ (other entries vanish), and the Kähler two-form reads

$$\Omega = -\frac{\mathrm{i}}{2}\left\{\delta_{a\bar b}\,\mathrm{d}z^a\wedge\mathrm{d}\bar z^{\bar b} + \frac{4\,R^4}{(R^2+y\bar y)^2}\,\mathrm{d}y\wedge\mathrm{d}\bar y\right\} = -\frac{\mathrm{i}}{2}\delta_{a\bar b}\,\mathrm{d}z^a\wedge\mathrm{d}\bar z^{\bar b} + \omega_{\vartheta\varphi}\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi. \tag{2.6}$$

For later use, we also note here the derivatives

$$\partial_{z^a} = \frac{1}{2}(\partial_{2a-1} + \mathrm{i}\partial_{2a}), \qquad \partial_{\bar z^{\bar a}} = \frac{1}{2}(\partial_{2a-1} - \mathrm{i}\partial_{2a}), \tag{2.7}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$ for $\mu = 1,\dots,2n$.

Classical field theory on the noncommutative deformation $\mathbb{R}^{2n}_\theta$ of $\mathbb{R}^{2n}$ may be realized in a star-product formulation or in an operator formalism. While the first approach alters the product of functions on $\mathbb{R}^{2n}$ the second one turns these functions f into linear operators $\hat f$ acting on the n-harmonic-oscillator Fock space $\mathcal{H}$.

[1] From now on we use the Einstein summation convention for repeated indices.

The noncommutative space $\mathbb{R}^{2n}_\theta$ may then be defined by declaring its coordinate functions $\hat{x}^1, \ldots, \hat{x}^{2n}$ to obey the Heisenberg algebra relations

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu} \tag{2.8}$$

with a constant antisymmetric tensor $\theta^{\mu\nu}$. The coordinates can be chosen in such a way that the matrix $(\theta^{\mu\nu})$ will be block-diagonal with nonvanishing components

$$\theta^{2a-1\,2a} = -\theta^{2a\,2a-1} =: \theta^a. \tag{2.9}$$

We assume that all $\theta^a \geqslant 0$; the general case does not hide additional complications. For the noncommutative version of the complex coordinates (2.3) we have

$$[\hat{z}^a, \hat{\bar{z}}^{\bar{b}}] = -2\delta^{a\bar{b}}\theta^a =: \theta^{a\bar{b}} = -\theta^{\bar{b}a} \leqslant 0, \quad \text{and all other commutators vanish.} \tag{2.10}$$

The Fock space $\mathcal{H}$ is spanned by the basis states

$$|k_1, k_2, \ldots, k_n\rangle = \prod_{a=1}^{n} \left(2\theta^a k_a!\right)^{-1/2} \left(\hat{z}^a\right)^{k_a} |0\rangle \quad \text{for } k_a = 0, 1, 2, \ldots, \tag{2.11}$$

which are connected by the action of creation and annihilation operators subject to

$$\left[\frac{\hat{\bar{z}}^{\bar{b}}}{\sqrt{2\theta^b}}, \frac{\hat{z}^a}{\sqrt{2\theta^a}}\right] = \delta^{a\bar{b}}. \tag{2.12}$$

We recall that, in the operator realization $f \mapsto \hat{f}$, derivatives of f get mapped according to

$$\partial_{z^a} f \mapsto \theta_{a\bar{b}}[\hat{\bar{z}}^{\bar{b}}, \hat{f}] =: \partial_{\hat{z}^a}\hat{f}, \qquad \partial_{\bar{z}^{\bar{a}}} f \mapsto \theta_{\bar{a}b}[\hat{z}^b, \hat{f}] =: \partial_{\hat{\bar{z}}^{\bar{a}}}\hat{f}, \tag{2.13}$$

where $\theta_{a\bar{b}}$ is defined via $\theta_{b\bar{c}}\theta^{\bar{c}a} = \delta^a_b$ so that $\theta_{a\bar{b}} = -\theta_{\bar{b}a} = \frac{\delta_{a\bar{b}}}{2\theta^a}$. Finally, we have to replace

$$\int_{\mathbb{R}^{2n}} \mathrm{d}^n x\, f \mapsto \left(\prod_{a=1}^{n} 2\pi\theta^a\right) \mathrm{Tr}_{\mathcal{H}}\, \hat{f}. \tag{2.14}$$

Tensoring $\mathbb{R}^{2n}_\theta$ with a commutative S^2 means extending the noncommutativity matrix θ by vanishing entries in the two new directions. A more detailed description of noncommutative field theories can be found in the review papers [2].

2.2. Donaldson–Uhlenbeck–Yau equations

Let M_{2q} be a complex $q = n+1$ dimensional Kähler manifold with some local real coordinates $x = (x^i)$ and a tangent space basis $\partial_i := \partial/\partial x^i$ for $i, j = 1, \ldots, 2q$, so that a metric and the Kähler two-form read $\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ and $\Omega = \Omega_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$, respectively. Consider a rank k complex vector bundle over M_{2q} with a gauge potential $\mathcal{A} = \mathcal{A}_i\,\mathrm{d}x^i$ and the curvature two-form $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ with components $\mathcal{F}_{ij} = \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_j]$. Both $\mathcal{A}_i$ and $\mathcal{F}_{ij}$ take values in the Lie algebra $\mathrm{u}(k)$. The Donaldson–Uhlenbeck–Yau (DUY) equations [16,17] on M_{2q} are

$$*\Omega \wedge \mathcal{F} = 0 \quad \text{and} \quad \mathcal{F}^{0,2} = 0, \tag{2.15}$$

where Ω is the Kähler two-form, $\mathcal{F}^{0,2}$ is the $(0,2)$ part of $\mathcal{F}$, and $*$ is the Hodge operator. In our local coordinates (x^i) we have $q!(*\Omega \wedge \mathcal{F}) = (\Omega, \mathcal{F})\Omega^q = \Omega^{ij}\mathcal{F}_{ij}\Omega^q$ where Ω^{ij} are defined via $\Omega^{ij}\Omega_{jk} = \delta^i_k$. Due to the anti-Hermiticity of $\mathcal{F}$, it follows that also $\mathcal{F}^{2,0} = 0$. For $q = 2$ the DUY equations (2.15) coincide with the anti-self-dual

Yang–Mills (ASDYM) equations

$$*\mathcal{F} = -\mathcal{F} \tag{2.16}$$

introduced in [3].

Specializing now M_{2q} to be $\mathbb{R}^{2n} \times S^2$, the DUY equations (2.15) in the local complex coordinates (z^a, y) take the form

$$\delta^{a\bar{b}}\mathcal{F}_{z^a\bar{z}^{\bar{b}}} + \frac{(R^2 + y\bar{y})^2}{4\,R^4}\mathcal{F}_{y\bar{y}} = 0, \qquad \mathcal{F}_{\bar{z}^{\bar{a}}\bar{z}^{\bar{b}}} = 0 \quad \text{and} \quad \mathcal{F}_{\bar{z}^{\bar{a}}\bar{y}} = 0, \tag{2.17}$$

where $a, b = 1, \dots, n$. Using formulae (2.4), we obtain

$$\mathcal{F}_{\bar{z}^{\bar{a}}\bar{y}} = \mathcal{F}_{\bar{z}^{\bar{a}}\vartheta}\frac{\partial\vartheta}{\partial\bar{y}} + \mathcal{F}_{\bar{z}^{\bar{a}}\varphi}\frac{\partial\varphi}{\partial\bar{y}} = \frac{1}{\bar{y}}(\sin\vartheta\,\mathcal{F}_{\bar{z}^{\bar{a}}\vartheta} - \mathrm{i}\,\mathcal{F}_{\bar{z}^{\bar{a}}\varphi}), \tag{2.18}$$

$$\mathcal{F}_{y\bar{y}} = \mathcal{F}_{\vartheta\varphi}\left|\frac{\partial(\vartheta,\varphi)}{\partial(y,\bar{y})}\right| = \frac{1}{2\mathrm{i}}\,\frac{\sin\vartheta}{y\bar{y}}\,\mathcal{F}_{\vartheta\varphi} = \frac{1}{2\mathrm{i}}\,\frac{(1+\cos\vartheta)^2}{R^2\sin\vartheta}\mathcal{F}_{\vartheta\varphi} \tag{2.19}$$

and finally write the Donaldson–Uhlenbeck–Yau equations on $\mathbb{R}^{2n} \times S^2$ in the alternative form

$$2\mathrm{i}\delta^{a\bar{b}}\mathcal{F}_{z^a\bar{z}^{\bar{b}}} + \frac{1}{R^2\sin\vartheta}\mathcal{F}_{\vartheta\varphi} = 0, \qquad \mathcal{F}_{\bar{z}^{\bar{a}}\bar{z}^{\bar{b}}} = 0, \qquad \sin\vartheta\,\mathcal{F}_{\bar{z}^{\bar{a}}\vartheta} - \mathrm{i}\,\mathcal{F}_{\bar{z}^{\bar{a}}\varphi} = 0. \tag{2.20}$$

The transition to the noncommutative DUY equations is trivially achieved by going over to operator-valued objects everywhere. In particular, the field strength components in (2.20) then read $\widehat{\mathcal{F}}_{ij} = \partial_{\hat{x}^i}\hat{\mathcal{A}}_j - \partial_{\hat{x}^j}\hat{\mathcal{A}}_i + [\hat{\mathcal{A}}_i, \hat{\mathcal{A}}_j]$, where, e.g., $\hat{\mathcal{A}}_i$ are simultaneously $\mathrm{u}(k)$ and operator valued. To avoid a cluttered notation, we drop the hats from now on.

3. Generalized vortex equations on $\mathbb{R}^{2n}_{\theta}$

3.1. *Noncommutative generalization of Taubes' ansatz*

Considering the particular case (2.16) of the SU(2) DUY equations on $\mathbb{R}^2 \times S^2$, Taubes introduced an SO(3)-invariant ansatz[2] for the gauge potential $\mathcal{A}$ which reduces the ASDYM equations (2.16) to the vortex equations on $\mathbb{R}^2$ [5] (see also [21]). Here we extend Taubes' ansatz to the higher-dimensional manifold $\mathbb{R}^{2n} \times S^2$ and reduce the noncommutative[3] U(2) Donaldson–Uhlenbeck–Yau equations (2.20) to generalized vortex equations on $\mathbb{R}^{2n}_{\theta}$, including their commutative ($\theta = 0$) limit. In Section 4, we will write down explicit solutions of the generalized noncommutative vortex equations on $\mathbb{R}^{2n}$ which determine multi-instanton solutions of the noncommutative YM equations on $\mathbb{R}^{2n} \times S^2$.

We begin with the u(2)-valued operator one-form $\mathcal{A}$ on $\mathbb{R}^{2n}_{\theta} \times S^2$. Imposing SO(3) invariance up to a gauge transformation, Taubes [5] found for $n = 1$ and $\theta = 0$ that the S^2 dependence of $\mathcal{A}$ must be collected in the su(2) matrix

$$Q = \mathrm{i}\begin{pmatrix} \cos\vartheta & \mathrm{e}^{-\mathrm{i}\varphi}\sin\vartheta \\ \mathrm{e}^{\mathrm{i}\varphi}\sin\vartheta & -\cos\vartheta \end{pmatrix} = \mathrm{i}(\sin\vartheta\cos\varphi\,\sigma_1 + \sin\vartheta\sin\varphi\,\sigma_2 + \cos\vartheta\,\sigma_3) \tag{3.1}$$

[2] Similarly, Witten's ansatz [19] for gauge fields on $\mathbb{R}^4$ reduces (2.16) to the vortex equations on the hyperbolic space H^2 (cf. [20] for the noncommutative $\mathbb{R}^4$).

[3] As it is well known [2], in the noncommutative case one should use U(2) instead of SU(2).

and its differential $\mathrm{d}Q$. Note that $Q^2=-1$ and $\frac{\partial Q}{\partial\varphi}=-\sin\vartheta\, Q\,\frac{\partial Q}{\partial\vartheta}$. Our slight generalization of his ansatz to $\mathbb{R}^{2n}_\theta\times S^2$ reads ($\mathbf{1}=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$)

$$\mathcal{A}=\frac{1}{2}\big\{(\mathrm{i}Q-\gamma\mathbf{1})A+(\phi_1-1)Q\,\mathrm{d}Q+\phi_2\,\mathrm{d}Q\big\}, \tag{3.2}$$

where the constant γ parametrizes the additional u(1) piece. The one-form $A=A_\mu(x)\,\mathrm{d}x^\mu$ with $A_\mu\in\mathrm{u}(1)\cong\mathrm{i}\mathbb{R}$ and $\mu=1,\dots,2n$ is anti-Hermitian while $\phi_{1,2}=\phi_{1,2}(x)\in\mathbb{R}$ are Hermitian, all being operators in $\mathcal{H}$ only. Note that this form reduces the non-Abelian connection $\mathcal{A}$ to the Abelian objects (A,ϕ_1,ϕ_2) whose noncommutative character thus does not interfere with the u(2) structure. Calculation of the curvature

$$\begin{aligned}\mathcal{F}&=\mathrm{d}\mathcal{A}+\mathcal{A}\wedge\mathcal{A}=\frac{1}{2}\mathcal{F}_{ij}\,\mathrm{d}x^i\wedge\mathrm{d}x^j\\&=\frac{1}{2}\mathcal{F}_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu+\mathcal{F}_{\mu\vartheta}\,\mathrm{d}x^\mu\wedge\mathrm{d}\vartheta+\mathcal{F}_{\mu\varphi}\,\mathrm{d}x^\mu\wedge\mathrm{d}\varphi+\mathcal{F}_{\vartheta\varphi}\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\end{aligned} \tag{3.3}$$

for $\mathcal{A}$ of the form (3.2) yields

$$2\mathcal{F}_{\mu\nu}=\mathrm{i}Q\big(\partial_\mu A_\nu-\partial_\nu A_\mu-\gamma[A_\mu,A_\nu]\big)-\gamma\mathbf{1}\left(\partial_\mu A_\nu-\partial_\nu A_\mu-\frac{1+\gamma^2}{2\gamma}[A_\mu,A_\nu]\right), \tag{3.4}$$

$$4\mathcal{F}_{\mu\vartheta}=\big\{Q\big(2\partial_\mu\phi_1+\mathrm{i}A_\mu\phi_2+\mathrm{i}\phi_2A_\mu-\gamma[A_\mu,\phi_1]\big)+\mathbf{1}\big(2\partial_\mu\phi_2-\mathrm{i}A_\mu\phi_1-\mathrm{i}\phi_1A_\mu-\gamma[A_\mu,\phi_2]\big)\big\}\frac{\partial Q}{\partial\vartheta}, \tag{3.5}$$

$$4\mathcal{F}_{\mu\varphi}=\big\{Q\big(2\partial_\mu\phi_1+\mathrm{i}A_\mu\phi_2+\mathrm{i}\phi_2A_\mu-\gamma[A_\mu,\phi_1]\big)+\mathbf{1}\big(2\partial_\mu\phi_2-\mathrm{i}A_\mu\phi_1-\mathrm{i}\phi_1A_\mu-\gamma[A_\mu,\phi_2]\big)\big\}\frac{\partial Q}{\partial\varphi}, \tag{3.6}$$

$$2\mathcal{F}_{\vartheta\varphi}=\big\{Q\big(1-\phi_1^2-\phi_2^2\big)+\mathbf{1}[\phi_1,\phi_2]\big\}\sin\vartheta. \tag{3.7}$$

In the complex coordinates (2.3) with $A_{z^a}=\frac{1}{2}(A_{2a-1}+\mathrm{i}A_{2a})$ and $A^\dagger_{\bar z^{\bar a}}=-A_{z^a}$ we have

$$\mathcal{F}_{2a-1\,2a}=-Q\big(\partial_{z^a}A_{\bar z^{\bar a}}-\partial_{\bar z^{\bar a}}A_{z^a}-\gamma[A_{z^a},A_{\bar z^{\bar a}}]\big)-\mathrm{i}\gamma\mathbf{1}\left(\partial_{z^a}A_{\bar z^{\bar a}}-\partial_{\bar z^{\bar a}}A_{z^a}-\frac{1+\gamma^2}{2\gamma}[A_{z^a},A_{\bar z^{\bar a}}]\right) \tag{3.8}$$

which agrees with $2\mathrm{i}\,\mathcal{F}_{z^a\bar z^{\bar a}}$.

3.2. Vortex-type equations in $\mathbb{R}^{2n}_\theta$

Introducing $\phi:=\phi_1+\mathrm{i}\phi_2$ and substituting (3.7) and (3.8) into the first equation from (2.20), we obtain

$$\begin{aligned}&-\delta^{a\bar b}\left\{Q\big(\partial_{z^a}A_{\bar z^{\bar b}}-\partial_{\bar z^{\bar b}}A_{z^a}-\gamma[A_{z^a},A_{\bar z^{\bar b}}]\big)+\mathrm{i}\gamma\mathbf{1}\left(\partial_{z^a}A_{\bar z^{\bar b}}-\partial_{\bar z^{\bar b}}A_{z^a}-\frac{1+\gamma^2}{2\gamma}[A_{z^a},A_{\bar z^{\bar b}}]\right)\right\}\\&\quad+\frac{1}{4R^2}\big(Q\big(2-\phi\phi^\dagger-\phi^\dagger\phi\big)+\mathrm{i}\mathbf{1}\big[\phi,\phi^\dagger\big]\big)=0\end{aligned} \tag{3.9}$$

which splits into the two equations

$$\delta^{a\bar b}\big\{\partial_{z^a}A_{\bar z^{\bar b}}-\partial_{\bar z^{\bar b}}A_{z^a}-\gamma[A_{z^a},A_{\bar z^{\bar b}}]\big\}=\frac{1}{4R^2}\big(2-\phi\phi^\dagger-\phi^\dagger\phi\big), \tag{3.10}$$

$$\gamma\,\delta^{a\bar b}\left\{\partial_{z^a}A_{\bar z^{\bar b}}-\partial_{\bar z^{\bar b}}A_{z^a}-\frac{1+\gamma^2}{2\gamma}[A_{z^a},A_{\bar z^{\bar b}}]\right\}=\frac{1}{4R^2}\big[\phi,\phi^\dagger\big] \tag{3.11}$$

after separating into the su(2) (proportional to Q) and u(1) (proportional to $\mathrm{i}\mathbf{1}$) components.

The second equation from (2.20) can be written as

$$Q\big(\partial_{\bar z^{\bar a}} A_{\bar z^{\bar b}} - \partial_{\bar z^{\bar b}} A_{\bar z^{\bar a}} - \gamma [A_{\bar z^{\bar a}}, A_{\bar z^{\bar b}}]\big) + \mathrm{i}\gamma \mathbf{1}\left(\partial_{\bar z^{\bar a}} A_{\bar z^{\bar b}} - \partial_{\bar z^{\bar b}} A_{\bar z^{\bar a}} - \frac{1+\gamma^2}{2\gamma}[A_{\bar z^{\bar a}}, A_{\bar z^{\bar b}}]\right) = 0. \tag{3.12}$$

After some algebra, using (3.5) and (3.6), we find that the third equation from (2.20) is equivalent to

$$2\partial_{\bar z^{\bar a}}\phi + (1-\gamma)A_{\bar z^{\bar a}}\phi + (1+\gamma)\phi A_{\bar z^{\bar a}} = 0. \tag{3.13}$$

Let us consider the commutative case $\theta^{\mu\nu} = 0$ and put $\gamma = 0$. Then the Donaldson–Uhlenbeck–Yau equations on $\mathbb{R}^{2n} \times S^2$ for $\mathcal{A}$ defined in (3.2) reduce to

$$\delta^{a\bar b}\{\partial_{z^a} A_{\bar z^{\bar b}} - \partial_{\bar z^{\bar b}} A_{z^a}\} = \frac{1}{2R^2}(1-\phi\bar\phi), \tag{3.14}$$

$$\partial_{\bar z^{\bar a}} A_{\bar z^{\bar b}} - \partial_{\bar z^{\bar b}} A_{\bar z^{\bar a}} = 0, \tag{3.15}$$

$$\partial_{\bar z^{\bar a}}\phi + A_{\bar z^{\bar a}}\phi = 0, \tag{3.16}$$

where $\bar\phi$ is the complex conjugate of the scalar field ϕ. Eqs. (3.14)–(3.16) generalize the vortex equations [5] on $\mathbb{R}^2$ to the higher-dimensional space $\mathbb{R}^{2n}$.

For the noncommutative case $\theta^{\mu\nu} \neq 0$ we choose $\gamma = -1$. Comparing (3.10) and (3.11), we obtain a constraint equation on the field ϕ,

$$2 - \phi\phi^\dagger - \phi^\dagger\phi = -[\phi, \phi^\dagger] \quad \Longrightarrow \quad \phi^\dagger\phi = 1, \tag{3.17}$$

and the following noncommutative generalization of the vortex equations in $2n$ dimensions:

$$\delta^{a\bar b} F_{z^a \bar z^{\bar b}} := \delta^{a\bar b}\big\{\partial_{z^a} A_{\bar z^{\bar b}} - \partial_{\bar z^{\bar b}} A_{z^a} + [A_{z^a}, A_{\bar z^{\bar b}}]\big\} = \frac{1}{4R^2}\big(1-\phi\phi^\dagger\big), \tag{3.18}$$

$$F_{\bar z^{\bar a}\bar z^{\bar b}} := \partial_{\bar z^{\bar a}} A_{\bar z^{\bar b}} - \partial_{\bar z^{\bar b}} A_{\bar z^{\bar a}} + [A_{\bar z^{\bar a}}, A_{\bar z^{\bar b}}] = 0, \tag{3.19}$$

$$\partial_{\bar z^{\bar a}}\phi + A_{\bar z^{\bar a}}\phi = 0. \tag{3.20}$$

These equations and their antecedent DUY equations on $\mathbb{R}^{2n}_\theta \times S^2$ are not integrable even for $n = 1$. Therefore, neither dressing nor splitting approaches, developed in [22] for integrable equations on noncommutative spaces, can be applied. The modified ADHM construction [6] also does not work in this case. However, some special solutions can be obtained by choosing a proper ansatz as we shall see next.

4. Multi-instanton solutions on $\mathbb{R}^{2n}_\theta \times S^2$

4.1. Solutions of the constrained vortex-type equations

We are going to present explicit solutions to the noncommutative generalized vortex equations (3.18)–(3.20) subject to the constraint (3.17). The latter can be solved by putting

$$\phi = S_N, \qquad \phi^\dagger = S_N^\dagger, \tag{4.1}$$

where S_N is an order-N shift operator acting on the Fock space $\mathcal{H}$, i.e.,

$$S_N^\dagger S_N = 1 \quad \text{while} \quad S_N S_N^\dagger = 1 - P_N, \tag{4.2}$$

with P_N being a Hermitian rank-N projector: $P_N^2 = P_N = P_N^\dagger$.

It is convenient to introduce the operators

$$X_{z^a} = A_{z^a} + \theta_{a\bar b}\,\bar z^{\bar b}, \qquad X_{\bar z^{\bar a}} = A_{\bar z^{\bar a}} + \theta_{\bar a b}\, z^b \tag{4.3}$$

in terms of which

$$F_{z^a\bar z^{\bar b}} = [X_{z^a}, X_{\bar z^{\bar b}}] + \theta_{a\bar b}, \qquad F_{\bar z^{\bar a}\bar z^{\bar b}} = [X_{\bar z^{\bar a}}, X_{\bar z^{\bar b}}]. \tag{4.4}$$

We now employ the shift-operator ansatz (see, e.g., [7,23])

$$X_{z^a} = \theta_{a\bar b}\, S_N\, \bar z^{\bar b}\, S_N^\dagger, \qquad X_{\bar z^{\bar a}} = \theta_{\bar a b}\, S_N\, z^b\, S_N^\dagger \tag{4.5}$$

for which

$$F_{z^a\bar z^{\bar b}} = \theta_{a\bar b}\, P_N = \delta_{a\bar b}\,\frac{P_N}{2\theta^a}, \qquad F_{\bar z^{\bar a}\bar z^{\bar b}} = 0 \tag{4.6}$$

since $\theta_{a\bar b} = \frac{\delta_{a\bar b}}{2\theta^a}$. After substituting (4.1) and (4.6) into the first vortex equation (3.18), we obtain the condition

$$\delta^{a\bar b}\theta_{a\bar b}\, P_N = \frac{1}{4R^2} P_N \quad\Longleftrightarrow\quad \frac{1}{\theta^1} + \cdots + \frac{1}{\theta^n} = \frac{1}{2R^2}. \tag{4.7}$$

The remaining vortex equations (3.19) and (3.20) are identically satisfied by (4.1) and (4.6).

Hence, for $\gamma = -1$ we have established on $\mathbb{R}^{2n}$ a whole class of noncommutative constrained vortex-type configurations

$$A_{z^a} = \theta_{a\bar b}\big(S_N\, \bar z^{\bar b}\, S_N^\dagger - \bar z^{\bar b}\big), \qquad \phi = S_N, \tag{4.8}$$

parametrized by shift operators S_N. Our particular form (3.2) for $\mathcal{A}$ then yields a plethora of solutions to the noncommutative DUY equations on $\mathbb{R}^{2n} \times S^2$. These configurations generalize U(2) multi-instantons from $\mathbb{R}^2 \times S^2$ to $\mathbb{R}^{2n}_\theta \times S^2$. To substantiate this interpretation we finally calculate their topological charge.

4.2. Topological charge

For $\gamma = -1$, from (3.7) and (3.8) we get

$$\mathcal{F}_{\vartheta\varphi} = \frac{1}{4}(Q - \mathrm{i}\mathbf{1})\sin\vartheta\, P_N, \qquad \mathcal{F}_{2a-1\,2a} = (\mathrm{i}\mathbf{1} - Q)F_{z^a\bar z^{\bar a}} = (Q - \mathrm{i}\mathbf{1})\,\frac{P_N}{2\theta^a}. \tag{4.9}$$

Employing

$$(Q - \mathrm{i}\mathbf{1})^{n+1} = (-2\mathrm{i})^n (Q - \mathrm{i}\mathbf{1}), \qquad \mathrm{tr}_{2\times 2}(Q - \mathrm{i}\mathbf{1}) = -2\mathrm{i} \tag{4.10}$$

we have

$$\begin{aligned}\mathrm{tr}_{2\times 2}\underbrace{\mathcal{F}\wedge\cdots\wedge\mathcal{F}}_{n+1} &= (n+1)!\,\mathrm{tr}_{2\times 2}\,\mathcal{F}_{12}\mathcal{F}_{34}\ldots\mathcal{F}_{2n-1\,2n}\mathcal{F}_{\vartheta\varphi}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^{2n}\wedge\mathrm{d}\vartheta\wedge\mathrm{d}\varphi \\ &= (n+1)!\,\frac{(-2\mathrm{i})^{n+1}}{2^{n+2}}\,\frac{P_N}{\prod_{a=1}^n \theta^a}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^{2n}\wedge\sin\vartheta\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi.\end{aligned} \tag{4.11}$$

With this, the topological charge indeed becomes

$$\begin{aligned}\mathcal{Q} &:= \frac{1}{(n+1)!}\left(\frac{\mathrm{i}}{2\pi}\right)^{n+1}\left(\prod_{a=1}^n 2\pi\theta^a\right)\mathrm{Tr}_{\mathcal{H}}\int_{S^2}\mathrm{tr}_{2\times 2}\underbrace{\mathcal{F}\wedge\cdots\wedge\mathcal{F}}_{n+1} \\ &= \left(\frac{\mathrm{i}}{2\pi}\right)^{n+1}\frac{(-2\mathrm{i})^{n+1}}{2^{n+2}}\left(\prod_{a=1}^n 2\pi\theta^a\right)\left(\mathrm{Tr}_{\mathcal{H}}\,\frac{P_N}{\prod_{a=1}^n\theta^a}\right)\int_{S^2}\sin\vartheta\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi \\ &= \frac{1}{4\pi}(\mathrm{Tr}_{\mathcal{H}}\, P_N)\int_{S^2}\sin\vartheta\,\mathrm{d}\vartheta\wedge\mathrm{d}\varphi = N.\end{aligned} \tag{4.12}$$

5. Concluding remarks

By solving the noncommutative Donaldson–Uhlenbeck–Yau equations we have presented explicit U(2) multi-instantons on $\mathbb{R}^{2n}_\theta \times S^2$ which are uniquely determined by Abelian vortex-type configurations on $\mathbb{R}^{2n}_\theta$. The existence of these solutions required the condition (4.7) relating the S^2-radius R to θ via $R = (2\sum_{a=1}^{n}\frac{1}{\theta^a})^{-1/2}$. We see that any commutative limit ($\theta^a \to 0$) forces $R \to 0$ as well, and the configuration becomes localized in $\mathbb{R}^{2n}$ (for $n = 1$) or disappears (for $n > 1$). The moduli space of our N-instanton solutions is that of rank-N projectors in the n-oscillator Fock space.

Since standard instantons localize all compact coordinates in the ambient space they have been interpreted as sub-branes inside Dp-branes [1,2,9–12]. The presence of an NS background B-field deforms such configurations noncommutatively. In the same vein, the solutions presented in this Letter may be viewed as a collection of N sub-branes of codimension $2n$, i.e., as D$(p-2n)$-branes located inside two coincident Dp-branes, with all branes sharing a common two-sphere S^2_R.

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