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Sigma models with $\mathcal{N}=8$ supersymmetries in $2+1$ and $1+1$ dimensions

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Abstract

We introduce an $\mathcal{N}=8$ supersymmetric extension of the Bogomolny-type model for Yang–Mills–Higgs fields in $2+1$ dimensions related with twistor string theory. It is shown that this model is equivalent to an $\mathcal{N}=8$ supersymmetric $\mathrm{U}(n)$ chiral model in $2+1$ dimensions with a Wess–Zumino–Witten-type term. Further reduction to $1+1$ dimensions yields $\mathcal{N}=(8,8)$ supersymmetric extensions of the standard $\mathrm{U}(n)$ chiral model and Grassmannian sigma models.
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1. Introduction and summary

Nonlinear sigma models in k dimensions describe mappings of a k-dimensional manifold X into a manifold Y (target space). In particular, as target spaces one can consider Lie groups G (chiral models) and homogeneous spaces G/H for closed subgroups $H\subset G$. Sigma models and their $\mathcal{N}$-extended supersymmetric generalizations play an important role both in physics and mathematics (see e.g. [1,2]). For instance, two-dimensional sigma models serve as a theoretical laboratory for the study of more complicated (quantum) super-Yang–Mills theory since they share many of its features such as asymptotic freedom, nontrivial topological structure, the existence of instantons, ultraviolet finiteness for the $\mathcal{N}=4$ supersymmetric case, etc. [3]. Moreover, supersymmetric two-dimensional sigma models are the building blocks for superstring theories [3,4].

Recall that for two-dimensional nonlinear sigma models admitting a Lagrangian formulation the number of supersymmetries is intimately related to the geometry of the target space. Namely, it was argued that Lagrangian $\mathcal{N}=1$ models can be defined for any target space Y, for $\mathcal{N}=2$ the target space must be Kähler, for $\mathcal{N}=4$ it must be hyper-Kähler, and no Lagrangian models were introduced for $\mathcal{N}>4$ [5,6]. Similar results hold for sigma models in three dimensions. In particular, this means that a target space Y admits no more than $\mathcal{N}=1$ supersymmetry in the case of (non-Kähler) group manifolds G and $\mathcal{N}\leqslant 2$ supersymmetries for homogeneous Kähler spaces G/H.

The field equations of the standard G and G/H sigma models in $1+1$ and $2+0$ dimensions can be obtained by dimensional reduction of the self-dual Yang–Mills (SDYM) equations in $2+2$ dimensions, with a gauge group G [7]. Concretely, the SDYM model reduced to two dimensions is *equivalent* to the sigma model with G-valued scalar fields, while the G/H sigma model arises after imposing additional algebraic constraints. Similar reduction to $2+1$ dimensions yields a modified integrable chiral model [8]. Recall that the SDYM model in $2+2$ dimensions can be endowed with up to four supersymmetries [9,10]. Reducing the $\mathcal{N}$-extended supersymmetric SDYM equations in $2+2$ dimensions to $2+1$ and $1+1$ dimensions yields models which have twice as many supersymmetries (cf. [11] for reductions from $3+1$ dimensions). We will show that for $G=\mathrm{U}(n)$ and $\mathcal{N}=4$ these models are equivalent to $\mathrm{U}(n)$ chiral models with $\mathcal{N}=8$ supersymmetries. These new supersymmetric sigma models in $2+1$ and $1+1$ dimensions are well defined on the level of equations of motion, but their Lagrangian formulation is not known yet.

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In this Letter we concentrate on the reduction of the $\mathcal{N}=4$ SDYM equations (instead of arbitrary $\mathcal{N}\leqslant 4$) in $2+2$ dimensions since for this case a Lagrangian can be written down at least in terms of the component fields of a reduced Yang–Mills-type supermultiplet. Moreover, it was shown by Witten [12] that the $\mathcal{N}=4$ SDYM model appears in twistor string theory, which is a B-type topological string with the supertwistor space $\mathbb{C}P^{3|4}$ as a target space.[1] This fact gives additional arguments in favour of introducing $\mathcal{N}=8$ supersymmetric sigma models in $2+1$ and $1+1$ dimensions related with twistor string theory and of studying their properties.

2. $\mathcal{N}=4$ supersymmetric SDYM equations in $2+2$ dimensions

Superspace $\mathbb{R}^{4|16}$. Let us consider the four-dimensional space $\mathbb{R}^{2,2}:=(\mathbb{R}^4,g)$ with the metric

$$ds^2=g_{\mu\nu}\,dx^\mu\,dx^\nu=\det\big(dx^{\alpha\dot\alpha}\big)=dx^{1\dot1}\,dx^{2\dot2}-dx^{2\dot1}dx^{1\dot2} \tag{2.1}$$

with $(g_{\mu\nu})=\text{diag}(-1,+1,+1,-1)$. Here $\mu,\nu,\ldots=1,\ldots,4$ are vector indices and $\alpha=1,2$, $\dot\alpha=\dot1,\dot2$ are spinor indices. We choose the real coordinates[2] $(x^\mu)=(x^a,\tilde t)=(t,x,y,\tilde t)$ with $a,b,\ldots=1,2,3$ such that

$$x^{1\dot1}=\frac12(t-y),\qquad x^{1\dot2}=\frac12(x+\tilde t),\qquad x^{2\dot1}=\frac12(x-\tilde t)\quad\text{and}\quad x^{2\dot2}=\frac12(t+y). \tag{2.2}$$

On the space $\mathbb{R}^{2,2}$ one can introduce real Majorana–Weyl spinors and extend $\mathbb{R}^{2,2}$ to a space with additional anticommuting (Grassmann) coordinates $\theta^{i\alpha}$ and $\eta_i^{\dot\alpha}$ of helicity $+\frac12$ and $-\frac12$, respectively. Here index $i=1,\ldots,4$ parametrizes fundamental and its conjugate representations of the R-symmetry group SL(4, $\mathbb{R}$) [9]. Thus, $(x^{\alpha\dot\alpha},\eta_i^{\dot\alpha},\theta^{i\alpha})$ are coordinates on superspace $\mathbb{R}^{4|16}$.

Supersymmetry algebra. The $\mathcal{N}=4$ supersymmetry algebra in $2+2$ dimensions is generated by $P_{\alpha\dot\alpha}=\partial_{\alpha\dot\alpha}=\partial/\partial x^{\alpha\dot\alpha}$ and 16 real supercharges

$$Q_{i\alpha}:=\partial_{i\alpha}-\eta_i^{\dot\alpha}\partial_{\alpha\dot\alpha}\quad\text{and}\quad Q^i_{\dot\alpha}:=\partial^i_{\dot\alpha}-\theta^{i\alpha}\partial_{\alpha\dot\alpha}, \tag{2.3}$$

with $\partial_{i\alpha}:=\partial/\partial\theta^{i\alpha}$ and $\partial^i_{\dot\alpha}:=\partial/\partial\eta_i^{\dot\alpha}$. The only nontrivial (anti)commutators in this superalgebra read

$$\big\{Q_{i\alpha},Q^j_{\dot\alpha}\big\}=-2\delta_i^j\,\partial_{\alpha\dot\alpha}. \tag{2.4}$$

In what follows we will also need superderivatives

$$D_{i\alpha}:=\partial_{i\alpha}+\eta_i^{\dot\alpha}\partial_{\alpha\dot\alpha}\quad\text{and}\quad D^i_{\dot\alpha}:=\partial^i_{\dot\alpha}+\theta^{i\alpha}\partial_{\alpha\dot\alpha}, \tag{2.5}$$

which anticommute with the operators (2.3) and satisfy

$$\big\{D_{i\alpha},D^j_{\dot\beta}\big\}=2\delta_i^j\,\partial_{\alpha\dot\beta}. \tag{2.6}$$

Antichiral superspace. On the superspace $\mathbb{R}^{4|16}$ we can introduce spin-tensor fields depending on both bosonic and fermionic coordinates (superfields) and impose on them various constraints. In particular, on any superfield $\mathcal{A}$ one can impose the so-called antichirality conditions $\mathcal{L}_{D_{i\alpha}}\mathcal{A}=0$, where $\mathcal{L}_Z$ denotes the Lie derivative along a vector superfield Z. One can easily solve these equations by using a coordinate transformation on superspace $\mathbb{R}^{4|16}$,

$$\big(x^{\alpha\dot\alpha},\eta_i^{\dot\alpha},\theta^{i\alpha}\big)\to\big(\tilde x^{\alpha\dot\alpha}=x^{\alpha\dot\alpha}-\theta^{i\alpha}\eta_i^{\dot\alpha},\eta_i^{\dot\alpha},\theta^{i\alpha}\big), \tag{2.7}$$

under which $\partial_{\alpha\dot\alpha}$, $D_{i\alpha}$ and $D^i_{\dot\alpha}$ transform to the operators

$$\tilde\partial_{\alpha\dot\alpha}=\partial_{\alpha\dot\alpha},\qquad \tilde D_{i\alpha}=\partial_{i\alpha},\qquad \tilde D^i_{\dot\alpha}=\partial^i_{\dot\alpha}+2\theta^{i\alpha}\partial_{\alpha\dot\alpha}. \tag{2.8}$$

The antichirality conditions then mean that a superfield $\mathcal{A}$ satisfies the equations

$$\tilde D_{i\alpha}\mathcal{A}=0 \tag{2.9}$$

meaning that $\mathcal{A}$ is defined on superspace $\mathbb{R}^{4|8}\subset\mathbb{R}^{4|16}$ called antichiral superspace with coordinates $(x^{\alpha\dot\alpha},\eta_i^{\dot\alpha})$. Note that for transformed supercharges we have

$$\tilde Q_{i\alpha}=\partial_{i\alpha}-2\eta_i^{\dot\alpha}\partial_{\alpha\dot\alpha}\quad\text{and}\quad \tilde Q^i_{\dot\alpha}=\partial^i_{\dot\alpha}. \tag{2.10}$$

In the following we will often omit the tilde when dealing with the antichiral superspace.

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1 For other variants of twistor string models see [13].

2 Our conventions are chosen to match those of [14] after reduction to the space $\mathbb{R}^{2,1}$ with coordinates (t,x,y).

$\mathcal{N}=4$ *SDYM in superfields*. The field content of $\mathcal{N}=4$ supersymmetric SDYM is given by a supermultiplet $(A_{\alpha\dot\alpha}, \chi^{i\alpha}, \phi^{ij}, \tilde\chi_i^{\dot\alpha}, G_{\dot\alpha\dot\beta})$ of fields on $\mathbb{R}^{2,2}$ of helicities $(+1, +\frac{1}{2}, 0, -\frac{1}{2}, -1)$. Here $A_{\alpha\dot\alpha}$ are the components of a gauge potential with the field strength $F_{\alpha\dot\alpha,\beta\dot\beta} = \partial_{\alpha\dot\alpha}A_{\beta\dot\beta} - \partial_{\beta\dot\beta}A_{\alpha\dot\alpha} + [A_{\alpha\dot\alpha}, A_{\beta\dot\beta}]$. Note that the scalars ϕ^{ij} are antisymmetric in ij and all the fields, including the fermionic ones $\chi^{i\alpha}$ and $\tilde\chi_i^{\dot\alpha}$, live in the adjoint representation of the gauge group $\mathrm{U}(n)$.

The $\mathcal{N}=4$ SDYM equations [9,15] can be written in terms of superfields on antichiral superspace $\mathbb{R}^{4|8}$ [9,16]. Namely, all fields from the above $\mathcal{N}=4$ supermultiplet can be combined into superfields $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$ on $\mathbb{R}^{4|8}$ in terms of which the $\mathcal{N}=4$ SDYM equations read

$$[\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] + [\nabla_{\alpha\dot\beta}, \nabla_{\beta\dot\alpha}] = 0, \qquad [\nabla^i_{\dot\alpha}, \nabla_{\beta\dot\beta}] + [\nabla^i_{\dot\beta}, \nabla_{\beta\dot\alpha}] = 0, \qquad \{\nabla^i_{\dot\alpha}, \nabla^j_{\dot\beta}\} + \{\nabla^i_{\dot\beta}, \nabla^j_{\dot\alpha}\} = 0, \tag{2.11}$$

where we have introduced the covariant derivatives

$$\nabla_{\alpha\dot\alpha} := \partial_{\alpha\dot\alpha} + \mathcal{A}_{\alpha\dot\alpha} \quad \text{and} \quad \nabla^i_{\dot\alpha} := \partial^i_{\dot\alpha} + \mathcal{A}^i_{\dot\alpha}. \tag{2.12}$$

Note that (2.11) can be combined into the manifestly supersymmetric equations

$$\{\tilde\nabla^i_{\dot\alpha}, \tilde\nabla^j_{\dot\beta}\} + \{\tilde\nabla^i_{\dot\beta}, \tilde\nabla^j_{\dot\alpha}\} = 0 \tag{2.13}$$

with

$$\tilde\nabla^i_{\dot\alpha} := \nabla^i_{\dot\alpha} + 2\theta^{i\alpha}\nabla_{\alpha\dot\alpha} = \tilde D^i_{\dot\alpha} + \tilde{\mathcal{A}}^i_{\dot\alpha} \quad \text{and} \quad \tilde{\mathcal{A}}^i_{\dot\alpha} := \mathcal{A}^i_{\dot\alpha} + 2\theta^{i\alpha}\mathcal{A}_{\alpha\dot\alpha}, \tag{2.14}$$

where $\mathcal{A}_{\alpha\dot\alpha}$ and $\mathcal{A}^i_{\dot\alpha}$ depend only on $x^{\alpha\dot\alpha}$ and $\eta_i^{\dot\alpha}$.

It is not difficult to see that Eqs. (2.13) are the compatibility conditions for the linear system of differential equations

$$\lambda^{\dot\alpha}_\pm (\tilde D^i_{\dot\alpha} + \tilde{\mathcal{A}}^i_{\dot\alpha})\psi_\pm = 0, \tag{2.15}$$

where $\lambda^{\dot\alpha}_\pm = \varepsilon^{\dot\alpha\dot\beta}\lambda^\pm_{\dot\beta}$, $(\lambda^+_{\dot\beta}) = (1\ \lambda_+)^\top$, $(\lambda^-_{\dot\beta}) = (\lambda_-\ 1)^\top$ and the extra (local) coordinates $\lambda_\pm$ lie on patches $U_\pm$ covering the Riemann sphere $\mathbb{C}P^1 = U_+ \cup U_-$ (see e.g. [17]). Here $\psi_\pm$ are $n\times n$ matrices depending not only on $x^{\alpha\dot\alpha}$ and $\eta_i^{\dot\alpha}$ but also (holomorphically) on $\lambda_\pm \in U_\pm$.

The field equations of the $\mathcal{N}=4$ SDYM model in the component fields read

$$F_{\dot\alpha\dot\beta} = 0, \qquad D_{\alpha\dot\alpha}\chi^{i\alpha} = 0, \qquad D_{\alpha\dot\alpha}D^{\alpha\dot\alpha}\phi^{ij} + \{\chi^{i\alpha}, \chi^j_\alpha\} = 0, \tag{2.16a}$$

$$D_{\alpha\dot\alpha}\tilde\chi_i^{\dot\alpha} + [\chi^j_\alpha, \phi_{ij}] = 0, \qquad \varepsilon^{\dot\alpha\dot\gamma}D_{\alpha\dot\alpha}G_{\dot\gamma\dot\beta} - \frac{1}{2}\{\chi^i_\alpha, \tilde\chi_{i\dot\beta}\} - \frac{1}{4}[\phi_{ij}, D_{\alpha\dot\beta}\phi^{ij}] = 0, \tag{2.16b}$$

where $F_{\dot\alpha\dot\beta} := -\frac{1}{2}\varepsilon^{\alpha\beta}F_{\alpha\dot\alpha,\beta\dot\beta}$, $D_{\alpha\dot\alpha} := \partial_{\alpha\dot\alpha} + [A_{\alpha\dot\alpha}, \cdot]$ and $\phi_{ij} := \frac{1}{2!}\varepsilon_{ijkl}\phi^{kl}$. These equations can be extracted from (2.11) by using η-expansions and Bianchi identities (see e.g. [16]). We will not reproduce this derivation. Note only that (2.16) follows from the Lagrangian [9,12]

$$\mathcal{L} = \mathrm{tr}\big(G^{\dot\alpha\dot\beta}F_{\dot\alpha\dot\beta} + \tilde\chi_i^{\dot\alpha}D_{\alpha\dot\alpha}\chi^{i\alpha} + \phi_{ij}D_{\alpha\dot\alpha}D^{\alpha\dot\alpha}\phi^{ij} + \phi_{ij}\chi^{i\alpha}\chi^j_\alpha\big). \tag{2.17}$$

3. $\mathcal{N}=8$ supersymmetric sigma models in $2+1$ dimensions

Reduction and spinors on $\mathbb{R}^{2,1}$. The $\mathcal{N}=8$ supersymmetric Bogomolny-type equations in $2+1$ dimensions are obtained from the described $\mathcal{N}=4$ super SDYM equations by the dimensional reduction $\mathbb{R}^{2,2}\to\mathbb{R}^{2,1}$. Namely, we impose the ∂_4-invariance condition on all the fields $(A_{\alpha\dot\alpha}, \chi^{i\alpha}, \phi^{ij}, \tilde\chi_i^{\dot\alpha}, G_{\dot\alpha\dot\beta})$ from the $\mathcal{N}=4$ supermultiplet. Also, the components A_μ of a gauge potential split into the components A_a in $2+1$ dimensions and the Lie-algebra valued scalar field $\varphi := A_4$ (Higgs field). To see how this splitting looks in spinor notation, we briefly discuss spinors in $2+1$ dimensions.

Recall that $\mathcal{N}=4$ SDYM theory on $\mathbb{R}^{2,2}$ has $\mathrm{SL}(4,\mathbb{R}) \cong \mathrm{Spin}(3,3)$ as an R-symmetry group [9]. Analogously to the case of standard $\mathcal{N}=4$ super-Yang–Mills (SYM) in Minkowski space with the Spin(6) R-symmetry, the appearance of the group Spin(3,3) can be interpreted via a reduction of $\mathcal{N}=1$ SYM theory on space $\mathbb{R}^{5,5}\cong\mathbb{R}^{2,2}\times\mathbb{R}^{3,3}$ to $\mathbb{R}^{2,2}$ with internal space $\mathbb{R}^{3,3}$ [10]. Furthermore, after reduction from $\mathbb{R}^{2,2}$ to $\mathbb{R}^{2,1}$ the R-symmetry group becomes Spin(4,4) and supersymmetry gets enlarged to $\mathcal{N}=8$ with Spin(4,3) as the manifest R-symmetry group (cf. [11] for Minkowski and [18] for Euclidean signatures). Roughly speaking, this happens due to no distinction between dotted and undotted spinor indices in three dimensions. Recall that the rotation group SO(2,2) of $\mathbb{R}^{2,2}$ is locally isomorphic to $\mathrm{SU}(1,1)_L\times\mathrm{SU}(1,1)_R \cong \mathrm{Spin}(2,1)_L\times\mathrm{Spin}(2,1)_R \cong \mathrm{Spin}(2,2)$. Upon dimensional reduction to $2+1$ dimensions, the rotation group of $\mathbb{R}^{2,1} = (\mathbb{R}^3, g)$ with $g = (g_{ab}) = \mathrm{diag}(-1,+1,+1)$ is locally $\mathrm{SU}(1,1)\cong\mathrm{Spin}(2,1)$, which is the diagonal subgroup of $\mathrm{Spin}(2,1)_L\times\mathrm{Spin}(2,1)_R\cong\mathrm{Spin}(2,2)$. Therefore, the distinction between dotted and undotted indices disappear.

Coordinates and derivatives on $\mathbb{R}^{3|16}$. The ∂_4-invariance reduces superspace $\mathbb{R}^{4|16}$ with coordinates $x^\mu, \eta_i^{\dot\alpha}$ and $\theta^{i\alpha}$ to $\mathbb{R}^{3|16}$ with coordinates x^a, η_i^α and $\theta^{i\alpha}$. Furthermore, x^a and η_i^α parametrize reduced antichiral superspace $\mathbb{R}^{3|8}$. For bosonic coordinates

$x^{\alpha\dot\beta} \to x^{\alpha\beta}$ in spinor notation we have

$$x^{\alpha\beta} = \frac{1}{2}\big(x^{\alpha\beta} + x^{\beta\alpha}\big) + \frac{1}{2}\big(x^{\alpha\beta} - x^{\beta\alpha}\big) = x^{(\alpha\beta)} + x^{[\alpha\beta]}. \tag{3.1}$$

Thus, we have coordinates

$$y^{\alpha\beta} := x^{(\alpha\beta)} \quad \text{with } y^{11} = x^{11} = \frac{1}{2}(t-y),\ y^{12} = \frac{1}{2}\big(x^{12}+x^{21}\big) = \frac{1}{2}x,\ y^{22} = x^{22} = \frac{1}{2}(t+y) \tag{3.2}$$

on $\mathbb{R}^{2,1}$ and $x^{[\alpha\beta]} = -\varepsilon^{\alpha\beta}x^4 = -\varepsilon^{\alpha\beta}\tilde t$, where $\varepsilon^{12} = -\varepsilon^{21} = 1$.

For derivatives we obtain

$$\partial_{\alpha\beta} = \frac{1}{2}(\partial_{\alpha\beta} + \partial_{\beta\alpha}) + \frac{1}{2}(\partial_{\alpha\beta} - \partial_{\beta\alpha}) = \partial_{(\alpha\beta)} - \varepsilon_{\alpha\beta}\partial_4 = \partial_{(\alpha\beta)} - \varepsilon_{\alpha\beta}\partial_{\tilde t}, \tag{3.3}$$

where $\varepsilon_{12} = -\varepsilon_{21} = -1$ and

$$\partial_{(11)} = \frac{\partial}{\partial y^{11}} = \partial_t - \partial_y, \qquad \partial_{(12)} = \partial_{(21)} = \frac{1}{2}\frac{\partial}{\partial y^{12}} = \partial_x, \qquad \partial_{(22)} = \frac{\partial}{\partial y^{22}} = \partial_t + \partial_y. \tag{3.4}$$

For the operators (2.8) acting on $\tilde t$-independent superfields we have

$$\hat D_{i\alpha} = \partial_{i\alpha} \quad \text{and} \quad \hat D^i_\alpha = \partial^i_\alpha + 2\theta^{i\beta}\partial_{(\alpha\beta)}. \tag{3.5}$$

Similarly, supercharges (2.10) reduce to the operators

$$\hat Q_{i\alpha} = \partial_{i\alpha} - 2\eta^\beta_i\partial_{(\alpha\beta)} \quad \text{and} \quad \hat Q^i_\alpha = \partial^i_\alpha, \tag{3.6}$$

anticommuting with (3.5).

$\mathcal N = 8$ supersymmetric Bogomolny-type equations on $\mathbb{R}^{2,1}$. After imposing the condition of $\tilde t$-independence on all fields in the linear system (2.15), we obtain the equations

$$\lambda^\alpha_\pm\big(\hat D^i_\alpha + \hat{\mathcal A}^i_\alpha\big)\psi_\pm = 0 \tag{3.7}$$

with

$$\hat{\mathcal A}^i_\alpha = \mathcal A^i_\alpha + 2\theta^{i\beta}(\mathcal A_{(\alpha\beta)} - \varepsilon_{\alpha\beta}\tilde\varphi), \tag{3.8}$$

and $\hat D^i_\alpha$ given in (3.5). Here $\mathcal A^i_\alpha$, $\mathcal A_{(\alpha\beta)}$ and $\tilde\varphi$ are superfields depending only on $y^{\alpha\beta}$ and η^β_i.

The compatibility conditions for the linear system (3.7) read

$$\big\{\hat D^i_\alpha + \hat{\mathcal A}^i_\alpha,\ \hat D^j_\beta + \hat{\mathcal A}^j_\beta\big\} + \big\{\hat D^i_\beta + \hat{\mathcal A}^i_\beta,\ \hat D^j_\alpha + \hat{\mathcal A}^j_\alpha\big\} = 0. \tag{3.9}$$

As usual, these manifestly $\mathcal N = 8$ supersymmetric equations are equivalent to equations in component fields,

$$f_{\alpha\beta} + D_{\alpha\beta}\varphi = 0, \qquad D_{\alpha\beta}\chi^{i\beta} + \varepsilon_{\alpha\beta}\big[\varphi, \chi^{i\beta}\big] = 0, \tag{3.10a}$$

$$D_{\alpha\beta}D^{\alpha\beta}\phi^{ij} + 2\big[\varphi, \big[\varphi, \phi^{ij}\big]\big] + \big\{\chi^{i\alpha}, \chi^j_\alpha\big\} = 0, \tag{3.10b}$$

$$D_{\alpha\beta}\ \tilde\chi^\beta_i - \varepsilon_{\alpha\beta}\big[\varphi, \tilde\chi^\beta_i\big] + \big[\chi^j_\alpha, \phi_{ij}\big] = 0, \tag{3.10c}$$

$$\varepsilon^{\gamma\delta}D_{\alpha\gamma}G_{\delta\beta} + [\varphi, G_{\alpha\beta}] - \frac{1}{2}\big\{\chi^i_\alpha, \tilde\chi_{i\beta}\big\} - \frac{1}{4}\big[\phi_{ij}, D_{\alpha\beta}\phi^{ij}\big] - \frac{1}{4}\varepsilon_{\alpha\beta}\big[\phi_{ij}, \big[\phi^{ij}, \varphi\big]\big] = 0, \tag{3.10d}$$

where $D_{\alpha\beta} := \partial_{(\alpha\beta)} + [A_{(\alpha\beta)}, \cdot]$, $f_{\alpha\beta} := -\frac{1}{2}\varepsilon^{\gamma\delta}[D_{\alpha\gamma}, D_{\beta\delta}]$ and $\varphi := A_4 = A_{\tilde t}$. Obviously, these equations are ∂_4-reduction of (2.16).

Supersymmetric sigma models. Note that matrices $\psi_\pm$ in (3.7) are defined up to a gauge transformation generated by a matrix which does not depend on $\lambda_\pm$ and therefore one can choose a gauge such that

$$\psi_+ = \Phi^{-1} + \mathcal O(\lambda_+) \quad \text{and} \quad \psi_- = \mathbf 1_n + \lambda_-\Upsilon + \mathcal O\big(\lambda_-^2\big), \tag{3.11}$$

where Φ is a U(n)-valued superfield and Υ is a $u(n)$-valued superfield both depending only on $y^{\alpha\beta}$ and η^α_i. For this gauge, from (3.7) we obtain

$$\hat{\mathcal A}^i_1 = 0 \quad \text{and} \quad \hat{\mathcal A}^i_2 = \Phi^{-1}\hat D^i_2\Phi, \tag{3.12}$$

and from (3.8) we have

$$\mathcal A^i_1 = 0, \qquad \mathcal A_{(11)} = 0, \qquad \mathcal A_{(12)} - \tilde\varphi = 0, \tag{3.13a}$$

$$\mathcal{A}_2^i = \Phi^{-1}\partial_2^i\Phi, \qquad \mathcal{A}_{(12)} + \tilde{\varphi} = \Phi^{-1}\partial_{(12)}\Phi \quad \text{and} \quad \mathcal{A}_{(22)} = \Phi^{-1}\partial_{(22)}\Phi. \tag{3.13b}$$

Substituting (3.12) into (3.9), we obtain equations

$$\hat{D}_1^i\left(\Phi^{-1}\hat{D}_2^j\,\Phi\right) + \hat{D}_1^j\left(\Phi^{-1}\hat{D}_2^i\,\Phi\right) = 0 \tag{3.14}$$

which after using (3.5) and (3.13) read

$$\partial_x\left(\Phi^{-1}\partial_x\Phi\right) + \partial_y\left(\Phi^{-1}\partial_y\Phi\right) - \partial_t\left(\Phi^{-1}\partial_t\Phi\right) + \partial_y\left(\Phi^{-1}\partial_t\Phi\right) - \partial_t\left(\Phi^{-1}\partial_y\Phi\right) = 0, \tag{3.15}$$

$$\partial_1^i\left(\Phi^{-1}\partial_x\Phi\right) - \partial_t\left(\Phi^{-1}\partial_2^i\Phi\right) + \partial_y\left(\Phi^{-1}\partial_2^i\Phi\right) = 0, \qquad \partial_1^i\left(\Phi^{-1}\partial_t\Phi\right) + \partial_1^i\left(\Phi^{-1}\partial_y\Phi\right) - \partial_x\left(\Phi^{-1}\partial_2^i\Phi\right) = 0, \tag{3.16}$$

$$\partial_1^i\left(\Phi^{-1}\partial_2^j\Phi\right) + \partial_1^j\left(\Phi^{-1}\partial_2^i\Phi\right) = 0. \tag{3.17}$$

Note that the last two terms in (3.15) are the Wess–Zumino–Witten terms which spoil the standard Lorentz invariance but yield an integrable U(n) chiral model in $2+1$ dimensions. For reduction to $1+1$ dimensions one should simply put $\partial_y\Phi = 0$ in (3.15)–(3.17) obtaining an $\mathcal{N} = 8$ supersymmetric extensions of the standard U(n) chiral model in two dimensions with field equations

$$\partial_t\left(\Phi^{-1}\partial_t\Phi\right) - \partial_x\left(\Phi^{-1}\partial_x\Phi\right) = 0, \qquad \partial_1^i\left(\Phi^{-1}\partial_2^j\Phi\right) + \partial_1^j\left(\Phi^{-1}\partial_2^i\Phi\right) = 0, \tag{3.18a}$$

$$\partial_1^i\left(\Phi^{-1}\partial_x\Phi\right) - \partial_t\left(\Phi^{-1}\partial_2^i\Phi\right) = 0, \qquad \partial_1^i\left(\Phi^{-1}\partial_t\Phi\right) - \partial_x\left(\Phi^{-1}\partial_2^i\Phi\right) = 0. \tag{3.18b}$$

For Φ taking values in the Grassmannian manifold $\mathrm{Gr}(k,n) \subset \mathrm{U}(n)$, Eqs. (3.15)–(3.17) and (3.18) describe correspondingly supersymmetric Grassmannian sigma models in $2+1$ and $1+1$ dimensions.

There is not yet a Lagrangian description of Eqs. (3.15) to (3.17) or (3.18). However, using the equivalence of Eqs. (3.10)–(3.14), one can write explicitly a Lagrangian in terms of fields $(A_{(\alpha\beta)},\ \chi^{i\alpha},\ \varphi,\ \phi^{ij},\ \tilde{\chi}_i^\alpha,\ G_{\alpha\beta})$. The proper Lagrangians follow from (2.17) by reduction to $2+1$ and $1+1$ dimensions. It is a challenging task to find Lagrangians in terms of the U(n)-valued superfield Φ.

Supersymmetry transformations. For brevity, we consider only $2+1$ dimensions, where the 16 supercharges have the form (3.6). Further reduction to $1+1$ dimensions does not create any problem. From (3.6) we obtain

$$\left\{\hat{Q}_{i\alpha}, \hat{Q}_\beta^j\right\} = -2\delta_i^j\,\partial_{(\alpha\beta)}. \tag{3.19}$$

On a (scalar) superfield Σ an infinitesimal supersymmetry transformation $\hat{\delta}$ acts by

$$\hat{\delta}\Sigma := \epsilon^{i\alpha}\hat{Q}_{i\alpha}\Sigma + \epsilon_i^\alpha\hat{Q}_\alpha^i\Sigma, \tag{3.20}$$

where $\epsilon^{i\alpha}$ and ϵ_i^α are 16 Grassmann parameters. In particular, for coordinates $y^{\alpha\beta}$ and η_i^β on the antichiral superspace $\mathbb{R}^{3|8}$ we have $\hat{\delta}y^{\alpha\beta} = -2\epsilon^{i(\alpha}\eta_i^{\beta)}$ and $\hat{\delta}\eta_i^\alpha = \epsilon_i^\alpha$.

It is obvious that the sigma model field equations (3.14) are invariant under the supersymmetry transformations (3.20) because the operators $\hat{D}_\alpha^i$ as well as $\hat{D}_{i\alpha}$ anticommute with the supersymmetry generators $\hat{Q}_{i\alpha}$ and $\hat{Q}_\beta^j$. Note that these $\mathcal{N} = 8$ supersymmetric extensions of the U(n) and $\mathrm{Gr}(k,n) = \mathrm{U}(n)/\mathrm{U}(k) \times \mathrm{U}(n-k)$ sigma models in $2+1$ and $1+1$ dimensions are not the standard ones defined only for $\mathcal{N} \leqslant 1$ and $\mathcal{N} \leqslant 2$, respectively. It will be interesting to study this new kind of sigma models in more detail.

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