Handling Handles: Nonplanar Integrability in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

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We propose an integrability setup for the computation of correlation functions of gauge-invariant operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at higher orders in the large N_c genus expansion and at any order in the 't Hooft coupling $g_{\mathrm{YM}}^2 N_c$. In this multistep proposal, one polygonizes the string world sheet in all possible ways, hexagonalizes all resulting polygons, and sprinkles mirror particles over all hexagon junctions to obtain the full correlator. We test our integrability-based conjecture against a nonplanar four-point correlator of large 1/2 Bogomol'nyi-Prasad-Sommerfield operators at one and two loops.

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Introduction.—*Integrable theories* are rather special 2D quantum field theories where the scattering of fundamental excitations factorizes into a sequence of two-body scattering events. This simplification often translates into solvability. The world-sheet theory describing superstrings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ is integrable [1,2]. Exploiting integrability machinery, the full *finite-size spectrum* has been obtained at any value of the coupling [3–5], yielding the energy spectra of single strings in this curved background or—equivalently—the spectra of anomalous dimensions of single-trace operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in the planar limit.

Beyond the planar limit, we are dealing with world sheets with handles. These induce nonlocal interactions in the two-dimensional theory, wormholes of sorts, which also appear in the gauge theory spin-chain description, see Fig. 1. One would guess that such nonlocal interactions could ruin integrability. Indeed, known degeneracies in the spectrum of the weakly coupled gauge theory—related to the hidden higher charges of the integrable theory—are lifted as one takes nonplanar corrections into account [6], and fermionic T duality—responsible for dual conformal symmetry, which in turn is closely related to integrability in the usual sense—is not a symmetry of string theory at a higher genus [7,8]. Because of all this, it has been common lore that integrability would not be useful beyond the planar limit [9]. See [13] for a very nice summary.

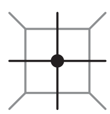
On the other hand, numerous other *planar* quantities have been explored at finite coupling using integrability, from scattering amplitudes or Wilson loops [14] to structure constants [15], higher-point correlation functions [16–18], and even mixed quantities involving correlation functions in the presence of Wilson loops [19]. Underlying all these computations is the idea of taming complicated string topologies by cutting the string into smaller and simpler patches (hexagonal or pentagonal), which are then glued back together. This is implemented by so-called branch-point twist field operators [20,21], whose expectation values can be bootstrapped.

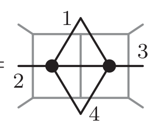
All these works strongly suggest that, instead of thinking about the nonplanar effects as nonlocal corrections to the planar world sheet, we should from the get go consider the theory in more general topologies, treat handles using the twist operators mentioned above, and keep everything else as local as possible, see Fig. 1. Following this philosophy, in this Letter, we propose a framework for computing correlation functions at any higher-genus order and any value of the 't Hooft coupling using integrability.

The data.—Our *experimental data*—against which we will test our integrability predictions—are the four-point correlation functions of single-trace 1/2 Bogomol'nyi-Prasad-Sommerfield (BPS) operators $\mathcal{Q}_i^k \equiv \mathrm{tr}([\alpha_i \cdot \Phi(x_i)]^k)$ studied in Refs. [22,23] in the simplifying configuration $\alpha_1 \cdot \alpha_4 = \alpha_2 \cdot \alpha_3 = 0$. Here, α_i is a null vector, and $\Phi = (\phi_1, \dots, \phi_6)$ are the scalar fields of $\mathcal{N} = 4$ SYM theory. The loop correlator $G_k \equiv \langle \mathcal{Q}_1^k \mathcal{Q}_2^k \mathcal{Q}_3^k \mathcal{Q}_4^k \rangle - \langle \mathcal{Q}_1^k \mathcal{Q}_2^k \mathcal{Q}_3^k \mathcal{Q}_4^k \rangle^{\mathrm{tree}}$ can then be decomposed according to the propagator structures that connect the operators as

$$G_k = \sum_{m=0}^{k} \left(\sum_{l=1}^{\infty} g^{2l} \mathcal{F}_{k,m}^{(l)} \right) X^m Y^{k-m}, \qquad (1)$$

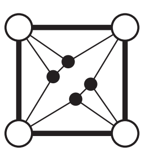
where $X \equiv (\alpha_1 \cdot \alpha_2)(\alpha_3 \cdot \alpha_4)/x_{12}^2 x_{34}^2$, $Y \equiv X|_{2\leftrightarrow 3}$ are the R-charge and space-time propagators, and $g^2 = g_{\mathrm{YM}}^2 N_c / 16\pi^2$. The quantum corrections dressing the propagator structures depend on the conformally invariant cross ratios $|z|^2 = x_{12}^2 x_{34}^2 / x_{13}^2 x_{24}^2$ and $|1-z|^2 = x_{23}^2 x_{14}^2 / x_{13}^2 x_{24}^2$. The one- and two-loop contributions were computed in Refs. [22,23]. A key ingredient are the conformal box and double-box functions

$$F^{(1)}(z, \bar{z}) = \frac{x_{13}^2 x_{24}^2}{\pi^2} \int \frac{\mathrm{d}^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} = $$  ,

$$\frac{F^{(2)}}{x_{14}^2} = \frac{x_{13}^2 x_{24}^2}{(\pi^2)^2} \int \frac{\mathrm{d}^4 x_5 \, \mathrm{d}^4 x_6}{x_{15}^2 x_{25}^2 x_{45}^2 x_{56}^2 x_{16}^2 x_{36}^2 x_{46}^2} = $$ 

. (2)

Other key players are the so-called color factors, which consist of color contractions of four symmetrized traces from the four operators, dressed with insertions of gauge group structure constants. For instance [24],

$C_m^{\mathrm{c}} =$ 

$$= \frac{f_{abe} f_{cd}{}^{e} f_{pqt} f_{rs}{}^{t}}{2m!^2 (k-m-2)!^2} \times$$
$$\times \, \mathrm{tr}((d_1 \dots d_{k'} a_1 \dots a_m bd)) \, \mathrm{tr}((a_1 \dots a_m b_1 \dots b_{k'} ar))$$
$$\times \, \mathrm{tr}((d_1 \dots d_{k'} c_1 \dots c_m cp)) \, \mathrm{tr}((c_1 \dots c_m b_1 \dots b_{k'} qs)) \, , \qquad (3)$$

where $k' = k - m - 2$. We explicitly performed the contractions with *Mathematica*, for up to $k = 8$ or 9 and various values of m. Then, we used the fact that—by their combinatorial nature—the various color factors should be quartic polynomials in k and m (up to boundary cases at extremal values of k or m), which we can fit using the data points at finite k and m. At the end of the day, one finds $C_m^c / N_c^{2k} k^4 = 2k^4 + \mathcal{P}^c / 6N_c^2 + \mathcal{O}(N_c^{-4})$, where $\mathcal{P}^c = k^4 + 4k^3 m + 42k^2 m^2 - 92km^3 + 46m^4 + \cdots$, and similar expressions for all other color factors.

We consider the further simplification of large external operators with $k \gg 1$ and $m/k \equiv r + 1/2$ held fixed. Putting the above ingredients together and keeping only the leading large k result at each genus order, we finally obtain our much desired experimental data:

$$\mathcal{F}_{k,m}^{(1)} = \frac{-2k^2}{N_c^2} \left(1 + \frac{k^4 (\frac{17r^4}{6} - \frac{7r^2}{4} + \frac{11}{32})}{N_c^2} \right) |z-1|^2 F^{(1)},$$
$$\mathcal{F}_{k,m}^{(2)} = \frac{4k^2}{N_c^2} \left[\left(1 + \frac{k^4 (\frac{17r^4}{6} - \frac{7}{4} r^2 + \frac{11}{32})}{N_c^2} \right) |z-1|^2 F^{(2)} + \left(1 + \frac{k^4 (\frac{29r^4}{6} - \frac{11}{4} r^2 + \frac{15}{32})}{N_c^2} \right) \frac{|z-1|^4}{4} (F^{(1)})^2 \right]. \qquad (4)$$

Integrability proposal.—We propose that the connected part of any correlator in the $\mathrm{U}(N_c)$ theory, including the full expansion in $1/N_c$, can be recovered from integrability via the formula

$$\langle \mathcal{Q}_1 \dots \mathcal{Q}_n \rangle = \mathcal{S} \circ \sum_{\substack{\text{skeleton} \\ \text{graphs}}} \sum_{\text{labelings}} \sum_{\substack{\text{bridge} \\ \text{fillings}}} \begin{pmatrix} R\text{-charge}\,\& \\ \text{space-time} \\ \text{propagators} \end{pmatrix} \times \sum_{\substack{\text{mirror} \\ \text{states}}} \begin{pmatrix} \text{mirror} \\ \text{propagation} \\ \text{factors} \end{pmatrix} \times \prod_{\text{Hexagons}} \begin{pmatrix} \text{hexagon} \\ \text{form} \\ \text{factors} \end{pmatrix}. \qquad (5)$$

The outermost sum runs over all graphs with n vertices, including all topologies, planar and nonplanar. Each edge (bridge) stands for a collection of one or more (planar, noncrossing) propagators connecting two operators. Hence, parallel edges must be identified, and this defines a skeleton graph; see Fig. 3 for examples of such graphs.

Next, we sum over all vertex labelings (distributions of operators on the vertices) and over all (nonzero) bridge fillings (numbers of propagators on each edge) compatible with the charges of the operators. All this combinatorial process is what we call *polygonization*.

Next follows what we call *hexagonalization*: After inserting the operators, all faces of the skeleton graphs are hexagons or higher polygons. For the latter, we pick a subdivision into hexagons by inserting zero-length bridges (ZLBs). Each hexagon gives home to one hexagon form factor whose expression was determined in Ref. [15]. Finally, we cut the graphs at the zero-length and non-zero-length bridges, and we insert a complete basis of mirror states; i.e., we sum over mirror excitations, on each bridge. The mirror propagation factors depend on the normalization and flavor of the mirror-particle states as well as on the bridge length; their expressions can be found in Refs. [15,16]. This last step we denote as *sprinkling*.

These three main processes are represented in Fig. 2 and discussed in detail below. For illustration and simplicity, in

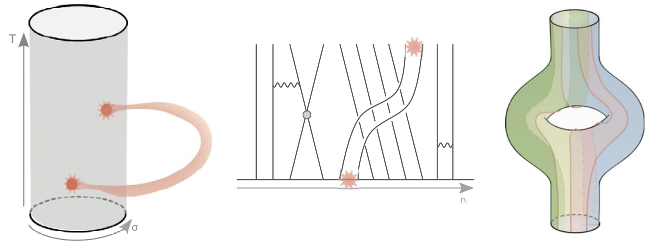


FIG. 1. (a) Nonplanar effects include handles in the string world sheet inducing nontrivial nonlocal effects. (b) The same effect can be seen in the gauge theory; nonplanar processes induce nonlocal interactions in the effective spin chain. (c) From a hexagonalization point of view, we tessellate higher-genus string topologies, always maintaining locality.

this Letter, we restrict ourselves to $n = 4$ large BPS operators computed up to the first subleading correction in $1/N_c^2$ (i.e., genus 0 and genus 1).

There is one last step represented by the seemingly innocuous $\mathcal{S}$ in (5) which stands for subtractions or *stratification*. The point is that the sum over polygonizations discretizes the integration over the moduli space of the Riemann surface, whose boundary contains degeneration points: At its boundary, a torus degenerates into a sphere, for instance. $\mathcal{S}$ stands for the appropriate subtractions which remove these boundary contributions; see, e.g., [25]. In this Letter, we will consider four large BPS operators on the torus, which are controlled by configurations where all cycles of the torus will be populated by many propagators. The relevant world sheets are thus very far from the boundary of the moduli space, and we can ignore $\mathcal{S}$ altogether. We will come back to it in Ref. [26], but the essential idea is that, to obtain the correct result at a given genus g, we must include the contributions of graphs with a genus smaller than g embedded on a genus g surface and subtract all the degenerations of that surface that do not affect the embedded graph.

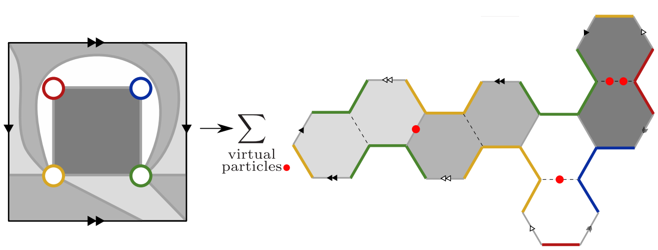


FIG. 2. We sum over all polygonizations of the torus with four operators or holes (left figure). Each polygonization is then broken apart into hexagons (right figure, edges with the same arrow marks are identified). Finally, we dress the hexagon junctions with mirror particles (sprinkling). In this example, we sprinkle $1 + 1 + 2$ mirror particles on one such hexagonalization. The two particles on a zero-length bridge (right) and the single particle on a bridge of nonzero length l (left) kick in at 4 and at $l + 1$ loops, respectively, and are thus highly suppressed (for large bridges). The remaining (middle) contribution with a single particle on a zero-length bridge is the only one relevant for this Letter; it kicks in at one loop already.

(Large k) polygonization.—As indicated by the first line of (5), the polygonization proceeds in three steps: (A) construct all inequivalent graphs with n vertices on the given topology, (B) sum over all inequivalent labelings of the vertices, and (C) for each labeled graph, sum over all possible distributions of propagators on the edges (bridges) of the graph such that each edge carries at least one propagator.

In a generic graph on the torus, any two operators will be connected by one or more bridges. In this work, we are interested in the leading contribution for large operator weights $k \gg 1$ with m/k finite. In this limit, graphs with a nonmaximal number of bridges will be suppressed by combinatorial powers of $1/k$. Namely, distributing $n \sim k$ propagators on j bridges comes with a factor

$$\sum_{\substack{1 \le n_1, \dots, n_j \le n \\ \sum_i n_i = n}} 1 = \frac{n^{j-1}}{(j-1)!} + \mathcal{O}(n^{j-2}). \tag{6}$$

In the leading term, all bridges carry many propagators. We consider operator polarizations that disallow propagator structures of the type $Z \equiv (\alpha_1 \cdot \alpha_4)(\alpha_2 \cdot \alpha_3)/x_{14}^2 x_{23}^2$; see (1). Hence, only graphs where the four operators are connected cyclically, as in 1-2-3-4-1, will contribute. Under this constraint, one easily finds that the maximal power from combinatorial factors (6) is k^4. We have classified all graphs contributing to this order and have found the six cases shown in Fig. 3.

For these six graphs, we have to consider all possible inequivalent operator labelings. In addition, each labeled graph comes with a combinatorial factor (6). We list all inequivalent labelings for the relevant graphs as well as their combinatorial factors in Table I [27].

(Large k) hexagonalization.—Next, we further decompose all polygons in Fig. 3—which are bounded by the finite bridges—into hexagons by adding ZLBs. There are

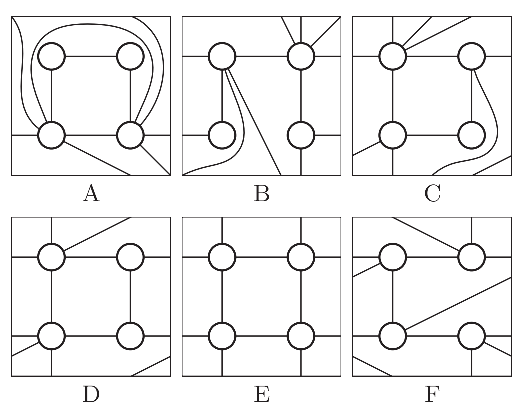


FIG. 3. Bridge configurations on the torus that contribute to the leading term in $1/k$ for correlators of the type (1).

TABLE I. All inequivalent operator labelings for the graphs that contribute to leading order in $1/k$, together with their combinatorial factors according to (6). The order of the labels runs from top to bottom, left to right in the graphs in Fig. 3.

Case	Inequivalent labelings	Combinatorial factor
A	1234,3412	$m^4/24$
A	1324,2413	$(k-m)^4/24$
B	1234,2143,3412,4321	$m^3(k-m)/6$
B	1324,3142,2413,4231	$m(k-m)^3/6$
C	1234,3412,2143,4321	$m^2/2 \cdot (k-m)^2/2$
D	1234,2143,1324,3142	$m^2(k-m)^2/2$
E	1234	$m^2(k-m)^2/2$
F	1234	$m^2(k-m)^2$

typically various ways of adding these ZLBs, and they are all equivalent. The independence on the tessellation (chosen set of additional ZLBs) was verified explicitly in the case of the octagon and decagon in Refs. [16,18] and represents a consistency check of the hexagonalization. We can easily see that all graphs in Fig. 3 are made out of four octagons; hence, we simply need to split each of those octagons into two hexagons. A hexagonalization of case A is illustrated in Fig. 2. The physical operators correspond to the thick colorful lines, the solid gray lines are the large bridges, and the dashed lines are the ZLBs.

(Large k) sprinkling.—Finally, we have to sprinkle mirror particles on the hexagonalizations of the previous section. Our large k result is given by a set of octagons separated by large bridges. Putting particles on those bridges is very costly in the perturbation theory, as the mirror-particle contribution is coupling suppressed by the corresponding bridge length. Hence, we can only put particles on ZLBs inside each octagon. Furthermore, putting two particles on the same bridge is also very costly (appearing at four loops only), so up to two loops only two contributions will matter: a single particle placed on a ZLB and two particles placed simultaneously on two distinct ZLBs. This latter contribution is essentially the square of the former one. The single-particle contribution has been studied in Ref. [16] and yields

$$\mathcal{M}^{(1)} = \left[z+\bar{z}-(\alpha+\bar{\alpha})\frac{\alpha\bar{\alpha}+z\bar{z}}{2\alpha\bar{\alpha}}\right](g^2F^{(1)}-2g^4F^{(2)}), \quad (7)$$

where, for the correlators considered here, the R-charge cross-ratios α and $\bar{\alpha}$ are given, respectively, by

$$\alpha = z\bar{z}X/Y \quad \text{and} \quad \bar{\alpha} = 1. \quad (8)$$

To get the g^4 term in (7), we simply expanded the integrand in Ref. [16] to one more order in the perturbation theory.

The above factors of X and Y are contained in the first factors in parentheses in the second line of (5) and combine with the propagator factors in the first line of that formula. Hence, to read off particular coefficients of monomials in X and Y to compare with perturbation theory predictions such as (4), we often need to consider the contribution of a few "neighboring" graphs.

Consider for illustration the particular case A in Fig. 2. There are four octagons to be considered, as shown in Fig. 4. The first two contain pairs of physical edges associated to the same external operator and thus give a vanishing contribution, as can be easily seen by taking the limit of two coinciding points for a generic octagon. For each of the labelings in Table I, the resulting expressions for the remaining two octagons are summarized in Table II. Accounting for the labeling and combinatorial factors listed in Table I, we can then read off the coefficient of X^mY^{k-m} as

$$\text{case A}|_{X^mY^{k-m}\text{coeff}} = k^4\left[\frac{(r+1/2)^4+(r-1/2)^4}{24} \times (4\mathcal{M}+2\mathcal{M}^2) \times \sum_{a=-2}^{2} X^{m+a}Y^{k-m-a}\right]_{X^mY^{k-m}\text{coeff}},$$

where we have used that $\mathcal{M}^{(1)}(1/z) = \mathcal{M}^{(1)}(z) \equiv \mathcal{M}$. As explained above, this coefficient receives contributions from a few neighboring polygonizations, accounted for by the sum in the last line. The remaining cases follow in complete analogy. When we sum them all, we obtain a perfect match with (4).

Conclusions and overlook.—We proposed here a novel formalism for computing correlation functions of local gauge-invariant operators in $\mathcal{N}=4$ SYM theory at any genus and any order in the coupling in the large N_c 't Hooft expansion.

In this Letter, we already performed one very nontrivial check of our conjecture. We reproduced the first nonplanar correction to the correlation function of four large BPS operators at one loop and two loops from integrability. At the end of the day, this computation is rather simple and uses only formulas for a single mirror particle already worked out in Ref. [16]. In an upcoming paper [26], we perform numerous other checks that probe all steps in our

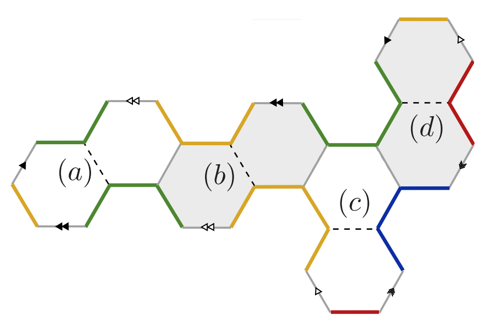


FIG. 4. The four octagons of case A.

TABLE II. Contributions for the one-particle octagons for each distinct operator labeling of case A.

Labeling	Octagon (c)	Octagon (d)
1234	$\mathcal{M}^{(1)}(z)$	$\mathcal{M}^{(1)}(z)$
1324	$\mathcal{M}^{(1)}(1/z)$	$\mathcal{M}^{(1)}(1/z)$
2413	$\mathcal{M}^{(1)}(1/z)$	$\mathcal{M}^{(1)}(1/z)$
3412	$\mathcal{M}^{(1)}(z)$	$\mathcal{M}^{(1)}(z)$

proposal in great detail: the polygonization, the hexagonalization, the sprinkling, and the stratification. These include finite-size corrections to the computation above, correlators at a strict finite size, higher-genus examples, and subtleties related to the choice of the gauge group. Through the operator product expansion of the obtained correlators, we can read off conformal data of non-BPS operators beyond the planar limit.

One of the advantages of dealing with BPS external operators (as considered in this Letter) is avoiding the subtlety of double-trace mixing. It would be interesting to study the mixing effects. (See [28] for very interesting first explorations in this direction.) It would also be important to better understand the integrand one obtains after sprinkling the hexagons with a few mirror particles. As we increase the number of mirror particles, it quickly becomes monstrous. How do we tame it efficiently? Another interesting problem—which can be realistically addressed only once we progress with the former—concerns going to strong coupling and making contact with the recent exciting developments on the bootstrap approach to loop corrections in AdS [29–36]. One can then explore various interesting questions such as the emergence of bulk locality [37,38]. Will we find higher-genus subtleties in our integrability-based formalism akin to the complications with supermoduli integrations recently observed in the Ramond-Neveu-Schwarz formalism [39–41]?

Finally, a fun project would be to resum the ’t Hooft expansion—perhaps starting with some simplifying kinematic limits. What awaits us there, and what can we learn about string (field) theory?

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[1] J. A. Minahan and K. Zarembo, The Bethe-ansatz for $\mathcal{N}=4$ super Yang-Mills, J. High Energy Phys. 03 (2003) 013.
[2] I. Bena, J. Polchinski, and R. Roiban, Hidden symmetries of the $\mathrm{AdS}_5 \times S^5$ superstring, Phys. Rev. D **69**, 046002 (2004).
[3] N. Beisert *et al.*, Review of AdS/CFT integrability: An overview, Lett. Math. Phys. **99**, 3 (2012).
[4] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, Quantum Spectral Curve for Planar $\mathcal{N}=4$ Super-Yang-Mills Theory, Phys. Rev. Lett. **112**, 011602 (2014).
[5] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, Quantum spectral curve for arbitrary state/operator in $\mathrm{AdS}_5/\mathrm{CFT}_4$, J. High Energy Phys. 09 (**2015**) 187.
[6] N. Beisert, C. Kristjansen, and M. Staudacher, The dilatation operator of $\mathcal{N}=4$ conformal super Yang-Mills theory, Nucl. Phys. **B664**, 131 (2003).
[7] N. Berkovits and J. Maldacena, Fermionic T-duality, dual superconformal symmetry, and the amplitude/Wilson loop connection, J. High Energy Phys. 09 (2008) 062.
[8] N. Beisert, R. Ricci, A. A. Tseytlin, and M. Wolf, Dual superconformal symmetry from $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring integrability, Phys. Rev. D **78**, 126004 (2008).
[9] For operators whose dimension scale as $\Delta \sim \mathcal{O}(N_c)$, integrability at the nonplanar level was discussed previously in Refs. [10–12].
[10] W. Carlson, R. de Mello Koch, and H. Lin, Nonplanar integrability, J. High Energy Phys. 03 (2011) 105.
[11] R. de Mello Koch, B. A. Elseid Mohammed, and S. Smith, Nonplanar integrability: Beyond the SU(2) sector, Int. J. Mod. Phys. A **26**, 4553 (2011).
[12] R. de Mello Koch, G. Kemp, B. A. Elseid Mohammed, and S. Smith, Nonplanar integrability at two loops, J. High Energy Phys. 10 (2012) 144.
[13] C. Kristjansen, Review of AdS/CFT integrability, Chapter IV.1: Aspects of non-planarity, Lett. Math. Phys. **99**, 349 (2012).
[14] B. Basso, A. Sever, and P. Vieira, Spacetime and Flux Tube S-Matrices at Finite Coupling for $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory, Phys. Rev. Lett. **111**, 091602 (2013).

[15] B. Basso, S. Komatsu, and P. Vieira, Structure constants and integrable bootstrap in planar $\mathcal{N} = 4$ SYM theory, arXiv:1505.06745.

[16] T. Fleury and S. Komatsu, Hexagonalization of correlation functions, J. High Energy Phys. 01 (2017) 130.

[17] B. Eden and A. Sfondrini, Tessellating cushions: Four-point functions in $\mathcal{N} = 4$ SYM, J. High Energy Phys. 10 (2017) 098.

[18] T. Fleury and S. Komatsu, Hexagonalization of correlation functions II: Two-particle contributions, J. High Energy Phys. 02 (2018) 177.

[19] M. Kim and N. Kiryu, Structure constants of operators on the Wilson loop from integrability at weak coupling, J. High Energy Phys. 11 (2017) 116.

[20] J. L. Cardy, O. A. Castro-Alvaredo, and B. Doyon, Form factors of branch-point twist fields in quantum integrable models and entanglement entropy, J. Stat. Phys. **130**, 129 (2008).

[21] O. A. Castro-Alvaredo, B. Doyon, and D. Fioravanti, Conical twist fields and null polygonal Wilson loops, Nucl. Phys. **B931**, 146 (2018).

[22] G. Arutyunov and E. Sokatchev, On a large N degeneracy in $\mathcal{N} = 4$ SYM and the AdS/CFT correspondence, Nucl. Phys. **B663**, 163 (2003).

[23] G. Arutyunov, S. Penati, A. Santambrogio, and E. Sokatchev, Four point correlators of BPS operators in $\mathcal{N} = 4$ SYM at order g^4, Nucl. Phys. **B670**, 103 (2003).

[24] Here, $\mathrm{tr}((a_1 \ldots a_k)) \equiv \mathrm{tr}(T^{(a_1} \ldots T^{a_k)})$ denotes a totally symmetrized trace of adjoint gauge group generators T^a.

[25] L. Chekhov, Matrix models and geometry of moduli spaces, arXiv:hep-th/9509001.

[26] T. Bargheer, J. Caetano, T. Fleury, S. Komatsu, and P. Vieira, Handling handles. Part II. Stratification and data analysis, J. High Energy Phys., 11 (2018) 095.

[27] Case E has an extra symmetry: Every pair of operators is connected by a pair of bridges. Exchanging the members of all pairs simultaneously amounts to a cyclic rotation of the four operators and thus leaves the configuration invariant. The naive sum over bridge lengths thus has to be corrected by a factor of $1/2$.

[28] B. Eden, Y. Jiang, D. le Plat, and A. Sfondrini, Colour-dressed hexagon tessellations for correlation functions and non-planar corrections, J. High Energy Phys. 02 (2018) 170.

[29] L. Rastelli and X. Zhou, Mellin Amplitudes for Supergravity on $\mathrm{AdS}_5 \times S^5$, Phys. Rev. Lett. **118**, 091602 (2017).

[30] O. Aharony, L. F. Alday, A. Bissi, and E. Perlmutter, Loops in AdS from conformal field theory, J. High Energy Phys. 07 (2017) 036.

[31] L. F. Alday and A. Bissi, Loop Corrections to Supergravity on $\mathrm{AdS}_5 \times S^5$, Phys. Rev. Lett. **119**, 171601 (2017).

[32] F. Aprile, J. M. Drummond, P. Heslop, and H. Paul, Quantum gravity from conformal field theory, J. High Energy Phys. 01 (2018) 035.

[33] F. Aprile, J. M. Drummond, P. Heslop, and H. Paul, Unmixing supergravity, J. High Energy Phys. 02 (2018) 133.

[34] L. Rastelli and X. Zhou, How to succeed at holographic correlators without really trying, J. High Energy Phys. 04 (2018) 014.

[35] L. F. Alday and S. Caron-Huot, Gravitational S-matrix from CFT dispersion relations, arXiv:1711.02031.

[36] F. Aprile, J. M. Drummond, P. Heslop, and H. Paul, Loop corrections for Kaluza-Klein AdS amplitudes, J. High Energy Phys. 05 (2018) 056.

[37] I. Heemskerk, J. Penedones, J. Polchinski, and J. Sully, Holography from conformal field theory, J. High Energy Phys. 10 (2009) 079.

[38] J. Maldacena, D. Simmons-Duffin, and A. Zhiboedov, Looking for a bulk point, J. High Energy Phys. 01 (2017) 013.

[39] E. Witten, Notes on supermanifolds and integration, arXiv:1209.2199.

[40] E. Witten, Notes on super Riemann surfaces and their moduli, arXiv:1209.2459.

[41] E. Witten, Superstring perturbation theory revisited, arXiv:1209.5461.