symmetry

MDPI

Review

Deformed $\mathcal{N} = 8$ Supersymmetric Mechanics

Evgeny Ivanov [1], Olaf Lechtenfeld [2] and Stepan Sidorov [1,*]

[1] Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russia; eivanov@theor.jinr.ru

[2] Institut für Theoretische Physik and Riemann Center for Geometry and Physics, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany; lechtenf@itp.uni-hannover.de

* Correspondence: sidorovstepan88@gmail.com

Received: 23 December 2018; Accepted: 21 January 2019; Published: 26 January 2019

Abstract: We give a brief review of deformed $\mathcal{N} = 8$ supersymmetric mechanics as a generalization of SU(2|1) mechanics. It is based on the worldline realizations of the supergroups SU(2|2) and SU(4|1) in the appropriate $\mathcal{N} = 8$, $d = 1$ superspaces. The corresponding models are deformations of the standard $\mathcal{N} = 8$ mechanics models by a mass parameter m.

Keywords: supersymmetry; superfields; supersymmetric quantum mechanics

1. Introduction

Recently, there has been a growth in interest in deformed models of supersymmetric quantum mechanics (SQM) based on some semisimple superalgebras treated as deformations of flat $d = 1$ supersymmetries with the same number of supercharges. This interest was mainly motivated by the study of higher-dimensional models with "curved" rigid supersymmetries (see e.g., [1]) derived within the localization method [2] as a powerful tool allowing one to compute non-perturbatively quantum objects, such as partition functions, Wilson loops, etc. Reduction of rigid supersymmetric field theories to the deformed SQM ones can be used for computation of the superconformal index [3], the vacuum (Casimir) energy [4], etc.

The simplest deformed $\mathcal{N} = 4$ SQM models with worldline realizations of SU(2|1) supersymmetry were considered in [3,5,6] and in [7] (named there "weak $d = 1$ supersymmetry"). The relevant deformation parameter is the mass-dimension parameter m. When m goes to zero, the standard "flat" $\mathcal{N} = 4, d = 1$ supersymmetry is recovered. The corresponding worldline multiplets were $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ and $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ (Multiplets of the standard and deformed $\mathcal{N} = 4$, $d = 1$ supersymmetry are denoted as $(\mathbf{k}, \mathbf{4}, \mathbf{4}-\mathbf{k})$ with $\mathbf{k} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$. These numbers correspond to the numbers of bosonic physical fields, fermionic physical fields and bosonic auxiliary fields, respectively. $\mathcal{N} = 8$, $d = 1$ multiplets are denoted in the same way as $(\mathbf{k}, \mathbf{8}, \mathbf{8}-\mathbf{k})$, where $\mathbf{k} = \mathbf{0}, \mathbf{1}, \ldots, \mathbf{8}$.). The systematic superfield approach to SU(2|1) supersymmetry was worked out in [8–11]. The models built on the multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$, $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ and $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ were studied at the classical and quantum levels. Recently, SU(2|1) invariant versions of super Calogero–Moser systems were constructed and quantized [12–14].

The common features of all these models can be summarized as:

- The oscillator-type Lagrangians for the bosonic fields, with m^2 as the oscillator strength.
- The appearance of the Wess–Zumino type terms for the bosonic fields, of the type $\sim im(\dot{z}\bar{z} - z\dot{\bar{z}})$.
- At the lowest energy levels, wave functions form *atypical* SU(2|1) multiplets, with unequal numbers of the bosonic and fermionic states and vanishing values of the Casimir operators. The energy spectrum involves an essential dependence on the deformation parameter m.

It was of obvious interest to advance further and to consider SQM models which exhibit deformations of $\mathcal{N} = 8$, $d = 1$ Poincaré superalgebra. As distinct from the $\mathcal{N} = 4$ case, there exist two different types of the $\mathcal{N}=8$ deformations. These are associated with worldline realizations of the supergroups SU(2|2) and SU(4|1), the odd sectors of which have the same dimension 8. The present review of deformed $\mathcal{N}=8$ supersymmetric mechanics is to a large extent based on our recent papers [15,16]. Here, we focus our attention mainly on the off-shell *chiral* multiplets of deformed $\mathcal{N}=8$ supersymmetric mechanics.

SU(2|1) *Supersymmetric Mechanics*

The systematic study of SU(2|1) supersymmetric mechanics initiated in [8] was based on the deformation (In fact, the standard $\mathcal{N}=2, d=1$ Poincaré superalgebra is written as $\{Q,\bar{Q}\}=2H$ and, together with the U(1) automorphism generator, can be identified with the superalgebra $u(1|1)$. In this interpretation, the standard Hamiltonian H appears as a central charge operator.)

$$\mathcal{N}=4, d=1 \text{ Poincaré} \qquad \Rightarrow \qquad su(2|1), \tag{1}$$

where the superalgebra $su(2|1)$ is given by the following non-vanishing relations

$$\begin{aligned}
&\{Q^i,\bar{Q}_j\} = 2mI^i_j + 2\delta^i_j\tilde{H}, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,,\\
&\left[I^i_j,\bar{Q}_l\right] = \frac{1}{2}\,\delta^i_j\bar{Q}_l - \delta^i_l\bar{Q}_j\,, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\,\delta^i_j Q^k,\\
&\left[\tilde{H},\bar{Q}_l\right] = \frac{m}{2}\,\bar{Q}_l\,, \qquad \left[\tilde{H}, Q^k\right] = -\,\frac{m}{2}\,Q^k.
\end{aligned} \tag{2}$$

The dimensionless generators $I^{ij}=I^{ji}$ form SU(2) symmetry (The doublet indices $i=1,2$ are raised and lowered in the standard way by the antisymmetric ε^{ij} symbols, e.g.,

$$Q_i = \varepsilon_{ij}Q^j, \qquad \varepsilon^{21}=\varepsilon_{12}=1.$$

while the generator $\tilde{H}$ of the dimension of mass is treated as a U(1) symmetry generator. The supercharges have the dimension $m^{1/2}$ and they carry SU(2) doublet indices. The mass-dimension parameter m plays the role of deformation parameter. In the limit $m=0$, the generators I^i_j become the SU(2) automorphism generators of the standard $\mathcal{N}=4$, $d=1$ superalgebra with the Hamiltonian $\tilde{H}$.

One can extend the algebra (2) by an external U(1) automorphism symmetry generator F which has non-zero commutation relations only with the supercharges [1,9]:

$$\left[F,\bar{Q}_l\right] = -\frac{1}{2}\,\bar{Q}_l\,, \qquad \left[F, Q^k\right] = \frac{1}{2}\,Q^k. \tag{3}$$

The redefinition $\tilde{H}\equiv H-mF$ brings the extended superalgebra $su(2|1)\Subset u(1)$ to the form in which it looks as a centrally extended superalgebra $\hat{su}(2|1)$:

$$\begin{aligned}
&\{Q^i,\bar{Q}_j\} = 2m\left(I^i_j - \delta^i_j F\right) + 2\delta^i_j H\,, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,,\\
&\left[I^i_j,\bar{Q}_l\right] = \frac{1}{2}\delta^i_j\bar{Q}_l - \delta^i_l\bar{Q}_j\,, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,\\
&\left[F,\bar{Q}_l\right] = -\frac{1}{2}\,\bar{Q}_l\,, \qquad \left[F, Q^k\right] = \frac{1}{2}\,Q^k.
\end{aligned} \tag{4}$$

In the new basis, the generator F becomes the internal U(1) generator, while H commutes with all generators and so can be treated as a central charge. In the limit $m=0$, the generators I^i_j and F decouple and become the U(2) automorphism generators of the standard $\mathcal{N}=4$, $d=1$ superalgebra, with H as the Hamiltonian.

2. Two Deformations of the Standard $\mathcal{N}=8$, $d=1$ Poincaré Superalgebra

In contrast to $\mathcal{N}=4$ supersymmetry, in the $\mathcal{N}=8$ case, there are two different options for the deformation due to the existence of two different superalgebras with eight supercharges:

(1) $\mathcal{N}=8, d=1$ Poincaré $\Rightarrow$ $su(2|2)$,
(2) $\mathcal{N}=8, d=1$ Poincaré $\Rightarrow$ $su(4|1)$.

Both deformed $\mathcal{N}=8$ superalgebras contain the superalgebra $su(2|1)$ as a subalgebra.

2.1. The Superalgebra $su(2|2)$

The first choice was in details studied in [15]. The non-vanishing (anti)commutators of the superalgebra $su(2|2)$ can be written as

$$\begin{aligned}
&\left\{Q^{ia},\bar{Q}_{jb}\right\}=2m\left(\delta^a_b I^i_j-\delta^i_j F^a_b\right)+2\delta^i_j\delta^a_b H,\\
&\left\{Q^{ia},Q^{jb}\right\}=-2i\,\varepsilon^{ij}\varepsilon^{ab}C,\qquad \left\{\bar{Q}^{ia},\bar{Q}^{jb}\right\}=2i\,\varepsilon^{ij}\varepsilon^{ab}\bar{C},\\
&\left[I^i_j,I^k_l\right]=\delta^k_j I^i_l-\delta^i_l I^k_j\,,\qquad [F^a_b,F^c_d]=\delta^c_b F^a_d-\delta^a_d F^c_b\,,\\
&\left[I^i_j,Q^{ka}\right]=\delta^k_j Q^{ia}-\frac{1}{2}\,\delta^i_j Q^{ka},\qquad \left[I^i_j,\bar{Q}_{la}\right]=\frac{1}{2}\,\delta^i_j\bar{Q}_{la}-\delta^i_l\bar{Q}_{ja}\,,\\
&\left[F^a_b,Q^{ic}\right]=\delta^c_b Q^{ia}-\frac{1}{2}\,\delta^a_b Q^{ic},\qquad [F^a_b,\bar{Q}_{ic}]=\frac{1}{2}\,\delta^a_b\bar{Q}_{ic}-\delta^a_c\bar{Q}_{ib}\,.
\end{aligned}\tag{5}$$

The superalgebra $su(2|2)$ contains in general three central charges C, $\bar{C}$ and H. The generators $I^{ij}=I^{ji}$, $F^{ab}=F^{ba}$ form two mutually commuting $su(2)$ algebras. The doublet indices $i=1,2$ and $a=1,2$ are raised and lowered in the standard way by two independent sets of ε^{ij} and ε^{ab} symbols. In the limit $m\to 0$, a centrally extended flat $\mathcal{N}=8$ superalgebra is reproduced, with two extra central charges C and $\bar{C}$ and SO(8) automorphisms broken to SU(2)×SU(2). The $su(2|1)$ subalgebra generators (4) can be distinguished as

$$Q^i=Q^{i1},\quad \bar{Q}_{j1}=\bar{Q}_j\,,\quad I^i_j=I^i_j\,,\quad F^1_1=F,\quad H=H.\tag{6}$$

The basic world-line realizations of SU$(2|2)$ supersymmetry are defined in the complex $\mathcal{N}=8$, $d=1$ superspace identified with the following supercoset of the supergroup with the superalgebra (5):

$$\frac{\left\{Q^{ia},\bar{Q}_{ia},I^i_j,F^a_b,C,\bar{C},H\right\}}{\left\{I^i_j,F^a_b,C,\bar{C}\right\}}\sim\left\{t,\theta^{ia},\bar{\theta}^{jb}\right\},\quad \overline{(\theta_{ia})}=\bar{\theta}^{ia}.\tag{7}$$

2.2. The Superalgebra $su(4|1)$

The superalgebra $su(4|1)$ is given by the following non-vanishing (anti)commutators:

$$\begin{aligned}
&\left\{Q^I,\bar{Q}_J\right\}=2m\,L^I_J+2\delta^I_J\mathcal{H},\qquad \left[L^I_J,L^K_L\right]=\delta^K_J L^I_L-\delta^I_L L^K_J,\\
&\left[L^I_J,Q^K\right]=\delta^K_J Q^I-\frac{1}{4}\,\delta^I_J Q^K,\qquad \left[L^I_J,\bar{Q}_L\right]=\frac{1}{4}\,\delta^I_J\bar{Q}_L-\delta^I_L\bar{Q}_J\,,\\
&\left[\mathcal{H},Q^K\right]=-\frac{3m}{4}\,Q^K,\qquad [\mathcal{H},\bar{Q}_L]=\frac{3m}{4}\,\bar{Q}_L\,.
\end{aligned}\tag{8}$$

Here, L^I_J are the generators of the R-symmetry group SU(4), and the capital indices I,J,K,L ($I=1,2,3,4$) refer to the SU(4) fundamental and anti-fundamental representations. The Hamiltonian is associated with the U(1) generator $\mathcal{H}$ (One could redefine the Hamiltonian to make it a central charge by the introduction of the external U(1) generator as in the Equation (3)).

In the contraction limit $m = 0$, the above superalgebra goes over to the SU(4) covariant form of the standard $\mathcal{N}=8$, $d=1$ superalgebra. This limiting superalgebra actually possesses an enhanced R-symmetry group SO(8) which mixes Q^I with $\bar{Q}_J$.

In the superalgebra (8), we can extract an SU(2) doublet from the SU(4) fundamental and anti-fundamental representations of the supercharges by restricting the SU(4) index I as $I \to i\ (i = 1, 2)$. In this way, one singles out in (8) a subalgebra containing twice as less supercharges. It coincides with the superalgebra (2) under the following identification of the SU(2|1) generators

$$Q^I \to Q^i, \quad \bar{Q}_J \to \bar{Q}_j, \qquad F = L^1_1 + L^2_2, \quad I^i_j = L^i_j - \frac{1}{2}\,\delta^i_j F, \quad \tilde{H} = \mathcal{H} + \frac{m}{2}\,F. \tag{9}$$

The basic SU$(4|1)$, $d=1$ superspace is defined as the coset superspace

$$\frac{\text{SU}(4|1)}{\text{SU}(4)} \sim \frac{\left\{Q^I, \bar{Q}_J, L^I_J, \mathcal{H}\right\}}{\left\{L^I_J\right\}} \sim \{\zeta\} = \left\{t, \theta_I, \bar{\theta}^J\right\}, \ \overline{(\theta_I)} = \bar{\theta}^I . \tag{10}$$

3. The SU(2|2) Chiral Multiplet $(\mathbf{5}, \mathbf{8}, \mathbf{3})$

As an example of the off-shell SU(2|2) $d = 1$ multiplets, we consider the multiplet $(\mathbf{5}, \mathbf{8}, \mathbf{3})$. It has a natural description in a chiral SU(2|2) superspace. We introduce the left chiral subspace parametrized by the complex coordinates

$$\zeta_{\text{L}} = \left\{t_{\text{L}}, \theta^{ia}\right\}, \tag{11}$$

with the following transformation properties

$$\delta\theta^{ia} = \epsilon^{ia} + 2m\,\theta^{ib}\theta^{ja}\left(\epsilon_{jb} - \bar{\epsilon}_{jb}\right), \qquad \delta t_{\text{L}} = -\,2i\theta^{ia}\bar{\epsilon}_{ia} + \frac{4im}{3}\,\theta^{ib}\theta^{ja}\theta_{jb}\bar{\epsilon}_{ia}\,. \tag{12}$$

Specializing in the ϵ^{i2}-transformations in the Equation (12) yields the odd transformations corresponding to the SU$(2|1)$ subgroup. The set (11) can be identified with the following complex supercoset of SU$(2|2)$:

$$\frac{\left\{Q^{ia}, \bar{Q}^{jb}, I^i_j, F^a_b, C, \bar{C}, H\right\}}{\left\{\bar{Q}^{jb}, I^i_j, F^a_b, C, \bar{C}\right\}}\,. \tag{13}$$

The invariant measure of integration over (11), $d\zeta_{\text{L}}$, is defined as

$$d\zeta_{\text{L}} = dt_{\text{L}}\,d^4\theta, \qquad \delta\left(d\zeta_{\text{L}}\right) = 0\,. \tag{14}$$

The SU(2|2) supermultiplet $(\mathbf{5}, \mathbf{8}, \mathbf{3})$ is described by SU(2|2) chiral superfield subjected to some extra constraints.

We start with an unconstrained complex superfield Ψ which is defined on the chiral subspace (11) and is given by the general θ-expansion

$$\begin{aligned}
&Z\left(t_{\text{L}}, \theta^{ia}\right) = z + \sqrt{2}\,\theta^{ia}\psi_{ia} + \theta^{ia}\theta^j_a A_{ij} + \theta^{ia}\theta^b_i B_{ab} + \frac{2\sqrt{2}}{3}\,\theta^{ib}\theta^{ja}\theta_{jb}\,\pi_{ia} + \frac{1}{3}\,\theta^{ib}\theta^{ja}\theta_{jb}\theta_{ia}D, \\
&B_{ab} = B_{ba}\,, \qquad A_{ij} = A_{ji}\,.
\end{aligned} \tag{15}$$

The superfield Z has no external indices with respect to the R-symmetry SU(2)×SU(2) subgroup and transforms as $\delta Z = 0$. This implies the following component transformations:

$$
\begin{aligned}
&\delta z = -\sqrt{2}\,\epsilon^{ia}\psi_{ia}\,, \qquad \delta\psi_{ia} = -\sqrt{2}\left(\epsilon^{j}_{a}A_{ij} + \epsilon^{b}_{i}B_{ab} - i\bar{\epsilon}_{ia}\dot{z}\right),\\
&\delta A_{ij} = -\sqrt{2}\,\epsilon^{a}_{(j}\left[\pi_{i)a} + m\,\psi_{i)a}\right] - \sqrt{2}\,\bar{\epsilon}^{a}_{(j}\left[i\dot{\psi}_{i)a} - m\,\psi_{i)a}\right],\\
&\delta B_{ab} = -\sqrt{2}\,\epsilon_{i(b}\left[\pi^{i}_{a)} + m\,\psi^{i}_{a)}\right] + \sqrt{2}\,\bar{\epsilon}_{i(b}\left[i\dot{\psi}^{i}_{a)} + m\,\psi^{i}_{a)}\right],\\
&\delta\pi_{ia} = \sqrt{2}\left(-i\bar{\epsilon}^{j}_{a}\dot{A}_{ij} + i\bar{\epsilon}^{b}_{i}\dot{B}_{ab} - \epsilon_{ia}D\right) + \sqrt{2}\,m\left[\left(\epsilon^{j}_{a} - \bar{\epsilon}^{j}_{a}\right)A_{ij} + \left(\epsilon^{b}_{i} - \bar{\epsilon}^{b}_{i}\right)B_{ab} - i\bar{\epsilon}_{ia}\dot{z}\right],\\
&\delta D = \sqrt{2}\,i\bar{\epsilon}^{ia}\left(\dot{\pi}_{ia} + m\,\dot{\psi}_{ia}\right).
\end{aligned} \tag{16}
$$

Their Lie brackets are easily checked to form SU$(2|2)$ symmetry. The chiral superfield (15) contains 16 bosonic and 16 fermionic fields and so is in fact reducible. To single out the multiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$, we can impose (by hand) the following extra SU$(2|2)$ covariant constraints (In [15], these extra constraints were written in terms of SU$(2|2)$ superfields living on the full superspace (7).)

$$
\begin{aligned}
&A_{ij} = \sqrt{2}\left(-i\dot{v}_{ij} + m\,v_{ij}\right), \qquad \pi_{ia} = -i\dot{\bar{\psi}}_{ia} + m\,\bar{\psi}_{ia} - m\,\psi_{ia}\,, \qquad D = \ddot{\bar{z}} + im\dot{\bar{z}}\,,\\
&\overline{(z)} = \bar{z}\,, \qquad \overline{(\psi_{ia})} = \bar{\psi}^{ia}, \qquad \overline{(v_{ij})} = v^{ij}, \qquad \overline{(B_{ab})} = B^{ab} = B^{ba}.
\end{aligned} \tag{17}
$$

The $d = 1$ field content now precisely matches with the multiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$. The deformed transformations (16) are rewritten for the remaining independent fields as

$$
\begin{aligned}
&\delta z = -\sqrt{2}\,\epsilon^{ia}\psi_{ia}\,, \qquad \delta\bar{z} = \sqrt{2}\,\bar{\epsilon}^{ia}\bar{\psi}_{ia}\,, \qquad \delta v_{ij} = -\epsilon^{a}_{(j}\bar{\psi}_{i)a} + \bar{\epsilon}^{a}_{(j}\psi_{i)a}\,,\\
&\delta\psi_{ia} = 2i\epsilon^{j}_{a}\dot{v}_{ij} - 2m\,\epsilon^{j}_{a}v_{ij} - \sqrt{2}\,\epsilon^{b}_{i}B_{ab} + \sqrt{2}\,i\bar{\epsilon}_{ia}\dot{z}\,,\\
&\delta\bar{\psi}_{ia} = -2i\bar{\epsilon}^{j}_{a}\dot{v}_{ij} - 2m\,\bar{\epsilon}^{j}_{a}v_{ij} - \sqrt{2}\,\bar{\epsilon}^{b}_{i}B_{ab} - \sqrt{2}\,i\epsilon_{ia}\dot{\bar{z}}\,,\\
&\delta B_{ab} = \sqrt{2}\,\epsilon_{i(b}\left[i\dot{\bar{\psi}}^{i}_{a)} - m\,\bar{\psi}^{i}_{a)}\right] + \sqrt{2}\,\bar{\epsilon}_{i(b}\left[i\dot{\psi}^{i}_{a)} + m\,\psi^{i}_{a)}\right].
\end{aligned} \tag{18}
$$

The SU$(2|2)$ invariant deformed action can be written as an integral over chiral subspaces, like in the case of flat $\mathcal{N}=8$ supersymmetry [17]:

$$
S_{(\mathbf{5},\mathbf{8},\mathbf{3})} = \frac{1}{4}\int d\zeta_{\rm L}\, f(Z) + \frac{1}{4}\int d\zeta_{\rm R}\,\bar{f}(\bar{Z}) = \int dt\,\mathcal{L}_{(\mathbf{5},\mathbf{8},\mathbf{3})}\,. \tag{19}
$$

The relevant component Lagrangian reads

$$
\begin{aligned}
\mathcal{L}_{(\mathbf{5},\mathbf{8},\mathbf{3})} \;=\;& g\left[\dot{\bar{z}}\dot{z} + \dot{v}_{ij}\dot{v}^{ij} + \frac{i}{2}\left(\psi_{ia}\dot{\bar{\psi}}^{ia} - \dot{\psi}_{ia}\bar{\psi}^{ia}\right) - m\,\psi_{ia}\bar{\psi}^{ia} - m^{2}v_{ij}v^{ij} + \frac{1}{2}\,B_{ab}B^{ab}\right]\\
&- im\left(\dot{\bar{z}}\partial_{z}f - \dot{z}\partial_{\bar{z}}\bar{f}\,\right) + \frac{i}{2}\left(\dot{\bar{z}}g_{\bar{z}} - \dot{z}g_{z}\right)\psi_{ia}\bar{\psi}^{ia} - \frac{1}{2}\left(g_{z}\,\psi^{i}_{a}\psi_{ib} + g_{\bar{z}}\,\bar{\psi}^{i}_{a}\bar{\psi}_{ib}\right)B^{ab}\\
&+ \frac{i}{\sqrt{2}}\left(g_{z}\,\psi_{ia}\psi^{a}_{j} - g_{\bar{z}}\,\bar{\psi}_{ia}\bar{\psi}^{a}_{j}\right)\dot{v}^{ij} - \frac{m}{\sqrt{2}}\left(g_{z}\,\psi_{ia}\psi^{a}_{j} + g_{\bar{z}}\,\bar{\psi}_{ia}\bar{\psi}^{a}_{j}\right)v^{ij}\\
&- \frac{1}{12}\left(g_{zz}\,\psi^{ib}\psi^{ja}\psi_{jb}\psi_{ia} + g_{\bar{z}\bar{z}}\,\bar{\psi}^{ib}\bar{\psi}^{ja}\bar{\psi}_{jb}\bar{\psi}_{ia}\right).
\end{aligned} \tag{20}
$$

Here, g is a special Kähler metric defined as

$$
g(z,\bar{z}) = \partial_{z}\partial_{z}f(z) + \partial_{\bar{z}}\partial_{\bar{z}}\bar{f}(\bar{z})\,, \qquad g_{z} = \frac{\partial g(z,\bar{z})}{\partial z}\,, \quad g_{\bar{z}} = \frac{\partial g(z,\bar{z})}{\partial\bar{z}}\,, \quad \text{etc.} \tag{21}
$$

As compared to the undeformed case, we observe the appearance of the oscillator-type fermionic ($\sim m$) and bosonic ($\sim m^{2}$) potential terms, as well as the internal bosonic Wess–Zumino term accompanied by some new Yukawa-type couplings.

The free action $S^{\text{free}}_{(\mathbf{5},\mathbf{8},\mathbf{3})}$ corresponds to the simplest choice $f(Z) = Z^2/4$. The corresponding component off-shell Lagrangian reads

$$\mathcal{L}^{\text{free}}_{(\mathbf{5},\mathbf{8},\mathbf{3})} = \dot{\bar{z}}\dot{z} + \dot{v}_{ij}\dot{v}^{ij} + \frac{i}{2}\left(\psi_{ia}\dot{\bar{\psi}}^{ia} - \dot{\psi}_{ia}\bar{\psi}^{ia}\right) - m\,\psi_{ia}\bar{\psi}^{ia} - \frac{i}{2}\,m\left(\dot{\bar{z}}z - \bar{z}\dot{z}\right) - m^2 v_{ij}v^{ij} + \frac{1}{2}\,B_{ab}B^{ab}. \tag{22}$$

SU(2|1) *Superfield Approach*

The supergroup SU(2|2) contains as a subgroup the supergroup SU(2|1). Hence, SU(2|2) supersymmetric mechanics can be equivalently viewed as SU(2|1) supersymmetric mechanics [8–11] associated with a few irreducible SU(2|1) multiplets forming a given SU(2|2) multiplet. The multiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$ can be split into SU(2|1) multiplets as $(\mathbf{4},\mathbf{4},\mathbf{0}) \oplus (\mathbf{1},\mathbf{4},\mathbf{3})$ or $(\mathbf{2},\mathbf{4},\mathbf{2}) \oplus (\mathbf{3},\mathbf{4},\mathbf{1})$. Following [15], we restrict our consideration to the second option.

Generally, the SU(2|2) invariant Lagrangian can be written in terms of SU(2|1) superfields. We define SU(2|1) as a subgroup of SU(2|2), such that the relevant $su(2|1)$ subalgebra is composed of the generators (6). We require the SU(2|1) Lagrangian to be invariant under the second SU(2) subgroup associated with F^a_b. Then, the closure of SU(2|1) transformations and SU(2) transformations necessarily yields the whole supersymmetry SU(2|2). Skipping details, the final metric $g := g\left(z,\bar{z},v_{ij}\right)$ of the target space is expressed as

$$g\left(z,\bar{z},v_{ij}\right) = -\,\frac{\partial^2\mathcal{F}\left(z,\bar{z},v_{ij}\right)}{\partial v^{ij}\,\partial v_{ij}} = \frac{4\,\partial^2\mathcal{F}\left(z,\bar{z},v_{ij}\right)}{\partial z\,\partial\bar{z}}\,, \tag{23}$$

where $\mathcal{F}$ is an arbitrary real scalar function of SU(2|1) superfields satisfying the five-dimensional Laplace equation [18,19]:

$$\left(\frac{4\,\partial^2}{\partial z\,\partial\bar{z}} + \frac{\partial^2}{\partial v^{ij}\,\partial v_{ij}}\right)\mathcal{F} = 0. \tag{24}$$

Equation (24) is none other than the conditions of the SU(2|2) supersymmetry. The metric (21) is the most general solution of (24) for those functions $\mathcal{F}$ which are restricted to the two-dimensional target space as $g \equiv g\left(z,\bar{z}\right)$. The actions for the SU(2|2) multiplet $(\mathbf{5},\mathbf{8},\mathbf{3})$ prove to be massive deformations of those for the same multiplet in the flat case [18].

The more general case with the target metric $g\left(z,\bar{z},v_{ij}\right)$ can also be worked out. Here, we give a simple example of such a model. We consider solutions involving dependence on the triplet v^{ij}. The most general solution with $g \equiv g\left(v_{ij}\right)$ yields the Lagrangian

$$\begin{aligned}
\mathcal{L}^{*}_{(\mathbf{5},\mathbf{8},\mathbf{3})} \;=\; & \frac{1}{2|v|}\Bigg[\dot{\bar{z}}\dot{z} + \dot{v}_{ij}\dot{v}^{ij} + \frac{i}{2}\left(\psi_{ia}\dot{\bar{\psi}}^{ia} - \dot{\psi}_{ia}\bar{\psi}^{ia}\right) + \frac{1}{2}\,B^{ab}B_{ab} + \frac{i}{|v|^2}\,\psi^{(i}_{a}\bar{\psi}^{j)a}v_{ik}\dot{v}^{k}_{j} \\
& + \frac{v^{ij}}{2\sqrt{2}\,|v|^2}\left(2\,\psi^a_i\bar{\psi}^b_j\,B_{ab} + i\psi_{ia}\psi^a_j\,\dot{\bar{z}} + i\bar{\psi}_{ia}\bar{\psi}^a_j\,\dot{z}\right) - \frac{3v_{(ij}v_{kl)}}{8|v|^4}\,\psi^a_i\psi_{ja}\,\bar{\psi}^b_k\bar{\psi}_{lb} \\
& - \frac{m}{2}\,\psi_{ia}\bar{\psi}^{ia} - m^2 v_{ij}v^{ij}\Bigg],
\end{aligned} \tag{25}$$

with

$$g\left(v_{ij}\right) = \frac{1}{2|v|}\,, \qquad |v| = \sqrt{v_{ij}v^{ij}}\,. \tag{26}$$

4. SU(4|1) Chiral Multiplets

The off-shell multiplets $(\mathbf{2},\mathbf{8},\mathbf{6})$ and $(\mathbf{8},\mathbf{8},\mathbf{0})$ of the flat $\mathcal{N}=8$, $d=1$ supersymmetry cannot be promoted to the SU(2|2) case. Such a deformation becomes possible in the case of SU(4|1) supersymmetry where the appropriate analogs of these multiplets are described by SU(4|1) chiral superfields subjected to some extra constraints. The extra constraints can be defined in terms of superfields living on the full superspace (10). We avoid calculation of the deformed covariant derivatives $\mathcal{D}^I$, $\bar{\mathcal{D}}_J$ (they in general involve complicated $\mathrm{U}(4)$ connection terms) and consider the multiplets $(\mathbf{2},\mathbf{8},\mathbf{6})$ and $(\mathbf{8},\mathbf{8},\mathbf{0})$ using the chiral superspace description. It should be noted, however, that the most general construction for the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ is achieved within the $\mathrm{SU}(2|1)$ superfield formulation [16].

The supergroup $\mathrm{SU}(4|1)$ admits two mutually conjugated complex supercosets which can be identified with the left and right chiral subspaces:

$$\{t_{\mathrm{L}},\theta_I\}\,,\qquad \left\{t_{\mathrm{R}},\bar{\theta}^J\right\}. \tag{27}$$

The left subspace is identified with the following coset space:

$$\zeta_{\mathrm{L}}=\{t_{\mathrm{L}},\theta_I\}\sim\frac{\left\{Q^I,\bar{Q}_J,L^I_J,\mathcal{H}\right\}}{\left\{\bar{Q}_J,L^I_J\right\}}. \tag{28}$$

The coordinate set ζ_{L} is closed under the relevant $\mathrm{SU}(4|1)$ supersymmetry transformations

$$\delta\theta_I=\epsilon_I+2m\,\bar{\epsilon}^K\theta_K\theta_I\,,\qquad \delta t_{\mathrm{L}}=2i\bar{\epsilon}^K\theta_K\,. \tag{29}$$

The invariant left chiral measure is defined as

$$d\zeta_{\mathrm{L}}:=dt_{\mathrm{L}}\,d^4\theta\,e^{-3imt_{\mathrm{L}}},\qquad \delta\left(d\zeta_{\mathrm{L}}\right)=0\,,\qquad \int d\zeta_{\mathrm{L}}\,\theta_I\theta_J\theta_K\theta_L\,e^{3imt_{\mathrm{L}}}=\varepsilon_{IJKL}\,. \tag{30}$$

Note that the Hamiltonian in this parametrization cannot be identified with the pure time derivative. In order to achieve this natural representation, one needs to pass to the new parametrization of superspace, changing the coordinates as

$$\tilde{\theta}_I=\theta_I\,e^{3imt_{\mathrm{L}}/4},\qquad t_{\mathrm{L}}=t_{\mathrm{L}}\,. \tag{31}$$

Just in this parametrization, the Hamiltonian takes the standard form $\mathcal{H}=i\partial_{t_{\mathrm{L}}}$. On the other hand, the advantage of the parametrization (28) is the simplest form of the transformations (29). Therefore, in what follows, it will be convenient to deal with such a simple parametrization. Due to the non-standard form of the Hamiltonian in this parametrization, all transformations and θ-expansions of the $\mathrm{SU}(4|1)$ superfields will be accompanied by the factors like $e^{3imt_{\mathrm{L}}/4}$. We consider the chiral superfield Φ given by the general θ-expansion

$$\begin{aligned}\Phi\left(t_{\mathrm{L}},\theta_I\right) \;=\;& \phi+\sqrt{2}\,\theta_K\chi^K e^{3imt_{\mathrm{L}}/4}+\theta_I\theta_J A^{IJ}e^{3imt_{\mathrm{L}}/2}+\frac{\sqrt{2}}{3}\,\theta_I\theta_J\theta_K\xi^{IJK}e^{9imt_{\mathrm{L}}/4}\\ &+\frac{1}{4}\,\varepsilon^{IJKL}\,\theta_I\theta_J\theta_K\theta_L B\,e^{3imt_{\mathrm{L}}},\quad A^{IJ}\equiv A^{[IJ]},\quad \xi^{IJK}\equiv\xi^{[IJK]}.\end{aligned} \tag{32}$$

The superfield Φ transforms as a singlet of the stability subgroup $\mathrm{SU}(4)$, i.e., $\delta_{su(4)}\Phi=0$. Taking into account (29), the transformations of its components under the odd generators look as:

$$
\begin{aligned}
&\delta\phi = -\sqrt{2}\,\epsilon_K\chi^K e^{3imt/4},\\
&\delta\chi^I = \sqrt{2}\,\bar\epsilon^I\,(i\dot\phi)\,e^{-3imt/4} - \sqrt{2}\,\epsilon_K A^{IK} e^{3imt/4},\\
&\delta A^{IJ} = 2\sqrt{2}\,\bar\epsilon^{[I}\left(i\dot\chi^{J]} + \frac{m}{4}\,\chi^{J]}\right)e^{-3imt/4} - \sqrt{2}\,\epsilon_K\xi^{IJK}e^{3imt/4},\\
&\frac{\sqrt{2}}{3}\,\delta\xi^{IJK} = 2\,\bar\epsilon^{[K}\left(i\dot A^{IJ]} + \frac{m}{2}\,A^{IJ]}\right)e^{-3imt/4} - \varepsilon^{IJKL}\,\epsilon_L B\,e^{3imt/4},\\
&\varepsilon^{IJKL}\,\delta B = \frac{8\sqrt{2}}{3}\,\bar\epsilon^{[L}\left(i\dot\xi^{IJK]} + \frac{3m}{4}\,\xi^{IJK]}\right)e^{-3imt/4}.
\end{aligned}
\tag{33}
$$

The general supersymmetric action can be written as a sum of integrals over chiral subspaces [17] as

$$
S_{\text{chiral}} = \int dt\,\mathcal{L}_{\text{chiral}} = -\frac{1}{4}\left[\int d\zeta_{\text{L}}\,K(\Phi) + \int d\zeta_{\text{R}}\,\bar K(\bar\Phi)\right],
\tag{34}
$$

where the overall coefficient $-1/4$ is chosen for further convenience. The component form of this SU$(4|1)$ invariant is given by the expression

$$
\begin{aligned}
S_{\text{chiral}} \;=\; & -\frac{1}{4}\int dt\left\{6B\,\partial_\phi K + \varepsilon_{IJKL}\left[\frac{2}{3}\,\chi^L\xi^{IJK} + \frac{1}{2}\,A^{IJ}A^{KL}\right](\partial_\phi)^2 K\right.\\
& \left. -\,\varepsilon_{IJKL}\,A^{IJ}\chi^K\chi^L\,(\partial_\phi)^3 K + \frac{1}{6}\,\varepsilon_{IJKL}\,\chi^I\chi^J\chi^K\chi^L\,(\partial_\phi)^4 K + \text{c.c.}\right\}.
\end{aligned}
\tag{35}
$$

This invariant does not display the kinetic term of the fields in (32) and so must be treated as a kind of "pre-action" for the multiplets $(\mathbf{2},\mathbf{8},\mathbf{6})$ and $(\mathbf{8},\mathbf{8},\mathbf{0})$. The genuine action appears after imposing some extra SU$(4|1)$ covariant conditions on the $d=1$ component fields in (32). Of course, they should follow from the appropriate superfield constraints [16], but it is much easier to guess their form directly at the component level, requiring the final field content to be $(\mathbf{2},\mathbf{8},\mathbf{6})$ or $(\mathbf{8},\mathbf{8},\mathbf{0})$ and resorting to the SU$(4|1)$-covariance reasonings.

4.1. The Multiplet $(\mathbf{2},\mathbf{8},\mathbf{6})$

The multiplets $(\mathbf{2},\mathbf{8},\mathbf{6})$ is described by the chiral superfield (32) subjected to the additional constraints

$$
\begin{aligned}
&\overline{(A^{IJ})} = A_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\,A^{KL}, \qquad B = \frac{2}{3}\left(\ddot{\bar\phi} + im\dot{\bar\phi}\right),\\
&\xi^{IJK} = -\,\varepsilon^{IJKL}\left(i\dot{\bar\chi}_L - \frac{m}{4}\,\bar\chi_L\right), \qquad \overline{(\chi^I)} = \bar\chi_I\,.
\end{aligned}
\tag{36}
$$

Substituting this into the transformation rules (33) gives the following odd transformations of the components of the multiplet $(\mathbf{2},\mathbf{8},\mathbf{6})$:

$$
\begin{aligned}
&\delta\phi = -\sqrt{2}\,\epsilon_K\chi^K e^{3imt/4}, \qquad \delta\bar\phi = \sqrt{2}\,\bar\epsilon^K\bar\chi_K\,e^{-3imt/4},\\
&\delta A^{IJ} = 2\sqrt{2}\,\bar\epsilon^{[I}\left(i\dot\chi^{J]} + \frac{m}{4}\chi^{J]}\right)e^{-3imt/4} + \sqrt{2}\,\varepsilon^{IJKL}\,\epsilon_{[K}\left(i\dot{\bar\chi}_{L]} - \frac{m}{4}\,\bar\chi_{L]}\right)e^{3imt/4},\\
&\delta\chi^I = \sqrt{2}\,\bar\epsilon^I\,(i\dot\phi)\,e^{-3imt/4} - \sqrt{2}\,\epsilon_K\,A^{IK}e^{3imt/4},\\
&\delta\bar\chi_I = -\sqrt{2}\,\epsilon_I\left(i\dot{\bar\phi}\right)e^{3imt/4} - \sqrt{2}\,\bar\epsilon^K A_{IK}\,e^{-3imt/4}.
\end{aligned}
\tag{37}
$$

The component Lagrangian (35), up to total time derivative, becomes

$$
\begin{aligned}
\mathcal{L}_{(\mathbf{2},\mathbf{8},\mathbf{6})} \;=\; & \left[\dot{\phi}\dot{\bar{\phi}} - \frac{1}{4}\,A^{IJ}A_{IJ} + \frac{i}{2}\left(\chi^K\dot{\bar{\chi}}_K - \dot{\chi}^K\bar{\chi}_K\right) - \frac{m}{4}\,\chi^K\bar{\chi}_K\right] g \\
& + im\left(\dot{\phi}\,\partial_{\bar{\phi}}\bar{K} - \dot{\bar{\phi}}\,\partial_{\phi}K\right) - \frac{i}{2}\left(\dot{\phi}\,\partial_{\phi}g - \dot{\bar{\phi}}\,\partial_{\bar{\phi}}g\right)\chi^K\bar{\chi}_K \\
& + \frac{1}{2}\,A_{IJ}\,\chi^I\chi^J\,\partial_{\phi}g - \frac{1}{2}\,A^{IJ}\bar{\chi}_I\bar{\chi}_J\,\partial_{\bar{\phi}}g \\
& - \frac{1}{24}\left[\varepsilon_{IJKL}\,\chi^I\chi^J\chi^K\chi^L\,\partial_z\partial_{\phi}g + \varepsilon^{IJKL}\,\bar{\chi}_I\bar{\chi}_J\bar{\chi}_K\bar{\chi}_L\,\partial_{\bar{\phi}}\partial_{\bar{\phi}}g\right], \qquad (38)
\end{aligned}
$$

where

$$
g\,(\phi,\bar{\phi}\,) = \partial_{\phi}\partial_{\phi}K\,(\phi) + \partial_{\bar{\phi}}\partial_{\bar{\phi}}\bar{K}\,(\bar{\phi})\,. \qquad (39)
$$

We observe that the complex fields ϕ parametrizes, as $f(z)$ in (21), a special Kähler manifold. The Lagrangian (38) is a deformation of the most general Lagrangian which was constructed in terms of $\mathcal{N}=4$ superfields in [20] and reproduced later in terms of $\mathcal{N}=8$ superfields in [17].

4.2. The Multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$

Proceeding in a similar way, we find that, in this case, the components of the chiral superfield (32) must be subjected to the following additional constraints:

$$
\begin{aligned}
& A^{IJ} = \sqrt{2}\left(i\dot{y}^{IJ} - \frac{m}{2}\,y^{IJ}\right), \qquad \overline{(y^{IJ})} = y_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\,y^{KL}, \\
& \xi^{IJK} = -\,\varepsilon^{IJKL}\left(i\dot{\bar{\chi}}_L - \frac{5m}{4}\,\bar{\chi}_L\right), \qquad \overline{(\chi^I)} = \bar{\chi}_I\,, \\
& B = \frac{2}{3}\left(\ddot{\bar{\phi}} + 2im\dot{\bar{\phi}}\right). \qquad (40)
\end{aligned}
$$

The odd SU$(4|1)$ transformations are realized on the minimal set of independent $8+8$ fields as:

$$
\begin{aligned}
& \delta\phi = -\sqrt{2}\,\epsilon_I\chi^I e^{3imt/4}, \qquad \delta\bar{\phi} = \sqrt{2}\,\bar{\epsilon}^I\bar{\chi}_I\,e^{-3imt/4}, \\
& \delta y^{IJ} = -\,2\,\bar{\epsilon}^{[I}\chi^{J]}e^{-3imt/4} + \varepsilon^{IJKL}\epsilon_K\bar{\chi}_L\,e^{3imt/4}, \\
& \delta\chi^I = \sqrt{2}\,\bar{\epsilon}^I\,(i\dot{\phi})\,e^{-3imt/4} - 2\,\epsilon_J\left(i\dot{y}^{IJ} - \frac{m}{2}\,y^{IJ}\right)e^{3imt/4}, \\
& \delta\bar{\chi}_I = -\sqrt{2}\,\epsilon_I\left(i\dot{\bar{\phi}}\right)e^{3imt/4} + 2\,\bar{\epsilon}^J\left(i\dot{y}_{IJ} + \frac{m}{2}\,y_{IJ}\right)e^{-3imt/4}. \qquad (41)
\end{aligned}
$$

They are consistent with the transformations (33) and leave invariant the constraints (40).

Substituting the constraints (40) into the pre-action (35), we find the component Lagrangian of the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$

$$
\begin{aligned}
\mathcal{L}_{(\mathbf{8},\mathbf{8},\mathbf{0})} \;=\; & g\left[\dot{\phi}\dot{\bar{\phi}} + \frac{1}{2}\,\dot{y}^{IJ}\dot{y}_{IJ} + \frac{i}{2}\left(\chi^K\dot{\bar{\chi}}_K - \dot{\chi}^K\bar{\chi}_K\right) - \frac{5m}{4}\,\chi^K\bar{\chi}_K - \frac{m^2}{8}\,y^{IJ}y_{IJ}\right] \\
& - \frac{im}{4}\left(\dot{\phi}\,\partial_{\phi}g - \dot{\bar{\phi}}\,\partial_{\bar{\phi}}g\right)y^{IJ}y_{IJ} + 2im\left(\dot{\phi}\,\partial_{\bar{\phi}}\bar{K} - \dot{\bar{\phi}}\,\partial_{\phi}K\right) \\
& + \frac{1}{\sqrt{2}}\left(i\dot{y}_{IJ} - \frac{m}{2}\,y_{IJ}\right)\chi^I\chi^J\,\partial_{\phi}g + \frac{1}{\sqrt{2}}\left(i\dot{y}^{IJ} + \frac{m}{2}\,y^{IJ}\right)\bar{\chi}_I\bar{\chi}_J\,\partial_{\bar{\phi}}g \\
& - \frac{i}{2}\left(\dot{\phi}\,\partial_{\phi}g - \dot{\bar{\phi}}\,\partial_{\bar{\phi}}g\right)\chi^K\bar{\chi}_K - \frac{1}{24}\,\varepsilon^{IJKL}\,\bar{\chi}_I\bar{\chi}_J\bar{\chi}_K\bar{\chi}_L\,\partial_{\bar{\phi}}\partial_{\bar{\phi}}g \\
& - \frac{1}{24}\,\varepsilon_{IJKL}\,\chi^I\chi^J\chi^K\chi^L\,\partial_{\phi}\partial_{\phi}g. \qquad (42)
\end{aligned}
$$

The target space metric coincides with the metric (39) of the special Kähler manifold. More general solutions involving some extra dependence on y^{IJ}, i.e., $g := g\left(\phi, \bar{\phi}, y^{IJ}\right)$, were obtained in [16] using the SU$(2|1)$ superfield formulation. By analogy to Section 3, there is an arbitrary real scalar function $f\left(\phi, \bar{\phi}, y^{IJ}\right)$ of SU(2|1) superfields which must satisfy the eight-dimensional Laplace equation. In the same way, the metric (39) is a particular solution of the eight-dimensional Laplace equation restricted to the two-dimensional target space, $g := g\left(\phi, \bar{\phi}\right)$.

5. Conclusions

Using $d = 1$ superfield approaches, we explained, on a few specific examples, how to construct models of the SU(2|2) supersymmetric mechanics based on the multiplet $(\mathbf{5}, \mathbf{8}, \mathbf{3})$, as well as models of SU(4|1) supersymmetric mechanics based on the multiplets $(\mathbf{2}, \mathbf{8}, \mathbf{6})$ and $(\mathbf{8}, \mathbf{8}, \mathbf{0})$. All three multiplets are described by chiral superfields subjected to some extra SU(2|2) and SU(4|1) covariant constraints. Their $\mathcal{N} = 8$ invariant deformed actions were written as integrals over chiral subspaces that leads to the target space metric corresponding to two-dimensional specical Kähler manifolds. In the case of the deformed multiplets $(\mathbf{5}, \mathbf{8}, \mathbf{3})$ and $(\mathbf{8}, \mathbf{8}, \mathbf{0})$, we have also pointed out that the most general construction of their invariant actions is achieved in terms of SU(2|1) superfields.

In [21], Berenstein, Maldacena and Nastase proposed an M-theory matrix model with 16 supercharges, which spurred investigations of massive super Yang–Mills mechanics. Since their matrix model has SU(4|2) supersymmetry, the SU(2|1), SU(2|2) and SU(4|1) supersymmetric mechanics models are expected to describe particular reductions of this general matrix system, with four and eight supercharges, respectively. In particular, the massive matrix models based on the multiplets $(\mathbf{5}, \mathbf{8}, \mathbf{3})$ and $(\mathbf{8}, \mathbf{8}, \mathbf{0})$, in the case of the simplest target space metric $g = 1$ (i.e., for the free model), were studied at the component level in [22,23], respectively. Our approach allows one to generate non-trivial interactions, which hopefully may be interpreted as effective actions with quantum corrections taken into account.

Author Contributions: The contributions of all authors are the same. All of them have worked together to develop the present manuscript.

Funding: The research of E.I. and S.S. was supported by the RFBR Grant No. 18-02-01046. Their research was also supported in parts by the Ministry of Science and Higher Education of Russian Federation, project No. 3.1386.2017.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

SQM Supersymmetric quantum mechanics

References

1. Festuccia, G.; Seiberg, N. Rigid supersymmetric theories in curved superspace. *JHEP* **2011**, *1106*, 114. [CrossRef]
2. Pestun, V. Localization of gauge theory on a four-sphere and supersymmetric Wilson loops. *Commun. Math. Phys.* **2012**, *313*, 71. [CrossRef]
3. Römelsberger, C. Counting chiral primaries in $N = 1$, $d = 4$ superconformal field theories. *Nucl. Phys. B* **2006**, *747*, 329–353. [CrossRef]
4. Assel, B.; Cassani, D.; Di Pietro, L.; Komargodski, Z.; Lorenzen, J.; Martelli, D. The Casimir Energy in Curved Space and its Supersymmetric Counterpart. *JHEP* **2015**, *7*, 043. [CrossRef]
5. Bellucci, S.; Nersessian, A. (Super)oscillator on CP(N) and constant magnetic field. *Phys. Rev. D* **2003**, *67*, 065013. [CrossRef]
6. Bellucci, S.; Nersessian, A. Supersymmetric Kähler oscillator in a constant magnetic field. In Proceedings of the 5th International Workshop on Supersymmetries and Quantum Symmetries (SQS'03), Dubna, Russia, 24–29 July 2003; pp. 379–384.

7. Smilga, A.V. Weak supersymmetry. *Phys. Lett. B* **2004**, *585*, 173. [CrossRef]
8. Ivanov, E.; Sidorov, S. Deformed supersymmetric mechanics. *Class. Quant. Grav.* **2014**, *31*, 075013. [CrossRef]
9. Ivanov, E.; Sidorov, S. Super Kähler oscillator from $SU(2|1)$ superspace. *J. Phys. A* **2014**, *47*, 292002. [CrossRef]
10. Ivanov, E.; Sidorov, S.; Toppan, F. Superconformal mechanics in $SU(2|1)$ superspace. *Phys. Rev. D* **2015**, *91*, 085032. [CrossRef]
11. Ivanov, E.; Sidorov, S. $SU(2|1)$ mechanics and harmonic superspace. *Class. Quant. Grav.* **2016**, *33*, 055001. [CrossRef]
12. Fedoruk, S.; Ivanov, E. Gauged spinning models with deformed supersymmetry. *JHEP* **2016**, *1611*, 103. [CrossRef]
13. Fedoruk, S.; Ivanov, E.; Sidorov, S. Deformed supersymmetric quantum mechanics with spin variables. *JHEP* **2018**, *1801*, 132. [CrossRef]
14. Fedoruk, S.; Ivanov, E.; Lechtenfeld, O.; Sidorov, S. Quantum $SU(2|1)$ supersymmetric Calogero–Moser spinning systems. *JHEP* **2018**, *1804*, 043. [CrossRef]
15. Ivanov, E.; Lechtenfeld, O.; Sidorov, S. $SU(2|2)$ supersymmetric mechanics. *JHEP* **2016**, *1611*, 031. [CrossRef]
16. Ivanov, E.; Lechtenfeld, O.; Sidorov, S. Deformed $\mathcal{N}=8$ mechanics of $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplets. *JHEP* **2018**, *1808*, 193. [CrossRef]
17. Bellucci, S.; Krivonos, S.; Nersessian, A. $\mathcal{N}=8$ supersymmetric mechanics on special Kähler manifolds. *Phys. Lett. B* **2005**, *605*, 181–184. [CrossRef]
18. Ivanov, E.; Smilga, A.V. Symplectic sigma models in superspace. *Nucl. Phys. B* **2004**, *694*, 473–492. [CrossRef]
19. Diaconescu, D.-E.; Entin, R. A non-renormalization theorem for the $d=1$, $N=8$ vector multiplet. *Phys. Rev. D* **1997**, *56*, 8045–8052. [CrossRef]
20. Bellucci, S.; Krivonos, S.; Nersessian, A.; Lechtenfeld, O. ABC of $N=8$, $d=1$ supermultiplets. *Nucl. Phys. B* **2004**, *699*, 226–252. [CrossRef]
21. Berenstein, D.; Maldacena, J.; Nastase, H. Strings in flat space and pp waves from $N=4$ super Yang–Mills. *JHEP* **2002**, *2002*, 013. [CrossRef]
22. Kim, N.; Park, J.-H. Massive super Yang–Mills quantum mechanics: classification and the relation to supermembrane. *Nucl. Phys. B* **2006**, *759*, 249–282. [CrossRef]
23. Motl, L.; Neitzke, A.; Sheikh-Jabbari, M.M. Heterotic plane wave matrix models and giant gluons. *JHEP* **2003**, *306*, 058. [CrossRef]