Published for SISSA by Springer

Received: *August 11, 2018*
Accepted: *August 28, 2018*
Published: *August 30, 2018*

Deformed $\mathcal{N}=8$ mechanics of $(8,8,0)$ multiplets

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Abstract: We construct new models of "curved" SU(4|1) supersymmetric mechanics based on two versions of the off-shell multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ which are "mirror" to each other. The worldline realizations of the supergroup SU(4|1) are treated as a deformation of flat $\mathcal{N}=8$, $d=1$ supersymmetry. Using SU(4|1) chiral superfields, we derive invariant actions for the first-type $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplet, which parametrizes special Kähler manifolds. Since we are not aware of a manifestly SU(4|1) covariant superfield formalism for the second-type $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplet, we perform a general construction of SU(4|1) invariant actions for both multiplet types in terms of SU(2|1) superfields. An important class of such actions enjoys superconformal OSp(8|2) invariance. We also build off-shell actions for the SU(4|1) multiplets $(\mathbf{6},\mathbf{8},\mathbf{2})$ and $(\mathbf{7},\mathbf{8},\mathbf{1})$ through appropriate substitutions for the component fields in the $(\mathbf{8},\mathbf{8},\mathbf{0})$ actions. The $(\mathbf{6},\mathbf{8},\mathbf{2})$ actions are shown to respect superconformal SU(4|1,1) invariance.

Keywords: Extended Supersymmetry, Superspaces, Space-Time Symmetries

ArXiv ePrint: 1807.11804

https://doi.org/10.1007/JHEP08(2018)193

Contents

1 Introduction

In recent years, mainly motivated by the study of higher-dimensional models with "curved" rigid supersymmetries (see e.g. [1]), there was a growth of activity in supersymmetric mechanics (SM) models underlain by some semi-simple superalgebras treated as deformations of flat one-dimensional supersymmetries with the same number of supercharges. The simplest superalgebra of this kind is $su(2|1)$ (and its central-charge extension $\widehat{su}(2|1)$), which is a deformation of rigid $\mathcal{N}=4, d=1$ supersymmetry by a mass-dimension parameter m. The first examples with a worldline realization of $su(2|1)$ supersymmetry were considered more than 10 years ago (prior to [1] and related works) in [2, 3] and in [4] (where it was named "weak $d=1$ supersymmetry"). The corresponding worldline $su(2|1)$ multiplets had $d=1$ field contents $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ and $(\mathbf{1}, \mathbf{4}, \mathbf{3})$.[1]

A systematic superfield approach to $su(2|1)$ supersymmetry was worked out in [6–8] and [9]. The models built on the multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$, $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ and $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ were studied at the classical and quantum level. Recently, $su(2|1)$ invariant versions of super Calogero-Moser systems were constructed and quantized [10–12]. The common notable features of all these models are:

- Oscillator-type Lagrangians for the bosonic fields, with m^2 as the oscillator strength,
- Wess-Zumino type terms for the bosonic fields, of the type $\sim im(\dot{z}\bar{z} - z\dot{\bar{z}})$,
- At the lowest energy levels, wave functions form *atypical* $su(2|1)$ multiplets, with unequal numbers of the bosonic and fermionic states.

It was of obvious interest to move one step further and to consider mechanics models with analogous deformations of $\mathcal{N}=8, d=1$ supersymmetry. In contrast to $\mathcal{N}=4$ supersymmetry, in the $\mathcal{N}=8$ case there exist two different possibilities for deformation due to the existence of two different superalgebras with eight supercharges: $su(2|2)$ and $su(4|1)$, with R-symmetry algebra $su(2) \oplus su(2)$ or $su(4) \oplus u(1)$, respectively.[2] The $su(2|2)$ models have been considered in [13] by analogy with the $su(2|1)$ case, on the basis of the appropriate superfield worldline formalism, as deformations of flat $\mathcal{N}=8$ SM models [14–18]. They were built on the off-shell multiplets $(\mathbf{3}, \mathbf{8}, \mathbf{5})$, $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ and $(\mathbf{5}, \mathbf{8}, \mathbf{3})$. One class of $(\mathbf{5}, \mathbf{8}, \mathbf{3})$ actions represents a massive deformation for the same multiplet in the flat case [19, 20]. Another class enjoys superconformal $\mathrm{OSp}(4^*|4)$ invariance. Remarkably, the superconformal group $\mathrm{OSp}(4^*|4)$ is a closure of its two different $\mathrm{SU}(2|2)$ subgroups, with deformation parameters m and $-m$. So any $\mathrm{SU}(2|2)$ invariant action involving only even powers of m is automatically superconformal. Based on this observation, the general $\mathrm{SU}(2|2)$ action of the multiplet $(\mathbf{3}, \mathbf{8}, \mathbf{5})$ was shown to be superconformal.

[1] Our notation follows ref. [5]: bold numerals denote, respectively, the number of physical bosonic, physical fermionic and auxiliary bosonic degrees of freedom in the given supermultiplet.

[2] In the $su(2|2)$ case one can also add two central charges.

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It turns out that some admissible multiplets of flat $\mathcal{N}=8$ supersymmetry do not have SU(2|2) analogs, most importantly the so called "root" $\mathcal{N}=8$ multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$. The significance of this root multiplet derives from the fact that all other flat $\mathcal{N}=8$ multiplets and their invariant actions can be obtained from the root one and its general actions through appropriate covariant substitution of the auxiliary fields (or Hamiltonian reductions, in the Hamiltonian formalism) [17] as a generalization of the phenomenon found in [5] at the linearized level.[3] Deforming the flat $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplet has remained an open problem.

In the present paper we show that the latter becomes possible within the alternative SU(4|1) deformation. Interestingly, there exist two such root SU(4|1) multiplets, which are complementary to each other in the sense that the SU(4) assignments of their fermionic and bosonic components are interchanged. Namely, in one multiplet, the bosonic d=1 fields are in $\underline{\mathbf{1}} \oplus \underline{\mathbf{1}}^* \oplus \underline{\mathbf{6}}$ of SU(4) (eight real fields) and the fermionic fields in $\underline{\mathbf{4}} \oplus \underline{\mathbf{4}}^*$ (4 complex fields), while in the other multiplet the bosonic fields are in $\underline{\mathbf{4}} \oplus \underline{\mathbf{4}}^*$ and the fermionic fields in $\underline{\mathbf{1}} \oplus \underline{\mathbf{1}}^* \oplus \underline{\mathbf{6}}$. In the "flat" $\mathcal{N}=8, d=1$ limit they go over to two different 8-dimensional multiplets of the SO(8) R-symmetry related by triality (see, e.g., [22, 23]). These two multiplets are analogs of the mutually "mirror" $\mathcal{N}=4$ multiplets $(\mathbf{4},\mathbf{4},\mathbf{0})$, for which bosonic and fermionic components form doublets with respect to different SU(2) factors of the SO(4) R-symmetry. For this reason it is natural to treat the two root SU(4|1) $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplets as "mirror" to each other.

The main incentive of our paper is constructing invariant actions for both types of the $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplets. To this end, we will use a manifestly SU(4|1) covariant superspace formalism along with the SU(2|1) superfield approach, in which the extra SU(4|1)/SU(2|1) transformations are realized in a hidden way. In some cases, it is simplest to use the component approach. The point is that SU(4|1) possesses many non-equivalent worldline supercosets, including the harmonic ones [24], and it is not easy to decide which superfield formalism is most adequate for one or another SU(4|1) multiplet. We utilize several versions of such an extended superfield approach for constructing invariant actions.

The paper is organized as follows. In section 2 we present the superalgebra $su(2|1)$ and describe the relevant worldline supercosets. In section 3, on the example of flat $\mathcal{N}=8, d=1$ supersymmetry, we discuss three possible $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplets, which are not equivalent if the SO(8) R-symmetry is broken, and argue that only two of them can be extended to the deformed SU(4|1) case. The various superfield and component descriptions of the first version of the SU(4|1) $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplet are the subject of section 4. We find three different classes of invariant actions for this multiplet, including an OSp(8|2) invariant one, with an R-symmetry enhanced to SO(8). The analogous treatment of the second version of the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ is given in section 5. We show that its general invariant action is superconformal and equivalent to the superconformal action of the first version. Summary and outlook are given in section 6. An appendix A contains details of calculating the invariant actions in the appropriate harmonic SU(4|1) superspaces, and in appendices B and C the off-shell actions for the SU(4|1) multiplets $(\mathbf{6},\mathbf{8},\mathbf{2})$ and $(\mathbf{7},\mathbf{8},\mathbf{1})$ are presented. The full set of (anti)commutation relations of the conformal superalgebra $osp(8|2)$ is given in appendix D.

[3] As an aside, the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ has a puzzling relationship with the octonion algebra [21].

2 Supergroup SU(4|1) and its worldline realizations

We consider SU(4|1) supersymmetry as a deformation of the standard $\mathcal{N}=8$, $d=1$ supersymmetry [14–17]. The superalgebra $su(4|1)$ is given by the following non-vanishing (anti)commutators:

$$\begin{aligned}
&\{Q^I, \bar{Q}_J\} = 2m\, L^I_J + 2\delta^I_J \mathcal{H}, && [L^I_J, L^K_L] = \delta^K_J L^I_L - \delta^I_L L^K_J\,,\\
&[L^I_J, Q^K] = \delta^K_J Q^I - \frac{1}{4}\,\delta^I_J Q^K, && [L^I_J, \bar{Q}_L] = \frac{1}{4}\,\delta^I_J \bar{Q}_L - \delta^I_L \bar{Q}_J\,,\\
&[\mathcal{H}, Q^K] = -\frac{3m}{4}\, Q^K, && [\mathcal{H}, \bar{Q}_L] = \frac{3m}{4}\, \bar{Q}_L\,.
\end{aligned} \tag{2.1}$$

Here, L^I_J are the generators of the R-symmetry group SU(4), and the capital indices I, J, K, L $(I=1,2,3,4)$ refer to the SU(4) fundamental ("quark") representation and its conjugate. $\mathcal{H}$ is the U(1) generator. In the contraction limit $m=0$ the above superalgebra goes over to the SU(4) covariant form of the flat $\mathcal{N}=8, d=1$ superalgebra. This limiting superalgebra actually possesses an enhanced R-symmetry group SO(8) which mixes Q^I with $\bar{Q}_J$ (they are joined into SO(8) spinor). In what follows we will not need the explicit form of these enhanced SO(8)/SU(4) transformations, except for their realizations on the covariant "flat' spinor derivatives.

The basic real SU(4|1), $d=1$ superspace is defined as the coset superspace

$$\frac{\mathrm{SU}(4|1)}{\mathrm{SU}(4)} \sim \frac{\{Q^I, \bar{Q}_J, L^I_J, \mathcal{H}\}}{\{L^I_J\}}\,, \tag{2.2}$$

with the coset parameters being the superspace coordinates:

$$\zeta = \left\{t, \theta_I, \bar{\theta}^J\right\}, \qquad \overline{(\theta_I)} = \bar{\theta}^I. \tag{2.3}$$

One could define these coordinates within the standard exponential parametrization of the supercoset. However, it will be more convenient to use another parametrization, the one associated with the purely fermionic coset SU(n|1)/U(n) defined in [30] (see also [31]). We uplift the U(1) group from the stability subgroup U(4) into the numerator and consider an extension of the SU(4|1)/U(4) coordinate set by a time coordinate t. Thus this U(1) generator is associated with the Hamiltonian. Following to [30], one can then write generators of (2.1) acting on the extended coset (2.2) as

$$\begin{aligned}
&Q^I = \frac{\partial}{\partial\theta_I} - 2m\,\bar{\theta}^I\bar{\theta}^K \frac{\partial}{\partial\bar{\theta}^K} + i\bar{\theta}^I\partial_t\,, \qquad \bar{Q}_J = \frac{\partial}{\partial\bar{\theta}^J} + 2m\,\theta_J\theta_K \frac{\partial}{\partial\theta_K} + i\theta_J\partial_t\,,\\
&L^I_J = \left(\bar{\theta}^I\frac{\partial}{\partial\bar{\theta}^J} - \theta_J\frac{\partial}{\partial\theta_I}\right) - \frac{\delta^I_J}{4}\left(\bar{\theta}^K\frac{\partial}{\partial\bar{\theta}^K} - \theta_K\frac{\partial}{\partial\theta_K}\right),\\
&\mathcal{H} = i\partial_t - \frac{3m}{4}\left(\bar{\theta}^K\frac{\partial}{\partial\bar{\theta}^K} - \theta_K\frac{\partial}{\partial\theta_K}\right).
\end{aligned} \tag{2.4}$$

Then, odd transformations corresponding to these supercharges are given by

$$\delta\theta_I = \epsilon_I + 2m\,\bar{\epsilon}^K\theta_K\theta_I\,, \qquad \delta\bar{\theta}^J = \bar{\epsilon}^J - 2m\,\epsilon_K\bar{\theta}^K\bar{\theta}^J, \qquad \delta t = i\left(\bar{\epsilon}^K\theta_K + \epsilon_K\bar{\theta}^K\right). \tag{2.5}$$

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According to [30], one can define the integration measure as

$$d\zeta := dt\, d^4\theta\, d^4\bar{\theta}\left(1 + 2m\,\bar{\theta}^K\theta_K\right)^3. \tag{2.6}$$

It is easily checked to be invariant under the transformations (2.5).

Note that the Hamiltonian in (2.4) is not a pure time derivative. One could pass to the new parametrization of superspace as

$$\tilde{\theta}_I = \theta_I\, e^{3imt/4}, \qquad \tilde{\bar{\theta}}^I = \bar{\theta}^I e^{-3imt/4}, \qquad t = t\,, \tag{2.7}$$

in which the Hamiltonian takes the standard form $\mathcal{H} = i\partial_t$. The advantage of the parametrization (2.3) is the simplest form of the transformations (2.5). So, in what follows it will be convenient to deal with such a simple parametrization. Due to the non-standard form of the Hamiltonian in this parametrization, all transformations and θ-expansions of the SU(4|1) superfields will be accompanied by the factors like $e^{\pm 3imt/4}$.

2.1 Chiral superspaces

The supergroup SU(4|1) admits two mutually conjugated complex supercosets which can be identified with the left and right chiral subspaces:

$$\zeta_{\rm L} = \left(t_{\rm L}, \theta_I\right), \qquad \zeta_{\rm R} = \left(t_{\rm R}, \bar{\theta}^J\right). \tag{2.8}$$

The left coordinate $t_{\rm L}$ is related to the real time coordinate t via

$$t_{\rm L} = t + \frac{i}{2m}\log\left(1 + 2m\,\bar{\theta}^K\theta_K\right). \tag{2.9}$$

Then we check that the left chiral space $\zeta_{\rm L}$ is closed under the supersymmetry transformations

$$\delta\theta_I = \epsilon_I + 2m\,\bar{\epsilon}^K\theta_K\theta_I\,, \qquad \delta t_{\rm L} = 2i\bar{\epsilon}^K\theta_K\,. \tag{2.10}$$

The invariant left chiral measure is defined as

$$d\zeta_{\rm L} := dt_{\rm L}\, d^4\theta\, e^{-3imt_{\rm L}}, \qquad \delta\left(d\zeta_{\rm L}\right) = 0\,, \qquad \int d\zeta_{\rm L}\,\theta_I\theta_J\theta_K\theta_L\, e^{3imt_{\rm L}} = \varepsilon_{IJKL}\,. \tag{2.11}$$

2.2 Reduction to SU(2|1), $d = 1$ superspace

One can consider reduction of the superspace (2.2) to the SU(2|1) superspace. It is performed on the superspace coordinates (2.3) as

$$\left\{t, \theta_i, \bar{\theta}^i\right\}, \qquad \overline{\left(\theta_i\right)} = \bar{\theta}^i, \quad i = 1, 2. \tag{2.12}$$

Limiting to the ϵ_1 and ϵ_2 transformations in (2.5), we obtain the reduced SU(2|1) supersymmetric transformations which coincide with those found in [6]:

$$\delta\theta_i = \epsilon_i + 2m\,\bar{\epsilon}^k\theta_k\theta_i\,, \quad \delta\bar{\theta}^j = \bar{\epsilon}^j - 2m\,\epsilon_k\bar{\theta}^k\bar{\theta}^j, \quad \delta t = i\left(\bar{\epsilon}^k\theta_k + \epsilon_k\bar{\theta}^k\right). \tag{2.13}$$

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Respectively, the superalgebra (2.1) contains as a subalgebra the extended $su(2|1) \niplus u(1)$ superalgebra:

$$
\begin{aligned}
&\left\{Q^i,\bar{Q}_j\right\}=2m\,I^i_j+m\,\delta^i_jF+2\delta^i_j\mathcal{H}, && \left[I^i_j,I^k_l\right]=\delta^k_jI^i_l-\delta^i_lI^k_j,\\
&\left[I^i_j,Q^k\right]=\delta^k_kQ^i-\frac{1}{2}\,\delta^i_jQ^k, && \left[I^i_j,\bar{Q}_l\right]=\frac{1}{2}\,\delta^i_j\bar{Q}_l-\delta^i_l\bar{Q}_j\,,\\
&\left[\mathcal{H},Q^k\right]=-\frac{3m}{4}\,Q^k, && \left[\mathcal{H},\bar{Q}_l\right]=\frac{3m}{4}\,\bar{Q}_l\,,\\
&\left[F,Q^k\right]=\frac{1}{2}\,Q^k, && \left[F,\bar{Q}_l\right]=-\frac{1}{2}\,\bar{Q}_l\,.
\end{aligned}
\tag{2.14}
$$

Here, SU(2) generators of SU(2|1) are defined as

$$
I^i_j=L^i_j-\frac{1}{2}\,\delta^i_jF. \tag{2.15}
$$

The combination $\mathcal{H}+\dfrac{m}{2}\,F$ can be identified with the internal U(1) generator of SU(2|1), while F becomes an external R-symmetry U(1) generator.

The explicit expressions for the covariant spinor derivatives $\mathcal{D}^k, \bar{\mathcal{D}}^k$ corresponding to the basic real coset of SU(2|1) defined in [8] and parametrized by the coordinates (2.12) with the transformation properties (2.13) are given by

$$
\begin{aligned}
\mathcal{D}^i&=e^{-3imt/4}\bigg\{\left[1+m\,\bar{\theta}^k\theta_k-\frac{3m^2}{8}\,(\theta)^2\left(\bar{\theta}\,\right)^2\right]\frac{\partial}{\partial\theta_i}-m\,\bar{\theta}^i\theta_j\frac{\partial}{\partial\theta_j}-i\bar{\theta}^i\partial_t\\
&\quad-\frac{m}{2}\,\bar{\theta}^i\tilde{F}-m\,\bar{\theta}^j\left(1-m\,\bar{\theta}^k\theta_k\right)\tilde{I}^i_j\bigg\},\\
\bar{\mathcal{D}}_j&=e^{3imt/4}\bigg\{-\left[1+m\,\bar{\theta}^k\theta_k-\frac{3m^2}{8}\,(\theta)^2\left(\bar{\theta}\,\right)^2\right]\frac{\partial}{\partial\bar{\theta}^j}+m\,\bar{\theta}^k\theta_j\frac{\partial}{\partial\bar{\theta}^k}+i\theta_j\partial_t\\
&\quad+\frac{m}{2}\,\theta_j\tilde{F}+m\,\theta_k\left(1-m\,\bar{\theta}^l\theta_l\right)\tilde{I}^k_j\bigg\},
\end{aligned}
\tag{2.16}
$$

where

$$
\begin{aligned}
&\tilde{I}^i_j\bar{\mathcal{D}}_l=\delta^i_l\bar{\mathcal{D}}_j-\frac{1}{2}\,\delta^i_j\bar{\mathcal{D}}_l\,, && \tilde{I}^i_j\mathcal{D}^k=\frac{1}{2}\,\delta^i_j\mathcal{D}^k-\delta^k_j\mathcal{D}^i,\\
&\tilde{F}\bar{\mathcal{D}}_l=\frac{1}{2}\,\bar{\mathcal{D}}_l\,, && \tilde{F}\mathcal{D}^k=-\frac{1}{2}\,\mathcal{D}^k.
\end{aligned}
\tag{2.17}
$$

In what follows we will avoid using the explicit form of the SU(4|1) counterparts of these derivatives, though they can be straightforwardly constructed by applying the standard coset (super)space machinery.

3 SU(4) covariant formulations of $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplet in flat $\mathcal{N}=8$ supersymmetry

Prior to the discussion of the superfield description of the root $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplets in SU(4|1) supersymmetry, we will consider SU(4) covariant form of its defining constraints in the standard flat $\mathcal{N}=8$ superspace, bearing in mind that the deformation to SU(4|1) mechanics must respect R-symmetry SU(4).

Such constraints can be written in the two superfield forms, both preserving not only SU(4) but also a non-manifest SO(8) R-symmetry.[4]

In the first formulation one deals with a chiral superfield Φ and an antisymmetric tensor superfield Y^{IJ} satisfying the constraints[5]

$$\begin{aligned} &\bar{D}_J\,\Phi = 0\,, \qquad D^I\,\bar{\Phi} = 0\,, \qquad \bar{D}_I\bar{D}_J\,\bar{\Phi} = \frac{1}{2}\,\varepsilon_{IJKL}\,D^KD^L\,\Phi\,, \\ &\sqrt{2}\,D^I\,Y^{JK} = -\varepsilon^{IJKL}\,\bar{D}_L\,\bar{\Phi}\,, \qquad \sqrt{2}\,\bar{D}_J\,Y_{KL} = \varepsilon_{IJKL}\,D^I\,\Phi\,, \\ &\overline{(Y^{IJ})} = Y_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\,Y^{KL}, \qquad \overline{(\Phi)} = \bar{\Phi}\,, \end{aligned} \tag{3.1}$$

where the flat covariant derivatives are defined as

$$D^I = \frac{\partial}{\partial\theta_I} - i\bar{\theta}^I\partial_t\,, \qquad \bar{D}_J = -\frac{\partial}{\partial\bar{\theta}^J} + i\theta_J\partial_t\,. \tag{3.2}$$

It is straightforward to check that (3.1) is covariant under the non-manifest SO(8)/SU(4) symmetry transformations realized as

$$\delta D^I = -\sqrt{2}\,\Lambda^{IJ}\bar{D}_J + i\lambda\,D^I, \qquad \delta\bar{D}_J = \sqrt{2}\,\bar{\Lambda}_{IJ}D^I - i\lambda\,\bar{D}_J\,, \tag{3.3}$$

$$\begin{aligned} &\delta\Phi = -\bar{\Lambda}^{IJ}Y_{IJ} - 2i\lambda\,\Phi\,, \qquad \delta\bar{\Phi} = -\Lambda^{IJ}Y_{IJ} + 2i\lambda\,\bar{\Phi}\,, \\ &\delta Y_{IJ} = \Lambda_{IJ}\,\Phi + \bar{\Lambda}_{IJ}\,\bar{\Phi}\,, \end{aligned} \tag{3.4}$$

where the antisymmetric complex 4×4 matrix

$$\Lambda_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\,\Lambda^{KL}, \qquad \bar{\Lambda}_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\,\bar{\Lambda}^{KL}, \tag{3.5}$$

accommodates just 12 real parameters of the coset SO(8)/U(4) and λ is the real $\mathrm{U}(1)\sim\mathrm{SO}(2)$ parameter. One can check that indeed

$$\Phi\bar{\Phi} + \frac{1}{2}\,Y^{IJ}Y_{IJ} = \text{inv}\,. \tag{3.6}$$

Another form of the SU(4) covariant superfield description of the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ involves the general superfield V^I which is subject to the constraints

$$\begin{aligned} &D^IV^J = \frac{1}{2}\,\varepsilon^{IJKL}\,\bar{D}_K\bar{V}_L\,, \qquad D^{(I}\,V^{J)} = 0\,, \qquad \bar{D}_{(K}\,\bar{V}_{L)} = 0\,, \\ &D^I\bar{V}_J = \frac{1}{4}\,\delta^I_JD^K\bar{V}_K\,, \qquad \bar{D}_JV^I = \frac{1}{4}\,\delta^I_J\bar{D}_KV^K \qquad \overline{(V^I)} = \bar{V}_I\,. \end{aligned} \tag{3.7}$$

The non-manifest SO(8)/SU(4) transformations of V^I leaving covariant the system (3.7) are written this time as

$$\delta V^I = \sqrt{2}\,\bar{\Lambda}^{IJ}\bar{V}_J - i\lambda\,V^I, \qquad \delta\bar{V}_J = -\sqrt{2}\,\Lambda_{IJ}V^I + i\lambda\,\bar{V}_J\,. \tag{3.8}$$

[4]In general, flat constraints defining the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ can be given many equivalent forms. For instance, in [15], they were written in $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$ covariant form. The common feature of all these formulations is the hidden covariance of the constraints under the full R-symmetry group of $\mathcal{N}=8$ superalgebra, the group SO(8).

[5]For further use, we introduce the antisymmetric tensor $\varepsilon^{IJKL} \equiv \varepsilon^{[IJKL]}$, such that

$$\varepsilon^{1234} = \varepsilon_{1234} = 1\,, \qquad \varepsilon^{IJKL}\varepsilon_{IJKL} = 24\,.$$

These transformations, together with the transformations of the covariant derivatives (3.3), preserve the constraints (3.7). One can also see that

$$V^I \bar{V}_I = \text{inv.} \tag{3.9}$$

It is rather easy to check that the constraints (3.1) leave in the bosonic sector of Φ, Y^{IJ} just the complex bosonic field $\phi(t)$ and tensorial field $y^{IJ}(t)$ which are first components of these superfields and transform as $\underline{\mathbf{1}}$ and $\underline{\mathbf{6}}$ of SU(4). The physical fermions are defined as $D^I \Phi|_{\theta=0}$ and transform as $\underline{\mathbf{4}}$ of SU(4). In the case of the constraints (3.9) the SU(4) assignment of the physical fields changes to the opposite: the physical bosons are the first components of V^J and transform as $\underline{\mathbf{4}}$, while fermions are defined as $\bar{D}_K V^K|_{\theta=0}$, $D^K \bar{V}_K|_{\theta=0}$, $D^{[I} V^{J]}|_{\theta=0}$ and transform as $\underline{\mathbf{1}} \oplus \underline{\mathbf{1}}^* \oplus \underline{\mathbf{6}}$. Thus, two $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets have "inverted" SU(4) contents: the contents of bosons and fermions of the first version coincide with those of fermions and bosons in the second one.

In order to better understand the interplay between the two forms of the $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplet, we note that the fermionic superfield $D^I \Phi$ transforms precisely as V^I. It is easy to check that it satisfies the constraints (3.7) as a consequence of (3.1). Analogously, the fermionic superfields $-2\sqrt{2}\, D^I V^J$ and $D^K \bar{V}_K$ possess the same transformation properties as Y^{IJ} and $\bar{\Phi}$, respectively. It is also straightforward to check that such fermionic superfields satisfy (3.1) as a consequence of (3.7). In other words, by the first multiplet one can construct the "derivative" fermionic multiplet satisfying the Grassmann-odd version of the second multiplet constraints (3.7). After establishing this correspondence, we could consider (3.7) for some new independent Grassmann-even superfield V^I and so come to the system (3.7) as an alternative description of the $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplet with the same Grassmann parities for the component fields as in the first version, but with "inverted" SU(4) assignments of these components. Its fermionic "derivative" satisfies the constraints (3.1).

This interplay between two $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets resembles a similar feature of "mirroring" of $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets in the standard (flat) $\mathcal{N}=4$ mechanics [25, 26]. The bosonic and fermionic components of the mutually mirror $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets form doublets with respect to different SU(2) factors of the full SO(4) R-symmetry group and are equivalent up to switching the roles of these two commuting SU(2) groups. However, there is an essential difference. In the $\mathcal{N}=4$ case the bosonic fields of the mutually mirror $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets are doublets of *different* R-symmetry SU(2) groups (the same is true for fermionic fields). As is seen from (3.6) and (3.9), in the $\mathcal{N}=8$ case the relevant fields form 8-dimensional irreps of the *same* full R-symmetry SO(8) and differ only in their assignments with respect to the fixed $\mathrm{U}(4) \subset \mathrm{SO}(8)$. So these two descriptions are associated with different embeddings of U(4) into SO(8). The first version corresponds to splitting $\mathrm{SO}(8) \to \mathrm{SO}(2) \times \mathrm{SO}(6)$ and representing the SO(8)-multiplet of superfields as a sum of SO(2) and SO(6) vectors. Then SU(4) is identified with SO(6), the additional R-symmetry U(1) with SO(2), while Φ and Y^{IK} with the corresponding SO(2) and SO(6) vectors. The second version corresponds to splitting $\mathrm{SO}(8) \to \mathrm{SO}(4) \times \mathrm{SO}(4)'$ and representing the relevant SO(8) superfield set as a sum of two 4-vectors. The diagonal SO(4) is identified with the "minimally embedded" $\mathrm{SO}(4) \subset \mathrm{SU}(4)$, and two 4-vectors are joined into a complex fundamental spinor V^I of SU(4).

Actually, the hidden SO(8) symmetry reveals the triality [22] between bosonic fields, fermionic fields and covariant derivatives. This triality interrelates the three irreducible fundamental representations of SO(8), *viz.* the vector representation and two spinorial ones.[6] All three representations can be written in the $\mathrm{SU}(4)\times\mathrm{U}(1)\sim\mathrm{SO}(6)\times\mathrm{SO}(2)$ notation [23] as

$$\begin{aligned} &\text{vector} \quad \underline{\mathbf{1}}_{1}\oplus\underline{\mathbf{1}}^{*}_{-1}\oplus\underline{\mathbf{6}}_{0}\,,\\ &\text{spinor} \quad \underline{\mathbf{4}}_{1/2}\oplus\underline{\mathbf{4}}^{*}_{-1/2}\,,\\ &\text{spinor} \quad \underline{\mathbf{4}}_{-1/2}\oplus\underline{\mathbf{4}}^{*}_{1/2}\,, \end{aligned} \tag{3.10}$$

where the subscript index refers to the U(1) charge. Comparing this with the U(4) assignments of the bosonic and fermionic fields of the $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplets, as well as of the covariant derivatives, we observe that just these SO(8) representations are realized on the quantities in question.

Supposing that the roles of two spinor representations can be switched, in flat $\mathcal{N}=8$, $d=1$ supersymmetry we can introduce yet a *third* multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ living on a different superspace, with the covariant derivatives defined as

$$\tilde{D}_{IJ}=\frac{1}{2}\,\varepsilon_{IJKL}\,\tilde{D}^{KL},\quad \overline{\left(\tilde{D}_{IJ}\right)}=\tilde{D}^{IJ},\qquad \tilde{D},\quad \bar{\tilde{D}}=\overline{\left(\tilde{D}\right)}, \tag{3.11}$$

and so belonging to the vector representation. However, an SU(4) covariant formulation of this third $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplet is beyond our purpose because the SU(4|1) covariant derivatives are SU(4) spinors by definition. So, this third option does not admit a generalization to SU(4|1) supersymmetry, in contrast to the first two.

In the case of the constraints (3.1), the bosonic fields belong to the SO(8) vector representation, and the fermionic fields form SO(8) spinor. For the multiplet given by (3.7) the picture is reversed, that is, the bosonic fields form an SO(8) spinor and the fermionic fields are combined into SO(8) vector. So, from the standpoint of SO(8) R-symmetry, due to the triality property, both $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplets can be considered as equivalent, once the spinorial representation of the covariant spinor derivatives has been fixed and one deals with SO(8) invariant actions for these multiplets (for more detail, see section 5.4).

The crucial point of the equivalence just discussed is the hidden SO(8) covariance of both sets of constraints (3.1) and (3.7). In the case of SU(4|1)-deformed mechanics there is no SO(8) R-symmetry, only U(4) remains. For this reason one cannot expect the corresponding counterparts of the two "flat" $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplets to be equivalent to one another.[7]

[6] To be more exact, the triality property is inherent to the group Spin(8).

[7] In the flat case the $\mathcal{N}=8$ supersymmetric Lagrangians are not obliged to simultaneously respect the full SO(8) symmetry. So for SO(8) non-invariant Lagrangians the equivalency of different $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplets may be broken in the flat case too.

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4 The SU(4|1) multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$: first version

The first version of the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ is defined by the SU(4|1) covariant constraints

$$
\begin{aligned}
&\bar{\mathcal{D}}_J\,\Phi = 0\,, \qquad \mathcal{D}^I\,\bar{\Phi} = 0\,, \qquad \bar{\mathcal{D}}_I\bar{\mathcal{D}}_J\,\bar{\Phi} = \frac{1}{2}\,\varepsilon_{IJKL}\,\mathcal{D}^K\mathcal{D}^L\,\Phi\,,\\
&\sqrt{2}\,\mathcal{D}^I\,Y^{JK} = -\varepsilon^{IJKL}\,\bar{\mathcal{D}}_L\,\bar{\Phi}\,, \qquad \sqrt{2}\,\bar{\mathcal{D}}_J\,Y_{KL} = \varepsilon_{IJKL}\,\mathcal{D}^I\,\Phi\,,\\
&\overline{(Y^{IJ})} = Y_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\,Y^{KL}, \qquad \overline{(\Phi)} = \bar{\Phi}\,,
\end{aligned}
\tag{4.1}
$$

where Φ is a chiral superfield and Y^{IJ} is an antisymmetric tensor superfield. In the flat limit, when $m \to 0$, $\bar{\mathcal{D}}_J \to \bar{D}_J$, $\mathcal{D}^I \to D^I$, this set of constraints becomes the set of superfield constraints (3.1) defining the standard $\mathcal{N}=8$, $d=1$ multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ [15], such that only $\mathrm{SU}(4) \subset \mathrm{SO}(8)$ is manifest.

In what follows, we avoid calculation of the deformed covariant derivatives $\mathcal{D}^I$, $\bar{\mathcal{D}}_J$ (they in general involve complicated U(4) connection terms) and consider the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ in the chiral superspace description, harmonic superspace description and SU(2|1) superfield approach.

4.1 Chiral superfield

We consider the chiral superfield Φ given by the general θ-expansion

$$
\begin{aligned}
\Phi\left(t_{\mathrm{L}},\theta_I\right) = {}& \phi + \sqrt{2}\,\theta_K\chi^K e^{3imt_{\mathrm{L}}/4} + \theta_I\theta_J A^{IJ} e^{3imt_{\mathrm{L}}/2} + \frac{\sqrt{2}}{3}\,\theta_I\theta_J\theta_K\xi^{IJK} e^{9imt_{\mathrm{L}}/4}\\
&+ \frac{1}{4}\,\varepsilon^{IJKL}\,\theta_I\theta_J\theta_K\theta_L B\,e^{3imt_{\mathrm{L}}}, \quad A^{IJ} \equiv A^{[IJ]}, \quad \xi^{IJK} \equiv \xi^{[IJK]}.
\end{aligned}
\tag{4.2}
$$

The superfield Φ transforms as a singlet of the stability subgroup SU(4), i.e. $\delta_{su(4)}\Phi = 0$. Taking into account (2.10), we can find the transformations of its components under the odd generators:

$$
\begin{aligned}
\delta\phi &= -\sqrt{2}\,\epsilon_K\chi^K e^{3imt/4},\\
\delta\chi^I &= \sqrt{2}\,\bar{\epsilon}^I\left(i\dot{\phi}\right)e^{-3imt/4} - \sqrt{2}\,\epsilon_K A^{IK} e^{3imt/4},\\
\delta A^{IJ} &= 2\sqrt{2}\,\bar{\epsilon}^{[I}\left(i\dot{\chi}^{J]} + \frac{m}{4}\,\chi^{J]}\right)e^{-3imt/4} - \sqrt{2}\,\epsilon_K\xi^{IJK} e^{3imt/4},\\
\frac{\sqrt{2}}{3}\,\delta\xi^{IJK} &= 2\,\bar{\epsilon}^{[K}\left(i\dot{A}^{IJ]} + \frac{m}{2}\,A^{IJ]}\right)e^{-3imt/4} - \varepsilon^{IJKL}\,\epsilon_L B\,e^{3imt/4},\\
\varepsilon^{IJKL}\,\delta B &= \frac{8\sqrt{2}}{3}\,\bar{\epsilon}^{[L}\left(i\dot{\xi}^{IJK]} + \frac{3m}{4}\,\xi^{IJK]}\right)e^{-3imt/4}.
\end{aligned}
\tag{4.3}
$$

The general supersymmetric action can be written as a sum of integrals over chiral subspaces [13, 18] as

$$
S_{\mathrm{chiral}} = \int dt\,\mathcal{L}_{\mathrm{chiral}} = -\frac{1}{4}\left[\int d\zeta_{\mathrm{L}}\,K\left(\Phi\right) + \int d\zeta_{\mathrm{R}}\,\bar{K}\left(\bar{\Phi}\,\right)\right],
\tag{4.4}
$$

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where the overall coefficient $-1/4$ is chosen for further convenience. The component form of this SU(4|1) invariant is found to be

$$S_{\rm chiral} = -\frac{1}{4}\int dt \Big\{ 6B\,\partial_\phi K + \varepsilon_{IJKL}\Big[\frac{2}{3}\,\chi^L\xi^{IJK} + \frac{1}{2}\,A^{IJ}A^{KL}\Big](\partial_\phi)^2 K - \varepsilon_{IJKL}\,A^{IJ}\chi^K\chi^L\,(\partial_\phi)^3 K + \frac{1}{6}\,\varepsilon_{IJKL}\,\chi^I\chi^J\chi^K\chi^L\,(\partial_\phi)^4 K + \text{c.c.}\Big\}. \tag{4.5}$$

This invariant does not display the kinetic term of the fields in (4.2) and so must be treated as a kind of "pre-action" for the $(\mathbf{8},\mathbf{8},\mathbf{0})$ multiplet. The genuine action appears after imposing some extra SU(4|1) covariant conditions on the components in (4.2). Of course they should follow from the rest of the superfield constraints (4.1), but it is easier to guess their form directly at the component level, requiring the final field content to be $(\mathbf{8},\mathbf{8},\mathbf{0})$ and resorting to the SU(4|1) covariance reasonings.

In this way we find that the components of the chiral superfield (4.2) must be subjected to the following additional constraints

$$\begin{aligned}
A^{IJ} &= \sqrt{2}\left(i\dot{y}^{IJ} - \frac{m}{2}\,y^{IJ}\right), &\qquad \overline{(y^{IJ})} &= y_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\,y^{KL},\\
\xi^{IJK} &= -\varepsilon^{IJKL}\left(i\dot{\bar{\chi}}_L - \frac{5m}{4}\,\bar{\chi}_L\right), &\qquad \overline{(\chi^I)} &= \bar{\chi}_I\,,\\
B &= \frac{2}{3}\left(\ddot{\bar{\phi}} + 2im\dot{\bar{\phi}}\right).
\end{aligned} \tag{4.6}$$

The odd SU(2|1) transformations are realized on this minimal set of fields as:

$$\begin{aligned}
\delta\phi &= -\sqrt{2}\,\epsilon_I\chi^I e^{3imt/4}, \qquad \delta\bar{\phi} = \sqrt{2}\,\bar{\epsilon}^I\bar{\chi}_I\,e^{-3imt/4},\\
\delta y^{IJ} &= -2\,\bar{\epsilon}^{[I}\chi^{J]}e^{-3imt/4} + \varepsilon^{IJKL}\epsilon_K\bar{\chi}_L\,e^{3imt/4},\\
\delta\chi^I &= \sqrt{2}\,\bar{\epsilon}^I\left(i\dot{\bar{\phi}}\right)e^{-3imt/4} - 2\,\epsilon_J\left(i\dot{y}^{IJ} - \frac{m}{2}\,y^{IJ}\right)e^{3imt/4},\\
\delta\bar{\chi}_I &= -\sqrt{2}\,\epsilon_I\left(i\dot{\phi}\right)e^{3imt/4} + 2\,\bar{\epsilon}^J\left(i\dot{y}_{IJ} + \frac{m}{2}\,y_{IJ}\right)e^{-3imt/4}.
\end{aligned} \tag{4.7}$$

They are consistent with the transformations (4.3) and leave invariant the constraints (4.6).

4.2 The final action

Substituting the constraints (4.6) into the pre-action (4.5), we find the correct component Lagrangian in the form

$$\begin{aligned}
\mathcal{L}_{\rm SK} = {}& g_1\left[\dot{\phi}\dot{\bar{\phi}} + \frac{1}{2}\,\dot{y}^{IJ}\dot{y}_{IJ} + \frac{i}{2}\left(\chi^K\dot{\bar{\chi}}_K - \dot{\chi}^K\bar{\chi}_K\right) - \frac{5m}{4}\,\chi^K\bar{\chi}_K - \frac{m^2}{8}\,y^{IJ}y_{IJ}\right]\\
&- \frac{im}{4}\left(\dot{\phi}\,\partial_\phi g_1 - \dot{\bar{\phi}}\,\partial_{\bar{\phi}}g_1\right)y^{IJ}y_{IJ} + 2im\left(\dot{\phi}\,\partial_{\bar{\phi}}\bar{K} - \dot{\bar{\phi}}\,\partial_\phi K\right)\\
&+ \frac{1}{\sqrt{2}}\left(i\dot{y}_{IJ} - \frac{m}{2}\,y_{IJ}\right)\chi^I\chi^J\,\partial_\phi g_1 + \frac{1}{\sqrt{2}}\left(i\dot{y}^{IJ} + \frac{m}{2}\,y^{IJ}\right)\bar{\chi}_I\bar{\chi}_J\,\partial_{\bar{\phi}}g_1\\
&- \frac{i}{2}\left(\dot{\phi}\,\partial_\phi g_1 - \dot{\bar{\phi}}\,\partial_{\bar{\phi}}g_1\right)\chi^K\bar{\chi}_K - \frac{1}{24}\,\varepsilon^{IJKL}\,\bar{\chi}_I\bar{\chi}_J\bar{\chi}_K\bar{\chi}_L\,\partial_{\bar{\phi}}\partial_{\bar{\phi}}g_1\\
&- \frac{1}{24}\,\varepsilon_{IJKL}\,\chi^I\chi^J\chi^K\chi^L\,\partial_\phi\partial_\phi g_1\,.
\end{aligned} \tag{4.8}$$

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We observe that the complex fields ϕ parametrizes a special Kähler (SK) manifold with the metric

$$g_1\left(\phi,\bar{\phi}\right)=\partial_{\phi}\partial_{\phi}K\left(\phi\right)+\partial_{\bar{\phi}}\partial_{\bar{\phi}}\bar{K}\left(\bar{\phi}\right). \tag{4.9}$$

4.3 Supercharges

The matrix models based on the multiplet under consideration, in the case of the simplest target space metric $g=1$ (i.e for the free model), were studied in [28]. Here, we consider a one-particle model generalized to the general SK metric (4.9) and find the relevant classical SU(4|1) supercharges. Poisson (Dirac) brackets are written as

$$\begin{aligned}
&\{\phi,p_{\phi}\}=1,\qquad \{\bar{\phi},p_{\bar{\phi}}\}=1,\qquad \{y^{KL},p_{IJ}\}=\frac{1}{2}\left(\delta_I^K\delta_J^L-\delta_I^L\delta_J^K\right),\\
&\{\chi^I,\bar{\chi}_J\}=-i\,\delta_J^I\,(g_1)^{-1},\\
&\{p_{\phi},\chi^I\}=\frac{1}{2}\,(g_1)^{-1}\,\partial_{\phi}g_1\,\chi^I,\qquad \{p_{\bar{\phi}},\chi^I\}=\frac{1}{2}\,(g_1)^{-1}\,\partial_{\bar{\phi}}g_1\,\chi^I,\\
&\{p_{\phi},\bar{\chi}_J\}=\frac{1}{2}\,(g_1)^{-1}\,\partial_{\phi}g_1\,\bar{\chi}_J\,,\qquad \{p_{\bar{\phi}},\bar{\chi}_J\}=\frac{1}{2}\,(g_1)^{-1}\,\partial_{\bar{\phi}}g_1\,\bar{\chi}_J\,.
\end{aligned} \tag{4.10}$$

Then the Noether supercharges are given by

$$\begin{aligned}
Q^I=e^{3imt/4}\bigg\{&2\bar{\chi}_K\left[p^{IK}+\frac{i}{2}\,m\,g_1\,y^{IK}-\frac{i}{6\sqrt{2}}\,\varepsilon^{IKLM}\bar{\chi}_L\bar{\chi}_M\,\partial_{\bar{\phi}}g_1\right]\\
&-\sqrt{2}\,\chi^I\left(p_{\phi}-2im\,\partial_{\bar{\phi}}\bar{K}+\frac{i}{4}\,m\,\partial_{\phi}g_1\,y^{KL}y_{KL}+\frac{i}{2}\,\partial_{\phi}g_1\,\chi^K\bar{\chi}_K\right)\bigg\},\\
\bar{Q}_J=e^{-3imt/4}\bigg\{&2\chi^K\left[p_{JK}-\frac{i}{2}\,m\,g_1\,y_{JK}-\frac{i}{6\sqrt{2}}\,\varepsilon_{JKLM}\chi^L\chi^M\,\partial_{\phi}g_1\right]\\
&-\sqrt{2}\,\bar{\chi}_J\left(p_{\bar{\phi}}+2im\,\partial_{\phi}K-\frac{i}{4}\,m\,\partial_{\bar{\phi}}g_1\,y^{KL}y_{KL}-\frac{i}{2}\,\partial_{\bar{\phi}}g_1\,\chi^K\bar{\chi}_K\right)\bigg\}.
\end{aligned} \tag{4.11}$$

Taking into account the brackets (4.10), these supercharges close on the following bosonic generators

$$\begin{aligned}
\mathcal{H}_{\rm SK}=(g_1)^{-1}&\left(p_{\phi}-2im\,\partial_{\bar{\phi}}\bar{K}+\frac{i}{4}\,m\,\partial_{\phi}g_1\,y^{IJ}y_{IJ}+\frac{i}{2}\,\partial_{\phi}g_1\,\chi^K\bar{\chi}_K\right)\\
&\times\left(p_{\bar{\phi}}+2im\,\partial_{\phi}K-\frac{i}{4}\,m\,\partial_{\bar{\phi}}g_1\,y^{IJ}y_{IJ}-\frac{i}{2}\,\partial_{\bar{\phi}}g_1\,\chi^K\bar{\chi}_K\right)\\
&+\frac{1}{2g_1}\left[p^{IJ}-\frac{i}{\sqrt{2}}\left(\chi^I\chi^J\partial_{\phi}g_1+\frac{1}{2}\,\varepsilon^{IJKL}\bar{\chi}_K\bar{\chi}_L\,\partial_{\bar{\phi}}g_1\right)\right]\\
&\times\left[p_{IJ}-\frac{i}{\sqrt{2}}\left(\bar{\chi}_I\bar{\chi}_J\,\partial_{\bar{\phi}}g_1+\frac{1}{2}\,\varepsilon_{IJMN}\,\chi^M\chi^N\partial_{\phi}g_1\right)\right]\\
&+g_1\left[\frac{5m}{4}\,\chi^K\bar{\chi}_K+\frac{m^2}{8}\,y^{IJ}y_{IJ}\right]+\frac{m}{2\sqrt{2}}\left(y_{IJ}\chi^I\chi^J\,\partial_{\phi}g_1-y^{IJ}\bar{\chi}_I\bar{\chi}_J\,\partial_{\bar{\phi}}g_1\right)\\
&+\frac{1}{24}\,\varepsilon^{IJKL}\,\bar{\chi}_I\bar{\chi}_J\bar{\chi}_K\bar{\chi}_L\,\partial_{\bar{\phi}}\partial_{\bar{\phi}}g_1+\frac{1}{24}\,\varepsilon_{IJKL}\,\chi^I\chi^J\chi^K\chi^L\,\partial_{\phi}\partial_{\phi}g_1\,,
\end{aligned} \tag{4.12}$$

$$L^I_J=2i\left(y^{IK}p_{JK}-\frac{\delta^I_J}{4}\,y^{KL}p_{KL}\right)+g_1\left(\chi^I\bar{\chi}_J-\frac{\delta^I_J}{4}\,\chi^K\bar{\chi}_K\right), \tag{4.13}$$

in full agreement with the superalgebra (2.1). The quantum version of these SU(4|1) (super)charges can be straightforwardly constructed and will be presented elsewhere.

4.4 Harmonic superspace description

We consider the harmonic coset of SU(4|1) with the harmonic part $\frac{\mathrm{SU}(4)}{[\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)]}$ [32]. The relevant harmonic variables are $u_a^{(+)I}$, $u_I^{(+)i}$, $u_I^{(-)a}$, $u_i^{(-)I}$ where $i=1,2$ and $a=1,2$ are the indices of the fundamental representations of the subgroup $\mathrm{SU}(2)\times\mathrm{SU}(2)$. The unitarity and unimodularity conditions are written as

$$
\begin{aligned}
&u_K^{(+)i}u_j^{(-)K}=\delta_j^i\,,\quad u_K^{(-)a}u_b^{(+)K}=\delta_b^a\,, && u_J^{(-)a}u_a^{(+)I}+u_J^{(+)i}u_i^{(-)I}=\delta_J^I\,,\\
&u_K^{(-)a}u_j^{(-)K}=u_K^{(+)i}u_b^{(+)K}=0\,, && \varepsilon^{IJKL}\varepsilon_{ij}\,u_K^{(+)i}u_L^{(+)j}+2\,\varepsilon^{ab}u_a^{(+)I}u_b^{(+)J}=0\,.
\end{aligned}\tag{4.14}
$$

Defining the harmonic projections of the SU(4|1) Grassmann coordinates as

$$
\begin{aligned}
&\theta_a^{(+)}=\theta_I\left(u_a^{(+)I}+m\,\bar\theta^{(+)k}\theta_a^{(+)}u_k^{(-)I}\right), && \theta_i^{(-)}=\theta_I\,u_i^{(-)I},\\
&\bar\theta^{(+)i}=\bar\theta^J\left(u_J^{(+)i}+m\,\bar\theta^{(+)i}\theta_c^{(+)}u_J^{(-)c}\right), && \bar\theta^{(-)a}=\bar\theta^J u_J^{(-)a},\\
&t_{\rm A}=t+i\left(\bar\theta^{(-)a}\theta_a^{(+)}-\bar\theta^{(+)i}\theta_i^{(-)}\right)\left[1-m\left(\bar\theta^{(-)a}\theta_a^{(+)}+\bar\theta^{(+)i}\theta_i^{(-)}\right)\right],
\end{aligned}\tag{4.15}
$$

one can find that they transform as

$$
\begin{aligned}
\delta\theta_i^{(-)}&=\epsilon_i^{(-)}+2m\left[\bar\epsilon^{(-)c}\theta_c^{(+)}\left(1+m\,\bar\theta^{(+)k}\theta_k^{(-)}\right)+\bar\epsilon^{(+)k}\theta_k^{(-)}\right]\theta_i^{(-)},\\
\delta\bar\theta^{(-)a}&=\bar\epsilon^{(-)a}-2m\left[\epsilon_c^{(+)}\bar\theta^{(-)c}+\epsilon_k^{(-)}\bar\theta^{(+)k}\left(1+m\,\bar\theta^{(-)c}\theta_c^{(+)}\right)\right]\bar\theta^{(-)a},\\
\delta\theta_a^{(+)}&=\epsilon_a^{(+)}+m\,\epsilon_k^{(-)}\bar\theta^{(+)k}\theta_a^{(+)}+2m\,\bar\epsilon^{(-)c}\theta_c^{(+)}\theta_a^{(+)},\\
\delta\bar\theta^{(+)i}&=\bar\epsilon^{(+)i}-m\,\bar\epsilon^{(-)c}\theta_c^{(+)}\bar\theta^{(+)i}-2m\,\epsilon_k^{(-)}\bar\theta^{(+)k}\bar\theta^{(+)i},\\
\delta u_I^{(+)i}&=-\Lambda_b^{(+2)i}u_I^{(-)b},\qquad \delta u_i^{(-)I}=0\,,\\
\delta u_b^{(+)I}&=\Lambda_b^{(+2)i}u_i^{(-)I},\qquad \delta u_I^{(-)b}=0\,,\\
\delta t_{\rm A}&=2i\left(\epsilon_k^{(-)}\bar\theta^{(+)k}+\bar\epsilon^{(-)c}\theta_c^{(+)}\right),
\end{aligned}\tag{4.16}
$$

where

$$
\begin{aligned}
&\Lambda_a^{(+2)i}=m\left(\epsilon_a^{(+)}\bar\theta^{(+)i}+\bar\epsilon^{(+)i}\theta_a^{(+)}\right)+m^2\left(\epsilon_k^{(-)}\bar\theta^{(+)k}+\bar\epsilon^{(-)c}\theta_c^{(+)}\right)\bar\theta^{(+)i}\theta_a^{(+)},\\
&\epsilon_i^{(-)}=\epsilon_I\,u_i^{(-)I},\quad \epsilon_a^{(+)}=\epsilon_I\,u_a^{(+)I},\quad \bar\epsilon^{(+)i}=\bar\epsilon^J u_J^{(+)i},\quad \bar\epsilon^{(-)a}=\bar\epsilon^J u_J^{(-)a}.
\end{aligned}\tag{4.17}
$$

We observe the existence of the analytic subspace closed under the SU(4|1) supersymmetry

$$
\zeta_{\rm A}=\left\{t_{\rm A},\theta_a^{(+)},\bar\theta^{(+)i},u_a^{(+)I},u_I^{(+)i},u_I^{(-)a},u_i^{(-)I}\right\}.\tag{4.18}
$$

Its integration measure is given by

$$
\begin{aligned}
&d\zeta_{\rm A}^{(-4)}=dt_{\rm A}\,du\,d^2\theta^{(+)}\,d^2\bar\theta^{(+)}\quad\Rightarrow\\
\Rightarrow\quad &\delta\left(d\zeta_{\rm A}^{(-4)}\right)=2\,d\zeta_{\rm A}^{(-4)}\Lambda^{(0)},\qquad \Lambda^{(0)}=m\left(\epsilon_k^{(-)}\bar\theta^{(+)k}-\bar\epsilon^{(-)c}\theta_c^{(+)}\right).
\end{aligned}\tag{4.19}
$$

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The only harmonic derivative $\mathcal{D}_a^{(+2)i}$ preserving the analytic subspace reads

$$\begin{aligned}\mathcal{D}_a^{(+2)i} &= u_a^{(+)K}\,\frac{\partial}{\partial u_i^{(-)K}} - u_K^{(+)i}\,\frac{\partial}{\partial u_K^{(-)a}} - 2i\bar{\theta}^{(+)i}\theta_a^{(+)}\partial_{\rm A} \\ &\quad + m\,\bar{\theta}^{(+)i}\theta_a^{(+)}\left(\bar{\theta}^{(+)k}\,\frac{\partial}{\partial\bar{\theta}^{(+)k}} - \theta_c^{(+)}\,\frac{\partial}{\partial\theta_c^{(+)}}\right) \\ &\quad + \frac{m^2}{4}\,\varepsilon^{ij}\varepsilon_{ab}\left(\theta^{(+)}\right)^4\left(u_j^{(-)K}\,\frac{\partial}{\partial u_b^{(+)K}} - u_K^{(-)b}\,\frac{\partial}{\partial u_K^{(+)j}}\right),\end{aligned} \tag{4.20}$$

where

$$\left(\theta^{(+)}\right)^4 = \left(\bar{\theta}^{(+)}\right)^2\left(\theta^{(+)}\right)^2 = \bar{\theta}^{(+)k}\bar{\theta}_k^{(+)}\theta_c^{(+)}\theta^{(+)c}. \tag{4.21}$$

The remaining harmonic covariant derivatives prove undeformed:

$$\begin{aligned}\mathcal{D}^{(0)} &= u_K^{(+)k}\,\frac{\partial}{\partial u_K^{(+)k}} + u_c^{(+)K}\,\frac{\partial}{\partial u_c^{(+)K}} - u_k^{(-)K}\,\frac{\partial}{\partial u_k^{(-)K}} - u_K^{(-)c}\,\frac{\partial}{\partial u_K^{(-)c}} \\ &\quad + \theta_c^{(+)}\,\frac{\partial}{\partial\theta_c^{(+)}} + \bar{\theta}^{(+)k}\,\frac{\partial}{\partial\bar{\theta}^{(+)k}}\,, \\ \mathcal{D}_j^i &= u_K^{(+)i}\,\frac{\partial}{\partial u_K^{(+)j}} - u_j^{(-)K}\,\frac{\partial}{\partial u_i^{(-)K}} + \bar{\theta}^{(+)i}\,\frac{\partial}{\partial\bar{\theta}^{(+)j}} \\ &\quad - \frac{\delta_j^i}{2}\left(u_K^{(+)k}\,\frac{\partial}{\partial u_K^{(+)k}} - u_k^{(-)K}\,\frac{\partial}{\partial u_k^{(-)K}} + \bar{\theta}^{(+)k}\,\frac{\partial}{\partial\bar{\theta}^{(+)k}}\right), \\ \mathcal{D}_b^a &= u_K^{(-)a}\,\frac{\partial}{\partial u_K^{(-)b}} - u_b^{(+)K}\,\frac{\partial}{\partial u_a^{(+)K}} - \theta_b^{(+)}\,\frac{\partial}{\partial\theta_a^{(+)}} \\ &\quad - \frac{\delta_b^a}{2}\left(u_K^{(-)c}\,\frac{\partial}{\partial u_K^{(-)c}} - u_c^{(+)K}\,\frac{\partial}{\partial u_c^{(+)K}} - \theta_c^{(+)}\,\frac{\partial}{\partial\theta_c^{(+)}}\right).\end{aligned} \tag{4.22}$$

One can check that

$$\begin{aligned}&\Lambda_a^{(+2)i} = \mathcal{D}_a^{(+2)i}\Lambda^{(0)}, \qquad \varepsilon^{ab}\varepsilon_{ij}\,\mathcal{D}_b^{(+2)j}\Lambda_a^{(+2)i} = m^2\,\delta\left(\theta^{(+)}\right)^4, \\ &\delta\left(\theta^{(+)}\right)^4 = 2\left(\delta\bar{\theta}^{(+)k}\right)\bar{\theta}_k^{(+)}\theta_c^{(+)}\theta^{(+)c} + 2\,\bar{\theta}^{(+)k}\bar{\theta}_k^{(+)}\theta_c^{(+)}\left(\delta\theta^{(+)c}\right),\end{aligned} \tag{4.23}$$

and

$$\delta\mathcal{D}_a^{(+2)i} = \Lambda_c^{(+2)i}\mathcal{D}_a^c - \Lambda_a^{(+2)k}\mathcal{D}_k^i - \frac{\Lambda_a^{(+2)i}}{2}\,\mathcal{D}^{(0)}. \tag{4.24}$$

4.4.1 Analytic harmonic superfield

The relevant analytic harmonic superfield is defined by the conditions

$$\mathcal{D}_a^{(+2)i}Y^{(+2)} = 0\,, \qquad \mathcal{D}_j^iY^{(+2)} = \mathcal{D}_b^aY^{(+2)} = 0\,, \tag{4.25}$$

and it transforms as

$$\delta Y^{(+2)} = \Lambda^{(0)}Y^{(+2)}. \tag{4.26}$$

It can be obtained by the "harmonization" of the superfield Y^{IJ} satisfying the constraints

$$\mathcal{D}^{(K}Y^{I)J} = 0\,, \qquad \bar{\mathcal{D}}_{(K}\,Y_{I)J} = 0\,. \tag{4.27}$$

These constraints in fact define the multiplet $(\mathbf{6}, \mathbf{8}, \mathbf{2})$. On the other hand, they are part of the full set of the constraints (4.1) defining the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$. Indeed, the solution of (4.25)

$$\begin{aligned}
Y^{(+2)} = y^{(+2)} &+ 2i\bar{\theta}^{(+)i}\theta^{(+)a}\dot{y}_{ia} - \bar{\theta}^{(+)k}\bar{\theta}^{(+)}_{k}\theta^{(+)}_{a}\theta^{(+)a}\ddot{y}^{(-2)} \\
&+ \bar{\theta}^{(+)}_{i}u^{(+)i}_{K}\chi^{K}e^{-3imt_{\mathrm{A}}/4} + \bar{\theta}^{(+)i}\bar{\theta}^{(+)}_{i}\theta^{(+)}_{a}u^{(-)a}_{K}\left(i\dot{\chi}^{K} + \frac{m}{4}\,\chi^{K}\right)e^{-3imt_{\mathrm{A}}/4} \\
&+ \theta^{(+)a}u^{(+)K}_{a}\bar{\chi}_{K}\,e^{3imt_{\mathrm{A}}/4} + \theta^{(+)}_{a}\theta^{(+)a}\bar{\theta}^{(+)i}u^{(-)K}_{i}\left(i\dot{\bar{\chi}}_{K} - \frac{m}{4}\,\bar{\chi}_{K}\right)e^{3imt_{\mathrm{A}}/4} \\
&+ \frac{1}{\sqrt{2}}\left(\bar{\theta}^{(+)i}\bar{\theta}^{(+)}_{i}D\,e^{-3imt_{\mathrm{A}}/2} + \theta^{(+)}_{a}\theta^{(+)a}\bar{D}\,e^{3imt_{\mathrm{A}}/2}\right),
\end{aligned} \tag{4.28}$$

reveals the field content $(\mathbf{6}, \mathbf{8}, \mathbf{2})$, where

$$\begin{aligned}
y^{(+2)} &= \frac{1}{2}\,\varepsilon^{ab}\,u^{(+)I}_{a}u^{(+)J}_{b}y_{IJ} + \frac{m^2}{4}\left(\theta^{(+)}\right)^4 y^{(-2)}\,, \\
y_{ia} &= u^{(+)I}_{a}u^{(-)J}_{i}y_{IJ}\,, \quad y^{(-2)} = \frac{1}{2}\,\varepsilon^{ij}\,u^{(-)I}_{i}u^{(-)J}_{j}y_{IJ}\,.
\end{aligned} \tag{4.29}$$

The component fields transform as

$$\begin{aligned}
\delta D &= -\sqrt{2}\,\epsilon_{I}\left(i\dot{\chi}^{I} - \frac{3m}{4}\,\chi^{I}\right)e^{3imt/4}, \qquad \delta\bar{D} = -\sqrt{2}\,\bar{\epsilon}^{I}\left(i\dot{\bar{\chi}}_{I} + \frac{3m}{4}\,\bar{\chi}_{I}\right)e^{-3imt/4}, \\
\delta y^{IJ} &= -2\,\bar{\epsilon}^{[I}\chi^{J]}e^{-3imt/4} + \varepsilon^{IJKL}\epsilon_{K}\bar{\chi}_{L}\,e^{3imt/4}, \\
\delta\chi^{I} &= \sqrt{2}\,\bar{\epsilon}^{I}D\,e^{-3imt/4} - 2\,\epsilon_{J}\left(i\dot{y}^{IJ} - \frac{m}{2}\,y^{IJ}\right)e^{3imt/4}, \\
\delta\bar{\chi}_{I} &= \sqrt{2}\,\epsilon_{I}\bar{D}\,e^{3imt/4} + 2\,\bar{\epsilon}^{J}\left(i\dot{y}_{IJ} + \frac{m}{2}\,y_{IJ}\right)e^{-3imt/4}.
\end{aligned} \tag{4.30}$$

The substitution $D = i\dot{\phi}$ in these transformations gives just the transformations (4.7) of the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$. Thus this substitution ensures the validity of the additional constraints imposed on the superfield Y^{IJ}. We conclude that the SU$(4|1)$ multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ admits an alternative description within harmonic SU$(4|1)$ superspace.

4.4.2 Invariant action via harmonic superspace

Introducing the shifted superfield

$$\begin{aligned}
Y^{(+2)} &= \hat{Y}^{(+2)} + c^{(+2)}, \\
c^{(+2)} &= \frac{1}{2}\,\varepsilon^{ab}\,u^{(+)I}_{a}u^{(+)J}_{b}c_{IJ} + \frac{m^2}{4}\left(\theta^{(+)}\right)^4 c^{(-2)}, \qquad c^{(-2)} = \frac{1}{2}\,\varepsilon^{ij}\,u^{(-)I}_{i}u^{(-)J}_{j}c_{IJ}\,, \\
\delta\hat{Y}^{(+2)} &= \Lambda^{(0)}\left(\hat{Y}^{(+2)} + c^{(+2)}\right) + \varepsilon^{ab}\varepsilon_{ij}\,\Lambda^{(+2)i}_{a}\left(\mathcal{D}^{(+2)j}_{b}c^{(-2)}\right) \\
&\quad - \frac{1}{4}\,\varepsilon^{ab}\varepsilon_{ij}\,c^{(-2)}\left(\mathcal{D}^{(+2)j}_{b}\Lambda^{(+2)i}_{a}\right),
\end{aligned} \tag{4.31}$$

we calculate the invariant action (see appendix A) as

$$
\begin{aligned}
&S_{(\mathbf{6},\mathbf{8},\mathbf{2})} = \frac{1}{16}\int d\zeta_{\rm A}^{(-4)} L^{(+4)}, \\
&L^{(+4)} = \frac{\hat{Y}^{(+2)}\hat{Y}^{(+2)}}{\left(1+c^{(-2)}\hat{Y}^{(+2)}\right)^4} + \frac{m^2}{2}\left(\theta^{(+)}\right)^4\left[1-\frac{1-c^{(-2)}\hat{Y}^{(+2)}}{\left(1+c^{(-2)}\hat{Y}^{(+2)}\right)^5}\right].
\end{aligned}
\tag{4.32}
$$

This action of the multiplet $(\mathbf{6},\mathbf{8},\mathbf{2})$ is in fact superconformal with respect to the supergroup SU$(4|1,1)$ (see appendix B). The relevant metric is SO(6) invariant and given by

$$
g_2 = \left[\frac{1}{2}\,y^{IJ}y_{IJ}\right]^{-2}. \tag{4.33}
$$

Substituting $D = i\dot{\phi}$, one can finally find the bosonic truncation of the component Lagrangian for the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$:

$$
\mathcal{L}_{\rm bos.} = g_2\left(\dot{\phi}\dot{\bar{\phi}} + \frac{1}{2}\,\dot{y}^{IJ}\dot{y}_{IJ} - \frac{m^2}{8}\,y^{IJ}y_{IJ}\right). \tag{4.34}
$$

Calculation of all terms in harmonic superspace is rather complicated. We skip all these calculations and write the full component Lagrangian (4.50) in the next subsection by employing SU$(2|1)$ superfields.

4.5 SU(2|1) superfield approach

To simplify the construction of SU$(4|1)$ invariant actions, it will be convenient to employ SU$(2|1)$ superfield approach elaborated in [6–9]. We split the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ into SU$(2|1)$ multiplets as a sum of the conventional multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ and the "mirror" multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ [9]. To obtain such a decomposition, we need to single out the ϵ_1 and ϵ_2 subvariety of the transformations of (4.7) corresponding to the SU$(2|1)$ superspace transformations (2.13). The SU$(2|1)$ covariant constraints given below involve the covariant derivatives (2.16).

4.5.1 The standard multiplet (4, 4, 0)

Introducing the new notations

$$
\begin{aligned}
&x^{11} := y^{14}, \qquad x^{12} := y^{13}, \qquad x^{21} := y^{24}, \qquad x^{22} := y^{23}, \\
&\xi^{1} := \bar{\chi}_3\,, \qquad \xi^{2} := -\bar{\chi}_4\,, \qquad \bar{\xi}_1 := \chi^3, \qquad \bar{\xi}_2 := -\chi^4, \\
&\overline{(x^{ia})} = \varepsilon_{ab}\,\varepsilon_{ij}\,x^{jb}, \quad \overline{(\xi^a)} = \bar{\xi}_a\,, \qquad \overline{(\xi^a)} = \bar{\xi}_a\,,
\end{aligned}
\tag{4.35}
$$

we obtain the same deformed transformations as in [9]:

$$
\begin{aligned}
&\delta x^{ia} = -\left(\epsilon^i\xi^a e^{3imt/4} + \bar{\epsilon}^i\bar{\xi}^a e^{-3imt/4}\right), \\
&\delta\xi^a = \bar{\epsilon}^k\left(2i\dot{x}^a_k + m\,x^a_k\right)e^{-3imt/4}, \qquad \delta\bar{\xi}^a = \epsilon_k\left(2i\dot{x}^{ka} - m\,x^{ka}\right)e^{3imt/4}.
\end{aligned}
\tag{4.36}
$$

The indices $i=1,2$ and $a=1,2$ correspond to the fundamental representations of the subgroup $\mathrm{SU}(2)\times\mathrm{SU}(2)\subset\mathrm{SU}(4)$.

The corresponding superfield q^{ia} obeys the SU(2|1) covariant constraints

$$\mathcal{D}^{(k}q^{i)a}=\bar{\mathcal{D}}^{(k}q^{i)a}=0\,,\qquad \tilde{F}q^{ia}=0\,,\qquad \overline{(q^{ia})}=q_{ia}\,. \tag{4.37}$$

These constraints are solved by

$$\begin{aligned} q^{ia}=&\left[1+\frac{m}{2}\,\bar{\theta}^k\theta_k-\frac{5m^2}{16}\,(\bar{\theta}\,)^2\,(\theta)^2\right]x^{ia}+\left(1+\frac{m}{4}\,\bar{\theta}^k\theta_k\right)\left(\theta^i\xi^a e^{3imt/4}+\bar{\theta}^i\bar{\xi}^a e^{-3imt/4}\right)\\ &+i\left(\bar{\theta}^k\theta^i\dot{x}^a_k-\bar{\theta}^i\theta_k\dot{x}^{ka}\right)-i\bar{\theta}^k\theta_k\left(\theta^i\dot{\xi}^a e^{3imt/4}-\bar{\theta}^i\dot{\bar{\xi}}^a e^{-3imt/4}\right)+\frac{1}{4}\,(\bar{\theta}\,)^2\,(\theta)^2\,\ddot{x}^{ia}, \end{aligned} \tag{4.38}$$

where the following conventions for the Grassmann monomials were employed: $(\theta)^2=\theta_i\theta^i$, $(\bar{\theta}\,)^2=\bar{\theta}^i\bar{\theta}_i$.

4.5.2 The mirror multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$

The "mirror" $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet is defined by the transformations

$$\begin{aligned} &\delta z=-\epsilon_k\psi^k e^{3imt/4}, &&\delta\bar{z}=\bar{\epsilon}^k\bar{\psi}_k\,e^{-3imt/4},\\ &\delta y=-\epsilon_k\bar{\psi}^k e^{3imt/4}, &&\delta\bar{y}=-\bar{\epsilon}^k\psi_k\,e^{-3imt/4},\\ &\delta\psi^i=\bar{\epsilon}^i\,(2i\dot{z})\,e^{-3imt/4}+\epsilon^i\,(2i\dot{\bar{y}}-m\bar{y})\,e^{3imt/4}, \\ &\delta\bar{\psi}_i=-\epsilon_i\,(2i\dot{\bar{z}})\,e^{3imt/4}+\bar{\epsilon}_i\,(2i\dot{y}+my)\,e^{-3imt/4}, \end{aligned} \tag{4.39}$$

where

$$\begin{aligned} &\sqrt{2}\,z:=\phi\,, &&\sqrt{2}\,\bar{z}:=\bar{\phi}\,, &&y:=y^{34}, &&\bar{y}:=y^{12},\\ &\psi^1:=\chi^1, &&\psi^2:=\chi^2, &&\bar{\psi}_1:=\bar{\chi}_1\,, &&\bar{\psi}_2:=\bar{\chi}_2\,. \end{aligned} \tag{4.40}$$

These transformations differ from those given in [9]. In the present case, Pauli-Gürsey SU(2) symmetry is broken. For this case the SU(2|1) superfield constraints defining the mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet are written as

$$\begin{aligned} &\bar{\mathcal{D}}^i Z=\bar{\mathcal{D}}^i Y=0\,, &&\mathcal{D}^i\bar{Z}=\mathcal{D}^i\bar{Y}=0\,,\\ &\mathcal{D}^i Z=-\bar{\mathcal{D}}^i\bar{Y}, &&\mathcal{D}^i Y=\bar{\mathcal{D}}^i\bar{Z},\\ &\tilde{F}Z=0\,, &&\tilde{F}Y=Y. \end{aligned} \tag{4.41}$$

Their solution reads

$$\begin{aligned} Z=&\;z+\theta_i\psi^i e^{3imt/4}+i\bar{\theta}^j\theta_j\,\dot{z}-(\theta)^2\left(i\dot{\bar{y}}-\frac{m}{2}\,\bar{y}\right)e^{3imt/2}+\bar{\theta}^j\theta_j\theta_i\left(i\dot{\psi}^i-\frac{3m}{4}\,\psi^i\right)e^{3imt/4}\\ &-\frac{1}{4}\,(\bar{\theta}\,)^2\,(\theta)^2\,(\ddot{z}+2im\dot{z})\,,\\ Y=&\;y+\theta_i\bar{\psi}^i e^{3imt/4}+\bar{\theta}^j\theta_j\left(i\dot{y}+\frac{m}{2}\,y\right)+i\,(\theta)^2\,\dot{\bar{z}}\,e^{3imt/2}+\bar{\theta}^j\theta_j\theta_i\left(i\dot{\bar{\psi}}^i-\frac{m}{4}\,\bar{\psi}^i\right)e^{3imt/4}\\ &-\frac{1}{4}\,(\bar{\theta}\,)^2\,(\theta)^2\left(\ddot{y}+im\dot{y}+\frac{3m^2}{4}\,y\right). \end{aligned} \tag{4.42}$$

4.5.3 SU(2|1) superfield action

The construction of SU(4|1) invariant actions in terms of the SU(2|1) superfields (4.38), (4.42) goes as follows. The general SU(2|1) superfield action can be written as

$$S = \int dt\, d^2\theta\, d^2\bar{\theta} \left(1 + 2m\,\bar{\theta}^k\theta_k\right) f\left(Z, \bar{Z}, Y\bar{Y}, q^{ia}q_{ia}\right). \tag{4.43}$$

The target space metric g is defined according to [17] as

$$\begin{gathered} g = \Delta_2 f = -\Delta_1 f\,, \qquad f = f\left(z, \bar{z}, y\bar{y}, x^{ia}x_{ia}\right), \qquad g = g\left(z, \bar{z}, y\bar{y}, x^{ia}x_{ia}\right), \\ \Delta_1 f + \Delta_2 f = 0 \quad \Rightarrow \quad \Delta_1 g + \Delta_2 g = 0\,, \end{gathered} \tag{4.44}$$

where

$$\begin{gathered} \partial_{ia} = \partial/\partial x^{ia}, \qquad\qquad \Delta_1 = \varepsilon^{ik}\varepsilon^{ab}\partial_{ia}\partial_{kb}\,, \\ \partial_z = \frac{\partial}{\partial z}\,, \quad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}\,, \quad \partial_y = \frac{\partial}{\partial y}\,, \quad \partial_{\bar{y}} = \frac{\partial}{\partial \bar{y}}\,, \qquad \Delta_2 = 2\left(\partial_z\partial_{\bar{z}} + \partial_y\partial_{\bar{y}}\right). \end{gathered} \tag{4.45}$$

Since SU(2|1) supersymmetry implies SU(2) × U(1) symmetry, the function f and g are functions of the following coordinate monomials: $z, \bar{z}, y\bar{y}, x^{ia}x_{ia}$.

Requiring SU(4) invariance of the corresponding component action amounts to the constraints:

$$\begin{aligned} m\left(\bar{y}g + 2\partial_y f + x^{ia}\partial_{ia}\partial_y f\right) = 0 \quad &\Rightarrow \quad m\left(x_{ia}\partial_y - \bar{y}\partial_{ia}\right) g = 0\,, \\ m\left(y g + 2\partial_{\bar{y}} f + x^{ia}\partial_{ia}\partial_{\bar{y}} f\right) = 0 \quad &\Rightarrow \quad m\left(x_{ia}\partial_{\bar{y}} - y\partial_{ia}\right) g = 0\,. \end{aligned} \tag{4.46}$$

These constraints admit three different solutions:

1) Special Kähler manifold metric (4.9)

$$\begin{aligned} f_1 &= \frac{1}{2}\left[\bar{z}\partial_z K(z) + z\partial_{\bar{z}}\bar{K}(\bar{z})\right] - \frac{1}{16}\left(x^{ia}x_{ia} + 4y\bar{y}\right)\left[\partial_z\partial_z K(z) + \partial_{\bar{z}}\partial_{\bar{z}}\bar{K}(\bar{z})\right], \\ g_1 &= \frac{1}{2}\left[\partial_z\partial_z K(z) + \partial_{\bar{z}}\partial_{\bar{z}}\bar{K}(\bar{z})\right] \quad \Longrightarrow \quad g_1 = \partial_\phi\partial_\phi K(\phi) + \partial_{\bar{\phi}}\partial_{\bar{\phi}}\bar{K}\left(\bar{\phi}\right). \end{aligned} \tag{4.47}$$

2) SO(6)-invariant metric (4.33)

$$\begin{aligned} f_2 &= \frac{1}{4}\left(x^{ia}x_{ia}\right)^{-1}\log\left(2y\bar{y} + x^{ia}x_{ia}\right), \\ g_2 &= \left(2y\bar{y} + x^{ia}x_{ia}\right)^{-2} \quad \Longrightarrow \quad g_2 = \left[\frac{1}{2}y^{IJ}y_{IJ}\right]^{-2}. \end{aligned} \tag{4.48}$$

3) SO(8)-invariant metric

$$\begin{aligned} f_3 &= -\frac{1}{8}\left(x^{ia}x_{ia}\right)^{-1}\left(2z\bar{z} + 2y\bar{y} + x^{ia}x_{ia}\right)^{-1}, \\ g_3 &= \left(2z\bar{z} + 2y\bar{y} + x^{ia}x_{ia}\right)^{-3} \quad \Longrightarrow \quad g_3 = \left[\phi\bar{\phi} + \frac{1}{2}y^{IJ}y_{IJ}\right]^{-3}. \end{aligned} \tag{4.49}$$

The first solution (4.47) reproduces the Lagrangian (4.8) with the metric (4.9). Other solutions correspond to new SU(4|1) invariant actions.

The second solution (4.48) gives the Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\mathrm{SO}(6)} = g_2 \Big[& \dot{\phi}\dot{\bar{\phi}} + \frac{1}{2}\,\dot{y}^{IJ}\dot{y}_{IJ} + \frac{i}{2}\left(\chi^K\dot{\bar{\chi}}_K - \dot{\chi}^K\bar{\chi}_K\right) - \frac{m}{4}\,\chi^K\bar{\chi}_K - \frac{m^2}{8}\,y^{IJ}y_{IJ}\Big] \\
& - \frac{i}{\sqrt{2}}\,\dot{\bar{\phi}}\,\partial_{IJ}g_2\,\chi^I\chi^J - \frac{i}{\sqrt{2}}\,\dot{\phi}\,\partial^{IJ}g_2\,\bar{\chi}_I\bar{\chi}_J + i\left(\dot{y}_{IK}\,\partial^{JK}g_2 - \dot{y}^{JK}\,\partial_{IK}g_2\right)\chi^I\bar{\chi}_J \\
& - \frac{1}{2}\,\partial_{IJ}\partial^{KL}g_2\,\chi^I\chi^J\bar{\chi}_K\bar{\chi}_L\,, \qquad (4.50)
\end{aligned}
$$

where

$$
\partial_{IJ} = \frac{\partial}{\partial y^{IJ}}\,, \qquad \partial_{IJ}y^{KL} = \frac{1}{2}\left(\delta_I^K\delta_J^L - \delta_I^L\delta_J^K\right), \qquad \partial_{IJ}\left(y_{KL}\right) = \frac{1}{2}\,\varepsilon_{IJKL}\,. \qquad (4.51)
$$

Substitution $i\dot{\phi} = D$ gives SU(4|1) invariant Lagrangian for the multiplet $(\mathbf{6},\mathbf{8},\mathbf{2})$, which is in fact superconformal, with the relevant group SU(4|1,1) (see appendix B).

The third solution (4.49) exhibits an invariance under the maximal R-symmetry group SO(8) and produces the component Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\mathrm{SO}(8)} = g_3 \Big[& \dot{\phi}\dot{\bar{\phi}} + \frac{1}{2}\,\dot{y}^{IJ}\dot{y}_{IJ} + \frac{i}{2}\left(\chi^K\dot{\bar{\chi}}_K - \dot{\chi}^K\bar{\chi}_K\right) + \frac{m}{4}\,\chi^K\bar{\chi}_K - \frac{m^2}{8}\,y^{IJ}y_{IJ}\Big] \\
& - \frac{i}{\sqrt{2}}\,\dot{\bar{\phi}}\,\partial_{IJ}g_3\,\chi^I\chi^J - \frac{i}{\sqrt{2}}\,\dot{\phi}\,\partial^{IJ}g_3\,\bar{\chi}_I\bar{\chi}_J + i\left(\dot{y}_{IK}\,\partial^{JK}g_3 - \dot{y}^{JK}\,\partial_{IK}g_3\right)\chi^I\bar{\chi}_J \\
& + \frac{1}{\sqrt{2}}\left(i\dot{y}_{IJ} - \frac{m}{2}\,y_{IJ}\right)\partial_{\phi}g_3\,\chi^I\chi^J + \frac{1}{\sqrt{2}}\left(i\dot{y}^{IJ} + \frac{m}{2}\,y^{IJ}\right)\partial_{\bar{\phi}}g_3\,\bar{\chi}_I\bar{\chi}_J \\
& - \frac{i}{2}\left(\dot{\phi}\,\partial_{\phi}g_3 - \dot{\bar{\phi}}\,\partial_{\bar{\phi}}g_3\right)\chi^K\bar{\chi}_K - \frac{im}{2}\left(\dot{\phi}\,\bar{\phi} - \dot{\bar{\phi}}\,\phi\right)g_3 \\
& + \frac{m}{4}\left(\phi\,\partial_{\phi}g_3 + \bar{\phi}\,\partial_{\bar{\phi}}g_3\right)\chi^K\bar{\chi}_K - \frac{1}{\sqrt{2}}\left(\chi^I\chi^J\,\partial_{IJ}\partial_{\phi}g_3 + \bar{\chi}_I\bar{\chi}_J\,\partial^{IJ}\partial_{\bar{\phi}}g_3\right)\chi^K\bar{\chi}_K \\
& - \frac{1}{24}\left(\varepsilon_{IJKL}\,\chi^I\chi^J\chi^K\chi^L\,\partial_{\phi}\partial_{\phi}g_3 + \varepsilon^{IJKL}\,\bar{\chi}_I\bar{\chi}_J\bar{\chi}_K\bar{\chi}_L\,\partial_{\bar{\phi}}\partial_{\bar{\phi}}g_3\right) \\
& - \frac{1}{2}\,\partial_{IJ}\partial^{KL}g_3\,\chi^I\chi^J\bar{\chi}_K\bar{\chi}_L + \frac{1}{2}\,\partial_{\phi}\partial_{\bar{\phi}}g_3\,\chi^I\bar{\chi}_I\chi^J\bar{\chi}_J\,. \qquad (4.52)
\end{aligned}
$$

4.6 Superconformal symmetry

Redefining the component fields in (4.52) as

$$
\begin{aligned}
&\phi \to \phi\,e^{-imt/2}, \qquad \chi^I \to \chi^I\,e^{-imt/4}, \\
&\bar{\phi} \to \bar{\phi}\,e^{imt/2}, \qquad \bar{\chi}_I \to \bar{\chi}_I\,e^{imt/4}, \qquad (4.53)
\end{aligned}
$$

we eliminate all the deformed terms proportional to m and write the Lagrangian in SO(8) invariant formulation:

$$
\begin{aligned}
\mathcal{L}_{\text{conf}} = g_3 &\left[\dot{\phi}\dot{\bar{\phi}} + \frac{1}{2}\,\dot{y}^{IJ}\dot{y}_{IJ} + \frac{i}{2}\left(\chi^K\dot{\bar{\chi}}_K - \dot{\chi}^K\bar{\chi}_K\right) - \frac{m^2}{4}\left(\phi\bar{\phi} + \frac{1}{2}\,y^{IJ}y_{IJ}\right)\right] \\
&- \frac{i}{\sqrt{2}}\,\dot{\bar{\phi}}\,\partial_{IJ}g_3\,\chi^I\chi^J - \frac{i}{\sqrt{2}}\,\dot{\phi}\,\partial^{IJ}g_3\,\bar{\chi}_I\bar{\chi}_J + i\left(\dot{y}_{IK}\,\partial^{JK}g_3 - \dot{y}^{JK}\,\partial_{IK}g_3\right)\chi^I\bar{\chi}_J \\
&+ \frac{i}{\sqrt{2}}\left(\dot{y}_{IJ}\,\chi^I\chi^J\,\partial_\phi g_3 + \dot{y}^{IJ}\bar{\chi}_I\bar{\chi}_J\,\partial_{\bar{\phi}}g_3\right) - \frac{i}{2}\left(\dot{\phi}\,\partial_\phi g_3 - \dot{\bar{\phi}}\,\partial_{\bar{\phi}}g_3\right)\chi^K\bar{\chi}_K \\
&- \frac{1}{24}\left(\varepsilon_{IJKL}\,\chi^I\chi^J\chi^K\chi^L\,\partial_\phi\partial_\phi g_3 + \varepsilon^{IJKL}\,\bar{\chi}_I\bar{\chi}_J\bar{\chi}_K\bar{\chi}_L\,\partial_{\bar{\phi}}\partial_{\bar{\phi}}g_3\right) \\
&- \frac{1}{\sqrt{2}}\left(\chi^I\chi^J\,\partial_{IJ}\partial_\phi g_3 + \bar{\chi}_I\bar{\chi}_J\,\partial^{IJ}\partial_{\bar{\phi}}g_3\right)\chi^K\bar{\chi}_K \\
&- \frac{1}{2}\,\partial_{IJ}\partial^{KL}g_3\,\chi^I\chi^J\bar{\chi}_K\bar{\chi}_L + \frac{1}{2}\,\partial_\phi\partial_{\bar{\phi}}g_3\,\chi^I\bar{\chi}_I\chi^J\bar{\chi}_J\,.
\end{aligned}
\tag{4.54}
$$

As a result, we obtain OSp(8|2) superconformal Lagrangian of the trigonometric type[8] that contains only m^2 terms. Since the new Lagrangian (4.54) is an even function of m, it is invariant under two types of SU(4|1) transformations, with the deformation parameters m and $-m$:

$$
\begin{aligned}
\delta\phi &= -\sqrt{2}\,\epsilon_I\chi^I e^{imt}, \qquad\qquad \delta\bar{\phi} = \sqrt{2}\,\bar{\epsilon}^I\bar{\chi}_I\,e^{-imt}, \\
\delta y^{IJ} &= -2\,\bar{\epsilon}^{[I}\chi^{J]}e^{-imt} + \varepsilon^{IJKL}\epsilon_K\bar{\chi}_L\,e^{imt}, \\
\delta\chi^I &= \sqrt{2}\,\bar{\epsilon}^I\left(i\dot{\phi} + \frac{m}{2}\,\phi\right)e^{-imt} - 2\,\epsilon_J\left(i\dot{y}^{IJ} - \frac{m}{2}\,y^{IJ}\right)e^{imt}, \\
\delta\bar{\chi}_I &= -\sqrt{2}\,\epsilon_I\left(i\dot{\bar{\phi}} - \frac{m}{2}\,\bar{\phi}\right)e^{imt} + 2\,\bar{\epsilon}^J\left(i\dot{y}_{IJ} + \frac{m}{2}\,y_{IJ}\right)e^{-imt},
\end{aligned}
\tag{4.55}
$$

$$
\begin{aligned}
\delta\phi &= -\sqrt{2}\,\eta_I\chi^I e^{-imt}, \qquad\qquad \delta\bar{\phi} = \sqrt{2}\,\bar{\eta}^I\bar{\chi}_I\,e^{imt}, \\
\delta y^{IJ} &= -2\,\bar{\eta}^{[I}\chi^{J]}e^{imt} + \varepsilon^{IJKL}\eta_K\bar{\chi}_L\,e^{-imt}, \\
\delta\chi^I &= \sqrt{2}\,\bar{\eta}^I\left(i\dot{\phi} - \frac{m}{2}\,\phi\right)e^{imt} - 2\,\eta_J\left(i\dot{y}^{IJ} + \frac{m}{2}\,y^{IJ}\right)e^{-imt}, \\
\delta\bar{\chi}_I &= -\sqrt{2}\,\eta_I\left(i\dot{\bar{\phi}} + \frac{m}{2}\,\bar{\phi}\right)e^{-imt} + 2\,\bar{\eta}^J\left(i\dot{y}_{IJ} - \frac{m}{2}\,y_{IJ}\right)e^{imt}.
\end{aligned}
\tag{4.56}
$$

In the closure of these transformations, we obtain superconformal algebra $osp(8|2)$ spanned by 16 supercharges and 31 bosonic generators (see appendix D),[9] where the conformal Hamiltonian $\mathcal{H}_{\text{conf}}$ is defined as

$$
\mathcal{H}_{\text{conf}} = \mathcal{H} - \frac{m}{2}\,F. \tag{4.57}
$$

The generators F^{IJ} and $\bar{F}_{IJ}$ produce SO(8)/U(4) transformations realized as

$$
\begin{aligned}
\delta\chi^I &= \sqrt{2}\,\bar{\Lambda}^{IJ}\bar{\chi}_J\,, \qquad \delta\bar{\chi}_I = \sqrt{2}\,\Lambda_{IJ}\chi^J, \\
\delta\phi &= -\bar{\Lambda}^{IJ}y_{IJ}\,, \qquad\quad \delta\bar{\phi} = -\Lambda^{IJ}y_{IJ}\,, \qquad \delta y_{IJ} = \Lambda_{IJ}\phi + \bar{\Lambda}_{IJ}\bar{\phi}.
\end{aligned}
\tag{4.58}
$$

[8] Here we follow the terminology suggested in [33].

[9] In the limit m=0 the Lagrangian (4.54) goes into the one invariant under the "parabolic" realization of OSp(8|2), as it was given in [34].

5 The SU(4|1) multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$: second version

The second version of the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ is described by a complex bosonic superfield V^I satisfying

$$
\begin{aligned}
&\mathcal{D}^I V^J = \frac{1}{2}\,\varepsilon^{IJKL}\,\bar{\mathcal{D}}_K \bar{V}_L\,, \qquad \mathcal{D}^{(I}\, V^{J)} = 0\,, \qquad \bar{\mathcal{D}}_{(K}\,\bar{V}_{L)} = 0\,,\\
&\mathcal{D}^I \bar{V}_J = \frac{1}{4}\,\delta^I_J \mathcal{D}^K \bar{V}_K \qquad \bar{\mathcal{D}}_J\, V^I = \frac{1}{4}\,\delta^I_J \bar{\mathcal{D}}_K V^K \qquad \overline{(V^I)} = \bar{V}_I\,.
\end{aligned} \tag{5.1}
$$

In the flat superspace limit $m \to 0$, these constraints go over to the SU(4) covariant constraints (3.7) specifying another form of the flat $\mathcal{N}=8$, $d=1$ multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$.

To avoid calculation of the deformed covariant derivatives $\mathcal{D}^I$ and $\bar{\mathcal{D}}_J$, we instead consider harmonization of part of these constraints, *viz.*

$$
\bar{\mathcal{D}}_{(K}\,\bar{V}_{L)} = 0\,, \qquad \mathcal{D}^I \bar{V}_J = \frac{1}{4}\,\delta^I_J \mathcal{D}^K \bar{V}_K\,, \tag{5.2}
$$

with the rest of constraints being solved at the component level.

5.1 Harmonic superspace

The option for harmonic superspace relevant to the given case uses the harmonic variables on SU(4)/[SU(3) × U(1)] [32]. The set of these harmonic variables is given by $u^{(+)\alpha}_I$, $u^{(+3)I}$, $u^{(-)I}_\beta$, $u^{(-3)}_I$, where the index $\alpha = 1,2,3$ refers to the SU(3) fundamental representation. The harmonics satisfy the following unitarity and unimodularity conditions:

$$
\begin{aligned}
&u^{(-3)}_I u^{(+3)I} = 1\,, \qquad u^{(+)\alpha}_I u^{(-)I}_\beta = \delta^\alpha_\beta\,, \qquad u^{(+)\alpha}_J u^{(-)I}_\alpha + u^{(-3)}_J u^{(+3)I} = \delta^I_J\,,\\
&u^{(+)\alpha}_I u^{(+3)I} = u^{(-3)}_I u^{(-)I}_\alpha = 0\,, \qquad \varepsilon^{IJKL} u^{(+)\alpha}_I u^{(+)\beta}_J u^{(+)\gamma}_K u^{(-3)}_L = \varepsilon^{\alpha\beta\gamma}.
\end{aligned} \tag{5.3}
$$

As in the previous case, we define the new coordinates

$$
\begin{aligned}
&\theta^{(+3)} = \theta_I \left(u^{(+3)I} + m\,\bar{\theta}^{(+)\alpha}\theta^{(+3)}u^{(-)I}_\alpha\right), \qquad \theta^{(-)}_\alpha = \theta_I\, u^{(-)I}_\alpha,\\
&\bar{\theta}^{(+)\alpha} = \bar{\theta}^J \left(u^{(+)\alpha}_J + m\,\bar{\theta}^{(+)\alpha}\theta^{(+3)}u^{(-3)}_J\right), \qquad \bar{\theta}^{(-3)} = \bar{\theta}^J u^{(-3)}_J,\\
&t_{\rm A} = t + i\bar{\theta}^{(-3)}\theta^{(+3)} - i\bar{\theta}^{(+)\alpha}\theta^{(-)}_\alpha \left[1 - m\,\bar{\theta}^{(+)\beta}\theta^{(-)}_\beta + \frac{4m^2}{3}\left(\bar{\theta}^{(+)\beta}\theta^{(-)}_\beta\right)^2\right].
\end{aligned} \tag{5.4}
$$

They transform as

$$
\begin{aligned}
\delta\theta^{(-)}_\alpha &= \epsilon^{(-)}_\alpha + 2m\left[\bar{\epsilon}^{(+)\beta}\theta^{(-)}_\beta + \bar{\epsilon}^{(-3)}\theta^{(+3)}\left(1 + m\,\bar{\theta}^{(+)\beta}\theta^{(-)}_\beta\right)\right]\theta^{(-)}_\alpha,\\
\delta\bar{\theta}^{(-3)} &= \bar{\epsilon}^{(-3)} - 2m\,\epsilon^{(-)}_\beta\bar{\theta}^{(+)\beta}\bar{\theta}^{(-3)},\\
\delta\theta^{(+3)} &= \epsilon^{(+3)} + m\,\epsilon^{(-)}_\alpha\bar{\theta}^{(+)\alpha}\theta^{(+3)},\\
\delta\bar{\theta}^{(+)\alpha} &= \bar{\epsilon}^{(+)\alpha} + m\,\bar{\epsilon}^{(-3)}\bar{\theta}^{(+)\alpha}\theta^{(+3)} - 2m\,\epsilon^{(-)}_\beta\bar{\theta}^{(+)\beta}\bar{\theta}^{(+)\alpha},\\
\delta u^{(+3)I} &= \Lambda^{(+4)\alpha}u^{(-)I}_\alpha \qquad \delta u^{(-3)}_I = 0\,,\\
\delta u^{(+)\alpha}_I &= -\Lambda^{(+4)\alpha}u^{(-3)}_I \qquad \delta u^{(-)I}_\beta = 0\,,\\
\delta t_A &= 2i\left(\epsilon^{(-)}_\alpha\bar{\theta}^{(+)\alpha} + \bar{\epsilon}^{(-3)}\theta^{(+3)}\right),
\end{aligned} \tag{5.5}
$$

where

$$
\begin{aligned}
\Lambda^{(+4)\alpha} &= m\left(\epsilon^{(+3)}\bar{\theta}^{(+)\alpha} + \bar{\epsilon}^{(+)\alpha}\theta^{(+3)}\right) + m^2\epsilon^{(-)}_{\beta}\bar{\theta}^{(+)\beta}\bar{\theta}^{(+)\alpha}\theta^{(+3)},\\
\epsilon^{(-)}_{\alpha} &= \epsilon_I\, u^{(-)I}_{\alpha}, \quad \epsilon^{(+3)} = \epsilon_I\, u^{(+3)I}, \quad \bar{\epsilon}^{(+)\alpha} = \bar{\epsilon}^J u^{(+)\alpha}_J, \quad \bar{\epsilon}^{(-3)} = \bar{\epsilon}^J u^{(-3)}_J.
\end{aligned}
\tag{5.6}
$$

It is straightforward to see that the analytic subspace

$$
\zeta_{\rm A} = \left\{t_{\rm A}, \theta^{(+3)}, \bar{\theta}^{(+)\alpha}, u^{(+)\alpha}_I, u^{(+3)I}, u^{(-)I}_{\beta}, u^{(-3)}_I\right\}, \tag{5.7}
$$

is closed under the transformations (5.5). Its integration measure

$$
d\zeta^{(-6)}_{\rm A} = dt_{\rm A}\, du\, d\theta^{(+3)}\, d^3\bar{\theta}^{(+)}\, e^{3imt_{\rm A}/2} \tag{5.8}
$$

transforms as

$$
\delta\left(d\zeta^{(-6)}_{\rm A}\right) = m\, d\zeta^{(-6)}_{\rm A}\left(\epsilon^{(-)}_{\alpha}\bar{\theta}^{(+)\alpha} - 3\,\bar{\epsilon}^{(-3)}\theta^{(+3)}\right). \tag{5.9}
$$

The harmonic derivatives are found to be

$$
\begin{aligned}
\mathcal{D}^{(+4)\alpha} &= \partial^{(+4)\alpha} - 2i\bar{\theta}^{(+)\alpha}\theta^{(+3)}\partial_{\rm A} - \frac{m}{6}\,\bar{\theta}^{(+)\alpha}\theta^{(+3)}\mathcal{D}^0 + m\,\bar{\theta}^{(+)\alpha}\theta^{(+3)}\bar{\theta}^{(+)\beta}\frac{\partial}{\partial\bar{\theta}^{(+)\beta}}\,,\\
\mathcal{D}^{\alpha}_{\beta} &= \partial^{\alpha}_{\beta} + \bar{\theta}^{(+)\alpha}\frac{\partial}{\partial\bar{\theta}^{(+)\beta}} - \frac{\delta^{\alpha}_{\beta}}{3}\,\bar{\theta}^{(+)\gamma}\frac{\partial}{\partial\bar{\theta}^{(+)\gamma}}\,,\\
\mathcal{D}^0 &= \partial^0 + \bar{\theta}^{(+)\alpha}\frac{\partial}{\partial\bar{\theta}^{(+)\alpha}} + 3\,\theta^{(+3)}\frac{\partial}{\partial\theta^{(+3)}}\,,
\end{aligned}
\tag{5.10}
$$

where

$$
\begin{aligned}
\partial^{(+4)\alpha} &= u^{(+3)K}\frac{\partial}{\partial u^{(-)K}_{\alpha}} - u^{(+)\alpha}_K\frac{\partial}{\partial u^{(-3)}_K}\,,\\
\partial^{\alpha}_{\beta} &= u^{(+)\alpha}_K\frac{\partial}{\partial u^{(+)\beta}_K} - u^{(-)K}_{\beta}\frac{\partial}{u^{(-)K}_{\alpha}} - \frac{\delta^{\alpha}_{\beta}}{3}\left(u^{(+)\gamma}_K\frac{\partial}{\partial u^{(+)\gamma}_K} - u^{(-)K}_{\gamma}\frac{\partial}{u^{(-)K}_{\gamma}}\right),\\
\partial^0 &= u^{(+)\alpha}_K\frac{\partial}{\partial u^{(+)\alpha}_K} - u^{(-)K}_{\alpha}\frac{\partial}{u^{(-)K}_{\alpha}} + 3\left(u^{(+3)K}\frac{\partial}{\partial u^{(+3)K}} - u^{(-3)}_K\frac{\partial}{\partial u^{(-3)}_K}\right).
\end{aligned}
\tag{5.11}
$$

Note that

$$
\mathcal{D}^{(+4)\alpha}\Lambda = \Lambda^{(+4)\alpha}, \qquad \Lambda = m\left(\epsilon^{(-)}_{\alpha}\bar{\theta}^{(+)\alpha} - \bar{\epsilon}^{(-3)}\theta^{(+3)}\right). \tag{5.12}
$$

The harmonic analytic superfield $\bar{V}^{(+3)}$ defined on (5.7) satisfies the harmonic constraints

$$
\mathcal{D}^{(+4)\alpha}\bar{V}^{(+3)} = 0\,, \qquad \mathcal{D}^{\alpha}_{\beta}\bar{V}^{(+3)} = 0\,, \qquad \mathcal{D}^0\bar{V}^{(+3)} = 3\bar{V}^{(+3)}. \tag{5.13}
$$

It can be treated as a harmonization of the superfield $\bar{V}_I$ defined by (5.2), where Grassmann analyticity constraints are provided by

$$
u^{(+3)K}u^{(+3)L}\bar{\mathcal{D}}_{(K}\,\bar{V}_{L)} = \bar{\mathcal{D}}^{(+3)}\,\bar{V}^{(+3)} = 0\,, \quad u^{(+)\alpha}_I u^{(+3)J}\mathcal{D}^I\bar{V}_J = \mathcal{D}^{(+)\alpha}\bar{V}^{(+3)} = 0\,. \tag{5.14}
$$

The full set of the constraints (5.1) operates with the set of superfields $V^{(+)\alpha}$, $V^{(-3)}$, $\bar{V}^{(+3)}$ and $\bar{V}^{(-)}_{\alpha}$ living on the full harmonic superspace (5.4). Here we consider just the superfield $\bar{V}^{(+3)}$ treated as an unconstrained deformed harmonic superfield satisfying the analyticity conditions (5.13). The rest of constraints on $\bar{V}^{(+3)}$ will be imposed below "by hand" at the component level, like in the previous cases.

The general expansion of $\bar{V}^{(+3)}$ reads

$$\begin{aligned}
\bar{V}^{(+3)} = {} & \bar{z}_I\, u^{(+3)I} + \sqrt{2}\,\theta^{(+3)}\chi\, e^{3imt/4} + 2\,\bar{\theta}^{(+)\alpha}\chi_{JI}\, u^{(+3)J}u^{(-)I}_{\alpha}\, e^{-3imt/4} \\
& + 2\,\bar{\theta}^{(+)\alpha}\theta^{(+3)}\left(i\dot{\bar{z}}_K + \frac{m}{4}\,\bar{z}_K\right)u^{(-)K}_{\alpha} \\
& + \varepsilon_{IJKL}\,\bar{\theta}^{(+)\alpha}\bar{\theta}^{(+)\beta}C^K u^{(+3)L}u^{(-)I}_{\alpha}u^{(-)J}_{\beta}\, e^{-3imt/2} \\
& + \varepsilon_{\alpha\beta\gamma}\,\bar{\theta}^{(+)\alpha}\bar{\theta}^{(+)\beta}\bar{\theta}^{(+)\gamma}\pi\, e^{-9imt/4} \\
& - 2\,\bar{\theta}^{(+)\alpha}\bar{\theta}^{(+)\beta}\theta^{(+3)}\left(i\dot{\chi}_{IJ} + \frac{m}{2}\,\chi_{IJ}\right)u^{(-)I}_{\alpha}u^{(-)J}_{\beta}e^{-3imt/4} \\
& - \frac{2}{3}\,\varepsilon_{IJKL}\,\bar{\theta}^{(+)\alpha}\bar{\theta}^{(+)\beta}\bar{\theta}^{(+)\gamma}\theta^{(+3)}\left(i\dot{C}^L + \frac{3m}{4}\,C^L\right)u^{(-)I}_{\alpha}u^{(-)J}_{\beta}u^{(-)K}_{\gamma}\, e^{-3imt/2},
\end{aligned} \tag{5.15}$$

where

$$\chi_{IJ} \equiv \chi_{[IJ]}\,. \tag{5.16}$$

Taking into account the transformation rule

$$\begin{aligned}
\delta\mathcal{D}^{(+4)\alpha} = {} & -\left(\frac{1}{3}\,\Lambda^{(+4)\alpha}\mathcal{D}^0 + \Lambda^{(+4)\beta}\mathcal{D}^{\alpha}_{\beta}\right) - \frac{m}{6}\left(\bar{\epsilon}^{(+)\alpha}\theta^{(+3)} - \epsilon^{(+3)}\bar{\theta}^{(+)\alpha}\right)\mathcal{D}^0 \\
& + \frac{m^2}{6}\,\epsilon^{(-)}_{\beta}\bar{\theta}^{(+)\beta}\bar{\theta}^{(+)\alpha}\theta^{(+3)}\mathcal{D}^0,
\end{aligned} \tag{5.17}$$

the superfield $\bar{V}^{(+3)}$ transforms as

$$\delta\bar{V}^{(+3)} = \Lambda\bar{V}^{(+3)} - \frac{m}{2}\left(\epsilon^{(-)}_{\alpha}\bar{\theta}^{(+)\alpha} + \bar{\epsilon}^{(-3)}\theta^{(+3)}\right)\bar{V}^{(+3)}. \tag{5.18}$$

This superfield transformation law amounts to the following component transformations

$$\begin{aligned}
\delta\bar{z}_J &= -2\,\bar{\epsilon}^K\chi_{JK}\, e^{-3imt/4} - \sqrt{2}\,\epsilon_J\chi\, e^{3imt/4}, \\
\delta\chi &= \sqrt{2}\,\bar{\epsilon}^K\left(i\dot{\bar{z}}_K + \frac{3m}{4}\,\bar{z}_K\right)e^{-3imt/4}, \\
\delta\chi_{IJ} &= \varepsilon_{IJKL}\,\bar{\epsilon}^K C^L\, e^{-3imt/4} - 2\,\epsilon_{[I}\left(i\dot{\bar{z}}_{J]} - \frac{m}{4}\,\bar{z}_{J]}\right)e^{3imt/4}, \\
\delta C^I &= \varepsilon^{IJKL}\epsilon_J\left(i\dot{\chi}_{KL} - \frac{m}{2}\,\chi_{KL}\right)e^{3imt/4} - 3\,\bar{\epsilon}^I\pi\, e^{-3imt/4}, \\
\delta\pi &= \frac{2}{3}\,\epsilon_K\left(i\dot{C}^K - \frac{3m}{4}\,C^K\right)e^{3imt/4}.
\end{aligned} \tag{5.19}$$

From the transformation properties of $\bar{V}^{(+3)}$ one can draw the conclusion that the construction of a "pre-action" similar to (4.5) cannot be performed within the analytic harmonic superspace. We conjecture that such a construction could become possible after taking account of the additional set of c constraints defining the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$. Then the action can probably be constructed in the full harmonic superspace approach (see [19]).

At the component level, the rest of the constraints (5.1) impose the relations

$$\overline{(\bar z_I)} = z^I, \quad \overline{(\chi)} = \bar\chi\,, \quad \overline{(\chi_{IJ})} = \frac{1}{2}\,\varepsilon^{IJKL}\chi_{KL} = \chi^{IJ},$$
$$C^I = i\dot z^I + \frac{m}{4}\,z^I, \qquad\qquad \pi = -\frac{\sqrt{2}}{3}\,(i\dot{\bar\chi} + m\,\bar\chi)\,. \tag{5.20}$$

The final form of the deformed transformations is

$$\begin{aligned}
\delta z^I &= 2\,\epsilon_K\chi^{IK}\,e^{3imt/4} + \sqrt{2}\,\bar\epsilon^I\bar\chi\,e^{-3imt/4}\,,\\
\delta\bar z_J &= -2\,\bar\epsilon^K\chi_{JK}\,e^{-3imt/4} - \sqrt{2}\,\epsilon_J\chi\,e^{3imt/4},\\
\delta\chi &= \sqrt{2}\,\bar\epsilon^K\left(i\dot{\bar z}_K + \frac{3m}{4}\,\bar z_K\right)e^{-3imt/4},\\
\delta\bar\chi &= -\sqrt{2}\,\epsilon_K\left(i\dot z^K - \frac{3m}{4}\,z^K\right)e^{3imt/4},\\
\delta\chi^{IJ} &= 2\,\bar\epsilon^{[I}\left(i\dot z^{J]} + \frac{m}{4}\,z^{J]}\right)e^{-3imt/4} - \varepsilon^{IJKL}\epsilon_K\left(i\dot{\bar z}_L - \frac{m}{4}\,\bar z_L\right)e^{3imt/4},
\end{aligned} \tag{5.21}$$

where

$$\overline{(z^I)} = \bar z_I\,, \qquad \overline{(\chi)} = \bar\chi\,, \qquad \overline{(\chi^{IJ})} = \chi_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\,\chi^{KL}. \tag{5.22}$$

5.2 SU(2|1) superfield formulation

Once again, we split the given multiplet into SU(2|1) multiplets as $(\mathbf{4},\mathbf{4},\mathbf{0}) \oplus (\mathbf{4},\mathbf{4},\mathbf{0})$.

The first multiplet is associated with the fields

$$x^{i1} := z^i, \quad x^2_i := \bar z_i\,, \qquad \xi^1 := 2\chi^{12}, \quad \bar\xi_1 := 2\chi_{12}\,, \quad \xi^2 := \sqrt{2}\,\chi\,, \quad \bar\xi_2 := \sqrt{2}\,\bar\chi\,, \tag{5.23}$$

such that

$$\begin{aligned}
\delta x^{iA} &= -\epsilon^i\xi^A\,e^{3imt/4} - \bar\epsilon^i\bar\xi^A\,e^{-3imt/4},\\
\delta\xi^1 &= 2\,\bar\epsilon^k\left(i\dot x^1_k + \frac{m}{4}\,x^1_k\right)e^{-3imt/4}, \qquad \delta\bar\xi_1 = 2\,\epsilon_k\left(i\dot x^k_1 - \frac{m}{4}\,x^k_1\right)e^{3imt/4},\\
\delta\xi^2 &= 2\,\bar\epsilon^k\left(i\dot x^2_k - \frac{3m}{4}\,x^2_k\right)e^{-3imt/4}, \qquad \delta\bar\xi_2 = 2\,\epsilon_k\left(i\dot x^k_2 + \frac{3m}{4}\,x^k_2\right)e^{3imt/4}.
\end{aligned} \tag{5.24}$$

This first multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ is accommodated by a superfield q^{iA} obeying the SU(2|1) covariant constraints

$$\mathcal{D}^{(k}q^{i)A} = 0\,, \quad \bar{\mathcal{D}}^{(k}q^{i)A} = 0\,, \qquad \tilde F q^{iA} = -\frac{1}{2}\,(\sigma_3)^A_B\,q^{iB}\,, \qquad \overline{(q^{iA})} = q_{iA}\,. \tag{5.25}$$

As distinct from (4.37), Pauli-Gürsey SU(2) symmetry is broken. Taking into account (2.16), we solve these constraints as

$$\begin{aligned}
q^{iA} = {} & \left[1 + \frac{m}{2}\,\bar\theta^k\theta_k - \frac{5m^2}{16}\,(\bar\theta)^2\,(\theta)^2\right]x^{iA} - i\varepsilon_{kl}\left(\bar\theta^i\theta^l + \bar\theta^l\theta^i\right)\left(\dot x^{kA} + \frac{im}{4}\,(\sigma_3)^A_B\,x^{kB}\right)\\
& - i\bar\theta^k\theta_k\left(\theta^i\dot\xi^A\,e^{3imt/4} - \bar\theta^i\dot{\bar\xi}^A\,e^{-3imt/4}\right)\\
& + \left(1 + \frac{m}{4}\,\bar\theta^k\theta_k\right)\left(\theta^i\xi^A\,e^{3imt/4} + \bar\theta^i\bar\xi^A\,e^{-3imt/4}\right)\\
& + \frac{m}{4}\,\bar\theta^k\theta_k\left(\theta^i\xi^B\,e^{3imt/4} - \bar\theta^i\bar\xi^B\,e^{-3imt/4}\right)(\sigma_3)^A_B\\
& + \frac{1}{4}\,(\bar\theta)^2\,(\theta)^2\left(\ddot x^{iA} + \frac{im}{2}\,(\sigma_3)^A_B\,\dot x^{iB} - \frac{m^2}{16}\,x^{iA}\right).
\end{aligned} \tag{5.26}$$

The second (mirror) multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ is formed by the fields

$$y^1 := z^4, \quad y^2 := z^3, \quad \bar{y}_1 := \bar{z}_4\,, \quad \bar{y}_2 := \bar{z}_3\,, \qquad \psi^{i1} := 2\chi^{i4}, \quad \psi^{i2} := 2\chi^{i3}, \tag{5.27}$$

with the SU(2|1) transformations

$$\begin{aligned}
&\delta y^a = -\epsilon_i \psi^{ia}\, e^{3imt/4}, \qquad \delta \bar{y}^a = -\bar{\epsilon}_i \psi^{ia}\, e^{-3imt/4}, \\
&\delta \psi^{ia} = 2\,\bar{\epsilon}^i \left(i\dot{y}^a + \frac{m}{4}\, y^a\right) e^{-3imt/4} - 2\,\epsilon^i \left(i\dot{\bar{y}}^a - \frac{m}{4}\, \bar{y}^a\right) e^{3imt/4}.
\end{aligned} \tag{5.28}$$

The superfield SU(2|1) constraints defining the mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet are written as

$$\begin{aligned}
&\bar{\mathcal{D}}^i Y^a = \mathcal{D}^i \bar{Y}^a = 0\,, \qquad \mathcal{D}^i Y^a = \bar{\mathcal{D}}^i \bar{Y}^a, \\
&\tilde{F} Y^a = \frac{1}{2}\, Y^a, \qquad \tilde{F} \bar{Y}^a = -\frac{1}{2}\, \bar{Y}^a, \qquad \overline{(Y^a)} = \bar{Y}_a\,.
\end{aligned} \tag{5.29}$$

They are solved by

$$\begin{aligned}
Y^a =& \left[1 + \frac{m}{4}\, \bar{\theta}^k \theta_k - \frac{7m^2}{64}\, (\bar{\theta})^2\, (\theta)^2\right] y^a + i\dot{y}^a \left[\bar{\theta}^k \theta_k - \frac{3m}{8}\, (\bar{\theta})^2\, (\theta)^2\right] - \frac{1}{4}\, (\bar{\theta})^2\, (\theta)^2\, \ddot{y}^a \\
&+ \theta_k \theta^k \left(i\dot{\bar{y}}^a - \frac{m}{4}\, \bar{y}^a\right) e^{3imt/2} + \left[\left(1 - \frac{m}{2}\, \bar{\theta}^k \theta_k\right) \theta_i \psi^{ia} + i\bar{\theta}^k \theta_k \theta_i \dot{\psi}^{ia}\right] e^{3imt/4}, \\
\bar{Y}^a =& \left[1 + \frac{m}{4}\, \bar{\theta}^k \theta_k - \frac{7m^2}{64}\, (\bar{\theta})^2\, (\theta)^2\right] \bar{y}^a - i\dot{\bar{y}}^a \left[\bar{\theta}^k \theta_k - \frac{3m}{8}\, (\bar{\theta})^2\, (\theta)^2\right] - \frac{1}{4}\, (\bar{\theta})^2\, (\theta)^2\, \ddot{\bar{y}}^a \\
&+ \bar{\theta}^k \bar{\theta}_k \left(i\dot{y}^a + \frac{m}{4}\, y^a\right) e^{-3imt/2} + \left[\left(1 - \frac{m}{2}\, \bar{\theta}^k \theta_k\right) \bar{\theta}_i \psi^{ia} - i\bar{\theta}^k \theta_k \bar{\theta}_i \dot{\psi}^{ia}\right] e^{-3imt/4}.
\end{aligned} \tag{5.30}$$

5.3 Invariant Lagrangian

The general SU(2|1) invariant action is written as

$$S = \int dt\, \mathcal{L} = \frac{1}{2} \int dt\, d^2\theta\, d^2\bar{\theta} \left(1 + 2m\, \bar{\theta}^k \theta_k\right) f\left(Y^a \bar{Y}_a, q^{iA} q_{iA}\right). \tag{5.31}$$

Requiring it to be SU(4) invariant produces the following conditions:

$$\begin{aligned}
&\Delta_y = -2\,\varepsilon^{ab} \partial_a \bar{\partial}_b\,, \qquad \partial_a = \partial/\partial y^a, \qquad \bar{\partial}_b = \partial/\partial \bar{y}^b, \\
&\Delta_x = \varepsilon^{ij} \varepsilon^{AB} \partial_{iA} \partial_{jB}\,, \qquad \partial_{iA} = \partial/\partial x^{iA}, \\
&G := \Delta_y f = -\Delta_x f \quad \Rightarrow \quad (\Delta_y + \Delta_x)\, G = 0\,,
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
&m\left(2\partial_a f + \bar{y}_a G + x^{iA} \partial_{iA} \partial_a f\right) = 0 \quad \Rightarrow \quad m\,(\bar{y}_a \partial_{iA} - x_{iA} \partial_a)\, G = 0\,, \\
&m\left(2\bar{\partial}_a f - y_a G + x^{iA} \partial_{iA} \bar{\partial}_a f\right) = 0 \quad \Rightarrow \quad m\left(y_a \partial_{iA} + x_{iA} \bar{\partial}_a\right) G = 0\,.
\end{aligned} \tag{5.33}$$

The unique solution of these equations is given by

$$\begin{aligned}
&f = \frac{1}{4}\, (y^a \bar{y}_a)^{-1} \left(y^a \bar{y}_a + \frac{1}{2}\, x^{iA} x_{iA}\right)^{-1} + c_1\, (y^a \bar{y}_a)^{-1} \left(x^{iA} x_{iA}\right)^{-1} + c_2 \left(x^{iA} x_{iA}\right)^{-1} \Rightarrow \\
&\Rightarrow G = \left(y^a \bar{y}_a + \frac{1}{2}\, x^{iA} x_{iA}\right)^{-3}.
\end{aligned} \tag{5.34}$$

Here, the terms with the constants c_1 and c_2 do not affect the metric G, since it is a harmonic function. One can check that these terms drop out of the component Lagrangian, which is finally written as

$$\begin{aligned}\mathcal{L} = &\left[\dot{z}^I\dot{\bar{z}}_I + \frac{i}{2}\,\chi^{IJ}\dot{\chi}_{IJ} + \frac{i}{2}\,(\chi\dot{\bar{\chi}} - \dot{\chi}\bar{\chi}) - \frac{im}{4}\,(\dot{z}^I\bar{z}_I - z^I\dot{\bar{z}}_I) + \frac{m}{4}\,\chi\bar{\chi} - \frac{3m^2}{16}\,z^I\bar{z}_I\right] G\\ &+ i\left(\dot{z}^I\partial_J G - \dot{\bar{z}}_J\bar{\partial}^I G\right)\chi_{IK}\chi^{JK} - \frac{m}{4}\left(z^I\partial_J G + \bar{z}_J\bar{\partial}^I G\right)\chi_{IK}\chi^{JK}\\ &+ \frac{i}{2}\left(\dot{z}^I\partial_I G - \dot{\bar{z}}_I\bar{\partial}^I G\right)\chi\bar{\chi} + \partial_J\bar{\partial}^I G\,\chi_{IK}\chi^{JK}\chi\bar{\chi} + \frac{1}{3}\,\partial_J\bar{\partial}^I G\,\chi_{IK}\chi^{LK}\chi_{LM}\chi^{JM}\\ &- \frac{\sqrt{2}}{3}\,\partial_I\partial_J G\,\bar{\chi}\,\chi^{IK}\chi^{JL}\chi_{KL} - \frac{\sqrt{2}}{3}\,\bar{\partial}^I\bar{\partial}^J G\,\chi\,\chi_{IK}\chi_{JL}\chi^{KL},\end{aligned} \tag{5.35}$$

where

$$G = \left(z^I\bar{z}_I\right)^{-3}. \tag{5.36}$$

5.4 Superconformal symmetry

By analogy with the section 4.6, one can redefine the component fields as

$$z^I \to z^I e^{-imt/4}, \qquad \bar{z}_I \to \bar{z}_I\, e^{imt/4}, \qquad \chi \to \chi\, e^{imt/2}, \qquad \bar{\chi} \to \bar{\chi}\, e^{-imt/2}, \tag{5.37}$$

after which the Lagrangian (5.35) becomes an even function of m. As a result, we obtain OSp(8|2) superconformal Lagrangian that is equivalent to (4.54):

$$\begin{aligned}\mathcal{L}_{\rm conf} = &\left[\dot{z}^I\dot{\bar{z}}_I + \frac{i}{2}\,\chi^{IJ}\dot{\chi}_{IJ} + \frac{i}{2}\,(\chi\dot{\bar{\chi}} - \dot{\chi}\bar{\chi}) - \frac{m^2}{4}\,z^I\bar{z}_I\right] G + i\left(\dot{z}^I\partial_J G - \dot{\bar{z}}_J\bar{\partial}^I G\right)\chi_{IK}\chi^{JK}\\ &+ \frac{i}{2}\left(\dot{z}^I\partial_I G - \dot{\bar{z}}_I\bar{\partial}^I G\right)\chi\bar{\chi} + \partial_J\bar{\partial}^I G\,\chi_{IK}\chi^{JK}\chi\bar{\chi} + \frac{1}{3}\,\partial_J\bar{\partial}^I G\,\chi_{IK}\chi^{LK}\chi_{LM}\chi^{JM}\\ &- \frac{\sqrt{2}}{3}\,\partial_I\partial_J G\,\bar{\chi}\,\chi^{IK}\chi^{JL}\chi_{KL} - \frac{\sqrt{2}}{3}\,\bar{\partial}^I\bar{\partial}^J G\,\chi\,\chi_{IK}\chi_{JL}\chi^{KL}.\end{aligned} \tag{5.38}$$

In the same way, this Lagrangian is invariant under two types of ϵ_I and η_I transformations which close on the superalgebra $osp(8|2)$ (D.1)–(D.3):

$$\begin{aligned}\delta z^I &= 2\,\epsilon_K\chi^{IK}\,e^{imt} + \sqrt{2}\,\bar{\epsilon}^I\bar{\chi}\,e^{-imt}, &\delta\bar{z}_J &= -2\,\bar{\epsilon}^K\chi_{JK}\,e^{-imt} - \sqrt{2}\,\epsilon_J\chi\,e^{imt},\\ \delta\chi &= \sqrt{2}\,\bar{\epsilon}^K\left(i\dot{\bar{z}}_K + \frac{m}{2}\,\bar{z}_K\right)e^{-imt}, &\delta\bar{\chi} &= -\sqrt{2}\,\epsilon_K\left(i\dot{z}^K - \frac{m}{2}\,z^K\right)e^{imt},\\ \delta\chi^{IJ} &= 2\,\bar{\epsilon}^{[I}\left(i\dot{z}^{J]} + \frac{m}{2}\,z^{J]}\right)e^{-imt} - \varepsilon^{IJKL}\epsilon_K\left(i\dot{\bar{z}}_L - \frac{m}{2}\,\bar{z}_L\right)e^{imt}, && \end{aligned} \tag{5.39}$$

$$\begin{aligned}\delta z^I &= 2\,\eta_K\chi^{IK}\,e^{-imt} + \sqrt{2}\,\bar{\eta}^I\bar{\chi}\,e^{imt}, &\delta\bar{z}_J &= -2\,\bar{\eta}^K\chi_{JK}\,e^{imt} - \sqrt{2}\,\eta_J\chi\,e^{-imt},\\ \delta\chi &= \sqrt{2}\,\bar{\eta}^K\left(i\dot{\bar{z}}_K - \frac{m}{2}\,\bar{z}_K\right)e^{imt}, &\delta\bar{\chi} &= -\sqrt{2}\,\eta_K\left(i\dot{z}^K + \frac{m}{2}\,z^K\right)e^{-imt},\\ \delta\chi^{IJ} &= 2\,\bar{\eta}^{[I}\left(i\dot{z}^{J]} - \frac{m}{2}\,z^{J]}\right)e^{imt} - \varepsilon^{IJKL}\eta_K\left(i\dot{\bar{z}}_L + \frac{m}{2}\,\bar{z}_L\right)e^{-imt}. && \end{aligned} \tag{5.40}$$

We see that the Lagrangians (4.54) and (5.38) have conformally flat metrics g_3 and G which both depend on the quadratic SO(8) invariants of the same power -3. The fields z^I and $\bar{z}_J$ can be reexpressed, by a linear transformation, through the bosonic fields $y^{I'J'}$, ϕ and $\bar{\phi}$ of the first multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$, where I' and J' label the fundamental representation of a different SU(4)$'$ subgroup of the SO(8) symmetry, such that it intersects with the first SU(4) in a common SU(3) subgroup. After an analogous linear transformation of the fermionic fields, the Lagrangian (5.38) will coincide with (4.54). So both superconformal Lagrangians are indeed equivalent. This feature of equivalence of $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets in the presence of exact SO(8) symmetry was already noted in the end of section 3.

6 Summary and outlook

We have shown the existence of two non-equivalent "root" multiplets $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ of deformed $\mathcal{N}=8$ supersymmetry associated with the supergroup SU(4|1). We described them in multiple ways with worldline superfields and in components and derived invariant actions for them. Some of these actions are superconformally OSp(8|2) invariant. For a non-trivially interacting example we gave the explicit form of the (classical) SU(4|1) supercharges. We also obtained the SU(4|1) invariant actions for the off-shell multiplets $(\mathbf{6}, \mathbf{8}, \mathbf{2})$ and $(\mathbf{7}, \mathbf{8}, \mathbf{1})$ (in appendices B and C) from the $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ actions, reconfirming the root interpretation of the $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets for $\mathcal{N}=8$ mechanics. The $(\mathbf{6}, \mathbf{8}, \mathbf{2})$ action was shown to exhibit superconformal SU(4|1,1) invariance.

As for further applications of these results, the most appropriate arena might be provided by supersymmetric matrix models (see, e.g., [35–37]). These possess SU(4|2) invariance, hence multi-particle mechanics based on SU(2|2) $\subset$ SU(4|2) or SU(4|1) $\subset$ SU(4|2) may appear as some truncation of such matrix models. The matrix models studied so far lead to free worldline multiplets and actions. Our approach allows one to generate nontrivial interactions, which hopefully may be interpreted as effective actions with quantum corrections taken into account. An important ingredient of matrix models is a gauging of appropriate isometries by non-propagating gauge multiplets. To promote this to the SU(4|1) superfield language, one needs to define suitable gauge superfields generalizing those used in [38, 39] or [10].

Another problem for the future is finding an action including both types of deformed $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ multiplets and inquiring the ensuing target-space geometry.

Acknowledgments

We thank Sergey Fedoruk, Armen Nersessian and Francesco Toppan for interest in this work and for valuable comments. This research was supported by the Heisenberg-Landau program and the joint DFG project LE 838/12. The work of E.I. and S.S. was supported by the Russian Foundation for Basic Research, project No. 18-02-01046. They thank the directorate of the Institute of Theoretical Physics at Leibniz University of Hannover for kind hospitality extended to them several times during this project.

A Some calculations

Here we collect the necessary identities for calculation of the function $L^{(+4)}$ in the invariant action (4.32). We represent it as an infinite series:

$$L^{(+4)}=\sum_{n=0}^{\infty}a_n\left(c^{(-2)}\right)^n\left(\hat{Y}^{(+2)}\right)^{n+2}+m^2\left(\theta^{(+)}\right)^4\sum_{n=1}^{\infty}b_n\left(c^{(-2)}\hat{Y}^{(+2)}\right)^n. \tag{A.1}$$

All the identities below are given up to terms with a total harmonic derivative $\mathcal{D}_a^{(+2)i}$. In addition, one must take into account the definitions (4.31). Each term in the variation of the series (A.1) also contains transformations compensating the measure transformations (4.19), i.e.

$$\begin{aligned}\delta\left(\hat{Y}^{(+2)}\right)^2=&\,4\Lambda^{(0)}\left(\hat{Y}^{(+2)}\right)^2+2\Lambda^{(0)}c^{(+2)}\hat{Y}^{(+2)}+2\,\varepsilon^{ab}\varepsilon_{ij}\,\Lambda_a^{(+2)i}\mathcal{D}_b^{(+2)j}c^{(-2)}\hat{Y}^{(+2)}\\&-\frac{1}{2}\,\varepsilon^{ab}\varepsilon_{ij}\,c^{(-2)}\hat{Y}^{(+2)}\left(\mathcal{D}_b^{(+2)j}\Lambda_a^{(+2)i}\right),\end{aligned}\tag{A.2}$$

$$\begin{aligned}\delta\left(c^{(-2)}\right)^n\left(\hat{Y}^{(+2)}\right)^{n+2}=&\,(n+4)\,\Lambda^{(0)}\left(c^{(-2)}\right)^n\left(\hat{Y}^{(+2)}\right)^{n+2}\\&+(n+2)\,\Lambda^{(0)}c^{(+2)}\left(c^{(-2)}\right)^n\left(\hat{Y}^{(+2)}\right)^{n+1}\\&+\frac{(n+2)}{(n+1)}\,\varepsilon^{ab}\varepsilon_{ij}\,\Lambda_a^{(+2)i}\mathcal{D}_b^{(+2)j}\left(c^{(-2)}\right)^{n+1}\left(\hat{Y}^{(+2)}\right)^{n+1}\\&-\frac{(n+2)}{4}\,\varepsilon^{ab}\varepsilon_{ij}\left(\mathcal{D}_b^{(+2)j}\Lambda_a^{(+2)i}\right)\left(c^{(-2)}\right)^{n+1}\left(\hat{Y}^{(+2)}\right)^{n+1}\\=&\,(n+4)\,\Lambda^{(0)}\left(c^{(-2)}\right)^n\left(\hat{Y}^{(+2)}\right)^{n+2}\\&+n\,\Lambda^{(0)}\left(c^{(-2)}\right)^{n-1}\left(\hat{Y}^{(+2)}\right)^{n+1}\\&-\frac{(n+3)\,(n+4)}{4\,(n+1)}\,\varepsilon^{ab}\varepsilon_{ij}\left(\mathcal{D}_b^{(+2)j}\Lambda_a^{(+2)i}\right)\left(c^{(-2)}\hat{Y}^{(+2)}\right)^{n+1}\end{aligned}\tag{A.3}$$

where

$$\begin{aligned}\Lambda^{(0)}\left(c^{(-2)}\right)^n c^{(+2)}=&-\frac{1}{4}\,\Lambda^{(0)}\left(c^{(-2)}\right)^n\left(\varepsilon^{ab}\varepsilon_{ij}\,\mathcal{D}_a^{(+2)i}\mathcal{D}_b^{(+2)j}c^{(-2)}\right)\\=&\,\frac{1}{4}\,\Lambda_a^{(+2)i}\left(c^{(-2)}\right)^n\left(\varepsilon^{ab}\varepsilon_{ij}\,\mathcal{D}_b^{(+2)j}c^{(-2)}\right)\\&+\frac{n}{4}\,\Lambda^{(0)}\left(c^{(-2)}\right)^{n-1}\left(\varepsilon^{ab}\varepsilon_{ij}\,\mathcal{D}_b^{(+2)j}c^{(-2)}\right)\left(\mathcal{D}_a^{(+2)i}c^{(-2)}\right)\\=&\,\frac{1}{4\,(n+1)}\,\Lambda_a^{(+2)i}\,\varepsilon^{ab}\varepsilon_{ij}\,\mathcal{D}_b^{(+2)j}\left(c^{(-2)}\right)^{n+1}\\&+\frac{n}{4}\,\Lambda^{(0)}\left(c^{(-2)}\right)^{n-1}\left(\varepsilon^{ab}\varepsilon_{ij}\,\mathcal{D}_b^{(+2)j}c^{(-2)}\right)\left(\mathcal{D}_a^{(+2)i}c^{(-2)}\right).\end{aligned}\tag{A.4}$$

We also use the identity ($c^{IJ}c_{IJ}=4$)

$$\Lambda^{(0)}\left[c^{(-2)}c^{(+2)}+\frac{1}{2}\,\varepsilon^{ab}\varepsilon_{ij}\left(\mathcal{D}_b^{(+2)j}c^{(-2)}\right)\left(\mathcal{D}_a^{(+2)i}c^{(-2)}\right)\right]=\frac{1}{4}\,c^{IJ}c_{IJ}\,\Lambda^{(0)}=\Lambda^{(0)}. \tag{A.5}$$

Thus, the series is invariant when

$$a_0 = 1\,, \qquad a_1 = -4a_0\,, \qquad a_2 = \frac{a_0\, 5!}{2!\, 3!}\,, \qquad a_n = \frac{(-1)^n\,(n+3)!}{n!\, 3!}\,,$$
$$b_n = \frac{(n+2)\,(n+3)\,a_{n-1}}{4n} = -\frac{(-1)^n\,(n+2)\,(n+3)!}{n!\, 4!}\,. \tag{A.6}$$

Using these relations, the above series can be summed up to the expression

$$L^{(+4)} = \frac{\hat{Y}^{(+2)}\hat{Y}^{(+2)}}{\left(1+c^{(-2)}\hat{Y}^{(+2)}\right)^4} + \frac{m^2}{2}\left(\theta^{(+)}\right)^4\left[1-\frac{1-c^{(-2)}\hat{Y}^{(+2)}}{\left(1+c^{(-2)}\hat{Y}^{(+2)}\right)^5}\right]. \tag{A.7}$$

For calculating the component Lagrangians, we use the following identities (up to total harmonic derivatives):

$$\begin{aligned}
c^{(-2)}\hat{y}^{(+2)} &= \frac{1}{12}\, c^{IJ}\hat{y}_{IJ}\,,\\
\left(c^{(-2)}\hat{y}^{(+2)}\right)^2 &= \frac{1}{80}\left[\left(c^{IJ}\hat{y}_{IJ}\right)^2 - \frac{2}{3}\,\hat{y}^{IJ}\hat{y}_{IJ}\right],\\
\left(c^{(-2)}\hat{y}^{(+2)}\right)^3 &= \frac{1}{400}\left[\left(c^{IJ}\hat{y}_{IJ}\right)^3 - \frac{3}{2}\,\hat{y}^{KL}\hat{y}_{KL}\, c^{IJ}y_{IJ}\right],\\
\left(c^{(-2)}\hat{y}^{(+2)}\right)^{2n} &= \frac{12\,(2n)!\,(2n)!}{2^{2n}\,(2n+2)!\,(2n+3)!}\\
&\quad\times\sum_{k=0}^{n}\frac{(-1)^k\,(2n-k+1)!}{k!\,(2n-2k)!}\left(\hat{y}^{KL}\hat{y}_{KL}\right)^k\left(c^{IJ}\hat{y}_{IJ}\right)^{2n-2k},\\
\left(c^{(-2)}\hat{y}^{(+2)}\right)^{2n+1} &= \frac{12\,(2n+1)!\,(2n+1)!}{2^{2n+1}\,(2n+3)!\,(2n+4)!}\\
&\quad\times\sum_{k=0}^{n}\frac{(-1)^k\,(2n-k+2)!}{k!\,(2n-2k+1)!}\left(\hat{y}^{KL}\hat{y}_{KL}\right)^k\left(c^{IJ}\hat{y}_{IJ}\right)^{2n-2k+1}.
\end{aligned} \tag{A.8}$$

Using them, one obtains

$$\begin{aligned}
\frac{\partial^2}{\partial\hat{y}^{(+2)}\partial\hat{y}^{(+2)}}\left[\frac{\hat{y}^{(+2)}\hat{y}^{(+2)}}{\left(1+c^{(-2)}\hat{y}^{(+2)}\right)^4}\right] &= \sum_{n=0}^{\infty}(n+1)\,(n+2)\,a_n\left(c^{(-2)}\hat{y}^{(+2)}\right)^n\\
&= 2\left(1+\frac{c^{IJ}\hat{y}_{IJ}}{2}+\frac{\hat{y}^{IJ}\hat{y}_{IJ}}{4}\right)^{-2}\\
&= 8\left[\frac{1}{2}\,y^{IJ}y_{IJ}\right]^{-2},\\
\sum_{n=1}^{\infty} b_n\left(c^{(-2)}\hat{Y}^{(+2)}\right)^n &= -\frac{1}{2}\left(1+\frac{c^{IJ}\hat{y}_{IJ}}{2}+\frac{\hat{y}^{IJ}\hat{y}_{IJ}}{4}\right)^{-1}\\
&= -\left[\frac{1}{2}\,y^{IJ}y_{IJ}\right]^{-1}.
\end{aligned} \tag{A.9}$$

B The multiplet $(\mathbf{6},\mathbf{8},\mathbf{2})$

As was already mentioned, the multiplets $(\mathbf{6},\mathbf{8},\mathbf{2})$ and $(\mathbf{8},\mathbf{8},\mathbf{0})$ are related to each other by the substitution $i\dot{\phi} = D$. The same substitution is admissible in the Lagrangian (4.50).

Then, the SU(4|1) invariant action (4.32) is given by

$$\begin{aligned}\mathcal{L}_{(\mathbf{6},\mathbf{8},\mathbf{2})} = g_2 &\left[\frac{1}{2}\,\dot{y}^{IJ}\dot{y}_{IJ} + \frac{i}{2}\left(\chi^K\dot{\bar{\chi}}_K - \dot{\chi}^K\bar{\chi}_K\right) + D\bar{D} - \frac{m}{4}\,\chi^K\bar{\chi}_K - \frac{m^2}{8}\,y^{IJ}y_{IJ}\right] \\ &+ \frac{1}{\sqrt{2}}\,\bar{D}\,\partial_{IJ}g_2\,\chi^I\chi^J - \frac{1}{\sqrt{2}}\,D\,\partial^{IJ}g_2\,\bar{\chi}_I\bar{\chi}_J + i\left(\dot{y}_{IK}\,\partial^{JK}g_2 - \dot{y}^{JK}\,\partial_{IK}g_2\right)\chi^I\bar{\chi}_J \\ &- \frac{1}{2}\,\partial_{IJ}\partial^{KL}g_2\,\chi^I\chi^J\bar{\chi}_K\bar{\chi}_L\,, \qquad g_2 = \left[\frac{1}{2}\,y^{IJ}y_{IJ}\right]^{-2}. \end{aligned} \tag{B.1}$$

One can obtain a superconformal Lagrangian by making the following substitutions of the component fields in (B.1)

$$D \to D\,e^{imt/2}, \qquad \bar{D} \to \bar{D}\,e^{-imt/2}, \qquad \chi^I \to \chi^I\,e^{imt/4}, \qquad \bar{\chi}_I \to \bar{\chi}_I\,e^{-imt/4}. \tag{B.2}$$

The resulting Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{conf}} = g_2 &\left[\frac{1}{2}\,\dot{y}^{IJ}\dot{y}_{IJ} + \frac{i}{2}\left(\chi^K\dot{\bar{\chi}}_K - \dot{\chi}^K\bar{\chi}_K\right) + D\bar{D} - \frac{m^2}{8}\,y^{IJ}y_{IJ}\right] \\ &+ \frac{1}{\sqrt{2}}\,\bar{D}\,\partial_{IJ}g_2\,\chi^I\chi^J - \frac{1}{\sqrt{2}}\,D\,\partial^{IJ}g_2\,\bar{\chi}_I\bar{\chi}_J + i\left(\dot{y}_{IK}\,\partial^{JK}g_2 - \dot{y}^{JK}\,\partial_{IK}g_2\right)\chi^I\bar{\chi}_J \\ &- \frac{1}{2}\,\partial_{IJ}\partial^{KL}g_2\,\chi^I\chi^J\bar{\chi}_K\bar{\chi}_L \end{aligned} \tag{B.3}$$

is an even function of m and so is superconformal. Its form is typical for Lagrangians with the trigonometric realizations of superconformal groups.

It can be shown that the superconformal group of (B.3) is SU(4|1,1). On the one hand, the transformations (4.30) become

$$\begin{aligned} &\delta D = -\sqrt{2}\,\epsilon_I\left(i\dot{\chi}^I - m\,\chi^I\right)e^{imt/2}, \qquad \delta\bar{D} = -\sqrt{2}\,\bar{\epsilon}^I\left(i\dot{\bar{\chi}}_I + m\,\bar{\chi}_I\right)e^{-imt/2}, \\ &\delta y^{IJ} = -2\,\bar{\epsilon}^{[I}\chi^{J]}e^{-imt/2} + \varepsilon^{IJKL}\epsilon_K\bar{\chi}_L\,e^{imt/2}, \\ &\delta\chi^I = \sqrt{2}\,\bar{\epsilon}^I B\,e^{-imt/2} - 2\,\epsilon_J\left(i\dot{y}^{IJ} - \frac{m}{2}\,y^{IJ}\right)e^{imt/2}, \\ &\delta\bar{\chi}_I = \sqrt{2}\,\epsilon_I\bar{B}\,e^{imt/2} + 2\,\bar{\epsilon}^J\left(i\dot{y}_{IJ} + \frac{m}{2}\,y_{IJ}\right)e^{-imt/2}, \end{aligned} \tag{B.4}$$

and leave the Lagrangian (B.3) invariant. On the other hand, new SU(4|1) transformations leaving invariant (B.3) are defined by replacing $m \to -m$ in (B.4):

$$\begin{aligned} &\delta D = -\sqrt{2}\,\eta_I\left(i\dot{\chi}^I + m\,\chi^I\right)e^{-imt/2}, \qquad \delta\bar{D} = -\sqrt{2}\,\bar{\eta}^I\left(i\dot{\bar{\chi}}_I - m\,\bar{\chi}_I\right)e^{imt/2}, \\ &\delta y^{IJ} = -2\,\bar{\eta}^{[I}\chi^{J]}e^{imt/2} + \varepsilon^{IJKL}\eta_K\bar{\chi}_L\,e^{-imt/2}, \\ &\delta\chi^I = \sqrt{2}\,\bar{\eta}^I B\,e^{imt/2} - 2\,\eta_J\left(i\dot{y}^{IJ} + \frac{m}{2}\,y^{IJ}\right)e^{-imt/2}, \\ &\delta\bar{\chi}_I = \sqrt{2}\,\eta_I\bar{B}\,e^{-imt/2} + 2\,\bar{\eta}^J\left(i\dot{y}_{IJ} - \frac{m}{2}\,y_{IJ}\right)e^{imt/2}. \end{aligned} \tag{B.5}$$

Introducing the conformal Hamiltonian as

$$\mathcal{H}'_{\text{conf}} = \mathcal{H} + \frac{m}{2}\,F, \tag{B.6}$$

the superconformal algebra $su(4|1,1)$ amounts to the following set of (anti)commutators

$$\begin{aligned}
&\{Q^I,\bar Q_J\}=2\delta^I_J\mathcal{H}'_{\rm conf}+2m\,L^I_J-m\,\delta^I_JF,\\
&\{S^I,\bar S_J\}=2\delta^I_J\mathcal{H}'_{\rm conf}-2m\,L^I_J+m\,\delta^I_JF,\\
&\{Q^I,\bar S_J\}=2\delta^I_JT', \qquad \{S^I,\bar Q_J\}=2\delta^I_J\bar T',
\end{aligned}\tag{B.7}$$

$$\begin{aligned}
&[L^I_J,L^K_L]=\delta^K_J L^I_L-\delta^I_L L^K_J,\\
&[T',\bar T']=2m\,\mathcal{H}'_{\rm conf}\,, \qquad [\mathcal{H}'_{\rm conf},T']=-m\,T', \qquad [\mathcal{H}'_{\rm conf},\bar T']=m\,\bar T',
\end{aligned}\tag{B.8}$$

$$\begin{aligned}
&[L^I_J,Q^K]=\delta^K_J Q^I-\frac{1}{4}\,\delta^I_J Q^K, \qquad [L^I_J,\bar Q_L]=\frac{1}{4}\,\delta^I_J\bar Q_L-\delta^I_L\bar Q_J\,,\\
&[L^I_J,S^K]=\delta^K_J S^I-\frac{1}{4}\,\delta^I_J S^K, \qquad [L^I_J,\bar S_L]=\frac{1}{4}\,\delta^I_J\bar S_L-\delta^I_L\bar S_J\,,\\
&[F,Q^I]=\frac{1}{2}\,Q^I, \qquad [F,\bar Q_J]=-\frac{1}{2}\,\bar Q_J\,, \quad [F,S^I]=\frac{1}{2}\,S^I, \quad [F,\bar S_J]=-\frac{1}{2}\,\bar S_J\,,\\
&[\mathcal{H}'_{\rm conf},Q^I]=-\frac{m}{2}\,Q^I, \qquad [\mathcal{H}'_{\rm conf},\bar Q_J]=\frac{m}{2}\,\bar Q_J\,,\\
&[\mathcal{H}'_{\rm conf},S^I]=\frac{m}{2}\,S^I, \qquad [\mathcal{H}'_{\rm conf},\bar S_J]=-\frac{m}{2}\,\bar S_J\,,\\
&[T',\bar Q_J]=m\,\bar S_J\,, \qquad [T',S^I]=m\,Q^I,\\
&[\bar T',\bar S_J]=-m\,\bar Q_J\,, \qquad [\bar T',Q^I]=-m\,S^I.
\end{aligned}\tag{B.9}$$

Note that the "parabolic" realization of the superconformal group SU$(4|1,1)$ on the undeformed multiplet $(\mathbf{6},\mathbf{8},\mathbf{2})$ was given in [40]. The corresponding superconformal Lagrangian can be obtained as the m=0 limit of (B.3).

C The multiplet $(\mathbf{7},\mathbf{8},\mathbf{1})$

The substitution $\sqrt{2}\,i\dot\phi = C - i\dot x$ in (4.7) gives SU$(4|1)$ transformations of the multiplet $(\mathbf{7},\mathbf{8},\mathbf{1})$:

$$\begin{aligned}
\delta C &= -\epsilon_I\left(i\dot\chi^I-\frac{3m}{4}\,\chi^I\right)e^{3imt/4}-\bar\epsilon^I\left(i\dot{\bar\chi}_I+\frac{3m}{4}\,\bar\chi_I\right)e^{-3imt/4},\\
\delta x &= \epsilon_I\chi^I e^{3imt/4}-\bar\epsilon^I\bar\chi_I\,e^{-3imt/4},\\
\delta y^{IJ} &= -2\,\bar\epsilon^{[I}\chi^{J]}e^{-3imt/4}+\varepsilon^{IJKL}\epsilon_K\bar\chi_L\,e^{3imt/4},\\
\delta\chi^I &= \bar\epsilon^I\,(C-i\dot x)\,e^{-3imt/4}-2\,\epsilon_J\left(i\dot y^{IJ}-\frac{m}{2}\,y^{IJ}\right)e^{3imt/4},\\
\delta\bar\chi_I &= \epsilon_I\,(C+i\dot x)\,e^{3imt/4}+2\,\bar\epsilon^J\left(i\dot y_{IJ}+\frac{m}{2}\,y_{IJ}\right)e^{-3imt/4}.
\end{aligned}\tag{C.1}$$

The same substitution is admissible in the Lagrangian (4.50). In this way we obtain the SU$(4|1)$ invariant Lagrangian for the multiplet $(\mathbf{7},\mathbf{8},\mathbf{1})$:

$$\begin{aligned}
\mathcal{L}_{(\mathbf{7},\mathbf{8},\mathbf{1})} &= g_2\left[\frac{\dot x^2}{2}+\frac{1}{2}\,\dot y^{IJ}\dot y_{IJ}+\frac{i}{2}\left(\chi^K\dot{\bar\chi}_K-\dot\chi^K\bar\chi_K\right)+\frac{C^2}{2}-\frac{m}{4}\,\chi^K\bar\chi_K-\frac{m^2}{8}\,y^{IJ}y_{IJ}\right]\\
&\quad+\frac{C}{2}\left(\partial_{IJ}g_2\,\chi^I\chi^J-\partial^{IJ}g_2\,\bar\chi_I\bar\chi_J\right)+\frac{i\dot x}{2}\left(\partial_{IJ}g_2\,\chi^I\chi^J+\partial^{IJ}g_2\,\bar\chi_I\bar\chi_J\right)\\
&\quad+i\left(\dot y_{IK}\,\partial^{JK}g_2-\dot y^{JK}\,\partial_{IK}g_2\right)\chi^I\bar\chi_J-\frac{1}{2}\,\partial_{IJ}\partial^{KL}g_2\,\chi^I\chi^J\bar\chi_K\bar\chi_L\,,\\
g_2 &= \left[\frac{1}{2}\,y^{IJ}y_{IJ}\right]^{-2}.
\end{aligned}\tag{C.2}$$

The corresponding action is not invariant with respect to the superconformal group F(4) inherent to the multiplet $(\mathbf{7},\mathbf{8},\mathbf{1})$ [41] because (C.2) cannot be brought to a form in which it depends only on m^2. On the other hand, since F(4) includes SU(4|1) as a subgroup, we expect the existence of an alternative, F(4) superconformal, action for the SU(4|1) multiplet $(\mathbf{7},\mathbf{8},\mathbf{1})$. At the component level, such a system has recently been constructed, without giving an action however [42].

D Superconformal algebra $osp(8|2)$

Superconformal algebra $osp(8|2)$ is given by the following non-vanishing (anti)commutators:

$$
\begin{aligned}
&\{Q^I,\bar{Q}_J\} = 2\delta^I_J\mathcal{H}_{\rm conf} + 2m\,L^I_J + m\,\delta^I_J F, \\
&\{S^I,\bar{S}_J\} = 2\delta^I_J\mathcal{H}_{\rm conf} - 2m\,L^I_J - m\,\delta^I_J F, \\
&\{Q^I,S^J\} = 2m\,F^{IJ}, \qquad \{\bar{Q}_I,\bar{S}_J\} = 2m\,\bar{F}_{IJ}, \\
&\{Q^I,\bar{S}_J\} = 2\delta^I_J T, \qquad \{S^I,\bar{Q}_J\} = 2\delta^I_J \bar{T},
\end{aligned}
\tag{D.1}
$$

$$
\begin{aligned}
&\left[L^I_J, L^K_L\right] = \delta^K_J L^I_L - \delta^I_L L^K_J, \\
&\left[L^I_J, F^{KL}\right] = \delta^K_J F^{IL} + \delta^L_J F^{KI} - \frac{1}{2}\,\delta^I_J F^{KL}, \\
&\left[L^I_J, \bar{F}_{KL}\right] = \frac{1}{2}\,\delta^I_J \bar{F}_{KL} - \delta^I_K \bar{F}_{JL} - \delta^I_L \bar{F}_{KJ}, \\
&\left[F^{IJ}, \bar{F}_{KL}\right] = \delta^I_K L^J_L - \delta^I_L L^J_K + \delta^J_L L^I_K - \delta^J_K L^I_L + \left(\delta^I_K\delta^J_L - \delta^I_L\delta^J_K\right) F, \\
&\left[F, F^{IJ}\right] = F^{IJ}, \qquad \left[F, \bar{F}_{IJ}\right] = -\bar{F}_{IJ}, \\
&\left[T, \bar{T}\right] = 4m\,\mathcal{H}_{\rm conf}, \qquad \left[\mathcal{H}_{\rm conf}, T\right] = -2m\,T, \qquad \left[\mathcal{H}_{\rm conf}, \bar{T}\right] = 2m\,\bar{T},
\end{aligned}
\tag{D.2}
$$

$$
\begin{aligned}
&\left[L^I_J, Q^K\right] = \delta^K_J Q^I - \frac{1}{4}\,\delta^I_J Q^K, \qquad \left[L^I_J, \bar{Q}_L\right] = \frac{1}{4}\,\delta^I_J \bar{Q}_L - \delta^I_L \bar{Q}_J, \\
&\left[L^I_J, S^K\right] = \delta^K_J S^I - \frac{1}{4}\,\delta^I_J S^K, \qquad \left[L^I_J, \bar{S}_L\right] = \frac{1}{4}\,\delta^I_J \bar{S}_L - \delta^I_L \bar{S}_J, \\
&\left[\bar{F}_{IJ}, Q^K\right] = \delta^K_I \bar{S}_J - \delta^K_J \bar{S}_I, \qquad \left[F^{IJ}, \bar{Q}_L\right] = \delta^J_L S^I - \delta^I_L S^J, \\
&\left[\bar{F}_{IJ}, S^K\right] = \delta^K_I \bar{Q}_J - \delta^K_J \bar{Q}_I, \qquad \left[F^{IJ}, \bar{S}_L\right] = \delta^J_L Q^I - \delta^I_L Q^J, \\
&\left[F, Q^I\right] = \frac{1}{2}\,Q^I, \qquad \left[F, \bar{Q}_J\right] = -\frac{1}{2}\,\bar{Q}_J, \\
&\left[F, S^I\right] = \frac{1}{2}\,S^I, \qquad \left[F, \bar{S}_J\right] = -\frac{1}{2}\,\bar{S}_J, \\
&\left[\mathcal{H}_{\rm conf}, Q^I\right] = -m\,Q^I, \qquad \left[\mathcal{H}_{\rm conf}, \bar{Q}_J\right] = m\,\bar{Q}_J, \\
&\left[\mathcal{H}_{\rm conf}, S^I\right] = m\,S^I, \qquad \left[\mathcal{H}_{\rm conf}, \bar{S}_J\right] = -m\,\bar{S}_J, \\
&\left[T, \bar{Q}_J\right] = 2m\,\bar{S}_J, \qquad \left[T, S^I\right] = 2m\,Q^I, \\
&\left[\bar{T}, \bar{S}_J\right] = -2m\,\bar{Q}_J, \qquad \left[\bar{T}, Q^I\right] = -2m\,S^I.
\end{aligned}
\tag{D.3}
$$

The supercharges Q^I, $\bar{Q}_J$ together with the generators L^I_J and $H = \mathcal{H}_{\rm conf} + \frac{m}{2}F$ form the subalgebra $su(4|1) \niplus u(1)$ in $osp(8|2)$, with F being an additional external R-symmetry U(1) generator. The second set of SU(4|1) supercharges S^I, $\bar{S}_J$ extends this subalgebra to the full superconformal algebra $osp(8|2)$. The latter involves twelve additional R-symmetry generators $F^{IJ} \equiv F^{[IJ]}$, $\bar{F}_{IJ} \equiv \bar{F}_{[IJ]}$ which, together with the U(4) generators L^I_J, F, form the full R-symmetry algebra $o(8)$. Additional conformal generators are $\bar{T}$, T, such that three bosonic generators $\mathcal{H}_{\rm conf}$, $\bar{T}$ and T constitute the conformal d=1 subalgebra $o(2,1)$.

Actually, the parameter m drops out from the superconformal algebra after performing redefinitions similar to those made in [8] for the case of the $\mathcal{N}=4, d=1$ superconformal algebra $D(2,1;\alpha)$.

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