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Skyrme–Faddeev model from 5d super-Yang–Mills

Olaf Lechtenfeld [a,b,*], Alexander D. Popov [a]

[a] Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany
[b] Riemann Center for Geometry and Physics, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany

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ABSTRACT

We consider 5d Yang–Mills–Higgs theory with a compact ADE-type gauge group G and one adjoint scalar field on $\mathbb{R}^{3,1} \times \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$ is the half-line. The maximally supersymmetric extension of this model, with five adjoint scalars, appears after a reduction of 6d $\mathcal{N}=(2,0)$ superconformal field theory on $\mathbb{R}^{3,1} \times \mathbb{R}_+ \times S^1$ along the circle S^1. We show that in the low-energy limit, when momenta along $\mathbb{R}^{3,1}$ are much smaller than along $\mathbb{R}_+$, the 5d Yang–Mills–Higgs theory reduces to a nonlinear sigma model on $\mathbb{R}^{3,1}$ with a coset G/H as its target space. Here H is a closed subgroup of G determined by the Higgs-field asymptotics at infinity. The 4d sigma model describes an infinite tower of interacting fields, and in the infrared it is dominated by the standard two-derivative kinetic term and the four-derivative Skyrme–Faddeev term.

1. Introduction and summary

Quantum chromodynamics (QCD) as well as Yang–Mills theory are strongly coupled in the infrared limit, and hence the perturbative expansion for them breaks down. In the absence of a quantitative understanding of non-perturbative QCD, convenient alternatives at low energy are provided by effective models among which nonlinear sigma models play an important role. A first model of this kind was introduced by Skyrme [1] for describing baryons as point-like solitons (see e.g. [2] for a review and references). The standard Skyrme model encodes pion degrees of freedom with an SU(2)-valued function on $\mathbb{R}^{3,1}$. Its action contains the standard two-derivative sigma-model term as well as the four-derivative Skyrme term which stabilizes solitons against scaling.

A related model was introduced by Faddeev [3]. This is a sigma model on $\mathbb{R}^{3,1}$ with a coset space $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ as its target space, and it also contains a four-derivative Skyrme-type term. Static Skyrme–Faddeev solitons are maps from $\mathbb{R}^3 \cup \{\infty\} = S^3$ to the target space S^2 and thus characterized by their homotopy class, the Hopf invariant. The cores of Skyrme–Faddeev solitons, sometimes called Hopfions, are twisted and knotted circles, in contrast to point-like cores of Skyrmions [4–8]. It is believed that the Skyrme model and its extension to other mesons provides a low-energy description of baryons, and that the Skyrme–Faddeev model may describe glueballs [9] or stable closed vortices in various areas of physics (see e.g. [10–12] and references therein). Both the Skyrme and the Skyrme–Faddeev model have been generalized to an arbitrary compact Lie group G and coset G/H, respectively (see e.g. [13–15]).

A classical problem of the standard Skyrme model was its difficulty to incorporate other mesons besides pions. This shortcoming was overcome recently with an extended 4d Skyrme model obtained from 5d Yang–Mills theory derived from D-brane configurations in string theory and the holographic approach [16] (see e.g. [17–19] for reviews and references). This extended Skyrme model can also be reached from 6d $\mathcal{N}=(2,0)$ superconformal field theory compactified on a circle to 5d super-Yang–Mills (SYM) theory on $\mathbb{R}^{3,1} \times \mathcal{I}$, where $\mathcal{I} = [-R, R]$ is a finite-length interval [20], upon forgetting the five adjoint scalar fields.

Here, we show that, like the extended Skyrme model, also an extended 4d Skyrme–Faddeev model can emerge in a low-energy limit of 5d SYM theory with Dirichlet boundary conditions [21, 22]. In contrast to the extended Skyrme model, for the extended Skyrme–Faddeev model one needs to keep one of the five adjoint scalars and also to modify the fifth dimension from $\mathcal{I} = [-R, R]$ to the half-line $\mathbb{R}_+ = [0, \infty) \ni x^4$. The boundary conditions required for the reduction to $\mathbb{R}^{3,1}$ are encoded in Nahm equations along the fifth dimension [21,22]. In our case, the latter become "baby" Nahm equations on $\mathbb{R}_+$ for the remaining adjoint scalar $\phi \in \mathfrak{g} = \mathrm{Lie}\, G$. Solutions to these equations were studied in [23]. The scalar is taken to approach an element τ of the Cartan subalgebra of $\mathfrak{g}$ in the limit $x^4 \to \infty$. The moduli space $\mathcal{M}_\tau$ of solutions to the

* Corresponding author.
E-mail addresses: olaf.lechtenfeld@itp.uni-hannover.de (O. Lechtenfeld), alexander.popov@itp.uni-hannover.de (A.D. Popov).

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baby Nahm equation then becomes the adjoint orbit of τ. In other words, $\mathcal{M}_\tau = G/H = \{g\,\tau\,g^{-1}\,|\,g \in G\}$, where H is the stabilizer of τ in G. This coset G/H becomes the target space for our 4d effective sigma model.

We start with 5d SYM theory on $\mathbb{R}^{3,1} \times \mathbb{R}_+$ and show how an extended Skyrme–Faddeev model appears rather naturally in the low-energy limit. Our derivation employs the adiabatic approach [24–30] based on Manton's seminal paper [24]. This might give a clue to the construction of a 4d supersymmetric Skyrme–Faddeev model, which seems to have not yet been completed. To this end one should keep three of the five adjoint scalars obeying Nahm equations on $\mathbb{R}_+$. To summarize, we demonstrate that not only the Skyrme model but also the Skyrme–Faddeev model as well as their extended versions emerge from the M5-brane system of M-theory.

2. Yang–Mills and Higgs fields in five dimensions

Gauge fields and adjoint scalars Let M^d be an oriented smooth manifold of dimension d, G a compact ADE-type Lie group with $\mathfrak{g}$ its Lie algebra, P a principal G-bundle over M^d, $\mathcal{A}$ a connection one-form on P and $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ its curvature. We consider also the bundle of groups $\mathrm{Int}P = P \times_G G$ (G acts on itself by internal automorphisms $h \mapsto ghg^{-1}$ with $h, g \in G$) associated with P and the bundle of Lie algebras $\mathrm{Ad}P = P \times_G \mathfrak{g}$ (adjoint action of G on $\mathfrak{g}$). These associated bundles inherit their connection $\mathcal{A}$ from P. Besides $\mathcal{A}$ we will also consider $\mathfrak{g}$-valued scalar fields ϕ on M^d, they are sections of the bundle $\mathrm{Ad}P$.

We denote by $\mathcal{G}$ the infinite-dimensional group of gauge transformations,

$$\mathcal{G} \ni f:\ \mathcal{A} \mapsto \mathcal{A}^f = f^{-1}\mathcal{A}\,f + f^{-1}\mathrm{d}f \qquad \text{and}$$
$$\phi \mapsto \phi^f = f^{-1}\phi\, f\ , \tag{2.1}$$

which can be identified with the space of global sections of the bundle $\mathrm{Int}P$. Correspondingly, the infinitesimal action of $\mathcal{G}$ is defined by global sections ϵ of the bundle $\mathrm{Ad}P$,

$$\mathrm{Lie}\,\mathcal{G} \ni \epsilon:\ \delta_\epsilon\mathcal{A} = D_{\mathcal{A}}\epsilon = \mathrm{d}\epsilon + [\mathcal{A}, \epsilon] \qquad \text{and}$$
$$\delta_\epsilon\phi = [\phi, \epsilon]\ . \tag{2.2}$$

The moduli space of pairs $(\mathcal{A}, \phi)$ is defined as the quotient of the space of all such pairs by the action (2.2) of the gauge group $\mathcal{G}$.

Space $\mathbb{R}^{3,1} \times \mathbb{R}_+$ We consider d=5 and Yang–Mills–Higgs theory on $M^5 = \mathbb{R}^{3,1} \times \mathbb{R}_+$ with coordinates $(x^\mu) = (x^a, x^4)$ for $a = 0, 1, 2, 3$, where $x^a \in \mathbb{R}^{3,1}$ and $x^4 \in \mathbb{R}_+ = [0, \infty)$. We introduce a family of flat metrics,

$$\mathrm{d}s^2_\varepsilon = g^\varepsilon_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu = \eta_{ab}\,\mathrm{d}x^a\mathrm{d}x^b + \varepsilon^2(\mathrm{d}x^4)^2\ , \tag{2.3}$$

where $(\eta_{ab}) = \mathrm{diag}(-1, 1, 1, 1)$ and $\varepsilon > 0$ is a dimensionless parameter regulating the transition to the low-energy limit. Namely, for $\varepsilon = 1$ one has the standard Yang–Mills–Higgs theory on $\mathbb{R}^{3,1} \times \mathbb{R}_+$. For small ε, momenta along $\mathbb{R}_+$ are much larger than momenta along $\mathbb{R}^{3,1}$, and Yang–Mills–Higgs theory on $\mathbb{R}^{3,1} \times \mathbb{R}_+$ reduces to a non-linear sigma model on $\mathbb{R}^{3,1}$ that will be described below.

Note that the definition of infrared limit hiddenly introduces an arbitrary scale L into the model. In five dimensions this scale is provided by e^2, where e is the (dimensionful) 5d gauge coupling. For physical application it is to be matched, e.g. to the nuclear scale. Here, the infrared region is defined by $\varepsilon \ll 1$ for convenience. For a dimensionful variant, one may absorb the length dimension of x^4 into ε and take the infrared domain as $\varepsilon \ll L$.

Action functional For a $\mathfrak{g}$-valued gauge potential (connection) $\mathcal{A}$ and its gauge field (curvature) $\mathcal{F}$ on the principal bundle P over $\mathbb{R}^{3,1} \times \mathbb{R}_+$ we have the obvious splitting

$$\mathcal{A} = \mathcal{A}_a\,\mathrm{d}x^a + \mathcal{A}_4\,\mathrm{d}x^4 \qquad \text{and}$$
$$\mathcal{F} = \tfrac12\mathcal{F}_{ab}\,\mathrm{d}x^a\wedge\mathrm{d}x^b + \mathcal{F}_{a4}\,\mathrm{d}x^a\wedge\mathrm{d}x^4\ . \tag{2.4}$$

The fields $\mathcal{A}$, $\mathcal{F}$ and ϕ are taken in the adjoint representation of $\mathfrak{g} = \mathrm{Lie}\,G$. For the adjoint generators I_i of G we use the standard normalization $\mathrm{tr}(I_iI_j) = -2\delta_{ij}$ with $i, j = 1, \dots, \mathrm{dim}G$.

For the metric tensor (2.3) we have $(g^{\mu\nu}_\varepsilon) = (\eta^{ab}, \varepsilon^{-2})$ and $\det(g^\varepsilon_{\mu\nu}) = -\varepsilon^2$. We denote by $\mathcal{F}^{\mu\nu}_\varepsilon$ the contravariant components raised from $\mathcal{F}_{\mu\nu}$ by $g^{\mu\nu}_\varepsilon$ and by $\mathcal{F}^{\mu\nu}$ those obtained by using $g^{\mu\nu} \equiv g^{\mu\nu}_{\varepsilon=1}$. We have $\mathcal{F}^{ab}_\varepsilon = \mathcal{F}^{ab}$ and $\mathcal{F}^{a4}_\varepsilon = \varepsilon^{-2}\mathcal{F}^{a4}$. We also rescale the Higgs field $\phi \mapsto \varepsilon^{-1}\phi$. The Yang–Mills–Higgs (YMH) action functional on $\mathbb{R}^{3,1} \times \mathbb{R}_+$ with the metric (2.3) then takes the form

$$\begin{aligned} S &= -\frac{1}{8e^2}\int_{\mathbb{R}^{3,1}\times\mathbb{R}_+} \mathrm{d}^5x\,\sqrt{|\det g^\varepsilon|}\ \mathrm{tr}\left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}_\varepsilon + \tfrac{2}{\varepsilon^2}D_\mu\phi D^\mu\phi\right) \\ &= -\frac{1}{8e^2}\int_{\mathbb{R}^{3,1}\times\mathbb{R}_+} \mathrm{d}^5x\ \mathrm{tr}\big(\varepsilon\,\mathcal{F}_{ab}\mathcal{F}^{ab} + \tfrac{2}{\varepsilon}\mathcal{F}_{a4}\mathcal{F}^{a4} + \tfrac{2}{\varepsilon}D_a\phi D^a\phi \\ &\quad + \tfrac{2}{\varepsilon^3}D_4\phi D_4\phi\big)\ . \end{aligned} \tag{2.5}$$

There is no potential for ϕ in (2.5), as in the standard action for monopoles. Instead, nontrivial geometry for ϕ will appear from asymptotic conditions at infinity. The action (2.5) follows from the bosonic action of maximally supersymmetric Yang–Mills theory in five dimensions (see e.g. [22]) after putting to zero four of the five adjoint scalar fields.

3. Boundary conditions and moduli space of vacua

Conditions at $x^4 = 0$ and at $x^4 \to \infty$ For convenience let us introduce a dimensionless fifth coordinate z and dimensionless field components,

$$z = x^4/L \qquad \text{and} \qquad \mathcal{A}_z = L\,\mathcal{A}_{x^4} \qquad \text{as well as} \qquad \varphi = L\,\phi\ . \tag{3.1}$$

The boundary of $M^5 = \mathbb{R}^{3,1} \times \mathbb{R}_+$ consists of Minkowski space $\mathbb{R}^{3,1}_0 = \partial M^5$ at $z = 0$. Infinity $z \to \infty$ is parametrized by Minkowski space $\mathbb{R}^{3,1}_\infty$ at $z = \infty$. For the $\mathfrak{g}$-valued fields $(\mathcal{A}, \varphi)$ on $\mathbb{R}^{3,1} \times \mathbb{R}_+$ we have to impose boundary (at $z = 0$) and asymptotic (for $z \to \infty$) conditions. We make the following choice [31],

$$\mathcal{A}_a(x^a, z{=}0) = 0 \qquad \text{and}$$
$$\left\{\partial_z\varphi(x^a, z) + [\mathcal{A}_z(x^a, z), \varphi(x^a, z)]\right\}\big|_{z=0} = 0\ , \tag{3.2}$$

$$\mathcal{A}_\mu(x^a, z{\to}\infty) = 0 \qquad \text{and} \qquad \varphi(x^a, z{\to}\infty) = \tau(x^a) \in \mathfrak{t} \subset \mathfrak{g}\ , \tag{3.3}$$

where $\mathfrak{t}$ is a Cartan subalgebra of the Lie algebra $\mathfrak{g}$. Such a class of conditions is parametrized by a Minkowski-space $\mathfrak{g}$-valued function τ and has been imposed e.g. in studies of the Nahm equations on $\mathbb{R}_+$ (see e.g. [32,33,23]).

Gauge group We employ the notation $\mathcal{I} := \mathbb{R}_+$ and consider the group $\mathcal{G} = C^\infty(\mathbb{R}^{3,1} \times \mathcal{I}, G)$ as well as its restriction $\mathcal{G}_\mathcal{I}$ to $\mathcal{I}$ obtained by fixing $x^a \in \mathbb{R}^{3,1}$ to some arbitrary value, i.e. $\mathcal{G}_\mathcal{I} \cong C^\infty(\mathcal{I}, G)$. The true group of gauge transformations has to preserve the chosen boundary and asymptotic conditions (3.2) and (3.3) (see e.g. [34]). This is not the case for $\mathcal{G}$ but for its subgroup

$$\mathcal{G}^0 \;=\; \big\{h \in \mathcal{G}: \quad h(x^a, z{=}0) = h(x^a, z{\to}\infty) = \mathrm{Id}\big\} \,. \tag{3.4}$$

In the following, we shall need two larger subgroups, which preserve (3.2) but not the asymptotics (3.3), namely

$$\mathcal{G}^1 \;=\; \big\{h \in \mathcal{G}: \quad h(x^a, z{=}0) = \mathrm{Id} \quad \text{but} \quad h(x^a, z{\to}\infty) \in G\big\} \quad \text{and} \tag{3.5}$$

$$\mathcal{G}^\tau \;=\; \big\{h \in \mathcal{G}: \quad h(x^a, z{=}0) = \mathrm{Id} \quad \text{and} \quad h(x^a, z{\to}\infty) \in H\big\}\,, \tag{3.6}$$

where H is the stabilizer of τ in $\mathfrak{t}$ under the adjoint action. Clearly, $\mathcal{G}^0 \subset \mathcal{G}^\tau \subset \mathcal{G}^1 \subset \mathcal{G}$, and the transformations from $\mathcal{G}^\tau$ respect the asymptotics (3.3) only for $\mathcal{A}_z$ and φ.

For the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of the Lie groups G and H, respectively, we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and choose $\mathfrak{m}$ to be orthogonal to $\mathfrak{h}$ with respect to the Cartan–Killing form. We assume that the adjoint orbit G/H is reductive, which means that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. When H is the maximal torus T in G, the coset space G/T is the orbit of maximal dimension.

We denote by $\mathcal{G}^1_\mathcal{I}$ and $\mathcal{G}^\tau_\mathcal{I}$ the restrictions of the groups $\mathcal{G}^1$ and $\mathcal{G}^\tau$ to the half-line $\mathcal{I} = \mathbb{R}_+$ by fixing $x^a \in \mathbb{R}^{3,1}$. It follows from the definitions of $\mathcal{G}_\mathcal{I}$, $\mathcal{G}^1_\mathcal{I}$ and $\mathcal{G}^\tau_\mathcal{I}$ that

$$\mathcal{G}_\mathcal{I}/\mathcal{G}^1_\mathcal{I} \;\cong\; G \qquad \text{and} \qquad \mathcal{G}^1_\mathcal{I}/\mathcal{G}^\tau_\mathcal{I} \;\cong\; G/H \tag{3.7}$$

since the elements of these groups differ only either at $z = 0$ or at $z = \infty$. Correspondingly, the definitions of $\mathcal{G}$, $\mathcal{G}^1$ and $\mathcal{G}^\tau$ imply that

$$\mathcal{G}/\mathcal{G}^1 \;\cong\; C^\infty(\mathbb{R}^{3,1}, G) \qquad \text{and} \qquad \mathcal{G}^1/\mathcal{G}^\tau \;\cong\; C^\infty(\mathbb{R}^{3,1}, G/H)\,. \tag{3.8}$$

Yang–Mills–Higgs model on $\mathbb{R}_+$ Our consideration of the low-energy limit $\varepsilon \to 0$ of the YMH model (2.5) is based on the adiabatic approach which for YMH theories was introduced in the seminal paper [24] by Manton (for brief reviews see e.g. [30, 35] and references therein). In the adiabatic approach one should firstly restrict the YMH theory (2.5) to $\mathcal{I}$ and classify solutions on $\mathcal{I}$ not depending on $x^a \in \mathbb{R}^{3,1}$ and secondly declare that their moduli, which parametrize such solutions, depend on $x^a \in \mathbb{R}^{3,1}$ and derive the effective action for these moduli functions.

From the action (2.5) it follows that for $\varepsilon \to 0$ and $\mathcal{A}_a = 0$ the equations of motion read

$$\partial_a \mathcal{A}_z = 0 = \partial_a \varphi \qquad \text{and} \qquad \partial_z \varphi + [\mathcal{A}_z, \varphi] = 0\,. \tag{3.9}$$

The conditions (3.2) and (3.3) become

$$\mathcal{A}_z(z{=}\infty) = 0 \qquad \text{and} \qquad \varphi(\infty) = \tau \in \mathfrak{t} \subset \mathfrak{g}\,, \tag{3.10}$$

while the boundary condition (3.2) at $z{=}0$ is satisfied due to (3.9). For regular elements τ (when H is the maximal torus T in G), solutions to (3.9) and (3.10) were described in [23]. We adapt the construction to non-regular τ.

Equation (3.9) is solved by

$$\mathcal{A}_z = h^{-1}\partial_z h \qquad \text{and} \qquad \varphi = h^{-1}\varphi(0)\,h \qquad \text{where} \quad h(z) \in \mathcal{G}_\mathcal{I}\,. \tag{3.11}$$

However, $h(z)$ and $h^{-1}(0)h(z)$ define the same solution, so we may impose $h(0) = \mathrm{Id}$ or, equivalently, take $h(z) \in \mathcal{G}^1_\mathcal{I}$. Then from (3.10) and (3.11) we obtain that

$$\varphi(\infty) = h^{-1}(\infty)\,\varphi(0)\,h(\infty) = \tau \qquad \Rightarrow \qquad \varphi(0) = h(\infty)\,\tau\,h^{-1}(\infty)\,. \tag{3.12}$$

As H is the stabilizer of τ under the adjoint G-action,

$$h_0\,\tau\,h_0^{-1} = \tau \quad \text{for} \quad h_0 \in H\,, \tag{3.13}$$

we may locally factorize

$$h(\infty) = m\,h_0 \qquad \Rightarrow \qquad \varphi(0) = m\,\tau\,m^{-1} \in G/H \qquad \text{for} \quad h_0 \in H \quad \text{and} \quad m \in G/H\,, \tag{3.14}$$

so that $\mathcal{M}_\mathcal{I} = G/H$ is the moduli space of solutions to (3.9).

Moduli space of vacua One arrives at the same vacuum moduli space $\mathcal{M}_\mathcal{I} = G/H$ for YMH theory on $\mathbb{R}_+$ by noting the one-to-one correspondence between $\mathcal{A}_z$ on $\mathbb{R}_+$ and $h(z) \in \mathcal{G}^1_\mathcal{I}$ given by the first formula in (3.11) and its "inverse"

$$h(z) \;=\; \mathcal{P}\exp\Big(\int_0^z \mathcal{A}_y \mathrm{d}y\Big)\,, \tag{3.15}$$

where $\mathcal{P}$ denotes path ordering. The gauge subgroup $\mathcal{G}^\tau_\mathcal{I}$ acts on $\mathcal{A}_z$ and $\varphi(z)$ (and hence on the solution space $\mathcal{G}^1_\mathcal{I} \ni h(z)$) by

$$\mathcal{G}^\tau_\mathcal{I} \ni f: \quad \mathcal{A}_z \mapsto \mathcal{A}_z^f = f^{-1}\mathcal{A}_z f + f^{-1}\partial_z f\,, \qquad \varphi \mapsto \varphi^f = f^{-1}\varphi\,f \quad \Rightarrow \quad h \mapsto h^f = hf\,. \tag{3.16}$$

Hence, the moduli space of solutions (3.11) is $\mathcal{M}_\mathcal{I} = \mathcal{G}^1_\mathcal{I}/\mathcal{G}^\tau_\mathcal{I} \cong G/H$, and one can define the principal $\mathcal{G}^\tau_\mathcal{I}$-bundle

$$q: \quad \mathcal{G}^1_\mathcal{I} \xrightarrow{\mathcal{G}^\tau_\mathcal{I}} G/H \qquad \text{with} \quad h(z) \mapsto m \tag{3.17}$$

for $m \in G/H$ defined in (3.14).

4. Infinitesimal change of solutions $(\mathcal{A}_z, \varphi)$

Linearized equations Suppose we have a solution $(\mathcal{A}_z, \varphi)$ to (3.9), which belongs to the moduli space $\mathcal{M}_\mathcal{I} = G/H$ from (3.17). Then $(\delta\mathcal{A}_z, \delta\varphi)$ will be a tangent vector to G/H at the point $(\mathcal{A}_z, \varphi)$ if

$$D_z\delta\varphi + [\delta\mathcal{A}_z, \varphi] = 0 \tag{4.1}$$

and

$$D_z\delta\mathcal{A}_z + [\varphi, \delta\varphi] = 0\,, \tag{4.2}$$

where $D_z = \partial_z + [\mathcal{A}_z, \cdot\,]$. Equation (4.1) means that $(\delta\mathcal{A}_z, \delta\varphi)$ belong to the tangent space $T_{(\mathcal{A}_z,\varphi)}\mathcal{G}^1_\mathcal{I}$ of the solution space $\mathcal{G}^1_\mathcal{I}$, and (4.2) says that $(\delta\mathcal{A}_z, \delta\varphi)$ is orthogonal to the gauge modes (cf. [26] for a similar discussion regarding the moduli space of monopoles in $\mathbb{R}^3$). Below we will explain this in more detail.

Geometry of G/H We consider the adjoint orbit (3.14). Let us choose a basis $\{I_i\}$ for the Lie algebra $\mathfrak{g}$ in such a way that $\{I_{\bar\imath}\}$ for $\bar\imath = 1,\dots,\dim G/H$ form a basis for $\mathfrak{m}$ and $\{I_{\hat\imath}\}$ for $\hat\imath = \dim G/H+1,\dots,\dim G$ provide a basis for $\mathfrak{h}$. For the total Lie algebra we have $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ and $\mathrm{tr}(I_{\bar\imath} I_{\hat\imath})=0$.

The space G/H consists of left cosets gH, and the natural projection $g\mapsto gH$ is denoted by

$$\pi:\quad G \xrightarrow{H} G/H\ . \tag{4.3}$$

On G/H there exists an orthonormal frame of left-invariant one-forms $\{e^{\bar\imath}\}$ which locally provides the G-invariant metric

$$\mathrm{d}s^2_{G/H} \;=\; \delta_{\bar\imath\bar\jmath}\, e^{\bar\imath} e^{\bar\jmath} \;=\; \delta_{\bar\imath\bar\jmath}\, e^{\bar\imath}_\alpha e^{\bar\jmath}_\beta\, \mathrm{d}X^\alpha \mathrm{d}X^\beta \;=:\; g_{\alpha\beta}\,\mathrm{d}X^\alpha \mathrm{d}X^\beta \quad \text{for}\quad \alpha,\beta=1,\dots,\dim G/H\ , \tag{4.4}$$

where X^α are local coordinates on G/H. The principal H-bundle (4.3) supports a unique G-invariant connection, the so called *canonical connection* [36–38],

$$\mathcal{A}_{G/H} = e^{\hat\imath} I_{\hat\imath} = e^{\hat\imath}_{\bar\imath} I_{\hat\imath}\, e^{\bar\imath} = e^{\hat\imath}_\alpha I_{\hat\imath}\, \mathrm{d}X^\alpha\ . \tag{4.5}$$

The one-forms $e^i=(e^{\bar\imath},e^{\hat\imath})$ obey the Maurer–Cartan equations,

$$\mathrm{d}e^{\bar\imath} = -f^{\bar\imath}_{\hat\jmath\bar k}\, e^{\hat\jmath}\wedge e^{\bar k} - \tfrac12 f^{\bar\imath}_{\bar\jmath\bar k}\, e^{\bar\jmath}\wedge e^{\bar k} \qquad\text{and}\qquad \mathrm{d}e^{\hat\imath} = -\tfrac12 f^{\hat\imath}_{\hat\jmath\hat k}\, e^{\hat\jmath}\wedge e^{\hat k} - \tfrac12 f^{\hat\imath}_{\bar\jmath\bar k}\, e^{\bar\jmath}\wedge e^{\bar k}\ . \tag{4.6}$$

The curvature of the canonical connection (4.5) follows as

$$\mathcal{F}_{G/H} \;=\; -\tfrac12 f^{\hat\imath}_{\bar\jmath\bar k}\, I_{\hat\imath}\, e^{\bar\jmath}\wedge e^{\bar k} \;=\; -\tfrac12 f^{\hat\imath}_{\bar\jmath\bar k}\, I_{\hat\imath}\, e^{\bar\jmath}_\alpha e^{\bar k}_\beta\, \mathrm{d}X^\alpha\wedge \mathrm{d}X^\beta\ . \tag{4.7}$$

Variation of $(\mathcal{A}_z,\varphi)$ Recall that the solution space $\mathcal{G}^1_{\mathcal{I}}$ to (3.9) is a group, and the moduli space $\mathcal{M}_{\mathcal{I}}=\mathcal{G}^1_{\mathcal{I}}/\mathcal{G}^\tau_{\mathcal{I}}$ is labelled locally by coset coordinates $X=\{X^\alpha\}$. Let us pick a coset representative $m(X)\in\mathcal{G}^1_{\mathcal{I}}$, which is a section of the bundle (3.17) over a point $X\in G/H$. Multiplication from the left by a group element $h\in\mathcal{G}^1_{\mathcal{I}}$ will generally carry $m(X)$ into a section $m(X')$ over another point X', so that

$$h\,m(X) = m(X')\,f \qquad\text{with}\qquad f\in\mathcal{G}^\tau_{\mathcal{I}}\ . \tag{4.8}$$

This yields formulae for the infinitesimal changes of $\mathcal{A}_z$ and φ, which live in $\mathrm{Lie}\,\mathcal{G}^1_{\mathcal{I}}=\mathfrak{m}\oplus\mathrm{Lie}\,\mathcal{G}^\tau_{\mathcal{I}}$,

$$\partial_\alpha\mathcal{A}_z \;=\; \delta_\alpha\mathcal{A}_z + \delta_{\epsilon_\alpha} A_z \;=\; \delta_\alpha\mathcal{A}_z + D_z\epsilon_\alpha \qquad\text{and}\qquad \partial_\alpha\varphi \;=\; \delta_\alpha\varphi + \delta_{\epsilon_\alpha}\varphi \;=\; \delta_\alpha\varphi + [\varphi,\epsilon_\alpha]\ , \tag{4.9}$$

where $\partial_\alpha=\partial/\partial X^\alpha$. The pair $(\delta_\alpha\mathcal{A}_z,\delta_\alpha\varphi)$ belongs to the tangent space $T_{(\mathcal{A}_z,\varphi)}\mathcal{M}_{\mathcal{I}}\cong\mathfrak{m}$, and ϵ_α are $\mathfrak{g}$-valued gauge parameters generating the infinitesimal gauge transformation $(\delta_{\epsilon_\alpha}A_z,\delta_{\epsilon_\alpha}\varphi)$ which represents the gauge part of the variation and sits in $\mathrm{Lie}\,\mathcal{G}^\tau_{\mathcal{I}}$. The orthogonality of $(\delta_\alpha\mathcal{A}_z,\delta_\alpha\varphi)$ and $(\delta_{\epsilon_\alpha}A_z,\delta_{\epsilon_\alpha}\varphi)$ is achieved by imposing the condition (4.2) for any $\alpha=1,\dots,\dim G/H$.

5. Skyrme–Faddeev model in the infrared limit of 5d YMH

Coset space sigma model We return to the YMH model (2.5) on $\mathbb{R}^{3,1}\times\mathbb{R}_+$ and non-vacuum fields $(\mathcal{A}_a,\mathcal{A}_z,\varphi)$. The adiabatic approach considers the collective coordinates $X=\{X^\alpha\}$ as dynamical fields, $X^\alpha=X^\alpha(x)$, where $x=\{x^a\}$. Their low-energy effective action is derived by expanding

$$\mathcal{A}_\mu \;=\; \mathcal{A}_\mu\big(X(x),z\big)+\dots \qquad\text{and}\qquad \varphi \;=\; \varphi\big(X(x),z\big)+\dots \tag{5.1}$$

and keeping only the first terms in the YMH action (2.5) [24–26,28,20]. Thereby one obtains an effective field theory which will be a non-linear sigma model describing maps $X:\mathbb{R}^{3,1}\to G/H$.

With the map X we pull back the adiabatic fields

$$\mathcal{A}_z \;=\; \mathcal{A}_z\big(X(x),z\big) \;=:\; \mathcal{A}_z(x,z) \qquad\text{and}\qquad \varphi \;=\; \varphi\big(X(x),z\big) \;=:\; \varphi(x,z) \tag{5.2}$$

by a slight abuse of notation from G/H to $\mathbb{R}^{3,1}$. Thus, we have to include a dependence on x^a in the formulae of Sections 3 and 4. In particular, multiplying (4.9) by $\partial_a X^\alpha$, we obtain

$$\partial_a\mathcal{A}_z \;=\; (\partial_a X^\alpha)\delta_\alpha\mathcal{A}_z + D_z\epsilon_a \qquad\text{and}\qquad \partial_a\varphi \;=\; (\partial_a X^\alpha)\delta_\alpha\varphi + [\varphi,\epsilon_a]\ , \tag{5.3}$$

where $\epsilon_a=(\partial_a X^\alpha)\,\epsilon_\alpha$ is the pull-back of ϵ_α to $\mathbb{R}^{3,1}$. From (5.3) it follows that

$$\mathcal{F}_{az} \;=\; \partial_a\mathcal{A}_z - D_z\mathcal{A}_a \;=\; (\partial_a X^\alpha)\delta_\alpha\mathcal{A}_z - D_z(\mathcal{A}_a{-}\epsilon_a)\ , \tag{5.4}$$

$$D_a\varphi \;=\; \partial_a\varphi + [\mathcal{A}_a,\varphi] \;=\; (\partial_a X^\alpha)\delta_\alpha\varphi - [\varphi,\mathcal{A}_a{-}\epsilon_a]\ . \tag{5.5}$$

In the moduli-space approximation, $\mathcal{F}_{a4}$ and $D_a\varphi$ are tangent to $\mathcal{M}_{\mathcal{I}}$ (see e.g. [24–26]). This can be achieved by putting

$$\mathcal{A}_a \;=\; \epsilon_a\big(X(x),z\big)\ . \tag{5.6}$$

Then, substituting (5.4) and (5.5) into the action (2.5) and remembering (3.1), we arrive at

$$S_{\mathrm{kin}} = -\frac{1}{4e^2\varepsilon}\int_{\mathbb{R}^{3,1}\times\mathbb{R}_+} \mathrm{d}^5x\ \eta^{ab}\ \mathrm{tr}\,(\mathcal{F}_{a4}\mathcal{F}_{b4}+D_a\phi D_b\phi) = \frac{1}{2e^2\varepsilon L}\int_{\mathbb{R}^{3,1}} \mathrm{d}^4x\ \eta^{ab}\ g_{\alpha\beta}\,\partial_a X^\alpha\partial_b X^\beta\ , \tag{5.7}$$

where

$$g_{\alpha\beta} \;=\; -\tfrac12\int_0^\infty \mathrm{d}z\ \mathrm{tr}\,\big(\delta_\alpha\mathcal{A}_z\delta_\beta\mathcal{A}_z+\delta_\alpha\varphi\delta_\beta\varphi\big) \;=\; \delta_{\bar\imath\bar\jmath}\, e^{\bar\imath}_\alpha e^{\bar\jmath}_\beta \tag{5.8}$$

are the components of the metric (4.4) on G/H pulled-back to $\mathbb{R}^{3,1}$. Thus, this part of the action (2.5) reduces to the standard non-linear sigma model on $\mathbb{R}^{3,1}$ with the coset G/H as target space.

Skyrme–Faddeev term The last term in the action (2.5) vanishes since $D_4\phi=0$ for any $x^a\in\mathbb{R}^{3,1}$ due to second equation in (3.9). It remains to evaluate the first term in the action (2.5). For this we notice that $\mathcal{A}_a=\epsilon_a=(\partial_a X^\alpha)\epsilon_\alpha$ depend on $X^\alpha(x)$ and z and that

$$\epsilon_a(z{=}0)=0 \qquad\text{and}\qquad \epsilon_a(z{=}\infty)\in\mathfrak{h}\ . \tag{5.9}$$

The asymptotics (5.9) at $z\to\infty$ does not agree with the asymptotic conditions (3.3) for the components $\mathcal{A}_a$. The reason is that, when we turn from YMH theory on $\mathbb{R}_+$ to YMH theory on $\mathbb{R}^{3,1}\times\mathbb{R}_+$, the group of gauge transformations are reduced from $\mathcal{G}^\tau$ to $\mathcal{G}^0$. To preserve (3.3) we switch from $\mathcal{A}_a$ to $\widehat{\mathcal{A}}_a$ via

$$\mathcal{A}_a \;=\; \epsilon_a \;=\; f^{-1}\,\widehat{\mathcal{A}}_a\,f + f^{-1}\,\partial_a f \qquad\text{with some}\quad f\in\mathcal{G}^\tau\ . \tag{5.10}$$

The conditions (5.9) for $\mathcal{A}_a$ translate to

$$\widehat{\mathcal{A}}_a(z{=}0)=0 \qquad\text{and}\qquad \widehat{\mathcal{A}}_a(z{=}\infty)=0 \tag{5.11}$$

since $f(z{=}0)=0$ and $f(z{=}\infty)\in H$ and $\mathcal{A}_a(z{=}\infty)=f^{-1}\,\partial_a f\in\mathfrak{h}$.

Recall that

$$\widehat{\mathcal{A}}_a = (\partial_a X^\alpha)\,\hat\epsilon_\alpha \qquad \text{where} \qquad \epsilon_\alpha = f^{-1}\,\hat\epsilon_\alpha\, f + f^{-1}\,\partial_\alpha f\,. \tag{5.12}$$

This $\hat\epsilon_\alpha\,\mathrm{d}X^\alpha$ is a one-form on the base G/H of the fibration with value in the Lie algebra Lie $\mathcal{G}^0$. One can always decompose $\hat\epsilon_\alpha$ as

$$\hat\epsilon_\alpha = \zeta(z)\,e^{\hat\imath}_\alpha\, I_{\hat\imath} + \epsilon^0_\alpha\,, \tag{5.13}$$

where $A_\alpha = e^{\hat\imath}_\alpha I_{\hat\imath}$ are the components of the unique G-equivariant connection (4.5) in the bundle (4.3), and $\zeta(z)$ is a real-valued function on $\mathbb{R}_+$ such that $\zeta(0)=0=\zeta(\infty)$. One can view (5.13) as a definition of ϵ^0_α. Then for $\widehat{\mathcal{A}}_a$ we have

$$\widehat{\mathcal{A}}_a = \zeta(z)(\partial_a X^\alpha)e^{\hat\imath}_\alpha I_{\hat\imath} + (\partial_a X^\alpha)\epsilon^0_\alpha = \zeta(z)A_a + \epsilon^0_a\,. \tag{5.14}$$

We remark that A_a is a composite field since the canonical connection (4.5) has a fixed dependence on the coordinates X^α on G/H, which is known explicitly if one chooses $G/H = \mathrm{SU}(n{+}1)/\mathrm{U}(n)$, $\mathrm{SU}(n{+}1)/\mathrm{U}(1)^n$ or similar. On the other hand, ϵ^0_a is not composite.

The curvature of $\widehat{\mathcal{A}}$ computes to

$$f\mathcal{F}f^{-1} = \widehat{\mathcal{F}} = \mathrm{d}\widehat{\mathcal{A}} + \widehat{\mathcal{A}}\wedge\widehat{\mathcal{A}} = F + \Sigma\,, \tag{5.15}$$

where

$$F = \zeta\,\mathrm{d}A + \zeta^2 A\wedge A = \tfrac12\,F_{ab}\,\mathrm{d}x^a\wedge\mathrm{d}x^b \qquad \text{and} \tag{5.16}$$

$$\Sigma = \mathrm{d}\epsilon^0 + \epsilon^0\wedge\epsilon^0 + \zeta(A\wedge\epsilon^0 + \epsilon^0\wedge A) = \tfrac12\,\Sigma_{ab}\,\mathrm{d}x^a\wedge\mathrm{d}x^b\,. \tag{5.17}$$

We will see in a moment that the term F in (5.15) yields a Skyrme–Faddeev type term for a generic coset space G/H. On the other hand, the curvature Σ in (5.15) describes $\mathfrak{g}$-valued one-forms with non-vanishing mass terms from the coupling to the composite field A in (5.17). A consideration of these fields is beyond the scope of our paper, which contents itself with identifying the Skyrme–Faddeev model as part of the low-energy limit of 5d SYM on $\mathbb{R}^{3,1}\times\mathbb{R}_+$. The discarded term Σ will yield corrections analogous to the tower of meson fields in the extended Skyrme model (see e.g. [16,19]).

For the components F_{ab} from (5.16) and (4.5)–(4.6) we obtain

$$F_{ab} = \big(\zeta(\zeta-1)f^{\hat\imath}_{\hat\jmath\hat k}\,e^{\hat\jmath}_\alpha e^{\hat k}_\beta - \zeta\, f^{\hat\imath}_{\bar\jmath\bar k}\,e^{\bar\jmath}_\alpha e^{\bar k}_\beta\big)\, I_{\hat\imath}\;\partial_a X^\alpha\,\partial_b X^\beta\,. \tag{5.18}$$

Substituting (5.15) into the action and discarding all Σ_{ab} terms, the first term in (2.5) produces

$$\begin{aligned} S_{\mathrm{SF}} &= -\frac{\varepsilon}{8e^2}\int\limits_{\mathbb{R}^{3,1}\times\mathbb{R}_+} \mathrm{d}^5x\;\mathrm{tr}\mathcal{F}_{ab}\mathcal{F}^{ab} \\ &= \frac{\varepsilon L}{4e^2}\int\limits_{\mathbb{R}^{3,1}} \mathrm{d}^4x\;\eta^{ac}\eta^{bd}\,\partial_a X^\alpha\,\partial_b X^\beta\,\partial_c X^\gamma\,\partial_d X^\delta \\ &\quad\times\Big\{a_1 f^{\hat\imath}_{\hat l\hat k} f^{\hat\jmath}_{\hat m\hat n}\, e^{\hat l}_\alpha e^{\hat k}_\beta e^{\hat m}_\gamma e^{\hat n}_\delta + a_2 f^{\hat\imath}_{\hat l\hat k} f^{\hat\jmath}_{\bar m\bar n}\, e^{\hat l}_\alpha e^{\hat k}_\beta e^{\bar m}_\gamma e^{\bar n}_\delta \\ &\quad + a_3 f^{\hat\imath}_{\bar l\bar k} f^{\hat\jmath}_{\bar m\bar n}\, e^{\bar l}_\alpha e^{\bar k}_\beta e^{\bar m}_\gamma e^{\bar n}_\delta\Big\}\delta_{\hat\imath\hat\jmath}\,, \end{aligned} \tag{5.19}$$

with numerical coefficients

$$a_1 = \int\limits_0^\infty \mathrm{d}z\;\zeta^2(\zeta-1)^2\,, \qquad a_2 = \int\limits_0^\infty \mathrm{d}z\;\zeta^2(\zeta-1) \qquad \text{and}$$

$$a_3 = \int\limits_0^\infty \mathrm{d}z\;\zeta^2 \tag{5.20}$$

The integrals (5.20) are finite for a suitably chosen function $\zeta(z)$ such as $\zeta(z) = \exp(-z)(1-\exp(-z))$. The expression (5.19) for the Skyrme–Fadeev-type term holds true for generic cosets G/H. It considerably simplifies when $H=T$ is the Cartan torus in G, because then $f^{\hat\imath}_{\hat\jmath\hat k}=0$, and one has only the a_3 term in (5.19). For $G/T = \mathrm{SU}(2)/\mathrm{U}(1)$, this term coincides with the standard Skyrme–Faddeev term of the $\mathbb{C}P^1$ sigma model.

To summarize, in the infrared limit the Yang–Mills–Higgs action (2.5) on $\mathbb{R}^{3,1}\times\mathbb{R}_+$ is reduced to the effective action of the Skyrme–Faddeev model

$$S_{\mathrm{eff}} = S_{\mathrm{kin}} + S_{\mathrm{SF}}\,, \tag{5.21}$$

where S_{kin} and S_{SF} are given by (5.7), (5.8) and (5.19).

Note added after review

By similar methods, the authors recently obtained the *standard* 4d Faddeev and Skyrme models in an infrared limit of 4d Yang–Mills–Higgs theory. Breaking the gauge group G to a subgroup H results in a Higgs vacuum manifold G/H, which coincides with the Faddeev sigma-model target. The coset may be chosen to be a group manifold, e.g. $G/H \simeq \mathrm{U}(N)$, in which case the standard $\mathrm{U}(N)$ Skyrme model emerges [39].

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