Existence and qualitative aspects of geophysical water waves

Von der Fakultät für Mathematik und Physik
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Zusammenfassung

Das Ziel der geophysikalischen Strömungsdynamik is das Studium der natürlich vorkommenden Ströme auf der Erde. Die meisten Probleme, die entsthen sind am oberen Ende der Skala wo entweder die Rotation der Erde oder Dichteunterschiede (warme und kalte Luftmassen, frische und Salzwasser) oder beide von Bedeutung sind. In dieser Hinsicht umfasst die geophysikalischen Strömungsdynamik rotierende-stratifizierte Flüssigkeitsströmungen.

Die vorligiende Dissertation setzt sich aus elf Forschungsarbeiten zusammen, die sich wiederum in zwei Teilen aufteilen lasen. Der erste Teil der Dissertation widmet sich dem Studium von nicht-linearen Kapillar-Gravitationswellen, die an der Oberfläche der Rotationsströmungen, mit rauen Verwirbellungen, oder an der oberfläche von Strömungen, mit einer vertikalen Schlichtung (stetig bzw. unstetig), wandern. Der Schwerpunkt liegt auf der Entwicklung einer exakte Existenztheorie für Lösungen.

Die im zweiten Teil der Disertation gesammelten Arbeiten untersuchen die Existenz und die qualitative Eigenschaften der äquatorialen geophysikalischen Wasserströmungen mit Coriolis Effekte in der sogenannten f-Ebene Approximation. Außerdem werden wir mehrere explizite Lösungen angeben, die äquatoriale Wasserströmungen in der f-Ebene Aproximation in verschiedene geophysikalischen Szenarien - Tiefwasserwellen, Randwellen, geschichtete Strömungen - beschreiben, und deren Eigenschaften analysieren.

Abstract

The object of geophysical fluid dynamics is the study of naturally occurring flows on the Earth. Most of the problems that arise are at the large-scale end, where either the rotation of the Earth or density differences (warm and cold air masses, fresh and saline waters) or both are of importance. In this regard, geophysical fluid dynamics comprises rotating-stratified fluid flows.

This thesis consists of eleven research papers which are grouped into two parts. The first part is dedicated to the study of nonlinear capillary-gravity water waves traveling at the surface of rotational flows with rough vorticities or of flows with a vertical layering (continuous resp. discontinuous) of density, the emphasis being on developing a rigorous existence theory.

The papers collected in the second part of the thesis investigate the existence and the qualitative properties of equatorial geophysical water flows with Coriolis effects in the so-called f-plane approximation. In this part we also present several explicit solutions describing equatorial water flows in the f-plane approximation in different geophysical scenarios −deep-water waves, edge waves, stratified flows− and we analyze their properties.

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Publications:

I. **Geophysical water flows without Coriolis effects**

1. D. Henry and A.–V. Matioc. Global bifurcation of capillary-gravity-stratified water waves, *Proc. Roy. Soc. Edinburgh Sect. A* **144** no. 4 (2014), 775–786.

2. A.–V. Matioc and B.–V. Matioc. Capillary-gravity water waves with discontinuous vorticity: existence and regularity results, *Commun. Math. Phys* **330** (2014), 859–886.

3. A.–V. Matioc. Steady internal water waves with a critical layer bounded by the wave surface, *J. Nonlinear Math. Phys.* **19** (1) (2012), 98–118.

II. **Geophysical water flows in the f-plane approximation**

4. A.–V. Matioc. An explicit solution for deep water waves with Coriolis effects, *J. Nonlinear Math. Phys.* **19** (1) (2012),43–50.

5. A.–V. Matioc. On the particle motion in geophysical deep water waves traveling over uniform currents, *Quart. Appl. Math.* **LXXII** no. 3 (2014), 455–469.

6. A.–V. Matioc. An exact solution for geophysical equatorial edge waves over a sloping beach, *J. Phys. A: Math. Theor.* **45** (2012) 365501 (10pp).

7. A.–V. Matioc. Exact geophysical waves in stratified fluids, *Applicable Analysis* **92** (11) (2013), 2254–2261.

8. A.–V. Matioc and B.–V. Matioc. On periodic water waves with Coriolis effects and isobaric streamlines, *J. Nonlinear Math. Phys.* **19** Suppl. 1 (2012),89–103.

9. D. Henry and A.–V. Matioc. On the existence of equatorial wind waves, *Nonlinear Anal.* **101** (2014), 113–123.

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11. D. Ionescu–Kruse and A.–V. Matioc. Small-amplitude equatorial water waves with constant vorticity: dispersion relations and particle trajectories, *Discrete Contin. Dyn. Syst.* **34** (8) (2014), 3045–3060.

Introduction

Geophysical fluid dynamics refers to all naturally occurring fluid motions. Two important features that are common to many of the phenomena studied in this field are the rotation of the fluid due to the Earth's rotation and the stratification. This thesis consists of eleven research papers which are grouped into two parts. The first part is dedicated to the study of nonlinear capillary-gravity water waves traveling at the surface of flows with rough vorticity or of flows with a vertical layering of density. The papers collected in the second part of the thesis investigate the existence and the qualitative properties of geophysical water flows with Coriolis effects in the so-called f-plane approximation. The waves that we consider are exact, some of them also explicit, solutions to the Euler equations. The derivation of the Euler equations in a reference frame which rotates with the Earth – and where the Coriolis and the centrifugal force naturally appear – is presented, for the sake of completeness, at the beginning of the second part of the thesis. Though the Coriolis force is quite small, for large-scale movement of water in the ocean its effects are noticeable while the effects due to the centrifugal force can be neglected.

The mathematical model considered in the first part of the thesis describes the propagation of periodic water waves over a rotational, inviscid and incompressible fluid, under the influence of gravity and capillary forces. Moreover, the water waves we are dealing with in Paper 1 are stratified, meaning that the fluid density varies with the height. Physically, the density of the fluid may vary due to several factors, such as the salinity, temperature, pressure, oxygenation (see the discussion in the references [19, 29] of Paper 1). Also the surface tension plays a key role for small- to medium-amplitude water waves, and in particular for wind waves. Using global bifurcation techniques we constructed in Paper 1 a global continuum of steady periodic stratified water waves, which is either unbounded or contains a wave of largest admissible amplitude, and which extends the local bifurcation curves found in the reference [19] of Paper 1. Furthermore, we obtained a description of the behavior of the stratified water waves solutions along the global continuum.

The study in Paper 2 was motivated on one hand by the physical setting of wind generated waves, which possess a thin layer of high vorticity (see the references [37, 39] of Paper 2) adjacent to wave surface. On the other hand, as a combined effect of the gravitational forces exerted by the Moon, Sun, and the rotation of the Earth, in the near-bed region of oceans there may exist strong tidal currents which are responsible for the transportation of sediments (see reference [38] of Paper 2). We established in this paper the existence of two-dimensional periodic capillarity-gravity water waves with an arbitrary bounded vorticity distribution. This was achieved by presenting a novel weak interpretation to the height function formulation of the water waves problem. This new formulation enabled us to establish the existence of weak solutions to the problem. Moreover, we proved that these solutions are in fact strong solutions to the problem, describing waves with a real-analytic free surface. Assuming merely integrability of the vorticity function, we showed that any weak solution corresponds to a flow having real-analytic streamlines.

In Paper 3 we considered two-dimensional internal periodic water waves traveling at the interface between two fluid layers with different, but constant, densities, under the rigid lid assumption. In this context, the fluids have constant vorticity and we constructed internal traveling waves with a critical layer and stagnation points in both gravity and capillary-gravity regimes. Besides, we proved, without excluding the presence of stagnation points, that if the vorticity function associated to each fluid in part is real analytic, bounded, and non-increasing, then capillary-gravity steady internal waves are a priori real-analytic. In particular, irrotational capillary-gravity water waves possess a real-analytic surface even if stagnation points are present, in contrast to the case of gravity waves with a stagnation point at their crest where the free surface is only Lipschitz continuous but not C^1 (see the reference [28] of Paper 5).

In the second part of the thesis we considered water flows influenced by the Coriolis force. We present firs the equation of motion in a coordinate frame rotating with uniform angular velocity we motivate that flows close to the Equator can be described by the so-called f-plane approximation of the geophysical model. This approximation will be used in all the remaining papers of this thesis.

1. The geophysical water wave problem and its f-plane approximation

The motion of a fluid layer located on the Earth's surface is also influenced by Earth's rotation around the polar axis. From a theoretical point of view, the equations governing geophysical fluid processes can be derived with respect to inertial reference frame (that are fixed with respect to distant stars). However, the most natural reference system to describe geophysical fluid motions is one which rotates with the Earth. The phenomena themselves are not affected by the choice of the frame, but the description of the phenomena depends on the chosen frame. For an observer in a rotating frame the objects fixed in the inertial frame will appear to rotate and to accelerate (due to the curvature of their apparent trajectory).

We now derive the equations of motion for a inviscid fluid located on the surface of the Earth entirely in terms of quantities directly observed from a rotating frame (as presented in [2, 3, 4]). The influence of the rotation of the attraction exerted by the Moon and Sun on the motion of the fluid are neglected so that we may fix an (orthogonal) inertial frame $OXYZ$ with O the center of mass of the Earth and the Z axis pointing towards the North Pole. The OXY plane coincides with the Earth's equatorial plane, see Figure 1. The Earth rotates with constant angular velocity

$$\omega = 73 \cdot 10^{-6} \mathrm{rad}/s,$$

and we let $\mathbf{\Omega} := (0, 0, \omega)$ denote the rotation vector of Earth round the polar axis toward east

The equations of motions of an inviscid fluid in this inertial reference frame are derived in most of the elementary fluid-dynamics books and consist

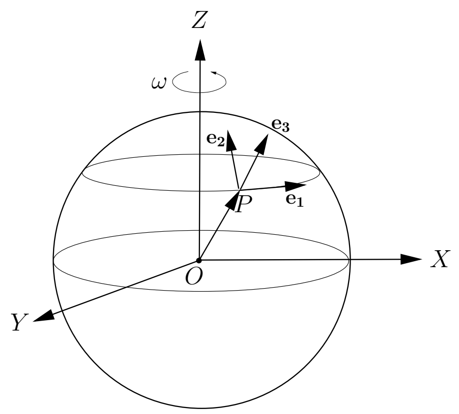

FIGURE 1

of the continuity equation

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{1}$$

and the conservation of momentum equation

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \rho\nabla\Phi \tag{2}$$

where $\rho\nabla\Phi$ is the body force (Φ is a potential by which conservative body forces, such as gravity, can be represented), and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$$

is the total derivative with respect to time in the inertial reference frame. We have denoted by ρ the density of the fluid, p is the pressure, and $\mathbf{u}$ is the velocity field.

We now choose, following [[4], page 15] a (orthogonal) reference frame $\{P, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ with P located on the Earth's surface which rotates with the Earth. We want to express the equations of motion (1)-(2) in this moving reference frame. To this end, we consider first a vector $\mathbf{A}$ having constant magnitude but rotating with angular velocity ω and denote by α the vector between $\mathbf{A}$ and $\mathbf{\Omega}$. In a small time Δt, $\mathbf{A}$ is rotated through the angle $\Delta\theta = \omega\Delta t$, see Figure 2.

It then follows that the small change in $\mathbf{A}$ is given by

$$\mathbf{A}(t+\Delta t) - \mathbf{A}(t) \equiv \Delta\mathbf{A} = \mathbf{n}\Delta\theta|\mathbf{A}|\sin\alpha + O((\Delta\theta)^2),$$

where

$$\mathbf{n} = \frac{\mathbf{\Omega}\times\mathbf{A}}{|\mathbf{\Omega}\times\mathbf{A}|}.$$

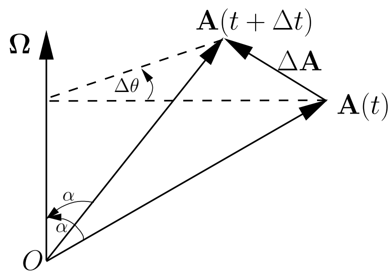

Figure 2

Letting $\Delta t \to 0$, we obtain

$$\lim_{\Delta t\to 0}\frac{\Delta \mathbf{A}}{\Delta t} = \frac{d\mathbf{A}}{dt} = |\mathbf{A}|\omega \sin\alpha \frac{\mathbf{\Omega}\times\mathbf{A}}{|\mathbf{\Omega}\times\mathbf{A}|},$$

and since $|\mathbf{\Omega}\times\mathbf{A}| = |\mathbf{A}|\omega\sin\alpha$ we get, for a vector of fixed magnitude, that

$$\frac{d\mathbf{A}}{dt} = \mathbf{\Omega}\times\mathbf{A}. \tag{3}$$

Both observers will see the same vector $\mathbf{A}$ because the definition of the vector $\mathbf{A}$ does not depend on the coordinate frame. However, an observer who is fixed in the rotating reference frame would see no change in $\mathbf{A}$, while an observer in a non-rotating frame would see the change in $\mathbf{A}$ as described by (3), that is

$$\left(\frac{d\mathbf{A}}{dt}\right)_I = 0 \qquad \text{and} \qquad \left(\frac{d\mathbf{A}}{dt}\right)_R = \mathbf{\Omega}\times\mathbf{A}.$$

The subscript I states for inertial frame and R for the rotating frame. We note that the rate of change of $\mathbf{\Omega}$ is the same in both frames because $\mathbf{\Omega}\times\mathbf{\Omega}$ vanishes identically.

We consider now an arbitrary vector $\mathbf{B}$. Following [[4], page 16], in the rotating frame the vector $\mathbf{B}$ can be written as

$$\mathbf{B} = B_1\mathbf{e_1} + B_2\mathbf{e_2} + B_3\mathbf{e_3},$$

where $B_j = \mathbf{B}\cdot\mathbf{e_j}$, $j=1,2,3$. Then, the rate of change of $\mathbf{B}$ in the rotating frame is given by

$$\left(\frac{d\mathbf{B}}{dt}\right)_R = \frac{dB_1}{dt}\mathbf{e_1} + \frac{dB_2}{dt}\mathbf{e_2} + \frac{dB_3}{dt}\mathbf{e_3},$$

since the unit vectors are fixed in length and direction. In the non-rotating reference frame, the components of $\mathbf{B}$ and the unit vectors are all changing with time, therewith the rate of change of $\mathbf{B}$ is

$$\begin{aligned}\left(\frac{d\mathbf{B}}{dt}\right)_I &= \frac{dB_1}{dt}\mathbf{e_1} + \frac{dB_2}{dt}\mathbf{e_2} + \frac{dB_3}{dt}\mathbf{e_3} + B_1\frac{d\mathbf{e_1}}{dt} + B_2\frac{d\mathbf{e_2}}{dt} + B_3\frac{d\mathbf{e_3}}{dt} \\ &= \left(\frac{d\mathbf{B}}{dt}\right)_R + B_1\mathbf{\Omega}\times\mathbf{e_1} + B_2\mathbf{\Omega}\times\mathbf{e_2} + B_3\mathbf{\Omega}\times\mathbf{e_3} \\ &= \left(\frac{d\mathbf{B}}{dt}\right)_R + \mathbf{\Omega}\times(B_1\mathbf{e_1} + B_2\mathbf{e_2} + B_3\mathbf{e_3}) \\ &= \left(\frac{d\mathbf{B}}{dt}\right)_R + \mathbf{\Omega}\times\mathbf{B},\end{aligned}$$

by (3). We have thus shown that

$$\left(\frac{d\mathbf{B}}{dt}\right)_I = \left(\frac{d\mathbf{B}}{dt}\right)_R + \mathbf{\Omega}\times\mathbf{B}. \tag{4}$$

As before, the rates of change with time of the same vector $\mathbf{B}$ are perceived differently by the observers in the rotating and non-rotating frame, respectively.

The conservation of momentum equation in the rotating frame. Let $\mathbf{r}$ be the position vector of an arbitrary fluid element. Then, from (4) we have

$$\left(\frac{d\mathbf{r}}{dt}\right)_I = \left(\frac{d\mathbf{r}}{dt}\right)_R + \mathbf{\Omega}\times\mathbf{r},$$

so that the velocity $\mathbf{u_I}$ seen in the non-rotating reference frame is equal to the velocity $\mathbf{u_R}$ observed in the rotating frame plus the velocity imparted to the fluid element by the Earth rotation $\mathbf{\Omega}\times\mathbf{r}$:

$$\mathbf{u_I} = \mathbf{u_R} + \mathbf{\Omega}\times\mathbf{r}. \tag{5}$$

Using (4) and (5) and the observation that the rate of change of $\mathbf{\Omega}$ is 0, we obtain that

$$\begin{aligned}\left(\frac{d\mathbf{u}_I}{dt}\right)_I &= \left(\frac{d\mathbf{u}_I}{dt}\right)_R + \mathbf{\Omega}\times\mathbf{u_I} \\ &= \left(\frac{d\mathbf{u_R}}{dt}\right)_R + \mathbf{\Omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_R + \mathbf{\Omega}\times(\mathbf{u_R} + \mathbf{\Omega}\times\mathbf{r}) \\ &= \left(\frac{d\mathbf{u_R}}{dt}\right)_R + 2\mathbf{\Omega}\times\mathbf{u_R} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}).\end{aligned}$$

Together with (2) we obtain that the conservation of momentum equation in the rotating frame is

$$\rho\Big(\frac{d\mathbf{u}}{dt} + 2\mathbf{\Omega}\times\mathbf{u}\Big) = -\nabla p + \rho\nabla\Phi - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}).$$

Compared to the momentum equation in the inertial frame there are two new terms that appear:

- the **Coriolis acceleration**[1] $-2\mathbf{\Omega}\times\mathbf{u}$ which always perpendicular to the velocity;
- the **centrifugal acceleration**[2]$-\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})=\omega^2\mathbf{r}_\perp$, whereby $\mathbf{r}_\perp$ denotes the perpendicular distance vector from the rotation axis to the position of the fluid particle at $\mathbf{r}$, cf. Figure 3. The centrifugal force depends only on the rotation rate and the distance of the particle to the rotation axis. Even at rest with respect to the rotating planet, particles experience an outward pull. In the absence of rotation, gravitational forces keep the matter together to form a spherical body. The outward pull caused by the centrifugal force distorts this spherical equilibrium and the planet assumes a slightly flattened shape. On the Earth the distortion is very slight because gravity by far exceeds the centrifugal force: the equatorial radius is 6378 km, slightly greater than the polar radius of 6357 km.

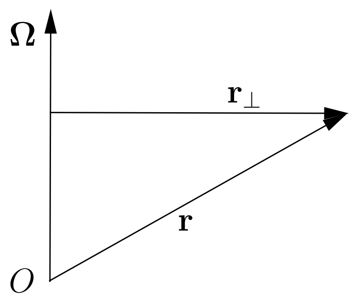

FIGURE 3

The f-plane approximation of the geophysical water wave problem. We now assume that the Earth is a perfect sphere of radius $R=6371$ km. Therefore we can choose spherical coordinates to parametrize the Earth's surface: $\phi\in[-\pi/2,\pi/2]$ is the latitude and $\theta\in[-\pi,\pi]$ is the longitude. Moreover, we choose the rotating reference frame such that P is located at latitude ϕ, $\mathbf{e_3}$ is the vector $OP/|OP|$, the vector $\mathbf{e_1}$ points horizontally due east and $\mathbf{e_2}$ horizontally due north. Letting (x,y,z) denote the coordinates in the frame $\{P,\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}\}$ and writing $\mathbf{u}=(u,v,w)$, we have that

$$\begin{aligned}\mathbf{\Omega} =&\omega\cos\phi\,\mathbf{e_2}+\omega\sin\phi\,\mathbf{e_3}\\ 2\mathbf{\Omega}\times\mathbf{u} =&2\omega(w\cos\phi-v\sin\phi)\,\mathbf{e_1}+2\omega u\sin\phi\,\mathbf{e_2}-2\omega u\cos\phi\,\mathbf{e_3},\\ \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) =&-\omega^2x\,\mathbf{e_1}+\omega^2\sin\phi((z+R)\cos\phi-y\sin\phi)\,\mathbf{e_2}\\ &-\omega^2\cos\phi((z+R)\cos\phi-y\sin\phi)\,\mathbf{e_3},\end{aligned}$$

as $\mathbf{r}=(x,y,z+R)$.

[1]"Gaspard Gustave Coriolis (1792-1843) was born in France and trained as an engineer. He began a career in teaching and research at age 24. Fascinated by the problems related to rotating machinery, he was led to derive the equations of motions in a rotating framework of references. The result of these studies was presented to the Académie des Sciences in the summer of 1831. In 1838, Coriolis stopped teaching to become director of studies at the Ecole Polytechnique, but his health declined quickly and he died a few short years later". (cited from reference [2].)

[2]Here we follow [2].

For equatorial motions, that is for $\phi = 0$, an eastward motion (that is, in the same direction as the rotation of the sphere) provides an upward acceleration known as the Eötvös effect as $-2\mathbf{\Omega} \times \mathbf{u} = 2\omega u\, \mathbf{e_3}$. Moreover, a particle at latitude ϕ in the Northern Hemisphere and moving eastwards will be pulled to the right (southwards) due by the Coriolis force as

$$-2\mathbf{\Omega} \times \mathbf{u} = -2\omega u \sin\phi\, \mathbf{e_2} + 2\omega u \cos\phi\, \mathbf{e_3}.$$

To write down the momentum equation in the f-plane approximation, we still have to express the gravity force in the rotating reference frame. A fluid particle positioned at (x, y, z) with respect to $\{P, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is attracted by the center of mass of the Earth, its gravity acceleration being defined as

$$\mathbf{g} = -\frac{Gm_E\mathbf{r}}{|\mathbf{r}|^3} = -gR^2\frac{Gm_E\mathbf{r}}{|\mathbf{r}|^3}$$

where G is the universal gravitational constant, m_E is the mass of the Earth, and $g = 9.81m/s^2$ is the gravity of Earth. Summarizing, the conservation of momentum equation reads as

$$\begin{cases} u_t + uu_x + vu_y + wu_z + 2\omega(w\cos\phi - v\sin\phi) = -\dfrac{p_x}{\rho} + \omega^2 x - \dfrac{gR^2x}{|\mathbf{r}|^3}, \\ v_t + uv_x + vv_y + wv_z + 2\omega u\sin\phi = -\dfrac{p_y}{\rho} - \omega^2\sin\phi((z+R)\cos\phi - y\sin\phi) \\ \qquad\qquad -\dfrac{gR^2y}{|\mathbf{r}|^3}, \\ w_t + uw_x + vw_y + ww_z - 2\omega u\cos\phi = -\dfrac{p_z}{\rho} - \dfrac{gR^2(z+R)}{|\mathbf{r}|^3} \\ \qquad\qquad +\omega^2\cos\phi((z+R)\cos\phi - y\sin\phi). \end{cases}$$

The terms

$$f := 2\omega\sin\phi \qquad \text{and} \qquad f_* := 2\omega\cos\phi$$

are called **Coriolis parameter** and **reciprocal Coriolis parameter**, respectively.

Let L be a characteristic length scale of the motion and U a horizontal velocity scale characteristic of the motion. The period it takes a fluid particle with velocity U to transverse the distance L is L/U. If that period of time is much less than the period of rotation of the Earth, the fluid can scarcely sense the Earth's rotation over the time scale of the motion. Hence, the rotation is important if $L/U \geq \omega^{-1}$, condition which can be expressed in terms of the **Rossby** ε as follows

$$\varepsilon := \frac{U}{2\omega L} \leq 1.$$

The Gulf Stream has velocities of order $U = 1m/s$, the characteristic horizontal scale being $L = 100km$, cf. [4]. The corresponding Rossby number is $\varepsilon = 0.07$, so that the Earth rotation influences the fluid motion. Another example where Coriolis forces are important is the tsunami of December 2004 which hat a wavelength of $\lambda = 100$ km, while the characteristic velocities were of order $U = 1km/h$ (it is the speed of the wave which is very large $720km/h$, cf. [1]), so that $\varepsilon = 0.019$.

For fluid flows with characteristic horizontal scale of $100km$ we can neglect all the terms in the right hand sides of the equations in the previous system which contain ω^2 as their are comparable to

$$\omega^2 R = 0.03m/s^2.$$

If the observer is within $1^\circ = \pi/180 = 0,017$ rad from the Equator (or equivalently 110 km), we can estimate

$$\sin\phi \approx \phi \approx 0, \quad \cos\phi \approx \cos 0 = 1,$$

so that $f =\approx 0$, $f_* \approx 2\omega$. Taking into account that

$$\frac{R^2|(x,y,z)|}{|\mathbf{r}|^3} \leq 0,045, \quad \frac{R^3}{|\mathbf{r}|^3} \approx 1,$$

for fluid motions near the Equator the geophysical water wave problem can be approximated by the following system

$$\begin{cases} u_t + uu_x + vu_y + wu_z + 2\omega w = -\dfrac{p_x}{\rho}, \\ v_t + uv_x + vv_y + wv_z = -\dfrac{p_y}{\rho}, \\ w_t + uw_x + vw_y + ww_z - 2\omega u = -\dfrac{p_z}{\rho} - g, \end{cases}$$

which is know (together with the conservation of mass equation) as the **f-plane approximation**.

In Paper 4 we present an exact explicit Gerstner-type solution describing geophysical equatorial periodic water waves over a rotational flow. In the Paper 5 it is shown that this solution can be modified to obtain a family of exact Gerstner-type solutions to the f-plane approximation describing waves over uniform horizontal currents. The particle paths in the presence and absence of the Coriolis force were also analyzed in dependence of the current strength.

In Paper 6 we provided an explicit exact solution to the edge wave problem in the f-plane approximation. This edge wave solution describes three-dimensional waves that propagate along the shoreline and whose amplitude decays rapidly offshore. In the Paper 7 we showed that the exact solutions presented in the Papers 4 and 6 can be modified to describe also geophysical waves on flows with a vertical density stratification.

The explicit Gerstner-type solutions whose flow was described in the Paper 5 have the property that the pressure is constant along each streamline. In Paper 8 we proved that any solution of the f-plane approximation for equatorial geophysical deep water waves which has the property that the streamlines are isobaric and do not possess stagnation points, belong to this family of Gerstner-type waves. Furthermore, for waves over a flat bed, we showed that there are only laminar flow solutions with these properties.

In the Papers 9 and 10 we dealt with periodic finite-depth equatorial wind waves in the f-plane approximation. Using local bifurcation theory we proved in Paper 9 the existence of steady, periodic two-dimensional surface water waves in the equatorial region which have a general underlying vorticity distribution. Furthermore, we derived explicit dispersion relations for the flow in the case where the vorticity is constant. Additionally, in the 10th Paper, we presented a symmetry result which states that the symmetric waves are characterized by the fact that the wave surface has only one crest per period.

In Paper 11 we considered the two-dimensional equatorial water-waves problem with constant vorticity in water of finite depth. Within the framework of small-amplitude waves, we derived the dispersion relation and we found analytic solutions to the nonlinear differential equation system describing the particle paths beneath such waves. Moreover, we shown that the solutions obtained are not close curves and we provided also some remarks on the stagnation points.

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