Hyperkähler manifolds of curves and l-hypercomplex structures

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To Stéphanie and my family
with my deepest gratitude for never letting me feel alone.

Abstract

Given a hyperkähler manifold M it is possible to construct a complex manifold Z, called its *twistor space* which codifies in its holomorphic structure all the Riemannian and complex properties of M.
Following the works of Bielawski [6] and Bielawski-Schwachhöfer [10], [9], this thesis is concerned with the study of manifolds of "higher degree" curves in twistor spaces that admit an intermediate fibration onto $T\mathbb{P}^1$.
After an introductory chapter, we fix our focus on such manifolds of curves and prove that they admit a hypercomplex (hyperkähler) structure. For each curve we construct a quadratic matrix polynomial and show that it satisfies some appropriate reality condition. We also identify this matrix polynomial with a *l-hypercomplex structure*, therefore justifying the title of this work.
We then make the choice of a complex structure on a manifold of curves and restate our study in the frame of transverse Hilbert scheme of points on a complex surface. We recover using elementary techniques a result of Beauville and show some interpretation in the spirit of holomorphic completely integrable systems.
After this, we shoe how the twistor space can be recovered once the manifold of curves and the matrix polynomial are known.
Finally we explore the link between the moduli space of monopoles of charge 2 and the theory of τ-invariant rank 2-bundles on elliptic curves, proving some partial result about the question whether such bundles be decomposable or not.

Key Words: Hyperkähler manifold, twistor space, transverse Hilbert scheme, matrix polynomial, spectral curve.

Zusammenfassung

Für eine gegebene hyperkählersche Mannigfaltigkeit M kann man eine komplexe Mannigfaltigkeit Z konstruieren, die *Twistor Raum* von M genannt wird, die die Riemannsche und komplexe Eigenschaften von M in seiner holomorphen Struktur entschlüsselt.
Nach den Artikeln von Bielawski [6] und Bielawski-Schwachhöfer [10], [9], untersucht diese Arbeit Mannigfaltigkeiten von Kurven höheren Grades, in Twistor Räumen die eine mittlere Faserung auf $T\mathbb{P}^1$ zulassen.
Nach einem einführenden Kapitel konzentrieren wir uns auf solche Mannigfaltigkeiten von Kurven und beweisen wir, dass sie eine hyperkählersche Struktur zulassen. Jeder Kurve weisen wir ein quadratisches Matrixpolynom zu und zeigen, dass es eine entsprechende Realitätsbedingung erfüllt. Außerdem identifizieren wir dieses Polynom mit einer *l-hyperkomplexe Struktur* und motivieren damit den Titel dieser Arbeit.
Danach wählen wir eine komplexe Struktur auf einer Mannigfaltigkeit von Kurven und beschreiben unsere Untersuchungen im Rahmen von transversale Hilbert Schemata von Punkten auf einer komplexen Fläche. Mit elementarer Technik bekommen wir wieder ein Resultat von Beauville und zeigen damit eine Verbindung mit der Theorie komplett integrierbarer holomorphen Systemen.
Nachher zeigen wir wie der Twistor Raum erneut konstruiert werden kann, falls uns die Mannigfaltigkeit von Kurven und das Matrixpolynom bekannt sind.
Schließlich studieren wir die Verbindung zwischen Moduliräumen magnetischer Monopolen der Ladung 2 und der Theorie von τ-invarianten Bündel von Rang 2 auf elliptischen Kurven und beweisen ein partielles Resultat über die Frage, ob solche Vektorbündel zerlegbar sind oder nicht.

Schlüsselwörter: Hyperkählersche Mannigfaltigkeit, Twistor Space, Transversale Hilbert Schema, Matrixpolynom, Spektrale Kurve.

Contents

Introduction

The twistor construction ([33], [24]) has proven itself an essential tool in the description of the geometry of hyperkähler manifolds. In fact, it allows us to holomorphically encode in one complex manifold Z, called the *twistor space*, all the complex and smooth data that are defined on a hyperkähler manifold M. Although from the $\mathcal{C}^\infty$ point of view Z is merely the Cartesian product $M \times \mathbb{P}^1$, from the holomorphic point of view Z is far from being a product space: it is, actually, a complex manifold that comes with a holomorphic fibration $\tilde{p}\colon Z \to \mathbb{P}^1$ and a *real structure* $\sigma\colon Z \to Z$ that covers the antipodal map of the Riemann sphere. As described in [24], every point m of the hyperkähler manifold M corresponds to a section $\mathbb{P}^1 \to Z$ of $\tilde{p}$ which is *real*, i.e. invariant under the real structure σ. In other words every point of m describes a copy of $\mathbb{P}^1$ inside Z and M can be recovered as a $4n$-real-dimensional of such sections.
In the frame and notation of [6], such copies of $\mathbb{P}^1$ inside Z are real curves $C \subset Z$ of *degree* one which, having all normal bundle equal to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$, satisfy the cohomological stability condition $H^*(C, N_{C/Z}(-2)) = 0$. In [6] Bielawski considers the natural questions of what happens when trying to describe the parameter M space of real cohomologically stable curves C of generic degree d inside a twistor space $Z \to \mathbb{P}^1$. It turns out that such space M is again a smooth manifold and admits a hypercomplex structure or, under natural symplectic assumptions on the fibres of $Z \to \mathbb{P}^1$, a full hyperkähler structure.
Moreover, in [10] and [9] Bielawski and Schwachhöfer introduced a wider geometry, which they called *pluricomplex geometry*, of which the hypercomplex (hence the hyperkähler) geometry is a particular example. Given a $2n$-dimensional real vector space V, they define a pluricomplex structure on V as an immersion $K\colon \mathbb{P}^1 \to \mathcal{J}(V)$ of the Riemann sphere into the space of all complex structures of V, such that the following hold:

- $K^*(\mathcal{V}^{0,1}) \cong \mathbb{C}^n \otimes \mathcal{O}(-1)$
- $V^{\mathbb{C}}/K^*(\mathcal{V}^{0,1}) \cong \mathbb{C}^n \otimes \mathcal{O}(1)$.

Adding the assumption

$$K\left(-\frac{1}{\bar{\zeta}}\right) = -K(\zeta)$$

for all ζ in $\mathbb{P}^1$, then we are exactly defining a hypercomplex structure on V.
When there exists a map $A\colon \mathbb{P}^1 \to T\mathcal{J}(V)$ that lifts a hypercomplex structure K, i.e. such that the composition of A with the canonical projection $T\mathcal{J}(V) \to V$ is a hypercomplex structure, we call A a *l-hypercomplex structure*.
In this thesis we deal with those twistor spaces $\tilde{p}\colon Z \to \mathbb{P}^1$ that come with an intermediate holomorphic fibration $p\colon Z \to T\mathbb{P}^1$. We consider the parameter space M of all real cohomologically stable curves $C \subset Z$ of *degree* d (again in the sense of [6]) such that the restriction $p|_C\colon C \to \bar{C} = p(C) \subset T\mathbb{P}^1$ is a biholomorphism. After the introductory chapter, where we recall from [6] the existence of a hypercomplex (hyperkähler) structure on M, we show that both a l-hypercomplex structure and a family of endomorphisms of the tangent space to M (i.e. a matrix polynomial once a basis has been chosen) which commute with all complex structures naturally arise on M. We briefly show the interplay between such two objects and then focus to the case when Z is three-dimensional. We continue by fixing one endomorphism in the family and describing the related geometry by means of the *transverse Hilbert scheme* construction in the spirit of the theory of *completely integrable systems*. We wish to remark here that the connection between twistor theory and algebraically completely integrable systems dates back to Hitchin's works such as [23] and [22]. Further, making use of the results obtained so far, we explain the geometric construction yielding back the fibration $p\colon Z \to T\mathbb{P}^1$ once the hypercomplex manifold M is known. Finally we recall from [5] the theory of vector bundles on curves induced by matrix polynomials and focus on the case of spectral curves of (strongly centred) magnetic monopoles of charge 2. Such curves are elliptic and the induced vector bundle has rank 2: in view of the Atiyah classification [2] we focus on understanding whether in our case the induced bundle is or not a decomposable one. The body of the thesis is divided into five chapters, whose precise development is as follows.
In the **first chapter** we introduce all the foundational material that is necessary to our study. We start by defining hypercomplex and hyperkähler manifolds, then we give a detailed review of Hitchin's approach to the hyperkähler twistor theory and expose the hyperkähler quotient construction and its links with twistor theory. After this, we focus our attention on [6] and introduce Bielawski's results about hyperkähler manifolds of higher degree curves in twistor spaces, already focusing on twistor space fibring over $T\mathbb{P}^1$. We conclude the chapter with a review of l-hypercomplex structures following [10] and [9] focusing on their link with spectral curves in $T\mathbb{P}^1$.
In **Chapter 2** we focus on the manifold M of real, cohomologically stable, *degree* d curves C (in

the sense of [6]) such that the restriction $p|_C \colon C \to \bar{C} = p(C) \subset T\mathbb{P}^1$ is a biholomorphism. For each such curve we construct a matrix polynomial as a section of $\mathrm{End}\left(H^0(C, N_{C/Z}(-1))\right) \otimes \mathcal{O}(2)$ which we interpret as a l-hypercomplex structure. We state then a reality condition for $A(\zeta)$ and, letting the curve C vary in M, prove that for every fixed $\zeta \in \mathbb{P}^1$ we can interpret $A(\zeta)$ as and endomorphism of $\left(T^{1,0}M\right)$ given by a section of $\mathrm{End}\left(H^0(C, N_{C/Z}(-1))\right)$, seen as a bundle on M, which is holomorphic with respect to the complex structure I_ζ.
Chapter 3 is devoted to investigating the geometry related to the endomorphism $A(\zeta)$ once a complex structure I_ζ, i.e. a point $\zeta \in \mathbb{P}^1$ is fixed. By means of Proposition 1.23 we can perform the study in full generality and consider the generic case of a surface S projecting onto $\mathbb{C}$ via a map p which is a holomorphic submersion outside a discrete set of points. We construct an endomorphism A of the tangent space $TS_p^{[d]}$ to the Hilbert scheme of d points of S transverse to p, which represents the analogous of $A(\zeta)$, and prove the following proposition characterizing all manifolds that arise as transverse Hilbert schemes of surfaces with a projection $p \colon S \to \mathbb{C}$ showing the above properties.

Theorem. *Let W^{2d} be a complex manifold of complex dimension $2d$ with the following properties.*

(i) W comes with an endomorphism $A \colon TW \to TW$ such that at every point the eigenspaces have complex dimension 2 and the characteristic polynomial is the square of the minimal polynomial

(ii) Assume that the induced projection $\mu \colon W \to X := \mathbb{C}^{[d]}$ is a surjective submersion on a subset $N \subset W$ such that $W \setminus N$ has codimension at least 2 and that A is compatible with μ so that the diagram

$$\begin{array}{ccc} TW & \xrightarrow{A} & TW \\ \downarrow{\scriptstyle d\mu} & & \downarrow{\scriptstyle d\mu} \\ TX & \xrightarrow[\bar{A}]{} & TX. \end{array} \tag{0.0.1}$$

is defined

(iii) The distribution $D := Im(z - A)$ is integrable on the incidence manifold

$$\tilde{W} = \{(z, w) \in \mathbb{C} \times W |\ z \text{ is an eigenvalue of } A_w\}.$$

Then $p \colon S := \tilde{W}/D \to \mathbb{C}$ is a surface projecting on $\mathbb{C}$ for which W is the length d Hilbert scheme of points transverse to the projection.

If, moreover, we assume the surface S to carry a holomorphic symplectic form ω, then we prove that ω induces a holomorphic symplectic form Ω on $S_p^{[d]}$ recovering with more elementary methods a result of Beauville. Finally, in Section 3.3 we complete our characterisation with the following proposition.

Theorem. *Let W be a complex manifold of complex dimension $2d$ endowed with an holomorphic endomorphism of the tangent space TW as in the previous theorem and let $N \subset W$ be the subset where the induced projection $\mu\colon W \to \mathbb{C}^{[d]}$ is a surjective submersion. Assume also that W possesses a symplectic form Ω such that at every point of N*

- $\Omega(A\cdot, \cdot) = \Omega(\cdot, A\cdot)$
- *the vertical subbundle $\ker(d\mu)$ is maximal Ω-isotropic.*

Then $S = \tilde{W}/D$ has a symplectic form induced by ω.

Our aim in **Chapter 4** is to describe how we can recover the fibred manifold $p\colon Z \to T\mathbb{P}^1$ when we are given the couple $(M, A(\zeta))$ as initial data, as long as $A(\zeta)$ satisfies some appropriate integrability condition involving the distribution $D = Im(\eta - A(\zeta))$ on the incidence manifold $Y \subset Z^{HK} \boxtimes T\mathbb{P}^1$, where Z^{HK} is the hyperkähler twistor space of M. The main result of the chapter is then as follows.

Theorem. *Let M be a hypercomplex manifold of complex dimension $2d$ with the following properties.*

1. *M is equipped with an aquaternionic endomorphisms of its real tangent bundle such that the associated real section $A(\zeta)$ of $\mathcal{O}(2)^{\oplus 4d^2}$ yields, for every $\zeta \in \mathbb{P}^1$, a holomorphic endomorphism of $(T^{1,0}M)_{I_\zeta}$ with 2-dimensional eigenspaces and such that its characteristic polynomial is the square of the minimal polynomial*
2. *For every ζ there exists a subset $N_\zeta \subset M$ such that $(M \setminus N_\zeta)$ has codimension at least two, the projection $\mu_\zeta\colon W \to \mathbb{C}^{[d]}$ is a surjective submersion on N_ζ and $A(\zeta)$ is* compatible *with μ_ζ.*
3. *For every ζ and every $X \in \ker d\mu_\zeta$ the condition $\nabla_X A(\zeta) = 0$ holds on N_ζ, being ∇ the Obata connection of W.*

Then $p\colon Z = Y/D \to T\mathbb{P}^1$ is a complex 3-dimensional manifold projecting on $T\mathbb{P}^1$, endowed with a real structure σ covering the antipodal map, for which M is the manifold of σ-invariant cohomologically stable degree d curves. The converse is also true, i.e. every such manifold of curves in a twistor space $p\colon Z \to T\mathbb{P}^1$ fulfils shows the above properties 1., 2., 3.

Finally we recall in **Chapter 5** some basic facts about the moduli spaces of monopole spectral curves and show that, in the case of a curve that lies in the (double covering of) Atiyah-Hitchin manifold, the endomorphism $A(\zeta)$ can be written in a particular block-anti-diagonal form. From this observation we are motivated to recall the *Beauville isomorphism* between conjugacy classes of "regular" matrix polynomials and acyclic bundles of rank 2 and to consider the so-called τ-*bundles* on curves inside the total space $|\mathcal{O}(2)|$ of the vector bundle $\mathcal{O}(2)$ on $\mathbb{P}^1$. Due to their importance as spectral curves of strongly centred charge 2 monopoles, we restrict our attention to elliptic curves $\bar{C}$ of the form $\big(\lambda^2 - a_2(\zeta) = 0\big)$ and consider the case of rank 2 acyclic vector bundles defined on such curves by matrix polynomials of degree 2 in the form

$$X(\zeta) = \begin{pmatrix} 0 & P(\zeta) \\ Q(\zeta) & 0 \end{pmatrix}.$$

Consider a monopole spectral curve $\bar{C}$ in $T\mathbb{P}^1$ and its lift C inside the total space of $L^2\backslash_{\{0\}}$. The twist $N_{C/L^2}(-2)$ of the normal bundle to C by $\mathcal{O}(-2)$ is, in the charge 2 case, one such bundle and (see [29]) it is related to the restriction of the monopole bundle $\tilde{E}$ to the curve via $N(-2) \cong \tilde{E}L|_{\bar{C}}$ once we identify C and $\bar{C}$. Now, as bundles of rank 2 on elliptic curves are completely classified, we turn our interest into understanding whether $\tilde{E}$ and, in general, the τ-bundles of rank 2 be or not decomposable vector bundles. Anyway, the study of vector bundles of rank 2 appears to be more difficult to approach than the case of line bundle, hence we achieve here only the following partial results.

Theorem. *Let Δ_P (respectively Δ_Q) denote the divisor of zeros of $\det(P(\zeta))$ (respectively $\det(Q(\zeta)))$ on $\bar{C}$ taken with single multiplicity and assume them to be disjoint. Then sheaf of sections $\mathcal{F}$ of the vector bundle induced by $X(\zeta)$ is $\mathcal{F} \cong \pi^*\mathcal{O}(-1)^{\oplus 2} \otimes [\Delta_Q] \cong \pi^*\mathcal{O}(-1)^{\oplus 2} \otimes [\Delta_P]$, where π is the projection $|\mathcal{O}(2)| \to |\mathcal{O}(4)|$ given by $(\zeta, \lambda) \mapsto (\zeta, \lambda^2)$ restricted to the curve $\bar{C}$.*

Unfortunately, the restriction of the monopole bundle is induced by a block anti-diagonal matrix polynomial $X(\zeta)$ with $\Delta_P = \Delta_Q$. In this case we have the following partial result.

Theorem. *Let $\zeta_1, \ldots, \zeta_4$ be the zeros of $\det(P(\zeta))$ (and of $\det(Q(\zeta))$. If either*

- $Im(P(\zeta_i))$ *all coincide*
- $Im(P(\zeta_1)) = Im(P(\zeta_2)) = Im(P(\zeta_3)) \neq Im(P(\zeta_4))$
- $Im(P(\zeta_1)) = Im(P(\zeta_2))$ *and* $Im(P(\zeta_2)) \neq Im(P(\zeta_3)) \neq Im(P(\zeta_4))$

then $\mathcal{F}$ is the sheaf of sections of an indecomposable vector bundle of rank 2.
If $Im(P(\zeta_1)) = Im(P(\zeta_2)) \neq Im(P(\zeta_3)) = Im(P(\zeta_4))$ then $\mathcal{F}$ is decomoposable if and only

if also $Im(Q(\zeta_1)) = Im(Q(\zeta_2)) \neq Im(Q(\zeta_3)) = Im(Q(\zeta_4))$*, otherwise it is indecomposable. In the case when* $Im(P(\zeta_i))$ *are four distinct lines in* $\mathbb{C}^2$ *the question is still an open one.*

Chapter 1

Hyperkähler Manifolds and their Twistor Theory

Contents

This introductory chapter is devoted to briefly collecting the basic results concerning the theory of *hypercomplex* and *hyperkähler manifold* and their *twistor spaces*, which will then play a central role in the following chapters.

1.1 Hypercomplex and Hyperkähler Manifolds

We start the section with by defining the central objects to our exposition, that is *hypercomplex* and *hyperkähler manifolds*.

Definition 1.1. *Let M be a smooth manifold of real dimension $4n$ equipped with a triple of (almost) complex structure I_1, I_2, I_3 that satisfy the quaternionic relations $I_1^2 = I_2^2 = I_3^2 = -\mathbb{1}$, $I_1 I_2 = I_3$. We call the quadruple (M, I_1, I_2, I_3) a (almost)* hypercomplex manifold
If, moreover, (M, I_1, I_2, I_3) is equipped with a Riemannian metric g such that it is (almost) Kähler for each (almost) complex structure then it is called a (almost) hyperkähler manifold.

An immediate remark is the following.

Remark 1.2. The triple of (almost) complex structures gives each tangent space to M the structure of a *quaternionic vector space*, hence the requirement of the dimension of M to be a multiple of 4 in the definition.

A celebrated theorem of Obata contained in [31] states the following.

Theorem 1.3. *Let M be a hypercomplex manifold. There exists a unique torsion-free affine connection on M preserving all three complex structures $I_i, i = 1, 2, 3$.*

Such a connection is called the *Obata connection* of M. The following property of the Obata connection holds true (cf.[26])

Proposition 1.4. *The holonomy of the Obata connection is contained in $GL_n(\mathbb{H})$.*

We now recall some equivalent conditions for a manifold M to be hyperkähler

1. Each *Kähler form* $\omega_i = g(I_i\cdot, \cdot), i = 1, 2, 3$ is closed
2. Each complex structure I_i is parallel for the Levi-Civita connection ∇, that is $\nabla I_i = 0$, $i = 1, 2, 3$.
3. The Riemannian holonomy of M is contained in $Sp(n)$.

Remark 1.5. Since both hypercomplex and hyperkähler manifolds belong to the wider class of *quaternionic manifolds* described by Salamon in [34] and [35], for such manifolds we can write $T^{\mathbb{C}}M \cong E \otimes H$ where, in particular, E and H are globally defined vector bundles with fibres isomorphic to $\mathbb{C}^{2n}$ and $\mathbb{C}^2$ respectively.

1.2 Hyperkähler Twistor Theory

In the present section we review in some detail the description of the *twistor space* as presented in the classical article by Hicthin-Karlhede-Lindström-Roček [24].

Let M be a real $4n$-dimensional hyperkähler manifold with complex structures I_1, I_2, I_3. An immediate observation is that if $(a, b, c) \in S^2 \subset \mathbb{R}^3$ then, thanks to the quaternionic relations that the I_i's satisfy, $aI_1 + bI_2 + cI_3$ is also a complex structure on M. This means that on a hyperkähler manifolds a whole 2-sphere of complex structures is actually defined, each of whose is compatible with the Riemannian metric and the Levi-Civita connection. The aim of the twistor construction is then to define a somewhat "larger" manifold, namely the *twistor space* of M, that encodes all the information about the complex structures and the metric of M into its complex and holomorphic structure. The first step in the construction of the twistor space is to see the sphere S^2 as the complex projective line $\mathbb{P}^1$, covered by the usual two patches U_0, U_1 with holomorphic coordinates ζ and $\tilde{\zeta}$ that satisfy $\tilde{\zeta} = 1/\zeta$ on the intersection $U_0 \cap U_1$. By means of the stereographic projection, we express a point of $S^2 \subset \mathbb{R}^3$ as

$$(a, b, c) = \frac{1}{1 + \zeta\bar{\zeta}} \left(1 - \zeta\bar{\zeta}, -i(\zeta - \bar{\zeta}), -(\zeta + \bar{\zeta})\right). \tag{1.2.1}$$

We define now the twistor space of M to be, from the $\mathcal{C}^\infty$ point of view, the Cartesian product $Z = M \times \mathbb{P}^1$, which we equip with an almost complex structure $\underline{I}$ defined as follows: if we express the tangent space at $(m, \zeta) \in Z$ as $T_{(m,\zeta)}Z = T_m M \oplus T_\zeta \mathbb{P}^1$, then we define

$$\underline{I} = \left(\frac{1 - \zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}} I_1 - \frac{i(\zeta - \bar{\zeta})}{1 + \zeta\bar{\zeta}} I_2 - \frac{(\zeta + \bar{\zeta})}{1 + \zeta\bar{\zeta}} I_3, I_0\right) = (I_\zeta, I_0), \tag{1.2.2}$$

where I_0 is the usual complex structure on the sphere given as multiplication by i on the holomorphic tangent space $T_\zeta \mathbb{P}^1$. At this point, the theorem of Newlander-Niremberg [30] shows the following.

Proposition 1.6. *The almost complex structure $\underline{I}$ on Z is integrable, therefore Z is a complex manifold of complex dimension $2n + 1$ and admits a system of complex coordinates.*

The projection $\tilde{p} \colon Z \to \mathbb{P}^1$ is then a holomorphic map and each map $P_m \colon \zeta \mapsto (m, \zeta)$ is a section of p.

Definition 1.7. *The sections P_m are called* twistor lines.

Remark 1.8. From a $\mathcal{C}^\infty$ point of view $Z = M \times S^2$, therefore the normal bundle to a twistor line P_m defined as $N_{P_m} = TZ|_{P_m}/TP_M$ is simply the product $S^2 \times T_m M$. Anyway, from the holomorphic point of view, N_{P_m} is not trivial and it is essential in order to invert the twistor construction to determine what it is.

If we represent the I_1, I_2 and I_3 as acting on $T_m M \cong \mathbb{C}^{2n}$ via the complex matrices

$$\begin{pmatrix} i\mathbb{1}_n & 0 \\ 0 & -i\mathbb{1}_n \end{pmatrix}, \qquad \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -i\mathbb{1}_n \\ -i\mathbb{1}_n & 0 \end{pmatrix} \tag{1.2.3}$$

then we can write I_ζ as

$$\frac{1}{1 + \zeta\bar{\zeta}} \begin{pmatrix} i(1 - \zeta\bar{\zeta}) & 2i\zeta \\ 2i\bar{\zeta} & -i(1 - \zeta\bar{\zeta}). \end{pmatrix} \tag{1.2.4}$$

The above matrix describes the complex defined on T_mM by a point $\zeta \in \mathbb{P}^1$. Every vector in the $+i$-eigenspace over ζ can therefore be written as $(X + i\bar{\zeta}I_3(X))$ where X is a $+i$-eigenvector for I_1. Moreover, if φ is a 1-form of type $(1,0)$ for the complex structure I_1, then $(\varphi + i\zeta I_3(\varphi))$ has type $(1,0)$ for the complex structure defined by $\zeta \in \mathbb{P}^1$. A quick computation of the transition functions on $U_0 \cap U_1$ shows that the normal bundle to any twistor line is holomorphically equivalent to the tensor product $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$, which we will denote $\mathbb{C}^{2n}(1)$. We will also keep the notation $\mathcal{O}(k)$ to denote the pullback of $\mathcal{O}(k)$ from $\mathbb{P}^1$ to Z via the projection p.
It is easy to see that the 2-form $\omega_+ = \omega_2 + i\omega_3$ has type $(2,0)$ with respect to complex structure I_1. Choosing an appropriate local basis $\{\varphi_i\}$ of forms of type $(1,0)$ for I_1, we can write

$$\omega_+ = 2\sum \varphi_i \wedge \varphi_{i+n} \tag{1.2.5}$$

from which we can construct the 2-form

$$\omega = 2\sum (\varphi_i - i\zeta I_3(\varphi_i)) \wedge (\varphi_{i+n} - i\zeta I_3(\varphi_{i+n}))\,. \tag{1.2.6}$$

Of course ω has type $(2,0)$ with respect to the complex structure given by ζ and, by expanding in powers of ζ and using the definitions of ω_i in terms of the metric g and the complex structure I_i it can be proved that

$$\omega = (\omega_2 + i\omega_3) + 2i\zeta\omega_1 + \zeta^2(\omega_2 - i\omega_3)\,. \tag{1.2.7}$$

Remark 1.9. For every $\zeta \in \mathbb{P}^1$ we have defined a holomorphic complex symplectic form ω on each fibre of the projection p. Such forms are all covariantly constant and depend quadratically of ζ. We can therefore describe ω as a section of the vector bundle $\bigwedge^2 T_F^* \otimes \mathcal{O}(2)$ over Z, where we mean

$$T_F = \ker(d\tilde{p})\colon TZ \to T\mathbb{P}^1 \tag{1.2.8}$$

to be the tangent along the fibres of the twistor projection. From now on we adopt the notation $\bigwedge^2 T_F^*(2)$ for $\bigwedge^2 T_F^* \otimes \mathcal{O}(2)$.

Finally, the antipodal map on $\zeta \mapsto -1/\bar{\zeta}$ defines an antiholomorphic involution on Z as

$$\begin{aligned} \sigma\colon M \times \mathbb{P}^1 &\to M \times \mathbb{P}^1 \\ (m,\zeta) &\mapsto \left(m, -\frac{1}{\bar{\zeta}}\right) \end{aligned} \tag{1.2.9}$$

that takes the complex structure $\underline{I}$ to its opposite $-\underline{I}$. Such a map is called a *real structure* and all the holomorphic data encoded so far are compatible with it.

Now, the main feature of twistor theory is that it allows us to reconstruct the hyperkähler manifold from the holomorphic properties of its twistor space. This is summarized in the following theorem, the proof of which we also recall from [24].

Theorem 1.10. *Let Z be a complex manifold of complex dimension $2n + 1$ with the following properties:*

- *Z comes with a holomorphic fibration $\tilde{p}\colon Z \to \mathbb{P}^1$ onto the complex projective line*
- *The fibration $\tilde{p}$ admits a family of holomorphic section each of whose has normal bundle isomorphic to $\mathbb{C}^{2n}(1)$.*
- *There exists a holomorphic section ω of $\bigwedge^2 T_F^*(2)$ which defines a holomorphic symplectic form on each fibre.*
- *Z possesses a real structure σ compatible with the previous data and covering the antipodal map of $\mathbb{P}^1$.*

Then the parameter space of M real sections is a manifold of real dimension $4n$ equipped with a natural hyperkähler metric and Z is its twistor space.

Remark 1.11. The above stated Theorem 1.10 does not give exactly the inverse procedure to the construction of the twistor space of a hyperkähler manifold. Indeed, if we start from such a complex $2n+1$-dimensional manifold Z, produce M and construct its twistor space we recover Z. On the other hand, if we start from a hyperkähler manifold M, construct its twistor space Z and then perform the procedure of Theorem 1.10 we might end up with a real $4n$-dimensional hyperkähler manifold M', possibly consisting of several connected components and such that $M \subset M'$. This consideration is somehow expressed in [36].

Proof of Theorem 1.10. The steps in order to prove the claim are first to show that the parameter space of all real sections, that we will again call twistor lines, is a smooth manifold and to compute its dimension, then to construct a metric and finally to prove the hyperkählerness of the latter.
Let P_m denote a twistor line corresponding to a point $m \in M$. According to Kodaira's deformation theory (see [27]), an infinitesimal deformation of the section P_m of $\tilde{p}\colon Z \to \mathbb{P}^1$ can be thought as a holomorphic section of the normal bundle N to P_m in Z. Precisely, if the cohomology group $H^1(\mathbb{P}^1, N)$ vanishes, then every holomorphic section of the normal bundle N comes from a deformation of the twistor line P_m , thus making the space of all holomorphic sections of $\tilde{p}$ a complex manifold whose holomorphic tangent space at each point m is isomorphic to the vector space $H^0(P_m, N)$ of all global sections of N on P_m. Since by assumptions we have

$N \cong \mathbb{C}^{2n}(1)$, the group $H^1(\mathbb{P}^1, N)$ is isomorphic to $\mathbb{C}^{2n} \otimes H^1(\mathbb{P}^1, \mathcal{O}(1))$ which of course vanishes.
Now, a global section of $\mathcal{O}(1)$ on the complex projective line is defined on U_0 by a linear polynomial in ζ, hence a holomorphic section s of $N \cong \mathbb{C}^{2n}(1)$ on P_m is given as

$$s(\zeta) = a + b\zeta \qquad \text{where } a, b \in \mathbb{C}^{2n} \tag{1.2.10}$$

Since these sections form a vector space of complex dimension $4n$ then, by the theory of Kodaira, the twistor line P_m belongs to a $4n$-dimensional family. The *real* (i.e. σ-invariant) twistor lines are then parametrized by a real submanifold M of real dimension $4n$ whose tangent space $T_m M$ at m satisfies

$$T_m^{\mathbb{C}} M \cong H^0(P_m, N) \cong H^0(P_m, T_F), \tag{1.2.11}$$

where, as usual, T_F stands for the tangent bundle to the fibres of $\tilde{p}$, i.e. the vertical bundle with respect to $\tilde{p}$. We have therefore completed the first step by constructing the real manifold M and describing its tangent space. We now have to define a metric g on M and prove it is hyperkähler.
First of all, since on P_m the isomorphism $T_F \cong \mathbb{C}^{2n}(1)$ holds true, then $T_F(-1) \cong \mathbb{C}^{2n}$. Therefore (1.2.11) can be rewritten as

$$T_m^{\mathbb{C}} M \cong H^0(P_m, T_F(-1)) \otimes H^0(P_m, \mathcal{O}(1)) \cong \mathbb{C}^{2n} \otimes \mathbb{C}^2. \tag{1.2.12}$$

Then, by hypothesis, we have that $\omega \in H^0(Z, \bigwedge^2 T_F^*(2))$ can be regarded as a 2-form with values in $\mathcal{O}(2)$, therefore it defines a non degenerate skew-from on $H^0(P_m, T_F(-1))$. Moreover, the space $H^0(P_m, \mathcal{O}(1)) \cong H^0(\mathbb{P}^1, \mathcal{O}(1))$ is naturally endowed with a symplectic form $\langle \cdot, \cdot \rangle$ defined by

$$\langle a_1 + b_1\zeta, a_2 + b_2\zeta \rangle = a_1 b_2 - a_2 b_1. \tag{1.2.13}$$

The tensor product of ω and $\langle \cdot, \cdot \rangle$ together with (1.2.12) define a complex scalar product g on $T_m^{\mathbb{C}} M$ given as

$$g(a + b\zeta, a + b\zeta) = 2\omega(a, b). \tag{1.2.14}$$

We now need to describe the tangent space to real twistor lines, i.e. the space of real tangent vectors. A small digression on *real* and *quaternionic structures* on vector spaces is here required. Analogously to what we said before, a real structure on a complex vector space V is a complex-antilinear (therefore real-linear) involution $t\colon V \to V$, while a quaternionic structure is a map $j\colon V \to V$ which is complex-antilinear and squares to $-\mathbb{1}$. As i and j anti-commute on V and $ij = k$, an action of the quaternions is given on V. An immediate remark is that the tensor product of two quaternionic structure on two vector spaces gives a real structure on the tensor product of the vector spaces. If we now apply this observation to (1.2.12), we can define a real structure on $T_m^{\mathbb{C}}M$ simply by tensoring two quaternionic structures defined respectively on $H^0(P_m, T_F(-1))$ and $H^0(P_m, \mathcal{O}(1))$. The quaternionic structure on $H^0(P_n, \mathcal{O}(1))$ is given as $j(a+b\zeta) = -\bar{b}+\bar{a}\zeta$ and comes as the unique antilinear action on $\mathcal{O}(1)$ that covers the antipodal map $\zeta \mapsto -1/\bar{\zeta}$ on $\mathbb{P}^1$. The real structure σ of Z preserves the vector bundle T_F as well as the decomposition (1.2.12) hence induces a quaternionic structure j on the space $H^0(P_m, T_F(-1))$. The real structure on $H^0(P_m, T_F)$ is therefore

$$\tau(a + b\zeta) = j(b) - j(a)\zeta, \quad \text{where } a, b \in H^0(P_m, T_F(-1)). \tag{1.2.15}$$

A real tangent vector is then of the form

$$X = a - j(a)\zeta, \quad \text{where } a \in H^0(P_m, T_F(-1)) \tag{1.2.16}$$

and the metric is given via (1.2.14) as $g(X, X) = -2\omega(j(a), j(a))$. The compatibility of ω with σ implies the positive definiteness of g. As a last step we must prove the hyperkähler property of g. In order to do so it will be helpful to identify the $4n$-dimensional Riemannian manifold that we have constructed with an open set in any of the fibres of $\tilde{p}\colon Z \to \mathbb{P}^1$. Consider then a real holomorphic section of the normal bundle of P_m in Z, which we write as $X = a - j(a)\zeta$, and assume it vanishes at some point $\zeta = \zeta_0 \in \mathbb{P}^1$. Then, from the definition of g, we get

$$g(X, X) = -2\omega(\zeta_0 j(a), j(a)) = -2\zeta_0\omega(j(a), j(a)) = 0 \tag{1.2.17}$$

which means, as g is positive definite, that X is identically zero. Therefore a an infinitesimal deformation of a real twistor line cannot vanish anywhere along the line. Geometrically this means that real twistor lines "separate" points in each fibre of Z so neighbouring real lines can only intersect the fibres of $\tilde{p}$ in distinct points. As a consequence of this, M can be identified with any of the fibres of $\tilde{p}$. We now choose to identify M with the fibre Z_0 of $\tilde{p}$ over $\zeta = 0$ in $\mathbb{P}^1$. A real tangent vector $X = a - j(a)\zeta$ is then identified with the element $a \in H^0(P_m, T_F(-1))$ and, since $T_F(-1)$ along P_m is isomorphic to the trivial bundle $\mathbb{C}^{2n}$, a is determined by its value at any point $\zeta \in \mathbb{P}^1$ so, in particular, at $\zeta = 0$. So, if we read T_F as the tangent bundle to the fibre Z_0, the map $X \mapsto a$ is the differential of the identification of M with Z_0. Clearly Z_0 is a complex manifold, hence the identification of M with Z_0 defines a complex structure I_1 on M which amounts to multiplying a by i. Hence

$$\begin{aligned} g(I_1(X), Y) &= -\omega(ia, j(b)) - \omega(b, j(ia)) = -i\omega(a, j(b)) + i\omega(b, j(a)) \\ &= \omega(a, j(ib)) + \omega(ib, j(a)) = -g(X, I_1(Y)), \end{aligned} \tag{1.2.18}$$

i.e. the metric g is Hermitian with respect to the complex structure I_1.
Consider now the form ω on the fibre over $\zeta = -1$: this yields a form φ_{-1} on M defined as

$$\varphi_{-1}(X, Y) = \omega(a + j(a), b + j(b)). \tag{1.2.19}$$

The same operation over $\zeta = 1$ yields

$$\varphi_{+1}(X, Y) = \omega(a - j(a), b - j(b)), \tag{1.2.20}$$

and

$$\frac{1}{2}\left(\varphi_{-1} - \varphi_{+1}\right)(X, Y) = \omega(j(a), b) + \omega(a, j(b)) = ig(I_1(X), Y)). \tag{1.2.21}$$

Since $\varphi_{-1}, \varphi_{+1}$ are both closed also $g(I\cdot, \cdot)$ is closed, hence g is Kähler with respect to I_1. The same procedure repeated for the complex structures I_2 and I_3 show g to be hyperkähler. ∎

Remark 1.12. According to Remark 1.5, the splitting (1.2.12) allows us to write $T^{\mathbb{C}}M$ as $E \otimes H$ where H is the trivial $\mathbb{C}^2$ bundle on M and E is the vector bundle whose fibre at each $m \in M$ is $E_m = H^0(P_m, T_F(-1))$.

1.3 Hyperkähler Quotient Construction

Another central tool in the theory of hyperkähler manifolds we are going to recall is the so-called *hyperkähler quotient* which extends the well-known symplectic (or Kähler) quotient construction due to Marsden-Weinstein (cf. [12], [19]).

Assume that M is a $4n$-dimensional hyperkähler manifold, i.e. it is equipped with a Riemannian metric g compatible with three covariantly constant complex structures I_1, I_2, I_3 that obey the quaternionic relations. Assume also that a compact Lie group G acts freely on M by isometries, preserving the complex structures. Then G preserves the Kähler forms $\omega_i, i = 1, 2, 3$ corresponding to the complex structures I_i hence three moment maps $\mu_i \colon M \to \mathfrak{g}^* = (Lie(G))^*$ can be defined, which can be invariantly described as a single map

$$\mu \colon M \to \mathfrak{g}^* \otimes \mathbb{R}^3. \tag{1.3.1}$$

The following theorem yields a generalization of Marsden-Weinstein's result.

Theorem 1.13. *The quotient metric on $\mu^{-1}(0)$ is hyperkähler.*

Sketch of the Proof. We consider one complex structure, say I_1, with its Kähler form ω_1. The complex function

$$\mu_+ = \mu_2 + i\mu_3 \colon M \to \mathfrak{g}^* \otimes \mathbb{C} \tag{1.3.2}$$

has the property

$$d\mu_+^X(I_1(Y)) = id\mu_+^X(Y) \tag{1.3.3}$$

for every fundamental vector field X and every tangent vector field Y, which imply that μ_+^X is a holomorphic function for I_1. Therefore $N = \mu_+^{-1}(0) = \mu_2^{-1}(0) \cap \mu_3^{-1}(0)$ is complex submanifold of M with respect to the complex structure I_1 and the induced metric on M is Kähler. The group G acts again on N: such action preserves the Kähler form of N and its moment map is clearly the restriction of μ_1 to N. So, by the usual Marsden-Weinstein theorem on symplectic reduction, the quotient metric on $N \cap \mu_1^{-1}(0)/G = \mu^{-1}(0)/G$ is Kähler with respect to I_1. The same argument, repeated for the complex structures I_2 and I_3 completes the proof. ∎

Remark 1.14. The form $\omega_+ = \omega_2 + i\omega_3$ has, as we said, type $(2,0)$ with respect to I_1. Being covariantly constant, ω_+ is holomorphic and can be proven to be complex non-degenerate, i.e. $\omega_+^n \neq 0$. Now, the action of the complexification $G^{\mathbb{C}}$ preserves ω_+ and admits μ_+ as a moment map with values in $\mathfrak{g} \otimes \mathbb{C}$. This means that the hyperkähler quotient $\mu^{-1}(0)/G$ is the usual symplectic quotient, this time in the holomorphic category.

There is of course some interplay between the hyperkähler quotient and the hyperkähler twistor constructions: in fact, a very natural question that can be asked is what kind of relation one has between the twistor space Z of a hyperkähler manifold M and the one $\hat{Z}$ of some quotient $\hat{M}$ of M by the action of some Lie group G. If G acts by isometries that preserve I_1, I_2, I_3, then the fundamental vector fields for the action of G are actually holomorphic on $Z = M \times S^2$ with the complex structure $\underline{I}$. Assuming that the action extends to one of the complexified Lie group $G^{\mathbb{C}}$, the quotient construction yields the twistor space $\hat{Z}$ from Z by taking the holomorphic symplectic quotient along each fibre of $Z \to \mathbb{P}^1$ with respect to the form ω of Theorem 1.10. Since ω is a section of $\bigwedge^2 T_F^*(2)$, the holomorphic moment map will be a holomorphic section of $\mathfrak{g}^* \otimes \mathcal{O}(2)$ over Z.

Example 1.1. A simple example of such an interplay is provided when a hyperkähler manifold M is acted on by the group S^1 In this case, assuming that the action complexifies to an action of $\mathbb{C}^*$ we can define, as explained above, a holomorphic moment map on each fibre of the twistor space Z of M and encode all such maps into a holomorphic section of $\mathcal{O}(2)$ over Z. The toy example is given by the following S^1-action on $\mathbb{C}^2$ defined as

$$\begin{aligned} \psi\colon S^1 \times \mathbb{C}^2 &\to \mathbb{C}^2 \\ \big(e^{it}, (z_1, z_2)\big) &\mapsto (e^{it} z_1, e^{-it} z_2), \end{aligned} \tag{1.3.4}$$

where $\mathbb{C}^2$ has been equipped with the complex structures I_1, I_2, I_3 as in (1.2.3) and z_1, z_2 are the holomorphic coordinates for the complex structure I_1. The I_1-holomorphic moment map for this action is then iz_1z_2. From (1.2.2) and (1.2.4) we deduce that every vector of type $(0,1)$ for the complex structure I_ζ is given as $X - i\zeta I_3(X)$ where X is a vector of type $(0,1)$ for I_1. Since giving the space of vectors of type $(0,1)$ for a complex structure is equivalent to giving the complex structure itself, we notice that we are mapping the two-sphere inside the complex Grassmannian $Gr_{\mathbb{C}}(1,2)$, which is also $\mathbb{P}^1_{\mathbb{C}}$, via $F\colon \zeta \mapsto [(-\zeta, 1)]$. By choosing the appropriate orientation on $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$, i.e. setting $\zeta = a/b$ we have that F is a holomorphic map. Now, in the spirit of (1.2.12) we write $T^{\mathbb{C}}\mathbb{C}^2 \cong E \otimes H \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ where we read the right $\mathbb{C}^2$ factor as $H^0(\mathbb{P}^1, \mathcal{O}(1))$ via $(a,b) \mapsto a + b\zeta$. The $(0,1)$ vectors for I_ζ consist then of the subspace $\mathbb{C}^2 \otimes \{\text{sections of } \mathcal{O}(1) \text{ vanishing at } \zeta\}$. and we deduce from this that the forms of type $(1,0)$ for each complex structure I_ζ are given by $dz_1{+}\zeta d\bar{z}_2$ and $dz_2{-}\zeta d\bar{z}_1$. The triple $(\zeta, z_1{+}\zeta\bar{z}_2, z_2{-}\zeta\bar{z}_1)$ therefore gives a system of coordinates on the twistor space $Z := \mathbb{P}^1 \times \mathbb{C}^2$ holomorphic for the twisted complex structure $\underline{I}_{(\zeta,x)} = \Big((I_\zeta)_x, (I_0)_\zeta\Big)$ of Z. Straightforward computations show that the holomorphic section μ of $\mathcal{O}(2)$ over Z is given by

$$\mu(\zeta) = i(z_1 + \zeta\bar{z}_2)(z_2 - \zeta\bar{z}_1), \tag{1.3.5}$$

i.e. on each fibre of Z, $\mu(\zeta)$ is i times the product of the ζ-holomorphic coordinates. We remark here that $-i\mu(\zeta)$ is a section of $\mathcal{O}(2)$ which is real for the real structure

$$\begin{aligned} \mathcal{O}(2) &\to \mathcal{O}(2) \\ (\zeta, \eta) &\mapsto \left(-\frac{1}{\bar{\zeta}}, -\frac{\bar{\eta}}{\bar{\zeta}^2}\right). \end{aligned} \tag{1.3.6}$$

Remark 1.15. The more general theory of *pluricomplex geometry* described in [10] tells us that pullback $F^*\mathcal{V}^{0,1}$ of the $-i$-tautological eigenbundle on $Gr(1,2)$ is isomorphic to $\mathbb{C}^2 \otimes \mathcal{O}(-1)$. In general for a hypercomplex manifold

$$\begin{aligned} H^0(P_m, T_F(-1)) \otimes \mathcal{O}(-1) &\to H^0(P_m, T_F) \\ (s, (a,b)) &\mapsto (-a + b\zeta)s \end{aligned} \tag{1.3.7}$$

is an isomorphism.

1.4 Manifolds of higher degree curves in Twistor Spaces

Let Z be the twistor space of some hyperkähler manifold M: according to Definition 1.7 a twistor line is an immersed $\mathbb{P}^1$ inside Z. In this section we will consider a complex manifold Z of complex dimension $n+1$, which by analogy we will keep calling a *twistor space*, endowed with a fibration $p\colon Z \to T\mathbb{P}^1$ and a *real structure* σ that covers the antipodal map of the Riemann sphere $\mathbb{P}^1$ and study the relevant geometric properties of the manifold M of "higher degree" curves in Z that satisfy appropriate cohomological conditions of stability. We now make the setting precise, following [6] as main source.
As usual let $\mathbb{P}^1$ be the complex projective line and consider a complex manifold Z, of complex dimension $\dim_{\mathbb{C}} Z = n+1$, fibring over the holomorphic tangent bundle to $\mathbb{P}^1$ via a holomorphic map $p\colon Z \to T\mathbb{P}^1$. Assume also that a *real structure*, that is a antiholomorphic involution, be defined on Z and that it covers the antipodal map of $\mathbb{P}^1$. By *curve* $C \subset Z$ we shall now mean a compact, complex, one-dimensional subspace whose fibres over $\mathbb{P}^1$ all have equal length (we are interested in those subspace which have fibres of *finite* length). Let $d \in \mathbb{N}\backslash_{\{0\}}$ and M be the space of all curves $C \subset Z$ that satisfy the following requirements:

- C is invariant under σ
- $p|_C$ is a biholomorphism $C \to \bar{C} = p|_C(C) \subset T\mathbb{P}^1$
- $\bar{C}$ is of degree d
- The cohomology $H^*(C, N_{C/Z}(-2))$ vanishes and $H^1(C, N_{C/Z}) = 0$ as well,

where we adopt the following conventions and notations:

- $N_{C/Z}$ stands for the sheaf of sections of the normal bundle of C in Z.
- $N_{C/Z}(-2)$ stands for $N_{C/Z} \otimes \mathcal{O}_Z(-2)$, where $\mathcal{O}_Z(-2)$ is the pullback of $\mathcal{O}_{\mathbb{P}^1}(-2)$ over Z via the projection $\tilde{p}\colon Z \to \mathbb{P}^1$ given by composing p with the base map $T\mathbb{P}^1 \to \mathbb{P}^1$.
- we say that $\bar{C}$ has degree d meaning that the projection $\bar{C} \to \mathbb{P}^1$ is generically $d\colon 1$. This is equivalent to saying that $\bar{C}$ is the vanishing locus of a global section of $\mathcal{O}_{T\mathbb{P}^1}(2d)$, i.e of a polynomial of the form $\eta^d + a_1(\zeta)\eta^{d-1} + \cdots + a_d(\zeta)$ with (ζ, η) the usual local coordinates on $T\mathbb{P}^1$ and $a_i(\zeta)$ a polynomial in ζ of degree $2i$ for each $i = 1, \ldots, d$.

We are now ready to describe the geometry of the manifold M.
Let $M^{\mathbb{C}}$ denote the *complexification* of M, which is described exactly as above simply by removing the hypothesis that the curves C be σ-invariant and let $C \in M$. By a general result of

deformation theory (see [18]) there is a canonical isomorphism $T_C\left(M^{\mathbb{C}}\right) \cong H^0(C, N_{C/Z})$ and the condition $H^1(C, N_{C/Z}) = 0$ implies that M is smooth at C, hence we will call M a *manifold* from now on. Here we denote by $T_C\left(M^{\mathbb{C}}\right)$ the holomorphic tangent space to $M^{\mathbb{C}}$ at C, hence $T_C\left(M^{\mathbb{C}}\right) \cong T_C M \otimes \mathbb{C}$. Now, recalling the *Euler sequence*

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \otimes \mathbb{C}^2 \to \mathcal{O} \to 0 \tag{1.4.1}$$

and the fact that $H^0(\mathbb{P}^1, \mathcal{O}(1)) \cong \mathbb{C}^2$, we get the analogous exact sequence

$$0 \to N_{C/Z}(-2) \to N_{C/Z}(-1) \otimes H^0(\mathbb{P}^1, \mathcal{O}(1)) \to N_{C/Z} \to 0. \tag{1.4.2}$$

Taking the cohomology of the sequence, together with the vanishing of $H^*(C, N_{C/Z}(-2))$, yields the isomorphism

$$H^0(C, N_{C/Z}) \cong H^0(C, N_{C/Z}(-1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(1)) \cong H^0(C, N_{C/Z}(-1)) \otimes \mathbb{C}^2. \tag{1.4.3}$$

We can therefore define an action of the unit quaternions on $T^{\mathbb{C}}M$ as in the previous chapter. Let I_1, I_2, I_3 be the complex structures on $\mathbb{C}^2$ defined as in (1.2.3) by

$$I_1(a,b) = (ia, -ib), \quad I_2(a,b) = (-b, a), \quad I_3(a,b) = (-ia, -ib). \tag{1.4.4}$$

These correspond to left multiplication by the unit quaternions i, j, k under the identification $\mathbb{C}^2 \cong \mathbb{H}$ by $(a,b) \mapsto a + jb$. A whole 2-sphere of complex structures is therefore defined on $\mathbb{C}^2$ by

$$I_\zeta = \frac{1}{1+\zeta\bar{\zeta}}\left((1-\zeta\bar{\zeta})I_1 - i(\zeta - \bar{\zeta})I_2 - (\zeta + \bar{\zeta})I_3\right). \tag{1.4.5}$$

With some abuse of notation we define the $\mathbb{P}^1$ of complex structures on $T^{\mathbb{C}}M$ by acting on the $\mathbb{C}^2$ factor in the splitting (1.4.3) by

$$\begin{aligned} I_\zeta \colon H^0(C, N_{C/Z}(-1)) \otimes \mathbb{C}^2 &\to H^0(C, N_{C/Z}(-1)) \otimes \mathbb{C}^2 \\ e \otimes (a,b) &\mapsto e \otimes I_\zeta(a,b). \end{aligned} \tag{1.4.6}$$

The subspace $\left(T_C^{0,1}M\right)_{I_\zeta} \subset T_C^{\mathbb{C}}M$ consisting of all vectors tangent to M at C of type $(0,1)$ is therefore the tensor product $H^0(C, N_{C/Z}(-1)) \otimes \left(T^{0,1}\mathbb{C}^2\right)_{I_\zeta}$. If we proceed as in Example 1.1 we obtain that $\left(T_C^{0,1}M\right)_{I_\zeta} \cong H^0(C, N_{C/Z}(-1)) \otimes (-a, b)$, i.e. by virtue of (1.4.3) and of the isomorphism $\mathbb{C}^2 \cong H^0(\mathbb{P}^1, \mathcal{O}(1))$, $(a,b) \mapsto a + b\zeta$ the $(0,1)$ tangent space at C is exactly the space $H^0(C, N_{C/Z}[-C_\zeta])$ consisting of all sections of $H^0(C, N_{C/Z})$ that vanish on the divisor $C_\zeta = C \cap Z_\zeta$, where Z_ζ is the fibre of Z over the point $\zeta \in \mathbb{P}^1$ along the map $Z \to T\mathbb{P}^1 \to \mathbb{P}^1$. Following [6] we will denote by this distribution by Q_ζ and state the next result.

Proposition 1.16. *The distribution Q_ζ defined above is integrable*

Proof. The proof is given in [6][Proposition 3.4] ∎

A hypercomplex structure arises therefore naturally on the manifold M. We now turn our attention to computing the dimension of M.
Fix $\zeta \in \mathbb{P}^1$ and a curve $C \in M$ and let $s \in H^0(\mathbb{P}^1, \mathcal{O}(1))$ such that $s(\zeta) = 0$. Taking the cohomology of the following exact sequence

$$0 \longrightarrow N(-2) \xrightarrow{\cdot s} N(-1) \xrightarrow{ev_\zeta} N(-1)|_{C_\zeta} \longrightarrow 0 \tag{1.4.7}$$

yields an isomorphism

$$H^0(C, N_{C/Z}(-1)) \cong H^0(C_\zeta, N_{C/Z}(-1)|_{C_\zeta}). \tag{1.4.8}$$

Since each fibre of $N_{C/Z}$ has rank n, then $h^0(C, N_{C/Z}(-1)) = dn$. Therefore (1.4.3) implies that $T_C^{\mathbb{C}}M$ is a vector space of complex dimension $2dn$ which means that $T_C M$ is a real vector space of real dimension $2dn$, i.e. when M is not empty then $\dim_{\mathbb{R}} M = 2dn$.
We can therefore summarize all the above results and reproduce [6][Theorem 3.5] as follows.

Theorem 1.17. *Let Z be a $(n+1)$-complex dimensional complex manifold endowed with a holomorphic fibration $p\colon Z \to T\mathbb{P}^1$ and a real structure σ covering the antipodal map of $\mathbb{P}^1$. The manifold M of σ-invariant curves of degree d that satisfy the cohomological conditions $H^*(C, N_{C/Z}(-2)) = 0 = H^1(C, N_{C/Z})$ is, if not empty, a hypercomplex manifold of real dimension $2dn$.*

Remark 1.18. Let $\mathcal{J}(T^{\mathbb{C}}M)$ denote the space of all possible complex structures on TM. The definition (1.4.6) of the 2-sphere of complex structures on M gives at every $C \in M$ a holomorphic embedding $F\colon \mathbb{P}^1 \to \mathcal{J}(T_C^{\mathbb{C}}M)$ or, equivalently, $F\colon \mathbb{P}^1 \to Gr_{dn}(C^{2dn})$ as giving a complex structure is equivalent to specifying its $-i$-eigenbundle. Let then $\mathcal{V}^{0,1}$ denote the tautological $-i$-eigenbundle on $Gr_{dn}(C^{2dn})$. In analogy to Remark 1.15 and [10], the pullback $F^*\mathcal{V}^{0,1}$ is isomorphic to $H^0(C, N_{C/Z}(-1)) \otimes \mathcal{O}(-1)$ via the map

$$\begin{aligned} H^0(C, N_{C/Z}(-1)) \otimes \mathcal{O}(-1) &\to H^0(C, N_{C/Z}) \otimes \mathcal{O} \\ (t, (a,b)) &\mapsto (-a + b\zeta)t \end{aligned} \tag{1.4.9}$$

where $(a,b) \in l$ and l is the fibre of $\mathcal{O}(-1)$ over $[l]$.

Remark 1.19. Thanks to (1.4.3) we have been able to equip M with a hypercomplex structure: if we now construct the twistor space Z^{HK} of M following Hitchin's hyperkähler theory, we have that at every $m \in M$

$$T_m^{\mathbb{C}} M \cong H^0(c_m, N_{c_m/Z^{HK}}) \cong H^0(c_m, T_F(-1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(1)), \tag{1.4.10}$$

where c_m denotes the section $\mathbb{P}^1 \to Z^{HK}$ associated to m, reconstructs the same hypercomplex structure on M. Therefore we get a natural isomorphism

$$H^0(C, N_{C/Z}(-1)) \cong H^0(c_m, N_{c_m/Z^{HK}}(-1)) \cong H^0(c_m, T_F(-1)), \tag{1.4.11}$$

where C is as usual the curve in $Z \to T\mathbb{P}^1$ corresponding to $m \in M$. Therefore, since $T_F(-1)$ is trivial on every twistor line, we have an induced vector bundle E on M, whose fibre at $m \in M$ is $H^0(c_m, T_F(-1)) \cong H^0(C, N_{C/Z}(-1))$ (see for example [25]).
Now, following [24], we recall here that the only (anti)linear action on $\mathcal{O}(1)$ that covers the antipodal map is the natural *quaternionic structure*, which induces the map

$$\begin{aligned} j_H \colon H^0(\mathbb{P}^1, \mathcal{O}(1)) &\to H^0(\mathbb{P}^1, \mathcal{O}(1)) \\ (a + b\zeta) &\mapsto (-\bar{b} + \bar{a}\zeta) \end{aligned} \tag{1.4.12}$$

that is also a quaternionic structure. Since the real structure σ on Z induces a real structure, that we still denote by σ, on $H^0(C, N_{C/Z})$ and $H^0(C, N_{C/Z}) \cong H^0(C, N_{C/Z}(-1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(1))$, then there exists a quaternionic structure $j_E \colon H^0(C, N_{C/Z}(-1)) \to H^0(C, N_{C/Z}(-1))$ such that $\sigma = j_E \otimes j_H$ as a map of $H^0(C, N_{C/Z}(-1)) \otimes \mathbb{C}^2$ to itself.
This gives a description of the real tangent space to M at C as the space $H^0(C, N_{C/Z})^\sigma$ of all σ-invariant global sections of $N_{C/Z}$. In terms of vector bundles over M, then, $T^{\mathbb{R}} M \cong (E \otimes H)^\sigma$, where H is the trivial $\mathbb{C}^2$ bundle on M.

Observe now that for any $C \in M$ and $\zeta \in \mathbb{P}^1$, the divisor C_ζ is a complex space of complex dimension 0 and length d, i.e. consists of d points, counting multiplicity. This suggests the upcoming quick digression about (transverse) *Hilbert schemes of points*. For a complex manifold X we define the full Hilbert scheme of d points in the following way.

Definition 1.20. *Let X be a complex manifold and $d \in \mathbb{N}\backslash_0$. The Hilbert scheme $X^{[d]}$ of d points of X is the set of all 0-dimensional analytic subspaces of X of length d. In other words (see [1]), an element of $X^{[d]}$ is by definition either*

- *an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ with $\dim \mathcal{O}_x/\mathcal{I} = d$*
- *a sheaf $\mathcal{S}$ of $\mathcal{O}_X$-modules with finite support, cyclic and with $\dim H^0(X, \mathcal{S}) = d$.*

Following and adopting now the notation of [8], let X be a complex manifold, C a complex one-dimensional manifold and $p\colon X \to C$ a surjective holomorphic map. We define the Hilbert scheme of d points in X *transverse* to p as follows.

Definition 1.21. *The length d Hilbert scheme of X transverse to p is the (open) subset $X_p^{[d]}$ of the full Hilbert scheme $X^{[d]}$ consisting of all 0-dimensional subschemes Z of length d such that the map $p|_Z\colon Z \to p(Z)$ is an isomorphism onto the scheme-theoretic image.*

Remark 1.22. A practical interpretation of the transversality condition is the following: taken $Z \in X_p^{[d]}$, if $z \in p(Z) \subset C$ is a point of multiplicity k, then it will correspond via p to a point $x \in p^{-1}(z)$ also of multiplicity k. A scenario with distinct points $x_1, \ldots, x_j \in p^{-1}(z)$ of multiplicity $k_1, \ldots, k_j$ respectively, with $\sum_{i=1}^{j} k_i = k$ is excluded.

If, going back to our notation, we consider the fibre Z_ζ of the twistor space Z over ζ as playing the role of the manifold X of Definition 1.21 and $T_\zeta\mathbb{P}^1 \cong \mathbb{C}$ for the role of the one-dimensional manifold C, then the same argument as in [8][Proposition3.7] yields the following result.

Proposition 1.23. *Let the couple (M, I_ζ) denote the manifold M endowed with the complex structure I_ζ for a chosen $\zeta \in \mathbb{P}^1$. The map*

$$\begin{aligned} \Psi_\zeta\colon (M, I_\zeta) &\to (Z_\zeta)_p^{[d]} \\ C &\mapsto C_\zeta \end{aligned} \tag{1.4.13}$$

is holomorphic and it is a local diffeomorphism of the smooth locus of $(Z_\zeta)_p^{[d]}$.

Remark 1.24. By Fogarty's results exposed in [15], [16], if Z_ζ is 2-dimensional then $(Z_\zeta)^{[d]}$ is everywhere smooth, hence so is $(Z_\zeta)_p^{[d]}$.

We conclude the section by showing that, under some natural hypothesis on Z, a hyperkähler metric can be defined on M.
The composition $\tilde{p}$ of the projection $p\colon Z \to T\mathbb{P}^1$ with the base map $T\mathbb{P}^1 \to \mathbb{P}^1$ gives a map $Z \to \mathbb{P}^1$. We denote by $T_{Z/\mathbb{P}^1}$ the tangent space to the fibres of such map and state the analogue to [6][Theorem 3.11].

Theorem 1.25. *Assume Z be a complex manifold of complex dimension 3 endowed with a fibration $p\colon Z \to T\mathbb{P}^1$ and an antiholomorphic involuction σ that covers the real structure of $\mathbb{P}^1$. Assume, also, that a symplectic form $\omega(\zeta)$ with values in $\mathcal{O}(2)$ is given on the bundle $T_{Z/\mathbb{P}^1}$ and that ω is compatible with the real structure σ, inducing a a symplectic structure on each fibre Z_ζ. Then the hypercomplex manifold M has a (pseudo)-hyperkähler metric.*

We now return once more to the splitting $T^{\mathbb{C}}M \cong E \otimes H$, where E is the bundle on M defined in Remark 1.5. A result of Pedersen-Poon stated in [32] for hyperkähler manifolds and improved by Feix in [14] for hypercomplex manifold states that a correspondence exists between hyperholomorphic bundles on a hyperkähler (hypercomplex) manifold and holomorphic bundles over its twistor space that fulfil the condition of being trivial on every twistor line. Therefore, since the bundle $T_F(-1)$ on Z^{HK} satisfies such condition, it can be endowed with a (unique) hyperholomorphic connection ∇^E. Moreover, since $T^{\mathbb{C}}M \cong E \otimes H$, we can consider the tensor product connection $\nabla = \nabla^E \otimes \nabla^H$, where ∇^H is the trivial flat connection on H. By definition the complex structures I_i, $i = 1, 2, 3$ are preserved by ∇ and [14][Proposition 6] states the following.

Proposition 1.26. *The affine connection $\nabla = \nabla^E \otimes \nabla^H$ has vanishing torsion.*

Therefore ∇ is the (complexified) *Obata-connection* of the manifold M.
Since E is the vector bundle over M induced by the bundle $T_F(-1)$ over Z^{HK}, we can also apply [6][Proposition 6.1] and get the following statement.

Proposition 1.27. *The vector bundle $E \to M$ is equipped with a canonical linear connection ∇'^E such that, for every $\zeta \in \mathbb{P}^1$ and point $m \in M$, if u is a local section of E with $du(X) = 0$ for every $X \in Q_\zeta$ then also $\nabla_X u = 0$ for every such X. Moreover, since Z^{HK} has a real structure covering the antipodal map of $\mathbb{P}^1$, the connection ∇'^E satisfies $\left(\nabla'^E\right)^{0,1}_\zeta = \bar{\partial}_\zeta$ for every $\zeta \in \mathbb{P}^1$.*

Remark 1.28. Let $\Pi \subset M$ be the subset consisting of all twistor lines that run through a given point $z \in Z^{HK}$. Comparing the results of Pedersen-Poon [32] and Bielawski [6], it is immediate to verify that the connections ∇^E and ∇'^E have the same horizontal distribution on Π. Since ∇^E is extended in a unique way to all M, it follows that $\nabla^E = \nabla'^E$. Therefore the Obata connection is

$$\nabla = \nabla^E \otimes \nabla^H = \nabla'^E \otimes \nabla^H. \tag{1.4.14}$$

Finally, we show that the dual connection $(\nabla'^E)^*$ has the same property of ∇'^E.

Proposition 1.29. *Let E^* be the dual bundle to E and $(\nabla^E)^*$ the dual connection induced by ∇^E on E^*. Then, if X is a vector in Q_ζ and φ a local section of E^* for which $d\varphi(X) = 0$ then $(\nabla'^E)^*_X \varphi = 0$.*

Proof. We recall that, by definition, if u is a local section of E

$$\begin{aligned} \left((\nabla'^{E})^{*}_{X}\varphi\right)(u) &= d_X(\varphi(u)) - \varphi\left(\nabla'^{E}_{X}u\right) \\ &= (d_X\varphi)(u) + \varphi(d_Xu) - \varphi(\nabla'^{E}_{X}u) \\ &= \varphi(d_Xu - \nabla'^{E}_{X}u). \end{aligned} \tag{1.4.15}$$

Now, let $(e_1, \dots, e_{2d})$ be a local frame for E such that $de_i = 0$, that is in particular $d_Xe_i = 0$. Writing $u = \sum u_ie_i$, we get

$$\left((\nabla'^{E})^{*}_{X}\varphi\right)(u) = \sum \varphi\left(u_i\left(d_X - \nabla'^{E}_{X}\right)e_i\right) = 0, \tag{1.4.16}$$

hence the claim is verified. ∎

By the above proposition we conclude that the connection induced by ∇'^{E} on the vector bundles generated by E via direct sum, tensor product and dualisation all share the same defining property as in Proposition 1.27.

1.5 l-Hypercomplex structures

Hypercomplex structures happen to carry more information than just the triple of quaternion-like behaving complex structures as in Definition 1.1. As we have already anticipated in Remarks 1.15 and 1.18, the study of hypercomplex structures is contained in the more general frame of that of pluricomplex structures. Although here we are not interested in the latter, we adopt its point of view for this section and give a new and equivalent definition of hypercomplex structure on a vector space.

Definition 1.30. *Let V be a vector space of real dimension $2n$. Denote by $\mathcal{J}(V)$ the space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ of all complex structures of V, which we call the* twistor space *of V and view as an open dense subset of the Grassmannian $Gr_n(V^{\mathbb{C}})$. A* hypercomplex structure *on V is an immersion*

$$K : \mathbb{P}^1 \to \mathcal{J}(V) \tag{1.5.1}$$

that satisfies the following conditions

- $K^*(\mathcal{V}^{0,1}) \cong \mathcal{O}(-1) \otimes \mathbb{C}^n$, *where $\mathcal{V}^{0,1}$ is the tautological $-i$-eigenbundle on $Gr_n(V^{\mathbb{C}})$*
- $V^{\mathbb{C}}/K^*(\mathcal{V}^{0,1}) \cong \mathcal{O}(1) \otimes \mathbb{C}^n$

- $K(\zeta) = -K(-1/\bar{\zeta})$ *for every* $\zeta \in \mathbb{P}^1$.

These conditions force n to be even. In the spirit of the above definition we recall the following definition from [9].

Definition 1.31. *Let V be a real vector space of dimension $2n$. A* l-hypercomplex structure *on V is a map $A\colon \mathbb{P}^1 \to T\mathcal{J}(V)$ such that $\pi \circ A$ is a hypercomplex structure, where we denoted by π the projection $T\mathcal{J}(V) \to \mathcal{J}(V)$.*

The tangent space at a point $J \in \mathcal{J}$ is described as

$$T_J\mathcal{J}(V) = \{X \in \mathrm{End}(V) |\, XJ + JX = 0\}\,, \tag{1.5.2}$$

i.e. it is the set of all endomorphisms of V that anti-commute with the complex structure J. This means that for a l-hypercomplex structure A, every $A(\zeta)$ anti-commutes with $K(\zeta)$.

Remark 1.32. A l-hypercomplex structure is a lift of a hypercomplex structure to the tangent bundle $T\mathcal{J}(V)$. Every hypercomplex structure defines naturally the trivial l-hypercomplex structure as the zero section but, as shown in [9], l-hypercomplex structures are usually non-trivial.

Consider now a non-trivial l-hypercomplex structure A, denote by K the underlying hypercomplex structure $K = \pi \circ A$ and take $\zeta \in \mathbb{P}^1$. If we extend $A(\zeta)$ to the complexification $V^{\mathbb{C}}$ of V by linearity, we can consider the map $\tilde{A}(\zeta)\colon V_\zeta^{0,1} \to V^{\mathbb{C}}/V_\zeta^{0,1}$ given by restricting $A(\zeta)$ to $V_\zeta^{0,1}$ and then taking the natural projection to the quotient. As usual $V_\zeta^{0,1}$ stands for the space of vectors of type $(0,1)$ for $K(\zeta)$. Varying ζ, we have a bundle map $\tilde{A}\colon \mathcal{V}^{0,1} \to V^{\mathbb{C}}/\mathcal{V}^{0,1}$ over the image space $K(\mathbb{P}^1) \subset \mathcal{J}(V)$.
If we set $L_\zeta = \mathrm{Span}\left\{K(\zeta') \in K(\mathbb{P}^1) |\, K(\zeta') \circ K(\zeta) + K(\zeta) \circ K(\zeta') = 0\right\}$ then L_ζ is the fibre of a holomorphic line bundle L over $K(\mathbb{P}^1)$, with $L \cong T\mathbb{P}^1$. Define $Tot(L)$ to be the total space of L and pull the bundles $\mathcal{V}^{0,1}$ and $V^{\mathbb{C}}/\mathcal{V}^{0,1}$ back to $Tot(L)$. We will abuse notation and still denote such pullbacks by $\mathcal{V}^{0,1}$ and $V^{\mathbb{C}}/\mathcal{V}^{0,1}$. Since every element $K(\zeta') \in L_\zeta$ anti-commutes with $K(\zeta)$, it defines a map $V_\zeta^{0,1} \to V^{\mathbb{C}}/V_\zeta^{0,1}$. Therefore, on $Tot(L) \cong T\mathbb{P}^1$ we obtain a tautological map $\mathcal{I}\colon \mathcal{V}^{0,1} \to V^{\mathbb{C}}/\mathcal{V}^{0,1}$. The situation is summarized by the following short exact sequence (of sheaves)

$$0 \longrightarrow \mathcal{V}^{0,1} \xrightarrow{\tilde{A}-\mathcal{I}} V^{\mathbb{C}}/\mathcal{V}^{0,1} \longrightarrow \mathcal{F} \longrightarrow 0. \tag{1.5.3}$$

Finally, let I be a locally trivializing section of $L \cong T\mathbb{P}^1$ and η the fibre coordinate relative to it. Then (1.5.3) can be seen as a sequence of sheaves on $T\mathbb{P}^1$ given as

$$0 \longrightarrow \mathcal{O}_{T\mathbb{P}^1}(-1) \otimes \mathbb{C}^n \xrightarrow{\tilde{A}(\zeta)-\eta I} \mathcal{O}_{T\mathbb{P}^1}(1) \otimes \mathbb{C}^n \longrightarrow \mathcal{F} \longrightarrow 0 \tag{1.5.4}$$

where, as usual, $\mathcal{O}_{T\mathbb{P}^1}(1)$ denotes the pullback of $\mathcal{O}(1)$ via the projection $T\mathbb{P}^1 \to \mathbb{P}^1$.
Observe that $\tilde{A}$ is compatible with the natural real structure $(\zeta,\eta) \mapsto (-1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2)$, that is $\tilde{A}(1/\bar{\zeta}) = -\overline{\tilde{A}(\zeta)}/\bar{\zeta}^2$. Moreover, if we consider the generic fibre of $\mathcal{V}^{0,1}$ and $V^{\mathbb{C}}/\mathcal{V}^{0,1}$ over a point $(\zeta,\eta) \in T\mathbb{P}^1$, for an appropriate choice of basis $\mathcal{I}$ is represented by the identity matrix and $\tilde{A}(\zeta)$ by an $n \times n$ complex matrix. Therefore $\tilde{A}(\zeta)$ can have at most n different eigenvalues, which means that $\tilde{A}(\zeta) - \eta\mathcal{I}$ is an isomorphism of the generic fibre i.e. the sheaf $\mathcal{F}$, which we call the *characteristic sheaf* of the l-hypercomplex structure A, is supported on the curve

$$C = \left\{(\zeta,\eta) \in T\mathbb{P}^1 |\ \ker(\tilde{A}(\zeta) - \eta\mathcal{I}) \neq 0\right\}. \tag{1.5.5}$$

We call C the *characteristic curve* of A and its degree the degree of A. As remarked in [9], the sheaf $\mathcal{F}$ satisfies the conditions $h^0(\mathcal{F}) = 2n$ and $H^*(\mathcal{F} \otimes \mathcal{O}_{T\mathbb{P}^1}(-2)) = 0$ and its space of sections is equipped with an antilinear automorphism σ that squares to the identity and covers the real structure of $T\mathbb{P}^1$. The significance of the sheaf $\mathcal{F}$ comes from the possibility of recovering the l-hypercomplex structure $A(\zeta)$ by knowing $\mathcal{F}$, applying the following strategy. First we identify the vector space V with the σ-invariant sections of $\mathcal{F}$. Then we observe that, due to the vanishing of the cohomology $H^*(\mathcal{F} \otimes \mathcal{O}_{T\mathbb{P}^1}(-2))$, for any choice of two elements $\zeta_0, \zeta_1 \in \mathbb{P}^1$, we have the isomorphism $H^0(\mathcal{F}) \cong H^0(D_{\zeta_0}, \mathcal{F}) \oplus H^0(D_{\zeta_1}, \mathcal{F})$, where D_{ζ_i} stands for the divisor cut out by $(\zeta - \zeta_i)$. In turn, this implies that the space $H^0(\mathcal{F})^\sigma$ of σ-invariant sections of $\mathcal{F}$ be isomorphic to $H^0(D_{\zeta_0}, \mathcal{F})$ by evaluation at a point $\zeta_0 \in \mathbb{P}^1$ and this determines the complex structure J_{ζ_0} corresponding to $\zeta_0 \in \mathbb{P}^1$. Assume now for simplicity that $C_{\zeta_0} = C \cap D_{\zeta_0}$ consists of d distinct points $u_1, \ldots, u_d$, for $d = deg(C)$, and identify the tangent space $T_{\zeta_0}\mathbb{P}^1$ with the complex line spanned by all complex structures J_ζ that anti-commute with J_{ζ_0}. Let J' be a complex structure fixed among those anti-commuting with J_{ζ_0}: then we can identify the points $u_1, \ldots, u_d$ with the elements $\eta_1 J', \ldots, \eta_d J'$ for some $\eta_i \in \mathbb{C}, i = 1, \ldots, d$. We have therefore that

$$V \cong H^0(D_{\zeta_0}, \mathcal{F}) \cong H^0(C_{\zeta_0}, \mathcal{F}) \cong \bigoplus_{i=1}^{d} H^0(\{u_i\}, \mathcal{F}). \tag{1.5.6}$$

Let π_i be the projection to the i-th summand in (1.5.6). Since every summand is invariant under J_{ζ_0}, they can be thought as complex subspaces of V for the complex structure J_{ζ_0}. Therefore, for every $v \in V$, we recover $A(\zeta)$ as

$$A(\zeta_0)(v) = \sum_{i}^{d} \eta_i \pi_i(J'v). \tag{1.5.7}$$

Remark 1.33. For $d = 1$ the l-hypercomplex structure is just an ordinary hypercomplex structure and we have $V \cong H^0(u, \mathcal{F})$, with $u = C_{\zeta_0}$ for the characteristic curve C of the sheaf $\mathcal{F}$.

Therefore, after fixing ζ_0 and choosing a complex structure J' which anti-commutes with J_{ζ_0}

$$A(\zeta_0)(v) = \eta J'. \tag{1.5.8}$$

Chapter 2

A Matrix Polynomial

Contents

In this chapter we consider the manifold M introduced in Section 1.4 and, for every element $C \in M$, we construct a matrix polynomial $A(\zeta)$ as a holomorphic section of the bundle $\mathrm{End}(H^0(C, N_{C/Z}(-1))) \otimes \mathcal{O}(2)$. We show that this can be interpreted as a l-hypercomplex structure according to description of Section 1.5. We prove then that $A(\zeta)$ is real for the natural real structure of $\mathcal{O}(2)$ and, recalling Remark 1.19, we show that it induces an endomorphism of the bundle E on M which, for every fixed $\zeta \in \mathbb{P}^1$, is holomorphic with respect to the complex structure I_ζ. We will still call this endomorphism $A(\zeta)$.

2.1 Construction

As in Section 1.4 let $p\colon Z \to T\mathbb{P}^1$ be our complex $(n+1)$-dimensional twistor space, σ its real structure covering the antipodal map of $\mathbb{P}^1$ and M the manifold of σ-invariant degree d curves in Z that satisfy the stability conditions $H^1(C, N_{C/Z}) = H^*(C, N_{C/Z}(-2)) = 0$. We now put the isomorphism (1.4.8) to use and apply a construction of Adams-Harnad-Hurtubise exposed in [1] which yields a linear endomorphism $A(\zeta)$ of $H^0(C, N_{C/Z}(-1))$ defined by a quadratic matricial polynomial.

Let $\zeta \in \mathbb{P}^1$. Multiplication by η on $H^0(C_\zeta, N_{C/Z}(-1)|_{C_\zeta})$ yields the commutative diagram

$$\begin{array}{ccc} H^0(C, N_{C/Z}(-1)) & \longrightarrow & H^0(C_\zeta, N_{C/Z}(-1)|_{C_\zeta}) \\ A(\zeta)\Big\downarrow & & \Big\downarrow \eta \\ H^0(C, N_{C/Z}(-1)) & \longrightarrow & H^0(C_\zeta, N_{C/Z}(-1)|_{C_\zeta}) \end{array} \tag{2.1.1}$$

If z_i is a point of C_ζ of multiplicity one, with $p(z_i) = (\zeta, \eta_i)$, then clearly we have $(\eta(s))(z_i) = \eta(z_i)s(z_i) = \eta_i s(z_i)$. If, instead, z_i is a point of higher multiplicity in C_ζ, the section s is given by its truncated power series centred in z_i and $(\eta(s))$ will be given by the truncated power series of the product ηs centred in z_i.
Therefore, if C_ζ consists of d distinct points (z_i) with $p(z_i) = (\zeta, \eta_i)$, then $A(\zeta)$ will be diagonalizable with eigenvalues $\eta_i, i = 1, \dots, d$ and sections of $N_{C/Z}(-1)$ vanishing at $z_j, j \neq i$ will be eigenvectors for η_i. If, instead, C_ζ has multiple points then $A(\zeta)$, given in this case via multiplication on truncated power series, will have a non-diagonal Jordan form.

Remark 2.1. Let $P(\zeta, \eta)$ be the polynomial defining the curve $\bar{C} = p(C) \subset T\mathbb{P}^1$ and denote by $ch_{A(\zeta)}(\eta)$ the characteristic polynomial of the endomorphism $A(\zeta)$ of $H^0(C, N_{C/Z}(-1))$. By the above construction we observe that

$$ch_{A(\zeta)}(\eta) = P(\zeta, \eta). \tag{2.1.2}$$

We now have to show that $A(\zeta)$ can be expressed as a quadratic polynomial with matricial coefficients. To achieve this, we first need the following technical proposition.

Proposition 2.2. *For $\zeta \in \mathbb{P}^1$ let $C_\zeta^{(j)}$ denote the j-th* formal neighbourhood *of C_ζ. The restriction map yields the isomorphism*

$$H^0(C, N_{C/Z}(j-1)) \cong H^0(C_\zeta^{(j)}, N_{C/Z}(j-1)|_{C_\zeta}). \tag{2.1.3}$$

Proof. Exactly as in [1], it is sufficient to tensor

$$0 \longrightarrow \mathcal{O}_C(-1) \xrightarrow{\zeta^{j+1}} \mathcal{O}_C(j) \longrightarrow \mathcal{O}_{C_\zeta^{(j)}}(j) \longrightarrow 0 \tag{2.1.4}$$

with $N_{C/Z}(-1)$ and take the long cohomology sequence. Again the result is a consequence of $H^*(C, N_{C/Z}(-2)) = 0$. ∎

Mimicking the proof of [1][Proposition 2.5] we are now able to show the following.

Proposition 2.3. $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$, *where $A_i\colon H^0(C, N_{C/Z}(-1)) \to H^0(C, N_{C/Z}(-1))$ is a linear endomorphism.*

Proof. The space $H^0(T\mathbb{P}^1, \mathcal{O}_{T\mathbb{P}^1}(2))$ is generated by $1, \zeta, \zeta^2, \eta$. Considering the Taylor expansion $A(\zeta) = \sum_{i=1}^2 A_i\zeta^i + R(\zeta)$, we have that the truncated series $\sum_{i=1}^2 A_i\zeta_i$ defines a map $H^0(C, N_{C/Z}(-1)) \to H^0(C, N_{C/Z}(1))$. In the same way, also multiplication by η defines a mapping $H^0(C, N_{C/Z}(-1)) \to H^0(C, N_{C/Z}(1))$. Taking the restriction to $C_0^{(2)}$ we have two maps $H^0(C, N_{C/Z}(-1)) \to H^0(C_0^{(2)}, N_{C/Z}(1))$ that coincide due to their definition in (2.1.1). Proposition 2.2 implies that $R(\zeta) = 0$, therefore the multiplication by η and $\sum_{i=1}^2 A_i\zeta^i$ coincide over C. ∎

Remark 2.4. By considering a section s of $\mathcal{O}(2)$ vanishing at ζ_1, ζ_2, in analogy to (1.4.7) we have

$$0 \longrightarrow N(-2) \xrightarrow{\cdot s} N \longrightarrow N|_{C_{\zeta_1}} \oplus N|_{C_{\zeta_2}} \longrightarrow 0 \tag{2.1.5}$$

from which we have the cohomology isomorphism

$$H^0(C, N_{C/Z}) \cong H^0\left(C_{\zeta_1}, N_{C/Z}|_{C_{\zeta_1}}\right) \oplus H^0\left(C_{\zeta_1}, N_{C/Z}|_{C_{\zeta_2}}\right). \tag{2.1.6}$$

This implies that we can identify

$$\left(T_C^{1,0}M\right)_{I_\zeta} \cong H^0\left(C, N_{C/Z}[-C_{-1/\bar{\zeta}}]\right) \cong H^0(C_\zeta, N_{C/Z}(-1)|_{C_\zeta}) \tag{2.1.7}$$

and consider $A(\zeta)$ as a endomorphism of $\left(T_C^{1,0}M\right)_{I_\zeta}$.

Remark 2.5. The construction of $A(\zeta)$ at C yields a vector bundle endomorphism of the vector bundle E defined in Remark 1.19, that we still denote $A(\zeta)$.

Example 2.1. We recall from example 1.1 that in the case of S^1 acting on a hyperkähler manifold M we get an $\mathcal{O}(2)$-twisted moment map μ. Therefore we can define a projection p from the the twistor space Z of M to $T\mathbb{P}^1 \cong \mathcal{O}(2)$ by setting $p = -i\mu$. Now, the manifold M can be recovered as parameter space of real curves in Z of degree one that satisfy our stability conditions, as this is exactly equivalent to the hypothesis of Theorem 1.10. This means that for any curve C corresponding to a point $m \in M$, the divisor $p(C_\zeta)$ only consists of the point $(\zeta, -i\mu_m(\zeta))$. For every m, then the endomorphism $A(\zeta)$ of the corresponding two-dimensional space $H^0(C, N_{C/Z}(-1))$ is then simply given by the diagonal matrix $-i\mu_m(\zeta)\mathbb{1}_2$.

2.2 A l-hypercomplex structure

We now see how $A(\zeta)$ has an interpretation in terms of l-hypercomplex structures. Assume that $\dim_{\mathbb{C}} Z = 3$ and consider, for the moment, the manifold M of curves of degree $d = 1$ that satisfy the usual requirements of invariance and stability. We know that M is naturally a hypercomplex manifold of dimension 4 and we have for $C \in M$

$$\begin{aligned} A(\zeta)\colon H^0(C, N_{C/Z}(-1)) \otimes \mathcal{O}(-1) \to H^0(C, N_{C/Z}(-1)) \otimes \mathcal{O}(1) \\ e \otimes \alpha \mapsto A(\zeta)(e) \otimes \alpha. \end{aligned} \tag{2.2.1}$$

Fix $\zeta_0 \in \mathbb{P}^1$ and assume for simplicity and without loss of generality that $\zeta_0 = 0$, hence $I_{\zeta_0} = I_1$. We identify the map

$$\begin{aligned} H^0(C, N_{C/Z}(-1)) \otimes (0,1) \to H^0(C, N_{C/Z}(-1)) \otimes (-1,0) \\ e \otimes (0,1) \mapsto e \otimes (-1,0) \end{aligned} \tag{2.2.2}$$

with the complex structure I_2. Now, since we have fixed $d = 1$, the divisor $C_{\zeta_0} = C \cap Z_{\zeta_0}$ consists of only one point and projects via p to the point $(\zeta_0, \eta_0) \in \bar{C} = p(C)$. Setting $\eta_0 = x_0 + iy_0$, the map

$$A(\zeta_0)\colon H^0(C, N_{C/Z}(-1)) \to H^0(C, N_{C/Z}(-1)) \tag{2.2.3}$$

is given by $A(\zeta_0) = \eta_0 \mathbb{1}$ and from this we construct a map

$$\begin{aligned} \tilde{A}(\zeta_0)\colon H^0(C, N_{C/Z}(-1)) \otimes (0,1) \to H^0(C, N_{C/Z}(-1)) \otimes (-1,0) \\ e \otimes (0,1) \mapsto (x_0 + iy_0) e \otimes (-1,0) \end{aligned} \tag{2.2.4}$$

which corresponds to applying $(x_0 + iy_0)I_2$. Therefore, comparing with Remark 1.33, we see that $\tilde{A}(\zeta)$ defines a degree 1 l-hypercomplex structure on $T^{\mathbb{R}}_C M \cong H^0(C, N_{C/Z}(-1))$.
Keeping $\dim_{\mathbb{C}} Z = 3$, consider now the case of a generic $d > 1$. We recall from (1.4.8) that for every fixed $\zeta_0 \in \mathbb{P}^1$ the isomorphism $H^0(C, N_{C/Z}(-1)) \cong H^0(C_{\zeta_0}, N_{C/Z}(-1)|_{C_{\zeta_0}})$ holds. If we assume for simplicity that the divisor C_{ζ_0} consists of all distinct points, then we write $p(C_{\zeta_0}) = \{(\zeta_0, \eta_i)\}, i = 1, \ldots, d$ and get

$$H^0(C_{\zeta_0}, N_{C/Z}(-1)|_{C_{\zeta_0}}) \cong \bigoplus_{i=1}^{d} H^0((\zeta_0, \eta_i), N_{C/Z}(-1)). \tag{2.2.5}$$

Let π_i be the projection to the i-th summand in (2.2.5). Then $A(\zeta_0)$ is given as

$$A(\zeta_0)(s) = \sum_{i=1}^{d} \eta_i \pi_i(s), \tag{2.2.6}$$

for every $s \in H^0(C, N_{C/Z}(-1)) \cong H^0(C_{\zeta_0}, N_{C/Z}(-1)|_{C_{\zeta_0}})$. As before, assume without loss of generality that $\zeta_0 = 0$, i.e. the corresponding complex structure be $I_{\zeta_0} = I_1$ and identify the map

$$\begin{aligned} H^0(C, N_{C/Z}(-1)) \otimes (0,1) \to H^0(C, N_{C/Z}(-1)) \otimes (-1,0) \\ e \otimes (0,1) \mapsto e \otimes (-1,0) \end{aligned} \tag{2.2.7}$$

with the complex structure I_2. Then the map

$$\begin{aligned}\tilde{A}(\zeta_0)\colon H^0(C, N_{C/Z}(-1)) \otimes (0,1) &\to H^0(C, N_{C/Z}(-1)) \otimes (-1,0) \\ e \otimes (0,1) &\mapsto A(\zeta_0)e \otimes (-1,0)\end{aligned} \tag{2.2.8}$$

is described as

$$v \mapsto \sum_{i=1}^{d} \eta_i \pi_i I_2(v), \quad v \in T_C^{\mathbb{R}} M \tag{2.2.9}$$

after the identification

$$T_C^{\mathbb{R}} M \cong T_C^{0,1} M \cong H^0(C, N_{C/Z}(-1)) \otimes (0,1). \tag{2.2.10}$$

In other words, starting from $A(\zeta)$ we can construct $\tilde{A}(\zeta)$ as $A(\zeta) \otimes j_H$, where j_H is the standard quaternionic structure of $H = H^0(\mathcal{O}(1))$ given by $(a + b\zeta) \mapsto -\bar{b} + \bar{a}\zeta$. This map $\tilde{A}(\zeta)$ clearly descends to $T^{\mathbb{R}}M \cong (E \otimes H)^\sigma$ (cfr. Remark 1.19) and defines, for every $\zeta \in \mathbb{P}^1$, an endomorphism of $T^{\mathbb{R}}M$ that anticommutes with the complex structure I_ζ, i.e. a l-hypercomplex structure of degree d. We remark here that another map can be obtained from $A(\zeta)$, simply by taking the tensor product $A(\zeta) \otimes \mathbb{1}$ with the identity of H. In this case what we get is an endomorphism of $E \otimes H$ that descends to $(E \otimes H)^\sigma$ and commutes with every complex structure. This is the object we are going to focus on in the following sections, as it plays a central role in the geometric construction we are about to explain.

2.3 The Reality Condition

Throughout this section we shall assume again $\dim_{\mathbb{C}} Z = 3$ and consider the *reality condition* for the endomorphism $A(\zeta)$. From (2.2.5) we clearly have $H^0(C, N_{C/Z}(-1)) \cong \mathbb{C}^{2d}$ and we can identify the quaternionic structure j_E defined in Remark 1.19 with the natural quaternionic structure

$$\begin{aligned}\mathbb{C}^{2d} &\to \mathbb{C}^{2d} \\ (\mathbf{z}, \mathbf{w}) &\mapsto (-\bar{\mathbf{w}}, \bar{\mathbf{z}}).\end{aligned} \tag{2.3.1}$$

The quaternionic structure j_E induces in turn a *real* structure σ_E on $\operatorname{End}(H^0(C, N_{C/Z}(-1)))$, which is defined by

$$\sigma_E(A) = -j_E \circ A \circ j_E, \quad A \in \operatorname{End}(H^0(C, N_{C/Z}(-1))). \tag{2.3.2}$$

Therefore σ_E represents a "conjugation" on $\mathrm{End}(H^0(C, N_{C/Z}(-1)))$ and the following reality condition for the family of endomorphisms $A(\zeta)$ naturally arises from the twistor construction proposed in (2.1.1)

$$A(-1/\bar{\zeta}) = -\frac{\sigma_E\left(A(\zeta)\right)}{\bar{\zeta}^2} = -\frac{-j_E \circ A(\zeta) \circ j_E}{\bar{\zeta}^2}. \tag{2.3.3}$$

We know that $A(\zeta)$ is given as a section of $\mathrm{End}\, H^0(C, N_{C/Z}(-1)) \otimes \mathcal{O}(2)$, hence it can be represented by a $2d$-dimensional square matrix whose entries are holomorphic sections of $\mathcal{O}(2)$. In other words, $A(\zeta)$ can be written in the form

$$A(\zeta) = \begin{pmatrix} a(\zeta) & b(\zeta) \\ c(\zeta) & d(\zeta), \end{pmatrix} \tag{2.3.4}$$

where $a(\zeta) = a_0 + a_1\zeta + a_2\zeta^2,\ a_i \in Mat(\mathbb{C}, d \times d)$ and analogously for $b(\zeta), c(\zeta)$ and $d(\zeta)$. The reality condition then translates into the equation

$$\frac{1}{\bar{\zeta}^2}\begin{pmatrix} -\bar{d}(\zeta) & \bar{c}(\zeta) \\ \bar{b}(\zeta) & -\bar{a}(\zeta) \end{pmatrix} = \begin{pmatrix} a\left(-\frac{-1}{\zeta}\right) & b\left(-\frac{-1}{\zeta}\right) \\ c\left(-\frac{-1}{\zeta}\right) & d\left(-\frac{-1}{\zeta}\right) \end{pmatrix} \tag{2.3.5}$$

from which we deduce that

$$\begin{cases} d_0 = -\overline{a_2} \\ d_1 = \overline{a_1} \\ d_2 = -\overline{a_0} \end{cases} \qquad \begin{cases} c_0 = \overline{b_2} \\ c_1 = -\overline{b_1} \\ c_2 = \overline{b_0}. \end{cases} \tag{2.3.6}$$

Therefore $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$ where the A_i's are $2d$-dimensional complex square matrices defined by

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ \bar{b}_2 & -\bar{a}_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} a_1 & b_1 \\ -\bar{b}_1 & \bar{a}_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & b_2 \\ \bar{b}_0 & -\bar{a}_0 \end{pmatrix} \tag{2.3.7}$$

and satisfy

$$\begin{aligned} \sigma_E(A_0) &= -A_2 \\ \sigma_E(A_1) &= A_1. \end{aligned} \tag{2.3.8}$$

Taking

$$\begin{aligned} L_1 &= \frac{1}{2}A_1 \\ L_2 &= \frac{1}{2}(A_0 - A_2) \\ L_3 &= \frac{i}{2}(A_0 + A_2) \end{aligned} \tag{2.3.9}$$

we get a triple of endomorphisms of $H^0(C, N_{C/Z}(-1))$ that are invariant under σ_E, i.e. commute with the quaternionic structure j_E.
At this point we recall from (1.4.3) and Remark 1.19 that $T^{\mathbb{C}}M \cong E \otimes H$ and, from (1.4.6), that we have defined the three complex structures I_1, I_2, I_3 simply by acting only on the trivial $\mathbb{C}^2$ part of the tensor product. Observe now that

$$\operatorname{End} E \otimes H \cong (E \otimes H)^* \otimes (E \otimes H) \cong E^* \otimes E \otimes H^* \otimes H \cong \operatorname{End}(E) \otimes \operatorname{End}(H). \tag{2.3.10}$$

Therefore, when we let C vary in M, we get endomorphisms $L_i, i = 1, 2, 3$ of E which yield endomorphisms $L_i \otimes \mathbb{1}$ of the space $E \otimes H \cong T^{\mathbb{C}}M$ which, of course, commute with our triple of complex structure. It is straightforward to notice that every endomorphism of $E \otimes H$ that commutes with all of I_1, I_2, I_3 arises as a tensor product $A_E \otimes \mathbb{1}$ of an endomorphism of E with the identity of H. Again from Remark 1.19 recall that the real tangent bundle to M is described as $T^{\mathbb{R}}M \cong (E \otimes H)^\sigma$, that is as the space of elements of $E \otimes H$ that are invariant under the real structure $\sigma = j_E \otimes j_H$. We now have the following easy lemma.

Lemma 2.6. *The bundle of real endomorphisms of* $(E \otimes H)^\sigma$ *is*

$$\operatorname{End}\left((E \otimes H)^\sigma\right) \cong \operatorname{End}(E)^{\sigma_E} \otimes \operatorname{End} H^{\sigma_H}, \tag{2.3.11}$$

where $\operatorname{End}(E)^{\sigma_E}$ *is the bundle of all endomorphisms of* E *that are invariant under the real structure* σ_E *and analogously* $\operatorname{End}(H)^{\sigma_H}$, σ_H *being the real structure induced by* j_H.

Proof. We start by observing that

$$\begin{aligned} \dim\left(\operatorname{End}(E)^{\sigma_E}\right) &= \frac{1}{2}\dim_{\mathbb{R}}\left(\operatorname{End}(E)\right) = \left(\dim_{\mathbb{C}} E\right)^2 = 4d^2 \\ \dim\left(\operatorname{End}(H)^{\sigma_H}\right) &= \frac{1}{2}\dim_{\mathbb{R}}\left(\operatorname{End}(H)\right) = \left(\dim_{\mathbb{C}} H\right)^2 = 4 \end{aligned} \tag{2.3.12}$$

therefore $\operatorname{End}(E)^{\sigma_E} \otimes \operatorname{End} H^{\sigma_H}$ has real dimension equal to $16d^2$. Since the dimension of $\operatorname{End}\left((E \otimes H)^\sigma\right)$ is also equal to $16d^2$, it will be sufficient to set up an injective map

$$\operatorname{End}(E)^{\sigma_E} \otimes \operatorname{End} H^{\sigma_H} \to \operatorname{End}(E)^{\sigma_E} \otimes \operatorname{End} H^{\sigma_H}.$$

We achieve this simply by viewing a tensor product $A_E \otimes A_H \in \mathrm{End}(E)^{\sigma_E} \otimes \mathrm{End}\, H^{\sigma_H}$ as an endomorphism of $E \otimes H$. As A_E (respectively A_H) is invariant with respect to σ_E (respectively σ_H), then $A_E \otimes A_H$ preserves the σ-invariant subspace of $E \otimes H$ and descends to an endomorphism of $(E \otimes H)^\sigma$. The injectivity is then obvious, hence we have the claimed result. ∎

As a consequence of Lemma 2.6 we get that the previously defined endomorphisms $L_i \otimes \mathbb{1}$ of $E \otimes H$ descend to endomorphisms of $T^{\mathbb{R}}M \cong (E \otimes H)^\sigma$ that commute with all three complex structures. Such endomorphisms can be combined to build what is known in literature as an *aquaternionic map*. We briefly recall this notion, following [20]. Consider the space $\mathrm{End}\left((E \otimes H)^\sigma\right)$ and the map

$$\begin{aligned} C\colon\ \mathrm{End}\left((E \otimes H)^\sigma\right) &\to \mathrm{End}\left((E \otimes H)^\sigma\right) \\ X &\mapsto I_1 X I_1 + I_2 X I_2 + I_3 X I_3. \end{aligned} \tag{2.3.13}$$

Such a map C is proven to satisfy the equation $C^2 + 2C - 3 = 0$, hence has eigenvalues $+1$ and -3. We can therefore decompose $\mathrm{End}\left((E \otimes H)^\sigma\right)$ into $B + \oplus B_-$ where $B+$ and B_- denote respectively the $+1$ and -3-eigenspaces. If we denote by $\mathrm{End}_{\mathbb{H}}\left((E \otimes H)^\sigma\right)$ the space of all quaternion-linear endomorphisms of $\left((E \otimes H)^\sigma\right)$, that is all those that commute with all three complex structures I_1, I_2, I_3, it is possible to check by direct computation that the following inclusions hold true

$$\begin{aligned} \mathrm{End}_{\mathbb{H}}\left((E \otimes H)^\sigma\right) &\subset B_- \\ \mathrm{End}_{\mathbb{H}}\left((E \otimes H)^\sigma\right) \otimes \mathbb{R}^3 &\subset B_+, \end{aligned} \tag{2.3.14}$$

where the second one is realized via the map

$$X_1 \otimes e_1 + X_2 \otimes e_2 + X_3 \otimes e_3 \mapsto I_1 X_1 + I_2 X_2 + I_3 X_3. \tag{2.3.15}$$

By a quick dimension counting we see that the above inclusions are actually isomorphisms. As in [20], we can now give the definition of *aquaternionic endomorphism.*

Definition 2.7. *Let X be an element of* $\mathrm{End}\left((E \otimes H)^\sigma\right)$. *We say that X is* aquaternionic *if it satisfies the condition*

$$I_1 X I_1 + I_2 X I_2 + I_3 X I_3 = X, \tag{2.3.16}$$

that is X has no quaternion-linear part.

Since the maps $L_i \otimes \mathbb{1}$ belong to $\mathrm{End}_{\mathbb{H}}\left((E \otimes H)^\sigma\right)$, we can construct the map

$$X = I_1 \circ (L_1 \otimes \mathbb{1}) + I_2 \circ (L_2 \otimes \mathbb{1}) + I_3 \circ (L_3 \otimes \mathbb{1}) = L_1 \otimes I_1 + L_2 \otimes I_2 + L_3 \otimes I_3, \tag{2.3.17}$$

which, by the above arguments, is an aquaternionic endomorphism of $((E \otimes H)^{\sigma})$. The converse construction is also possible: if we start we an aquaternionic endomorphism X of $((E \otimes H)^{\sigma})$ and use the isomorphisms (2.3.14) we obtain a triple of quaternion linear endomorphisms of $((E \otimes H)^{\sigma})$, i.e. $L_i \otimes \mathbb{1}$ for $L_i \in \mathrm{End}(E)^{\sigma_E}, i = 1, 2, 3$. From these, setting

$$A_0 = L_2 + iL_3,\ A_1 = L_1,\ A_2 = -(L_2 - iL_3) \tag{2.3.18}$$

we construct the section $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$ of $\mathrm{End}(E) \otimes \mathcal{O}(2)$ and observe that it is real for the natural real structure of $\mathcal{O}(2)$.

Example 2.2. Let Z be the hyperkähler twistor space of $\mathbb{C}^2$. We recall from Example 1.1 that we have a projection

$$\begin{aligned} Z &\to T\mathbb{P}^1 \\ (\zeta, z_1, z_2) &\mapsto -i\mu(\zeta), \end{aligned} \tag{2.3.19}$$

where $\mu(\zeta) = i(z_1 + \zeta\bar{z}_2)(z_2 - \zeta\bar{z}_1)$ is the $\mathcal{O}(2)$-twisted moment map provided by the twistor construction exposed in Section 1.3. By Theorem 1.10 we can recover $\mathbb{C}^2$ as the manifold of real (i.e. σ-invariant) degree-one curves that satisfy our cohomological stability conditions. Therefore, following the construction we have exposed so far, we can represent our bundle endomorphism $A(\zeta)\colon E \to E$ by the diagonal matrix

$$A(\zeta) = -i\mu(\zeta)\mathbb{1}_2 = \begin{pmatrix} z_1z_2 + \zeta(|z_2|^2 - |z_1|^2) - \zeta^2\bar{z}_1\bar{z}_2 & 0 \\ 0 & z_1z_2 + \zeta(|z_2|^2 - |z_1|^2) - \zeta^2\bar{z}_1\bar{z}_2 \end{pmatrix}, \tag{2.3.20}$$

which of course yields

$$A_0 = \begin{pmatrix} z_1z_2 & 0 \\ 0 & z_1z_2 \end{pmatrix},\quad A_1 = \begin{pmatrix} (|z_2|^2 - |z_1|^2) & 0 \\ 0 & (|z_2|^2 - |z_1|^2) \end{pmatrix},\quad A_2 = \begin{pmatrix} \bar{z}_1\bar{z}_2 & 0 \\ 0 & \bar{z}_1\bar{z}_2 \end{pmatrix}. \tag{2.3.21}$$

Therefore we define L_1, L_2, L_3 to be

$$\begin{aligned} L_1 &= \frac{1}{2}A_1 \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{pmatrix},\quad L_2 = \frac{1}{2}(A_0 - A_2) = \begin{pmatrix} \mu_3 & 0 \\ 0 & \mu_3 \end{pmatrix} \\ L_3 &= \frac{i}{2}(A_0 + A_2) = \begin{pmatrix} -\mu_2 & 0 \\ 0 & -\mu_2 \end{pmatrix}. \end{aligned} \tag{2.3.22}$$

The quaternionic endomorphism X is hence represented by

$$X = \begin{pmatrix} i\mu_1 & \mu_3 - i\mu_2 \\ -\mu_3 + i\mu_2 & -i\mu_1 \end{pmatrix} \otimes \mathbb{1}_2. \tag{2.3.23}$$

2.4 Holomorphicity of $A(\zeta_0)$ on M

In this section we fix a point ζ_0 in $\mathbb{P}^1$ and show that the endomorphism $A(\zeta_0)$ of the bundle E is holomorphic with respect to the complex structure I_{ζ_0}. We then translate such a holomorphicity condition into an equivalent Cauchy-Riemann-type equation for the associated aquaternionic endomorphism X of $T^{\mathbb{R}}M$.
Once more, we recall from [27] that

$$T_C(M^{\mathbb{C}}) \cong H^0(C, N_{C/Z}) \cong H^0(C, N_{C/Z}(-1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(1)). \tag{2.4.1}$$

Moreover we remember the identifications

$$T^{\mathbb{R}}_C M \cong \left(T^{0,1}_C M\right)_{I_{\zeta_0}} \cong H^0(C, N_{C/Z}(-1)) \otimes (\zeta - \zeta_0) \cong H^0(C, N_{C/Z}[-C_{\zeta_0}]) \tag{2.4.2}$$

Fix now a point in M and denote by C the corresponding curve inside Z. As usual let p be the projection $Z \to T\mathbb{P}^1$. Let us focus our attention to the divisor $C_{\zeta_0} = C \cap Z_{\zeta_0}$ which we assume, for simplicity, consisting of d different points, i.e. $C_{\zeta_0} = \{(\zeta_0, \eta_i), i = 1, \dots, d\}$. If we write, as in Remark 2.1, $P(\zeta, \eta)$ for the polynomial that defines the curve $\bar{C} = p(C) \subset T\mathbb{P}^1$, then the divisor C_{ζ_0} is simply the zero locus of the polynomial

$$\prod_{i=1}^{d} (\eta - \eta_i) = P(\zeta_0, \eta). \tag{2.4.3}$$

Consider now a tangent vector $\mathbf{x} \in T^{\mathbb{C}}_C M \cong H^0(C, N_{C/Z})$. Our aim is to identify the deformation induced by $\mathbf{x}$ on the divisor C_{ζ_0} with a d-tuple $(\eta'_1, \dots, \eta'_d)$ so that such deformation of the divisor C_{ζ_0} be described by the equation

$$\prod_{i=1}^{d} (\eta - \eta_i(z)) = 0, \tag{2.4.4}$$

where $\eta_i(z) = \eta_i + \eta'_1 z + O(z^2)$. We achieve such description by considering the short exact sequence

$$0 \longrightarrow \left(T^{0,1}_C M\right)_{I_{\zeta_0}} \longrightarrow T^{\mathbb{C}}_C M \longrightarrow T^{\mathbb{C}}_C M / \left(T^{0,1}_C M\right)_{I_{\zeta_0}} \longrightarrow 0, \tag{2.4.5}$$

that gives

$$0 \longrightarrow H^0(C, N[-C_{\zeta_0}]) \longrightarrow H^0(C, N_{C/Z}) \longrightarrow H^0(C, N_{C/Z})/H^0(C, N[-C_{\zeta_0}]) \longrightarrow 0. \tag{2.4.6}$$

To each $\rho \in H^0(C, N_{C/Z})/H^0(C, N[-C_{\zeta_0}])$ we can therefore define the corresponding tuple $\eta' = \left(\eta'_j(\rho)\right) = (\rho(\zeta_0, \eta_j))$. We observe that, as we should expect, the d-tuple associated to a tangent vector to M at C of type $(0,1)$ for the complex structure I_{ζ_0} is the 0 tuple, as such deformation do fix the whole divisor C_{ζ_0}. Moreover, since we have assumed $H^1(C, N_{C/Z})$ to vanish, there is no obstruction to deformations which means that, for $|z|$ small enough, we have a family of curves $C^{(z)}$ with $C^{(0)} = C$ such that the divisor $C^{(z)}_{\zeta_0}$ is described by (2.4.3). At this point recalling the definition of $A(\zeta)$ at C, we can analogously define $A(\zeta_0)$ at $C^{(z)}$ as given by multiplication by $\eta(z)$ and denote it by $A(\zeta)(z)$. It is then natural to define the derivative $A'(\zeta_0)(0)$ at $C = C^{(0)}$ as

$$A'(\zeta_0)(0) = \lim_{|z|\to 0} \frac{\eta(z) - \eta(0)}{z} = \eta'(z). \tag{2.4.7}$$

This means that, once we pick a local frame (s_i) of the bundle E such that the evaluation isomorphism (1.4.8) at $\zeta = \zeta_0$ is represented by a diagonal matrix (i.e. $s_i(\zeta_0, \eta_j) = 0$ if $i \neq j$), we have

$$A(\zeta_0)\,(s_i(z)) = \eta_i(z) s_i(z). \tag{2.4.8}$$

If ∇ is a connection on the vector bundle E, then applying $\nabla_{\mathbf{x}}$ to both sides of (2.4.8) we obtain

$$(\nabla_{\mathbf{x}} A(\zeta_0))\, s_i(z) + A(\zeta_0)\,(\nabla_{\mathbf{x}} s_i(z)) = \left(\frac{d}{dz}\eta_i(z)\right) s_i(z) + \eta_i(z) \nabla_{\mathbf{x}} s_i(z). \tag{2.4.9}$$

If in addition we assume ∇ to be compatible with the complex structure I_{ζ_0}, that is $\nabla^{0,1}_{\zeta_0} = \bar{\partial}_{\zeta_0}$, and take the frame $s_i(z)$ to be holomorphic for I_{ζ_0}, we obtain

$$(\nabla_{\mathbf{x}} A(\zeta_0))\, s_i(z) = \left(\frac{d}{dz}\eta_i(z)\right) s_i(z) \tag{2.4.10}$$

which at $z = 0$, i.e. on $C = C^{(0)}$, yields

$$(\nabla_{\mathbf{x}} A(\zeta_0))\, s_i(0) = \eta'_i s_i(0). \tag{2.4.11}$$

So, if the vector $\mathbf{x}$ was chosen of type $(0,1)$ for I_{ζ_0}, then $\eta'_i = 0$ for all $i = 1, \ldots, d$ hence at $C \in M$ we get $(\nabla_{\mathbf{x}} A(\zeta_0))\, s_i(0) = 0$ for all i. Since this is a punctual condition, we deduce that $\nabla_{\mathbf{x}} A(\zeta_0) = 0$ for every vector $\mathbf{x}$ of type $(0,1)$ for I_{ζ_0} on the subset of M consisting of all curves C whose divisor C_{ζ_0} has d distinct points, which is dense in M.

Remark 2.8. We can always find a connection ∇ on E that satisfies the above requirement: in fact, since E is the bundle induced on M by the vector bundle $T_F(-1)$ over the hyperkähler twistor space Z^{HK} of M, by [6][Proposition 6.1] we are granted the existence of a connection ∇ on E such that for every $\zeta \in \mathbb{P}^1$, $\nabla_\zeta^{0,1} = \bar{\partial}_\zeta$.

We have therefore proven the following statement.

Proposition 2.9. *Fix $\zeta \in \mathbb{P}^1$ and consider the manifold M endowed with the complex structure I_ζ. Let E be the vector bundle on M induced by $T_F(-1)$ on Z^{HK}, with the holomorphic structure $\bar{\partial}_\zeta$ corresponding to I_ζ. Then $A(\zeta)$ is a section of* $\operatorname{End}(E)$ *holomorphic with respect to $\bar{\partial}_\zeta$.*

Recalling from (2.3.18) that

$$A(\zeta) = (L_2 + iL_3) + 2\zeta L_1 - \zeta^2(L_2 - iL_3) \tag{2.4.12}$$

and that the L_i's yield the aquaternionic endomorphisms

$$X = \sum_{i=1}^{3} L_i \otimes I_i \tag{2.4.13}$$

of the real tangent bundle to M, we translate the holomorphicity of $A(\zeta)$ with respect to I_ζ into a condition on the L_i's.

Proposition 2.10. *The σ_E-invariant endomorphisms L_i obey the equations*

$$\begin{aligned} \nabla_\xi L_2 &= \nabla_{I_1\xi} L_3 \\ 2\nabla_{I_3\xi} L_1 &= \nabla_\xi L_3 - \nabla_{I_1\xi} L_2, \end{aligned} \tag{2.4.14}$$

for every real tangent vector field ξ.

Proof. Recall that a vector field of type $(0,1)$ for I_ζ is given by $X - i\zeta I_3(X)$ for X of type $(0,1)$ for I_1. Therefore we write the holomorphicity condition for $A(\zeta)$ as

$$\begin{aligned} 0 &= \nabla_{X - i\zeta I_3(X)} \left((L_2 + iL_3) + 2\zeta L_1 - \zeta^2(L_2 - iL_3) \right) \\ &= \nabla_X (L_2 + iL_3) + 2\zeta \nabla_X L_1 - \zeta^2 \nabla_X (L_2 - iL_3) \\ &- i\zeta \nabla_{I_3(X)} (L_2 + iL_3) - 2i\zeta^2 \nabla_{I_3(X)} L_1 + i\zeta^3 \nabla_{I_3(Z)} (L_2 - iL_3) \\ &= \nabla_X (L_2 + iL_3) + \zeta \left(2\nabla_X L_1 - i\nabla_{I_3(X)} (L_2 + iL_3) \right) \\ &- \zeta^2 \left(\nabla_X (L_2 - iL_3) + 2i\nabla_{I_3(X)} L_1 \right) + i\zeta^3 \nabla_{I_3(X)} (L_2 - iL_3) . \end{aligned} \tag{2.4.15}$$

We obtain the following system of equations

$$\nabla_X (L_2 + iL_3) = 0 \tag{2.4.16}$$

$$\nabla_{I_3(X)} (L_2 - iL_3) = 0 \tag{2.4.17}$$

$$2\nabla_X L_1 + \nabla_{I_3(X)} (L_3 - iL_2) = 0 \tag{2.4.18}$$

$$2i\nabla_{I_3(X)} L_1 + \nabla_X (L_2 - iL_3) = 0, \tag{2.4.19}$$

which is redundant since (2.4.16) and (2.4.17) are equivalent as well as (2.4.18) and (2.4.19). Consider now (2.4.16). As X is of type $(0,1)$ for I_1, there exists a real vector field ξ such that $X = \xi + iI_1(\xi)$. With this substitution the equation becomes

$$\nabla_\xi L_2 + i\nabla_{I_1(\xi)} L_2 = -i \left(\nabla_\xi L_3 - i\nabla_{I_1(\xi)} L_3\right), \tag{2.4.20}$$

from which we get

$$\nabla_\xi L_2 = \nabla_{I_1(\xi)} L_3. \tag{2.4.21}$$

Finally, the same observation applied to (2.4.18) yields

$$2\nabla_{I_1(\xi)} L_1 = \nabla_{I_3(\xi)} L_2 - \nabla_{I_2(\xi)} L_3. \tag{2.4.22}$$

∎

Remark 2.11. The properties of $A(\zeta)$ that we have exposed so far show the same phenomena that we observe for the twisted symplectic form on the ordinary hyperkähler twistor space. Both $-i\omega(\zeta)$ and $A(\zeta)$, in fact, define $\mathcal{O}(2)$ twisted objects that are real for the natural real structure of $\mathcal{O}(2)$ and for fixed ζ they define holomorphic objects on (M, I_ζ).
Our next goal is to understand how the geometry of M is characterized by $A(\zeta)$. We will show in the following chapters that knowing M and $A(\zeta)$ will be sufficient to recover the fibration $Z \to T\mathbb{P}^1$ and M as manifold of curves in Z of degree d.

Chapter 3

Transverse Hilbert Schemes

Contents

The present chapter is devoted to the study of the geometry related to the endomorphism $A(\zeta)$ for a fixed complex structure $I_\zeta,\ \zeta \in \mathbb{P}^1$, assuming as usual $\dim_{\mathbb{C}} Z = 3$. Since it is defined by multiplication by the tautological section η of $\mathcal{O}_{T\mathbb{P}^1}(2)$ we see that, once we fix $\zeta_0 \in \mathbb{P}^1$, $A(\zeta_0)$ only depends on the projection $p\colon Z_{\zeta_0} \to T_{\zeta_0}\mathbb{P}^1$. Moreover we know from Proposition 1.23 that the manifold (M, I_{ζ_0}) is locally diffeomorphic to the smooth locus of the transverse Hilbert scheme $(Z_{\zeta_0})^{[d]}$, the latter being the whole transverse Hilbert scheme when, as in the case we consider, the fibre Z_{ζ_0} is complex two-dimensional. We are therefore invited to perform our study in the general frame of a complex surface S equipped with a holomorphic map $p\colon S \to \mathbb{C}$ which is a surjective submersion outside of a discrete subset $B \subset S$. We construct an endomorphism A, analogous to $A(\zeta_0)$, of the tangent space to the transverse Hilbert scheme $S_p^{[d]}$ with two-dimensional eigenspaces and such that its characteristic polynomial is the square of its minimal polynomial and provide the inverse construction that is, starting with a complex manifold W of complex dimension $2d$ with a tangent endomorphism $A\colon TW \to TW$ with the above properties, we recover the initial surface S with $W \cong S_p^{[d]}$.

It is noteworthy that the results of this chapter have an interpretation in terms of holomorphic completely integrable systems. As it is well known (see, for example, [12]), a Hamiltonian system is called *completely integrable* if and only if it possesses the maximal number of Poisson-commuting Hamiltonian functions. Here we work in the holomorphic category and associate to a complex surface endowed with a holomorphic symplectic form a holomorphic completely integrable system of complex dimension $2d$ that arises as its Hilbert scheme of d points transverse to p. Moreover, the inverse construction provides a full characterization of the holomorphic completely integrable systems that arise as transverse Hilbert schemes of points of a surface with the aforementioned properties.

3.1 An Endomorphism of the Tangent Space

Let S be a complex surface with a holomorphic projection $p\colon S \to \mathbb{C}$ which is a surjective submersion outside a discrete subset $B \subset S$. Recalling the definition 1.21 of transverse Hilbert scheme of d points with respect to p, we can define a natural endomorphism of the tangent space to $S_p^{[d]}$ as follows. Let $q(z)$ be the monic polynomial of degree d defining the image $p(Z)$ of $Z \in S_p^{[d]}$ via p and observe that $H^0(Z,\mathbb{C}) \cong \mathbb{C}[z]/(q(z))$, where the generator z stands as a preferred element. Recall now that, for every $Z \in S_p^{[d]}$, one has $T_Z S_p^{[d]} \cong H^0(Z, TS|_Z)$ due to the well-known theorem of Kodaira ([27]). Then we set A_Z to be the map

$$\begin{gathered} H^0(Z,TS|_Z) \longrightarrow H^0(Z,TS|_Z) \\ \sigma(z) \mapsto f(z)\sigma(z), \end{gathered} \tag{3.1.1}$$

where we take $f\colon Z \to \mathbb{C}$ to be the function $z \in H^0(Z,\mathbb{C})$.

Remark 3.1. If $\sigma \in H^0(Z,TS|_Z)$ and $p \in S$ is a point of Z with multiplicity one then $(A\sigma)(p) = z(p)\sigma(p)$. If, instead, p has multiplicity $k > 1$, the section σ is given as a power series in $(z - z(p))$ truncated at order k, that is

$$\sigma(z) = \sigma(z(p)) + \sigma'(z(p))(z - z(p)) + \cdots + \frac{\sigma^{(k)}(z(p))}{k!}(z - z(p))^k. \tag{3.1.2}$$

Then $A\sigma$ will be give the truncated power series of $z\sigma(z)$, that is

$$(z\cdot\sigma)(z) = z(p)\sigma(z(p)) + (\sigma(z(p)) + z(p)\sigma'(z(p)))(z - z(p)) + \dots \tag{3.1.3}$$

$$+ \frac{k\sigma^{(k-1)}(z(p)) + z(p)\sigma^{(k)}(z(p))}{k!}(z - z(p))^k. \tag{3.1.4}$$

Comparing (3.1.3) and (3.1.2) we deduce that the eigenspaces of A are of dimension 2. Also, the eigenvalues have even multiplicity each one equal to the dimension of the relative power expansion space. These two observations altogether yield, at each point of $S_p^{[d]}$, the Jordan canonical form of A and we deduce that the minimal polynomial of A is $q(z)$ and the characteristic polynomial is the square of $q(z)$.

Example 3.1 (The space of rational maps)**.** The machinery we have introduced so far allows us to build such an endomorphism A on the tangent space to the space of based rational maps of degree d. As an example we compute it for $d = 2$.

Let us define the complex surface $S = \mathbb{C} \times \mathbb{C}^*$ projecting onto $\mathbb{C}$ via p which we interpret as the moduli space of charge 1 monopoles and let $S_p^{[2]}$ be its Hilbert scheme of points of length 2 transverse to p. We identify (see [3]) $S_p^{[2]}$ with the space of all based rational maps of degree 2, defined by

$$R_2 = \left\{ \frac{p(z)}{q(z)} = \frac{a_1 z + a_0}{z^2 - q_1 z - q_0} \mid p(z) \text{ and } q(z) \text{ have no common roots} \right\}. \tag{3.1.5}$$

Observe that a tangent vector to R_2 at a point $(p(z), q(z))$ is given as a couple of degree 1 polynomials $(q'(z), p'(z))$ where we write $q'(z) = q'_1 z + q'_0$ and $p'(z) = p'_1 z + p'_0$. Applying the previous construction we get an endomorphism A of the tangent bundle to R_2 which on the tangent space to R_2 at each point $(p(z), q(z))$ operates as multiplication by z modulo $q(z)$. This means that

$$A_{(q(z),p(z))} \colon T_{(q(z),p(z))}R_2 \longrightarrow T_{(q(z),p(z))}R_2 \tag{3.1.6}$$

$$(q'_1 z + q_0, p'_1 z + p_0) \mapsto ((q_1 q'_1 + q'_0)z + q_0 q'_1, (q_1 p'_1 + p'_0)z + q_0 p'_1) \tag{3.1.7}$$

is represented at $(p(z), q(z))$ by the block-diagonal matrix

$$A_{(q(z),p(z))} := \begin{pmatrix} q_1 & 1 & 0 & 0 \\ q_0 & 0 & 0 & 0 \\ 0 & 0 & q_1 & 1 \\ 0 & 0 & q_0 & 0 \end{pmatrix} \tag{3.1.8}$$

where each block is the so-called companion matrix of the polynomial $q(z)$.

Let us now focus on the open dense subset of R_2 consisting of all based degree 2 rational maps with simple poles. If a map $p(z)/q(z)$ has distinct poles, i.e. the roots of q are distinct, then it can be identified with the point $X = ((\beta_1, p(\beta_1), (\beta_2, p(\beta_2))) \in S_p^{[2]}$, where the β_i's are the roots of q and $p(z)$ is recovered by Lagrange interpolation as the unique linear polynomial taking the values $p(\beta_i)$ at β_i. The projection $p \colon S \longrightarrow \mathbb{C}$ induces on every $X \in S_p^{[2]}$ a function $f \colon X \longrightarrow \mathbb{C}$ taking $(\beta_i, p(\beta_i))$ into $\beta_i \in \mathbb{C}$. Using the fact that $T_X S_p^{[2]} \cong H^0(X, TS|_X)$, the function f induces an endomorphism $H^0(X, TS|_X) \longrightarrow H^0(X, TS|_X)$ given by $\sigma(x) \longrightarrow f(x)\sigma(x)$ for $x \in X$. In the tangent frame provided by these coordinates, A at $(\beta_i, p(\beta_i))$ is represented by the diagonal matrix $\mathrm{diag}(\beta_1, \beta_1, \beta_2, \beta_2)$. Since on this open subset $q_0 = -\beta_1\beta_2, q_1 = \beta_1 + \beta_2$, a

computation shows that this diagonal matrix actually is the Jordan canonical form of (3.1.8).
We observe also that, when $q_1 = 2\beta$ and $q_0 = -\beta^2$ i.e. the rational map has a double pole at $z = \beta$, then the Jordan form of A is

$$\begin{pmatrix} \beta & 1 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & \beta \end{pmatrix} \tag{3.1.9}$$

Example 3.2. Let us consider the double cover of the Atiyah-Hitchin manifold. As described in [3] this is a surface $S \subset \mathbb{C}^3$ defined by $S = \{(z, x, y) |\, x^2 - zy^2 = 1\}$. We can therefore consider the Hilbert scheme of d points of S transverse to the projection p onto the first coordinate. We recall from [8] that it can be described as the set of triple of polynomials $x(z), y(z), q(z)$ such that $x(z)$ and $y(z)$ have degree $d-1$, $q(z)$ is monic of degree d and the equation $x^2(z) - zy^2(z) = 1$ modulo $q(z)$ is verified.
An alternative description (also explained in [8]), which we will use here, is obtained by considering the quadratic extension $z = u^2$. In this case the equation $x^2 - zy^2 = 1$ is rewritten as $(x + uy)(x - uy) = 1$ and we observe that $p(u) = x(u^2) + uy(u^2)$ is a polynomial of degree $2d - 1$ in u while $q(u^2)$ is a monic polynomial of degree $2d$ in u which has no odd terms. The Hilbert scheme $S_p^{[d]}$ is then described as the set of all couples of polynomials $(p(u), q(u^2))$ such that $p(u)p(-u) = 1$ modulo $q(u^2)$. Similarly to the previous example, a tangent element in $T_{(p(u),q(u^2))}S_p^{[d]}$ is given by a couple of polynomials of the form

$$p'(u) = p'_0 + p'_1 u + \cdots + p'_{2d-1}u^{2d-1} \tag{3.1.10}$$
$$q'(u^2) = q'_0 + q'_1 u^2 + \cdots + q'_{d-1}u^{2d-2} \tag{3.1.11}$$

such that

$$p'(u)p(-u) + p(u)p'(-u) = 0 \text{ modulo } q(u^2). \tag{3.1.12}$$

We finally produce the endomorphism $A\colon T_{(p(u),q(u^2))}S_p^{[d]} \to T_{(p(u),q(u^2))}S_p^{[d]}$ at every point $(p(u), q(u^2)) \in S_p^{[d]}$ as multiplication by u^2 modulo $q(u^2)$, after observing that it preserves the space of solutions to (3.1.12).

Remark 3.2. Consider the projection $\mu\colon S_p^{[d]} \to \mathbb{C}^{[d]}$ defined by sending $Z \in S_p^{[d]}$ to the minimal polynomial $q_Z(z)$ of A at Z. Whenever $p\colon S \to \mathbb{C}$ is a submersion, we can choose the standard coordinate z of $\mathbb{C}$ as first coordinate on S and use the coefficients of $q(z)$ as first d coordinates on $S_p^{[d]}$. Therefore on $N = (S \setminus B)_p^{[2]}$ the map μ is just the projection onto the first d coordinates and is therefore a submersion. We remark that, since B is discrete in S, then $S_p^{[d]} \setminus N$ has codimension at least two.

We now define the manifold $\tilde{S}_p^{[d]}$ as the set of all $(z, Z) \in \mathbb{C} \times S_p^{[d]}$ such that z is eigenvalue of A_Z and observe that it comes with a double projection

$$\begin{array}{ccc} \tilde{S}_p^{[d]} & & \\ \downarrow{\scriptstyle \rho} & \searrow^{\pi} & \\ \mathbb{C} & & S_p^{[d]} \end{array} \tag{3.1.13}$$

where π is a branched $d : 1$ covering of $S_p^{[d]}$. Also, for every $X \in S_p^{[d]}$, one can lift A_Z to an endomorphism $\pi^* T_Z S_p^{[d]} \longrightarrow \pi^* T_Z S_p^{[d]}$. Hence we draw the following diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & T_p^V S|_Z & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \pi^* T_Z S_p^{[d]} & \xrightarrow{z - A_Z} & \pi^* T_Z S_p^{[d]} & \longrightarrow & TS|_Z & \longrightarrow & 0 \\ & & & & & \searrow & \downarrow & & \\ & & & & & & T\mathbb{C} & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array} \tag{3.1.14}$$

Let β be the function defined by the dotted arrow. We see that $Im(z - A_Z)$ lies in the kernel of β. Also, one has that elements of $\pi^* T_Z S_p^{[d]}$ correspond to deformations of $S_p^{[d]}$ at Z and elements in $T_\rho^V \tilde{S}_p^{[d]}$ correspond to deformations fixing the eigenvalue. From this we get that $\ker \beta = T_\rho^V \tilde{S}_p^{[d]}$ and $Im(z - A) \subset T_\rho^V \tilde{S}_p^{[d]}$.

Remark 3.3. For every $z \in \mathbb{C}$ the holomorphic distribution $Im(z - A)$ defined on $S_p^{[d]}$ is clearly involutive on the dense subset of all length d 0-dimensional subschemes of S consisting of d points that are all distinct and such that none of them lies in $B \subset S$. Since this set in dense in $(S \setminus B)_p^{[d]}$ and the latter has codimension at least two in $S_p^{[d]}$, the integrability holds on all $S_p^{[d]}$. Therefore the distribution $D := Im(z - A)$ defined on $\widetilde{S_p^{[d]}}$ is involutive outside the nowhere dense branch locus of π, hence on all $\widetilde{S_p^{[d]}}$.

Construct now the double fibration

$$\begin{array}{ccc} & Y & \\ & \downarrow \quad \searrow & \\ & S \qquad S_p^{[d]} & \end{array} \tag{3.1.15}$$

the manifold Y being defined as $Y = \left\{(s, Z) \in S \times S_p^{[d]} | \ s \in Z\right\}$. So far we notice that

$$\begin{aligned} \tilde{S}_p^{[d]} &= \left\{(z, Z) \in \mathbb{C} \times S_p^{[d]} | \ z \text{ is eigenvalue of } A_Z\right\} \\ &= \left\{(z, Z) \in \mathbb{C} \times S_p^{[d]} | \ z = p(x) \text{ for some } x \in Z\right\} \\ &\cong \left\{(x, Z) \in S \times S_p^{[d]} | \ x \in Z\right\}. \end{aligned}$$

Hence we recover our initial surface as the space of leaves $S = Y/D \cong \tilde{S}_p^{[d]}/D$.
This suggests us the following *inverse construction*.

3.2 The Inverse Construction

Let us start with a complex manifold W of complex dimension $2d$ endowed with an endomorphism $A : TW \to TW$ of its holomorphic tangent bundle TW with two-dimensional eigenspaces and such that its characteristic polynomial is the square of its minimal polynomial. Set now $X := \mathbb{C}^{[d]}$ Hilbert scheme of d points of $\mathbb{C}$ and define a map $\mu \colon W \longrightarrow X$ which assigns to each point $w \in W$ the minimal polynomial of A at w which we denote $q_w(\lambda)$. Assume now μ to be a surjective submersion on a (open) subset N such that $(W \setminus N)$ has codimension at least two and define a vector field $V \in \mathfrak{X}(W)$ to be projectable for μ if, for every $x \in X$, $d\mu_w(V_w)$ does not depend of the choice of w in $\mu^{-1}(x)$. If we suppose that A preserves the vertical vectors and the projectable vector fields for the projection μ, then it descends to a map $\bar{A} \colon TX \longrightarrow TX$ which makes the following diagram commute

$$\begin{array}{ccc} TW & \xrightarrow{A} & TW \\ {\scriptstyle d\mu}\downarrow & & \downarrow{\scriptstyle d\mu} \\ TX & \xrightarrow[\bar{A}]{} & TX. \end{array} \tag{3.2.1}$$

Definition 3.4. *If an endomorphism A that satisfies the above conditions is such that at every point of N none of its generalized eigenspaces is fully contained in $\ker(d\mu)$, then we will call it* compatible *with the projection μ defined by its minimal polynomial.*

For $q(\lambda) \in X$ let us identify $T_{q(\lambda)}X \cong \mathbb{C}[\lambda]/(q(\lambda))$ and assume that A is compatible with μ. Then $\bar{A}$ is naturally given by multiplication by λ modulo $q(\lambda)$. Set also $W_z = \{w \in W |\, z \in \operatorname{Spec} A_w\}$, i.e. $W = \mu^{-1}(X_z)$ where X_z is the set of all monic polynomials of degree d for which z is a root. With these definitions we see for a tangent vector V that $V \in TW_z \Longleftrightarrow d\mu(V) \in TX_z$, where the tangent space to X_z at $q(\lambda)$ can be described as

$$T_q X_z = \{p(\lambda)|\ \deg p(\lambda) = d-1 \text{ and } p(z) = 0\}\,. \tag{3.2.2}$$

Take now a polynomial $q'(\lambda) \in T_{q(\lambda)}X$ and $z \in C$: the definition of $\bar{A}$ implies that $(z\mathbb{1} - \bar{A})(q'(\lambda))$ is a polynomial of $T_{q(\lambda)}X$ that vanishes at z that is, by the commutativity of the diagram, $Im(z\mathbb{1} - A) \subset TW_z$.

Define now $\tilde{W} = \{(z,w) \in \mathbb{C} \times W |\, z \text{ is an eigenvalue of } A_w\}$, which is a $d\colon 1$ covering of W, with two projections

$$\begin{array}{ccc} \tilde{W} & & \\ {\scriptstyle\rho}\downarrow & \searrow^{\pi} & \\ \mathbb{C} & & W. \end{array} \tag{3.2.3}$$

Then A can be lifted to an endomorphism of $T(\mathbb{C} \times W)$, which we will still denote by A, preserving the vertical subbundles of ρ and π. The previous observations imply that, at every point (z,w), A acts on the vertical subbundle of π as multiplication by z and that it descends to $T\tilde{W}$. Assuming now that the distribution $Im\,(z\mathbb{1} - A)$ is integrable, we see that it defines a subdistribution of the integral distribution $\ker d\rho$. We can therefore recover our initial surface S as the leaf space

$$S := \frac{\tilde{W}}{Im\,(z\mathbb{1} - A)} \tag{3.2.4}$$

The surface S comes with a natural projection $p\colon S \longrightarrow \mathbb{C}$ defined as $p([(z,w)]) = z$, which makes the following diagram commute

$$\begin{array}{ccc} S & \xleftarrow{proj} & \tilde{W} \\ {\scriptstyle p}\downarrow & \swarrow_{\rho} & \\ \mathbb{C} & & \end{array} \tag{3.2.5}$$

It is now sufficient to define $\mathcal{U} \subset S^{[d]}$ as $\mathcal{U} = \left\{proj(\pi^{-1}(w))|\, w \in W\right\}$ and Z as the element $Z(w) = proj(\pi^{-1}(w)) \in \mathcal{U}$ in order to apply the previously exposed construction for getting $A_X : T_X\mathcal{U} \longrightarrow T_X\mathcal{U}$ i.e. once more our endomorphism $A_w\colon T_wW \longrightarrow T_wW$ for every point w of W.

Hence, keeping the conventions that we have introduced so far, we have proven the following.

Proposition 3.5. *Let W^{2d} be a complex manifold of complex dimension $2d$ with the following properties.*

(i) W comes with an endomorphism $A\colon TW \to TW$ such that at every point the eigenspaces have complex dimension 2 and the characteristic polynomial is the square of the minimal polynomial

(ii) Assume that the induced projection $\mu\colon W \to X := \mathbb{C}^{[d]}$ is a surjective submersion on a subset $N \subset W$ such that $W \setminus N$ has codimension at least 2 and that A is compatible with μ in the sense of Definition 3.4, so that Diagram (3.2.1) *is defined*

(iii) The distribution $D := Im(z - A)$ is integrable on the incidence manifold

$$\tilde{W} = \{(z, w) \in \mathbb{C} \times W |\ z \text{ is an eigenvalue of } A_w\}.$$

Then $p\colon S := \tilde{W}/D \to \mathbb{C}$ is a surface projecting on $\mathbb{C}$ for which W is the length d Hilbert scheme of points transverse to the projection.

3.3 A Symplectic Form

In this section we shall assume the surface S to carry a symplectic form ω on its tangent bundle and revise the previously exposed construction. From now on we will often use the assumption that $p\colon S \to \mathbb{C}$ is a submersion outside a discrete subset $B \subset S$.

Remark 3.6. As anticipated in the introduction to the chapter, the results of Sections 3.2 and 3.3 show that the transverse Hilbert scheme $S_p^{[d]}$ has the structure of a holomorphic integrable system. The importance of A in distinguishing whether a given holomorphic integrable system arises a transverse Hilbert scheme of points of a holomorphic symplectic surface is well motivated by the following example.

Example 3.3 (Motivational Example)**.** Let us consider the complex $2d$-dimensional manifold $\mathbb{C}^{2d}$, with coordinates $(z_i, t_i), i = 1, \ldots, d$ and endowed with the standard symplectic form $\Omega_0 = \sum_i dz_i \wedge dt_i$, endowed with he projection $p_0\colon \mathbb{C}^{2d} \to \mathbb{C}^d$ given by $p_0(z_i, t_i) = (z_i)$. Observe that the coordinates z_i are d commuting Hamiltonian functions. Now, the projection p_0 induces an endomorphism $A\colon T\mathbb{C}^{2d} \to T\mathbb{C}^{2d}$ given, at every point of $\mathbb{C}^{2d}$, by the diagonal matrix $\operatorname{diag}(z_1, z_1, \ldots, z_d, z_d)$. The eigenspaces of this endomorphism, however, show a jump in dimension whenever two eigenvalues happen to coincide. As a result of this, although $(\mathbb{C}^{2d}, \Omega_0, z_1, \ldots, z_d)$ is a holomorphic completely integrable system, it does not arise via a transverse Hilbert scheme construction as A does not meet the necessary requirements.

We now consider a different $\mathbb{C}^{2d}$ from the one above, with a different projection. Namely we take the space X_d of all couples of polynomials $(Q(\lambda), P(\lambda))$ such that Q is monic of degree d and T has degree $d-1$. If we write $Q = \lambda^d - \sum Q_j \lambda^j$ and $t = \sum T_i \lambda^i$ then (Q_i, T_i) are global coordinates and $X \cong \mathbb{C}^{2d}$. Now, on the open dense subset $\mathcal{V}$ of X consisting of couples $(Q(\lambda), T(\lambda))$ such that Q has all distinct roots, we also have coordinates $(\beta_i, T(\beta_i))$ where β_i are the roots of Q and $T(\beta_i)$ the values of T on those roots. In these latter coordinates the form $\sum d\beta_i \wedge T(\beta_i)$ is defined and it can uniquely extended to a holomorphic symplectic 2-form Ω on the whole of X_d. One obsevese in fact that the form $Q_1 dT_1 \wedge dQ_1 + dT_1 \wedge dQ_0 + dT_0 \wedge dQ_1$ is globally defined on X_d and coincide with $\sum d\beta_i \wedge dT(\beta_i)$ on $\mathcal{V}$, therefore representing the unique extension of the latter to all X_d. The functions $Q_0, \ldots, Q_{d-1}$ will then be d commuting Hamiltonians with respect to Ω: in fact, on the open dense subset $\mathcal{V}$ they are just the elementary symmetric polynomials in the roots β_i, hence they commute with each other on $\mathcal{V}$, so on all X_d. We set then the projection $p\colon X_d \to \mathbb{C}^d$, $p(Q_i, T_i) = (Q_i)$ and observe that $(X_d, \Omega, Q_0, \ldots, Q_{d-1})$ is a holomorphic completely integrable system. The projection p defines here an endomorphism A of TX_d which is represented by

$$\begin{pmatrix} C_Q & 0 \\ 0 & C_Q \end{pmatrix}, \tag{3.3.1}$$

where C_Q is the so-called companion matrix of the polynomial $Q(\lambda)$ and meets our requirements. Therefore, thanks to our results of sections 3 and 4, we can recover the holomorphic completely integrable system $(X_d, \Omega, Q_0, \ldots, Q_{d-1})$ as the Hilbert scheme of d points of the surface $\mathbb{C} \times \mathbb{C}$ transverse to the projection onto the first coordinate.

In [4, Proposition 5] Beauville proves that the full Hilbert scheme $S^{[d]}$ of a complex symplectic surface (S, ω) has a symplectic form induced by ω. In the following Lemma we will explicitly recover his result on the transverse Hilbert scheme of d points $S_p^{[d]}$, which we know to be an open subset of the full Hilbert scheme. We remark that the existence of a symplectic form on the Hilbert scheme of d points in $\mathbb{C} \times \mathbb{C}^*$ transverse for the projection $p\colon \mathbb{C} \times \mathbb{C}^* \to \mathbb{C}$ onto the first coordinate was pointed out by Atiyah-Hitchin in [3, Chapter 2], where an explicit formula is only given on the subset $\mathcal{V} \subset (\mathbb{C} \times \mathbb{C}^*)_p^{[d]}$ of d-tuples consisting of all distinct points.

Lemma 3.7. *Let $p\colon S \to \mathbb{C}$ be a complex surface projecting onto $\mathbb{C}$ and assume that p is a submersion outside a discrete set $B \subset S$. Assume also that S possesses a holomorphic symplectic form ω Then ω induces a symplectic form Ω on the Hilbert scheme $S_p^{[d]}$ of d points in S transverse to p.*

Proof. We start by proving the Lemma in the case $d = 2$.
Fix $d = 2$ and define $N = (S \setminus B)_p^{[2]}$. Let $\mathcal{V} \subset S_p^{[2]}$ be the set of all 0-dimensional subschemes

of S consisting of two distinct points. For every $Z \in N \cap \mathcal{V}$, i.e. consisting of two different points p_1, p_2 of S, the isomorphism $T_Z N \cong H^0(Z, TS|_Z)$ easily yields the symplectic form on $N \cap \mathcal{V}$:the map $\psi \colon S \times S \to S_p^{[2]}$ has no ramification on $\mathcal{V}$ hence is a 2 to 1 covering. By breaking the $\mathfrak{S}_2$-symmetry and choosing a sheet of ψ, that is ordering the couple p_1, p_2, one splits $H^0(Z, TS|_Z) \cong T_{p_1}S \oplus T_{p_2}S$ and defines $\Omega_Z = \omega_{p_1} \oplus \omega_{p_2}$. Since the local coordinates (z, t) on $S \setminus B$ induce local coordinates (z_1, t_1, z_2, t_2) around each Z in $N \cap \mathcal{V}$ simply by evaluating on the points p_1 and p_2 of $Z \in N \cap \mathcal{V}$, we can locally write $\Omega = dz_1 \wedge dt_1 + dz_2 \wedge dt_2$ on $N \cap \mathcal{V}$. We now have to extend the form Ω to those elements $W \in N$ which consist of one point $s \in S \setminus B$ taken with double multiplicity. In order to do so we adapt a construction by Bielawski, [8], in the following way.

Let $W \in N$ be as above and observe that since p is a submersion on N, we can choose local coordinates (z, t) on a neighbourhood $\mathcal{U}$ around s such that the first one is the base coordinate of $\mathbb{C}$. Moreover, we can choose them in such a way that $\omega = dz \wedge dt$: if this was not the case, i.e. $\omega = \omega(z, t) dz \wedge dt$, we could define a Darboux coordinate chart (z, u) around s simply by choosing a new holomorphic fibre coordinate u such that $\partial u / \partial t = \omega(z, t)$. Of course such a u can always be found as it amounts to finding a primitive of a holomorphic function on a simply connected domain. We then describe the open set $(\mathcal{U})_p^{[2]} \subset N$ as the set of couples of polynomials $(q(z), t(z))$ such that q is monic of degree 2 and t is linear, that is $q(z) = z^2 - Q_1 z - Q_0$, $t(z) = T_0 + T_1 z$. On $(\mathcal{U})_p^{[2]} \cap \mathcal{V}$, i.e. where $q(z)$ has distinct roots z_1 and z_2 the polynomial $t(z)$ can be recovered by Lagrange interpolation from the values $t_1 = t(z_1)$ and $t_2 = t(z_2)$: this gives an equivalence between the two sets of coordinates (z_1, t_1, z_2, t_2) and (Q_0, Q_1, T_0, T_1). At this point we observe that the form Ω can be rewritten in the coordinates (Q_i, T_i) as $\Omega = Q_1 dT_1 \wedge dQ_1 + dT_1 \wedge dQ_0 + dT_0 \wedge dQ_1$, which is well defined, closed and non degenerate on the whole $\mathcal{U}_p^{[2]}$. Since, as we will prove in the next Lemma, this construction is independent of the choice of local coordinates ω induces a holomorphic symplectic form Ω on N. As B is discrete in S then $S_p^{[2]} \setminus N$ has codimension at least 2 in $S_p^{[2]}$ therefore Ω extends to the whole $S_p^{[2]}$ by Hartog's Theorem.

In the $d > 2$ case one proceeds exactly as above to get a form Ω defined on the set of all $Z \in S_p^{[d]}$ consisting either of d distinct points or of $(d-2)$ distinct points and one point which is taken with double multiplicity. Since the remaining subset has codimension greater than 2 in $S_p^{[d]}$ again the form Ω extends to the whole $S_p^{[d]}$ by Hartog's Theorem. ∎

We now show that this construction does not depend of the choice of coordinates.

Lemma 3.8. *The construction of Lemma 3.7 is independent of the choice of coordinates.*

Proof. Again we start from the $d = 2$ case, keeping the notation of the previous lemma. Let (z, t)

and (z, w) be two sets of Darboux coordinates adapted to the projection p on (S, ω). Denote by φ the change of coordinates between (z, t) and (z, w), so that $\omega' = (\varphi^{-1})^*\omega$ is defined. The same observation as in Lemma 3.7 yields a symplectic form Ω' on $N \cap \mathcal{V}$ and a change of coordinates Φ such that $\Omega' = \Phi^*\Omega$.
Let now $E \in N$ be an element of $S_p^{[2]}$ consisting of a point $s \in S$ taken with double multiplicity and let $\mathcal{U}'$ be a coordinate neighbourhood of s for the coordinates (z, w). The description of $(\mathcal{U}')_p^{[2]}$ is then

$$(\mathcal{U}')_p^{[2]} = \{(q(z), w(z)) | \deg q(z) = 2, \deg w(z) = 1, \ q \text{ monic } \}$$

where

$$q(z) = z^2 - Q_1 z - Q_0$$
$$w(z) = W_1 z + W_0.$$

Observe that (Q_i, W_i) are local coordinates on $(\mathcal{U}')_p^{[2]}$. By abuse of notation, we keep denoting by Φ the change of coordinates between (Q_i, T_i) of Lemma 3.7 and (Q_i, W_i) on the intersection $(\mathcal{U})_p^{[2]} \cap (\mathcal{U}')_p^{[2]}$. Then $(\Phi^{-1})^*\Omega$ is defined on all $(\mathcal{U})_p^{[2]} \cap (\mathcal{U}')_p^{[2]}$ and coincides with Ω' on $(\mathcal{U}')_p^{[2]} \cap (\mathcal{U})_p^{[2]} \cap \mathcal{V}$, therefore being its unique holomorphic extension. We conclude by extending $\Omega' = (\Phi^{-1})^*\Omega$ to the whole $S_p^{[2]}$ via Hartog's Theorem.
The generalization to greater d is again achieved by applying the $d = 2$ construction to the subset of all elements in $S_p^{[d]}$ consisting of d distinct points or $(d-2)$ distinct points and one double point and then by extension via Hartog's Theorem. ∎

Corollary 3.9. *The endomorphism A and the symplectic form Ω satisfy the condition $\Omega(A\cdot, \cdot) = \Omega(\cdot, A\cdot)$.*

Proof. It suffices to show the claim on the open dense subset $\mathcal{V} \subset S_p^{[d]}$ of elements consisting of all distinct points. But there we can use coordinates that are both Darboux for Ω and diagonalizing A, so the assertion is trivially verified. ∎

Corollary 3.10. *The transverse Hilbert scheme of points $S_p^{[d]}$ is a holomorphic completely integrable system*

Proof. This is an immediate consequence of Corollary 3.9: the coefficients Q_i of the minimal polynomial of A are d Poisson-commuting functions for the Poisson structure associated to Ω on the dense subset $\mathcal{V}$, hence on all $S_p^{[d]}$. ∎

Example 3.4. A basic example summarizing what we have done is given by taking $S = \mathbb{C} \times \mathbb{C}$, $p\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by $(x, y) \mapsto z = xy$ and $\omega = dx \wedge dy$. Of course p is a submersion away from the origin and $\{(0,0)\}$ is a codimension 2 subset of S. Hence on $(\mathbb{C} \times \mathbb{C})\backslash_{\{(0,0)\}} = \{x \neq 0\} \cup \{y \neq 0\} = \mathcal{U}_1 \cup \mathcal{U}_2$ we proceed exactly as in Example 3.1 and apply our construction taking coordinate (z, χ_1) on $\mathcal{U}_1$ and (z, χ_2) on $\mathcal{U}_2$ where $\chi_1 = -\log(x)$ and $\chi_2 = \log(y)$. Observe that on $\mathcal{U}_1$ we write $\omega = dz \wedge d\chi_1$ and $\omega = dz \wedge d\chi_2$ on $\mathcal{U}_2$ and that they agree on the overlap $\mathcal{U}_1 \cap \mathcal{U}_2$. Each patch can be described as the set

$$\{(q(z), p(z)) |\ q \text{ is monic of } \deg d,\ p \text{ is of degree } d-1 \text{ and } p(0) \neq 0 \text{ if } q(0) = 0\}. \tag{3.3.2}$$

The construction now yields the symplectic form Ω on the subset

$$((\mathbb{C} \times \mathbb{C}) \setminus B)_p^{[2]} = \{E \in (\mathbb{C} \times \mathbb{C})_p^{[2]} |\ (0,0) \notin E\}. \tag{3.3.3}$$

Since the complementary set to $((\mathbb{C} \times \mathbb{C}) \setminus B)_p^{[2]}$ has codimension 2, we get Ω on the whole $(\mathbb{C} \times \mathbb{C})_p^{[2]}$ applying Hartog's Theorem.

In the following proposition we work out the inverse construction in order to recover the holomorphic 2-form initially given on the surface S starting from the induced completely integrable system.

Proposition 3.11. *Let W be a complex manifold of complex dimension $2d$ endowed with an holomorphic endomorphism of the tangent space TW as in Proposition 3.5 and let $N \subset W$ be the subset where the induced projection $\mu\colon W \to \mathbb{C}^{[d]}$ is a surjective submersion. Assume also that W possesses a symplectic form Ω such that at every point of N*

- $\Omega(A\cdot, \cdot) = \Omega(\cdot, A\cdot)$
- *the vertical subbundle* $\ker(d\mu)$ *is maximal Ω-isotropic.*

Then $S = \tilde{W}/D$ has a symplectic form induced by ω

Proof. Our aim is to define a symplectic form τ on $\tilde{W}$ such that $\tau(X, Y) = 0$ for every $X \in T\tilde{W}$ and $Y \in D$, thus getting an induced form $\bar{\tau}$ on $S = \tilde{W}/D$. Consider the set $N \subset W$ on which μ is a surjective submersion: since $W \subset N$ has codimension at least two, the same also holds for its $d\colon 1$ covering $\tilde{N} \subset \tilde{W}$. Therefore it will suffice to define τ just on $\tilde{N}$. We will achieve this by setting $\tau = \rho^* dz \wedge \alpha_z$ (in the notation of Diagram 3.2.3) on $\tilde{N}$, where α_z is a 1-form on $\tilde{N}_z = \rho^{-1}(z) \cap \tilde{N} = \left\{(z, w) \in \tilde{W} |\ z \text{ is eigenvalue of } A_w,\ w \in N\right\}$. By the commutativity of Diagram 3.2.3 and the surjectivity of $d\mu$, at every point of N $\ker(z\mathbb{1} - A)$ maps surjectively onto $\ker(z\mathbb{1} - \bar{A})$. At this point we define $N_z = \pi(\tilde{N}_z)$ and we observe that the restriction

$\pi|_{\tilde{N}_z}: \tilde{N}_z \to N_z$ is obviously a diffeomorphism for every $z \in \mathbb{C}$ therefore N_z comes comes with a manifold structure. Using the surjectivity of $d\mu$ on N we can then choose a vector field $\mathbf{v}_z$ on $X_z = \mu(N_z)$ such that $\mathbf{v}_z \in \ker(z\mathbb{1} - \bar{A})$ at every point of X_z and lift it to a vector $\mathbf{V}_z$ tangent to N_z with $\mathbf{V}_z \in \ker(z\mathbb{1} - A)$. This lift is not uniquely determined: if $\mathbf{V}'_z \in \ker(z\mathbb{1} - A)$ is a second such lift, then $\mathbf{V}_z - \mathbf{V}'_z \in \ker(z\mathbb{1} - A) \cap \ker d\mu$.
Define now the 1-form $\alpha_z = \iota_{\mathbf{V}_z}\Omega$ on TW_z. This definition does not depend of the choice of $\mathbf{V}_z$. In fact, at every $w \in N_z$, one can split the tangent space to N_z as

$$T_w N_z \cong Im(z\mathbb{1} - A) \oplus \langle L \rangle \tag{3.3.4}$$

where $L \in \ker(z\mathbb{1} - A) \cap \ker(d\mu)$. Now, since $\Omega(A\cdot, \cdot) = \Omega(\cdot, A\cdot)$ we have that $\ker(z\mathbb{1} - A)$ and $Im(z\mathbb{1} - A)$ are Ω-orthogonal. This implies $\iota_{\mathbf{V}_z - \mathbf{V}'_z}\Omega(X) = 0$ for every $X \in Im(z\mathbb{1} - A$. Also, since $\ker(d\mu)$ is Lagrangian by assumption, we have $\iota_{\mathbf{V}_z - \mathbf{V}'_z}\Omega(L) = 0$. Hence $\iota_{\mathbf{V}_z - \mathbf{V}'_z}\Omega = \alpha_z - \alpha'_z = 0$ on all N_z, meaning α_z is well defined on N_z.
Since $\pi|_{\tilde{N}_z}$ is a diffeomorphism for every $z \in \mathbb{C}$, $d\pi$ is an isomorphism and we can therefore pull α_z back to $\tilde{N}_z$ via π and define $\tau = \rho^* dz \wedge \pi^* \alpha_z$ on $\tilde{N}$, hence on all $\tilde{W}$. As $D = Im(z\mathbb{1} - A)$ satisfies $\tau(\cdot, D) = 0$, the form τ descends to a form $\bar{\tau}$ on $S = \tilde{W}/D$. We now prove that $\bar{\tau}$ is symplectic. First of all, $d\bar{\tau} = 0$ as $\bar{\tau}$ is a 2-form on a 2-dimensional space. In order to prove its non-degeneracy we proceed as follows. First we observe that at every point $[(z, w)] \in S$ we have

$$T_{[(z,w)]}S \cong \langle Y \rangle \oplus T_w \tilde{W}_z / D_{(z,w)} \tag{3.3.5}$$

where Y is a vector in $T_w \tilde{W}$ such that $d\rho(Y) = \partial/\partial z$. Consider now $(z, w) \in \tilde{N}$, take $W \in T_w \tilde{W}_z$ such that $[W] \neq 0$ in $T_w \tilde{W}_z / D$ and compute

$$(\rho^* dz \wedge \pi^* \alpha_z)_{[(z,w)]}(Y, W) = \pi^* \alpha_z(W) = \Omega((\mathbf{V}_z)_w, d\pi_w(W)) \neq 0 \tag{3.3.6}$$

otherwise we would have $(\mathbf{V}_z)_w \in (T_w W_z)^\Omega$, where we denote with the superscript Ω the symplectic orthogonal complement. Now one observes that because both $Im(z\mathbb{1} - A)$ and $\ker(d\mu)$ are contained in $T_w W_z$ then $(T_w W_z)^\Omega \subseteq \ker(z\mathbb{1} - A) \cap \ker(d\mu)^\Omega = \ker(z\mathbb{1} - A) \cap \ker(d\mu)$ as $\ker(d\mu)$ is Lagrangian. By counting dimensions we actually have $(T_w W_z)^\Omega = \ker(z\mathbb{1} - A) \cap \ker(d\mu)$. But this would imply $\mathbf{V}_z \in \ker(d\mu)$ at w, which is in contrast with the fact that $\mathbf{V}_z$ was constructed as a lift of a vector field $\mathbf{v}_z$. Therefore on $\tilde{N}$ the form τ^2 gives an isomorphism $\bigwedge^2(T\tilde{W}/D) \cong \bigwedge^2(T\tilde{W}/D)^*$ which then extends to all $\tilde{W}$. Hence $\bar{\tau}$ is non degenerate on S.
As a last step we prove that when (W, Ω) is constructed as the transverse Hilbert scheme of a symplectic surface (S, ω) projecting onto $\mathbb{C}$ via p with the symplectic form Ω induced by ω then, once we recover S as $\tilde{W}/D$ we also get back the original symplectic form ω.
On $S \setminus B$ let us write $\omega = dz \wedge \varphi_z$, where φ_z is a 1-form defined on the fibre $p^{-1}(z)$.

Then on the usual open dense subset $\mathcal{V} \subset N$ of all d-tuples of distinct points in N we have $\Omega = \sum dz_i \wedge \varphi_{z_i}$. Let $\tilde{\mathcal{V}} = \{(z, w) \in \tilde{N} | z \text{ has multiplicity exactly } 2\}$ and note that $\tilde{\mathcal{V}}$ is open and dense in $\tilde{N}$. Call $r\colon \tilde{W} \to S$ the canonical projection onto the space of leaves: we have $cl(r(\tilde{\mathcal{V}})) = cl(r(cl(\tilde{\mathcal{V}})) = S$, where cl stands for the topological closure. Hence $r(\tilde{\mathcal{V}})$ is dense in S. Moreover, as the canonical projection onto the space of leaves of a foliation is always an open map [11, pag.47, Theorem 1], $r(\tilde{\mathcal{V}})$ is open in $S \setminus B$. Since on $\mathcal{V}$ we have $\mathbf{V}_{z_i} = \partial/\partial z_i$ for $i = 1, \ldots, d$, then $\iota_{\mathbf{V}_z}\Omega = \varphi_z$ for every z and it is clear that $\bar{\tau}$ agrees with ω on $r(\tilde{\mathcal{V}})$, hence ω and $\bar{\tau}$ coincide on $S \setminus B$, i.e. as claimed on the whole S. ∎

Chapter 4

The Geometry of l-hypercomplex structures

Contents

In this chapter we study how the fibration $p\colon Z \to T\mathbb{P}^1$ can by recovered once we are given $(M, A(\zeta))$ as long as $A(\zeta)$ satisfies some appropriate integrability assumption. The resulting construction, inverse to what we have exposed in Section 1.4 and Chapter 3, yields then a characterization of all hypercomplex manifold M equipped with a real endomorphism $A(\zeta) \in H^0(\mathrm{End}(E) \otimes \mathcal{O}(2))$, where E is defined as in 1.5, that arise as manifold of curve.

4.1 Geometric Considerations on $A(\zeta)$

Once more we consider the holomorphic fibration $p\colon Z \to T\mathbb{P}^1$, the manifold M of σ-invariant curves of degree d defined in Chapter 2 and denote by Z^{HK} the hyperkähler twistor space of M, equipped with its natural projection $\varphi\colon Z^{HK} \to \mathbb{P}^1$. Since also Z fibres on $\mathbb{P}^1$ via the map $\tilde{p}$ defined by composition of p and the base map of $T\mathbb{P}^1$, we can construct the incidence manifold $Y \subset Z \boxtimes_{\mathbb{P}^1} Z^{HK}$ defined as

$$Y = \left\{(u, z) \in Z \times Z^{HK} \,|\, \tilde{p}(u) = \varphi(z),\ u \in C_z\right\}, \tag{4.1.1}$$

where C_z is the curve in Z the corresponds to the point in M given by $z \in Z^{HK}$. Such incidence manifold Y comes naturally with a double fibration

$$\begin{array}{ccc} Y & & \\ {\scriptstyle\rho}\downarrow & \searrow^{\pi} & \\ Z & & Z^{HK} \end{array} \qquad (4.1.2)$$

for which we have

$$\rho^{-1}(u_0) = \{(u_0, z)|\ \varphi(z) = \tilde{p}(u_0),\ u_0 \in C_z\} \qquad (4.1.3)$$

and, of course, $T_{(u,z)}\rho^{-1}(u) = \ker d\rho_{(u,z)}$. Moreover we can set up the sequence

$$0 \longrightarrow T_u C \longrightarrow T_{(u,z)}Y \longrightarrow T_z\left(\varphi^{-1}(\zeta)\right) \longrightarrow 0, \qquad (4.1.4)$$

where we have set $\zeta = \tilde{p}(u)$ and z is the point corresponding to C in the fibre $\varphi^{-1}(\zeta) \subset Z^{HK}$. We can fit the above sequence into the following picture

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & T_u C & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & (\ker d\rho)_{(u,z)} & \overset{i}{\hookrightarrow} & T_{(u,z)}Y & \overset{d\rho}{\longrightarrow} & T_u Z & \longrightarrow & 0 \\
 & & & \searrow & \downarrow{\scriptstyle d\pi} & & & & \\
 & & & & T_z\left(\varphi^{-1}(\zeta)\right) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array} \qquad (4.1.5)$$

From the construction of the twistor space of a hyperkähler manifold exposed in Section 1.2 we immediately deduce that $T_z\left(\varphi^{-1}(\zeta)\right) \cong \left(T^{1,0}_C M\right)_{I_\zeta}$ where C is again the point of M corresponding to $z \in \varphi^{-1}(\zeta)$. Moreover, since $\left(T^{1,0}_C M\right)_{I_\zeta} \cong \left(T^{0,1}_C M\right)_{I_{-1/\bar{\zeta}}}$ and, as explained in Section 1.4, $\left(T^{0,1}_C M\right)_{I_{-1/\bar{\zeta}}} \cong H^0(C, N_{C/Z}[-C_{-1/\bar{\zeta}}])$ we have the identification

$$T_z\left(\varphi^{-1}(\zeta)\right) \cong H^0(C, N_{C/Z}[-C_{-1/\bar{\zeta}}]). \qquad (4.1.6)$$

We also observe that

$$(d\pi \circ i)\,(\ker d\rho)_{(u,z)} \subset H^0\left(C, N_{C/Z}[-C_{-1/\bar{\zeta}} - u]\right) \tag{4.1.7}$$

as the deformations that belong to $\ker d\rho$ at (u, z) fix the point $u \in C$ by definition. Actually we can improve the observation by proving the following.

Lemma 4.1. *The inclusion in Equation* (4.1.7) *is in fact an equality.*

Proof. First we note that the two vector space have the same dimension, hence it is enough to prove the injectivity of the map $d\pi \circ i$ on $\ker d\rho$ at the point (u, z). Assume then by contradiction that there exists a non zero element $v \in \ker d\rho$ at (u, z) such that $d\pi \circ i(v) = 0$ as a deformation in $H^0\left(C, N_{C/Z}[-C_{-1/\bar{\zeta}} - u]\right)$. Then, reading the vertical line of (4.1.5) we have that $v \in T_uC$, that means $v \in T_uC \cap \ker d\rho$, which cannot be when $v \neq 0$. Therefore $(d\pi \circ i)(\ker d\rho)_{(u,z)} \cong H^0\left(C, N_{C/Z}[-C_{-1/\bar{\zeta}} - u]\right)$, for every $(u, z) \in Y$. ∎

Remark 4.2. Define $D_{(u,z)} = H^0\left(C, N_{C/Z}[-C_{-1/\bar{\zeta}} - u]\right)$ where, as usual, $C \in M$ corresponds to $z \in \varphi^{-1}(\zeta)$ and $\zeta = \tilde{p}(u)$. Let $(\zeta, \eta) = p(u)$. With the identifications

$$T_z\varphi^{-1}(\zeta) \cong (T_C^{1,0}M)_{I_\zeta} \cong H^0\left(C, N_{C/Z}[-C_{-1/\bar{\zeta}}]\right) \cong H^0(C, N_{C/Z}(-1)) \tag{4.1.8}$$

and, since $A(\zeta)$ can now be lifted to each fibre of Y, it is then immediate to verify that $\ker(d\rho)_{(u,z)}$ coincides with the distribution $Im(\eta - A_C(\zeta))$. Therefore, having identified $Im(\eta - A_C(\zeta))$ with the set of all holomorphic deformations that fix the point $u \in C \subset Z$, the same proof of [6][Proposition 3.4] yields the integrability of the distribution.

The following observation will be of particular interest in the next section.

Remark 4.3. For every $\zeta \in \mathbb{P}^1$ we can consider the projection $\mu_\zeta \colon M \to \mathbb{C}^{[d]}$ that sends each $C \in M$ to the minimal polynomial $min_{A_C(\zeta)}(\lambda)$ of $A(\zeta)$ at the point C. Applying Proposition 1.23 and Remark 3.2 for every $\zeta \in \mathbb{P}^1$, we have that μ_ζ is a surjective submersion on a subset $N_\zeta \subset M$ such that $M \setminus N_\zeta$ is of codimension at least two. Moreover, a vector $X \in (\ker d\mu_\zeta)_C \subset (T_C^{1,0}M)_{I_\zeta}$ corresponds to a holomorphic deformation of the curve C that fixes the eigenvalues of $A(\zeta)$ at C. In other words it is an element X in $Q_{-1/\bar{\zeta}}$ for which $d_XA(\zeta) = 0$ at C. Let now ∇' be the connection on $\mathrm{End}(E) = E^* \otimes E$ obtained by the connection ∇'^E given in 1.27. Then $\nabla'_X(A(\zeta)) = 0$ for every X corresponding to a vector in $\ker d\mu_\zeta$. In turn, this means $\nabla_X\left(A(\zeta) \otimes \mathbb{1}\right) = 0$ for every $X \in \ker d\mu_\zeta$ for the Obata connection of M.

4.2 Reconstructing Z

In this section we perform the inverse construction, in order to recover the manifold Z and the fibration $p\colon Z \to T\mathbb{P}^1$ starting from the data on the hypercomplex manifold M.
Let us then start with a hypercomplex manifold of complex dimension $2d$ equipped with a aquaternionic endomorphism X of its real tangent space $T^{\mathbb{R}}M$. Recalling that on a hypercomplex manifold $T^{\mathbb{C}}M \cong E \otimes H$, we run the construction exposed in Section 2.3 backwards and get a matrix polynomial $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$, real for the natural real structure of $\mathcal{O}(2)$, which for every $\zeta \in \mathbb{P}^1$ yields and endomorphism of $T\varphi^{-1}(\zeta) \cong (T^{1,0}M)_{I_\zeta}$ on the fibres of the twistor space $\varphi\colon Z^{HK} \to \mathbb{P}^1$. Let us assume that for every $\zeta \in \mathbb{P}^1$ the eigenvalues of $A(\zeta)$ have eigenspaces of dimension two and that the characteristic polynomial of $A(\zeta)$ is the square of the minimal polynomial. If we also make the hypothesis that $A(\zeta)$ is holomorphic with respect to the complex structure I_ζ,we can consider for every $\zeta \in \mathbb{P}^1$ the projection to the Hilbert scheme $\mathbb{C}^{[d]}$ of d points of $\mathbb{C}$:

$$\begin{aligned} \mu_\zeta\colon M &\to \mathbb{C}^{[d]} \\ C \in M &\mapsto min_{A_C(\zeta)}(\lambda), \end{aligned} \tag{4.2.1}$$

which is then holomorphic for the complex structure I_ζ, i.e. we have a holomorphic projection on every fibre of $\varphi^{-1}(\zeta) \subset Z^{HK}$ to $\mathbb{C}^{[d]}$. For every ζ we assume that μ_ζ be a submersion on a subset N_ζ such that $M \setminus N_\zeta$ has codimension at least two and $A(\zeta)$ be compatible with μ_ζ in the sense of Definition 3.4. Finally we assume that, for every vector $X \in \ker d\mu_\zeta$, $\nabla_X A(\zeta) = 0$ for the Obata connection of M. With this corpus of hypotheses we can state the following proposition.

Proposition 4.4. *The Nijenhuis tensor of the restriction $A(\zeta)|_{\ker d\mu_\zeta}$ of $A(\zeta)$ to $\ker d\mu_\zeta$ vanishes.*

Proof. Since the Obata connection is torsion-free, for any couple of vectors $X, Y \in \ker d\mu_\zeta$ we have

$$\begin{aligned} N_C^{A(\zeta)}(X,Y) &= [A(\zeta)X, A(\zeta)Y] - A(\zeta)[A(\zeta)X, Y] - A(\zeta)[X, A(\zeta)Y] + A^2(\zeta)[X,Y] \\ &= \nabla_{A(\zeta)X}A(\zeta)Y - \nabla_{A(\zeta)Y}A(\zeta)X - A(\zeta)\nabla_{A(\zeta)X}Y + A(\zeta)\nabla_Y A(\zeta)X \\ &\quad -A(\zeta)\nabla_X A(\zeta)Y + A(\zeta)\nabla_{A(\zeta)Y}X + A^2(\zeta)\nabla_X Y - A^2(\zeta)\nabla_Y X, \end{aligned} \tag{4.2.2}$$

which vanishes since $\nabla_X(A(\zeta)(Y)) = A(\zeta)(\nabla_X Y)$ for X, Y as above. ∎

Let now $\mathcal{U}_\zeta$ be the open dense subset of N_ζ where $A(\zeta)$ has all distinct eigenvalues $\eta_1, \dots, \eta_d$ (and is, therefore, diagonalisable). Then on $\mathcal{U}_\zeta$ the nilpotent part of $A(\zeta)$ is trivial and its kernel is the full vertical tangent space $\ker d\mu_\zeta$. Therefore we can apply [38][Theorem1] and immediately get the following result.

Proposition 4.5. *For every $\zeta \in \mathbb{P}^1$ on every leaf of $\mu_\zeta|_{\mathcal{U}_\zeta}$ there exists coordinates $(\xi_1, \ldots, \xi_d)$ such that $A(\zeta)$ is represented by the diagonal matrix $\mathrm{diag}(\eta_1, \ldots, \eta_d)$ with respect to the frame $(\partial/\partial\xi_1, \ldots, \partial/\partial\xi_d)$.*

This means, in turn, that for every $\zeta \in \mathbb{P}^1$ there exist local coordinates $(\eta_1, \ldots, \eta_d, \xi_1, \ldots, \xi_d)$ on $\mathcal{U}_\zeta$ that are holomorphic for I_ζ and such that $A(\zeta)$ is represented by the matrix

$$\begin{pmatrix} \eta_1 \mathbb{1}_2 & & \\ & \ddots & \\ & & \eta_d \mathbb{1}_2 \end{pmatrix}. \tag{4.2.3}$$

Therefore on $\mathcal{U}_\zeta$ we have that $Im(\eta_i - A(\zeta)) = Span\,\{\partial/\partial\eta_j, \partial/\partial\xi_j\}_{j\neq i}$, which is of course an involutive, hence integrable, distribution. Since $\mathcal{U}_\zeta$ is dense in N_ζ and $M \setminus N_\zeta$ has codimension at least two, the distribution is integrable on all M.
Define now the incidence manifold $Y \subset T\mathbb{P}^1 \boxtimes Z^{HK}$ as

$$Y = \{(\zeta, \eta, z) | \eta \text{ is an eigenvalue of } A_C(\zeta)\} \tag{4.2.4}$$

where, as usual, C is the curve on M corresponding to $z \in Z^{HK}$. We have therefore the following diagram

$$\begin{array}{ccccc} & & Y \supset Y_\zeta & & \\ & \overset{\rho}{\swarrow} & & \overset{\pi}{\searrow} & \\ T\mathbb{P}^1 & & & & Z^{HK} \supset \varphi^{-1}(\zeta) \\ & \searrow & & \underset{\varphi}{\swarrow} & \\ & & \mathbb{P}^1 & & \end{array} \tag{4.2.5}$$

where we have denoted by Y_ζ the set $\{(\zeta, \eta, z) | \eta \text{ is eigenvalue of } A_C(\zeta)\}$), which is of course the $d\colon 1$ (branched) covering of $\varphi^{-1}(\zeta)$. At this point, since for every η in the spectrum of $A(\zeta)$ the distribution $Im(\eta - A(\zeta))$ is integrable on $(M, I_\zeta) \cong \varphi^{-1}(\zeta)$ we have that $D_{(\zeta,\eta,z)} = Im(\eta - A_C(\zeta))$ defines a distribution on Y which is contained in $\ker d\rho$ and is involutive outside the branch locus of π. Since the branch locus is nowhere dense, the integrability holds on all Y.
We can therefore define Z as the leaf space

$$Z := Y/D \tag{4.2.6}$$

and observe that it comes with a natural projection

$$\begin{aligned} p \colon Z &\to T\mathbb{P}^1 \\ [(\zeta, \eta, z)] &\mapsto (\zeta, \eta) \end{aligned} \tag{4.2.7}$$

that makes the following diagram commute.

$$\begin{array}{ccc} Z & \xleftarrow{proj} & Y \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle \rho} & \\ T\mathbb{P}^1 & & \end{array} \tag{4.2.8}$$

Finally, Z comes with a real structure covering the antipodal map of $\mathbb{P}^1$ naturally induced by the real structures of $T\mathbb{P}^1$ and Z^{HK}. We can summon the result we have proven up in the following proposition.

Proposition 4.6. *Let M be a hypercomplex manifold of complex dimension $2d$ with the following properties.*

1. *M is equipped with an aquaternionic endomorphisms of its real tangent bundle such that the associated real section $A(\zeta)$ of $\mathcal{O}(2)^{\oplus 4d^2}$ yields, for every $\zeta \in \mathbb{P}^1$, a holomorphic endomorphism of $(T^{1,0}M)_{I_\zeta}$ with 2-dimensional eigenspaces and such that its characteristic polynomial is the square of the minimal polynomial*
2. *For every ζ there exists a subset $N_\zeta \subset M$ such that $(M \setminus N_\zeta)$ has codimension at least two, the projection $\mu_\zeta \colon W \to \mathbb{C}^{[d]}$ is a surjective submersion on N_ζ and $A(\zeta)$ is compatible with μ_ζ in the sense of Definition 3.4*
3. *For every ζ and every $X \in \ker d\mu_\zeta$ the condition $\nabla_X A(\zeta) = 0$ holds on N_ζ, being ∇ the Obata connection of W.*

Then $p \colon Z = Y/D \to T\mathbb{P}^1$ is a complex 3-dimensional manifold projecting on $T\mathbb{P}^1$, endowed with a real structure σ covering the antipodal map, for which M is the manifold of σ-invariant cohomologically stable degree d curves.

This can be further improved when we assume M to be not just hypercomplex but hyperkähler.

Proposition 4.7. If M is taken to be hyperkähler then, on each fibre $\varphi^{-1}(\zeta)$ of its twistor space Z^{HK} a holomorphic symplectic 2-form $\omega(\zeta)$ is defined and, letting T_F stand for the tangent to the fibre of $\varphi \colon Z^{HK} \to \mathbb{P}^1$, the resulting section in $H^0\left(\bigwedge^2 T_F^* \otimes \mathcal{O}(2)\right)$, is real for the real structure of $\mathcal{O}(2)$. If we assume that for every $\zeta \in \mathbb{P}^1$ the differential form $\omega(\zeta)$ and the map

μ_ζ meet the requirements of Proposition 3.11, then on every fibre of the projection $\tilde{p}\colon Z \to \mathbb{P}^1$ we get a symplectic structure and such structures altogether describe a symplectic form on $\ker dp$ with values in $\mathcal{O}(2)$ which is compatible with the real structure σ.

Chapter 5

Rank 2 τ-bundles, monopoles and matrix polynomials

Contents

An interesting example of manifolds of curves that can be described as in Chapters 1 to 4 is provided by the moduli space of magnetic monopoles of charge k. In this chapter we give a very short review of the basic facts about such moduli spaces, then we focus on the case $k = 2$ and investigate some relation between matrix polynomials and rank 2 bundles on spectral curves in $|\mathcal{O}(2)|$.

5.1 Magnetic Monopoles in $\mathbb{R}^3$: the Basics

As a guideline for this section about the most important properties of magnetic monopoles we shall follow the classical article [21] by Hitchin and a more recent review [28] by Murray. We consider the Euclidean space $\mathbb{R}^3$ and denote by A a one-form with values in the Lie algebra $\mathfrak{su}(2)$, that is $A = \sum_{i=1}^{3} A_i dx^i$ where $A_i\colon \mathbb{R}^3 \to \mathfrak{su}(2)$ for every i. We interpret such matrix A as the matrix of one-forms of the connection $\nabla = d + A$ on a trivial $SU(2)$ bundle E over $\mathbb{R}^3$. Consider moreover an additional function $\Phi\colon \mathbb{R}^3 \to \mathfrak{su}(2)$, which takes the name of *Higgs field*. We are ready to define a $SU(2)$ magnetic monopole on $\mathbb{R}^3$ as follows.

Definition 5.1. *A magnetic monopole is a couple* (∇, Φ) *as above that satisfies the so-called*

Bogomolny equation

$$F_\nabla = \star\nabla\Phi \tag{5.1.1}$$

together with the appropriate boundary conditions.

Remark 5.2. The Bogomolny equation is gauge-invariant, therefore a monopole really defines an equivalence class of couples (A, Φ) under gauge transformations. Moreover, from the boundary conditions one can define a limiting value for Φ^∞ for the Higgs field by setting

$$\Phi^\infty(u) = \lim_{t\to\infty} \Phi(tu). \tag{5.1.2}$$

It can also be shown that the norm $|\Phi(u)|$ must be constant for all $u \in S^2$, hence one can normalise the Higgs field so that $|\Phi(u)| = 1$ for all directions u. Since $\mathfrak{su}(2)$ is three-dimensional, the Higgs field at infinity is then a map $\Phi^\infty\colon S^2 \to S^2$ and each such map belongs to a connected component of the space of all continuous maps $S^2 \to S^2$ and is labelled by a winding number k that we call the *charge* of the monopole.

As in [21], we view the tangent space $T\mathbb{P}^1$ to the Riemann sphere $\mathbb{P}^1$ as the space of oriented geodesics (i.e. straight lines) in $\mathbb{R}^3$. We define a vector bundle $\tilde{E}$ on $T\mathbb{P}^1$ by

$$\tilde{E}_z = \{s \in \Gamma(\gamma_z, E)|\, (\nabla_U - i\Phi)s = 0\}\,, \tag{5.1.3}$$

where U is the unit tangent vector along the oriented geodesic γ_z corresponding to $z \in T\mathbb{P}^1$. With this notation we recall Hitchin's result [21][Theorem 4.2].

Theorem 5.3. *If (∇, φ) is a solution of the Bogomolny equation $F_\nabla = \star\nabla\Phi$, then the vector bundle $\tilde{E}$ is in a natural way a holomorphic bundle on $T\mathbb{P}^1$ with the following properties*

1. *$\tilde{E}$ is trivial on every real section of $T\mathbb{P}^1$*
2. *$\tilde{E}$ has a symplectic structure*
3. *$\tilde{E}$ has a quaternionic structure*

$$E_z \to E_{\tau(z)}, \tag{5.1.4}$$

where τ denotes here the natural real structure of $T\mathbb{P}^1$.

On the converse, every such bundle on $T\mathbb{P}^1$ defines a solution to the Bogomolny equation.

Further detail about the nature of $\tilde{E}$ are provided by the following theorem [21][Theorem 6.3].

Theorem 5.4. *Let (∇, Φ) be a solution of the Bogomolny equation with the appropriate (see [21]) boundary conditions. Let L be the line bundle on $T\mathbb{P}^1$ with transition function $e^{-\eta/\zeta}$ and let L^+ (respectively L^-) denote the subbundle of $\tilde{E}$ consisting of all the solutions to $(\nabla_U - i\Phi)s = 0$ decaying when $t \to +\infty$. Then L^+ is a holomorphic subbundle isomorphic to $L(-k)$ and $\tilde{E}$ is given as the extension*

$$0 \longrightarrow L^+ \longrightarrow \tilde{E} \longrightarrow (L^+)^* \longrightarrow 0. \tag{5.1.5}$$

One can also prove

$$0 \longrightarrow L^- \longrightarrow \tilde{E} \longrightarrow (L^-)^* \longrightarrow 0. \tag{5.1.6}$$

and, by projecting L^- onto $(L^+)^*$ inside $\tilde{E}$ and using the fact that L^+ and L^- are holomorphically equivalent respectively to $L(-k)$ and $L^*(-k)$ one obtains a section of $H^0(T\mathbb{P}^1, \mathcal{O}(2k))$ the zero set of which describes the locus C of all $z \in T\mathbb{P}^1$ where $L_z^- = L_z^+$. Such locus C takes the name of *spectral curve* of the monopole and shows the following properties [21][Proposition 7.3].

Proposition 5.5. *Let C be the spectral curve of a $SU(2)$ monopole of charge k. Then the following hold true.*

1. *C is compact*
2. *C is defined by an equation*
$$P(\zeta, \eta) = \eta^k + a_i(\zeta)\eta^{k-1} + \cdots + a_k(\zeta) = 0, \tag{5.1.7}$$
where $\deg a_i = 2i$
3. *L^2 is holomorphically trivial on C, i.e. $L^2|_C \cong \mathcal{O}_C$*
4. *C is real for τ, i.e. $\tau(C) = C$.*

We observe that a trivialization $f(\zeta, \eta)$ of $L^2|_C$ embeds C as a curve $f(C)$ inside the three dimensional total space of L^2. Such curve is real for the real structure of $|L^2|$ and, by a result of Nash [29] satisfies our cohomological stability conditions on the normal bundle. In [21][Theorem 7.6] Hitchin finally proves that the spectral curve C fully determines the bundle $\tilde{E}$, therefore the two data are equivalent and a monopole is completely determined once we know its spectral curve. Finally, we recall the following fact due to Donaldson [13]. Let M_k be the moduli space of $SU(2)$ monopoles of charge k and fix a point $\zeta_0 \in \mathbb{P}^1$ and consider the map into the space Rat_k of based rational maps given by

$$\begin{aligned} \psi_{\zeta_0} \colon M_k &\to Rat_k \\ (P, f) &\mapsto \frac{f(\zeta_0, \eta)}{P(\zeta_0, \eta)}. \end{aligned} \tag{5.1.8}$$

Such map ψ_{ζ_0} is a then a diffeomorphism. This is of course a particular version of Proposition 1.23.In fact, the zero set $((\zeta_0,\eta_1),\dots,(\zeta_0,\eta_j))$ of $P(\zeta_0,\eta)$ together with the power expansions of $f(\zeta_0,\eta)$ around each point (ζ_0,η_j) describe exactly the divisor $C_{\zeta_0} = C\cap (L^2\backslash_0)_{\zeta_0}$ of the spectral curve C over the point $\zeta_0\in\mathbb{P}^1$. After this short introduction, we now consider the moduli space M_2 of charge 2 monopoles.

5.2 Monopoles of Charge 2

From the theory exposed in [3] we know that for every k the moduli space M_k of charge k monopoles is a hyperkähler manifold endowed with an action of $S^1\times\mathbb{R}^3$ such that $M_k = M_k^0\times S^1\times\mathbb{R}^3$, where M_k^0 stand for the space of centred monopoles of charge k. By choosing an identification of $S^1\times R^3$ with $\mathbb{C}\times\mathbb{C}^*$, that is fixing an element $\zeta_0\in\mathbb{P}^1$, we can describe such an action in term of rational maps as the map

$$\begin{aligned}(\mathbb{C}\times\mathbb{C}^*)\times Rat_k &\to Rat_k\\ \left((c,e^{\mu}),\frac{p(z)}{q(z)}\right) &\mapsto \left(\frac{e^{\mu}p(z)}{q(z+c)}\right).\end{aligned} \tag{5.2.1}$$

Let us fix now $k=2$ and define $S=\mathbb{C}\times\mathbb{C}^*$ with $p\colon S\to\mathbb{C}$ the projection onto the first factor. As before, let $M_2 = S_p^{[2]}$ stand for the space of charge 2 monopoles, which we identify with the space of based rational maps of degree 2 and let X be the double covering of the Atiyah-Hitchin manifold M_2^0, which sits in M_2 as

$$X = \left\{\frac{p(z)}{q(z)} \mid q_1 = 0,\ p(z)p(-z) = 1 \bmod q(z)\right\}, \tag{5.2.2}$$

where we write

$$\begin{aligned}p(z) &= p_0 + p_1 z\\ q(z) &= z^2 - q_1 z - q_0\end{aligned} \tag{5.2.3}$$

At every point $(p(z),q(z))$ of X the action (5.2.1) yields $TM_2|_X = TX\oplus V$ where V is generated by the action. Explicitly we have

$$TX = \left\{(p'(z),q'(z)) \mid q_1' = 0,\ p'(z)p(-z)+p'(-z)p(z) = 0 \bmod q(z)\right\} \tag{5.2.4}$$

and

$$V = \left\{(p'(z),q'(z)) \mid p'(z) = tp(z),\ q'(z) = uz,\ t,u\in\mathbb{C}\right\}. \tag{5.2.5}$$

. Using, as in Chapter 3, the coefficients (q_1, q_0, p_1, p_0) of $p(z)$ and $q(z)$ as global coordinates on M_2 and the induced frame of TM_2 we have at every point of X the following description

$$\begin{aligned} TX &= \mathrm{Span}\{\partial/\partial q_0, p_0\partial/\partial p_1 + q_0 p_1 \partial/\partial p_0\} \\ V &= \mathrm{Span}\{\partial/\partial q_1, p_1\partial/\partial p_1 + p_0\partial/\partial p_0\}. \end{aligned} \tag{5.2.6}$$

With respect to this generators, then the endomorphism $A(\zeta_0)$ is represented at every point of X by the matrix

$$A = \begin{pmatrix} 0 & 0 & q_0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & q_0 & 0 & 0 \end{pmatrix}. \tag{5.2.7}$$

The decomposition $TM_2|_X = TX \oplus V$ is actually a decomposition of real tangent spaces (see [3]), meaning that the vector space $E = H^0(C, N(-1)) \cong T_C^{\mathbb{R}} M_2$ splits into $E_1 \oplus E_2$ where, say, E_1 corresponds to TX and E_2 corresponds to the space generated by the $S^1 \times \mathbb{R}^3$ actions. Therefore we have that for every $\zeta \in \mathbb{P}^1$ the matrix polynomial $A_C(\zeta)$ has a block anti-diagonal form

$$A = \begin{pmatrix} 0 & P(\zeta) \\ Q(\zeta) & 0 \end{pmatrix}. \tag{5.2.8}$$

and the spectral curve C cut out by its minimal polynomial has equation $\eta^2 - a_2(\zeta) = 0$, where $a_2(\zeta)$ is a polynomial of degree 4. With this information we observe that if $(u, v) \in \mathbb{C}^4$ is an eigenvector of $A_C(\zeta)$ with eigenvalue λ then $P(\zeta)v = \lambda u$ and $Q(\zeta)u = \lambda v$. Applying again $A(\zeta)$ to the vector $(\lambda u, \lambda v)$ we deduce that $P(\zeta)Q(\zeta) = Q(\zeta)P(\zeta) = \lambda^2 \mathbb{1}_2 = a_2(\zeta)\mathbb{1}_2$. Now, this particular block anti-diagonal form of $A_C(\zeta)$, together with the results of [7], suggests that we consider in general the case of degree 2 matricial polynomials with block-anti-diagonal form as in (5.2.8), that have a spectral curve with equation $\lambda^2 - a_2(\zeta) = 0$ in $|\mathcal{O}(2)|$ with coordinates (ζ, λ).

5.3 Some partial results on rank 2 τ-bundles

The aim of this section is to provide a deeper understanding of those rank 2 vector bundles on elliptic curves that are induced by matricial polynomials of the form (5.2.8). A useful guideline

to be followed is Bielawski's article [7]. We start by setting some notation and recalling the so-called *Beauville isomorphism*. We then focus on the so called τ-bundles of rank two on elliptic curves and explain their relation with the spectral curves of magnetic monopoles of charge 2. We conclude the section by approaching, with some partial result, the issue whether such rank-2 bundles be or not decomposable as, in view of the famous Atiyah classification of bundles of rank 2 on elliptic curves (see [2]), this would be a crucial information.
Consider now a quadratic polynomial $X(\zeta) = X_0 + X_1\zeta + X_2\zeta^2$ with $X_i \in \mathfrak{gl}_k(\mathbb{C})$. Each such polynomial gives rise to a 1-dimensional acyclic sheaf on $\overline{T} = |\mathcal{O}(2)|$ by means of the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{T}}(-3)^{\oplus(k)} \xrightarrow{\lambda - X(\zeta)} \mathcal{O}_{\overline{T}}(-1)^{\oplus(k)} \longrightarrow \mathcal{F} \longrightarrow 0\,. \tag{5.3.1}$$

When $X(\zeta)$ has two-dimensional eigenspaces and its characteristic polynomial is the square of its minimal polynomial then the sheaf $\mathcal{F}$ is the sheaf of sections of a rank 2 vector bundle supported on the curve in $\overline{T}$ cut out by the minimal polynomial $P(\zeta,\lambda)$ of $X(\zeta)$. Taking into account the action of conjugation by elements of $GL_k(\mathbb{C})$ we have the following proposition, reproducing the well known result of [5].

Proposition 5.6. *Let d be a positive integer and $P(\zeta,\lambda) = \lambda^k + a_1(\zeta)\lambda^{k-1} + \cdots + a_k(\zeta)$ be a polynomial with $\deg a_i = di$. Let M_P be the variety defined by*

$$M_P = \{X(\zeta) \in \mathfrak{gl}_k(\mathbb{C}) |\ \deg X(\zeta) = d,\ \min(X(\zeta)) = P(\zeta,\lambda)\} \tag{5.3.2}$$

and denote by M'_P the submanifold of M_P consisting of all $X(\zeta)$ having eigenspaces of dimension two and whose characteristic polynomial is the square of the minimal polynomial. Then the exact sequence

$$0 \longrightarrow \mathcal{O}_T(-d-1)^{\oplus(k)} \xrightarrow{\lambda - X(\zeta)} \mathcal{O}_T(-1)^{\oplus(k)} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{5.3.3}$$

induces a one-to-one correspondence between $M'_P/GL_k(\mathbb{C})$ and the isomorphism classes of acyclic vector bundles of rank 2 defined on the curve $C \subset |\mathcal{O}(d)|$ cut out by the polynomial $P(\zeta,\lambda)$.

We shall refer to the above correspondence as to the *Beauville isomorphism*.
We now recall the description of a particular case of the Beauville isomorphism and describe the construction of the so-called τ*-sheaves* using matrix polynomials of degree 2. To this purpose, let $R_{m,n} \subset \mathfrak{gl}_{n+m}(\mathbb{C}) \otimes \mathbb{C}^3$ be the subset of triples of matrices of the form

$$C_i = \begin{pmatrix} 0 & P_i \\ Q_i & 0 \end{pmatrix}, \quad i = 0, 1, 2. \tag{5.3.4}$$

For any such triple (C_0, C_1, C_2) of square $(m+n) \times (m+n)$ matrices we can define a 1-dimensional sheaf on the space $\overline{T} = |\mathcal{O}(2)|$ via the sequence

$$0 \longrightarrow \mathcal{O}_{\overline{T}}(-3)^{\oplus(m+n)} \xrightarrow{\lambda - C(\zeta)} \mathcal{O}_{\overline{T}}(-1)^{\oplus(m+n)} \longrightarrow \mathcal{F} \longrightarrow 0\,, \tag{5.3.5}$$

where we have set $C(\zeta) = C_0 + C_1\zeta + C_2\zeta^2$. At this point we observe that the involution

$$\tau\colon |\mathcal{O}(2)| \to |\mathcal{O}(2)| \tag{5.3.6}$$

$$(\zeta, \lambda) \mapsto (\zeta, -\lambda) \tag{5.3.7}$$

can be defined on the total space of $\mathcal{O}(2)$, and that the quotient $|\mathcal{O}(2)|/\tau$ is exactly $|\mathcal{O}(4)|$. For an element $(C_0, C_1, C_2) \in R_{m,n}$, the polynomial $\det(\lambda - C(\zeta))$ is τ-invariant and has the form

$$\det(\lambda - C(\zeta)) = \lambda^{n-m}\left(\lambda^{2m} + a_1(\zeta)\lambda^{2m-2} + \cdots + a_{m-1}(\zeta)\lambda^2 + a_m(\zeta)\right), \tag{5.3.8}$$

where $\deg(a_i) = 2i$. Applying the Beauville isomorphism we have that $GL_n(\mathbb{C}) \times GL_m(\mathbb{C})$-orbits of elements in $R_{m,n}$ correspond to acyclic τ-invariant sheaves supported on the spectral curve of $C(\zeta)$. By τ-invariance we mean equivariance with respect to the action of τ, i.e. in the case when $\mathcal{F}$ is the sheaf of sections of a vector bundle we have a bundle involution $\bar{\tau}$ on the total space of $\mathcal{F}$ lifting τ. If we restrict to the subset of matrices $C(\zeta)$ in $R_{m,n}$ having minimal polynomial $P(\zeta, \lambda)$ and satisfying our regularity conditions on eigenspaces and characteristic and minimal polynomial we get, as before, a one-to-one correspondence between $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$-orbits of regular elements and acyclic τ-vector bundles of rank two on the curve in $|\mathcal{O}(2)|$ cut out by $P(\zeta, \lambda)$.

Our endomorphism $A(\zeta)$ clearly falls in the above description with $m = n = 2$ therefore, as anticipated, we will consider in general those matrix polynomials of the form

$$C(\zeta) = \begin{pmatrix} 0 & P(\zeta) \\ Q(\zeta) & 0 \end{pmatrix}, \tag{5.3.9}$$

which satisfy our regularity assumptions and whose spectral curve is given as the zero locus of a polynomial of the form $\lambda^2 - a_2(\zeta)$, where $a_2(\zeta)$ is a polynomial in ζ of degree 4. Now, as explained in [7], we note that the quotient of the spectral curve $\bar{C} \subset \overline{T} = |\mathcal{O}(2)|$ of $C(\zeta)$ by the involution τ is a curve $\hat{C}$ inside $\hat{T} = |\mathcal{O}(4)|$ and there is a correspondence between acyclic τ-invariant vector bundles of rank 2 on $\bar{C}$ and acyclic vector bundles of rank 2 on $\hat{C}$. With respect to (5.3.9), we write $P(\zeta) = P_0 + P_1\zeta + P_2\zeta^2$ and $Q(\zeta) = Q_0 + Q_1\zeta + Q_2\zeta^2$. Let $\overline{P}(\zeta, \lambda)$ denote the minimal polynomial of $C(\zeta)$ and $\hat{P}(\zeta, \eta)$ be the minimal polynomial of $P(\zeta)Q(\zeta)$. Since from $C^2(\zeta) = a_2(\zeta)\mathbb{1}_4$ we deduce that $P(\zeta)Q(\zeta) = Q(\zeta)P(\zeta) = a_2(\zeta)\mathbb{1}_2$, it is then clear that $\hat{P}(\zeta,\eta) = (\eta - a_2(\zeta))$ and that $\overline{P}(\zeta,\lambda) = \hat{P}(\zeta,\lambda^2)$ holds true. We denote by $\mathcal{F}$ the rank 2, τ-invariant, 1-dimensional acyclic sheaf defined by $C(\zeta)$ on the curve $\bar{C}$. and by $\mathcal{G}$ the rank 2 acyclic sheaf defined on the curve $\hat{C}$ by $P(\zeta)Q(\zeta)$.

Remark 5.7. When the curve $\bar{C}$ and the matrix polynomial $C(\zeta)$ are constructed as in the previous chapters, we observe that, by construction, $\mathcal{F}$ is the sheaf of sections of the vector bundle $N_{C/Z}(-2)$. In the case of spectral curves of monopoles we have by the results of Nash [29] that $N_{C/Z}(-2) \cong \tilde{E}L(k-2)|_C$, where (after identifying C and $\bar{C}$ as in Nash) $\tilde{E}$ is the *monopole bundle* provided by Hitchin's construction, L is the line bundle on $T\mathbb{P}^1$ with transition function $exp(\lambda/\zeta)$ and k is the charge of the monopole. In this spirit, the next discussion gives us some new hint concerning the decomposability of the monopole bundle $\tilde{E}$.

In the most desirable case, the zeros of $\det(P(\zeta))$ are distinct from those of $\det(Q(\zeta))$, i.e. the divisors defined by $\det(P(\zeta))$ and $\det(Q(\zeta))$ are disjoint. Denoting by Δ_P (respectively Δ_Q) the divisor of zeros of $\det(P(\zeta))$ (respectively $\det(Q(\zeta)) = 0$) on $\bar{C}$ taken with single multiplicity, we can prove the following statement.

Proposition 5.8. *With the above assumptions, we have* $\mathcal{F} \cong \pi^*\mathcal{O}(-1)^{\oplus 2} \otimes [\Delta_Q] \cong \pi^*\mathcal{O}(-1)^{\oplus 2} \otimes [\Delta_P]$*, where* π *is the projection* $\bar{C} \to \hat{C}$.

Proof. Again following [7] we start by considering the sheaves $\mathcal{F}(1)$ and $\mathcal{G}(1)$ respectively given as co-kernels of the maps

$$\begin{aligned} \lambda - C(\zeta)\colon \mathcal{O}(-2)^{\oplus 4} &\to \mathcal{O}^{\oplus 4} \\ \eta - P(\zeta)Q(\zeta)\colon \mathcal{O}(-4)^{\oplus 2} &\to \mathcal{O}^{\oplus 2}. \end{aligned} \tag{5.3.10}$$

It is immediate to observe that $\eta - P(\zeta)Q(\zeta)$ actually is the zero map, therefore we have $\mathcal{G}(1) = \mathcal{O} \oplus \mathcal{O}$. Now, looking at the sequence

$$0 \longrightarrow \mathcal{O}_{\hat{T}}(-4)^{\oplus 2} \xrightarrow{\eta - P(\zeta)Q(\zeta)} \mathcal{O}_{\hat{T}}^{\oplus 2} \longrightarrow \mathcal{G}(1) \longrightarrow 0 \tag{5.3.11}$$

we see that, since $H^0(\mathcal{O}^{\oplus 2}) = \mathbb{C}^2$ and $\mathcal{G}(1) = \mathcal{O}^{\oplus 2}$, every vector $u \in \mathbb{C}^2$ trivially defines a section s_u of $\mathcal{G}(1)$. Consider the sections s_1 and s_2 corresponding to the vectors $u_1 = (1,0)$ and $u_2 = (0,1)$ in $\mathbb{C}^2$. In general, by means of (5.3.5) tensored by $\mathcal{O}(1)$, every vector in $\mathbb{C}^4$ defines, by the projection to the quotient, a global section of $\mathcal{F}(1)$. Therefore for a fixed vector $u \in \mathbb{C}^2$, the vector $(u,0) \in \mathbb{C}^4$ defines a global section $\bar{s}_u$ of $\mathcal{F}(1)$. Assuming $u \neq 0$, one can easily check that $(u,0) \in Im(\lambda - C(\zeta))$ if and only if either $\lambda \neq 0$ and $u \in Im(\lambda^2 - P(\zeta)Q(\zeta))$, i.e. $u = 0$, or $\lambda = 0$, $\det Q(\zeta) = 0$ and $u \in Im(P(\zeta))$, where the last condition actually follows from $\lambda = 0$, $\det Q(\zeta) = 0$ since $\Delta_P \cap \Delta_Q = \emptyset$.Therefore the divisor of a section $\bar{s}_u$ is given as $(\bar{s}_u) = \Delta_Q$ when $u \neq 0$. Using the isomorphism $H^0(\overline{C}, \mathcal{F}(1)) \cong H^0(\overline{C}_\zeta, \mathcal{F}(1))$ we see that the sections $\bar{s}_1$ and $\bar{s}_2$ given by $(u_1, 0)$ and $(u_2, 0)$ are linearly independent in $H^0(\mathcal{F}(1))$ and span a two-dimensional subset consisting of sections which vanish on Δ_B. In other words, they yield two global sections $\bar{t}_1, \bar{t}_2$ of $\mathcal{F}(1) \otimes [-\Delta_Q]$, that is $h^0(\bar{C}, \mathcal{F}(1) \otimes [-\Delta_Q]) \geq 2$. At this point we observe that the genus of $\bar{C}$ is $g_{\bar{C}} = 1$, that is $\bar{C}$ is an elliptic curve, and that $\mathcal{F}(1) \otimes [-\Delta_Q]$ is a rank 2 vector bundle of degree zero on $\bar{C}$. Since its space of sections is two-dimensional, $\mathcal{F}(1) \otimes [-\Delta_Q]$ cannot be indecomposable, as the only indecomposable bundle of rank 2 and degree 0 with sections is the so-called *Atiyah bundle* and its space of sections is 1-dimensional [37][Proposition 4.6]. Therefore, since $\mathcal{F}(1)$ must decompose into the sum of two line bundles, we know from the above observations that it must be $\mathcal{F}(1) = \pi^*\mathcal{O}^{\oplus 2} \otimes [\Delta_Q]$, hence $\mathcal{F} = \pi^*\mathcal{O}(-1)^{\oplus 2} \otimes [\Delta_Q]$. ▎

Unfortunately, we observe in the following remark that in the case of spectral curves of magnetic monopoles the determinants of $P(\zeta)$ and $Q(\zeta)$ have exactly the same zeros.

Remark 5.9. We know from Section two that (5.2.8) is conjugated to (5.2.7) by an element of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$. Now, if there were a point ζ_0 such that $\det(Q(\zeta_0)) = 0$ whilst $\det(P(\zeta_0)) \neq 0$, then the regularity assumptions on A would force $Q(\zeta_0)$ to actually be the zero matrix. In this case the matrix $A(\zeta_0)$ would not be $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$-conjugated to

$$\begin{pmatrix} 0 & 0 & q_0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & q_0 & 0 & 0 \end{pmatrix}$$

and this would be a contradiction.

Therefore we investigate now the case when $\det(P(\zeta))$ and $\det(Q(\zeta)$ vanish at the same four points, which turns out to be more complicated than the previous one. Now, keeping in mind that the spectral curve $\bar{C}$ is an elliptic curve our goal, for the moment only partially fulfilled, is to try and understand whether and when the vector bundle defined by the matrix polynomial $C(\zeta)$ of (5.3.9) is decomposable or not. As in the previous proposition, let us consider the vector bundles associated to the sheaves $\mathcal{F}(1)$ and $\mathcal{G}(1)$, given as the co-kernels of the maps

$$\begin{aligned} \lambda - C(\zeta)&\colon \mathcal{O}(-2)^{\oplus 4} \to \mathcal{O}^{\oplus 4} \\ \eta - P(\zeta)Q(\zeta)&\colon \mathcal{O}(-4)^{\oplus 2} \to \mathcal{O}^{\oplus 2}. \end{aligned} \tag{5.3.12}$$

Again $\mathcal{G}(1) \cong \mathcal{O} \oplus \mathcal{O}$ and any vector $u \in \mathbb{C}^2$ trivially defines a section $s_u \in H^0(\hat{C}, \mathcal{G}(1))$ and we can consider the section $\bar{s}_u \in H^0(\bar{C}, \mathcal{F}(1))$ defined by the vector $(u, 0) \in \mathbb{C}^4$ exactly as before. Again, at a point $(\zeta, \lambda) \in \bar{C}$ with $\lambda \neq 0$, we have that $(u, 0)$ belongs to $Im(\lambda - C(\zeta))$ if and only if $u = 0$. At a point of $(\zeta, 0)$ of $\bar{C}$ we have that $(u, 0)$ belongs to $Im(\lambda - C(\zeta))$ if and only if $u \in Im(P(\zeta))$, but this last condition is not any more automatically satisfied. Let us now assume that the bundle $\mathcal{F}(1)$ be decomposable and, in analogy to the case of monopole spectral curves, that $h^1(\mathcal{F}(1)) = 0$. Then $\mathcal{F}(1)$ can only split

- as a sum of degree 0 and a degree 4 line bundle $L_0 \oplus L_4$. Since we know that $h^0(\mathcal{F}(1) = 4)$, both bundles must have sections, hence $L_0 \cong \mathcal{O}_{\bar{C}}$. This bundle has, anyway, a non-vanishing h^1 therefore this splitting cannot be realized
- as a sum of a degree 1 and a degree 3 line bundle $L_1 \oplus L_3$
- as a sum of two degree 2 lined bundles $L \oplus L'$.

Let now $\zeta_1, \ldots, \zeta_4$ be the zeros of $\det(P(\zeta))$ (and of $\det(Q(\zeta))$ and let us proceed case by case.

1. The first possibility is that $Im(P(\zeta_i)) \subset \mathbb{C}^2$ all coincide for $i = 1, \ldots, 4$. In this case, if we choose $u \in Im(P(\zeta_i))$, the section corresponding to the vector $(u, 0) \in \mathbb{C}^4$ will vanish at all four points $\zeta_1, \ldots, \zeta_4$. Now, if the splitting $L_1 \oplus L_3$ was realized, we would have, due to $h^1(\mathcal{F}(1)) = 0$ that $h^1(L_1) = 0 = h^1(L_3)$ and, in particular that $h^0(L_1) = 1$. Therefore all sections of L_1 with a zero would actually vanish at the same point. Therefore, since a section of $L_1 \oplus L_3$ is given as a couple (s_1, s_3) of sections $s_i \in H^0(L_i)$, it would be impossible to construct a section with 4 distinct zeros. Similarly, if $\mathcal{F}(1) \cong L \oplus L'$, the section corresponding to the above choice of $(u, 0)$ would be given as a couple of sections (s, s'), where $s \in H^0(L)$ and $s' \in H^0(L')$, with four common zeros. Since s, s' are sections of line bundles of degree 2 they can only have two zeros, therefore also this decomposition cannot be realized. Hence we can conclude that when the subspaces $Im(P(\zeta_i)) \subset \mathbb{C}^2$ all coincide for $i = 1, \ldots, 4$ the bundle $\mathcal{F}(1)$ is indecomposable.
2. Assume now that $Im(P(\zeta_1)) = Im(P(\zeta_2)) = Im(P(\zeta_3)) \neq Im(P(\zeta_4))$. The choice of a vector $u \in Im(P(\zeta_i)), i = 1, 2, 3$ yields a global section of $\mathcal{F}(1)$ vanishing at ζ_i, $i = 1, 2, 3$. As before, this excludes both the $L_1 \oplus L_3$ and $L \oplus L'$ possibilities, hence the bundle $\mathcal{F}(1)$ must be indecomposable also in this case.
3. As a third possibility, we focus on the situation when $Im(P(\zeta_1)) = Im(P(\zeta_2))$ and $Im(P(\zeta_2)) \neq Im(P(\zeta_3)) \neq Im(P(\zeta_4))$. Arguing as in the previous cases, we are here able to construct a section with zeros at $(\zeta_1, 0)$ and $(\zeta_2, 0)$, hence we can exclude the

$L_1 \oplus L_3$ case. If we assume then that $\mathcal{F}(1)$ decomposes as $L \oplus L'$, we have that such a section is given as (s, s'), with $s \in H^0(L)$ and $s' \in H^0(L')$ sharing the same two zeros $(\zeta_1, 0)$ and $(\zeta_2, 0)$. Therefore we would have that $L = L' = [(\zeta_1, 0), (\zeta_2, 0)]$. At this point, taking a vector $v \in Im(P(\zeta_3))$ we could construct a section only vanishing at $(\zeta_3, 0)$. Let this section be described as (t, t'), where $t, t' \in H^0(L)$. Then t and t' share $(\zeta_3, 0)$ as the only common zero and t/t' would be a function on $\bar{C}$ with exactly one zero and one pole. It is anyway known that no such function exists on an elliptic curve. This contradiction yields the indecomposability of the bundle $\mathcal{F}(1)$ also in this situation.

4. The fourth possibility is when $Im(P(\zeta_1)) = Im(P(\zeta_2)) \neq Im(P(\zeta_3)) = Im(P(\zeta_4))$. The decomposition of type $L_1 \oplus L_3$ is clearly immediately excluded as, also in this situation, we can construct sections that vanish at more than one point. If we assume $\mathcal{F}(1) \cong L \oplus L'$, we can argue as before an conclude that $L = L' = [(\zeta_1, 0), (\zeta_2, 0)] \cong [(\zeta_3, 0), (\zeta_4, 0)]$. From section $s = (s_1, s_2)$ of $L \oplus L$ vanishing at $(\zeta_1, 0)$ and $(\zeta_2, 0)$ we can clearly construct two sections $(s_1, 0)$ and $(0, s_2)$ with the same zeros, which are linearly independent in $H^0(L \oplus L)$. This means that we can construct two sections t_1 and t_2 that correspond to vectors $(u_1, 0)$ and $(0, v_2)$ and both vanish exactly at $(\zeta_1, 0)$ and $(\zeta_2, 0)$. From this we deduce that also for $Q(\zeta)$ we have $Im(Q(\zeta_1)) = Im(Q(\zeta_2)) \neq Im(Q(\zeta_3)) = Im(Q(\zeta_4))$. On the other hand, it is immediate to show that when $Im(P(\zeta_1)) = Im(P(\zeta_2)) \neq Im(P(\zeta_3)) = Im(P(\zeta_4))$ and $Im(Q(\zeta_1)) = Im(Q(\zeta_2)) \neq Im(Q(\zeta_3)) = Im(Q(\zeta_4))$ the bundle decomposes as $L \oplus L$. From this equivalence we also conclude that if $Im(P(\zeta_1)) = Im(P(\zeta_2)) \neq Im(P(\zeta_3)) = Im(P(\zeta_4))$ holds true but the condition $Im(Q(\zeta_1)) = Im(Q(\zeta_2)) \neq Im(Q(\zeta_3)) = Im(Q(\zeta_4))$ is not satisfied, then $\mathcal{F}(1)$ cannot be decomposable.

5. The last case is when $Im(P(\zeta_i))$ are four distinct subspaces of $\mathbb{C}^2$. Looking at the subspaces $Im(Q(\zeta_i)), i = 1, \ldots, 4$ at applying the above consideration, we immediately get the indecomposability of $\mathcal{F}(1)$ in all sub-cases, except for the one when $Im(Q(\zeta_i))$ are also all distinct. In this last situation we can exclude both a decomposition of type $L_1 \oplus L_3$, due to the fact that we can construct four sections with each with a different single zero, as well as one of type $L \oplus L$, again due to the possibility of constructing sections with single zeros. The question whether a decomposition of type $L \oplus L'$ be admissible instead of the indecomposability being the only option for this situation is at the moment still an open one.

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Curriculum vitae

Niccolò Lora Lamia Donin was born on December 21, 1990, in Biella, Italy. He grew up in Biella, where he attended the *Liceo Scientifico "Amedeo Avogadro"*. He obtained his High-school Diploma in Scientific Studies in July 2009.

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In October 2012, he enrolled as a first-year M.Sc. student at the *University of Turin* in Pure Mathematics, majoring in (Differential) Geometry. During his second year of courses, he was a winner of a money prize in support of international mobility for Master students. He graduated Summa cum Laude in Pure Mathematics in July 2014 under the joint supervision of Prof. Dr. Simon Salamon and Prof. Dr. Anna Maria Fino, with the thesis *"Kähler, Complex, Hermitian Geometry. Fubini distance in* $\mathbb{CP}^{n-1}$ *with computations in dimension* 2*"*, achieving the Master Degree and the academic title of *"Dottore Magistrale in Matematica"*.

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