Skyrme model from 6d $\mathcal{N} = (2,0)$ theory

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ABSTRACT

We consider 5d Yang–Mills theory with a compact ADE-type gauge group G on $\mathbb{R}^{3,1} \times \mathcal{I}$, where $\mathcal{I}$ is an interval. The maximally supersymmetric extension of this model appears after compactification on S^1 of 6d $\mathcal{N} = (2,0)$ superconformal field theory on $\mathbb{R}^{3,1} \times S_2^2$, where $S_2^2 \cong \mathcal{I} \times S^1$ is a two-sphere with two punctures. In the low-energy limit, when the length of $\mathcal{I}$ becomes small, the 5d Yang–Mills theory reduces to a nonlinear sigma model on $\mathbb{R}^{3,1}$ with the Lie group G as its target space. It contains an infinite tower of interacting fields whose leading term in the infrared is the four-derivative Skyrme term. A maximally supersymmetric generalization leading to a hyper-Kähler sigma-model target space is briefly discussed.

1. Introduction and summary

It is generally believed that the Skyrme model [1] describes low-energy QCD by interpreting baryons as solitons of a chiral model (see e.g. [2] for a review and references). The standard Skyrme model describes pion degrees of freedom, and it is not easy to include other mesons into the model. A possible resolution of this difficulty was proposed by Sakai and Sugimoto [3,4] who analyzed non-supersymmetric D4–D8–$\overline{\text{D8}}$ brane configurations in string theory and the holographic dual of this system. Starting from the DBI action they arrived at gauge theory on five-dimensional AdS-type manifold M^5 with Minkowski space $\mathbb{R}^{3,1}$ as a conformal boundary and fifth spatial coordinate $z \in \mathbb{R}$ in the additional holographic direction. Their further analysis of holographic QCD leads to an effective Skyrme model on $\mathbb{R}^{3,1}$.

The group-valued Skyrme field (parametrizing the pion) in the Sakai–Sugimoto model corresponds to the holonomy of a gauge connection,

$$g(x) = \mathcal{P} \exp\left(\int_{-\infty}^{\infty} \mathcal{A}_z(x,z)\,\mathrm{d}z \right) \quad \text{for} \quad x \in \mathbb{R}^{3,1}, \tag{1.1}$$

where the component $\mathcal{A}_z$ of a gauge potential along the holographic direction in M^5 corresponds to the quark condensate of QCD, and the holonomy (1.1) describes the space–time dependent fluctuations around the manifold of vacua governed by the Skyrme model [5]. Static skyrmion fields in this model correspond to Yang–Mills instantons on a Euclidean slice of M^5.

The Sakai–Sugimoto model [3,4] is currently the best known holographic model of hadron physics.[1] The Skyrme Lagrangian as seen by holography is modified – an infinite number of terms coupling the pion field to a tower of vector mesons naturally appears in the Lagrangian. Remarkably, the holographic description of static baryons as instantons via holonomies of type (1.1) was anticipated by Atiyah and Manton [7]. Sutcliffe introduced a simplified version [8,9] of the Sakai–Sugimoto model in which Yang–Mills theory is defined on flat $M^5 = \mathbb{R}^{3,1} \times \mathbb{R}$ but the holonomy (1.1) is again defined along $z \in \mathbb{R}$. Truncating this model one also gets the standard Skyrme model on $\mathbb{R}^{3,1}$.

Here, we show that the Skyrme model (and its extension by the tower of vector mesons) appears also from 6d $\mathcal{N} = (2,0)$ superconformal field theory with an ADE-type gauge group G and defined on $\mathbb{R}^{3,1} \times S_2^2$, where S_2^2 is a two-sphere with two punctures. This theory, which describes multiple M5-branes, is first compactified on a circle $S^1 \hookrightarrow S_2^2 \cong S^1 \times \mathcal{I}$, where $\mathcal{I}$ is a closed interval, to the five-dimensional maximally supersymmetric Yang–Mills theory. When $\mathcal{I}$ shrinks, it leads to a supersymmetric sigma model

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[1] See e.g. [5,6] for reviews and references.

on $\mathbb{R}^{3,1}$ [10,11]. The target space of this sigma model is the moduli space $M_{\mathcal{I}}^{hK}$ of Nahm's equations for adjoint scalar fields ϕ_A, $A = 1, 2, 3$, defined on the interval $\mathcal{I} = [-\pi, \pi]$. This $M_{\mathcal{I}}^{hK}$ is also the moduli space of vacua of the 5d SYM theory, with flat connection and other scalar fields vanishing. In the case of Dirichlet boundary conditions this moduli space $M_{\mathcal{I}}^{hK}$ coincides with the cotangent bundle $T^*G_{\mathbb{C}}$ carrying a hyper-Kähler metric [12,13,10]. Here, $G_{\mathbb{C}}$ is a complexification of the gauge group G. Topologically $T^*G_{\mathbb{C}} \cong G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$, where G describes the moduli space $M_{\mathcal{I}}$ of the gauge potential $\mathcal{A}_{\mathcal{I}}$ along the interval $\mathcal{I}$, and the triple product of $\mathfrak{g} = \text{Lie}\,G$ describes the moduli space of the three adjoint scalar fields ϕ_A of 5d SYM theory [10,11].

Since the action of the 6d superconformal field theory is not available, we begin with pure Yang–Mills in five dimensions and show how the Skyrme model appears in a low-energy limit under rather natural assumptions. In fact, the Skyrme term is the leading piece in a systematic expansion of the vector meson tower contributions. Our derivation is based on the adiabatic approach [14–20], which provides the expansion parameter and differs from the holographic approach used in [3,4,8,9]. Finally we briefly discuss a generalization of our results to the supersymmetric case with $\phi_A \neq 0$. There are many difficulties on the way to a supersymmetrization of the Skyrme model (see e.g. [21] and references therein). Our approach can give a clue to the construction of an $\mathcal{N}=2$ supersymmetric Skyrme model in four dimensions, which seems yet unknown. To summarize, we demonstrate that the extended Skyrme model (describing the pion plus the tower of vector mesons) emerges not only from a D-brane system of string theory but also from an M5-brane system of M-theory.

2. Action functional in five dimensions

Moduli space. Let M^d be an oriented smooth manifold of dimension d, G a compact ADE-type Lie group with $\mathfrak{g}$ as its Lie algebra, P a principal G-bundle over M^d, $\mathcal{A}$ a connection one-form on P and $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ its curvature. We denote by $\mathbb{A}$ the space of irreducible connections on P, by $\mathcal{G}$ the infinite-dimensional group of gauge transformations acting on $\mathbb{A}$ with the infinitesimal action of $\mathcal{G}$ defined by its Lie algebra $\text{Lie}\,\mathcal{G}$,

$$\mathcal{G} \ni f : \mathcal{A} \mapsto \mathcal{A}^f = f^{-1}\mathcal{A}f + f^{-1}\mathrm{d}f$$
$$\text{and} \quad \text{Lie}\,\mathcal{G} \ni \epsilon : \delta_\epsilon \mathcal{A} = D_{\mathcal{A}}\epsilon \,, \tag{2.1}$$

where $D_{\mathcal{A}}\epsilon := \mathrm{d}\epsilon + [\mathcal{A}, \epsilon]$. The *moduli space* of connections on P is defined as the quotient $\mathbb{A}/\mathcal{G}$, i.e. as the space of orbits of $\mathcal{G}$ in $\mathbb{A}$.

Space $\mathbb{R}^{3,1} \times \mathcal{I}$. Now we consider $d=5$ and Yang–Mills theory on the direct product manifold $M^5 = \mathbb{R}^{3,1} \times \mathcal{I}$ for $\mathcal{I} = [-\pi, \pi]$, with coordinates $(x^\mu) = (x^a, x^4)$, where $x^a \in \mathbb{R}^{3,1}$ and $x^4 \in \mathcal{I}$. We introduce a family of flat metrics,

$$\mathrm{d}s_R^2 = g_{\mu\nu}^R \mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{ab}\,\mathrm{d}x^a\mathrm{d}x^b + R^2(\mathrm{d}x^4)^2 \,, \tag{2.2}$$

where $(\eta_{ab}) = \text{diag}(-1, 1, 1, 1)$ with $a, b = 0, 1, 2, 3$, and the dimensionful coordinate $\tilde{x}^4 = Rx^4$ parametrizes the scaled interval $\mathcal{I}_R = [-\pi R, \pi R]$ of length $2\pi R$.

Gauge fields. Let us look at the principal G-bundle P over $\mathbb{R}^{3,1} \times \mathcal{I}$ with a gauge potential (connection) $\mathcal{A}$ and the gauge field (curvature) $\mathcal{F}$ both valued in the Lie algebra $\mathfrak{g}$ of the group G. On $\mathbb{R}^{3,1} \times \mathcal{I}$ we have the obvious splitting

$$\mathcal{A} = \mathcal{A}_a\,\mathrm{d}x^a + \mathcal{A}_4\,\mathrm{d}x^4$$
$$\text{and} \quad \mathcal{F} = \tfrac{1}{2}\mathcal{F}_{ab}\,\mathrm{d}x^a \wedge \mathrm{d}x^b + \mathcal{F}_{a4}\,\mathrm{d}x^a \wedge \mathrm{d}x^4 \,. \tag{2.3}$$

For the generators I_i in the adjoint representation of G we will use the standard normalization $\text{tr}(I_iI_j) = -2\delta_{ij}$ with $i, j = 1, \ldots, \dim G$.

For the metric tensor (2.2) we have $(g_R^{\mu\nu}) = (\eta^{ab}, R^{-2})$ and $\det(g_{\mu\nu}^R) = -R^2$. We denote by $\mathcal{F}_R^{\mu\nu}$ the contravariant components raised from $\mathcal{F}_{\mu\nu}$ by the tensor $g_R^{\mu\nu}$ and by $\mathcal{F}^{\mu\nu}$ those obtained by $g^{\mu\nu} \equiv g_{R=1}^{\mu\nu}$. We have $\mathcal{F}_R^{ab} = \mathcal{F}^{ab}$ and $\mathcal{F}_R^{a4} = R^{-2}\mathcal{F}^{a4}$.

Action. The standard Yang–Mills action functional takes the form

$$S = -\frac{1}{8e^2}\int_{\mathbb{R}^{3,1}\times\mathcal{I}}\mathrm{d}^5x\,\sqrt{|\det g^R|}\,\text{tr}\mathcal{F}_{\mu\nu}\mathcal{F}_R^{\mu\nu}$$
$$= -\frac{1}{8e^2R}\int_{\mathbb{R}^{3,1}\times\mathcal{I}}\mathrm{d}^5x\,\text{tr}\big(R^2\,\mathcal{F}_{ab}\mathcal{F}^{ab} + 2\mathcal{F}_{a4}\mathcal{F}^{a4}\big)\,, \tag{2.4}$$

where e is the gauge coupling constant. A supersymmetric extension of (2.4) can be found e.g. in [11]. We want to discuss the infrared region of the pure Yang–Mills model (2.4), because QCD is not a supersymmetric theory and we are after the Skyrme model as a description of hadrons. The infrared is reached by tuning down the parameter R, so the interval $\mathcal{I}_R$ becomes very short.

3. Moduli space of vacua

Gauge group. Consider the group $\mathcal{G} = C^\infty(\mathbb{R}^{3,1} \times \mathcal{I}, G)$ and its restriction $\mathcal{G}_{\mathcal{I}}$ to $\mathcal{I}$ by fixing $x^a \in \mathbb{R}^{3,1}$ to an arbitrary value. The boundary of our manifold $M^5 = \mathbb{R}^{3,1} \times \mathcal{I}$ consists of two Minkowski spaces at $x^4 = \pm\pi$. On manifolds M^d with nonempty boundary ∂M^d, the group of gauge transformations is naturally restricted to the identity when x^a reaches ∂M^d (see e.g. [22] and [10–13]) for our case). For our M^5, this means allowing only gauge-group elements $f \in \mathcal{G}$ obeying $f(\partial M^5) = \text{Id}$ on $\partial M^5 = \mathbb{R}^{3,1}_{x^4=\pm\pi}$. We denote this group by $\mathcal{G}^0$ and its restriction to $\mathcal{I}$ by $\mathcal{G}_{\mathcal{I}}^0$.

Vacua. Vacua of Yang–Mills theory (2.4) on $M^5 = \mathbb{R}^{3,1} \times \mathcal{I}$ are defined by the vanishing of the gauge fields, $\mathcal{F} = 0$. The components $\mathcal{F}_{ab} = 0$ can be solved by putting $\mathcal{A}_a = 0$, and from $\mathcal{F}_{a4} = 0$ one obtains

$$\mathcal{A}_z = \mathcal{A}_z(z) = h^{-1}\partial_z h\,, \quad \text{with} \quad z = x^4 \quad \text{and} \quad \mathcal{A}_z = \mathcal{A}_4 \tag{3.1}$$

for notational convenience. Here $h(z) \in \mathcal{G}_{\mathcal{I}}$ is not an element of the gauge group $\mathcal{G}_{\mathcal{I}}^0$. Therefore, $\mathcal{A}_z$ in (3.1) cannot be transformed to zero by an admissible gauge transformation. In fact,

$$h(-\pi) =: h_L \in G_L = \mathcal{G}_{\mathcal{I}}\big|_{z=-\pi} \cong G \quad \text{and}$$
$$h(\pi) =: h_R \in G_R = \mathcal{G}_{\mathcal{I}}\big|_{z=\pi} \cong G \,. \tag{3.2}$$

Holonomy. For the interval $\mathcal{I} = [-\pi, \pi]$ with coordinate z we denote by $\mathcal{A}_{\mathcal{I}} = \mathcal{A}_z\mathrm{d}z$ a connection one-form on the bundle $P_{\mathcal{I}} = \mathcal{I} \times G \to \mathcal{I}$ over $\mathcal{I}$, which is a restriction of the bundle $P = \mathbb{R}^{3,1} \times \mathcal{I} \times G \to \mathbb{R}^{3,1} \times \mathcal{I}$ to $\mathcal{I}$ by fixing an arbitrary point $x^a \in \mathbb{R}^{3,1}$. Then, given any connection $\mathcal{A}_{\mathcal{I}}$ on $\mathcal{P}_{\mathcal{I}}$ we have the differential equation (3.1) for h. The group of gauge transformations $\mathcal{G}_{\mathcal{I}}^0$ acts on $\mathcal{A}_{\mathcal{I}}$ and h by

$$\mathcal{G}_{\mathcal{I}}^0 \ni f : \mathcal{A}_{\mathcal{I}} \mapsto \mathcal{A}_{\mathcal{I}}^f = f^{-1}\mathcal{A}_{\mathcal{I}}f + f^{-1}\mathrm{d}_{\mathcal{I}}f$$
$$\text{and} \quad h \mapsto h^f = f^{-1}(\pi)h(z)f(z) = hf\,, \tag{3.3}$$

with $\mathrm{d}_{\mathcal{I}} = \mathrm{d}z\partial_z$ and $f(\pi) = \text{Id}$ for $f \in \mathcal{G}_{\mathcal{I}}^0$.

To solve (3.1) with Dirichlet boundary conditions [10,11] one has to choose an initial value for the G-valued function h on $\mathcal{I}$. From (3.1) it follows that h and $h_L^{-1}h$ define the same connection $\mathcal{A}_z\mathrm{d}z$, since h_L does not depend on $z \in \mathcal{I}$. Hence, the space of all flat connections $\mathcal{A}_{\mathcal{I}}$ on $P_{\mathcal{I}}$ (equivalently, the space of all solutions h to (3.1) with fixed initial condition) is the coset space $\mathcal{N}_{\mathcal{I}} = \mathcal{G}_{\mathcal{I}}/G_L$. The unique solution to the differential equation (3.1) can be written as

$$h(z) = \mathcal{P}\exp\left(\int_{-\pi}^{z} A_y\,\mathrm{d}y\right),\tag{3.4}$$

where $\mathcal{P}$ denotes path ordering. Notice that for (3.4) we have $h(z=-\pi) = \mathrm{Id}$, i.e. $h \in \mathcal{N}_{\mathcal{I}}$. The group element $g=h(z=\pi) \in G_R \cong G$ is the holonomy of $\mathcal{A}_{\mathcal{I}}$, which is not transformed under the group $\mathcal{G}_{\mathcal{I}}^0$ of gauge transformations, as follows from (3.3) and $f(\pi) = \mathrm{Id}$ for $f \in \mathcal{G}_{\mathcal{I}}^0$.

Gauge-equivalent vacua. We note that $\mathcal{G}_{\mathcal{I}} = \mathcal{G}_{\mathcal{I}}^0 \rtimes (G_L \times G_R)$, and the solution space of (3.1) is $\mathcal{N}_{\mathcal{I}} = \mathcal{G}_{\mathcal{I}}^0 \rtimes G$ from $G \cong G_R$. Thus the gauge group $\mathcal{G}_{\mathcal{I}}^0$ can be defined as the kernel of the projection (evaluation map)

$$q : \mathcal{N}_{\mathcal{I}} \overset{\mathcal{G}_{\mathcal{I}}^0}{\to} G \qquad \text{with} \qquad h(z) \mapsto h(\pi).\tag{3.5}$$

The action (3.3) of $\mathcal{G}_{\mathcal{I}}^0$ on $\mathcal{N}_{\mathcal{I}}$ is free, and the projection q is injective. Hence, (3.5) is the principal $\mathcal{G}_{\mathcal{I}}^0$-bundle over G. The base G of this bundle is the moduli space $\mathcal{M}_{\mathcal{I}}$ of vacua of Yang–Mills theory on $\mathbb{R}^{3,1} \times \mathcal{I}$.

4. Changes of $\mathcal{A}_{\mathcal{I}}$ under shifts on $\mathcal{M}_{\mathcal{I}}$

Connections on G. Introducing local coordinates $X = \{X^\alpha\}$ on G, the differential of the holonomy element $g = h(\pi) \in G$ can be expressed as $\mathrm{d}g = (\partial_\alpha g)\mathrm{d}X^\alpha$. Then, the canonical flat connection on the tangent bundle TG reads

$$\Gamma = g^{-1}\mathrm{d}g = (g^{-1}\partial_\alpha g)\,\mathrm{d}X^\alpha = e_\alpha^i I_i\,\mathrm{d}X^\alpha = e^i I_i,\tag{4.1}$$

where e^i are left-invariant one-forms on G. They satisfy the Maurer–Cartan equations

$$\mathrm{d}e^i + \tfrac{1}{2} f_{jk}^i e^j \wedge e^k = 0,\tag{4.2}$$

where f_{jk}^i are the structure constants of the group G. The collection $\{e^i\}$ forms an orthonormal basis on the cotangent bundle T^*G, and for a metric on G we have

$$\mathrm{d}s_G^2 = \delta_{ij}e^i e^j = \delta_{ij}e_\alpha^i e_\beta^j \,\mathrm{d}X^\alpha\mathrm{d}X^\beta =: g_{\alpha\beta}\,\mathrm{d}X^\alpha\mathrm{d}X^\beta.\tag{4.3}$$

Variation of $\mathcal{A}_{\mathcal{I}}$. Our fields g, h and $\mathcal{A}_{\mathcal{I}}$ are parametrized by the coordinates X^α of G. In general, $\mathcal{A}_{\mathcal{I}}$ belongs to the space $\mathcal{N}_{\mathcal{I}}$ described in Section 3 and fibred over G. We introduce the tangent bundle $T\mathcal{N}_{\mathcal{I}}$ of $\mathcal{N}_{\mathcal{I}}$ as the fibration

$$q_* : T\mathcal{N}_{\mathcal{I}} \to TG\tag{4.4}$$

with fibres $T_{\mathcal{A}_{\mathcal{I}}}\mathcal{G}_{\mathcal{I}}^0 \cong \mathrm{Lie}\,\mathcal{G}_{\mathcal{I}}^0$ at any point $\mathcal{A}_{\mathcal{I}} \in G$. Also, we have $T_{\mathcal{A}_{\mathcal{I}}}G \cong \mathfrak{g}$ and therefore

$$T_{\mathcal{A}_{\mathcal{I}}}\mathcal{N}_{\mathcal{I}} = q^*T_{\mathcal{A}_{\mathcal{I}}}G \oplus T_{\mathcal{A}_{\mathcal{I}}}\mathcal{G}_{\mathcal{I}}^0 \cong \mathfrak{g} \oplus \mathrm{Lie}\,\mathcal{G}_{\mathcal{I}}^0.\tag{4.5}$$

Note that even if $\mathcal{A}_{\mathcal{I}}$ belongs to the base G of the fibration (3.5) (after fixing a gauge), its derivative $\partial_\alpha\mathcal{A}_{\mathcal{I}}$ with respect to X^α belongs to the tangent space $T_{\mathcal{A}_{\mathcal{I}}}\mathcal{N}_{\mathcal{I}}$ and not necessarily to the

tangent space $T_{\mathcal{A}_{\mathcal{I}}}G$. However, $\partial_\alpha\mathcal{A}_{\mathcal{I}}$ can always be decomposed as

$$\partial_\alpha\mathcal{A}_z = \delta_\alpha\mathcal{A}_z + \delta_{\epsilon_\alpha}\mathcal{A}_z =: \xi_\alpha + D_z\epsilon_\alpha$$
$$\text{with} \quad \xi_\alpha \in T_{\mathcal{A}_{\mathcal{I}}}G \quad \text{and} \quad D_z\epsilon_\alpha \in T_{\mathcal{A}_{\mathcal{I}}}\mathcal{G}_{\mathcal{I}}^0,\tag{4.6}$$

where $T_{\mathcal{A}_{\mathcal{I}}}G \cong \mathfrak{g}$ and $T_{\mathcal{A}_{\mathcal{I}}}\mathcal{G}_{\mathcal{I}}^0 \cong \mathrm{Lie}\,\mathcal{G}_{\mathcal{I}}^0$. The $\mathfrak{g}$-valued gauge parameters ϵ_α generate infinitesimal gauge transformations which, after the ∂_α-shift, bring $\mathcal{A}_{\mathcal{I}}$ back to G.

Orthogonality of $\xi_\alpha = \delta_\alpha\mathcal{A}_z$ and $D_z\epsilon_\alpha = \delta_{\epsilon_\alpha}\mathcal{A}_z$ is achieved by imposing the condition

$$D_z\xi_\alpha = 0 \quad \Leftrightarrow \quad D_z^2\epsilon_\alpha = D_z\partial_\alpha\mathcal{A}_z.\tag{4.7}$$

From (3.1) and (4.7) one obtains

$$\xi_\alpha = e_\alpha^i h^{-1}I_i h = h^{-1}(g^{-1}\partial_\alpha g)h,\tag{4.8}$$

which shows that the z-dependence of ξ_α is located in $h(z)$ alone.

5. Skyrme model in the infrared limit of 5d Yang–Mills

Moduli-space approximation. After having described the moduli space $\mathcal{M}_{\mathcal{I}}$ of Yang–Mills theory on $\mathbb{R}^{3,1} \times \mathcal{I}$, we return to non-vacuum gauge fields. In the moduli-space approximation it is postulated that the collective coordinates X^α depend on $x^a \in \mathbb{R}^{3,1}$, so that $X^\alpha = X^\alpha(x^a)$ may be considered as dynamical fields, and that this captures the x^a dependence of "slow" full solutions. The low-energy effective action for X^α is derived by expanding

$$\mathcal{A}_\mu = \mathcal{A}_\mu(X^\alpha(x^a), x^4) + \dots,\tag{5.1}$$

where the first term depends on $x^a \in \mathbb{R}^{3,1}$ only via the coordinates $X^\alpha \in \mathcal{M}_{\mathcal{I}}$ [14,15,17,20]. Then for distances in $\mathbb{R}^{3,1}$ which are large in comparison with the length $2\pi R$ of the interval $\mathcal{I}_R$ (or, in other words, for small values of R) all terms in (5.1) beyond the first one are discarded. By substituting the leading term of (5.1) into the initial action (2.4), one obtains an effective field theory describing small fluctuations around the vacuum manifold.

Kinetic part of effective action. The gauge potential decomposes as

$$\mathcal{A} = \mathcal{A}_{\mathbb{R}^{3,1}} + \mathcal{A}_{\mathcal{I}} \qquad \text{with} \quad \mathcal{A}_{\mathbb{R}^{3,1}} = \mathcal{A}_a\mathrm{d}x^a \quad \text{and} \quad \mathcal{A}_{\mathcal{I}} = \mathcal{A}_z\mathrm{d}z.\tag{5.2}$$

For any fixed $x^a \in \mathbb{R}^{3,1}$, the part $\mathcal{A}_{\mathcal{I}}(X^\alpha(x^a), x^4)$ belongs to the space $\mathcal{N}_{\mathcal{I}}$ described in Sections 3 and 4. We now use the formulæ from these sections and include the dependence on x^a. In particular, multiplying (4.6) by $\partial_a X^\alpha$, we obtain

$$\partial_a\mathcal{A}_z = (\partial_a X^\alpha)\xi_\alpha + D_z\epsilon_a,\tag{5.3}$$

where $\epsilon_a = (\partial_a X^\alpha)\epsilon_\alpha$ is the pull-back of ϵ_α to $\mathbb{R}^{3,1}$.

We have provided the details of the $\mathcal{A}_{\mathcal{I}}$ part of the connection $\mathcal{A}$. On the other hand, the components $\mathcal{A}_a$ of the $\mathcal{A}_{\mathbb{R}^{3,1}}$ part are not yet fixed. To this end, we note that

$$\mathcal{F}_{a4} = \partial_a\mathcal{A}_z - D_z\mathcal{A}_a = (\partial_a X^\alpha)\xi_\alpha + D_z(\epsilon_a - \mathcal{A}_a).\tag{5.4}$$

In the moduli-space approximation, $\mathcal{F}_{a4}$ has to be tangent to $\mathcal{M}_{\mathcal{I}}$ (see e.g. [14,15]). Hence, the second term in (5.4) should vanish, i.e. $\epsilon_a - \mathcal{A}_b$ must lie in kernel of D_z, which according to (4.7) is proportional to ξ_α. Thus, we have

$$\mathcal{A}_a = \epsilon_a + A_a^\alpha\xi_\alpha = \epsilon_a + A_a^i(X)h^{-1}I_i h,\tag{5.5}$$

where A_a^i are arbitrary functions of the group coordinates $X^\alpha = X^\alpha(x^a)$. For simplicity we pick a gauge where $A_a^i = 0$, so that

$$\mathcal{A}_a = \epsilon_a$$

with boundary conditions $\epsilon_a(z=-\pi) = 0 = \epsilon_a(z=\pi)$.
$$(5.6)$$

Substituting

$$\mathcal{F}_{a4} = (\partial_a X^\alpha)\xi_\alpha = (\partial_a X^\alpha)e_\alpha^i h^{-1} I_i h = h^{-1}(g^{-1}\partial_a g)h \quad (5.7)$$

into the action (2.4), the second term becomes

$$S_{\mathrm{kin}} = -\frac{1}{8e^2 R}\int\limits_{\mathbb{R}^{3,1}\times\mathcal{I}} \mathrm{d}^5 x\, \eta^{ab}\, \mathrm{tr}\mathcal{F}_{a4}\mathcal{F}_{b4}$$
$$= -\frac{\pi}{4e^2 R}\int\limits_{\mathbb{R}^{3,1}} \mathrm{d}^4 x\, \eta^{ab}\, \mathrm{tr}(L_a L_b) , \qquad (5.8)$$

where we used (5.7) and the definition

$$L_a := g^{-1}\partial_a g . \qquad (5.9)$$

Thus, this part of the action reduces to a sigma model on $\mathbb{R}^{3,1}$ with $\mathcal{M}_\mathcal{I} \cong G$ as target space.

Skyrme term. For calculating the first term in the action (2.4) it is convenient to rewrite (5.6) as

$$\mathcal{A}_a = \epsilon_a = h\left(h^{-1}\epsilon_a h + h^{-1}\partial_a h\right)h^{-1} + h\,\partial_a h^{-1}$$
$$=: h\,\widehat{\mathcal{A}}_a h^{-1} + h\,\partial_a h^{-1} , \qquad (5.10)$$

where $\widehat{\mathcal{A}}_a$ depends on z. The boundary conditions (5.6) for ϵ_a translate to

$$\widehat{\mathcal{A}}_a(z=-\pi) = 0 \quad \text{and} \quad \widehat{\mathcal{A}}_a(z=\pi) = L_a \qquad (5.11)$$

since $h(z=-\pi) = \mathrm{Id}$ and $h(z=\pi) = g$. Therefore, we can expand $\widehat{\mathcal{A}}_a(z)$ on $\mathcal{I}$ as[2]

$$\widehat{\mathcal{A}}_a(z) = \frac{z+\pi}{2\pi}L_a + \sum_{n=1}^{\infty}B_a^{(n)}\sin nz , \qquad (5.12)$$

where L_a represents the pion degree of freedom and the $B_a^{(n)}$ describe the tower of mesons.

The curvature of $\widehat{\mathcal{A}}$ then computes to

$$h^{-1}\mathcal{F}_{ab}h = \widehat{\mathcal{F}}_{ab} = \partial_a\widehat{\mathcal{A}}_b - \partial_b\widehat{\mathcal{A}}_a + [\widehat{\mathcal{A}}_a, \widehat{\mathcal{A}}_b]$$
$$= \frac{z^2-\pi^2}{4\pi^2}[L_a, L_b] + B_{ab} , \qquad (5.13)$$

where the term B_{ab} contains the meson contributions. Substituting this into the action (2.4) and truncating to the pion, i.e. discarding all B_{ab} terms, we obtain

$$S_{\mathrm{Skyrme}} = -\frac{R}{8e^2}\int\limits_{\mathbb{R}^{3,1}\times\mathcal{I}} \mathrm{d}^5 x\, \mathrm{tr}\mathcal{F}_{ab}\mathcal{F}^{ab}$$
$$= -\frac{\pi R}{120e^2}\int\limits_{\mathbb{R}^{3,1}} \mathrm{d}^4 x\, \eta^{ac}\eta^{bd}\, \mathrm{tr}\left([L_a, L_b][L_c, L_d]\right) . \qquad (5.14)$$

Thus, in the infrared limit the Yang–Mills action on $\mathbb{R}^{3,1}\times\mathcal{I}$ is reduced to the effective action of the Skyrme model,

$$S_{\mathrm{eff}} = -\int\limits_{\mathbb{R}^{3,1}} \mathrm{d}^4 x\Bigg\{ \frac{f_\pi^2}{4}\,\eta^{ab}\,\mathrm{tr}(L_a L_b)$$
$$+ \frac{1}{32\varsigma^2}\,\eta^{ac}\eta^{bd}\,\mathrm{tr}\left([L_a, L_b][L_c, L_d]\right)\Bigg\} , \qquad (5.15)$$

where ς is the dimensionless Skyrme parameter and f_π may be interpreted as the pion decay constant. Their relation to the gauge coupling and the infrared scale R is

$$\frac{f_\pi^2}{4} = \frac{\pi}{4e^2 R} \quad \text{and} \quad \frac{1}{32\varsigma^2} = \frac{\pi R}{120e^2} . \qquad (5.16)$$

We see that the ratio of these parameters depends on the length of the interval $\mathcal{I}_R = [-\pi R, \pi R]$ characterizing the approach to the infrared.

Towards the supersymmetric model. What will change if we consider the infrared limit of maximally supersymmetric Yang–Mills theory (SYM)? 5d SYM contains five adjoint scalars, namely ϕ^A, ϕ^4, ϕ^5, and the moduli space $\mathcal{M}_\mathcal{I}^{hK}$ of this theory is defined as the moduli space of flat connections $\mathcal{F}_{\mu\nu} = 0$, which we considered, extended by the moduli space of solutions to the Nahm equations

$$\partial_z\phi^A + [\mathcal{A}_z, \phi^A] = \tfrac{1}{2}\varepsilon_{BC}^A[\phi^B, \phi^C] \quad \text{and} \quad \phi^4 = \phi^5 = 0 , \qquad (5.17)$$

on the scalar fields depending on $z\in\mathcal{I} = [-\pi, \pi]$ with all fermions vanishing [10,11]. This moduli space $\mathcal{M}_\mathcal{I}^{hK}$ depends essentially on the boundary conditions imposed on $\mathcal{A}_\mathcal{I}$ and ϕ^A and was discussed e.g. in [12,13] (see also references therein). For the simplest Dirichlet boundary conditions [10,12,13] and assuming regularity at $z = \pm\pi$, the moduli space $\mathcal{M}_\mathcal{I}^{hK}$ is the cotangent bundle $T^*G_\mathbb{C} \cong G_\mathbb{C}\times\mathfrak{g}_\mathbb{C} \cong G\times\mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}$ with a hyper-Kähler metric. The explicit form of the $\mathcal{N}=2$ supersymmetric sigma-model action for the hyper-Kähler vacuum moduli space $\mathcal{M}_\mathcal{I}^{hK}$ e.g. in [11]. Its derivation from 5d SYM in the infrared limit is similar to the one for the bosonic case.

Finally, the Skyrme term should get supersymmetrized. We had $X^\alpha \in \mathcal{M}_\mathcal{I} \cong G$. In terms of X^α and one-forms $e^i = e_\alpha^i \mathrm{d}X^\alpha$ on G the standard Skyrme term in the Lagrangian of (5.14) can be written as

$$\eta^{ac}\eta^{bd}\,\partial_a X^\alpha\,\partial_b X^\beta\,\partial_c X^\gamma\,\partial_d X^\delta\,e_\alpha^i e_\beta^j e_\gamma^k e_\delta^l\, R_{ijkl}$$
$$= \partial_a X^\alpha\,\partial_b X^\beta\,\partial^a X^\gamma\,\partial^b X^\delta\, R_{\alpha\beta\gamma\delta} , \qquad (5.18)$$

where $\partial^a = \eta^{ac}\partial_c$ and R_{ijkl} is the curvature of the connection on $\mathcal{M}_\mathcal{I} \cong G$. We expect that for the supersymmetric case (5.18) will have the same form but with $(X^\alpha, e^i, R_{ijkl})$ being defined on the hyper-Kähler moduli space $\mathcal{M}_\mathcal{I}^{hK}$. Additional fermionic and possibly auxiliary terms may also need to be deduced. However, this is beyond the scope of our paper.

Acknowledgements

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[2] The coefficient linear in z is just a convenient choice of a function interpolating between 0 and 1 on $\mathcal{I}$. It yields a family of metric-compatible linear connections which are non-flat inside $\mathcal{I}$, with torsion $T_{jk}^i = \frac{z}{\pi}f_{jk}^i$ and curvature $R_{ijkl} = \frac{z^2-\pi^2}{2\pi^2}\delta_{mr}f_{ij}^m f_{kl}^r$ (see e.g. [23] for a discussion). At $z=0$ one finds the Levi-Civita connection.

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