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$\mathcal{N}$-extended supersymmetric Calogero models

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ABSTRACT

We propose a new $\mathcal{N}$-extended supersymmetric $su(n)$ spin-Calogero model. Employing a generalized Hamiltonian reduction adopted to the supersymmetric case, we explicitly construct a novel rational n-particle Calogero model with an arbitrary even number of supersymmetries. It features $\mathcal{N}n^2$ rather than $\mathcal{N}n$ fermionic coordinates and increasingly high fermionic powers in the supercharges and the Hamiltonian.

1. Introduction

The original rational Calogero model of n interacting identical particles on a line [1], pertaining to the roots of $A_1 \oplus A_{n-1}$ and given by the classical Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j} \frac{g^2}{\left(x^i - x^j\right)^2}\,, \tag{1.1}$$

has often been the subject of "supersymmetrization". In this endeavor, *extended* supersymmetry has turned out to be surprisingly rich. After the straightforward formulation of $\mathcal{N}=2$ supersymmetric Calogero models by Freedman and Mende [2], a barrier was encountered at $\mathcal{N}=4$ [3]. An important step forward then was the explicit construction of the supercharges and the Hamiltonian for the $\mathcal{N}=4$ supersymmetric three-particle Calogero model [4,5], which introduced a second prepotential F besides the familiar prepotential U. However, it was found that quantum corrections modify the potential in (1.1), and that F is subject to intricate nonlinear differential equations, the WDVV equations, beyond the three-particle case. These results were then confirmed and elucidated in a superspace description [6]. Finally, extending the system by a single harmonic degree of freedom ($su(2)$ spin variables [7]) it was possible to write down a unique $osp(4|2)$ symmetric four-particle Calogero model [8].[1] A detailed discussion concerning the supersymmetrization of the Calogero models can be found in the review [9].

It seems that a guiding principle was missing for the construction of extended supersymmetric Calogero models. Indeed, while for $n \leq 3$ translation and (super-)conformal symmetry almost completely defines the system, the $n \geq 4$ cases admit a lot of freedom which cannot *a priori* be fixed. In the bosonic case, such a guiding principle exists [10]. The Calogero model as well as its different extensions (see, e.g. [11–13]) are closely related with matrix models and can be obtained from them by a reduction procedure (see [14] for first results and [15] for a review). If we want to employ this principle also for finding extended supersymmetric Calogero models, then the two main steps are

- supersymmetrization of a matrix model
- supersymmetrization of the reduction procedure or proper gauge fixing.

This idea is not new. It has successfully been employed in [16–19]. The resulting supersymmetric systems feature

- a large number of fermions – far more than the $4n$ fermions expected in an $\mathcal{N}=4$ n-particle system within the standard (but unsuccessful!) approach
- a rather complicated structure of the supercharges and the Hamiltonian, with fermionic polynomials of maximal degree
- a variety of bosonic potentials, including $su(2)$ *spin*-Calogero interactions

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[1] Here and in the above history, the goal is a bosonic potential exactly as in (1.1). Models with more general interactions can be found for any number of particles.

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but they do not contain a genuine $\mathcal{N}=4$ supersymmetric Calogero model, i.e. one with a mere pairwise inverse-square no-spin bosonic potential.

Here we use the same guiding principle and start with the bosonic $su(n)$ spin-Calogero model in the Hamiltonian approach. We then provide an $\mathcal{N}$-extended supersymmetrization of this system. It is important that we do *not a priori* fix a realization for the $su(n)$ generators. Finally we generalize the reduction procedure to the $\mathcal{N}$-extended system and find the first $\mathcal{N}$-extended supersymmetric Calogero model, for *any* even number of supersymmetries.

2. $\mathcal{N}$-extended supersymmetric Calogero model

2.1. Bosonic Calogero model from hermitian matrices

It is well known that the rational n-particle Calogero model [1] can be obtained by Hamiltonian reduction from the hermitian matrix model [10,14]. Adapted to our purposes, the procedure reads as follows. One starts from the $su(n)$ spin generalization [12] of the standard Calogero model, as given by

$$H=\frac{1}{2}\sum_{i=1}^{n}p_i^2+\frac{1}{2}\sum_{i\neq j}^{n}\frac{\ell_{ij}\ell_{ji}}{\left(x^i-x^j\right)^2}\,. \tag{2.1}$$

The particles are described by their coordinates x^i and momenta p_i together with their internal degrees of freedom encoded in the angular momenta $\left(\ell_{ij}\right)^\dagger=\ell_{ji}$ with $\sum_i \ell_{ii}=0$. The non-vanishing Poisson brackets are

$$\left\{x^i,p_j\right\}=\delta^i_j \qquad \text{and} \qquad \left\{\ell_{ij},\ell_{km}\right\}=\mathrm{i}\left(\delta_{im}\ell_{kj}-\delta_{kj}\ell_{im}\right)\,. \tag{2.2}$$

The Hamiltonian (2.1) follows directly from the free hermitian matrix model (for details see [15]).

To get the standard Calogero Hamiltonian (1.1) from (2.1) one has to reduce the angular sector of the latter, in two steps. Firstly, one (weakly) imposes the constraints

$$\ell_{11}\approx\ell_{22}\approx\ldots\approx\ell_{nn}\approx 0\,. \tag{2.3}$$

They commute with the Hamiltonian (2.1) and with each other, hence are of first class. To resolve them one introduces auxiliary complex variables v_i and $\bar v_i=(v_i)^\dagger$ obeying the Poisson brackets

$$\left\{v_i,\bar v_j\right\}=-\mathrm{i}\,\delta_{ij} \tag{2.4}$$

and realizes the $su(n)$ generators ℓ_{ij} as

$$\hat\ell_{ij}=-v_i\bar v_j+\frac{1}{n}\delta_{ij}\sum_k^n v_k\bar v_k\,. \tag{2.5}$$

Secondly, passing to polar variables r_i and ϕ_i defined as

$$v_i=r_ie^{\mathrm{i}\phi_i} \quad \text{and} \quad \bar v_i=r_ie^{-\mathrm{i}\phi_i} \quad \Rightarrow \quad \left\{r_i,\phi_j\right\}=\frac{1}{2r_i}\delta_{ij}\,, \tag{2.6}$$

the constraints (2.3) are resolved by putting

$$r_1\approx r_2\approx\ldots\approx r_n\,. \tag{2.7}$$

Plugging this solution into the Hamiltonian (2.1) one may additionally fix $n-1$ angles ϕ_i, say

$$\phi_1\approx\phi_2\approx\ldots\approx\phi_{n-1}\approx 0\,. \tag{2.8}$$

At this stage the $2n$ variables $\{r_i,\phi_i\}$ are reduced to the two variables r_n and ϕ_n. However, the reduced Hamiltonian does not depend on ϕ_n and has the form

$$H_{\rm red}=\frac{1}{2}\sum_{i=1}^{n}p_i^2+\frac{1}{2}\sum_{i\neq j}^{n}\frac{r_n^4}{\left(x^i-x^j\right)^2}\,. \tag{2.9}$$

Therefore

$$\{H_{\rm red},r_n\}\approx 0 \qquad \text{and} \qquad r_n^2\approx \text{const}=:g\,, \tag{2.10}$$

and the reduced Hamiltonian $H_{\rm red}$ coincides with the standard n-particle rational Calogero Hamiltonian. We note that in the bosonic case most reduction steps are not needed, because the Hamiltonian (2.1) does not depend on the angles ϕ_i at all. However, in the supersymmetric case all reduction steps will be important.

In what follows we will construct an $\mathcal{N}$-extended supersymmetric generalization of the Hamiltonian (2.1) and perform the supersymmetric version of the reduction just discussed, finishing with an $\mathcal{N}$-extended supersymmetric Calogero model, for $\mathcal{N}=2M$ and $M=1,2,3,\ldots$.

2.2. $\mathcal{N}$-extended supersymmetric $su(n)$ spin-Calogero model

On the outset we have to clarify what is the minimal number of fermionic variables necessary to realize an $\mathcal{N}=2M$ supersymmetric extension of the $su(n)$ spin-Calogero model (2.1). Clearly, as partners to the bosonic coordinates x^i one needs $\mathcal{N}n$ fermions ψ_i^a and $\bar\psi_{i\,a}$ with $a=1,2,\ldots M$. However, this is not enough to construct $\mathcal{N}$ supercharges Q^a and $\overline{Q}_b$ which must generate the $\mathcal{N}=2M$ superalgebra

$$\left\{Q^a,\overline{Q}_b\right\}=-2\mathrm{i}\,\delta^a_b\,H \qquad \text{and} \qquad \left\{Q^a,Q^b\right\}=\left\{\overline{Q}_a,\overline{Q}_b\right\}=0\,. \tag{2.11}$$

The reason is simple: to generate the potential term $\sum_{i\neq j}^n\frac{\ell_{ij}\ell_{ji}}{\left(x^i-x^j\right)^2}$ in the Hamiltonian, the supercharges Q^a and $\overline{Q}_b$ must contain the terms

$$\mathrm{i}\sum_{i\neq j}^{n}\frac{\ell_{ij}\rho^a_{ji}}{x^i-x^j} \qquad \text{and} \qquad -\,\mathrm{i}\sum_{i\neq j}^{n}\frac{\ell_{ji}\bar\rho_{ij\,a}}{x^i-x^j}\,, \tag{2.12}$$

respectively, where ρ^a_{ij} and $\bar\rho_{ij\,a}$ are some additional fermionic variables. These fermions cannot be constructed from ψ_i^a or $\bar\psi_{i\,a}$. Hence, we are forced to introduce $\mathcal{N}n(n-1)$ further independent fermions ρ^a_{ij} and $\bar\rho_{ij\,a}$ subject to $\rho^a_{ii}=\bar\rho_{ii\,a}=0$ for each value of the index i. In total, we thus utilize $\mathcal{N}n^2$ fermions of type ψ or ρ, which we demand to obey the following Poisson brackets,

$$\left\{\psi_i^a,\bar\psi_{j\,b}\right\}=-\mathrm{i}\,\delta^a_b\delta_{ij}\,,\quad \left\{\rho^a_{ij},\bar\rho_{km\,b}\right\}=-\mathrm{i}\,\delta^a_b\delta_{im}\delta_{jk}\,,$$
$$\text{with}\quad \left(\rho^a_{ij}\right)^\dagger=\bar\rho_{ji\,a} \quad\text{and}\quad \rho^a_{ii}=\bar\rho_{ii\,a}=0\,. \tag{2.13}$$

The next important ingredient of our construction is the composite object

$$\Pi_{ij}=\sum_{a=1}^{M}\Big[\left(\psi_i^a-\psi_j^a\right)\bar\rho_{ij\,a}+\left(\bar\psi_{i\,a}-\bar\psi_{j\,a}\right)\rho^a_{ij}+\sum_{k=1}^{n}\left(\rho^a_{ik}\bar\rho_{kj\,a}+\bar\rho_{ik\,a}\rho^a_{kj}\right)\Big] \quad\Rightarrow\quad \left(\Pi_{ij}\right)^\dagger=\Pi_{ji}\,. \tag{2.14}$$

One may check that, with respect to the brackets (2.13), the Π_{ij} form an $su(n)$ algebra just like the ℓ_{ij},

$$\left\{\Pi_{ij},\Pi_{km}\right\} = \mathrm{i}\left(\delta_{im}\Pi_{kj} - \delta_{kj}\Pi_{im}\right) , \tag{2.15}$$

and they Poisson-commute with the our fermions as follows,

$$\begin{aligned}
&\left\{\Pi_{ij},\psi_k^a\right\} = \mathrm{i}\left(\delta_{ik}-\delta_{jk}\right)\rho_{ij}^a ,\\
&\left\{\Pi_{ij},\rho_{km}^a\right\} = -\mathrm{i}\,\delta_{im}\delta_{jk}\left(\psi_i^a-\psi_j^a\right) - \mathrm{i}\delta_{jk}\rho_{im}^a + \mathrm{i}\delta_{im}\rho_{kj}^a ,\\
&\left\{\Pi_{ij},\bar\psi_{k\,a}\right\} = \mathrm{i}\left(\delta_{ik}-\delta_{jk}\right)\bar\rho_{ij\,a} ,\\
&\left\{\Pi_{ij},\bar\rho_{km\,a}\right\} = -\mathrm{i}\,\delta_{im}\delta_{jk}\left(\bar\psi_{i\,a}-\bar\psi_{j\,a}\right) - \mathrm{i}\delta_{jk}\bar\rho_{im\,a} + \mathrm{i}\delta_{im}\bar\rho_{kj\,a} .
\end{aligned} \tag{2.16}$$

It is a matter of straightforward calculation to check that the supercharges

$$\begin{aligned}
Q^a &= \sum_{i=1}^n p_i\psi_i^a + \mathrm{i}\sum_{i\neq j}^n \frac{\left(\ell_{ij}+\Pi_{ij}\right)\rho_{ji}^a}{x^i-x^j} \qquad \text{and}\\
\overline{Q}_b &= \sum_{i=1}^n p_i\bar\psi_{i\,b} - \mathrm{i}\sum_{i\neq j}^n \frac{\bar\rho_{ij\,b}\left(\ell_{ji}+\Pi_{ji}\right)}{x^i-x^j}
\end{aligned} \tag{2.17}$$

obey the $\mathcal{N}=2M$ superalgebra (2.11) with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^n p_i^2 + \frac{1}{2}\sum_{i\neq j}^n \frac{\left(\ell_{ij}+\Pi_{ij}\right)\left(\ell_{ji}+\Pi_{ji}\right)}{\left(x^i-x^j\right)^2} , \tag{2.18}$$

modulo the first-class constraints[2]

$$\chi_i := \ell_{ii} + \Pi_{ii} \approx 0 \quad \forall\, i , \tag{2.19}$$

with

$$\left\{Q^a,\chi_i\right\} \approx \left\{\overline{Q}_a,\chi_i\right\} \approx \left\{H,\chi_i\right\} \approx \left\{\chi_i,\chi_j\right\} \approx 0 . \tag{2.20}$$

Details of this computation can be found in the Appendix. The supercharges Q^a and $\overline{Q}_b$ in (2.17) and the Hamiltonian H in (2.18) describe the $\mathcal{N}=2M$ supersymmetric $su(n)$ spin-Calogero model.

For $\mathcal{N}=4$ it essentially coincides with the $osp(4|2)$ supersymmetric mechanics constructed in [16,17]. However, there are a few differences:

- The Hamiltonian (2.18) has no interaction for the center-of-mass coordinate $X=\sum_i x^i$. Correspondingly, the supercharges (2.17) do not include certain terms which appeared in [16,17].
- Working at the Hamiltonian level, we may keep the $su(n)$ generators ℓ_{ij} unspecified. Precisely this enables the minimal realization (2.5) with a minimal number of auxiliary variables $v_i,\bar v_i$. At the Lagrangian level this corresponds to using $(2,4,2)$ supermultiplets for the auxiliary bosonic superfields instead of $(4,4,0)$ superfields as in [16,17].

Now we are ready to reduce our $\mathcal{N}=2M$ $su(n)$ spin-Calogero model to a genuine $\mathcal{N}=2M$ Calogero model.

2.3. $\mathcal{N}$-extended supersymmetric (no-spin) Calogero models

As we can see from the previous subsection, the supersymmetric analogs (2.19) of the purely bosonic constraints (2.3) appear automatically. These constraints generate $n-1$ local U(1) transformations[3] of the variables $\{v_i,\bar v_i,\rho_{ij}^a,\bar\rho_{ij\,a}\}$. In terms of the $2n$ polar variables r_i and ϕ_i defined in (2.6), the constraints (2.19) can be easily resolved as

$$r_k^2 \approx r_n^2 + \Pi_{kk} - \Pi_{nn} \qquad \text{for} \quad k=1,\ldots,n-1 . \tag{2.21}$$

After fixing the residual gauge freedom as

$$\phi_1 \approx \phi_2 \approx \ldots \approx \phi_{n-1} \approx 0 , \tag{2.22}$$

we obtain the supercharges and Hamiltonian which still obey the $\mathcal{N}=2M$ superalgebra (2.11) and contain only the surviving pair (r_n,ϕ_n) of the originally $2n$ "angular" variables. One may check that the supercharges Q^a and $\overline{Q}_b$ and the Hamiltonian H, with the generators ℓ_{ij} replaced by $\hat\ell_{ij}$ and with the constraints (2.21) and (2.22) taken into account, perfectly commute with $r_n^2-\Pi_{nn}$. Thus, the final step of the reduction is to impose the constraint

$$r_n^2 - \Pi_{nn} \approx \text{const} =: g \tag{2.23}$$

and to fix the remaining U(1) gauge symmetry via

$$\phi_n \approx 0 . \tag{2.24}$$

The previous two relations are the supersymmetric analogs of (2.10). We conclude that the full set of the reduction constraints reads

$$r_i^2 \approx g + \Pi_{ii} \qquad \text{and} \qquad \phi_i \approx 0 \qquad \text{for} \quad i=1,\ldots,n . \tag{2.25}$$

With these constraints taken into account, our supercharges Q^a and $\overline{Q}_b$ and the Hamiltonian H acquire the form

$$\begin{aligned}
\widehat{Q}^a &= \sum_{i=1}^n p_i\psi_i^a \;-\; \mathrm{i}\sum_{i\neq j}^n \frac{\left(\sqrt{g+\Pi_{ii}}\sqrt{g+\Pi_{jj}}-\Pi_{ij}\right)\rho_{ji}^a}{x^i-x^j} ,\\
\widehat{\overline{Q}}_b &= \sum_{i=1}^n p_i\bar\psi_{i\,b} \;+\; \mathrm{i}\sum_{i\neq j}^n \frac{\bar\rho_{ij\,b}\left(\sqrt{g+\Pi_{ii}}\sqrt{g+\Pi_{jj}}-\Pi_{ji}\right)}{x^i-x^j} ,\\
\widehat{H} &= \frac{1}{2}\sum_{i=1}^n p_i^2\\
&\quad + \frac{1}{2}\sum_{i\neq j}^n \frac{\left(\sqrt{g+\Pi_{ii}}\sqrt{g+\Pi_{jj}}-\Pi_{ij}\right)\left(\sqrt{g+\Pi_{ii}}\sqrt{g+\Pi_{jj}}-\Pi_{ji}\right)}{\left(x^i-x^j\right)^2} .
\end{aligned} \tag{2.26}$$

It is matter of quite lengthy and tedious calculations to check that these supercharges and Hamiltonian form an $\mathcal{N}=2M$ superalgebra (2.11). The strategy is given in the Appendix. The main complication arises from the expressions $\sqrt{g+\Pi_{ii}}$ present in the supercharges and the Hamiltonian. Due to the nilpotent nature of Π_{ij}, the series expansion eventually terminates, but even in the two-particle case with $\mathcal{N}=4$ supersymmetry we encounter a lengthy expression,

$$\sqrt{g+\Pi_{11}} = \sqrt{g}\left(1+\tfrac{1}{2g}\Pi_{11}-\tfrac{1}{8g^2}\Pi_{11}^2+\tfrac{1}{16g^3}\Pi_{11}^3-\tfrac{5}{128g^4}\Pi_{11}^4\right) . \tag{2.27}$$

For n particles the series will end with a term proportional to $(\Pi_{ii})^{\mathcal{N}(n-1)}$. Clearly, these terms will generate higher-degree monomials in the fermions, both for the supercharges and for the

[2] The system with the Hamiltonian (2.18) and with $\ell_{ij}=0$ has been previously considered in [20].

[3] Due to the relation $\sum_i^n \chi_i = 0$ we have only $n-1$ independent constraints.

Hamiltonian. We can only speculate that the dread of such complexities impeded an earlier discovery of genuine $\mathcal{N}=4$ Calogero models.

2.4. Simplest example: $\mathcal{N}=2$ supersymmetric two-particle Calogero model

For $\mathcal{N}=2$ supersymmetry one has to put $M=1$ in the expressions (2.26) for the supercharges and Hamiltonian. This somewhat reduces their complexity compared to the $\mathcal{N}=4$ case, but the real simplification occurs for two particles. Indeed, for $n=2$ we get

$$\Pi_{22}=-\Pi_{11} \qquad \text{and} \qquad \Pi_{11}^3\equiv 0$$
$$\Rightarrow \quad \sqrt{g+\Pi_{11}}\sqrt{g-\Pi_{11}}=\left(g-\tfrac{1}{2g}\Pi_{11}^2\right) \qquad \text{for} \quad g\neq 0\,. \tag{2.28}$$

Moreover, the term Π_{11}^2 is of the maximal possible power in the ρ and $\bar\rho$ fermions and, therefore, disappears from the supercharges. Thus, we are left with

$$\widehat{Q}_{(2)}=\sum_{i=1}^{2}p_i\psi_i-\mathrm{i}\sum_{i\neq j}^{2}\frac{\left(g-\Pi_{ij}\right)\rho_{ji}}{x^i-x^j} \qquad \text{and}$$
$$\widehat{\overline{Q}}_{(2)}=\sum_{i=1}^{2}p_i\bar\psi_i+\mathrm{i}\sum_{i\neq j}^{2}\frac{\bar\rho_{ij}\left(g-\Pi_{ji}\right)}{x^i-x^j}\,, \tag{2.29}$$

which have the standard structure – linear and cubic in the fermions. The Hamiltonian $\widehat{H}_{(2)}$ reduces to

$$\widehat{H}_{(2)}=\frac{1}{2}\sum_{i=1}^{2}p_i^2+\frac{g^2-\Pi_{11}^2-g\left(\Pi_{12}+\Pi_{21}\right)+\Pi_{12}\Pi_{21}}{\left(x^1-x^2\right)^2}\,, \tag{2.30}$$

with the explicit expressions

$$\Pi_{11}=\rho_{12}\bar\rho_{21}+\bar\rho_{12}\rho_{21}\,,$$
$$\Pi_{12}=\left(\psi_1-\psi_2\right)\bar\rho_{12}+\left(\bar\psi_1-\bar\psi_2\right)\rho_{12}\,, \tag{2.31}$$
$$\Pi_{21}=\left(\psi_2-\psi_1\right)\bar\rho_{21}+\left(\bar\psi_2-\bar\psi_1\right)\rho_{21}\,.$$

This $\mathcal{N}=2$ supersymmetric two-particle Calogero model has been previously constructed and analyzed in [16] (for details see the review [9]). This demonstrates that our approach perfectly reproduces the unique known $\mathcal{N}=2$ example.

3. Conclusion

We propose a novel $\mathcal{N}$-extended supersymmetric $su(n)$ spin-Calogero model as a direct supersymmetrization of the bosonic $su(n)$ model [12]. In the case of $\mathcal{N}=4$ supersymmetry, our model resembles the one constructed in [16,17]. However, there are two main differences:

- the center of mass is free
- the $su(n)$ generators are not specified in a particular realization.

Thanks to these features, we were able to generalize the reduction procedure to the no-spin Calogero model from $\mathcal{N}=4$ supersymmetry to any number $\mathcal{N}=2M$ of supersymmetries. This led to the discovery of a genuine $\mathcal{N}=2M$ supersymmetric rational Calogero model for any number of particles.

Our models belong to same class which was proposed in [16,17]. Its main features are

- a huge number of fermionic coordinates, namely $\mathcal{N}n^2$ in number rather than the $\mathcal{N}n$ to be expected
- the supercharges and the Hamiltonian contain terms which a fermionic power much larger than three.

Clearly, these features merit a more careful and detailed analysis.

The following further developments come to mind:

- a superspace description of the constructed models, at least for $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetry, presumably with nonlinear chiral supermultiplets
- an extension to the Calogero–Sutherland inverse-sine-square model
- an extension to the Euler–Calogero–Moser system [11] and its reduction to the goldfish system [13], yielding a supersymmetric goldfish model upon reduction, to be compared with recent results from [21].

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Appendix A. Details of the calculations

We check that the $\mathcal{N}$ supercharges Q^a and $\overline{Q}_b$ in (2.17) generate the $\mathcal{N}=2M$ superalgebra (2.11) with the Hamiltonian (2.18) modulo the first-class constraints (2.19). The best way to perform this calculation is to introduce the composite object

$$L_{ij}=\ell_{ij}+\Pi_{ij}\,. \tag{A.1}$$

Like ℓ_{ij} and Π_{ij}, their sums also form an $su(n)$ algebra,

$$\left\{L_{ij},L_{km}\right\}=\mathrm{i}\left(\delta_{im}L_{kj}-\delta_{kj}L_{im}\right)\,, \tag{A.2}$$

and they Poisson-commute with the fermions exactly as Π_{ij} in (2.16),

$$\left\{L_{ij},\psi_k^a\right\}=\mathrm{i}\left(\delta_{ik}-\delta_{jk}\right)\rho_{ij}^a\,,$$
$$\left\{L_{ij},\rho_{km}^a\right\}=-\mathrm{i}\,\delta_{im}\delta_{jk}\left(\psi_i^a-\psi_j^a\right)-\mathrm{i}\delta_{jk}\rho_{im}^a+\mathrm{i}\delta_{im}\rho_{kj}^a\,,$$
$$\left\{L_{ij},\bar\psi_{k\,a}\right\}=\mathrm{i}\left(\delta_{ik}-\delta_{jk}\right)\bar\rho_{ij\,a}\,,$$
$$\left\{L_{ij},\bar\rho_{km\,a}\right\}=-\mathrm{i}\,\delta_{im}\delta_{jk}\left(\bar\psi_{i\,a}-\bar\psi_{j\,a}\right)-\mathrm{i}\delta_{jk}\bar\rho_{im\,a}+\mathrm{i}\delta_{im}\bar\rho_{kj\,a}\,. \tag{A.3}$$

It should be clear now that the closing of the superalgebra (2.11) for the supercharges

$$Q^a=\sum_{i=1}^{n}p_i\psi_i^a+\mathrm{i}\sum_{i\neq j}^{n}\frac{L_{ij}\,\rho_{ji}^a}{x^i-x^j} \qquad \text{and}$$
$$\overline{Q}_b=\sum_{i=1}^{n}p_i\bar\psi_{i\,b}-\mathrm{i}\sum_{i\neq j}^{n}\frac{\bar\rho_{ij\,b}\,L_{ji}}{x^i-x^j} \tag{A.4}$$

does not depend on the number of particles or the number of supersymmetries. It is based on the Poisson brackets (A.2) and (A.3) and the basic brackets (2.2) and (2.13), i.e.

$$\{x^i, p_j\} = \delta^i_j \qquad \text{and} \qquad \{\psi^a_i, \bar\psi_{j\,b}\} = -\mathrm{i}\,\delta^a_b\delta_{ij}\,, \qquad \left\{\rho^a_{ij}, \bar\rho_{km\,b}\right\} = -\mathrm{i}\,\delta^a_b\delta_{im}\delta_{jk}\,. \tag{A.5}$$

The computation makes repeated use of the all-important identity

$$\frac{1}{(x^i-x^j)(x^i-x^k)} + \frac{1}{(x^j-x^i)(x^j-x^k)} + \frac{1}{(x^k-x^i)(x^k-x^j)} = 0 \quad \text{for} \quad i\neq j\neq k. \tag{A.6}$$

Direct calculation then yields

$$\begin{aligned}
\{Q^a, Q^b\} &= \mathrm{i}\sum_{i\neq j}\frac{\rho^a_{ij}\rho^b_{ji}}{(x^i-x^j)^2}\left(L_{ii}-L_{jj}\right),\\
\{\overline{Q}_a, \overline{Q}_b\} &= \mathrm{i}\sum_{i\neq j}\frac{\bar\rho_{ij\,a}\bar\rho_{ji\,b}}{(x^i-x^j)^2}\left(L_{ii}-L_{jj}\right),\\
\{Q^a, \overline{Q}_b\} &= -2\mathrm{i}\,\delta^a_b H + \mathrm{i}\sum_{i\neq j}\frac{\rho^a_{ij}\bar\rho_{ji\,b}}{(x^i-x^j)^2}\left(L_{ii}-L_{jj}\right),
\end{aligned} \tag{A.7}$$

where

$$H = \frac{1}{2}\sum_{i=1}^n p_i^2 + \frac{1}{2}\sum_{i\neq j}^n \frac{L_{ij}L_{ji}}{\left(x^i-x^j\right)^2}\,. \tag{A.8}$$

Clearly, imposing the constraints (2.19),

$$\chi_i = \ell_{ii} + \Pi_{ii} = L_{ii} \approx 0 \quad \forall\, i \quad (\text{no sum})\,, \tag{A.9}$$

closes the superalgebra (A.7). Since $\sum_i L_{ii} = 0$ reduces $u(n)$ to $su(n)$, the constraints (A.9) cannot be relaxed to $L_{ii} \approx \alpha \neq 0$.

With the same strategy we can check that the supercharges $\widehat{Q}^a$ and $\widehat{\overline{Q}}_a$ and the Hamiltonian $\widehat{H}$ in (2.26) form the same superalgebra (2.11). We do not need to go inside the objects Π_{ij} or $\sqrt{g+\Pi_{ii}}$. Instead, we directly employ the Poisson brackets of the composites,

$$\begin{aligned}
&\{\Pi_{ij}, \Pi_{km}\} = \mathrm{i}\big(\delta_{im}\Pi_{kj} - \delta_{kj}\Pi_{im}\big),\\
&\{\Pi_{ij}, \sqrt{g+\Pi_{kk}}\} = \frac{\mathrm{i}}{2\sqrt{g+\Pi_{kk}}}\big(\delta_{ik}\Pi_{kj} - \delta_{kj}\Pi_{ik}\big),\\
&\{L_{ij}, \psi^a_k\} = \mathrm{i}\big(\delta_{ik}-\delta_{jk}\big)\rho^a_{ij}\,,\\
&\{L_{ij}, \rho^a_{km}\} = -\mathrm{i}\,\delta_{im}\delta_{jk}\big(\psi^a_i-\psi^a_j\big) - \mathrm{i}\delta_{jk}\rho^a_{im} + \mathrm{i}\delta_{im}\rho^a_{kj}\,,\\
&\{L_{ij}, \bar\psi_{k\,a}\} = \mathrm{i}\big(\delta_{ik}-\delta_{jk}\big)\bar\rho_{ij\,a}\,,\\
&\{L_{ij}, \bar\rho_{km\,a}\} = -\mathrm{i}\,\delta_{im}\delta_{jk}\big(\bar\psi_{i\,a}-\bar\psi_{j\,a}\big) - \mathrm{i}\delta_{jk}\bar\rho_{im\,a} + \mathrm{i}\delta_{im}\bar\rho_{kj\,a},\\
&\{\sqrt{g+\Pi_{ii}}, \psi^a_k\} = 0\,,\\
&\{\sqrt{g+\Pi_{ii}}, \rho^a_{km}\} = -\frac{\mathrm{i}}{2\sqrt{g+\Pi_{ii}}}\left(\delta_{ik}\rho^a_{im} - \delta_{im}\rho^a_{ki}\right),\\
&\{\sqrt{g+\Pi_{ii}}, \bar\psi_{k\,a}\} = 0\,,\\
&\{\sqrt{g+\Pi_{ii}}, \bar\rho_{km\,a}\} = -\frac{\mathrm{i}}{2\sqrt{g+\Pi_{ii}}}\left(\delta_{ik}\bar\rho_{im\,a} - \delta_{im}\bar\rho_{ij\,a}\right).
\end{aligned} \tag{A.10}$$

This drastically simplifies the evaluation and, by using again (A.6), one may readily convince oneself that indeed

$$\{\widehat{Q}^a, \widehat{\overline{Q}}_b\} = -2\mathrm{i}\,\delta^a_b\widehat{H} \qquad \text{and} \qquad \{\widehat{Q}^a, \widehat{Q}^b\} = \{\widehat{\overline{Q}}_a, \widehat{\overline{Q}}_b\} = 0\,. \tag{A.11}$$

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