$\mathcal{N} = 4$ supersymmetric mechanics on curved spaces

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We present $\mathcal{N} = 4$ supersymmetric mechanics on n-dimensional Riemannian manifolds constructed within the Hamiltonian approach. The structure functions entering the supercharges and the Hamiltonian obey modified covariant constancy equations as well as modified Witten–Dijkgraaf–Verlinde–Verlinde equations specified by the presence of the manifold's curvature tensor. Solutions of original Witten–Dijkgraaf–Verlinde–Verlinde equations and related prepotentials defining $\mathcal{N} = 4$ superconformal mechanics in flat space can be lifted to $so(n)$-invariant Riemannian manifolds. For the Hamiltonian this lift generates an additional potential term which, on spheres and (two-sheeted) hyperboloids, becomes a Higgs-oscillator potential. In particular, the sum of n copies of one-dimensional conformal mechanics results in a specific superintegrable deformation of the Higgs oscillator.

I. INTRODUCTION

The Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations were introduced a few decades ago in the context of two-dimensional topological field theories [1]. In their initial version they read

$$F_{ijm}^{(0)}\delta^{nm}F_{kln}^{(0)} = F_{ilm}^{(0)}\delta^{mn}F_{kjn}^{(0)} \quad \text{with} \quad F_{ijk}^{(0)} = \partial_i\partial_j\partial_k F^{(0)}(x). \tag{1.1}$$

Clearly, these equations are not covariant with respect to general coordinate transformations, so their covariantization requires introducing additional geometric structures. Finding solutions to the WDVV equations is a nontrivial task, which was considered in numerous papers in various contexts (see, e.g. [2–8] and references in them). The WDVV equations appear also when constructing $\mathcal{N} = 4$ supersymmetric/superconformal extensions of mechanics on n-dimensional Euclidean space (see [9–14] and references therein). Indeed, as it was firstly demonstrated in [9],

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if we insist that the $\mathcal{N} = 4$ supercharges Q^a and $\bar{Q}_b$ with a, $b = 1$, 2, in the form

$$Q^a = p_i\psi^{ai} + iW_i^{(0)}\psi^{ai} + iF_{ijk}^{(0)}\psi^{bi}\psi_b^j\bar{\psi}^{ak} \quad \text{and}$$

$$\bar{Q}_a = p_i\bar{\psi}_a^i - iW_i^{(0)}\bar{\psi}_a^i + iF_{ijk}^{(0)}\bar{\psi}_b^i\bar{\psi}^{bj}\psi_a^k, \tag{1.2}$$

obey the $\mathcal{N} = 4$ super Poincaré algebra

$$\{Q^a, Q^b\} = 0, \quad \{\bar{Q}_a, \bar{Q}_b\} = 0, \quad \{Q^a, \bar{Q}_b\} = \frac{i}{2}\delta_b^a H, \tag{1.3}$$

then the totally symmetric structure functions $F_{ijk}^{(0)}(x)$ entering the supercharges (1.2) have to satisfy (1.1), while the prepotential $W^{(0)}$ is found by solving

$$\partial_i\partial_j W^{(0)} + F_{ijk}^{(0)}\partial_k W^{(0)} = 0 \quad \text{and} \quad W_i^{(0)} = \partial_i W^{(0)}. \tag{1.4}$$

We should add that, when evaluating the brackets between the supercharges in (1.3), the basic variables were taken to obey the standard Dirac brackets

$$\{x^i, p_j\} = \delta_j^i \quad \text{and} \quad \{\psi^{ai}, \bar{\psi}_b^j\} = \frac{i}{2}\delta_b^a\delta^{ij}. \tag{1.5}$$

Exploiting this relation, one can postulate a general coordinate invariant version of the WDVV equations simply by constructing $\mathcal{N} = 4$ supersymmetric mechanics on arbitrary Riemannian spaces. This was achieved in our

recent paper [15]. To construct such mechanics, one has to generalize the ansatz for the supercharges (1.2) and the Poisson brackets (1.5) to be covariant under general coordinate transformations, and then to check the conditions on the structure functions implied by the $\mathcal{N} = 4$ super-Poincaré algebra relations (1.3) these supercharges should obey. While [15] accomplished this for the special case of $W_i = 0$—no potential term—its result for the structure functions F_{ijk} remains true in general:

$$\nabla_i F_{jkm} = \nabla_j F_{ikm} \quad \text{and}$$

$$F_{ikp} g^{pq} F_{jmq} - F_{jkp} g^{pq} F_{imq} + R_{ijkm} = 0. \quad (1.6)$$

These are the curved WDVV equations on spaces with a metric g_{ij}. It was also shown there that solutions of the flat WDVV equations (1.1) can be extended to solutions of the curved WDVV equations (1.6) on isotropic spaces.

The main goal of the present paper is to construct $\mathcal{N} = 4$ supersymmetric mechanics with a nonzero potential on arbitrary Riemannian spaces. As a first step, in Sec. II we introduce generalized Poisson brackets which are covariant with respect to general coordinate transformations. Then we write down the most general ansatz for the supercharges (linear and cubic in the fermionic variables, for the case with and without additional spin variables) and analyze the conditions on the structure functions. We obtain equations on the prepotentials which generalize (1.4). In Sec. III we specialize to isotropic spaces and extend the general solution found in [15] to include the prepotential W_i. In Sec. IV we present some explicit solutions for the most interesting spaces—spheres and pseudospheres. Finally, we conclude with a few comments and remarks.

II. SYMPLECTIC STRUCTURE, SUPERCHARGES AND HAMILTONIAN

We are going to construct the supercharges Q^a, $\bar{Q}_b, (a, b = 1, 2)$ forming an $\mathcal{N} = 4$, $d = 1$ Poincaré superalgebra[1]

$$\{Q^a, Q^b\} = 0, \qquad \{\bar{Q}_a, \bar{Q}_b\} = 0, \qquad \{Q^a, \bar{Q}_b\} = \frac{i}{2} \delta^a_b H$$

$$(2.1)$$

for n-dimensional systems in which each bosonic degree of freedom $x^i (i = 1, \ldots, n)$ is accompanied by four fermionic ones ψ^{ai}, $\bar{\psi}^j_b = (\psi^{bj})^\dagger$. The extended phase space, parametrized by the bosonic coordinates x^i and momenta p_i and the fermionic coordinates ψ^{ai}, $\bar{\psi}^j_b$, can be equipped by the symplectic structure

$$\Omega = dp_i \wedge dx^i + i d(\psi^{ia} g_{ij} D\bar{\psi}^j_a - \bar{\psi}^{ia} g_{ij} D\psi^j_a)$$

$$= dp_i \wedge dx^i + i R_{ijkl} \psi^{ia} \bar{\psi}^j_a dx^k \wedge dx^l + 2i g_{ij} D\psi^{ia} \wedge D\bar{\psi}^j_a,$$

$$(2.2)$$

where $D\psi^{ia} \equiv d\psi^{ia} + \Gamma^i_{jk} \psi^{ja} dx^k$, and Γ^i_{jk} and R^i_{jkl} are the components of the Levi-Civita connection and curvature of the metric $g_{ij}(x)$ defined in a standard way as

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} (\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}), \quad \text{and}$$

$$R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^m_{jl} \Gamma^i_{mk} - \Gamma^m_{jk} \Gamma^i_{ml}. \quad (2.3)$$

This symplectic structure is manifestly invariant with respect to the transformations

$$\tilde{x}^i = \tilde{x}^i(x), \quad \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} p_j, \quad \tilde{\psi}^{ia} = \frac{\partial \tilde{x}^i(x)}{\partial x^j} \psi^{ja}. \quad (2.4)$$

The Poisson brackets between the basic variables can be immediately extracted from the symplectic structure (2.2):

$$\{x^i, p_j\} = \delta^i_j, \qquad \{\psi^{ai}, \bar{\psi}^j_b\} = \frac{i}{2} \delta^a_b g^{ij},$$

$$\{p_i, p_j\} = -2i R_{ijkm} \psi^{ak} \bar{\psi}^m_a,$$

$$\{p_i, \psi^{aj}\} = \Gamma^j_{ik} \psi^{ak}, \qquad \{p_i, \bar{\psi}^j_a\} = \Gamma^j_{ik} \bar{\psi}^k_a. \quad (2.5)$$

To construct the supercharges Q^a, $\bar{Q}_b$ we have two possibilities.

(i) One may construct the standard supercharges in terms of the variables x^i, p_j, ψ^{ai}, $\bar{\psi}^j_b$ only, mainly following to the line of the paper [11].

(ii) One may extend the set of the basic variables by the additional bosonic spin variables $\{u^a, \bar{u}_a | a = 1, 2\}$ parametrizing an internal two-sphere and obeying the brackets

$$\{u^a, \bar{u}_b\} = -i \delta^a_b. \quad (2.6)$$

These new variables will appear in the supercharges only through the $su(2)$ currents [14,16]

$$J^{ab} = \frac{i}{2} (u^a \bar{u}^b + u^b \bar{u}^a) \Rightarrow \{J^{ab}, J^{cd}\}$$

$$= -\epsilon^{ac} J^{bd} - \epsilon^{bd} J^{ac}. \quad (2.7)$$

Let us consider these possibilities separately.

A. Standard supercharges

The most general ansatz for the standard supercharges reads:

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[1]We use the following convention for raising and lowering $su(2)$ indices: $\mathcal{A}_a = \epsilon_{ab} \mathcal{A}^b$, $\mathcal{A}^a = \epsilon^{ab} \mathcal{A}_b$, $\epsilon_{12} = \epsilon^{21} = 1$.

$$Q_W^a = p_i\psi^{ai} + iW_i\psi^{ai} + iF_{ijk}\psi^{bi}\psi_b^j\bar\psi^{ak} + iG_{ijk}\psi^{ai}\psi^{bj}\bar\psi_b^k,$$

$$(\bar Q_W)_a = p_i\bar\psi_a^i - iW_i\bar\psi_a^i + iF_{ijk}\bar\psi_b^i\bar\psi^{bj}\psi_a^k + iG_{ijk}\bar\psi_a^i\bar\psi_b^j\psi^{bk}.$$

$$(2.8)$$

Here, W_i, F_{ijk} and G_{ijk} are arbitrary, for the time being, real functions depending on n coordinates x^i. In addition, we assume that the functions F_{ijk} and G_{ijk} are symmetric and antisymmetric over the first two indices, respectively:

$$F_{ijk} = F_{jik}, \qquad G_{ijk} = -G_{jik}. \qquad (2.9)$$

The conditions that these supercharges span an $\mathcal{N} = 4$ super Poincaré algebra (2.1) result in the following equations on the functions involved:

$$G_{ijk} = 0, \quad F_{ijk} - F_{ikj} = 0 \Rightarrow F_{ijk} \text{ is totally symmetric,}$$

$$(2.10)$$

$$\nabla_i F_{jkm} - \nabla_j F_{ikm} = 0, \qquad (2.11)$$

$$F_{ikp}g^{pq}F_{jmq} - F_{jkp}g^{pq}F_{imq} + R_{ijkm} = 0, \qquad (2.12)$$

and

$$\nabla_i W_j - \nabla_j W_i = 0 \Rightarrow W_i = \partial_i W, \qquad (2.13)$$

$$\nabla_i \partial_j W + F_{ijk}g^{km}\partial_m W = 0, \qquad (2.14)$$

where, as usual,

$$\nabla_i W_j = \partial_i W_j - \Gamma_{ij}^k W_k,$$

$$\nabla_i F_{jkl} = \partial_i F_{jkl} - \Gamma_{ij}^m F_{klm} - \Gamma_{ik}^m F_{jlm} - \Gamma_{il}^m F_{jkm}. \qquad (2.15)$$

Once the equations (2.10)–(2.14) are satisfied, the Hamiltonian H acquires the form

$$H_W = g^{ij}p_ip_j + g^{ij}\partial_i W\partial_j W + 4\nabla_i\partial_j W\psi^{ci}\bar\psi_c^j$$
$$- 4[\nabla_m F_{ijk} + R_{imjk}]\psi^{ci}\bar\psi_c^m\psi^{dj}\bar\psi_d^k. \qquad (2.16)$$

B. Supercharges with spin variables

Following [14,16], the spin variables $\{u^a, \bar u_a\}$ may be utilized to slightly modify the prepotential term in the supercharges to be

$$Q_U^a = p_i\psi^{ia} + U_iJ^{ab}\psi_b^i - iF_{ijk}\psi^{bi}\psi_b^j\bar\psi^{ak} + iG_{ijk}\psi^{ai}\psi^{bj}\bar\psi_b^k,$$

$$(\bar Q_U)_a = p_i\bar\psi_a^i - U_iJ_{ab}\bar\psi^{bi} - iF_{ijk}\bar\psi_b^i\bar\psi^{bj}\psi_a^k + iG_{ijk}\bar\psi_a^i\bar\psi_b^j\psi^{bk}.$$

$$(2.17)$$

These supercharges form an $\mathcal{N} = 4$ super Poincaré algebra if the functions F_{ijk} and G_{ijk} obey the same constraints (2.10)–(2.12), while the constraints (2.13), (2.14) which include the prepotential are changed to

$$\nabla_i U_j - \nabla_j U_i = 0 \Rightarrow U_i = \partial_i U, \qquad (2.18)$$

$$\nabla_i \partial_j U - \partial_i U\partial_j U + F_{ijk}g^{km}\partial_m U = 0. \qquad (2.19)$$

When the constraints (2.10)–(2.12) and (2.18), (2.19) are satisfied, the Hamiltonian reads

$$H_U = g^{ij}p_ip_j + \frac{1}{2}J^{ab}J_{ab}g^{ij}\partial_i U\partial_j U - 4i\nabla_i\partial_j UJ_{ab}\psi^{ai}\bar\psi^{bj}$$
$$+ 4[\nabla_m F_{ijk} - R_{imjk}]\psi^{ci}\bar\psi_c^m\psi^{dj}\bar\psi_d^k. \qquad (2.20)$$

As we can see now, the $su(2)$ Casimir element $J_{ab}J^{ab}$ plays the role of the coupling constant.

Finally, note that the solutions of the Eqs. (2.13), (2.14) and (2.18), (2.19) are related as follows,

$$W = e^{-U}. \qquad (2.21)$$

Thus, any solution of the system (2.13), (2.14) generates a solution of the system (2.18), (2.19), and vice versa. It should be stressed, however, that the exact form of the bosonic potentials and terms quadratic in fermionic variables are very different in the Hamiltonians H_W (2.16) and H_U (2.20).

Equation (2.11) qualifies F_{ijk} as a so-called third-rank Codazzi tensor [17], while (2.12) is the curved WDVV equation proposed in [15], for which the Eqs. (2.14), (2.19) are the curved analogs of the flat equations on the prepotentials discussed in [11,14].

Summarizing, one may conclude that to construct $\mathcal{N} = 4$ supersymmetric n-dimensional mechanics with a given bosonic metric g_{ij} one has

(i) To solve the curved WDVV equations (2.11), (2.12) for the fully symmetric function F_{ijk},
(ii) To find admissible prepotentials as solutions of the Eqs. (2.13), (2.14) and/or (2.18), (2.19).

In what follows, we will use this procedure to construct $\mathcal{N} = 4$ supersymmetric n-dimensional mechanics.

III. ISOTROPIC SPACES

A. Solution of curved WDVV equation

In [15] a large class of solutions to the Eqs. (2.11), (2.12) has been constructed on isotropic spaces. Such spaces admit an $so(n)$-invariant metric with components

$$g_{ij} = \frac{1}{f(r)^2}\delta_{ij},$$

$$\Gamma_{ij}^k = -\frac{f'}{rf}(x^i\delta_j^k + x^j\delta_i^k - x^k\delta_{ij}),$$

$$r^2 = \delta_{ij}x^ix^j. \qquad (3.1)$$

The key point of the analysis performed in [15] is the following ansatz on the structure function F_{ijk}:

$$F_{ijk} = a(r)x^i x^j x^k + b(r)(\delta_{ij}x^k + \delta_{jk}x^i + \delta_{ik}x^j)$$
$$+ f(r)^{-2}F_{ijk}^{(0)}, \tag{3.2}$$

where $F_{ijk}^{(0)}$ is an arbitrary solution of the flat WDVV equation, i.e.,

$$F_{ikp}^{(0)}\delta^{pq}F_{jmq}^{(0)} - F_{jkp}^{(0)}\delta^{pq}F_{imq}^{(0)} = 0 \quad \text{with} \quad F_{ijk}^{(0)} = \partial_i\partial_j\partial_k F^{(0)}. \tag{3.3}$$

One may check that the linear equation (2.11) is satisfied if

$$r(rf' - f)a + 4f'b + fb' = 0 \quad \text{and} \quad x^i F_{ijk}^{(0)} = c\delta_{jk},$$
$$c = \text{const}, \tag{3.4}$$

where we fixed the scale of $F^{(0)}$. Here, prime means differentiation with respect to r. Thus, the symmetric tensor F_{ijk} (3.2) with the restrictions (3.3), (3.4) provides a third rank Codazzi tensor on isotropic spaces (3.1).

The curved WDVV equation (2.12) further imposes the quadratic conditions

$$f^2 b(r^2 a + b) + ca = -\frac{1}{rf^3}\left(\frac{f'}{r}\right)' \quad \text{and}$$
$$r^2 f^2 b^2 + 2cb = \frac{r^2 f'}{f^4}\left(\frac{f}{r^2}\right)'. \tag{3.5}$$

We note that these equations already imply the condition (3.4).

The equations (3.5) may be easily solved as

$$a = \frac{2cf\sqrt{c^2 f^2 - 2rff' + r^2(f')^2} \pm (2c^2 f^2 - 3rff' + r^2(f')^2 + r^2 ff'')}{r^4 f^3\sqrt{c^2 f^2 - 2rff' + r^2(f')^2}},$$
$$b = -\frac{cf \pm \sqrt{c^2 f^2 - 2rff' + r^2(f')^2}}{r^2 f^3}. \tag{3.6}$$

For $c = 1$, it simplifies to

$$a = \frac{2f(f - rf') \pm (2f^2 - 3rff' + r^2(f')^2 + r^2 ff'')}{r^4 f^3(f - rf')} \quad \text{and} \quad b = -\frac{f \pm (f - rf')}{r^2 f^3}. \tag{3.7}$$

It should be noted that the solution for a in (3.6) becomes $0/0$ indeterminate if

$$f_0(r) = \mu r^{1\pm\sqrt{1-c^2}}, \qquad \mu = \text{const}. \tag{3.8}$$

With such a metric the solution of the Eqs. (3.4), (3.5) reads

$$a = \frac{2c}{r^4 f_0^2}, \qquad b = -\frac{c}{r^2 f_0^2}. \tag{3.9}$$

Another exceptional case corresponds to $c = \pm 1$ and the metric function

$$f_1(r) = \mu r. \tag{3.10}$$

For this case, the function $a(r)$ is not restricted while $b(r)$ has the form

$$b = -\frac{c}{\mu^2 r^4}. \tag{3.11}$$

B. Searching for the prepotentials

Having at hands the solution for the curved WDVV equations (2.11), (2.12), one may try to solve the equations

for the prepotentials (2.13), (2.14) or (2.18), (2.19), respectively. It should be noted that even in the flat case, where a variety of the solutions to the WDVV equations is known [3–7], such a task is far from being completely solved. Nevertheless, many particular solutions are known for the standard supercharges [11–13] as well as for the case with spin variables [14,16]. Leaving the full analysis of the admissible prepotentials for the future, let us demonstrate that each prepotential found for the flat WDVV equations can be embedded into isotropic spaces with a metric (3.1).

1. Lifting a "flat" prepotential $W^{(0)}$

In this case we have to solve the equation (2.13) which for the metric (3.1) and for the F_{ijk} given in (3.2) acquires the form

$$\partial_i\partial_j W + \left(\frac{f'}{rf} + bf^2\right)(x^i\partial_j W + x^j\partial_i W)$$
$$+ \delta_{ij}\left(bf^2 - \frac{f'}{rf}\right)x^m\partial_m W + af^2 x^i x^j x^m\partial_m W$$
$$+ F_{ijm}^{(0)}\delta^{mn}\partial_n W = 0. \tag{3.12}$$

Let us also suppose that we know the solution $W^{(0)}$ of the flat equation

$$\partial_i \partial_j W^{(0)} + F^{(0)}_{ijm} \delta^{mn} \partial_n W^{(0)} = 0. \qquad (3.13)$$

All such solutions found in [11–13] obey the additional condition

$$x^i \partial_i W^{(0)} = \alpha = \text{const} \Rightarrow x^m F^{(0)}_{mjk} = \delta_{jk}, \qquad (3.14)$$

i.e., the parameter c in (3.6) fixed to be equal to one.

If we now choose the following ansatz for the prepotential W,

$$W = \tilde{W}(r) + W^{(0)}, \qquad (3.15)$$

then the "flat" prepotential $W^{(0)}$ will appear in the Eq. (3.12) only through the constant α, except for the second term

$$\sim \left(\frac{f'}{rf} + bf^2 \right)(x^i \partial_j W^{(0)} + x^j \partial_i W^{(0)}).$$

To kill this term we have to choose $b = -\frac{f'}{rf^3}$. This choice corresponds to the following solution in (3.7)

$$a = \frac{ff' - r(f')^2 - rff''}{r^3 f^3 (f - rf')}, \qquad b = -\frac{f'}{rf^3}. \qquad (3.16)$$

Finally, it is a matter of straightforward calculations to check that the prepotential

$$W = \tilde{W}(r) + W^{(0)} \quad \text{with} \quad \tilde{W}' = \alpha \frac{f'}{f - rf'} \qquad (3.17)$$

solves the Eq. (3.12). Correspondingly, the bosonic potential in the Hamiltonian (2.16) reads

$$g^{ij} \partial_i W \partial_j W = f^2 \left[\delta^{ij} \partial_i W^{(0)} \partial_j W^{(0)} + \alpha^2 \frac{f'(2f - rf')}{r(f - rf')^2} \right]. \qquad (3.18)$$

2. Lifting a "flat" prepotential with spin variables

For the supercharges with spin variables the solutions for the Eq. (2.19) may be found by using the relation (2.21). However, the additional constraint for the flat prepotential (3.14) we used above is quite unconventional. Indeed, in [14] the additional condition on the solution of the flat equation

$$\partial_i \partial_j U^{(0)} - \partial_i U^{(0)} \partial_j U^{(0)} + F^{(0)}_{ijk} \delta^{km} \partial_m U^{(0)} = 0 \quad (3.19)$$

reads

$$x^k \partial_k U^{(0)} = \alpha - 1 \Rightarrow x^k F^{(0)}_{ijk} = \alpha \delta_{ij}. \qquad (3.20)$$

Therefore, we need to reconsider the solution of (2.19) using an ansatz

$$U = \tilde{U}(r) + U^{(0)}. \qquad (3.21)$$

We will use the same ansatz for the F_{ijk} (3.2) with the same conditions on the functions $a(r)$, $b(r)$ (3.5) and the constraint (3.20).

Now, it is rather easy to check that the prepotential U (3.21) obeys the Eqs. (2.19) if

$$\tilde{U}' = \frac{f'}{f} + rf^2 b. \qquad (3.22)$$

Therefore, the resulting potential in the Hamiltonian (2.20) reads

$$V = \frac{1}{2} J^{ab} J_{ab} f^2 \left[\delta^{ij} \partial_i U^{(0)} \partial_j U^{(0)} + \frac{2(\alpha - 1)}{r} \left(\frac{f'}{f} + rf^2 b \right) \right.$$
$$\left. + \left(\frac{f'}{f} + rf^2 b \right)^2 \right], \qquad (3.23)$$

with b given in (3.6).

3. Lifting a vanishing "flat" prepotential

This case corresponds to the absence of the "flat" prepotential, i.e., to the case with $W^{(0)} = 0$. To simplify the analysis we suppose that the prepotential W depends on r only, while the "flat" WDVV solution $F^{(0)}_{ijk}$ still obeys the constraint (3.14), i.e.,

$$x^i F^{(0)}_{ijk} = \delta_{jk} \Rightarrow c = 1. \qquad (3.24)$$

With these assumptions the Eq. (2.13) reads

$$\delta^{ij} \frac{2f + br^2 f^3 - rf'}{rf} W'$$
$$+ x^i x^j \frac{rfW'' + (-f + (2b + ar^2)r^2 f^3 + 2rf')W'}{r^3 f} = 0. \qquad (3.25)$$

To kill the first term in (3.25) we have to choose $b = \frac{-2f + rf'}{r^2 f^3}$. This choice corresponds to the following solution in (3.7),

$$a = \frac{-rf' + f(4 + \frac{r^2 f''}{f - rf'})}{r^4 f^3}, \qquad b = \frac{-2f + rf'}{r^2 f^3}. \qquad (3.26)$$

With such functions a, b the equation which defines the prepotential acquires the form

$$rW'' + \left(-1 + \frac{3rf'}{f} + \frac{r^2 f''}{f - rf'} \right) W' = 0. \qquad (3.27)$$

This equation has the general solution

$$W = c_1\left(\frac{r^2}{f^2} + c_2\right), \qquad c_1, c_2 = \text{const.} \qquad (3.28)$$

The bosonic potential in the Hamiltonian (2.16) now reads

$$g^{ij}\partial_i W \partial_j W = 4c_1^2 \frac{r^2}{f^4}(f - rf')^2. \qquad (3.29)$$

It is interesting that, after passing to the prepotential $U = -\log(W)$ (2.21), the potential term in the Hamiltonian with spin variables (2.20) acquires the form

$$\frac{1}{2}J^{ab}J_{ab}g^{ij}\partial_i U\partial_j U = J^{ab}J_{ab}\frac{2r^2(f - rf')^2}{(r^2 + c_2 f^2)^2}. \qquad (3.30)$$

Thus, we see that the constant c_2 entering the solution (3.28) and having no impact on the Hamiltonian H_W (3.29) becomes quite important for the Hamiltonian H_U (2.20).

IV. EXAMPLES OF PREPOTENTIALS ON THE (PSEUDO)SPHERE

A. Supersymmetric black holes

In this section we present some interesting prepotentials on (pseudo)spheres which admit $\mathcal{N} = 4$ supersymmetry. As we can see from the previous section, any "flat" system obeying the constraints (3.14), i.e.,

$$x^i\partial_i W^{(0)} = \alpha = \text{const} \Rightarrow x^m F^{(0)}_{mjk} = \delta_{jk}, \qquad (4.1)$$

has its image on isotropic spaces with the potential (3.18)

$$\mathcal{V} = g^{ij}\partial_i W \partial_j W = f^2\left[\delta^{ij}\partial_i W^{(0)}\partial_j W^{(0)} + \alpha^2\frac{f'(2f - rf')}{r(f - rf')^2}\right]. \qquad (4.2)$$

In the case of a (pseudo)sphere with the metric

$$f = 1 + \epsilon r^2, \qquad \epsilon = \pm 1, \qquad (4.3)$$

the potential $\mathcal{V}$ is simplified to be

$$\mathcal{V} = (1 + \epsilon r^2)^2\delta^{ij}\partial_i W^{(0)}\partial_j W^{(0)} + 4\alpha^2\epsilon\left(\frac{1 + \epsilon r^2}{1 - \epsilon r^2}\right)^2. \qquad (4.4)$$

Thus we see that, besides getting multiplied with the standard factor $(1 + \epsilon r^2)^2$, the "flat" potential is shifted by the potential of a Higgs oscillator [18],

$$V_{\text{Higgs}} \equiv \left(\frac{1 + \epsilon r^2}{1 - \epsilon r^2}\right)^2. \qquad (4.5)$$

This means that even a mutually noninteracting system, being placed on the (pseudo)sphere, becomes interacting

via the Higgs potential. A prominent example comes from the sum of several $\mathcal{N} = 4$ supersymmetric mechanics on flat space with conformal prepotentials

$$W^{(0)} = \sum_i^n a_i \log(x^i), \qquad F^{(0)} = \frac{1}{2}\sum_i^n (x^i)^2 \log(x^i). \qquad (4.6)$$

With such almost "free" prepotentials we obtain the following potential for the system on the (pseudo)sphere,

$$\alpha = \sum_i^n a_i \Rightarrow \mathcal{V}_1 = (1 + \epsilon r^2)^2\left[\sum_i^n \frac{a_i^2}{(x^i)^2} + \frac{4\epsilon(\sum_i^n a_i)^2}{(1 - \epsilon r^2)^2}\right]. \qquad (4.7)$$

A system with such a potential was obtained as a reduced angular (compact) part of conformal mechanics describing the motion of a particle in a near-horizon Myers–Perry black-hole background with coinciding rotational parameters in [19], and its superintegrability was proven there as well.

B. (Pseudo)sphere image of a free system

This case is analogous to the one previously considered, but, unfortunately, we cannot just substitute $W^{(0)} = 0$ into (4.4) because this is in contradiction with the constraint $x^i\partial_i W^{(0)} = \alpha$. Thus, we have to use the general consideration in Sec. III B 3.

For the (pseudo)sphere with metric $f = 1 + \epsilon r^2$ the potential in (3.29) reads

$$\mathcal{V}_W = 4c_1^2\frac{r^2(1 - \epsilon r^2)^2}{(1 + \epsilon r^2)^4}. \qquad (4.8)$$

At the same time, the potential for the supercharges with spin variables can be easily obtained by passing to the prepotential U (2.21),

$$U = -\log(W) = -\log\left[c_1\left(\frac{r^2}{(1 + \epsilon r^2)^2} + c_2\right)\right], \qquad (4.9)$$

and by evaluating the potential for the case with spin variables in the Hamiltonian (2.20) we get

$$\mathcal{V}_U = \frac{1}{2}J^{ab}J_{ab}g^{ij}\partial_i U\partial_j U = 2J^{ab}J_{ab}\frac{r^2(1 - \epsilon r^2)^2}{(r^2 + c_2(1 + \epsilon r^2)^2)^2}. \qquad (4.10)$$

Choosing now

$$c_2 = -\frac{1}{4\epsilon} \qquad (4.11)$$

we obtain

$$\mathcal{V}_U = 32\epsilon^2 J^{ab} J_{ab} \frac{r^2}{(1-\epsilon r^2)^2}$$

$$= 8\epsilon J^{ab} J_{ab} \left[\frac{(1+\epsilon r^2)^2}{(1-\epsilon r^2)^2} - 1 \right]$$

$$\equiv 8\epsilon J^{ab} J_{ab} [V_{\text{Higgs}} - 1]. \tag{4.12}$$

Thus, the Higgs oscillator in the Hamiltonian with spin variables on the (pseudo)sphere is the image of a completely free system.

C. (Pseudo)sphere image of the isotropic harmonic oscillator

Due to the exceptional role the standard harmonic oscillator plays among integrable systems, it is interesting to find its image in the case of $\mathcal{N}=4$ supersymmetric mechanics on the (pseudo)sphere. Unfortunately, in this case our consideration of the previous sections does not help too much, because if we choose

$$W^{(0)} = mr^2 \quad \text{with} \quad m = \text{const}, \tag{4.13}$$

then the condition $x^i \partial_i W^{(0)} = \alpha$ we used previously will not be valid anymore. However, plugging the expression (4.13) into (3.13) we will get

$$\partial_i \partial_j W^{(0)} + F^{(0)}_{ijm} \delta^{mn} \partial_n W^{(0)} = 0 \Rightarrow x^m F^{(0)}_{ijm} = -\delta_{ij}. \tag{4.14}$$

Thus, in the general solution (3.2), (3.6) for the curved WDVV equation we have to substitute $c = -1$. Thus, everything is simplified, and we have two solutions for the (pseudo)sphere with $f = 1 + \epsilon r^2$:

$$a_1 = \frac{4\epsilon^2}{(1-\epsilon r^2)(1+\epsilon r^2)^3}, \qquad b_1 = \frac{2\epsilon}{(1+\epsilon r^2)^3}, \tag{4.15}$$

$$a_2 = -\frac{4}{r^4(1-\epsilon r^2)(1+\epsilon r^2)^3}, \qquad b_2 = \frac{2}{r^2(1+\epsilon r^2)^3}. \tag{4.16}$$

Now, the basic equation which defines the admissible prepotentials (3.12) for the prepotentials $\tilde{W} = \tilde{W}(r)$ acquires the form

$$0 = x^i x^j \left[\frac{\tilde{W}''}{r^2} - \frac{\tilde{W}'}{r^3} + \frac{4\epsilon + (2b + ar^2)(1+\epsilon r^2)^3}{r(1+\epsilon r^2)} (\tilde{W}' + 2mr) \right]$$

$$+ \delta_{ij} r(1+\epsilon r^2)^2 \left(b - \frac{2\epsilon}{(1+\epsilon r^2)^3} \right) (\tilde{W}' + 2mr). \tag{4.17}$$

Now we see that, to kill the δ_{ij} term, we have to choose the first solution (4.15). Therefore, the equation for the prepotential $\tilde{W}$ reads

$$r(1-\epsilon r^2)(1+\epsilon r^2)\tilde{W}'' - (1 - 8\epsilon r^2 + 3\epsilon^2 r^4)\tilde{W}'$$

$$+ 8m\epsilon r^3(2 - \epsilon r^2) = 0, \tag{4.18}$$

with the solution

$$\tilde{W} = -\frac{2m(1 + 15\epsilon r^2 + 3\epsilon^2 r^4 + \epsilon^3 r^6) + \epsilon r^2 M_1}{2\epsilon(1+\epsilon r^2)^2} + M_2, \tag{4.19}$$

where M_1 and M_2 are integration constants. Thus, the (pseudo)sphere image of the oscillator potential $W^{(0)}$ (4.13) reads

$$\mathcal{V}_W = f^2 \delta^{ij} \partial_i (\tilde{W} + W^{(0)}) \partial_j (\tilde{W} + W^{(0)})$$

$$= (24m + M_1)^2 r^2 \frac{(1-\epsilon r^2)^2}{(1+\epsilon r^2)^4}. \tag{4.20}$$

One may see that this potential coincides, modulo redefinition of the coupling constants, with the image of the "free" system (4.8), despite the fact that the prepotential (4.19) is different from the one in (3.28). This difference plays an essential role after passing to the Hamiltonian with spin variables, in which the corresponding potential acquires the form

$$U = -\log(W) \Rightarrow \mathcal{V}_U = J^{ab} J_{ab} \frac{2(24m + M_1)^2 \epsilon^2 r^2 (1-\epsilon r^2)^2}{(2\epsilon^2(m - \epsilon M_2)r^4 + \epsilon(28m + M_1 - 4\epsilon M_2)r^2 + 2(m - \epsilon M_2))^2}. \tag{4.21}$$

Special cases of this potential are known. For example,

$$M_2 = \frac{32m + M_1}{8\epsilon} \Rightarrow \mathcal{V}_U = J^{ab} J_{ab} 8\epsilon(V_{\text{Higgs}} - 1), \tag{4.22}$$

$$M_2 = \frac{m}{\epsilon} \Rightarrow \mathcal{V}_U = J^{ab} J_{ab} 2 \frac{(1-\epsilon r^2)^2}{r^2}. \tag{4.23}$$

V. CONCLUSIONS

We constructed n-dimensional $\mathcal{N} = 4$ supersymmetric mechanics on arbitrary spaces with a metric g_{ij}. Besides reproducing the curved WDVV equations already found in [15] for the first prepotential, we obtained the curved version of the equations defining the second prepotential for the cases of supercharges with and without spin variables. For any solution of the curved WDVV equations on $so(n)$-invariant conformally flat spaces we constructed admissible prepotentials on these spaces. A nice feature of our construction is the possibility to lift any "flat" prepotentials to isotropic spaces. Finally, we provided some interesting potentials for $\mathcal{N} = 4$ mechanics on the (pseudo)sphere.

A still unsolved task is a superspace description of our mechanics. To generalize the superspace approach developed in [20,21] clearly one will have to find new superspace irreducibility constraints for the (1,4,3) supermultiplets which are covariant under general coordinate transformations in the target space. A related task is to understand how the real target-space Kähler metric which necessarily appears in the superspace approach [20,21] is related with our component Hamiltonian description. These tasks will be considered elsewhere.

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