



A new construction of Calabi–Yau manifolds: Generalized CICYs

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Abstract

We present a generalization of the complete intersection in products of projective space (CICY) construction of Calabi–Yau manifolds. CICY three-folds and four-folds have been studied extensively in the physics literature. Their utility stems from the fact that they can be simply described in terms of a ‘configuration matrix’, a matrix of integers from which many of the details of the geometries can be easily extracted. The generalization we present is to allow negative integers in the configuration matrices which were previously taken to have positive semi-definite entries. This broadening of the complete intersection construction leads to a larger class of Calabi–Yau manifolds than that considered in previous work, which nevertheless enjoys much of the same degree of calculational control. These new Calabi–Yau manifolds are complete intersections in (not necessarily Fano) ambient spaces with an effective anticanonical class. We find examples with topology distinct from any that has appeared in the literature to date. The new manifolds thus obtained have many interesting features. For example, they can have smaller Hodge numbers than ordinary CICYs and lead to many examples with elliptic and K3-fibration structures relevant to F-theory and string dualities. © 2016 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

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1. Introduction

Calabi–Yau manifolds constructed as complete intersections of polynomial hypersurfaces in products of projective spaces, abbreviated here as ‘CICY’s, have proven to be a very useful class of compactification manifolds for string theory. The CICY three-folds were first discussed and classified in a series of articles in the 1980s [1–4]. Since that time, these manifolds have been used in a wide variety of different studies, including recent work on moduli stabilization and model building in string phenomenology (see [5–11] for some recent examples). Recently the CICY four-folds have been classified and studied [12–15] and work is underway to use this dataset in the context of F-theory compactifications to 4-dimensions. One of the reasons why this method of constructing Calabi–Yau manifolds has proven to be so useful is that it is exceptionally simple to study and work with. This simplicity stems from the fact that the complicated Calabi–Yau geometry is embedded within an extremely simple ambient space. Many of the geometric quantities of interest on the Calabi–Yau manifold can be worked out via relations to associated ambient space quantities – leading to an exceptional degree of computational control. In this paper, we will generalize the CICY construction in a manner which is applicable to any dimensionality of Calabi–Yau manifold.

A CICY can be described by a configuration matrix which encodes the data essential to the definition of the manifold. An example is as follows,¹

$$X_1 = \left[\begin{array}{c|ccc} \mathbb{P}^2 & 1 & 1 & 1 \\ \hline \mathbb{P}^4 & 3 & 1 & 1 \end{array} \right], \quad (1.1)$$

where this describes a manifold, X_1 , that is embedded in an ambient space $\mathbb{P}^2 \times \mathbb{P}^4$ as the complete intersection of the solution of three polynomial equations, one for each column of integers in (1.1). The integers themselves denote the degree of the defining polynomials in the homogeneous coordinates of the ambient projective factors (here one multi-degree (1, 3) and two multi-degree (1, 1) defining relations). Note that, in this example, X_1 is of complex dimension 3 since the ambient space is of dimension 6 and there are 3 polynomials that define this complete intersection. Equivalently, X_1 is defined as a complete intersection of global sections of three line bundles. That is, the common solutions of one element² of $H^0(\mathbb{P}^2 \times \mathbb{P}^4, \mathcal{O}(1, 3))$ and two elements of $H^0(\mathbb{P}^2 \times \mathbb{P}^4, \mathcal{O}(1, 1))$. Due to their interpretation as the degrees of polynomial equations, the integer entries in a CICY configuration matrix such as (1.1) are taken to be positive semi-definite. In the case of generalized CICYs, we will drop this requirement, in general allowing for negative degrees in the defining relations in the ambient space. In order to explain better the correspondence between the actual variety and the configuration matrix, we illustrate now an example of a generalized CICY.

1.1. A simple example

A generalized CICY, or “gCICY”, will be described by a configuration matrix, similar to (1.1), but in which negative integers are now permitted to be used. For example,

¹ Strictly speaking, configuration matrices describe an entire family of varieties and a variety is only determined once a choice has been made of its complex structure within the family. In this sense, it can be misleading to use the equality symbol in characterizing a specific variety by its configuration matrix. When no confusions arise, however, we will employ such a conventional abuse of notation.

² See Appendix A for a review of notation and tools to compute line bundle cohomology.

$$X_2 = \left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^5 & 3 & 1 \end{array} \middle| \begin{array}{cc} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{array} \right]. \tag{1.2}$$

In this example one cannot define X_2 as a simple complete intersection of sections of line bundles on the ambient space $\mathcal{A} \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^5$. This is due to the simple fact that $h^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^5, \mathcal{O}(1, -1, 1)) = 0$, for example. To state the problem in another way, one cannot have a polynomial in ambient space coordinates with negative degree! However if we apply the polynomial conditions sequentially from left to right there is no such problem. Define a manifold, \mathcal{M} , to be the complete intersection of the first two polynomials (columns) in the ambient space. Then the remaining two line bundles do have sections on \mathcal{M} and as such a manifold can be defined associated to the full configuration matrix in (1.2). Note that the sections of the last two line bundles should correspond to polynomials in appropriately chosen coordinates for \mathcal{M} although they are not polynomial in the homogeneous coordinates of the ambient space \mathcal{A} .

Let us look at what the defining equations of X_2 are explicitly in terms of the homogeneous coordinates of \mathcal{A} . The first two columns correspond to the vanishing of polynomial equations, $p_1 = 0$ and $p_2 = 0$ (which we can take to be generic polynomials of appropriate degrees in the homogeneous coordinates). The remaining two defining relations, $q_1 = 0$ and $q_2 = 0$, will each describe the vanishing of a specific rational function. Since we know that these rational functions should correspond to some polynomial defining relations in the coordinates of \mathcal{M} , they should not have poles on that surface (as described by $p_1 = p_2 = 0$). We therefore require that the poles in these rational functions do not lie on the solution set of the first two defining relations. Denoting the ambient homogeneous coordinates in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^5$ as $\mathbf{x}_1 = (x_1^0 : x_1^1)$, $\mathbf{x}_2 = (x_2^0 : x_2^1)$, and $\mathbf{x}_3 = (x_3^0 : x_3^1 : x_3^2 : x_3^3 : x_3^4 : x_3^5)$, respectively, one simple way to ensure this is the following. We write

$$p_2 = c_0(\mathbf{x}_1, \mathbf{x}_3)x_2^0 + c_1(\mathbf{x}_1, \mathbf{x}_3)x_2^1 = d_0(\mathbf{x}_2, \mathbf{x}_3)x_1^0 + d_1(\mathbf{x}_2, \mathbf{x}_3)x_1^1. \tag{1.3}$$

In this expression, c_0 and c_1 are generic polynomials of degree $(1, 0, 1)$ in the homogeneous coordinates $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ of the three ambient space factors. Likewise, d_0 and d_1 are generic degree $(0, 1, 1)$ polynomials in the same variables. Then the following rational expressions for q_1 and q_2 ,

$$\begin{aligned} q_1 &= \frac{d_1}{x_1^0} = -\frac{d_0}{x_1^1}, \\ q_2 &= \frac{c_1}{x_2^0} = -\frac{c_0}{x_2^1}, \end{aligned} \tag{1.4}$$

which are of the right multi-degree, can be used to algebraically embed X_2 in \mathcal{M} . The poles of these expressions miss the locus $p_2 = 0$, in the sense that, when the denominator vanishes, the numerator has to vanish to the same order. For example, if $x_1^0 = 0$ in (1.4), then $p_2 = 0$, as in (1.3), implies that $d_1 = 0$ as well (the two \mathbf{x}_1 homogeneous coordinates cannot simultaneously vanish). The fact that we can write two equivalent expressions for each of q_1 and q_2 is a direct consequence of (1.3), which implies the second equalities in each line of (1.4) if $p_2 = 0$.

The expressions (1.4) allow us to gain a very concrete picture of the manifold X_2 described by (1.2). On an open patch of the ambient space where $x_1^0 \neq 0$ and $x_2^0 \neq 0$ we observe that the manifold is defined by the solutions to the equations $p_1 = p_2 = d_1 = c_1 = 0$. Thus on such a coordinate patch the manifold is simply the common solution locus of a set of polynomial equations. Such a description does not hold globally, however. If we look instead at the patch where $x_1^1 \neq 0$ and $x_2^1 \neq 0$, the expression $\frac{d_1}{x_1^0}$ for the q_1 equation in (1.4) is not the most useful

one since there is a locus in this patch where the denominator x_1^0 vanishes. Using the second expression, $-\frac{d_0}{x_1}$, however, leads again to a polynomial description of our manifold by $p_1 = p_2 = d_0 = c_1 = 0$. Such a trick works for any given patch and we thus learn that the manifold is defined locally by polynomial equations, although the exact polynomials involved vary as one changes the coordinate chart one is considering. Note in particular that, since no single polynomial d can globally replace q_1 for instance, this manifold is *not* equivalent to a manifold of the form

$$\left[\begin{array}{l} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^5 \end{array} \middle| \begin{array}{cccc} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 1 \end{array} \right], \quad (1.5)$$

which one would naively obtain simply by setting a numerator of each rational defining relation to zero globally. This is just as well, as the configuration matrix (1.5) does not describe a Calabi–Yau manifold.

Constructions of the form we are discussing may seem a little strange at first to the reader who is used to dealing with ordinary CICYs. However it is simply an extreme example of a very well studied phenomenon. Many CICYs have so-called “non-polynomial” deformations [16–21]. That is, they have complex structure deformations which are not visible as changes to the polynomial defining relations in the ambient space coordinates. In the case of generalized CICYs we have taken a logical limit of this situation in which *all* of the complex structure associated to certain defining relations are non-polynomial. These new manifolds are also very similar to previous constructions in that they are defined simply as algebraic hypersurfaces (or complete intersections) in simple ambient spaces. However, unlike in previous constructions, here the ambient spaces themselves are not necessarily toric nor Fano. Instead, these ambient spaces themselves are defined as complete intersections in products of projective spaces and are characterized by the fact that their anticanonical divisor is *effective but not, in general, ample* (as in the Fano case). We refer the readers to Refs. [22–24] for a discussion of the importance of the form of the anticanonical divisor in recent mathematical classifications in the context of the Minimal Model program [25].

This class of Calabi–Yau manifolds shares many of the advantages of the ordinary CICYs, including much of the same ease in computation, a preponderance of elliptically fibered examples and so forth. It is, however, clearly a more general class of manifolds. In addition gCICYs have many interesting features; for example a tendency to lead to Calabi–Yau manifolds with smaller Hodge numbers than the former construction. Finally, the fact that, unlike the ordinary CICYs, not every gCICY is smooth, leads to the possibility of the dataset giving rise to simple, computable examples with, in the language of the F-theory literature, manifest non-Higgsable clusters [26–28]. Such global constructions of this phenomenon could greatly aid current explorations of the F-theory landscape of vacua. We hope to obtain a complete classification of the gCICY three- and four-folds in future work [29]. In the present work we generate an initial dataset of 2,761 three-fold geometries many of which are manifestly new CY manifolds with distinct topology not appearing before in existing datasets of Calabi–Yau three-folds. For comparison, we refer the reader to [2,30–45] for other classes of constructions of CY manifolds.

The rest of this paper is structured as follows. In Section 2 we describe the construction of gCICYs in detail. We begin with our conventions for defining a gCICY configuration matrix in Subsection 2.1. In Subsection 2.2 we describe how to compute the topology associated to such a gCICY configuration matrix, assuming that the variety itself is smooth. In Section 3 we

describe how to construct the gCICY geometry explicitly via its defining relations, and in the process detail how a smoothness check can be performed on the variety. In Section 4 we detail some of the redundancies that can occur relating different configuration matrices to the same Calabi–Yau manifold. Some of these redundancies mirror the standard CICY case and some do not. While all the techniques described in Sections 2–4 apply to gCICYs of an arbitrary dimension, in Section 5 we focus on Calabi–Yau three-folds and take the first steps towards a classification of gCICYs, by studying and classifying certain low co-dimensional sub-classes and detail their geometry. Finally in Section 6 we summarize the physics applications of this new dataset, including obvious elliptic fibration structures [15] of gCICYs, as well as fibration structures involving K3 and higher-dimensional Calabi–Yau varieties and provide an outlook of interesting future directions of investigation.

2. The construction and topology

2.1. The construction

Our construction of gCICYs is based on a multi-step process. We first impose all of the semi-positive defining relations to obtain an ordinary complete intersection, \mathcal{M} , of a set of polynomial relations in the ambient product of projective spaces, while the rest of the defining relations (including negative entries) are to be sequentially imposed on \mathcal{M} .

Let us be more specific and begin by constructing \mathcal{M} as the complete intersection of K polynomials, p_α where $\alpha = 1, \dots, K$, in a product of projective spaces, $\mathcal{A} = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$. We choose the dimension of \mathcal{M} ,

$$\dim_{\mathbb{C}} \mathcal{M} = \sum_{r=1}^m n_r - K, \tag{2.1}$$

such that it is strictly greater than the desired dimension, N , of the Calabi–Yau manifold to be constructed. Note that, as in a standard CICY, the polynomials p_α are sections of an appropriate line bundle $\mathcal{O}_{\mathcal{A}}(a_\alpha^1, \dots, a_\alpha^m)$, with $a_\alpha^r \geq 0$ specifying the non-negative homogeneous degree of p_α in the r -th projective piece. Here the indices $r, s, \dots = 1, \dots, m$ are used to label the projective ambient space factors \mathbb{P}^{n_r} , and the indices $\alpha, \beta, \dots = 1, \dots, K$, to label the polynomials, p_α . A family of such geometries can be characterized by a configuration matrix of the form,

$$[\mathbf{n} \parallel \{\mathbf{a}_\alpha\}] = \left[\begin{array}{c|ccc} \mathbb{P}^{n_1} & a_1^1 & \dots & a_K^1 \\ \mathbb{P}^{n_2} & a_1^2 & \dots & a_K^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}^{n_m} & a_1^r & \dots & a_K^r \end{array} \right], \tag{2.2}$$

with non-negative integer entries a_α^r .

Inside \mathcal{M} we define the generalized CICY, X , in terms of an ordered list of line bundles \mathcal{L}_μ where $\mu = 1, \dots, L$. These are used to iteratively describe hypersurfaces inside a nested sequence of spaces as follows. First, we demand that $h^0(\mathcal{M}, \mathcal{L}_1) \neq 0$ and define a hypersurface \mathcal{M}_1 inside \mathcal{M} as the vanishing locus of a global section in $H^0(\mathcal{M}, \mathcal{L}_1)$. Second, we demand that $h^0(\mathcal{M}_1, \mathcal{L}_2) \neq 0$ and define a hypersurface \mathcal{M}_2 inside \mathcal{M}_1 as the vanishing locus of a global section in $H^0(\mathcal{M}_1, \mathcal{L}_2)$. We sequentially repeat this procedure for all the line bundles \mathcal{L}_μ until the gCICY is obtained as $X = \mathcal{M}_L$. Note that, in general, one needs not have a section of \mathcal{L}_μ

on the ambient space \mathcal{A} , or even on \mathcal{M} for $\mu \geq 2$. We need only have sections of \mathcal{L}_μ on $\mathcal{M}_{\mu-1}$, i.e. we simply require that $h^0(\mathcal{M}_{\mu-1}, \mathcal{L}_\mu) \neq 0$. As such, as we have already discussed, *gCICYs are not described by the solution set of a system of simple polynomial equations in the ambient space \mathcal{A} .*

By writing $\mathcal{L}_\mu = \mathcal{O}_{\mathcal{M}_{\mu-1}}(b_\mu^1, \dots, b_\mu^m)$, $\mu = 1, \dots, L$, with $\mathcal{M}_0 = \mathcal{M}$, we present the defining data of a *gCICY* by the following matrix,

$$[\mathbf{n} \parallel \{\mathbf{a}_\alpha\} \mid \{\mathbf{b}_\mu\}] = \left[\begin{array}{c|ccc|ccc} \mathbb{P}^{n_1} & a_1^1 & \cdots & a_K^1 & b_1^1 & \cdots & b_L^1 \\ \mathbb{P}^{n_2} & a_1^2 & \cdots & a_K^2 & b_1^2 & \cdots & b_L^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}^{n_m} & a_1^m & \cdots & a_K^m & b_1^m & \cdots & b_L^m \end{array} \right]. \tag{2.3}$$

This looks very much like an ordinary CICY configuration matrix with just two important differences. Firstly, the entries b_μ^r can be negative, although the a_α^r are still positive semi-definite. Secondly, the order of the columns involving b 's is important and we establish the convention that one restricts to sections of the line bundles in an order taken from left to right in constructing the Calabi–Yau N -fold. Clearly, not every such matrix describes a *gCICY*, as the \mathcal{L}_μ must be chosen so as to have global sections over $\mathcal{M}_{\mu-1}$. However, by definition, every *gCICY* can be described by such a configuration matrix and we will see that much of the familiar utility of describing manifolds with such objects is preserved in this case.

Clearly, the number, L , of the line bundles on \mathcal{M} has to be,

$$L = \dim_{\mathbb{C}} \mathcal{M} - N = \sum_{r=1}^m n_r - K - N, \tag{2.4}$$

if we want the resulting subvariety, $X \subset \mathcal{M}$, to be an N -fold. The N -dimensional variety X has codimension $K + L$ in \mathcal{A} . However, since the K sections p_α and the L sections q_μ are of a different type, we say that X is of codimension (K, L) in this case. The condition that the resulting variety is Calabi–Yau remains unaltered: namely that the sum of the degrees of the entries in each row must equal $n_r + 1$ for its respective \mathbb{P}^{n_r} factor. This will be discussed in detail in the next subsection. In the case of a single hypersurface with $L = 1$, this is the familiar condition that X must be the anticanonical hypersurface of \mathcal{M} .

It should be noted that there is a sub-type of *gCICYs* which are particularly easy to describe and which display many of the interesting properties of the dataset. These are the cases where $h^0(\mathcal{M}, \mathcal{L}_\mu) \neq 0$ for all the μ 's. In this case it frequently does not matter in which order one restricts to sections of the \mathcal{L}_μ as they all have global sections already on \mathcal{M} .³ We refer to such a generalized CICY as a *minimally generalized CICY* or an “*mgCICY*.” Many of the concrete examples that we discuss in what follows will be of this type.

2.2. Determining the topology of generalized CICY manifolds

In this subsection tools are provided to compute topological properties of a *gCICY* from its configuration matrix. For the moment, we will make the assumption that the associated variety

³ Because the number of sections of a given line bundle may vary on the different \mathcal{M}_μ , it should be noted that the order in which the line bundles are placed may still be important in questions such as deciding smoothness of the resulting variety.

is smooth.⁴ Unlike in the CICY case, gCICYs are not always generically smooth and this property must be explicitly tested as will be explored in Section 3. Nevertheless we find it useful to describe the computation of topological properties of the dataset first and then proceed on to verifying smoothness in a second step. This allows us to preserve the separation between topology, as derived by integer manipulation of the configuration matrix, and geometry, as determined by the detail of the defining relations, that is seen in the discussion of standard CICYs. In any case, it is useful to determine which configuration matrices could have interesting topology, such as vanishing first Chern class, before proceeding on to the more detailed question of whether or not they are smooth.

For the rest of this section, we explore such defining topological quantities as the Hodge numbers, Chern classes, and triple intersection numbers of our new class of Calabi–Yau manifolds. We will also discuss the computation of other geometrical data such as the Kähler and Mori cones and line bundle cohomology which is crucial in order to extract the effective physics in a string compactification. One important omission that we do not consider here is a study of the first fundamental group, $\pi_1(X)$, of these geometries. Unlike in the case of ordinary CICY manifolds, it is not guaranteed that the gCICY manifolds are simply connected. Due to the absence of such useful tools as the Lefschetz Hyperplane theorem (employed in the case of CICYs [46]) we leave a complete study of this to a future classification.

Much of the formalism that we describe below applies to any Calabi–Yau three-fold described as an algebraic hypersurface inside some Kähler ambient space and is not unique to the construction at hand. For a more complete overview of these tools, we refer the reader to such useful references as [46,47]. Below we will review some of these general results and focus on how they can be implemented for generalized complete intersection manifolds.

2.2.1. The Koszul complex

To begin, consider a generalized complete intersection manifold, X , defined as a single hypersurface in the compact, Kähler ambient space \mathcal{M} . This construction will be iterated below to describe more general complete intersection manifolds. As described previously, these ambient spaces themselves are each constructed as complete intersection manifolds in products of projective spaces [1–4]. By Bertini’s theorem (see [48,49]), \mathcal{M} constructed in this way are generically smooth Kähler manifolds. As in many other constructions of CY manifolds, we begin by constructing X as an *anticanonical hypersurface* in \mathcal{M} , and thus we will demand that $\mathcal{K}_{\mathcal{M}}^{\vee}$ has global sections ($h^0(\mathcal{M}, \mathcal{K}_{\mathcal{M}}^{\vee}) > 0$). However, unlike in many constructions of CY manifolds, \mathcal{M} will not be required to be Fano (that is, $\mathcal{K}_{\mathcal{M}}^{\vee}$ is not necessarily ample).

To construct the hypersurface, X , we will impose on the coordinates of \mathcal{M} , collectively denoted by x , the algebraic equation $p(x) = 0$ in the class D (i.e. p is a global, holomorphic section of the line bundle $\mathcal{O}_{\mathcal{M}}(D)$ and $D = [p(x) = 0]$). Then we have the familiar Adjunction Formula [48,49]:

$$0 \rightarrow TX \rightarrow T\mathcal{M}|_X \rightarrow \mathcal{O}_{\mathcal{M}}(D)|_X \rightarrow 0. \quad (2.5)$$

As will be argued below, X will satisfy $c_1(X) = 0$ and thus, is a CY manifold when $\mathcal{O}(D) = \mathcal{K}_{\mathcal{M}}^{\vee}$ [50,51]. Using (2.5), the tangent bundle valued cohomology, $H^*(X, TX)$, can be com-

⁴ In fact we can somewhat relax this assumption, as we will describe in later sections.

puted by restriction from \mathcal{M} . The most useful tool to this end is the familiar Koszul short exact sequence [48,49]:

$$0 \rightarrow \mathcal{I}_{X|\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{M}|X} \rightarrow 0, \quad (2.6)$$

where $\mathcal{I}_{X|\mathcal{M}}$ is the ideal sheaf of $X \subset \mathcal{M}$. In the case the divisor D is reduced and irreducible,⁵ the ideal sheaf is invertible and a line bundle, leading to

$$0 \rightarrow \mathcal{O}_{\mathcal{M}}(-D) \rightarrow \mathcal{O}_{\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{M}|X} \rightarrow 0. \quad (2.7)$$

Moreover, for any vector bundle, $\pi : \mathcal{V} \rightarrow \mathcal{M}$, we can take the tensor product with the above exact sequence to produce another short exact sequence (i.e. the above sequence of sheaves/modules is *flat* [48]):

$$0 \rightarrow \mathcal{O}_{\mathcal{M}}(-D) \otimes \mathcal{V} \rightarrow \mathcal{V} \rightarrow \mathcal{V}|_X \rightarrow 0. \quad (2.8)$$

From this sequence we can use the cohomology of any bundle \mathcal{V} on \mathcal{M} to determine the cohomology of the restricted bundle $\mathcal{V}|_X$. It should be noted that in most known algebraic constructions of CY manifolds [1–4], this sequence is defined for *ample* divisors D . Such a choice guarantees that the variety X swept out by the condition $p(x) = 0$ is generically a smooth, irreducible variety. For the case at hand, however, we will simply require that D is effective (and frequently nef) and that the line bundle $\mathcal{O}_{\mathcal{M}}(D)$ has global sections. In general the variety X defined by the global sections of a non-Ample line bundle may be singular, reducible, or non-reduced. To proceed, we must first consider the global sections of $\mathcal{O}_{\mathcal{M}}(D)$ and whether or not they can be used to define smooth manifolds. The details of this analysis will be described in Section 3. For now, we will simply assume that such good sections exist and that X is smooth, but it is important to note that the following analysis is only valid for smooth X .

To determine the topology of X then, consider the long exact sequence in cohomology associated to (2.5):

$$0 \rightarrow H^0(X, TX) \rightarrow H^0(X, T\mathcal{M}|_X) \rightarrow H^0(X, \mathcal{O}(D)|_X) \rightarrow H^1(X, TX) \rightarrow \dots \quad (2.9)$$

where the cohomology of bundles restricted to X can be computed using (2.8) and standard techniques (see Appendix A).

At this point, determining the tangent bundle valued cohomology $h^*(X, TX)$ is as straightforward as any other construction of a Calabi–Yau manifold as a complete intersection manifold and the techniques employed are entirely analogous. This can be concisely summarized by the two-step procedure:

- 1) Compute bundle-valued cohomology on the smooth Kähler manifold \mathcal{M} .
- 2) Use the Koszul (2.7) and Adjunction (2.5) sequences as well as the long exact sequence (2.9) to determine bundle valued cohomology on X .

The first of these steps is now well-understood in the physics literature and the reader is referred to [46] for a review. The way that these standard results can be implemented for gCICYs in step 2) is most simply understood in the context of an example.

⁵ More specifically, \mathcal{I} is invertible if and only if D is cut out by a single equation, not vanishing at any associated point of \mathcal{M} . In this case, D is called an effective Cartier divisor [48,49].

2.2.2. An illustration of the determination of Hodge numbers

The calculation of Hodge numbers is best understood by illustration and we consider a simple example here. Let $\mathcal{M} = \left[\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^4 \end{array} \middle| \begin{array}{c} 3 \\ 2 \end{array} \right]$ and X be defined by a generic global section of the line bundle $\mathcal{O}_{\mathcal{M}}(-1, 3)$ (where $\mathcal{O}_{\mathcal{M}}(a, b)$ denotes the line bundle associated to the divisor $aH_1 + bH_2$ where H_r are the ambient hyperplanes restricted to \mathcal{M}). Note that on the ambient product of projective spaces $\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^4$ the line bundle $\mathcal{O}_{\mathcal{A}}(-1, 3)$ is not ample and has no global sections. However, on the hypersurface \mathcal{M} , $h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(-1, 3)) = 15$ and it is possible to consider an algebraic variety swept out by one such general global holomorphic section. As described above, in analogy to the configuration matrices used to describe ordinary CICY manifolds [1–4], we will schematically denote the final CY manifold X here by the following configuration matrix

$$X = \left[\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^4 \end{array} \middle| \begin{array}{c} 3 \\ 2 \end{array} \middle| \begin{array}{c} -1 \\ 3 \end{array} \right]. \tag{2.10}$$

Using the techniques described in [46,48,49] and the Euler sequence for the tangent bundle to projective space,

$$0 \rightarrow \mathcal{O}_{\mathcal{A}}^{\oplus 2} \rightarrow \mathcal{O}_{\mathcal{A}}(1, 0)^{\oplus 2} \oplus \mathcal{O}_{\mathcal{A}}(0, 1)^{\oplus 5} \rightarrow T\mathcal{A} \rightarrow 0, \tag{2.11}$$

and the key sequences (2.7) and (2.5):

$$0 \rightarrow \mathcal{O}_{\mathcal{A}}(-3, -2) \rightarrow \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{A}}|_{\mathcal{M}} \rightarrow 0, \tag{2.12}$$

$$0 \rightarrow T\mathcal{M} \rightarrow T\mathcal{A}|_{\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{A}}(3, 2)|_{\mathcal{M}} \rightarrow 0. \tag{2.13}$$

It is straightforward to determine the tangent bundle valued cohomology on \mathcal{M} to be

$$h^*(\mathcal{M}, T\mathcal{M}) = (0, 32, 0, 0, 0). \tag{2.14}$$

To move on to the second step described in the previous Subsection, consider next the adjunction formula (2.5) for the CY manifold itself:

$$0 \rightarrow TX \rightarrow T\mathcal{M}|_X \rightarrow \mathcal{O}_{\mathcal{M}}(-1, 3)|_X \rightarrow 0. \tag{2.15}$$

If the cohomologies, $h^*(X, T\mathcal{M}|_X)$ and $h^*(X, \mathcal{O}_{\mathcal{M}}(-1, 3)|_X)$ are each determined, then (2.9) can be used to find the Hodge numbers of X . First, to determine the cohomology of $\mathcal{O}_{\mathcal{M}}(-1, 3)|_X$ the Koszul sequence yields

$$0 \rightarrow \mathcal{O}_{\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{M}}(-1, 3) \rightarrow \mathcal{O}_{\mathcal{M}}(-1, 3)|_X \rightarrow 0. \tag{2.16}$$

From the long exact sequence in cohomology associated to this we have

$$h^*(X, \mathcal{O}(-1, 3)|_X) = (14, 0, 0, 0). \tag{2.17}$$

Likewise, for $T\mathcal{M}$:

$$0 \rightarrow T\mathcal{M} \otimes \mathcal{O}_{\mathcal{M}}(1, -3) \rightarrow T\mathcal{M} \rightarrow T\mathcal{M}|_X \rightarrow 0. \tag{2.18}$$

The cohomology of the twisted bundle $T\mathcal{M} \otimes \mathcal{O}_{\mathcal{M}}(1, -3)$ can be computed in the same manner as that of $T\mathcal{M}$ described above. Using

$$0 \rightarrow \mathcal{O}_{\mathcal{A}}(1, -3)^{\oplus 2} \rightarrow \mathcal{O}_{\mathcal{A}}(2, -3)^{\oplus 2} \oplus \mathcal{O}_{\mathcal{A}}(1, -2)^{\oplus 5} \rightarrow T\mathcal{A} \otimes \mathcal{O}_{\mathcal{A}}(1, -3) \rightarrow 0, \tag{2.19}$$

$$0 \rightarrow \mathcal{O}_{\mathcal{A}}(-3, -2) \rightarrow \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{A}}|_{\mathcal{M}} \rightarrow 0, \tag{2.20}$$

$$0 \rightarrow T\mathcal{M} \otimes \mathcal{O}_{\mathcal{M}}(1, -3) \rightarrow (T\mathcal{A} \otimes \mathcal{O}_{\mathcal{A}}(1, -3))|_{\mathcal{M}} \rightarrow \mathcal{O}_{\mathcal{M}}(4, -1) \rightarrow 0. \tag{2.21}$$

It can be verified that

$$h^*(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(4, -1)) = (0, 0, 0, 0, 0) , \tag{2.22}$$

$$h^*(\mathcal{M}, (T\mathcal{A} \otimes \mathcal{O}_{\mathcal{A}}(1, -3))|_{\mathcal{M}}) = (0, 0, 0, 2, 0) , \tag{2.23}$$

which can now be combined in the long exact sequence in cohomology associated to (2.18):

$$h^*(X, T\mathcal{M}|_X) = (0, 32, 2, 0) . \tag{2.24}$$

At last then, the pieces can be put together. The long exact sequence in cohomology associated to (2.15)

$$0 \rightarrow H^0(X, TX) \rightarrow H^0(X, T\mathcal{M}|_X) \rightarrow H^0(X, \mathcal{O}_{\mathcal{M}}(-1, 3)|_X) \rightarrow H^1(X, TX) \rightarrow \dots \tag{2.25}$$

simplifies to

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{O}_{\mathcal{M}}(-1, 3)|_X) \rightarrow H^1(X, TX) \rightarrow H^1(X, T\mathcal{M}|_X) \rightarrow 0 \\ \rightarrow H^2(X, TX) \rightarrow H^2(X, T\mathcal{M}|_X) \rightarrow 0 . \end{aligned} \tag{2.26}$$

Thus, the Hodge numbers of the CY are

$$h^{2,1}(X) = h^1(TX) = 46 , \tag{2.27}$$

$$h^{1,1}(X) = h^1(TX^\vee) = h^2(TX) = 2 . \tag{2.28}$$

An interesting observation at this stage is that once again, $h^1(TX) > h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(-1, 3))$. That is, this generalized complete intersection manifold is “non-polynomial” twice over! Not only can it not be realized as a complete intersection hypersurface in $\mathbb{P}^1 \times \mathbb{P}^4$ alone, but moreover, there are more complex structure moduli than are realized as coefficients of the defining relation $p(x) = 0$ on \mathcal{M} . This example illustrates how much general structure can potentially be overlooked by considering standard CICY manifolds.

Having obtained a gCICY manifold with $\chi = -88$ we are led now to consider the remaining topology of X . In the next section we will consider the Chern classes and triple intersection numbers of X and perform an independent check of the Euler number given above.

2.2.3. Chern classes and triple intersection numbers

Once again we begin by considering the co-dimension 1 case, in which X is defined as a single hypersurface in \mathcal{M} . Since $X \subset \mathcal{M}$ is a Kähler submanifold of a Kähler space \mathcal{M} , it is easiest to discuss the Chern classes of X via pullback. Let $i : X \hookrightarrow \mathcal{M}$ denote the embedding of X in \mathcal{M} . In this notation, the Adjunction formula (2.5) takes the form

$$0 \rightarrow TX \rightarrow i^*(T\mathcal{M}) \rightarrow i^*(\mathcal{K}^\vee_{\mathcal{M}}) \rightarrow 0 . \tag{2.29}$$

By the commutivity of pullback maps and Chern classes, for any bundle $\pi : \mathcal{V} \rightarrow \mathcal{M}$, we have $c(i^*\mathcal{V}) = i^*(c(\mathcal{V}))$. Thus, by the short exactness of (2.29),

$$i^*(c(T\mathcal{M})) = c(TX) \wedge i^*(c(\mathcal{K}^\vee_{\mathcal{M}})) . \tag{2.30}$$

For each of the Chern classes then, we have that

$$c_1(TX) = i^*(c_1(T\mathcal{M}) - c_1(\mathcal{K}^\vee_{\mathcal{M}})) = 0 , \tag{2.31}$$

$$c_2(TX) = i^*(c_2(T\mathcal{M})) , \tag{2.32}$$

$$c_3(TX) = i^*(c_3(\mathcal{M}) - c_2(T\mathcal{M}) \wedge c_1(\mathcal{K}^\vee_{\mathcal{M}})) . \tag{2.33}$$

The Chern classes on \mathcal{M} are given as simple functions of the entries in the configuration matrix [46]. For the total Chern class

$$c(\mathcal{M}) = c_1^r J_r + c_2^{rs} J_r J_s + c_3^{rst} J_r J_s J_t + \dots, \tag{2.34}$$

where J_r is a basis the Kähler (i.e. (1, 1)) forms in \mathbb{P}^{n_r} , the configuration matrix (2.2) implies that

$$\begin{aligned} c_1^r &= (n_r + 1) - \sum_{j=1}^K a_j^r, \\ c_2^{rs} &= \frac{1}{2} \left[-\delta^{rs} (n_r + 1) + \sum_{j=1}^K a_j^r a_j^s + c_1^r c_1^s \right], \\ c_3^{rst} &= \frac{1}{6} \left[\delta^{rst} (n_r + 1) - \sum_{j=1}^K a_j^r a_j^s a_j^t - c_2^{(rs)} c_1^t + c_1^r c_1^s c_1^t \right]. \end{aligned} \tag{2.35}$$

Using the formulas above and the pullback of the Kähler form, these formulas allow us to efficiently calculate the total Chern class of the Calabi–Yau three-fold, X . From (2.31) we see that the Calabi–Yau condition remains unchanged from the case of ordinary CICYs. *In order to define a CY manifold, the hypersurface defining the CY must be in the class of the anticanonical divisor of preceding rows of the configuration matrix.* As mentioned above, this amounts to the familiar condition that the sum of the entries of each row of (2.3) must add up to its respective \mathbb{P}^{n_r} dimension plus one.

The third and top Chern class determines the Euler number of the three-fold. This last quantity can be found with a notion of integration over X . As described in [46] (see Thm. 1.3), since X is a complex submanifold of \mathcal{M} (which is in turn a submanifold of a product of projective spaces), there is a closed (1, 1) form, μ , whose restriction to X represents the top Chern class of the normal bundle, $c_1(\mathcal{N})$. Then if ω is any closed (3, 3)-form on X , then

$$\int_X \omega = \int_{\mathcal{M}} \mu_X \wedge \omega. \tag{2.36}$$

Likewise, viewing \mathcal{M} as a complex subspace of the ambient multi-projective space, integration over \mathcal{M} can be defined with respect to a measure $\mu_{\mathcal{M}}$ and pulled back to a simpler integration over the ambient space $\mathcal{A} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \dots$:

$$\int_{\mathcal{M}} \cdot = \int_{\mathcal{A}} \mu_{\mathcal{M}} \wedge \cdot, \quad \mu_{\mathcal{M}} := \wedge_{j=1}^K \left(\sum_{r=1}^m a_r^j J_r \right). \tag{2.37}$$

Putting these pieces together then, we see that we can define integration over X via integration on the simple ambient space \mathcal{A} with a suitable choice of normal form:

$$\int_X \cdot = \int_{\mathcal{A}} \mu_X \wedge \mu_{\mathcal{M}} \wedge \cdot. \tag{2.38}$$

We will explore the exact form of μ_X in more detail in Section 3. As an application of the above formula, the triple intersection numbers can be computed as

$$d_{rst} = \int_X J_r \wedge J_s \wedge J_t = \int_{\mathcal{A}} J_r \wedge J_s \wedge J_t \wedge \mu_X \wedge \mu_{\mathcal{M}}. \quad (2.39)$$

An example As an example of the utility of these techniques, we can once again consider the example manifold given in Section 2.2.2. As observed in (2.27), the Hodge numbers of this manifold are $(h^{1,1}, h^{2,1}) = (2, 46)$. It is natural to ask, from this topology alone can we conclude whether or not (2.10) defines a previously unknown CY three-fold? A comparison with the literature shows that this Hodge number pair appears twice⁶ in the original CICY list [2] and not at all in the Kreuzer–Skarke list [30]. From the Hodge data alone then, we cannot decide if this is a new manifold. However, the further topology described above in this case can help us to distinguish this geometry. Using (2.32), the second Chern class can be written as

$$c_2(TX)^{rs} d_{rst} = (24, 46), \quad (2.40)$$

and from (2.39) the only non-vanishing intersection numbers are

$$d_{122} = 6, \quad d_{222} = 7. \quad (2.41)$$

With this in hand it is easy compare this second Chern class and triple intersection numbers to the two manifolds in the CICY list with the same Hodge numbers. We can ask whether there exists any basis change for which they can be made equivalent? A straightforward calculation verifies that no such basis change exists and in fact, this manifold is inequivalent to the other known CYs with this Hodge data. It consists of a previously unknown Calabi–Yau three-fold.

2.2.4. The Kähler and Mori cones

A key component of the CY geometry that is crucial to understand is the structure of the Kähler and Mori cones (i.e. the cone of ample divisors and that of effective curves). Practically, we define the Kähler cone of a Calabi–Yau three-fold to be defined by the requirements

$$\int_X J \wedge J \wedge J > 0, \quad \int_S J \wedge J > 0, \quad \int_C J > 0, \quad (2.42)$$

for all $S, C \subset X$ homologically non-trivial, irreducible, reduced proper surfaces and curves in X , respectively. The Kähler cone is associated to ample divisors on X (and the interior of the nef cone [49]). If the Kähler forms are well understood they can be used to define the Mori cone (or cone of effective curves) as the cone of curves intersecting positively with the divisors dual to $(1, 1)$ -forms in the Kähler cone.

An important case distinction must be made between those forms/divisors which descend from the ambient space – referred to as “favorable” in the context of [5,7,9,52,53] – and those that only exist on X . In general, for any construction of CY manifolds and $X \subset \mathcal{A}$, when $h^{1,1}(X) > h^{1,1}(\mathcal{A})$ very few general tools exist for explicitly determining the Kähler and Mori cone and each example must be handled on a case-by-case basis. Even in the case when X itself is fully favorable and $h^{1,1}(X) = h^{1,1}(\mathcal{A})$, there remain important questions that must be addressed

⁶ Note that this Hodge number pair also appears in [33] but the remaining topology (i.e. Chern classes and triple intersection numbers) has not been calculated for that construction of manifolds.

in determining the effective and ample cones. We restrict ourselves to the case of favorable manifolds in the following discussion.

In the case of ordinary CICYs, a key tool in determining the Kähler and Mori cones is the Lefschetz Hyperplane Theorem (see the statement given in [54]) which guarantees that given an ample divisor $D \subset \mathcal{A}$ it is possible to construct an isomorphism between the spaces of Kähler forms, $H^{1,1}(D)$ to $H^{1,1}(\mathcal{A})$. In the case at hand, gCICYs are defined via effective but not ample divisors in the ambient spaces \mathcal{M} and thus, we cannot always apply this standard theorem. To make progress then, we must directly explore the conditions in (2.42) directly.

To determine the Kähler and Mori cones of X , a necessary first step is the determination of these cones on the ambient space \mathcal{M} . To begin, it must be noted that in general, the cone of *effective divisors* of \mathcal{M} is *larger* than that of the ambient product of projective spaces. It is precisely this enhancement that provides the necessary freedom to build gCICYs. As an example, consider again the manifold given in (2.10) in Section 2.2.2:

$$X = \left[\begin{array}{c|c} \mathbb{P}^1 & \left| \begin{array}{c} 3 \\ 2 \end{array} \right. \\ \hline \mathbb{P}^4 & \left| \begin{array}{c} -1 \\ 3 \end{array} \right. \end{array} \right]; \tag{2.43}$$

On the ambient space $\mathcal{A} = \mathbb{P}^1 \times \mathbb{P}^4$ all effective divisors are in the class $aH_1 + bH_2$ with $a, b \geq 0$ where $H_i, i = 1, 2$ are the hyperplanes of the ambient projective space factors. Since the defining equation of $\mathcal{M} \subset \mathcal{A}$ is associated to the ample line bundle $\mathcal{O}_{\mathcal{A}}(3, 2)$, the Lefschetz hyperplane theorem [46] can be applied and $h^{1,1}(\mathcal{M}) = h^{1,1}(\mathcal{A}) = 2$. However, it is straightforward to verify that although the dimension of the effective cone stays the same in going from \mathcal{A} to \mathcal{M} , the mapping between these spaces increases the “width” of the effective cone.

An analysis of the line bundle cohomology on \mathcal{M} using the techniques described in Appendix A reveals that $h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(aH_1 + bH_2)) > 0$ for

$$\begin{aligned} & \{a, b \geq 0\} \quad || \quad \{a = -1, b \geq 2\} \quad || \\ & \{a \leq -2, b \geq \frac{1}{6}(-5 + 8|a|) + \frac{1}{6}\sqrt{-119 + 64|a| + 64a^2}\}. \end{aligned} \tag{2.44}$$

Thus, the effective cone has enhanced to include divisors with $a < 0$. Each of these may be used to define 3 (complex) dimensional subvarieties of \mathcal{M} . Those subvarieties of \mathcal{M} that satisfy $c_1(X) = 0$ and give rise to smooth Calabi–Yau three-folds are listed in Section 5. If the effective cone can change, it is natural to ask next, what about the ample cone?

Before testing (2.42) directly, it should be recalled that there is a useful standard cohomological criteria for ampleness of a divisor $D \subset \mathcal{M}$ [48]:

D is ample if and only if, for each coherent sheaf \mathcal{F} on \mathcal{M} , there is an integer, n_0 , depending on \mathcal{F} , such that, for each $i > 0$ and each $n \geq n_0$, $H^i(\mathcal{M}, \mathcal{F} \otimes \mathcal{O}_{\mathcal{M}}(nD)) = 0$.

While this is not of practical use to determine the ample cone (since we do not a priori know the full set of coherent sheaves on \mathcal{M}), it can be used to rapidly constrain the ample cone. For example, it clearly follows from this result that if D is ample, it must be the case that

$$h^i(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(nD)) = 0 \quad \text{for } i > 0, \tag{2.45}$$

for some $n > n_0$ (taking $\mathcal{F} = \mathcal{O}_{\mathcal{M}}$). It is easy to check that for the example given above, all the divisors determined by the conditions (2.44) also satisfy this necessary (but not sufficient) condition for ampleness.

To proceed further then, we must use our understanding of the effective cone to produce a basis of subvarieties of \mathcal{M} to test the equivalent of (2.42) directly. For \mathcal{M} a fourfold these conditions take the form

$$\int_{\mathcal{M}} J \wedge J \wedge J \wedge J > 0, \quad \int_D J \wedge J \wedge J > 0, \quad \int_S J \wedge J > 0, \quad \int_C J > 0. \tag{2.46}$$

For the example above, we will see that in fact, the ample cone of \mathcal{M} is identical to that of \mathcal{A} . Each divisor satisfying (2.44) can be tested directly. As a first step, we can consider a putative J associated to a line bundle in (2.44) and compare it to the subvarieties obtained by intersecting strictly positive divisors (i.e. $aH_1 + bH_2$ with $a, b > 0$, those descending from ample divisors on \mathcal{A}). In each case, we find that all of the divisors in (2.44) satisfy the positivity conditions of the ample cone. However, since the effective cone has enlarged there are *more* algebraic sub-varieties to be considered and hence in this example, we find that the Kähler cone can restrict back to the strictly positive range for a, b .

To illustrate this, the anticanonical class of \mathcal{M} itself, $K_{\mathcal{M}}^{-1} = \mathcal{O}(-1, 3)$, is a useful example. $D = -H_1 + 3H_2$ is an effective divisor and satisfies the conditions in (2.46) when compared to sub-varieties obtained via restriction from \mathcal{A} . However, there exist subvarieties of \mathcal{M} which do not descend from \mathcal{A} , over which the volume can fail to be positive. For example, $-H_1 + 2H_2$ is an effective divisor on \mathcal{M} (with $h^0(\mathcal{M}, \mathcal{O}(D)) = 3$). Consider the smooth surface defined by the intersection of two divisors in this class

$$S = \{p_1 = 0\} \cap \{p_2 = 0\}, \tag{2.47}$$

where $[p_i = 0] \in [-H_1 + 2H_2]$. Then denoting $J_{K^{-1}}$ to be the form dual to $K_{\mathcal{M}}^{-1}$ it can be verified that

$$\int_S J_{K^{-1}} \wedge J_{K^{-1}} = \int_{\mathcal{A}} J_{K^{-1}} \wedge J_{K^{-1}} \wedge \mu_S \wedge \mu_{\mathcal{M}} < 0, \tag{2.48}$$

with $\mu_S = (-J_1 + 2J_2) \wedge (-J_1 + 2J_2)$. Thus, in this case the anticanonical class of \mathcal{M} itself is effective but not ample.

Finally, to compute the Kähler and Mori cones of the Calabi–Yau three-fold itself, we proceed in a step-wise manner. Using the tools described above we first understand the change in the effective and ample cones in moving from \mathcal{A} to \mathcal{M} before finally exploring the same conditions on $X \subset \mathcal{M}$ itself. In each step, the restriction of Kähler forms are Kähler, but linear independence must be verified⁷ and the constraints on coefficients generically change as illustrated above.

3. Construction of sections and smoothness

In this section we return to the question of smoothness of gCICYs. As a precursor to discussing such issues, we must describe the defining relations of these manifolds in more detail than was required for the investigation of topology in the preceding section.

Given a configuration matrix of the form (2.3), one can construct the gCICY, X , in the manner described in Section 2.1. For this section we will, for simplicity, restrict ourselves to the case of

⁷ And indeed, in many cases independent Kähler forms on the ambient space can become dependent upon restriction to the hypersurface.

mgCICYs which can be constructed in two steps,

$$X \hookrightarrow \mathcal{M} \hookrightarrow \mathcal{A}, \tag{3.1}$$

where each of the two embeddings is a complete intersection. Much of what we will describe in this section generalizes to more complicated examples, however.

The embedding, $\mathcal{M} \hookrightarrow \mathcal{A}$, is manifestly algebraic in the coordinates of \mathcal{A} in that all the sections over \mathcal{A} involved in forming the complete intersection are a polynomial of the ambient homogeneous coordinates, $\mathbf{x}_r = (x_r^0 : x_r^1 : \dots : x_r^{n_r})$, $r = 1, \dots, m$. The other embedding, $X \hookrightarrow \mathcal{M}$, is also an algebraic complete intersection and as such must have a polynomial description when an appropriate coordinate system is used for \mathcal{M} . However, much of the computational power comes from relating various quantities to the simple ambient space geometry for \mathcal{A} . In this paper, therefore, the homogeneous coordinates, \mathbf{x}_r , $r = 1, \dots, m$, will be employed whenever possible, and all the global sections for this latter (gCICY) embedding, $X \hookrightarrow \mathcal{M}$, will be constructed in terms of them, using the idea of “tuning” in the following sense.

Let us start with a general set up and consider a section $q \in \Gamma(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(\mathbf{b}))$, with $\mathbf{b} = (b^1, \dots, b^m)$. In view of its zero and pole structure, q may take the rational form,⁸

$$q = \frac{N(\mathbf{x}_1, \dots, \mathbf{x}_m)}{D(\mathbf{x}_1, \dots, \mathbf{x}_m)}, \tag{3.2}$$

where N and D are a polynomial in $\mathbf{x}_1, \dots, \mathbf{x}_m$, of multi-degree $[\mathbf{b}]_+$ and $[\mathbf{b}]_-$, respectively. Here, $[\mathbf{b}]_+$ is a vector of length m , extracting only the positive entries from \mathbf{b} . Similarly, $[\mathbf{b}]_-$ only extracts the negative entries from \mathbf{b} , but with the signs inverted so that $\mathbf{b} = [\mathbf{b}]_+ - [\mathbf{b}]_-$ (e.g., for $\mathbf{b} = (1, 2, 0, -3)$ of length $m = 4$, we have $[\mathbf{b}]_+ = (1, 2, 0, 0)$ and $[\mathbf{b}]_- = (0, 0, 0, 3)$). Note that the expression, (3.2), for a general numerator N of the right multi-degree, is not regular on the divisor $D = 0$ of the manifold \mathcal{M} . However, q can remain regular on this divisor if N takes a specific form such that it also vanishes there. Thus, the key idea in making the rational expression a holomorphic object is to “tune” the coefficients of the numerator polynomial, N , so that vanishing of the denominator, D , is completely canceled by that of N . To be precise, we demand that

$$N \in \langle D \rangle \cap \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_m], \tag{3.3}$$

should hold, where $\langle D \rangle$ is the ideal generated by D in the homogeneous coordinate ring of \mathcal{M} ,

$$R(\mathcal{M}) := \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_m] / \langle p_1, \dots, p_K \rangle. \tag{3.4}$$

The regularity condition, (3.3), says that $q = N/D$ is in fact a polynomial section over \mathcal{M} .

Once we have all the sections, $q_\mu \in \Gamma(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(\mathbf{b}_\mu))$ written down for $\mu = 1, \dots, L$, the gCICY, X , is then constructed as the common vanishing locus,

$$X = \{x \in \mathcal{M} \mid q_1(x) = \dots = q_L(x) = 0\}. \tag{3.5}$$

⁸ To be precise, the line bundle degree rules the behavior of the entire object q and therefore, N and D may have a common degree shift. Thus, we are not guaranteed to find all the sections by demanding the rational form as in eq. (3.2). To make sure all the sections have been constructed, one should compare the number of linearly independent q 's and the dimension of the section space, $H^0(\mathcal{M}, \mathcal{O}(\mathbf{b}))$.

To be more explicit, let us illustrate the numerator tuning with the two earlier examples. For the first gCICY, (1.2), the configuration matrices for X and \mathcal{M} are, respectively,

$$X = \begin{bmatrix} \mathbb{P}^1 & \parallel & 1 & 1 & \mid & 1 & -1 \\ \mathbb{P}^1 & \parallel & 1 & 1 & \mid & -1 & 1 \\ \mathbb{P}^5 & \parallel & 3 & 1 & \mid & 1 & 1 \end{bmatrix}; \quad \mathcal{M} = \begin{bmatrix} \mathbb{P}^1 & \parallel & 1 & 1 \\ \mathbb{P}^1 & \parallel & 1 & 1 \\ \mathbb{P}^5 & \parallel & 3 & 1 \end{bmatrix}. \tag{3.6}$$

Using the methods described in Section 2.2.1, we can determine the Hodge number of this manifold to be $h^{1,1}(X_2) = 3$, $h^{1,2}(X_2) = 81$. The rational expressions for $q_1 \in H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(1, -1, 1))$ and $q_2 \in H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(-1, 1, 1))$ have already been written in eq. (1.4), where they have their numerators completely fixed by one of the defining equations, $p_2 \in H^0(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(1, 1, 1))$, for the embedding $\mathcal{M} \hookrightarrow \mathcal{A}$. Note that for the right multi-degree of the numerator polynomial of q_1 , for instance, 12 independent monomials that are bi-linear in $\mathbf{x}_1 = (x_1^0 : x_1^1)$ and $\mathbf{x}_3 = (x_3^0 : \dots : x_3^5)$ can be used. It is the holomorphicity of q_1 in \mathcal{M} that rules out all but one particular linear combination of them (up to scaling). Such a restriction of the numerator polynomial is the result of the aforementioned tuning. Indeed, one can compute the dimension of the line bundle cohomology:

$$h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(1, -1, 1)) = 1; \quad h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(-1, 1, 1)) = 1, \tag{3.7}$$

which shows that eq. (1.4) is the complete answer.

For the second gCICY, (3.2), whose topological properties have been computed in Section 2, the configuration matrices for X and \mathcal{M} are, respectively,

$$X = \begin{bmatrix} \mathbb{P}^1 & \parallel & 3 & \mid & -1 \\ \mathbb{P}^4 & \parallel & 2 & \mid & 3 \end{bmatrix}; \quad \mathcal{M} = \begin{bmatrix} \mathbb{P}^1 & \parallel & 3 \\ \mathbb{P}^4 & \parallel & 2 \end{bmatrix}. \tag{3.8}$$

Again, the two sections, $p_1 \in H^0(\mathcal{A}, \mathcal{O}_{\mathcal{A}}(3, 2))$ and $q_1 \in H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(-1, 3))$ can be sequentially constructed so that the rational expression for q_1 is determined by the choice of p_1 . For concreteness, let us write p_1 as,

$$p_1(\mathbf{x}_1, \mathbf{x}_2) = (x_1^0)^3 P_{11}(\mathbf{x}_2) + (x_1^0)^2 x_1^1 P_{12}(\mathbf{x}_2) + x_1^0 (x_1^1)^2 P_{13}(\mathbf{x}_2) + (x_1^1)^3 P_{14}(\mathbf{x}_2), \tag{3.9}$$

where P_{1a} , for $a = 1, 2, 3, 4$, are a quadric in \mathbf{x}_2 . According to the degree-splitting rule, (3.2), q_1 may have the rational form,

$$\frac{N(\mathbf{x}_1, \mathbf{x}_2)}{D(\mathbf{x}_1, \mathbf{x}_2)}.$$

We can generate a basis of the space of holomorphic sections describing q_1 by choosing the polynomials N and D of multi-degree $(0, 3)$ and $(1, 0)$, respectively. For the numerator tuning, we first need to choose a denominator. With the choice, $D(\mathbf{x}_1, \mathbf{x}_2) = x_1^0$, for instance, the corresponding numerator polynomial should vanish when $x_1^0 = 0$ in \mathcal{M} . On the other hand, since p_1 vanishes on \mathcal{M} , so it does on the divisor $x_1^0 = 0$ of \mathcal{M} . Therefore, substituting $x_1^0 = 0$ in eq. (3.9), we obtain an identically-vanishing section $P_{14}(\mathbf{x}_2) \equiv 0$ over that divisor. Now, P_{14} is a quadric in \mathbf{x}_2 while we need a cubic expression for the numerator, N . Filling in the missing degree with a linear polynomial in \mathbf{x}_2 , we obtain the following 5 global sections

$$s_i := \frac{P_{14}(\mathbf{x}_2) x_2^i}{x_1^0}, \quad i = 0, \dots, 4, \tag{3.10}$$

of the line bundle $\mathcal{O}_{\mathcal{M}}(-1, 3)$. On the other hand, by computing the dimension of the section space one sees that $h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(-1, 3)) = 15$, which shows that for a complete basis one needs

to obtain 10 more independent sections. This can be achieved by starting with different choices for the denominator polynomial. For instance, following the same procedure with $D(\mathbf{x}_1, \mathbf{x}_2) = x_0 - x_1$, one obtains 5 more sections

$$t_i := \frac{\sum_{a=1}^4 P_{1a}(\mathbf{x}_2) x_2^i}{x_1^0 - x_1^1}, \quad i = 0, \dots, 4, \tag{3.11}$$

and similarly, with $D(\mathbf{x}_1, \mathbf{x}_2) = x_0 + x_1$, one obtains

$$u_i := \frac{\sum_{a=1}^4 (-1)^a P_{1a}(\mathbf{x}_2) x_2^i}{x_1^0 + x_1^1}, \quad i = 0, \dots, 4, \tag{3.12}$$

as additional 5 global sections. A total of 15 sections have thus been obtained, as in eqs. (3.10), (3.11), and (3.12), each of which lies in the 15-dimensional section space of $\mathcal{O}_{\mathcal{M}}(-1, 3)$. Interestingly, given that p_1 is a generic polynomial, one can show that these 15 sections are linearly independent. Furthermore, any other rational expression for q_1 , obtained by starting with yet another choice of D , turns out to be expressible as their linear combination. Therefore, we come to the conclusion that these 15 sections do form a complete basis.

For certain gCICY configuration matrices, especially for those involving entries strictly smaller than -1 , the analytic construction of sections may become trickier than for the two example cases above. We have thus developed a method to obtain the sections numerically in Mathematica, details of which can be found in [Appendix C](#).

3.1. Smoothness

With such an explicit description for the defining relations of a gCICY we can now return to the question of its smoothness. Let us recall that for an ordinary CICY with b'_μ non-negative, X is embedded in \mathcal{A} as a smooth complete intersection if the $(K + L)$ -form,

$$\Theta_{X/\mathcal{A}} = dp_1 \wedge \dots \wedge dp_K \wedge dq_1 \wedge \dots \wedge dq_L, \tag{3.13}$$

is nowhere vanishing on X . Let us first illustrate how vanishing of this $(K + L)$ -form, $\Theta_{X/\mathcal{A}}$, remains the correct singularity criterion for a gCICY case, even when there is no algebraic embedding of X in \mathcal{A} and $\Theta_{X/\mathcal{A}}$ cannot be thought of as a normal form any longer.

The starting point is that the manifold \mathcal{M} , with its generic complex structure, is a smooth complete intersection of K polynomials in \mathcal{A} . In particular, this requires that the corresponding normal form,

$$\Theta_{\mathcal{M}/\mathcal{A}} = dp_1 \wedge \dots \wedge dp_K, \tag{3.14}$$

is nowhere vanishing on \mathcal{M} . The smoothness criterion for X then follows in terms of its own normal form,

$$\Theta_{X/\mathcal{M}} = dq_1 \wedge \dots \wedge dq_L. \tag{3.15}$$

Now, the claim we need to establish is that vanishing of $\Theta_{X/\mathcal{M}}$ at a point in X is equivalent to that of $\Theta_{X/\mathcal{A}}$. Note that at a practical level, we write these expressions in the coordinates \mathbf{x}_r , and in taking exterior derivatives of q_μ , some misleading components in dq_μ that do not lie in $\mathcal{T}_x^* \mathcal{M}$ can appear. However, when we eventually form the $(K + L)$ -form $\Theta_{X/\mathcal{A}}$, all these components

normal to $\mathcal{T}_x^* \mathcal{M}$ disappear due to the presence of the prefactor, $\Theta_{\mathcal{M}/\mathcal{A}}$. Furthermore, because that prefactor is nowhere vanishing on \mathcal{M} and in particular on X , we are led to the desired conclusion that at each point in X vanishing of $\Theta_{X/\mathcal{M}}$ implies that of $\Theta_{X/\mathcal{A}}$, and vice versa.

Let us illustrate this smoothness check based on the $(K + L)$ -form, $\Theta_{X/\mathcal{A}}$, with the two gCICYs, (3.6) and (3.8), for which the relevant global sections have already been constructed. For the former gCICY, (3.6), p_1 and p_2 are each a generic homogeneous polynomial of the right multi-degree and with respect to that choice, q_1 and q_2 are given as eq. (1.4). Given these 4 sections, p_1, p_2, q_1 and q_2 , one can form the system of equations,

$$p_1 = 0 = p_2 = q_1 = q_2 ; \quad dp_1 \wedge dp_2 \wedge dq_1 \wedge dq_2 = 0 , \tag{3.16}$$

whose solutions are to be found for the homogeneous coordinates,

$$\mathbf{x}_1 = (x_1^0 : x_1^1), \quad \mathbf{x}_2 = (x_2^0 : x_2^1), \quad \mathbf{x}_3 = (x_3^0 : \dots : x_3^5) . \tag{3.17}$$

Similarly, for the latter gCICY, (3.8), p_1 is a generic homogeneous polynomial of the bi-degree $(3, 2)$ and a generic q_1 can be obtained by linearly combining the 15 independent sections, s_i, t_i and u_i , for $i = 0, \dots, 4$, explicitly given in eqs. (3.10), (3.11) and (3.12), respectively. In other words, we take

$$q_1 = \sum_{i=0}^4 \alpha_i s_i + \sum_{i=0}^4 \beta_i t_i + \sum_{i=0}^4 \gamma_i u_i , \tag{3.18}$$

for a generic choice of the 15 coefficients, α_i, β_i and $\gamma_i, i = 0, \dots, 4$. Given these 2 sections, p_1 and q_1 , one can form the system of equations,

$$p_1 = 0 = q_1 ; \quad dp_1 \wedge dq_1 = 0 , \tag{3.19}$$

which have to be solved for the homogeneous coordinates,

$$\mathbf{x}_1 = (x_1^0 : x_1^1), \quad \mathbf{x}_2 = (x_2^0 : \dots : x_2^4) . \tag{3.20}$$

Given the system of constraints for a singularity, such as eqs. (3.16) and (3.19), one needs to test whether the system admits a solution or not. However, it can be extremely time-consuming on a computer if tackled in a brute-force manner. For the case of an ordinary CICY, this can be efficiently achieved by the Gröbner basis method, as implemented in the Mathematica package ‘‘Stringvacua’’ [55], for instance. For the case of a gCICY, however, due to the presence of negative entries in the configuration matrix, not every constraining equation is algebraic when written in terms of the \mathbf{x}_r and the Gröbner basis method does not apply as straight-forwardly. One way to avoid this obstacle is to separately analyze different regions of \mathcal{M} , by rewriting the system of constraints as an equivalent polynomial system in each region. More specifically, we subdivide \mathcal{M} into various regions according to which denominators that appear in the constraining equations vanish. For instance, the first region to consider is the one where none of the denominators vanish; in this region, we may clear the denominators to obtain an equivalent algebraic system of constraints. The next simplest regions are those where only a single denominator vanishes; any rational expressions involving this vanishing denominator can then be rewritten, in the coordinate ring of \mathcal{M} , as an alternative algebraic expression. One thereby obtains an equivalent algebraic system also in these regions, upon clearing the remaining denominators. Such a strategy works for the common zero loci of an arbitrary collection of denominator expressions, and that is how we subdivide \mathcal{M} into various regions, in each of which the usual Gröbner basis method may apply to efficiently test the existence of a singular point.

Again, let us illustrate this with our two examples. For the first gCICY example, (3.6), the system in question, (3.16), has rational expressions originating from $q_1 = d_1/x_1^0$ and $q_2 = c_1/x_2^0$, as in eq. (1.4). This case is particularly simple that one does not even have to clear denominators. However, the subdivision idea is still useful so let us proceed. We consider the subdivision of \mathcal{M} into the following four regions,

$$\begin{aligned}
 R_1 : & \quad x_1^0 \neq 0, \quad x_2^0 \neq 0, \\
 R_2 : & \quad x_1^0 = 0, \quad x_2^0 \neq 0, \\
 R_3 : & \quad x_1^0 \neq 0, \quad x_2^0 = 0, \\
 R_4 : & \quad x_1^0 = 0, \quad x_2^0 = 0,
 \end{aligned}
 \tag{3.21}$$

depending on which of the two denominators, x_1^0 and x_2^0 , vanish. Firstly, in the region R_1 , both x_1^0 and x_2^0 can be set to 1 using the scaling of the two \mathbb{P}^1 coordinates. Therefore, q_1 and q_2 both reduces to a polynomial, d_1 and c_1 , respectively, and the system becomes completely algebraic. In R_2 , one may still set x_2^0 to 1 so that $q_2 = c_1$ is a polynomial but q_1 involves a non-trivial denominator. However, as shown in eq. (1.4), $q_1 = d_1/x_1^0$ can also be written as $-d_0/x_1^1$ and x_1^1 can be set to 1 since x_1^0 and x_1^1 can not vanish simultaneously. Therefore, in this region, q_1 also reduces to the polynomial $-d_0$ and the system becomes algebraic again. Similarly, one can use $q_1 = d_1, q_2 = -c_0$ in the region R_3 , and $q_1 = -d_0, q_2 = -c_0$ in the region R_4 .

Now, for the second gCICY example, (3.8), let us first recall that the generic section q_1 in eq. (3.18) had 3 denominator expressions, $x_0, x_0 - x_1$ and $x_0 + x_1$. Thus, we consider the subdivision of \mathcal{M} into the following 4 regions,

$$\begin{aligned}
 R_1 : & \quad x_1^0 \neq 0, \quad x_1^0 - x_1^1 \neq 0, \quad x_1^0 + x_1^1 \neq 0, \\
 R_2 : & \quad x_1^0 = 0, \quad x_1^0 - x_1^1 \neq 0, \quad x_1^0 + x_1^1 \neq 0, \\
 R_3 : & \quad x_1^0 \neq 0, \quad x_1^0 - x_1^1 = 0, \quad x_1^0 + x_1^1 \neq 0, \\
 R_4 : & \quad x_1^0 \neq 0, \quad x_1^0 - x_1^1 \neq 0, \quad x_1^0 + x_1^1 = 0,
 \end{aligned}
 \tag{3.22}$$

depending again on which denominators vanish. Note that at most one denominator can be set to zero at a time on $\mathbb{P}^1_{x_1}$. In region R_1 , none of the denominators vanish and hence, one can clear all the denominators from the system (3.19) and test if it admits a solution. With the denominators cleared, however, naively testing the existence of a solution might lead to a point outside of R_1 and the test result would become ambiguous. To avoid such an ambiguity, we add to the system (3.19) three auxiliary variables, let us call them y_1, y_2 and y_3 , together with the three constraints,

$$y_1 x_1^0 - 1 = 0; \quad y_2 (x_1^0 - x_1^1) - 1 = 0; \quad y_3 (x_1^0 + x_1^1) - 1 = 0,
 \tag{3.23}$$

so that none of the denominators can vanish subject to the new system of algebraic equations. In region R_2 , on the other hand, s_i in eq. (3.10) have their denominator vanishing in that region, and therefore the part in q_1 that involves s_i need to be carefully taken care of. A proper way to deal with this is to rewrite them as,

$$\begin{aligned}
 s_i &= \frac{P_{14}(\mathbf{x}_2) x_2^i}{x_1^0} \\
 &= - \frac{(x_1^0)^2 P_{11}(\mathbf{x}_2) + x_1^0 x_1^1 P_{12}(\mathbf{x}_2) + (x_1^1)^2 P_{13}(\mathbf{x}_2)}{(x_1^1)^3} x_2^i,
 \end{aligned}
 \tag{3.24}$$

where we have made use of $p_1 = 0$, (3.9). Since $x_1^0 = 0$ in this region, the new denominator x_1^1 cannot vanish and hence, we may set $x_1^1 = 1$ using the scaling. Then the expressions for s_i become a polynomial and if we use them in eq. (3.18), the denominator clearing does not lead to any ambiguity once we consider two auxiliary variables y_1, y_2 , together with the two constraints,

$$y_1(x_1^0 - x_1^1) - 1 = 0; \quad y_2(x_1^0 + x_1^1) - 1 = 0. \tag{3.25}$$

To complete the story, let us also work out the details for regions R_3 and R_4 . In region R_3 , we rewrite t_i in eq. (3.11) as

$$\begin{aligned} t_i &= \frac{\sum_{a=1}^4 P_{1a}(\mathbf{x}_2) x_2^i}{x_1^0 - x_1^1} \\ &= - \frac{P_{11}(\mathbf{x}_2)((x_1^0)^2 + x_1^0 x_1^1 + (x_1^1)^2) + P_{12}(\mathbf{x}_2)(x_1^0 + x_1^1)x_1^1 + P_{13}(\mathbf{x}_2)(x_1^1)^2}{(x_1^1)^3} x_2^i, \end{aligned} \tag{3.26}$$

where vanishing of p_1 has been used. Since $x_1^1 \neq 0$ in this region, we may set $x_1^1 = 1$ by scaling and use the resulting polynomial expressions for t_i to rewrite q_1 in eq. (3.18). Again, we clear all the denominators and, to make sure only the points in region R_3 are analyzed, add two auxiliary variables y_1 and y_2 , together with the constraints,

$$y_1 x_1^0 - 1 = 0; \quad y_2(x_1^0 + x_1^1) - 1 = 0. \tag{3.27}$$

Finally in region R_4 , we rewrite u_i in eq. (3.12) as

$$\begin{aligned} u_i &= \frac{\sum_{a=1}^4 (-1)^a P_{1a}(\mathbf{x}_2) x_2^i}{x_1^0 + x_1^1} \\ &= - \frac{P_{11}(\mathbf{x}_2)((x_1^0)^2 - x_1^0 x_1^1 + (x_1^1)^2) + P_{12}(\mathbf{x}_2)(x_1^0 - x_1^1)x_1^1 + P_{13}(\mathbf{x}_2)(x_1^1)^2}{(x_1^1)^3} x_2^i, \end{aligned} \tag{3.28}$$

where vanishing of p_1 has been used. Since $x_1^1 \neq 0$ in this region, we may set $x_1^1 = 1$ by scaling and use the resulting polynomial expressions for u_i to rewrite q_1 in eq. (3.18). Again, we clear all the denominators and, to make sure only the points in region R_4 are analyzed, add two auxiliary variables y_1 and y_2 , together with the constraints,

$$y_1 x_1^0 - 1 = 0; \quad y_2(x_1^0 - x_1^1) - 1 = 0. \tag{3.29}$$

Therefore, the singularity criteria, in each of the subdivided regions of \mathcal{M} , have become a purely algebraic system and the techniques from numerical algebraic geometry applies straightforwardly to test the existence of a singular point in each region. This way, it has been shown that both of the above gCICY examples, (3.6) and (3.8), lead to a smooth Calabi–Yau three-fold.

For gCICY cases, it is by now clear that when constructing the sections for the line bundles involving a negative degree, one needs to go through the “tuning” process based on (3.3) and as a result, not every candidate expression satisfying the degree constraint can be a global holomorphic section. It is such a tuning that leads to distinctive features of gCICYs not observed for the ordinary CICY cases.

One interesting feature is that there may arise singular gCICYs even for a generic complex structure while the ordinary CICYs are all smooth [3,46]. Note that factorization of a defining

equation is a potential source for singularities. For CICY cases, a generic defining equation with a given multi-degree does not factorize. However, for gCICYs, even a generic defining equation may factorize (i.e. the associate divisor may not be base point free). Let us illustrate such a section-space factorization with the following example,

$$X = \left[\begin{array}{c|cc} \mathbb{P}^3 & 2 & 0 \\ \mathbb{P}^1 & 0 & 1 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 1 & 2 \end{array} \middle| \begin{array}{c} 2 \\ 1 \\ 0 \\ -1 \end{array} \right]; \quad \mathcal{M} = \left[\begin{array}{c|cc} \mathbb{P}^3 & 2 & 0 \\ \mathbb{P}^1 & 0 & 1 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 1 & 2 \end{array} \right], \tag{3.30}$$

for which we may write

$$p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = x_4^0 P_{11}(\mathbf{x}_1) + x_4^1 P_{12}(\mathbf{x}_1), \tag{3.31}$$

where P_{11} and P_{12} are a generic quadric in $\mathbf{x}_1 = (x_1^0 : \dots : x_1^3)$. Thus, the numerator tuning due to eq. (3.3) leads to the following holomorphic sections,

$$s_i = \frac{P_{12}(\mathbf{x}_1) x_2^i}{x_4^0}, \quad i = 0, 1, \tag{3.32}$$

of the line bundle $\mathcal{O}_{\mathcal{M}}(2, 1, 0, -1)$ over \mathcal{M} . Furthermore, the computation of line bundle cohomology results in $h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(2, 1, 0, -1)) = 2$, which means that the two sections s_1 and s_2 should form a complete basis for the section space. Therefore, a generic section q_1 should have the following form,

$$q_1 = \alpha_0 s_0 + \alpha_1 s_1 = \frac{P_{12}(\mathbf{x}_1)}{x_4^0} \cdot (\alpha_0 x_2^0 + \alpha_1 x_2^1), \tag{3.33}$$

where α_0 and α_1 are complex constants. Because both factors, P_{12}/x_4^0 and $\alpha_0 x_2^0 + \alpha_1 x_2^1$, are an algebraic section, this corresponds to the defining section q_1 being legitimately factorized. Furthermore, one can see that there indeed exists a point in X where both factors vanish simultaneously, thereby leading to a singularity.

3.2. Reducedness

Another consequence of the numerator tuning is that the coordinate ring does not have to be reduced. Again, let us illustrate this with a simple example how a non-reduced coordinate ring may arise. We consider,

$$X = \left[\begin{array}{c|cc} \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^4 & 1 & 1 \\ \mathbb{P}^1 & 1 & 4 \end{array} \middle| \begin{array}{c} 0 \\ 3 \\ -3 \end{array} \right]; \quad \mathcal{M} = \left[\begin{array}{c|cc} \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^4 & 1 & 1 \\ \mathbb{P}^1 & 1 & 4 \end{array} \right], \tag{3.34}$$

for which the first defining polynomial can be written as

$$p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = x_3^0 P_{11}(\mathbf{x}_2) + x_3^1 P_{12}(\mathbf{x}_2), \tag{3.35}$$

where P_{11} and P_{12} are a generic linear polynomial in $\mathbf{x}_2 = (x_2^0 : \dots : x_2^4)$. The numerator tuning for the section of the line bundle $\mathcal{O}_{\mathcal{M}}(0, 3, -3)$ over \mathcal{M} results in the following expression,

$$\left(\frac{P_{12}(\mathbf{x}_2)}{x_3^0} \right)^3. \tag{3.36}$$

Furthermore, computation of the dimension of the associated line bundle cohomology results in $h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(0, 3, -3)) = 1$ and thus, the most general section takes the form,

$$q_1 = \alpha \left(\frac{P_{12}(\mathbf{x}_2)}{x_3^0} \right)^3, \tag{3.37}$$

for a complex constant α . Note that this can be thought of as the cube of another holomorphic section,

$$s = \frac{P_{12}(\mathbf{x}_2)}{x_3^0}, \tag{3.38}$$

of the line bundle $\mathcal{O}_{\mathcal{M}}(0, 1, -1)$. Now, the coordinate ring of X defined naively as

$$R(X) = R(\mathcal{M}) / \langle s^3 \rangle, \tag{3.39}$$

is not reduced. However, since the tools developed in Subsection 2.2 for working out the topological properties assume that the coordinate ring is reduced, it is better to consider

$$R(X_{\text{red}}) = R(\mathcal{M}) / \langle s \rangle, \tag{3.40}$$

which describes X_{red} defined as the hypersurface $s = 0$ in \mathcal{M} . Set-theoretically, the gCICY configuration, (3.34), for X can be equivalently thought of as

$$X_{\text{red}} = \left[\begin{array}{c|cc} \mathbb{P}^1 & 0 & 2 & 0 \\ \mathbb{P}^4 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 4 & -1 \end{array} \right]; \quad \mathcal{M} = \left[\begin{array}{c|cc} \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^4 & 1 & 1 \\ \mathbb{P}^1 & 1 & 4 \end{array} \right], \tag{3.41}$$

and only with this reduced configuration can we directly apply the tools in Subsection 2.2 to compute the topological quantities of the resulting classical variety. One immediate observation from this reduction is that the gCICY, $X \sim X_{\text{red}}$, is not a Calabi–Yau manifold. For ordinary CICY cases, the Calabi–Yau criterion was that each row in the configuration matrix should have its entries sum up to match the degree of the anticanonical bundle of the corresponding ambient projective space. However, from the illustration above, we learn that this could be a misleading criterion for gCICYs when the configuration leads to a non-reduced coordinate ring.

Furthermore, in such a non-reduced case, the smoothness check has to be carefully performed, too. In the example, (3.34), if we naively form the system of singularity constraints,

$$p_1 = 0 = p_2 = q_1; \quad dp_1 \wedge dp_2 \wedge dq_1 = 0, \tag{3.42}$$

with $q_1 = \alpha s^3$, then we come to the incorrect conclusion that X is singular everywhere as $s = 0$ in \mathcal{M} is already sufficient to satisfy eq. (3.42).

4. Redundancies

Just as in the case of ordinary CICYs, configuration matrices of gCICYs also exhibit *redundancies*. A redundancy is simply a situation where two different configuration matrices describe the same manifold. In fact, the structure of known redundancies is even richer in the gCICY case than in that of the CICYs. Many of the same relations between configuration matrices seen in the simpler case still hold. In addition, new redundancies do arise, such as those relating seemingly good configuration matrices to a description of the empty set. In this section, we present possible sources of such redundancies, one in each subsection, emphasizing some distinctive features that are only seen for gCICYs.

4.1. Splitting transitions

Between ordinary CICY manifolds, there is an important process known as a *splitting transition* [2], which can essentially be thought of as a combination of deformation and blowup. The splitting, in its most general version, relates two configuration matrices of the following form,

$$\left[\begin{array}{c} \mathbb{P}^n \\ A \end{array} \left\| \begin{array}{cccc} 1 & \cdots & 1 & 0 \\ \mathbf{u}_1 & \cdots & \mathbf{u}_{n+1} & C \end{array} \right. \right] \longleftrightarrow \left[A \left\| \begin{array}{c} \sum_i^{n+1} \mathbf{u}_i \\ C \end{array} \right. \right], \tag{4.1}$$

where A is a product of projective spaces, \mathbf{u}_i 's are a column vector of degree entries each, C , a submatrix of an appropriate size, and 0 , a row of zeros. A simple example, as studied in Ref. [2], is the following pair of CICY three-folds,

$$\left[\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \left\| \begin{array}{cc} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{array} \right. \right] \longleftrightarrow \left[\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \left\| \begin{array}{c} 3 \\ 3 \end{array} \right. \right], \tag{4.2}$$

where the bi-cubic column of the right hand side is splitted via a \mathbb{P}^1 row of the left hand side. We may write the two defining equations for the splitted configuration as,

$$\begin{aligned} p_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= x_1^0 P_{11}(\mathbf{x}_2) + x_1^1 P_{12}(\mathbf{x}_2), \\ p_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= x_1^0 P_{21}(\mathbf{x}_3) + x_1^1 P_{22}(\mathbf{x}_3), \end{aligned} \tag{4.3}$$

where P_{11} and P_{12} are a cubic polynomial in \mathbf{x}_2 , and P_{21} and P_{22} are a cubic in \mathbf{x}_3 . In order for the system (4.3) to have a non-trivial solution in \mathbf{x}_1 , the determinantal expression,

$$\Delta = P_{11}(\mathbf{x}_2) P_{22}(\mathbf{x}_3) - P_{12}(\mathbf{x}_2) P_{21}(\mathbf{x}_3), \tag{4.4}$$

should vanish, leading to a bi-cubic relation in \mathbf{x}_2 and \mathbf{x}_3 , as indicated by the RHS of eq. (4.2). In general, however, there exist point-like singularities in the vanishing locus of Δ and one arrives, via a deformation, at a smooth bi-cubic three-fold.

In Ref. [2], splitting was called *effective* if the Euler numbers of the two manifolds differ and *ineffective* if they are the same. It was also argued there that one should consider two CICY three-fold configurations equivalent if they are related by an ineffective splitting because that guarantees the process is completely free of singularities. In the example (4.2), the Euler numbers of the LHS and the RHS are 0 and -162 , respectively, and thus the splitting is effective.

The idea of splitting and the consequence of its effectiveness straight-forwardly generalizes to gCICY manifolds. The rule for relating two gCICY configuration matrices remains the same as eq. (4.1), except that now the 1's of the \mathbb{P}^n row can either be in the non-negative sector or in the sector where negative entries are allowed. However, one needs to be careful about certain new aspects of the gCICY construction. Firstly, although the splitting process is most clear in terms of the configuration matrices, what we have in mind is how the corresponding gCICY geometries are related and it does not make sense, for the purpose of this paper, to naively relate two configuration matrices by the splitting of the form, eq. (4.1), when one or both of them lead to an empty set due, for instance, to absence of sections. Secondly, the Euler-number comparison only makes sense if both configurations correspond to a smooth gCICY three-fold and if we were to make use of the topology tools of Section 2 in computing the Euler number, both configuration matrices would have to result in a reduced geometry. Therefore, to be able to say that two gCICY configurations related by an ineffective splitting lead to the same manifold,

one first needs to verify that both are a nice configuration matrix, in that the corresponding geometry is a non-empty smooth three-fold with a reduced coordinate ring.⁹

4.2. Identities and reduction rules

In addition to the ineffective splitting, there are many other sources of redundancies. Most notably, as already described in Ref. [2], one can make use of the identities between various manifolds for part of the full configuration matrices; there, an exhaustive list is given of identities between one-folds and between two-folds, and also a partial list of those between three-folds and between manifolds of a higher dimension. For instance, the second identity in the list for one-folds [2] is

$$\left[\begin{array}{c|c} \mathbb{P}^1 & 1 \\ \hline \mathbb{P}^1 & 1 \end{array} \right] = \mathbb{P}^1, \quad (4.5)$$

which can be applied to show that

$$\left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & a \\ \hline \mathbb{P}^1 & 1 & b \\ A & \mathbf{0} & C \end{array} \right] = \left[\begin{array}{c|c} \mathbb{P}^1 & a+b \\ \hline A & C \end{array} \right], \quad (4.6)$$

where a and b are a row vector of degree entries, A is a product of projective spaces, $\mathbf{0}$, a column of zeros, and C , a submatrix of an appropriate size.

All such reduction rules are based on the identities between part of the two configuration matrices as well as the resulting relationship between the holomorphic line bundles on the two sides. Therefore, they straight-forwardly generalize to the gCICY cases and we may retrieve all the rules from Section 4 and Appendix of Ref. [2].

4.3. Trivial line bundles

The trivial line bundle over a connected base manifold has a unique global section, the constant function. For the ordinary CICY case, even when the configuration matrix is viewed as that of a gCICY, the line bundle corresponding to each column of the configuration is non-trivial on the sequentially constructed ambient space over which the sections are constructed. For the gCICY case, however, due to the presence of negative entries, one might end up describing a trivial line bundle by a non-trivial column. Let us begin with an illustrative toy example for which this happens:

$$X = \left[\begin{array}{c|c} \mathbb{P}^1 & -1 \\ \hline \mathbb{P}^1 & 1 \end{array} \right]; \quad \mathcal{M} = \left[\begin{array}{c|c} \mathbb{P}^1 & 1 \\ \hline \mathbb{P}^1 & 1 \end{array} \right]. \quad (4.7)$$

⁹ To be precise, there is another subtlety one needs to be careful about. As it is clear from the example, (4.3), the derivation of splitting relies on the fact that each of the sections from the n columns involved in the process has a linear dependence in the coordinates of the splitted \mathbb{P}^n . For ordinary CICYs, this is surely the case and even for gCICYs, any sections constructed as a rational form, eq. (3.2), will meet this property. However, because our gCICY construction bases on a sequential section construction, eq. (3.2) might not give rise to a most general section and sometimes one might need to investigate a common degree shift in the numerator and the denominator, in which case the derivation of splitting might not work. Although this does not seem to happen due to the controlled cohomology structure corresponding to the splitted columns of the configuration matrix, we do not have a complete proof of that and instead, for given configurations we explicitly ensure the linearity of sections.

The ambient space \mathcal{M} , where the line bundle $\mathcal{L} = \mathcal{O}_{\mathcal{M}}(-1, 1)$ is defined, is a \mathbb{P}^1 , which has a rank-one Picard lattice. Therefore, its configuration matrix in eq. (4.7) is non-favorable since the Picard lattice of its ambient space, $\mathbb{P}^1 \times \mathbb{P}^1$, is of rank two. There are several ways to see that \mathcal{L} is a trivial line-bundle over $\mathcal{M} = \mathbb{P}^1$; one can directly compute its cohomology dimensions, resulting in

$$h^0(\mathcal{M}, \mathcal{L}) = 1; \quad h^1(\mathcal{M}, \mathcal{L}) = 0, \tag{4.8}$$

which implies that $\mathcal{L} = \mathcal{O}_{\mathcal{M}}$, or alternatively, by constructing the unique holomorphic section of \mathcal{L} over \mathcal{M} in its ration form, eq. (3.2), one can also see that the vanishing locus is empty. An immediate consequence of this is that any gCICY configuration matrices with the two-by-two sub-block for $X = \left[\begin{array}{c|c} \mathbb{P}^1 & \begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix} \end{array} \right]$ appearing as

$$\left[\begin{array}{c|c} \vdots & \begin{matrix} \ddots & \mathbf{0} & \ddots \\ \dots & 1 & \dots \\ \vdots & \mathbf{0} & \ddots \\ \mathbb{P}^1 & \dots & \dots & \dots & -1 & \dots \\ \vdots & \mathbf{0} & \ddots \\ \mathbb{P}^1 & \dots & \dots & \dots & 1 & \dots \\ \vdots & \mathbf{0} & \ddots \end{matrix} \end{array} \right], \tag{4.9}$$

corresponds to an empty set, given that only a single section (up to scaling) arises from the mixed-sign column with -1 and 1 along the two \mathbb{P}^1 directions. Note that such reductions to an empty set can only happen for a non-favorable situation and hence, the minimal number of rows involved is 2. One can then see that the most general way in which an empty set arises via a two-by-two description of $\mathcal{O}_{\mathbb{P}^1}$, as in eq. (4.9), is either of the following two types,

$$\left[\begin{array}{c|c} \vdots & \begin{matrix} \ddots & \mathbf{0} & \ddots \\ \dots & 1 & \dots \\ \vdots & \mathbf{0} & \ddots \\ \mathbb{P}^1 & \dots & \dots & \dots & \pm 1 & \dots \\ \vdots & \mathbf{0} & \ddots \\ \mathbb{P}^1 & \dots & n & \dots & \mp n & \dots \\ \vdots & \mathbf{0} & \ddots \end{matrix} \end{array} \right]. \tag{4.10}$$

There are various other ways that lead to an empty-set configuration via the appearance of $\mathcal{O}_{\mathbb{P}^1}$ involving a bigger sub-block and even via that of $\mathcal{O}_{\mathcal{M}}$ for a more general \mathcal{M} . Although classification of all such possibilities can be an interesting direction to pursue further, the idea should already be clear from these toy instances, (4.9) and (4.10), and we do not attempt in this paper to complete the job.

4.4. Multiple components

Configuration matrices of a gCICY may describe multiple Calabi–Yau components and, in particular, multiple copies of a Calabi–Yau manifold. One way this can occur is through the gCICY configuration matrix of the following form,

$$\left[\begin{array}{c|ccc|ccc} \vdots & \ddots & \mathbf{0} & \ddots & \vdots & \ddots \\ \mathbb{P}^1 & \dots & n & \dots & \dots & \dots \\ \vdots & \ddots & \mathbf{0} & \ddots & \vdots & \ddots \end{array} \right], \tag{4.11}$$

where the column involving n gives rise to n points in the \mathbb{P}^1 and the rest of the configuration matrix, obtained by deleting that column as well as the \mathbb{P}^1 row, leads to a gCICY three-fold, for each of these n points on the \mathbb{P}^1 . An example can be found in the third entry in Table 1, which describes copies of a quintic three-fold.

Note that such a multiple-copy structure via configurations of the form, (4.11), could also have arisen for ordinary CICYs. Since negative entries are not allowed for them, however, the entry, n , in eq. (4.11) is either 0, 1, or 2, all of which can then be neglected for the following reason. The column involving n is redundant for $n = 0$ and $n = 1$ cases, as the former gives no defining relation at all and the latter only fixes a point in the \mathbb{P}^1 , which should then be substituted to the defining relation dependent on this \mathbb{P}^1 direction, resulting in the CICY three-fold whose configuration matrix is a reduced one from eq. (4.11) by deleting the column involving the $n = 1$ and the \mathbb{P}^1 row. Similarly, $n = 2$ case will lead to two copies of a Calabi–Yau three-fold, as the column involving the $n = 2$ gives a quadratic equation in the \mathbb{P}^1 coordinates, leading to two zero points. However, such case was neglected in Ref. [2] (see eq. (1.40) there) as it corresponds to a product manifold.

For a similar reason, gCICY configuration matrices of the form, (4.11), give n copies of a Calabi–Yau topology, each of which has a different shape in general. Nevertheless, if we only search for a connected geometry, such a multiple-copy case, or a multiple-component case in general, is another source of redundancy and can thus be excluded.

5. Beginning a classification of generalized CICYs

Although a full classification of gCICY manifolds is beyond the scope of the current work, in this section we take the first steps towards such a categorization and build an initial dataset of 2,761 configuration matrices. We proceed to systematically classify and/or scan several classes of low co-dimension gCICY three-folds.

5.1. Codimension (1, 1) generalized CICYs

The simplest type of generalized CICYs is a class given by the configuration matrix:

$$X_{(1,1)} = \left[\begin{array}{c|cc} \mathbb{P}^{n_1} & a^1 & b^1 \\ \mathbb{P}^{n_2} & a^2 & b^2 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^{n_m} & a^m & b^m \end{array} \right], \tag{5.1}$$

where we recall that the a^i 's are all semi-positive, whereas some of the b^i 's can take negative values. We call this class codimension (1, 1) gCICYs. To begin, it must be understood first what kind of new gCICY configuration matrices of this type are well-defined varieties – that is, under what conditions the second column of the configuration matrix defines a line bundle with global sections on a manifold defined by the first positive column in (5.1). We will impose the CY condition on the first Chern class in a second step. More concretely, we will study how many

negative entries we have in the second column, b^i 's, and how much negative they can go, by requiring the existence of a global section of the line bundle

$$\mathcal{L}_{\mathcal{M}} \equiv \mathcal{O}_{\mathcal{M}}(b^1, \dots, b^m), \tag{5.2}$$

where $\mathcal{L}_{\mathcal{M}}$ defines a hypersurface in a new ambient manifold, given by the complete intersection manifold \mathcal{M} ,

$$\mathcal{M} = \left[\begin{array}{c|c} \mathbb{P}^{n_1} & a^1 \\ \mathbb{P}^{n_2} & a^1 \\ \vdots & \vdots \\ n_m & a^m \end{array} \right]. \tag{5.3}$$

We will reduce the problem from \mathcal{M} to the original ambient space by using the twisted Koszul short-exact sequence

$$0 \longrightarrow \mathcal{N}^\vee \otimes \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_{\mathcal{M}} \longrightarrow 0, \tag{5.4}$$

where the normal bundle of \mathcal{M} is the line bundle $\mathcal{N}^\vee = \mathcal{O}(-a^1, \dots, -a^m)$ in the original ambient space. The values of b^i 's will be practically all constrained by the cohomology of $\mathcal{L}_{\mathcal{M}}$, which we evaluate using the long-exact sequence in cohomology associated to (5.4). In fact, all we need to do now is to check when the condition $h^0(\mathcal{M}, \mathcal{L}) \neq 0$ is satisfied.

5.1.1. A negative bound

The number of global sections, $h^0(\mathcal{M}, \mathcal{L})$, depends on the ambient space cohomology groups of $\mathcal{N}^\vee \otimes \mathcal{L} = \mathcal{O}(-a^1 + b^1, \dots, -a^m + b^m)$ and $\mathcal{L} = \mathcal{O}(b^1, \dots, b^m)$. The dimensions $h^*(\mathcal{A}, \mathcal{L})$ and $h^*(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L})$ can be evaluated with the Bott–Borel–Weyl formula for line bundle cohomology for products of projective spaces (see Appendix A and [46,56] for reviews) and by looking then at the long exact sequence in cohomology associated to (5.4). We will begin our analysis at this last sequence. Using the results of Appendix A it can be verified that non-ample line bundles on this class of \mathcal{M} will arise only from negative entries in \mathbb{P}^1 directions and as result, the only non-vanishing cohomology groups associated to the first two terms in (5.4) are $H^1(\mathcal{A}, \cdot)$. The map between them is given by the defining relation of \mathcal{M} :

$$q : h^1(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}) \rightarrow h^1(\mathcal{A}, \mathcal{L}). \tag{5.5}$$

In the long exact sequence associated to (5.4) this map is *generically* injective when $h^1(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}) < h^1(\mathcal{A}, \mathcal{L})$, and hence we can generically¹⁰ exclude this possibility since we demand that $h^0(\mathcal{M}, \mathcal{L}) \neq 0$. For this reason and for the fact that there is only one non-vanishing line bundle cohomology in an ambient space, which is a product of projective space, we have just two different possibilities for $h^0(\mathcal{M}, \mathcal{L}) \neq 0$,

1. $h^1(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}) = 0 \Rightarrow h^0(\mathcal{A}, \mathcal{L}) > h^0(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L})$, with $h^0(\mathcal{A}, \mathcal{L}) \neq 0$, however, if the zero-cohomology of \mathcal{L} is non-vanishing in the ambient space, it reduces to the case where all the integer entries of \mathcal{L} are positive: $b^i \geq 0, \forall i = 1, \dots, m$.

¹⁰ We have strong evidences that the map is injective when the defining relation of \mathcal{M} are general enough. We gain these evidences by computing explicitly the line bundle cohomologies and by analyzing their behaviors when the entries of \mathcal{L} and $\mathcal{N}^\vee \otimes \mathcal{L}$ change.

2. $h^0(\mathcal{A}, \mathcal{L}) = 0$ and $h^1(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}) \neq 0$, this last condition, together with the Bott–Borel–Weyl formula in (A.1) and (A.2) [46,56] yields the following sufficient¹¹ (and generically necessary) conditions for $h^0(\mathcal{M}, \mathcal{L}) > 0$:

$$\exists! i = 1 \mid n_1 = 1, \quad (b^1 - a^1) \leq -2, \quad \forall i \neq 1 \quad b^i > 0, \quad (b^i - a^i) \geq 0, \quad (5.6)$$

where we have set the first $n_1 = 1$ for convenience, and, moreover,

$$h^1(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}) > h^1(\mathcal{A}, \mathcal{L}). \quad (5.7)$$

We are of course interested in extensions of the standard class of CICYs, hence we will study further only the second case. Equation (5.6) tells us that in codimension (1, 1) only a negative entry, b^1 , in a \mathbb{P}^1 factor is possible. Finally, we expect that (5.7) gives a bound on the negative value for b^1 . We then apply the Bott–Borel–Weyl theorem [46,56], together with all the conditions in (5.6), and we obtain the following expressions for the line bundle first-cohomologies, h^1 , in the ambient space,

$$h^1(\mathcal{A}, \mathcal{L}) = (-b^1 - 1) \prod_{i=2}^m \binom{b^i + n_i}{n_i}, \quad (5.8)$$

$$h^1(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}) = (a^1 - b^1 - 1) \prod_{i=2}^m \binom{b^i - a^i + n_i}{n_i}. \quad (5.9)$$

Plugging in (5.8)–(5.9) into (5.7), we get the following bounding inequality:

$$\frac{(a^1 - b^1 - 1)}{(-b^1 - 1)} > R, \quad (5.10)$$

where we rearranged the expression and R is a ratio that depends on the line bundle cohomologies in \mathbb{P}^{n_i} , with $i > 1$,

$$R \equiv \frac{\prod_{i=2}^m \binom{b^i + n_i}{n_i}}{\prod_{i=2}^m \binom{b^i - a^i + n_i}{n_i}}. \quad (5.11)$$

Reducing (5.10) in terms of b^1 , we get

$$1 < R < 1 + a^1 \quad \& \quad -\frac{a^1}{R - 1} - 1 < b^1 < -2 \quad (5.12)$$

$$R = 1 \quad \& \quad b^1 < -2. \quad (5.13)$$

In the second case (5.13), we do not have any negative bound, however this apparently infinite class is actually finite. As will be shown in Section 5.1.3 these are just copies of a known CICY,

¹¹ These conditions are always sufficient and generically necessary for the existence of non-trivial global sections of a line bundle on \mathcal{M} for suitably “generic” defining relations in (5.3). However, for all configuration matrices (including ordinary CICYs) these conditions can be weakened and line bundle cohomology can “jump” [57–62] when special defining polynomials are chosen for \mathcal{M} . In these special cases new effective divisors exist. We will not consider such cases in the following discussion.

when we impose the right condition on the Chern-classes of the manifold $X_{(1,1)}$. From (5.12), we get an expression for the negative bound of b^1 in terms of R and a^1 .

$$b^1 = -\left[1 + \frac{a^1}{R-1}\right]. \tag{5.14}$$

Finally we want to impose the Calabi–Yau condition on the first Chern class $c_1(TX) = 0$, which restricts $b^1 = 2 - a^1$, and $b^i + a^i = n_i + 1$ with $i = 2 \dots m$. In this way the bound depends only on R . We can summarize all the possible cases in the following four cases:

- $R = 4$, the bound is $b^1 \geq -2$.
- $R > 4$, the bound is $b^1 = -1$.
- $R = 2$, there is no bound, and $b^i \leq -1$. Apparently this is another infinite class, we will show in Section 5.1.3 it is actually finite.
- $R = 1$, for this case as well, we have no bound, and the subclass appears infinite. However it turns out that the Euler characteristic is a multiple of a Euler characteristic of a known CY, \tilde{X} , as well as all the topological data. Moreover, looking in detail at this case, we can see that for $R = 1$ the configuration matrix takes the form

$$X_{(1,1)} = \left[\begin{array}{c|cc} \mathbb{P}^1 & 2+i & i \\ \mathbb{P}^{n_2} & 0 & b^2 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^{n_m} & 0 & b^m \end{array} \right], \quad i \in \mathbb{Z}_{>0}. \tag{5.15}$$

We will show in section 5.1.3 that $X_{(1,1)}$ corresponds to $(2 + i)$ copies of the known Calabi–Yau

$$\tilde{X} = \left[\begin{array}{c|c} \mathbb{P}^{n_2} & b^2 \\ \vdots & \vdots \\ \mathbb{P}^{n_m} & b^m \end{array} \right], \tag{5.16}$$

with all positive entries for definition. These infinite manifolds are all non-strictly Calabi–Yau, more specifically each of these is a manifold formed by multiple copies of an already known CICY.

5.1.2. The classification result

CICYs with codimension $(1, 1)$ are determined by two equations in a product of projective space, hence, we can encounter only 5 ambient space which are possible for this class of CICY three-folds. The complete list of generalized codimension $(1, 1)$ CICY is given according to the classification in 5.1.1. The Euler characteristic together with the Hodge numbers $(h^{(1,1)}, h^{(2,1)})$ are summarized in Tables 1–5. In the last column, we present the results of a smoothness check of the manifolds. In the tables below, we will use the symbol ● to denote a smooth manifold, ○ a singular manifold and ◐ will indicate a manifold for which smoothness is not yet determined. We remark that the last infinite class of every table cannot be strictly considered as CY manifolds:

- For each i in the last subclass of manifolds of Table 1, we have a manifold formed by $(2 + i)$ copies of $[\mathbb{P}^4 \parallel 5]$.
- For each i in the last subclass of manifolds of Table 2, we have a manifold formed by $(2 + i)$ copies of $\left[\begin{array}{c|c} \mathbb{P}^3 & 4 \\ \mathbb{P}^1 & 2 \end{array} \right]$.

Table 1
3 cases in $\mathbb{P}^4 \times \mathbb{P}^1$. ● indicates they are smooth.

X	R	i	χ	$(h^{1,1}(X), h^{1,2}(X))$	Infinite class	Smoothness
$\left[\begin{array}{c c} \mathbb{P}^4 & 2 \quad 3 \\ \mathbb{P}^1 & 3 \quad -1 \end{array} \right]$	7	N/A	-88	(2, 46)	N/A	●
$\left[\begin{array}{c c} \mathbb{P}^4 & 1 \quad 4 \\ \mathbb{P}^1 & 2+i \quad -i \end{array} \right]$	2	$i \in \mathbb{Z}_{>0}$	-168	(2, 86)	Type III	●
$\left[\begin{array}{c c} \mathbb{P}^4 & 0 \quad 5 \\ \mathbb{P}^1 & 2+i \quad -i \end{array} \right]$	1	$i \in \mathbb{Z}_{>0}$	$-200(i+2)$	$(i+2, 101(i+2))$	Type I	●

Table 2
6 cases in $\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$. ● and ○ indicate smooth and undetermined manifolds, respectively.

X	R	i	χ	$(h^{1,1}(X), h^{1,2}(X))$	Infinite class	Smoothness
$\left[\begin{array}{c c} \mathbb{P}^3 & 2 \quad 2 \\ \mathbb{P}^1 & 0 \quad 2 \\ \mathbb{P}^1 & 3 \quad -1 \end{array} \right]$	10	N/A	-56	(3, 31)	N/A	●
$\left[\begin{array}{c c} \mathbb{P}^3 & 2 \quad 2 \\ \mathbb{P}^1 & 1 \quad 1 \\ \mathbb{P}^1 & 3 \quad -1 \end{array} \right]$	20	N/A	-104	(3, 55)	N/A	●
$\left[\begin{array}{c c} \mathbb{P}^3 & 1 \quad 3 \\ \mathbb{P}^1 & 1 \quad 1 \\ \mathbb{P}^1 & 2+i \quad -i \end{array} \right]$	4	$i = 1, 2$	-72, -48	(3, 39), (3, 27)	N/A	●, ○
$\left[\begin{array}{c c} \mathbb{P}^3 & 1 \quad 3 \\ \mathbb{P}^1 & 0 \quad 2 \\ \mathbb{P}^1 & 2+i \quad -i \end{array} \right]$	2	$i \in \mathbb{Z}_{>0}$	-144	(3, 75)	Type III	●
$\left[\begin{array}{c c} \mathbb{P}^3 & 0 \quad 4 \\ \mathbb{P}^1 & 1 \quad 1 \\ \mathbb{P}^1 & 2+i \quad -i \end{array} \right]$	2	$i \in \mathbb{Z}_{>0}$	-168	(2, 86)	Type II	●
$\left[\begin{array}{c c} \mathbb{P}^3 & 0 \quad 4 \\ \mathbb{P}^1 & 0 \quad 2 \\ \mathbb{P}^1 & 2+i \quad -i \end{array} \right]$	1	$i \in \mathbb{Z}_{>0}$	$-168(i+2)$	$(i+2, 86(i+2))$	Type I	●

- For each i in the last subclass of manifolds of Table 3, we have a manifold formed by $(2+i)$ copies of $\left[\begin{array}{c|c} \mathbb{P}^2 & 3 \\ \mathbb{P}^2 & 3 \end{array} \right]$.
- For each i in the last subclass of manifolds of Table 4, we have a manifold formed by $(2+i)$ copies of $\left[\begin{array}{c|c} \mathbb{P}^2 & 3 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \end{array} \right]$.
- For each i in the last subclass of manifolds of Table 5, we have a manifold formed by $(2+i)$ copies of $\left[\begin{array}{c|c} \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \end{array} \right]$.

The list of (1, 1) manifolds is presented in Tables 1–5 given below.

Table 3

3 cases in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$. ● and ○ indicate smooth and undetermined manifolds, respectively.

X	R	i	χ	$(h^{1,1}(X), h^{1,2}(X))$	Infinite class	Smoothness
$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^1 \end{matrix} \left \begin{matrix} 1 & 2 \\ 1 & 2 \\ 2+i & -i \end{matrix} \right.$	4	$i = 1, 2$	-78, -60	(3, 42), (3, 33)	N/A	●, ○
$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^1 \end{matrix} \left \begin{matrix} 0 & 3 \\ 1 & 2 \\ 2+i & -i \end{matrix} \right.$	2	$i \in \mathbb{Z}_{>0}$	-144	(3, 75)	Type III	●
$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^1 \end{matrix} \left \begin{matrix} 0 & 3 \\ 0 & 3 \\ 2+i & -i \end{matrix} \right.$	1	$i \in \mathbb{Z}_{>0}$	-162(i + 2)	$(2(i + 2), 83(i + 2))$	Type I	●

Table 4

6 cases in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. ●, ○ and ○ indicate smooth, undetermined and singular manifolds, respectively.

X	R	i	χ	$(h^{1,1}(X), h^{1,2}(X))$	Infinite class	Smoothness
$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \left \begin{matrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \\ 3 & -1 \end{matrix} \right.$	8	N/A	-68	(4, 38)	N/A	●
$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \left \begin{matrix} 1 & 2 \\ 0 & 2 \\ 1 & 1 \\ 2+i & -i \end{matrix} \right.$	4	$i = 1, 2$	-56, -32	(4, 32), (4, 20)	N/A	●, ○
$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \left \begin{matrix} 0 & 3 \\ 1 & 1 \\ 1 & 1 \\ 2+i & -i \end{matrix} \right.$	4	$i = 1, 2$	-36, 0	(9, 27), (11, 11)	N/A	●, ○
$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \left \begin{matrix} 0 & 3 \\ 0 & 2 \\ 1 & 1 \\ 2+i & -i \end{matrix} \right.$	2	$i \in \mathbb{Z}_{>0}$	-144	(3, 75)	Type II	●
$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \left \begin{matrix} 1 & 2 \\ 0 & 2 \\ 0 & 2 \\ 2+i & -i \end{matrix} \right.$	2	$i \in \mathbb{Z}_{>0}$	-128	(4, 68)	Type III	●
$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \left \begin{matrix} 0 & 3 \\ 0 & 2 \\ 0 & 2 \\ 2+i & -i \end{matrix} \right.$	1	$i \in \mathbb{Z}_{>0}$	-144(i + 2)	$(3(i + 2), 75(i + 2))$	Type I	●

5.1.3. Apparently infinite classes

There are twelve apparently infinite families of Calabi–Yau manifolds in the list given in the previous section. Here we will analyze these cases further and demonstrate that these are, in each case, a set of redundant descriptions of the same, standard CICY. These infinite classes of configurations matrices can be grouped in to three distinct types which is illustrated in Tables 1–5.

Table 5

5 cases in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. ● and ○ indicate smooth and singular manifolds, respectively.

X	R	i	χ	$(h^{1,1}(X), h^{1,2}(X))$	Infinite class	Smoothness
$\left[\begin{array}{c cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 3 & -1 \end{array} \right]$	16	N/A	-80	(5, 45)	N/A	●
$\left[\begin{array}{c cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 3 & -1 \end{array} \right]$	8	N/A	-48	(5, 29)	N/A	●
$\left[\begin{array}{c ccc} \mathbb{P}^1 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 1 \\ \mathbb{P}^1 & 0 & 2 & 2 \\ \mathbb{P}^1 & 0 & 2 & 2 \\ \mathbb{P}^1 & 2+i & -i & -i \end{array} \right]$	4	$i = 1, 2$	-32, 0	(10, 26), (12, 12)	N/A	●, ○
$\left[\begin{array}{c ccc} \mathbb{P}^1 & 1 & 1 & 1 \\ \mathbb{P}^1 & 0 & 2 & 2 \\ \mathbb{P}^1 & 0 & 2 & 2 \\ \mathbb{P}^1 & 0 & 2 & 2 \\ \mathbb{P}^1 & 2+i & -i & -i \end{array} \right]$	2	$i \in \mathbb{Z}_{>0}$	-128	(4, 68)	Type II	●
$\left[\begin{array}{c ccc} \mathbb{P}^1 & 0 & 2 & 2 \\ \mathbb{P}^1 & 0 & 2 & 2 \\ \mathbb{P}^1 & 0 & 2 & 2 \\ \mathbb{P}^1 & 0 & 2 & 2 \\ \mathbb{P}^1 & 2+i & -i & -i \end{array} \right]$	1	$i \in \mathbb{Z}_{>0}$	$-128(i+2)$	$(4(i+2), 68(i+2))$	Type I	●

Type I The configurations matrices of Type I all describe multiple, disconnected, copies of a regular CICY. Let us exemplify the structure with the third example in Table 1,

$$\left[\begin{array}{c|cc} \mathbb{P}^1 & 2+i & -i \\ \mathbb{P}^4 & 0 & 5 \end{array} \right], \quad i \in \mathbb{Z}_{>0}. \tag{5.17}$$

We can solve the first equation, a degree $2 + i$ polynomial in \mathbb{P}^1 to find $2 + i$ points in that first ambient space projective factor. Substituting these into the rational functions described by the second column of the configuration matrix, we obtain $2 + i$ quintic polynomials in \mathbb{P}^4 . At a generic locus in complex structure modulus space, therefore, this configuration describes $2 + i$ disconnected quintics in \mathbb{P}^4 separated from one another by their location in \mathbb{P}^1 .

As a confirmation of this analysis, we find that $h^0(\mathcal{O}) = 2 + i$ in these cases and that the Euler number is $-(2 + i)(200)$, i.e. $(2 + i)$ times that of the quintic.

Type II The configuration matrices of Type II also all describe regular CICYs. We again exemplify the structure with a single case, in this instance the fifth example in Table 2,

$$\left[\begin{array}{c|ccc} \mathbb{P}^1 & 2+i & -i & 1 \\ \mathbb{P}^1 & 1 & 1 & 1 \\ \mathbb{P}^3 & 0 & 4 & 4 \end{array} \right], \quad i \in \mathbb{Z}_{>0}. \tag{5.18}$$

These configuration matrices all describe the same manifold as the regular CICY,

$$\left[\begin{array}{c|c} \mathbb{P}^1 & 2 \\ \hline \mathbb{P}^3 & 4 \end{array} \right], \tag{5.19}$$

via a relation analogous to what is called “ineffective splitting” in the CICY literature.

Let us write the two equations described by the configuration matrices (5.18) as follows

$$p_1 = P_{10}y^0 + P_{11}y^1 = 0, \tag{5.20}$$

$$p_2 = \frac{P_{20}}{D_{20}}y^0 + \frac{P_{21}}{D_{21}}y^1 = 0. \tag{5.21}$$

It is important for us to describe exactly what is meant by the various P polynomials which appear in these expressions. P_{10} and P_{11} are general degree $2 + i$ polynomials. The polynomials P_{20} , P_{21} , D_{20} and D_{21} are not generic examples of their degree, however. P_{20} and P_{21} are degree $\{0, 0, 4\}$ while D_{20} and D_{21} are degree $\{i, 0, 0\}$. However, the choice of the denominator and numerator polynomials are correlated (and also correlated with P_{10} and P_{11}) such that the poles in the associated rational functions miss the hypersurface p_1 . As described earlier, this structure is associated to the fact that the line bundle $\mathcal{O}(-i, 1, 4)$ only has global sections when taken to be a line bundle on the hypersurface defined by p_1 and not when interpreted as a line bundle on the ambient projective product. Nevertheless, the final Calabi–Yau three-fold can be described in terms of the intersection of a rational function and a polynomial on the ambient space, once a tuning of the freedom in the P ’s and D ’s of this form has been performed.

We may rewrite the ambient space equations (5.20) and (5.21) in a matrix form as follows

$$\begin{pmatrix} P_{10} & P_{11} \\ \frac{P_{20}}{D_{20}} & \frac{P_{21}}{D_{21}} \end{pmatrix} \begin{pmatrix} y^0 \\ y^1 \end{pmatrix} = 0. \tag{5.22}$$

These equations have a solution, remembering that the y^i are homogeneous coordinates on \mathbb{P}^1 , if and only if the determinant of the matrix vanishes,

$$\frac{P_{10}P_{21}}{D_{21}} - \frac{P_{11}P_{20}}{D_{20}} = 0. \tag{5.23}$$

These equations have “net degree” $\{2, 4\}$ in $\mathbb{P}^1 \times \mathbb{P}^3$. However, naively, (5.23) seems to be still a rational defining relation. In fact, due to the manner in which the singularities in p_2 miss the zero locus of p_1 , the numerators in the two terms of (5.23) contain factors of D_{20} and D_{21} respectively, leading to a polynomial, $\{2, 4\}$ defining relation in $\mathbb{P}^1 \times \mathbb{P}^3$.

To demonstrate this let us derive (5.23) in a slightly different manner (we performed the analysis above to emphasize the similarity with ineffective splits in the case of an ordinary CICY). First, we solve (5.20) for either y^0 or y^1 ,

$$y^0 = -\frac{P_{11}}{P_{10}}y^1, \tag{5.24}$$

$$y^1 = -\frac{P_{10}}{P_{11}}y^0. \tag{5.25}$$

For a generic choice of complex structure, P_{11} and P_{10} are two uncorrelated degree $2 + i$ polynomials in \mathbb{P}^1 . They can thus not both be vanishing simultaneously and thus at least one of the above solutions is always perfectly well defined.

Substituting these results into (5.21) we obtain the following two expressions

$$\left(-\frac{P_{20}}{D_{20}} \frac{P_{11}}{P_{10}} + \frac{P_{21}}{D_{21}}\right) y^1 = 0, \tag{5.26}$$

$$\left(\frac{P_{20}}{D_{20}} - \frac{P_{21}}{D_{21}} \frac{P_{10}}{P_{11}}\right) y^0 = 0. \tag{5.27}$$

If $P_{10} \neq 0$ then from (5.25) $y^1 \neq 0$ (as not both homogeneous coordinates can vanish simultaneously). We can divide (5.26) by y^1 and multiply up by P_{10} . We then obtain (5.23) once more (a similar argument can be made in the case where $P_{11} \neq 0$). However, by the definition of our procedure for forming the section of the second normal bundle factor, that is the rational function p_2 , we know that it can have no poles when $p_1 = 0$. We have just demonstrated that (5.23) is p_2 evaluated on the solution to p_1 and thus it can have no poles. The only way in which this can happen is if the numerators and denominators of the full expression have canceling factors leaving a simply polynomial expression. This can easily be verified explicitly in any given example.

Thus (5.18) describes the same manifold as (5.19) for any value of i and the infinite class can be viewed as generalized ineffective splittings of this standard CICY.

Type III The configuration matrices of Type III also all describe regular CICYs. As before, we exemplify the structure with a single case, in this instance the second example in Table 1,

$$\left[\begin{array}{c|c|c} \mathbb{P}^1 & 2+i & -i \\ \mathbb{P}^4 & 1 & 4 \end{array} \right], \quad i \in \mathbb{Z}_{>0}. \tag{5.28}$$

A series of operations on this infinite class of configuration matrices can demonstrate that (5.28) describes the same manifold for all i . We begin by performing an ineffective split as follows

$$\left[\begin{array}{c|c|c} \mathbb{P}^1 & 2+i & -i \\ \mathbb{P}^4 & 1 & 4 \end{array} \right] \rightarrow \left[\begin{array}{c|c|c} \mathbb{P}^1 & 0 & 2+i & -i \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^4 & 1 & 0 & 4 \end{array} \right]. \tag{5.29}$$

In performing such a split in the gCICY case, all of the subtleties mentioned in Section 4 must be considered. Once we have the configuration matrix in the form (5.29) we can use the following postulated identity [2]

$$\left[\begin{array}{c|c|c} \mathbb{P}^1 & 1 & a \\ \mathbb{P}^n & 1 & nb \\ Y & 0 & M \end{array} \right] \rightarrow \left[\begin{array}{c|c|c} \mathbb{P}^1 & a+b \\ \mathbb{P}^{n-1} & nb \\ Y & M \end{array} \right]. \tag{5.30}$$

This identity was used in the original investigations of CICY three-folds [2], although it has never, to our knowledge, been proven. Here we follow those authors in making use of the identity, leaving a rigorous proof of its validity to future work. Applying (5.30) in this case we find the following

$$\left[\begin{array}{c|c|c} \mathbb{P}^1 & 2+i & -i \\ \mathbb{P}^4 & 1 & 4 \end{array} \right] \rightarrow \left[\begin{array}{c|c|c} \mathbb{P}^1 & 0 & 2+i & -i \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^4 & 1 & 0 & 4 \end{array} \right] \rightarrow \left[\begin{array}{c|c|c} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^3 & 0 & 4 \\ \mathbb{P}^1 & 2+i & -i \end{array} \right]. \tag{5.31}$$

Finally, we may apply an ineffective contraction in the first \mathbb{P}^1 direction of (5.31) to arrive at the following chain

$$\left[\begin{array}{c|c|c} \mathbb{P}^1 & 2+i & -i \\ \hline \mathbb{P}^4 & 1 & 4 \end{array} \right] \rightarrow \left[\begin{array}{c|c|c} \mathbb{P}^1 & 0 & 2+i & -i \\ \hline \mathbb{P}^1 & 1 & 1 & 0 \\ \hline \mathbb{P}^4 & 1 & 0 & 4 \end{array} \right] \rightarrow \left[\begin{array}{c|c|c} \mathbb{P}^1 & 1 & 1 \\ \hline \mathbb{P}^3 & 0 & 4 \\ \hline \mathbb{P}^1 & 2+i & -i \end{array} \right] \rightarrow \left[\begin{array}{c|c} \mathbb{P}^3 & 4 \\ \hline \mathbb{P}^1 & 2 \end{array} \right]. \quad (5.32)$$

Thus, we finally see that the entire infinite class of configuration matrices (5.28) all simply describe a single ordinary CICY.

A very similar analysis applies to the other cases of Type III. The fourth example in Table 2 and second example in Table 3 turn out to be equivalent to the configuration matrix,

$$\left[\begin{array}{c|c} \mathbb{P}^1 & 2 \\ \hline \mathbb{P}^1 & 2 \\ \hline \mathbb{P}^2 & 3 \end{array} \right]. \quad (5.33)$$

The fifth example in Table 4 turns out to be equivalent to the tetraquadric:

$$\left[\begin{array}{c|c} \mathbb{P}^1 & 2 \\ \hline \mathbb{P}^1 & 2 \\ \hline \mathbb{P}^1 & 2 \\ \hline \mathbb{P}^1 & 2 \end{array} \right]. \quad (5.34)$$

With the demonstration that this last type of infinite set of configuration matrices actually corresponds to a finite number of Calabi–Yau three-folds we have proven that the gCICYs of codimension (1, 1) constitute a finite dataset of manifolds.

5.1.4. New Calabi–Yau three-folds

It is interesting to determine which of the manifolds in Tables 1–5 are genuinely new Calabi–Yau three-folds, never before seen in other datasets. Steps can be taken in towards this goal by computing topological invariants. We have studied the Hodge data and basis independent quantities computed from the Chern classes and intersection numbers (see Ref. [15] for an analogue of this for the four-fold cases). We find that eight of the codimension (1, 1) gCICYs have never appeared before, at least as far as our search of the literature could determine. They are certainly of a different topological type to anything seen in the CICY or the Kreuzer–Skarke datasets. The eight new manifolds are those with

$$(h_{1,1}, h_{2,1}) = (2, 46), (3, 31), (3, 39), (3, 27), (3, 42), (3, 33), (4, 20), (5, 29) \quad (5.35)$$

from Tables 1–5. Note that many of these Hodge pairs have certainly appeared in the literature before. The manifolds are distinguished by the more subtle topological properties mentioned above and illustrated in Subsection 2.2.3.

5.2. Codimension (2, 1) generalized CICYs

The codimension two examples in Section 5.1 can be straightforwardly generalized to codimension three. We once again focus on negative codimension one cases, i.e., codimension (2, 1) examples with the following configuration matrix:

$$X_{(2,1)} = \left[\begin{array}{c|cc} \mathbb{P}^{n_1} & a_1^1 & a_2^1 & b^1 \\ \mathbb{P}^{n_2} & a_1^2 & a_2^2 & b^2 \\ \vdots & \vdots & \vdots & \vdots \\ \mathbb{P}^{n_N} & a_1^m & a_2^m & b^m \end{array} \right], \tag{5.36}$$

where the first column $a_1^i, a_2^i \geq 0$, but b^i are allowed to assume negative integer values. As described previously, we require the third column to define an anticanonical hypersurface in \mathcal{M} . That is, $\mathcal{L} \equiv \mathcal{O}_{\mathcal{M}}(b_3^1, \dots, b_3^m) = K_{\mathcal{M}}^{-1}$ with

$$\mathcal{M} = \left[\begin{array}{c|cc} \mathbb{P}^{n_1} & a_1^1 & a_2^1 \\ \mathbb{P}^{n_2} & a_2^2 & a_2^2 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^{n_N} & a_1^m & a_2^m \end{array} \right]. \tag{5.37}$$

In defining these CY 3-folds there are 10 possible ambient spaces for this class of generalized CICY three-folds:

$$\begin{aligned} &\mathbb{P}^5 \times \mathbb{P}^1, \mathbb{P}^4 \times \mathbb{P}^2, \mathbb{P}^3 \times \mathbb{P}^3, \mathbb{P}^4 \times \mathbb{P}^1 \times \mathbb{P}^1, \\ &\mathbb{P}^3 \times \mathbb{P}^2 \times \mathbb{P}^1, \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2, \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \\ &\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1. \end{aligned} \tag{5.38}$$

In each of these embedding products of projective spaces, there are many more classes of generalized configuration matrices than in codimension (1, 1) case.

Beginning with these ambient spaces, we construct the generalized CICYs in several steps:

- To ensure global sections on \mathcal{M} , the negative number b^i can only appear in two \mathbb{P}^1 factors or one \mathbb{P}^2 (see Appendix A for an analysis of the Koszul sequences and the Bott Theorem (A.1) leading to this requirement). Moreover, since the last column corresponds to an algebraic constraint arising from the vanishing of a global section of a line bundle on \mathcal{M} , we explicitly check in each case that $h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(b^i)) > 0$.
- We choose $b^i < 0$ consistent with vanishing first Chern class for the gCICY. For example, in a configuration matrix, if one of the rows corresponds to an ambient \mathbb{P}^2 factor (for example $[\mathbb{P}^2 \parallel 0 \ 3 \ 0 \]$) we will construct a configuration to be $[\mathbb{P}^2 \parallel 0 \ 3 + n \ | \ -n \]$. For the present, preliminary scan we will restrict the integer range of n , allowing it to vary from 1 to 4, i.e. (5.39),

$$X_{(2,1)} = \left[\begin{array}{c|cc} \vdots & \vdots & \vdots \\ \mathbb{P}^2 & 0 & 3 \ 0 \\ \vdots & \vdots & \vdots \end{array} \right] \Rightarrow X'_{(2,1)} = \left[\begin{array}{c|cc} \vdots & \vdots & \vdots \\ \mathbb{P}^2 & 0 & 3 + n \ | \ -n \\ \vdots & \vdots & \vdots \end{array} \right], \tag{5.39}$$

the negative entries appearing in the generalized configuration is greater or equal to -4 . In this scan, since the negative entries appearing in the generalized configuration are greater or equal to -4 , for each n we generate 4 new generalized CICYs in one class of the generalized configuration matrices listed in Tables 7–12. In this way, 34,192 classes of generalized configuration matrices were generated for the initial codimension (2, 1) scan.

- To remove some singular/non-reduced, non-CY geometries, we also verify explicitly that the trivial line bundle cohomology on X is $h^*(X, \mathcal{O}) = \{1, 0, 0, 1\}$. So in this scanning, we have already ruled out the Type I infinite class discussed in Section 5.1.3 from the starting point.

Table 6

The distribution of codimension (2, 1) spaces embedded in products of projective spaces.

Embedding projective spaces	# of classes of generalized configuration matrices	# of spaces with positive χ	# of spaces with non-positive χ
$\mathbb{P}^5 \times \mathbb{P}^1$	168	0	28
$\mathbb{P}^4 \times \mathbb{P}^2$	210	0	6
$\mathbb{P}^4 \times \mathbb{P}^1 \times \mathbb{P}^1$	1,197	3	226
$\mathbb{P}^3 \times \mathbb{P}^2 \times \mathbb{P}^1$	1,800	2	261
$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$	550	0	12
$\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	4,410	17	528
$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$	5,235	9	511
$\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	12,180	16	754
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	8,442	10	350
Total	34,192	57	2,676

Besides the above requirements, there is another constraint that arises from the Koszul sequence, (B.3). This provides certain bounds on the magnitude of non-zero entries in the rows of \mathcal{M} and can be used to rule out certain non-CY configuration matrices. This additional constraint is described in Appendix B. Under all these constraints, we constructed all the classes of generalized configuration matrices of this type, which could be analyzed in a reasonable computing time.¹²

Unlike in the codimension (1, 1) case and the ordinary CICY dataset, in the case of codimension (2, 1) configuration matrices we find many manifolds with positive Euler number $\chi > 0$. This preliminary scan yielded 57 spaces with $\chi > 0$ and 2,676 spaces with $\chi \leq 0$ out of 34,192 classes of generalized configuration matrices (5.36), which satisfied all the requirements discussed above. The distribution of these spaces in terms of embedding projective spaces is classified in Table 6.¹³ Due to the rapidly increasing number of spaces in this class of configuration matrices, we leave a full classification of such geometries to future work.

Codimension (2, 1) spaces with positive Euler number From Table 6 we see that there is no single example with positive Euler number in $\mathbb{P}^5 \times \mathbb{P}^1$, $\mathbb{P}^4 \times \mathbb{P}^2$ and $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ which passes all the criteria (out of 168, 210 and 550 classes of generalized configuration matrices scanned respectively). The distribution of the 57 spaces is explicitly shown in Tables 7–12. Once again in these tables, ● indicates a smooth manifold, ○ a singular manifold and finally, ● indicates that the smoothness check has timed out in the current run. For all the generically singular geometries we found, the singularities are not of a minimal order in that the “normal form”, eq. (3.13), and its exterior derivative both vanish.

In terms of topological data, none of the positive Euler number examples appears in the regular CICY list [2], though many of the Hodge number pairs appear in the Kreuzer–Skarke list [30] (however, as seen already in co-dimension (1, 1) case, two spaces with the same Hodge number does not guarantee they have the same Chern classes or triple intersection numbers). In

¹² The primary time constraints arises from the calculation of line bundle cohomology. To complete the initial scan in finite time, we leave for future scans any line bundle cohomology whose calculation time was greater than five minutes.

¹³ There may still exist some redundancies in counting of these spaces like ineffective splits, some identity and reducedness, generally described in sections 4 and 5. Although some of these geometries are singular, we assume the method in determining the topological quantity described in section 2.2 still applied, and we assume all of them are Calabi–Yau when calculating Hodge number.

Table 7

3 results out of 1,197 classes of generalized configuration matrices scanned in $\mathbb{P}^4 \times \mathbb{P}^1 \times \mathbb{P}^1$. \circ , \bullet indicate singular and undetermined manifolds respectively. In the Hodge numbers, x denotes an undetermined non-negative integer.

X	χ	$(h^{1,1}(X), h^{1,2}(X))$	Smoothness
$\left[\begin{array}{c cc} \mathbb{P}^4 & 1 & 3 \\ \mathbb{P}^1 & 0 & 0 \\ \mathbb{P}^1 & 1 & -2 \end{array} \right]$	12	$(6 + x, x)$	\circ
$\left[\begin{array}{c cc} \mathbb{P}^4 & 1 & 3 \\ \mathbb{P}^1 & 0 & 0 \\ \mathbb{P}^1 & 1 & -3 \end{array} \right]$	60	$(30 + x, x)$	\circ
$\left[\begin{array}{c cc} \mathbb{P}^4 & 1 & 3 \\ \mathbb{P}^1 & 0 & 0 \\ \mathbb{P}^1 & 2 & -3 \end{array} \right]$	24	$(12 + x, x)$	\bullet

Table 8

2 results out of 1,800 classes of generalized configuration matrices scanned in $\mathbb{P}^3 \times \mathbb{P}^2 \times \mathbb{P}^1$. \circ , \bullet indicate singular and undetermined manifolds respectively. In the Hodge numbers, x denotes an undetermined non-negative integer.

X	χ	$(h^{1,1}(X), h^{1,2}(X))$	Smoothness
$\left[\begin{array}{c cc} \mathbb{P}^3 & 1 & 3 \\ \mathbb{P}^2 & 1 & -1 \\ \mathbb{P}^1 & 0 & 1 \end{array} \right]$	12	$(6 + x, x)$	\circ
$\left[\begin{array}{c cc} \mathbb{P}^3 & 1 & 3 \\ \mathbb{P}^2 & 0 & 1 \\ \mathbb{P}^1 & 2 & -3 \end{array} \right]$	18	$(9 + x, x)$	\bullet

some cases, we are not yet able to determine the Hodge pairs (again due to the slow computation of line bundle cohomology. The integer uncertainty in these Hodge pairs is denoted “ x ” in Tables 7–12).

Codimension (2, 1) spaces with non-positive Euler number The initial scan produced 2,676 co-dimension (2, 1) examples with non-positive Euler number out of the 34,192 classes of generalized configuration matrices considered. Of these, the exact Hodge number pairs could be determined for 2,469. For these geometries then, we can compare the Hodge number pairs with those found in the literature to date: we find 319 pairs with 162 different Euler number which are not in the regular CICY list [2] and among them, 129 pairs with 24 different Euler number not present in the Kreuzer–Skarke list [30]. Further more, there are 16 geometries with new Hodge numbers not appearing elsewhere in the literature [63]. These 16 Hodge pairs are distributed in 8 different Euler numbers. These examples with new Hodge number are listed in Table 13.

6. Physics applications and outlook

In this work we have introduced a new construction of Calabi–Yau spaces that has the potential to yield very large datasets of manifolds of a variety of different dimensions. We have seen already that this construction has yielded new manifolds as well as previously unknown Hodge number pairs. If we take one for each infinite family in the codimension (1, 1) dataset, we constructed 28 spaces. For codimension (2, 1) case, there are 57 spaces with positive Euler number and 2,676 with non-positive Euler number. In total we constructed 2,761 spaces.

Table 9

17 results out of 4,410 classes of generalized configuration matrices scanned in $\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. \circ , \bullet indicate singular and undetermined manifolds respectively. In the Hodge numbers, x denotes an undetermined non-negative integer.

X	χ	$(h^{1,1}(X), h^{1,2}(X))$	Smoothness
$\begin{bmatrix} \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$	24	$(12 + x, x)$	\circ
$\begin{bmatrix} \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 3 & -2 \end{bmatrix}$	32	$(16 + x, x)$	\circ
$\begin{bmatrix} \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 3 & -2 \end{bmatrix}$	72	$(36 + x, x)$	\circ
$\begin{bmatrix} \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 4 & -3 \end{bmatrix}$	144	$(72 + x, x)$	\circ
$\begin{bmatrix} \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & -2 \\ 1 & 1 & 0 \end{bmatrix}$	24	$(27, 15)$	\bullet
$\begin{bmatrix} \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 5 & -4 \end{bmatrix}$	16	$(69, 61)$	\bullet
$\begin{bmatrix} \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & 3 & -1 \end{bmatrix}$	72	$(36 + x, x)$	\bullet
$\begin{bmatrix} \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 5 & -3 \\ 1 & 1 & 0 \end{bmatrix}$	36	$(32, 14)$	\bullet
$\begin{bmatrix} \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 5 & -4 \end{bmatrix}$	36	$(32, 14)$	\bullet
$\begin{bmatrix} \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \\ 2 & 3 & -3 \end{bmatrix}$	72	$(36 + x, x)$	\bullet
$\begin{bmatrix} \mathbb{P}^3 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 6 & -4 \\ 1 & 1 & 0 \end{bmatrix}$	16	$(75, 67)$	\bullet

(continued on next page)

Table 9 (continued)

X	χ	$(h^{1,1}(X), h^{1,2}(X))$	Smoothness
$\left[\begin{array}{c c c} \mathbb{P}^3 & & \\ \hline \mathbb{P}^1 & 1 & 2 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 2 & 0 \\ \mathbb{P}^1 & 2 & 4 \\ \hline & & -4 \end{array} \right]$	32	$(16 + x, x)$	●
$\left[\begin{array}{c c c} \mathbb{P}^3 & & \\ \hline \mathbb{P}^1 & 1 & 3 \\ \mathbb{P}^1 & 0 & 1 \\ \mathbb{P}^1 & 0 & 5 \\ \mathbb{P}^1 & 2 & 1 \\ \hline & & -1 \end{array} \right]$	36	$(31, 13)$	●
$\left[\begin{array}{c c c} \mathbb{P}^3 & & \\ \hline \mathbb{P}^1 & 1 & 3 \\ \mathbb{P}^1 & 0 & 1 \\ \mathbb{P}^1 & 0 & 6 \\ \mathbb{P}^1 & 1 & 1 \\ \hline & & 0 \end{array} \right]$	72	$(48, 12)$	●
$\left[\begin{array}{c c c} \mathbb{P}^3 & & \\ \hline \mathbb{P}^1 & 2 & 2 \\ \mathbb{P}^1 & 0 & 1 \\ \mathbb{P}^1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 4 \\ \hline & & -3 \end{array} \right]$	24	$(27, 15)$	●
$\left[\begin{array}{c c c} \mathbb{P}^3 & & \\ \hline \mathbb{P}^1 & 1 & 3 \\ \mathbb{P}^1 & 0 & 4 \\ \mathbb{P}^1 & 1 & 0 \\ \mathbb{P}^1 & 2 & 1 \\ \hline & & -1 \end{array} \right]$	48	$(76, 52)$	●
$\left[\begin{array}{c c c} \mathbb{P}^3 & & \\ \hline \mathbb{P}^1 & 1 & 3 \\ \mathbb{P}^1 & 0 & 5 \\ \mathbb{P}^1 & 1 & 0 \\ \mathbb{P}^1 & 2 & 1 \\ \hline & & -1 \end{array} \right]$	96	$(116, 68)$	●

In the discussion below we turn briefly to some of the physics applications of this new dataset of manifolds. Many of the topics here deserve to be the subject of future work and we highlight the ways that this dataset will be particularly applicable to many physical problems, as well as areas where interesting open questions remain.

6.1. Fibration structure

The obvious fibration structures exhibited by generalized CICYs follow a very similar form to those seen in standard semi-positive configuration matrices. Consider a configuration matrix which can be put in the following form by row and column permutations:

$$X = \left[\begin{array}{c|c} \mathcal{A}_1 & \mathcal{F} \\ \hline \mathcal{A}_2 & \mathcal{T} \end{array} \right], \tag{6.1}$$

where \mathcal{A}_1 and \mathcal{A}_2 are two products of projective spaces, while \mathcal{F}, \mathcal{B} and \mathcal{T} are block submatrices. Such a configuration matrix, in the case of semi-positive configuration matrices, describes a fibration of the manifold given by $F = [\mathcal{A}_1 \parallel \mathcal{F}]$ over the base $B = [\mathcal{A}_2 \parallel \mathcal{B}]$, where the variation of the fiber over the base is determined by the matrix \mathcal{T} . It has been observed, for example, that almost all known CICY three- and four-folds are fibered with Calabi–Yau manifolds of every lower dimension multiple times [64].

As we have mentioned, a similar structure is exhibited by generalized CICYs. Consider, for example, the configuration (1.2) for a gCICY three-fold discussed in the introduction. In this example, we can rearrange the matrix and make the division as follows,

Table 10

9 results out of 5,235 classes of generalized configuration matrices scanned in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. \circ , \bullet indicate singular and undetermined manifolds respectively. In the Hodge numbers, x denotes an undetermined non-negative integer.

X	χ	$(h^{1,1}(X), h^{1,2}(X))$	Smoothness
$\left[\begin{array}{c c c} \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^2 \\ \hline 0 & 0 & 3 \\ 1 & 1 & 1 \\ 2 & 0 & 0 \\ 2 & 1 & -1 \end{array} \right]$	36	$(18 + x, x)$	\circ
$\left[\begin{array}{c c c} \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^2 \\ \hline 0 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 2 \\ 1 & 3 & -2 \end{array} \right]$	16	$(8 + x, x)$	\circ
$\left[\begin{array}{c c c} \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^2 \\ \hline 0 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 3 & -2 \end{array} \right]$	16	$(8 + x, x)$	\circ
$\left[\begin{array}{c c c} \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^2 \\ \hline 3 & 1 & -1 \\ 0 & 0 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right]$	18	$(9 + x, x)$	\bullet
$\left[\begin{array}{c c c} \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^2 \\ \hline 0 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 6 & -4 \\ 1 & 1 & 0 \end{array} \right]$	16	$(32, 24)$	\bullet
$\left[\begin{array}{c c c} \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^2 \\ \hline 0 & 0 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -2 \\ 2 & 0 & 0 \end{array} \right]$	36	$(18 + x, x)$	\bullet
$\left[\begin{array}{c c c} \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^2 \\ \hline 0 & 0 & 3 \\ 1 & 1 & 1 \\ 0 & 5 & -3 \\ 2 & 0 & 0 \end{array} \right]$	36	$(18 + x, x)$	\bullet
$\left[\begin{array}{c c c} \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^2 \\ \hline 1 & 0 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & -1 \\ 0 & 4 & -2 \end{array} \right]$	12	$(75, 69)$	\bullet
$\left[\begin{array}{c c c} \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^2 \\ \hline 1 & 0 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & -1 \\ 0 & 5 & -3 \end{array} \right]$	48	$(111, 87)$	\bullet

$$X = \left[\begin{array}{c|c|c|c} \mathbb{P}^5 & 3 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 1 & -1 \\ \mathbb{P}^1 & 1 & 1 & -1 & 1 \end{array} \right]. \tag{6.2}$$

Here the base is $B = \mathbb{P}^1$ and the fiber, F , is given as

$$F = \left[\begin{array}{c|c|c|c} \mathbb{P}^5 & 3 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 1 & -1 \end{array} \right], \tag{6.3}$$

Table 11

16 results out of 12,180 classes of generalized configuration matrices scanned in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. \circ , \bullet indicate singular and undetermined manifolds respectively. In the Hodge numbers, x denotes an undetermined non-negative integer.

X	χ	$(h^{1,1}(X), h^{1,2}(X))$	Smoothness
$\left[\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right]$	24	$(12 + x, x)$	\circ
$\left[\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right]$	32	$(16 + x, x)$	\circ
$\left[\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right]$	64	$(32 + x, x)$	\circ
$\left[\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right]$	32	$(29, 13)$	\bullet
$\left[\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right]$	32	$(16 + x, x)$	\bullet
$\left[\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right]$	32	$(16 + x, x)$	\bullet
$\left[\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right]$	24	$(24, 12)$	\bullet
$\left[\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right]$	16	$(40, 32)$	\bullet
$\left[\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right]$	36	$(33, 15)$	\bullet

Table 11 (continued)

X	χ	$(h^{1,1}(X), h^{1,2}(X))$	Smoothness
$\left[\begin{array}{c c c} \mathbb{P}^2 & & \\ \hline \mathbb{P}^1 & 0 & 0 & 3 \\ \hline \mathbb{P}^1 & 1 & 0 & 1 \\ \hline \mathbb{P}^1 & 1 & 0 & 1 \\ \hline \mathbb{P}^1 & 1 & 1 & 0 \\ \hline \mathbb{P}^1 & 1 & 5 & -4 \end{array} \right]$	72	(55, 19)	●
$\left[\begin{array}{c c c} \mathbb{P}^2 & & \\ \hline \mathbb{P}^1 & 3 & 1 & -1 \\ \hline \mathbb{P}^1 & 1 & 0 & 1 \\ \hline \mathbb{P}^1 & 0 & 1 & 1 \\ \hline \mathbb{P}^1 & 0 & 0 & 2 \\ \hline \mathbb{P}^1 & 0 & 0 & 2 \end{array} \right]$	16	$(8 + x, x)$	●
$\left[\begin{array}{c c c} \mathbb{P}^2 & & \\ \hline \mathbb{P}^1 & 1 & 0 & 2 \\ \hline \mathbb{P}^1 & 0 & 0 & 2 \\ \hline \mathbb{P}^1 & 0 & 1 & 1 \\ \hline \mathbb{P}^1 & 0 & 1 & 1 \\ \hline \mathbb{P}^1 & 1 & 5 & -4 \end{array} \right]$	32	(29, 13)	●
$\left[\begin{array}{c c c} \mathbb{P}^2 & & \\ \hline \mathbb{P}^1 & 1 & 0 & 2 \\ \hline \mathbb{P}^1 & 0 & 0 & 2 \\ \hline \mathbb{P}^1 & 0 & 6 & -4 \\ \hline \mathbb{P}^1 & 1 & 0 & 1 \\ \hline \mathbb{P}^1 & 1 & 1 & 0 \end{array} \right]$	16	(58, 50)	●
$\left[\begin{array}{c c c} \mathbb{P}^2 & & \\ \hline \mathbb{P}^1 & 0 & 0 & 3 \\ \hline \mathbb{P}^1 & 0 & 5 & -3 \\ \hline \mathbb{P}^1 & 1 & 0 & 1 \\ \hline \mathbb{P}^1 & 1 & 0 & 1 \\ \hline \mathbb{P}^1 & 1 & 1 & 0 \end{array} \right]$	36	(53, 35)	●
$\left[\begin{array}{c c c} \mathbb{P}^2 & & \\ \hline \mathbb{P}^1 & 0 & 0 & 3 \\ \hline \mathbb{P}^1 & 0 & 6 & -4 \\ \hline \mathbb{P}^1 & 1 & 0 & 1 \\ \hline \mathbb{P}^1 & 1 & 0 & 1 \\ \hline \mathbb{P}^1 & 1 & 1 & 0 \end{array} \right]$	72	(75, 39)	●
$\left[\begin{array}{c c c} \mathbb{P}^2 & & \\ \hline \mathbb{P}^1 & 1 & 0 & 2 \\ \hline \mathbb{P}^1 & 0 & 1 & 1 \\ \hline \mathbb{P}^1 & 0 & 1 & 1 \\ \hline \mathbb{P}^1 & 1 & 0 & 1 \\ \hline \mathbb{P}^1 & 1 & 4 & -3 \end{array} \right]$	24	(24, 12)	●

which is a K3 surface. A difference with the normal CICY situation is that we must check that any mixed-sign line bundles involved in defining the fiber and the base, $\mathcal{O}_{\mathcal{M}'}(1, -1)$ in this case, with $\mathcal{M}' = \left[\begin{array}{c|c|c} \mathbb{P}^5 & & \\ \hline \mathbb{P}^1 & 3 & 1 & 1 \\ \hline \mathbb{P}^1 & 1 & 1 & 1 \end{array} \right]$, have global sections such that the associated manifolds do indeed exist.

Computation of the line bundle cohomology indeed leads to $h^0(\mathcal{M}', \mathcal{O}_{\mathcal{M}'}(1, -1)) = 2 > 0$. As a further check, the Hodge number computation reveals that $h^{0,0}(F) = 1$ and $h^{1,1}(F) = 20$ as they should. The gCICY three-fold at hand is also elliptically fibered, which can similarly be seen from the division,

$$X = \left[\begin{array}{c|c|c|c} \mathbb{P}^5 & & & \\ \hline \mathbb{P}^1 & 3 & 1 & 1 \\ \hline \mathbb{P}^1 & 1 & 1 & -1 \\ \hline \mathbb{P}^1 & 1 & 1 & 1 \end{array} \right], \tag{6.4}$$

Table 12

10 results out of 8,441 classes of generalized configuration matrices scanned in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. \circ , \bullet indicate singular and undetermined manifolds respectively. In the Hodge numbers, x denotes an undetermined non-negative integer.

X	χ	$(h^{1,1}(X), h^{1,2}(X))$	Smoothness
$\left[\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right] \begin{array}{c} 2 \ 1 \\ 2 \ 0 \\ 1 \ 0 \\ 0 \ 1 \\ 0 \ 1 \\ 0 \ 1 \end{array} \begin{array}{c} -1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}$	16	$(8 + x, x)$	\circ
$\left[\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right] \begin{array}{c} 2 \ 1 \\ 2 \ 0 \\ 1 \ 0 \\ 0 \ 1 \\ 0 \ 1 \\ 0 \ 0 \\ 0 \ 0 \end{array} \begin{array}{c} -1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{array}$	32	$(16 + x, x)$	\circ
$\left[\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right] \begin{array}{c} 0 \ 0 \\ 0 \ 0 \\ 0 \ 5 \\ 1 \ 0 \\ 1 \ 0 \\ 1 \ 1 \end{array} \begin{array}{c} 2 \\ 2 \\ -3 \\ 1 \\ 1 \\ 0 \end{array}$	32	$(50, 34)$	\bullet
$\left[\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right] \begin{array}{c} 0 \ 1 \\ 0 \ 1 \\ 1 \ 0 \\ 1 \ 0 \\ 1 \ 0 \\ 1 \ 4 \end{array} \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -3 \end{array}$	16	$(25, 17)$	\bullet
$\left[\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right] \begin{array}{c} 0 \ 0 \\ 0 \ 1 \\ 0 \ 1 \\ 1 \ 0 \\ 1 \ 0 \\ 1 \ 4 \end{array} \begin{array}{c} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ -3 \end{array}$	32	$(26, 10)$	\bullet
$\left[\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right] \begin{array}{c} 0 \ 1 \\ 0 \ 4 \\ 1 \ 0 \\ 1 \ 0 \\ 1 \ 0 \\ 1 \ 1 \end{array} \begin{array}{c} 1 \\ -2 \\ 1 \\ 1 \\ 1 \\ 0 \end{array}$	16	$(25, 17)$	\bullet
$\left[\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \right] \begin{array}{c} 0 \ 0 \\ 0 \ 1 \\ 0 \ 4 \\ 1 \ 0 \\ 1 \ 0 \\ 1 \ 1 \end{array} \begin{array}{c} 2 \\ 1 \\ -2 \\ 1 \\ 1 \\ 0 \end{array}$	32	$(26, 10)$	\bullet

Table 12 (continued)

X	χ	$(h^{1,1}(X), h^{1,2}(X))$	Smoothness
$\left[\begin{array}{c c c} \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 4 & -3 \end{array} \right]$	32	(32, 16)	●
$\left[\begin{array}{c c c} \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 5 & -4 \end{array} \right]$	64	(52, 20)	●
$\left[\begin{array}{c c c} \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 6 & -4 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 1 & 0 \end{array} \right]$	64	(70, 38)	●

that makes manifest the fibration of the elliptic curve,

$$F = [\mathbb{P}^5 \parallel [3 \quad 1 \quad 1 \quad 1]], \tag{6.5}$$

over the base $B = \mathbb{P}^1 \times \mathbb{P}^1$. Note that the elliptic and K3 fibration structures here are nested.

As another example, the following gCICY four-fold,

$$X = \left[\begin{array}{c|c|c} \mathbb{P}^3 & 2 & 2 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 3 & -1 \end{array} \right], \tag{6.6}$$

similarly has a nested fibration structure of elliptic, K3, and Calabi–Yau three-fold fibers, over the bases $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1$, and \mathbb{P}^1 , respectively. The following divisions of the configuration (6.6) in turn make such fibrations manifest:

$$\left[\begin{array}{c|c|c} \mathbb{P}^3 & 2 & 2 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 3 & -1 \end{array} \right]; \quad \left[\begin{array}{c|c|c} \mathbb{P}^3 & 2 & 2 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 3 & -1 \end{array} \right]; \quad \left[\begin{array}{c|c|c} \mathbb{P}^3 & 2 & 2 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 3 & -1 \end{array} \right]. \tag{6.7}$$

It seems likely that the generalized CICYs exhibit the same rich fibration structure that has recently been investigated in other constructions [13–15,26,65–69]. Naturally, a rigorous confirmation of such a claim must await a full classification of the dataset.

6.2. Discrete symmetries, torsion and Wilson lines

One application of this simple, algebraic set of manifolds lies in the ease with which its discrete automorphisms can be classified. Discrete symmetries of CY manifolds have a wide number

Table 13

The Hodge pairs and configuration matrices of novel codimension (2, 1) examples. These new Hodge pairs do not appear in the regular CICY list [2], Kreuzer–Skarke list [30] or elsewhere in the known literature [63].

$(h^{1,1}(X), h^{1,2}(X))$	X
(1, 91)	$\begin{bmatrix} \mathbb{P}^2 & 1 & 1 & 1 \\ \mathbb{P}^2 & 0 & 3 & 0 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 1 & 2 & -1 \end{bmatrix}$
(1, 109)	$\begin{bmatrix} \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^2 & 0 & 3 & 0 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{bmatrix}$
(2, 98)	$\begin{bmatrix} \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 2 & 0 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 2 & 0 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{bmatrix}$
(6, 18)	$\begin{bmatrix} \mathbb{P}^2 & 0 & 1 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{bmatrix}, \begin{bmatrix} \mathbb{P}^2 & 0 & 1 & 2 \\ \mathbb{P}^1 & 0 & 3 & -1 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \mathbb{P}^2 & 0 & 1 & 2 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 3 & -2 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{bmatrix}$
(10, 19)	$\begin{bmatrix} \mathbb{P}^2 & 0 & 0 & 3 \\ \mathbb{P}^2 & 1 & 1 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{bmatrix}, \begin{bmatrix} \mathbb{P}^2 & 0 & 0 & 3 \\ \mathbb{P}^2 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 5 & -4 \end{bmatrix}, \begin{bmatrix} \mathbb{P}^2 & 0 & 0 & 3 \\ \mathbb{P}^2 & 1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 2 & 3 & -3 \end{bmatrix}, \begin{bmatrix} \mathbb{P}^2 & 0 & 0 & 3 \\ \mathbb{P}^2 & 2 & 2 & -1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{bmatrix}$
(9, 13)	$\begin{bmatrix} \mathbb{P}^3 & 2 & 0 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{bmatrix}, \begin{bmatrix} \mathbb{P}^3 & 2 & 0 & 2 \\ \mathbb{P}^1 & 0 & 3 & -1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 0 \end{bmatrix}$
(9, 15)	$\begin{bmatrix} \mathbb{P}^3 & 1 & 0 & 3 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{bmatrix}$
(10, 14)	$\begin{bmatrix} \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 3 & -2 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 3 & -1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 3 & -2 \end{bmatrix}$

of applications in string compactifications ranging from Type II orientifold actions [70] to heterotic Wilson lines [62,71–73] to discrete symmetries in F-theory (see for example [74–81] for recent work). The role of cohomological torsion in Mirror symmetry [82–84] is also an ongoing subject of investigation.

For the ordinary CICYs, the set of freely acting discrete symmetries was recently classified in [85]. The structure of this classification took a two step approach:

1. Classify discrete symmetries descending from the ambient projective space factors (i.e. whose action was manifest on the homogeneous coordinates of a product of projective spaces).
2. Consider CICYs that are invariant under those actions (i.e. equivariant normal bundles) and for which the induced symmetry action is fixed point free.

For the manifolds presented here, the same procedure directly applies. Once again, the ambient space symmetry actions are known and their action on the defining relations – even the “rational” sections described here – can be readily studied. Since the dataset includes many manifolds with small Hodge numbers already, it is intriguing future area of study to consider the quotient manifolds X/Γ under the already classified finite discrete groups Γ .

6.3. Computability and simple algebraic construction

The two most commonly used datasets of CY manifolds – the CICY list [2] and the Kreuzer–Skarke list [30] – have played an important role in the development of string theory largely due to the simple algebraic nature of the constructions. The arenas of ordinary projective or toric geometry provide a rich toolkit which can be used to calculate the necessary topological and cohomological structure of string backgrounds (i.e. bundle valued cohomology, Chern classes, intersection numbers, etc.) [86–92].

For illustration, one such recent application in heterotic string theory includes systematic, algorithmic searches for heterotic Standard Model vacua on smooth CY 3-folds [5,7,9,10,52,53,93–96]. These scans over literally hundreds of billions of heterotic vacua have only been possible due to the simple description of the CICY geometry and vector bundles over it and the suitability of these algebraic constructions for analysis using computational algebraic geometry. We emphasize here that this dataset is on an equal footing in terms of computational ease. Indeed, one could also follow a similar program of Calabi–Yau manifold construction to the one discussed here for the case of general Toric ambient spaces.

6.4. M-theory on CY 4-folds and instantons

The simple nature of ordinary CICYs in products of projective spaces have also allowed for the easy extraction of certain general properties of instanton physics in heterotic string theory, M-theory and F-theory (see for example [97,98]). We explore one aspect of this here involving the physics of instantons in 3-dimensional compactifications of M-theory.

In [99], Witten pointed out that for M-theory on a CY 4-fold there is a crucial condition that must be satisfied for any instanton to contribute non-trivially to the $N = 1$ superpotential in 3-dimensions. In particular, it is necessary that

$$\chi(D, \mathcal{O}_D) = 1. \quad (6.8)$$

In [99] it is argued that for ordinary CICYs in products of projective space, there are no D ’s with the required properties and thus, the superpotential is identically zero. This argument is based on fact that the set of effective divisors on the CICY consist of those that descend from the ambient product of projective spaces. As we have demonstrated in Section 2.2.4, in the case of gCICYs the effective cone of X is generically much larger than that of the ambient space $\mathcal{A} = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$. As a result, there are many more divisors available to satisfy the condition given above. A full exploration of such effects and the geometry of gCICY fourfolds is an exciting prospect for future study.

6.5. Relationship to GLSMs?

Positive CICYs have played an important role as the vacuum solutions of Gauged Linear Sigma Models (GLSMs) [100]. It is an intriguing question whether or not gCICYs could also be realized in this way? At first pass, it seems that the answer should in fact be no, since the generalization under consideration is precisely the fact that these manifolds cannot be viewed as the complete intersection of a set of strictly polynomial equations in an ambient product of projective spaces. Phrased differently, they are not realizable as complete intersection manifolds on the ambient product of projective spaces. Instead they only exhibit such a structure on \mathcal{M} (where all negative entry line bundles have global sections) which does not carry manifest toric/ $U(1)$ actions on its coordinates.

Despite this obstacle however, there remain open questions. While the manifolds cannot be realized as a complete intersection of *polynomial* equations on $\mathcal{A} = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$, they can be described by a complete intersection of (suitably non-singular) *rational* functions in these coordinates as described in detail in Section 3. The question arises then, could such a system of rational conditions arise from the F-terms associated to a holomorphic superpotential of a GLSM? If so, it would perhaps provide a novel generalization of the solutions studied to date in the literature. We leave this as an intriguing topic of future investigation.

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Appendix A. Line bundles and cohomology

An important tool in the computation of vector bundle-valued cohomology on CICYs or gCICYs is the calculation of line bundle cohomology. In this section we focus on the cohomology of $\mathcal{O}_X(D)$ where D is a divisor obtained by restriction from the ambient space (i.e. the divisor is “favorable” in the sense used in [5,7,9,52,53]).

The Bott–Borel–Weil Theorem (see [101] for the form used here) is a powerful tool that can be used to calculate the cohomology of V , a holomorphic homogeneous vector bundle over some Flag manifold, M . Here we will apply this technology in the simple case of projective space to obtain the so-called Bott-formula [48] for cohomology of line bundles on a single projective space:

$$h^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \begin{cases} \binom{k+n}{n} & q = 0 \quad k > -1 \\ 1 & q = n \quad k = -n - 1 \\ \binom{-k-1}{-k-n-1} & q = n \quad k < -n - 1 \\ 0 & \text{otherwise} \end{cases} \quad (\text{A.1})$$

The computation of line bundle cohomology described by the Bott–Borel–Weil theorem is easily generalized to products of projective space using the Künneth formula [46] which gives the

cohomology of bundles over a direct product of spaces. For products of projective spaces it states that:

$$\begin{aligned}
 H^n(\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}, \mathcal{O}(q_1, \dots, q_m)) \\
 = \bigoplus_{k_1 + \dots + k_m = n} H^{k_1}(\mathbb{P}^{n_1}, \mathcal{O}(q_1)) \times \dots \times H^{k_m}(\mathbb{P}^{n_m}, \mathcal{O}(q_m)) .
 \end{aligned}
 \tag{A.2}$$

With this in hand, we can compute the cohomology of line bundles over the ambient space.

Next, given the Koszul sequence (2.7), it is clear that the cohomology $H^*(X, L)$ can be determined in terms of $H^*(\mathcal{M}, L)$ for any line bundle L obtained by restriction from \mathcal{M} . Here we review briefly the techniques for calculating line bundle cohomology on a general (non-CY) complete intersection manifold \mathcal{M} of the form given in (2.3).

$$0 \rightarrow \mathcal{V} \otimes \wedge^K N_X^* \rightarrow \mathcal{V} \otimes \wedge^{K-1} N_X^* \rightarrow \dots \rightarrow \mathcal{V} \otimes N_X^* \rightarrow \mathcal{V} \rightarrow \mathcal{V}|_X \rightarrow 0 .
 \tag{A.3}$$

We can break the sequence (A.3) into a series of short exact sequences as

$$\begin{aligned}
 0 \rightarrow \mathcal{V} \otimes \wedge^K N_X^* \rightarrow \mathcal{V} \otimes \wedge^{K-1} N_X^* \rightarrow \mathcal{K}_1 \rightarrow 0 , \\
 0 \rightarrow \mathcal{K}_1 \rightarrow \mathcal{V} \otimes \wedge^{K-2} N_X^* \rightarrow \mathcal{K}_2 \rightarrow 0 , \\
 \dots \\
 0 \rightarrow \mathcal{K}_{K-1} \rightarrow \mathcal{V} \rightarrow \mathcal{V}|_X \rightarrow 0 ,
 \end{aligned}
 \tag{A.4}$$

and each of these short exact sequences will give rise to a long exact sequence in cohomology:

$$\begin{aligned}
 0 \rightarrow H^0(\mathcal{A}, \mathcal{V} \otimes \wedge^K N_X^*) \rightarrow H^0(\mathcal{A}, \mathcal{V} \otimes \wedge^{K-1} N_X^*) \rightarrow H^0(\mathcal{A}, \mathcal{K}_1) , \\
 0 \rightarrow H^0(\mathcal{A}, \mathcal{K}_1) \rightarrow H^0(\mathcal{A}, \mathcal{V} \otimes \wedge^{K-2} N_X^*) \rightarrow H^0(\mathcal{A}, \mathcal{K}_2) \rightarrow \dots , \\
 \dots \\
 0 \rightarrow H^0(\mathcal{A}, \mathcal{K}_{K-1}) \rightarrow H^0(\mathcal{A}, \mathcal{V}) \rightarrow H^0(X, \mathcal{V}|_X) \rightarrow \dots .
 \end{aligned}
 \tag{A.5}$$

To find $H^*(X, \mathcal{V}|_X)$ we must determine the various cohomologies in (A.5). It is easy to see that for higher co-dimensional spaces or tensor powers of bundles, this decomposition of sequences is a laborious process. Fortunately, the analysis of these arrays of exact sequences is dramatically simplified by the use of spectral sequences. Spectral sequences are completely equivalent to the collection of exact sequences described above, but designed for explicit cohomology computation. Since there are many good reviews of spectral sequence available in the literature [46,56, 102], we will only discuss the essential features in the following paragraphs.

To obtain the necessary cohomology of $\mathcal{V}|_X$ from (A.3), we define a tableaux

$$E_1^{j,k}(V) := H^j(\mathcal{A}, \mathcal{V} \otimes \wedge^k N_X^*), \quad k = 0, \dots, K; \quad j = 0, \dots, \dim(\mathcal{A}) = \sum_{i=1}^m n_i , \tag{A.6}$$

which forms the first term of a *spectral sequence* [48,49]. Here the spectral sequence of line bundle cohomology is a complex defined by differential maps $d_i : E_i^{j,k} \rightarrow E_i^{j-i+1,k-i}$ for $j = 1, 2, \dots$ *ad infinitum* where $d_i \circ d_i = 0$. The higher terms in the spectral sequence are defined by

$$E_{i+1}^{j,k}(V) = \frac{\ker(d_i : E_i^{j,k}(V) \rightarrow E_i^{j-i+1,k-i}(V))}{\text{Im}(d_i : E_i^{j+i-1,k+i}(V) \rightarrow E_i^{j,k}(V))} , \tag{A.7}$$

Since the number of terms in the Koszul sequence (A.3) is finite, there exists a limit to the spectral sequence. That is, the sequence of tableaux converge after a finite number of steps to $E_\infty^{j,k}(V)$. The actual cohomology of the bundle V is constructed from this limit tableaux:

$$h^q(X, V|_X) = \sum_{m=0}^K \text{rank} E_\infty^{q+m,m}(V), \tag{A.8}$$

where $h^q(X, V|_X) = \dim(H^q(X, V|_X))$. In practice, this sequence converges rapidly. For other discussions of the calculation of vector/line bundle cohomology in similar contexts, see [56, 86–88].

Appendix B. Trivial bundle cohomology and filtering out non-CY configuration matrices

As illustrated in Section 3, some configuration matrices of the form (2.3) do not give rise to Calabi–Yau manifolds – due to the fact that the zero-locus of the defining equations produces a non-reduced scheme whose classical variety¹⁴ does not satisfy the necessary CY conditions. In this section, we mention a simple criteria for configuration matrices that *always* give rise to such non-CY geometries. All such configuration matrices have been omitted from the scans carried out in this work.

A simple filter for “non-CY” configurations can be obtained by considering the cohomology of the trivial bundle. In the case of a CY manifold the cohomology is $h(\mathcal{O}_X) = (1, 0, 0, 1)$. Here we present a class of manifolds which will generically produce a trivial bundle with different cohomology. Any $(p, 1)$ configuration matrix of the form

$$[\mathbf{n} \parallel \{\mathbf{a}_\alpha\} \mid \{\mathbf{b}_\mu\}] = \left[\begin{array}{c|ccc|ccc} n_1 & a_1^1 & a_r^1 & \cdots & b_1^1 & & \\ n_2 & a_1^2 & 0 & \cdots & b_1^2 & & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \\ n_m & a_1^m & a_r^m & \cdots & b_1^m & & \end{array} \right], \tag{B.1}$$

which has a column containing a zero and *another entry* a_r^s *in the same column satisfying*

$$a_r^s \geq n_r + 1 \tag{B.2}$$

for its respective ambient \mathbb{P}^{n_r} dimension.

To see why such configuration matrices are problematic, as in Section 2.2.1, we can consider the Koszul sequence:

$$0 \rightarrow N_X^\vee \rightarrow \mathcal{O}_M \rightarrow \mathcal{O}_X \rightarrow 0, \tag{B.3}$$

where the cohomology of the line bundles on M is given in terms of the ambient product of projective spaces via

$$\begin{aligned} 0 \rightarrow \wedge^K N_M^* \rightarrow \wedge^{K-1} N_M^* \rightarrow \mathcal{K}_1 \rightarrow 0, \\ 0 \rightarrow \mathcal{K}_1 \rightarrow \wedge^{K-2} N_M^* \rightarrow \mathcal{K}_2 \rightarrow 0, \\ \dots \\ 0 \rightarrow \mathcal{K}_{K-2} \rightarrow N_M^* \rightarrow \mathcal{K}_{K-1}, \\ 0 \rightarrow \mathcal{K}_{K-1} \rightarrow \mathcal{O}_A \rightarrow \mathcal{O}|_M \rightarrow 0. \end{aligned} \tag{B.4}$$

¹⁴ I.e. the variety associated to the radical of the ideal [48].

The key observation here is that if the configuration matrix of \mathcal{M} contains a column as described above then generically $h^i(\mathcal{M}, \mathcal{O}_{\mathcal{M}}) \neq 0$ for some $i > 0$. When this result is combined with the long exact sequence associated to (B.3), it leads to trivial bundle cohomology on X that is incompatible with the CY condition.

This is most simply illustrated via an example. Consider the following configuration matrix:

$$X = \left[\begin{array}{c|cc} \mathbb{P}^4 & 1 & 0 & 4 \\ \mathbb{P}^1 & 1 & 2 & -1 \\ \mathbb{P}^1 & 0 & 3 & -1 \end{array} \right]; \tag{B.5}$$

Here the sequence (B.4) takes the form

$$0 \rightarrow \mathcal{O}(-1, -3, -3) \rightarrow \mathcal{O}(-1, -1, 0) \oplus \mathcal{O}(0, -2, -3) \rightarrow \mathcal{K}_1 \rightarrow 0, \tag{B.6}$$

$$0 \rightarrow \mathcal{K}_1 \rightarrow \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_M \rightarrow 0. \tag{B.7}$$

Unlike in the case of good configuration matrices, the line bundle $\mathcal{O}(0, -2, -3)$ does not have entirely vanishing or top cohomology. Rather, $H^2(\mathcal{A}, \mathcal{O}(0, -2, -3)) = 2$. As a result, the long exact sequence in cohomology associated to this sequence yields

$$h^*(M, \mathcal{O}_M) = (1, 2, 0, 0, 0). \tag{B.8}$$

Combining this with the Koszul sequence (B.3) for $X \subset M$

$$0 \rightarrow \mathcal{O}(-4, 1, 1) \rightarrow \mathcal{O}_M \rightarrow \mathcal{O}_X \rightarrow 0, \tag{B.9}$$

and considering the long exact sequence in cohomology, we arrive at the final conclusion:

$$h^*(X, \mathcal{O}_X) = (1, 2, 0, 3). \tag{B.10}$$

This fails to agree with the $h^*(X, \mathcal{O}_X) = (1, 0, 0, 1)$ of a CY manifold. It is straightforward to verify that similar problematic cohomology will arise for $h^*(X, \mathcal{O}_X)$ for any configuration of the form given in (B.1) and (B.2).

Appendix C. Numerical method for section construction

In this appendix, we discuss a numerical method to construct the global sections of a line bundle for defining a gCICY three-fold. For a configuration matrix, eq. (2.3), let us explain how the sections,

$$q_1 \in H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(\mathbf{b}_1)), \tag{C.1}$$

can be systematically constructed on a computer. For simplicity, only the construction of q_1 is discussed here, but the method straight-forwardly generalizes to sections coming from any other columns of the configuration matrix.

As described in Section 3, let us consider the rational form, (3.2),

$$q_1 = \frac{N(\mathbf{x}_1, \dots, \mathbf{x}_m)}{D(\mathbf{x}_1, \dots, \mathbf{x}_m)}, \tag{C.2}$$

for a section $q_1 \in H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(\mathbf{b}_1))$, where N and D are a polynomial in $\mathbf{x}_1, \dots, \mathbf{x}_m$, of multi-degree $[\mathbf{b}_1]_+$ and $[\mathbf{b}_1]_-$, respectively. We should then require that the numerator N vanishes on the divisor $D = 0$ of \mathcal{M} .

At the practical level on a computer, we first choose a generic denominator polynomial D and write N as a linear combination,

$$N(\mathbf{x}_1, \dots, \mathbf{x}_m) = \sum_{\text{deg } \mathbf{m}=[\mathbf{b}_1]_+} c_{\mathbf{m}} \cdot \mathbf{m}(\mathbf{x}_1, \dots, \mathbf{x}_m), \tag{C.3}$$

of all the monomials \mathbf{m} of the right multi-degree, with their coefficients, $c_{\mathbf{m}}$, being a free parameter. We then intersect the divisor $D = 0$ of \mathcal{M} , whose complex dimension is,

$$\dim_{\mathbb{C}} \mathcal{M} - 1 = L + 2, \tag{C.4}$$

with an appropriate number of generic multi-linear hypersurfaces,

$$h_i(\mathbf{x}_1, \dots, \mathbf{x}_m) = 0, \quad \text{for } i = 1, \dots, L + 2, \tag{C.5}$$

so that the resulting solution set upon the slicing may only consist of a finite number of points,

$$\mathcal{I}_{\mathbf{h}} = \{x \in A \mid D(x) = 0, p_{\alpha}(x) = 0 \text{ for } 1 \leq \alpha \leq K, h_i(x) = 0 \text{ for } 1 \leq i \leq L + 2\}, \tag{C.6}$$

which can be obtained numerically on a computer. Here, the subscript \mathbf{h} collectively denotes the choice of generic hypersurfaces, (C.5). Then we demand that the numerator polynomial, N , should vanish when evaluated at each point $x = (\mathbf{x}_1, \dots, \mathbf{x}_m) \in \mathcal{I}_{\mathbf{h}}$, thereby obtaining a system of linear equations on the coefficient parameters, $c_{\mathbf{m}}$. The procedure should be repeated with different generic slicing choices, \mathbf{h} , and the additional constraints should augment the system until it becomes big enough so that yet another \mathbf{h} choice does not give rise to any more independent constraints. Once the system saturates in the aforementioned sense, we end up with a subspace of the available linear combinations for N , each of which will lead to a globally holomorphic section, $q_1 = N/D$. Note that such a linear system on $c_{\mathbf{m}}$ leads to a necessary condition for q_1 to be a global section over \mathcal{M} . It is also sufficient, however, given the genericity of the slicing choices.

Let us illustrate our numerical method with the gCICY in eq. (3.30),

$$X = \left[\begin{array}{c|cc} \mathbb{P}^3 & 2 & 0 & 2 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 2 & 0 \\ \mathbb{P}^1 & 1 & 2 & -1 \end{array} \right]; \quad \mathcal{M} = \left[\begin{array}{c|cc} \mathbb{P}^3 & 2 & 0 \\ \mathbb{P}^1 & 0 & 1 \\ \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^1 & 1 & 2 \end{array} \right], \tag{C.7}$$

by constructing sections,

$$q \in H^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(2, 1, 0, -1)), \tag{C.8}$$

for the last column of the configuration matrix. To begin with, we fix the complex structure of \mathcal{M} by choosing two sections, p_1 and p_2 , as generic polynomials with the right multi-degree. Moving on to q for the remaining complex structure of X , let us start with the rational form, (C.2), where the multi-degrees of N and D are $(2, 1, 0, 0)$ and $(0, 0, 0, 1)$, respectively. Then we choose a generic denominator, say,

$$D = 3x_4^0 - 7x_4^1, \tag{C.9}$$

that is linear in \mathbf{x}_4 and write the numerator as a linear combination,

$$N = \sum_{A=1}^{20} c_A \cdot \mathbf{m}_A(\mathbf{x}_1, \mathbf{x}_2), \tag{C.10}$$

of the 20 monomials \mathbf{m}_A with bi-degree $(2, 1)$ in \mathbf{x}_1 and \mathbf{x}_2 . The last choice we should make is $L + 2 = 3$ generic multi-linear polynomials, h_1 , h_2 and h_3 , each of which is a generic linear combination of 32 multi-linear monomials. For each generic choice of $\mathbf{h} = (h_1, h_2, h_3)$, an 18-point set $\mathcal{Z}_{\mathbf{h}}$ is obtained numerically. Consequently, we are given the corresponding 18 linear constraints on the 20 coefficient parameters, c_A . However, these 18 are not linearly independent in that when the system is solved there remains 8 free parameters. Thus, we repeat the procedure with another generic choice of \mathbf{h} and add the resulting 18 linear constraints to the constraint system. Now the system has 36 equations on 20 variables and ends up leading to a 2-parameter family of solutions. By repeating it with yet another \mathbf{h} choice, the system now consists of 54 linear constraints. However, the solution set is not constrained any further and is still parametrized by the same 2 free parameters. This means that we end up with the 2-dimensional subspace of linear combinations for N , which in turn results in the corresponding 2-dimensional space of sections for $q = N/D$. An independent computation of line bundle cohomology leads to $h^0(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(2, 1, 0, -1)) = 2$, which indicates the completeness of our numerical section construction method for this example.

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