Fourier Integral Operators on Non-Compact Manifolds

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Abstract

We consider Fourier integral operators on non-compact manifolds and their applications, in particular in spectral theory. Fourier integral operators appear naturally as the solution operators of certain pseudodifferential evolution equations, such as the Schrödinger equation or the wave equation.

For Euclidean space there are two important global pseudodifferential calculi: First there is the isotropic calculus, which contains the quantum harmonic oscillator, its inverse, and similar operators. We consider the solution operator to the dynamical Schrödinger equation with an isotropic pseudodifferential operator of order two and show how singularities and growth evolve with time. Moreover we show that for generic lower order perturbations of the harmonic oscillator the eigenvalues are more equally distributed then in the case of the unperturbed operator.

The second important calculus, the scattering calculus, contains the Laplacian plus a bounded potential on asymptotically Euclidean manifolds. We define a class of geometric distributions that are related to the solution operators of the Klein-Gordon equation of quantum field theory and contain certain distributions that are appear in the scattering theory of the Laplacian. We show that these distributions have a symbol structure that admits an invariantly defined order and the existence of a principal symbol.

Zusammenfassung

Wir betrachten Fourier-Integraloperatoren auf nicht-kompakten Mannigfaltigkeiten und deren Anwendungen, insbesondere in der Spektraltheorie. Fourier-Integraloperator treten natürlicherweise als Lösungsoperatoren von bestimmten pseudodifferenziellen Entwicklungsgleichungen auf, wie der Schrödingergleichung oder der Wellengleichung.

Für den euklidischen Raum gibt es zwei wichtige globale Pseudodifferentialkalküle: Zunächst gibt es den isotropen Kalkül, der den quantenharmonischen Oszillator, dessen Inverses und verwandte Operatoren enthält. Wir betrachten den Lösungsoperator der dynamischen Schrödingergleichung mit einem isotropen Pseudodifferentialoperator zweiter Ordnung und zeigen, wie sich Singularitäten und Wachstumsverhalten mit der Zeit entwickeln. Weiterhin zeigen wir, dass für generische Störungen niedriger Ordnung des harmonischen Oszillators die Eigenwerte asymptotisch sehr viel gleichmäßiger verteilt sind als für den ungestörten Operator.

Der zweite wichtige Kalkül, der Streukalkül, enthält den Laplace-Operator plus ein beschränktes Potential auf asymptotisch euklidischen Mannigfaltigkeiten. Wir definieren eine Klasse an geometrischen Distributionen, die verwandt sind zu Lösungsoperatoren der Klein-Gordon-Gleichung aus der Quantenfeldtheorie und gewisse Distributionen enthalten, die in der Streutheorie des Laplace-Operators auftaucht. Wir zeigen, dass diese Distributionen eine Symbolstruktur besitzen, die eine invariant definierte Ordnung implizieren, und dass ein Hauptsymbol existiert.

Keywords. Fourier integral operators, harmonic oscillator, Weyl asymptotics, propagation of singularities.

Contents

CHAPTER 1

Introduction

In this thesis, we consider pseudodifferential operators and Fourier integral operators on non-compact manifolds and associated problems in spectral theory and microlocal analysis. In particular, we prove an improved remainder estimate for the Weyl law for the perturbed harmonic oscillator and define Lagrangian distributions on asymptotically Euclidean manifolds.

First, we will review the spectral theory of the Laplacian Δ_g on a compact, connected, and oriented Riemannian manifold (M, g); if M has a boundary ∂M (which is assumed to be smooth), we impose Dirichlet boundary conditions, $u|_{\partial M} = 0$. We use the definition

$$\Delta_g u = -|g|^{-1/2} \sum_{j,k} \partial_j \left(|g|^{1/2} g^{jk} \partial_k \right) u,$$

where $|g|$ is the determinant and g^{jk} is the inverse of the metric tensor.

It is well-known that the Laplacian is a self-adjoint operator and has discrete spectrum consisting of eigenvalues of finite multiplicity which accumulates at infinity. The Laplacian is positive on $L^2(M)$, therefore the spectrum is contained in the positive reals and we may write the eigenvalues (counted with multiplicity) as

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \to \infty.$$

It is rarely possible to calculate the eigenvalues explicitly. Therefore, we would like to understand the asymptotic behavior of the eigenvalues λ_k as k tends to infinity. Define $N(\lambda)$ to be the number of eigenvalues that are smaller than λ.

Asymptotics for $N(\lambda)$ are interesting for several different reasons. The eigenvalues of the Laplacian with Dirichlet boundary conditions can be thought of as the overtones (harmonics) of an idealized drum with shape $\Omega \subset \mathbb{R}^2$ a bounded domain with regular boundary. The eigenfunction u_k of the eigenvalue λ_k represents the displacement of the membrane of the drum and a solution of the wave-equation is given by $u(t, x) = \cos(t\sqrt{\lambda_k})\, u_k(x)$. In the three-dimensional case, one may think of the air vibrating in a wind instrument, such as a clarinet or a flute. If M is a three-dimensional rectangular cuboid, then Lord Rayleigh in "The Theory of Sound" (1877) showed that $N(\lambda)$ is asymptotic to $\mathrm{vol}(M)\lambda^3$ as $\lambda \to \infty$, where $\mathrm{vol}(M)$ denotes the volume of the cuboid.

A completely different physical problem leads to the same mathematical concepts. Namely, in thermodynamics the amount of energy emitted by a body is determined by the high-energy spectrum of the electromagnetical waves. The electromagnetical waves correspond to the soundwaves and the high-energy spectrum is the equivalent to the high overtones of a musical instrument. From physical intuition and experiments of black-body radiation it is clear that the asymptotic $N(\lambda) \sim V\lambda^3$ should be independent of the precise shape of the body. In 1911, Hermann Weyl proved that this conjecture is true just one year after it was posed by Hendrik Lorentz and it is now known as *Weyl's law*.

There was significant effort to extend this result to a very general setting, in particular to compact manifolds with boundary of arbitrary dimension. Stated slightly different, it was shown that

$$N(\lambda) = \mathrm{vol}(M)\lambda^{d/2} + o(\lambda^{d/2}),^{1} \qquad \text{as } \lambda \to \infty. \tag{1.1}$$

It is now natural to ask what the sharpest possible error estimate in (1.1) is. In the case M is a closed manifold, meaning that $\partial M = \emptyset$, Hörmander [26] used the theory of microlocal analysis and in particular Fourier integral operators to show that the error estimate is

$$N(\lambda) = \mathrm{vol}(M)\lambda^{d/2} + O(\lambda^{(d-1)/2})$$

and this result is sharp by considering the (explicit) eigenvalues on the sphere (see also Levitan [36]). Assuming that the set of periodic geodesics has measure zero, Duistermaat and Guillemin [13] even improved this further to $o(\lambda^{(d-1)/2})$.

The basic idea of microlocal analysis is to relate properties of differential operators to properties of a classical mechanical system. In the case of the Laplacian Δ_g, one is for instance interested in the spectrum, regularity of the associated wave equation or asymptotics of the heat equation. The classical system is defined by the principal symbol $p(x, \xi) = |\xi|^2_{g(x)}$ of the Laplacian and one can analyze for instance the set of periodic orbits of the Hamiltonian flow associated to p. In the case of the Weyl asymptotic sketched above, Duistermaat–Guillemin used that the Hamiltonian flow of the principal symbol p of the Laplacian Δ_g is nothing but the geodesic flow on T^*M.

One of the main ingredients to such results is the definition of a suitable calculus of *pseudodifferential operators*. These are naively defined by taking a "good" function $a \in \mathcal{C}^\infty(T^*M)$ and replacing ξ by $-i\partial_x$. Of course this cannot work in general, since multiplication is commutative, whereas differentiation and multiplication do not commute. The Fourier transform $\mathcal{F}$ turns differentiation into multiplication and therefore, we may define

$$a(x, D)u(x) = \mathcal{F}^{-1}_{\xi\to x} a(x, \xi)\mathcal{F}_{y\to\xi} u(y).$$

It turns out that there are better ways to *quantize* symbols. Now, we want to discuss, what we mean by a "good" symbol. Differential operators (with smooth coefficients) correspond to symbols of the form

$$a(x, \xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha,$$

[1] For the notation used, see Index of Notation.

where a_α are smooth functions. The natural generalization is to allow functions that have an asymptotic expansion in homogeneous terms in ξ. If we define such a calculus of pseudodifferential operators, the inverses of certain differential operators such as the Laplacian are also in this calculus and properties of the inverses can be deduced from constructing approximate inverses (parametrices).

The Schwartz kernel of such a pseudodifferential operator is of the form

$$(2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} a(x,\xi)\, d\xi$$

and it can be shown that these operators do not increase the singular support (the set of points, where a function is not smooth). If we want to consider evolution equations such as the wave equation it is clear that these operators do not suffice. Already the solution operator to the transport equation $(\partial_t - \partial_x)u(t,x) = 0$ with initial data $u(0,x) = u_0(x)$ is given by

$$u(t,x) = u_0(x+t).$$

Written as an integral operator, the solution operator is given by

$$u(t,x) = \int_{\mathbb{R}^d} e^{i(x-y+t)\xi} u_0(y)\, dy\, d\xi.$$

We see that we have to allow more general phase functions ϕ than just $(x-y)\xi$ to describe the solution operators to evolution equations. These considerations led to the development of *Fourier integral operators* (cf. [14, 27]), which are more involved to define because the phase functions are not canonical and one has to identify the correct underlying geometric objects. Roughly speaking, Fourier integral operators are operators defined by integral kernels such that the wavefront set[2] is contained in a Lagrangian submanifold on T^*M. The theory of pseudodifferential operators and Fourier integral operators is well-known for compact manifolds M. On non-compact manifolds various problems arise.

On $\mathbb{R}^d$, the Laplacian does not have discrete spectrum because the embedding of Sobolev spaces $H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is not compact anymore. There are two ways to deal with this fact: First, we may instead consider differential operators with certain growth at infinity, with the prototypical example being the harmonic oscillator

$$H_0 = \frac{1}{2}(\Delta + |x|^2),$$

which models a quantum particle, such as an electron, confined in a potential $V(x) = |x|^2$. In this case the eigenvalues can be explicitly calculated and they are given by sums of integers,

$$\lambda_\alpha = \sum_{j=1}^{d} \alpha_j + d/2,$$

[2]This is a refinement of the singular support also measuring the directions in which functions are not smooth.

where $\alpha \in \mathbb{N}^d$ is a multiindex. The high multiplicities of the eigenvalues correspond to the symmetry of the underlying Hamiltonian system with Hamilton function

$$p_2(x, \xi) = \frac{1}{2}(|\xi|^2 + |x|^2).$$

The classical calculus of pseudodifferential operators is not suited for studying problems on $\mathbb{R}^d$ and we have to use a different calculus. For problems related to the harmonic oscillator we use symbols that admit an asymptotic expansion not in ξ, but jointly in (x, ξ). This can be viewed as radially compactifying $T^*\mathbb{R}^d$ in all variables. This idea leads to the *isotropic* or Shubin calculus of pseudodifferential operators. The corresponding class of Fourier integral operators was defined by Helffer–Robert [22–24].

It is also possible to analyze the continuous spectrum of the operator Δ on a non-compact manifold (M, g). The first challenge is to identify a suitable class of non-compact manifolds. This is usually done by compactifying the manifold and assuming that the Riemannian metric is degenerate in a very specific way at the boundary. The manifolds we will be interested in are asymptotically Euclidean (or *scattering manifolds*) in this sense, meaning that near the boundary the metric is given by

$$g_M = \frac{d\rho^2}{\rho^4} + \frac{h(\rho, x)}{\rho^2}$$

where h is symmetric and restricts to a Riemannian metric on the boundary $\partial M = \{\rho = 0\}$ and $\rho \geq 0$ is a boundary defining function. The notion of scattering manifolds and the corresponding calculus of pseudodifferential operator on scattering manifolds was introduced by Melrose [42] and used to prove meromorphic continuation of the resolvent of the Laplacian. Melrose and Zworski [45] defined a class of distributions, which encode lack of decay near spatial infinity. They used these *Legendrian distributions* to prove that the scattering matrix is a Fourier integral operator on the boundary.

The case of the harmonic oscillator and related operators is discussed in the Chapters 2-4. In Chapter 2 we give an introduction to the calculus of isotropic pseudodifferential operators and the spectral theory of perturbations of the harmonic oscillator. In Chapter 3 we prove a refined remainder estimate for perturbations of the isotropic harmonic oscillator, meaning that all frequencies are the same and finally in Chapter 4 we discuss the propagation or rather recurrence of classical singularities for second order isotropic pseudodifferential operators.

The scattering calculus and the natural class of Lagrangian distributions are discussed in the Chapters 5-6. In Chapter 5 we define the class of asymptotically Euclidean manifolds and the calculus of scattering pseudodifferential operators. In Section 5.4 we illustrate with two examples possible applications of an Fourier integral operator calculus based on Lagrangian distributions in the setting of scattering manifolds. We develop the theory of Lagrangian distributions on asymptotically Euclidean manifolds in Chapter 6.

We present results that have been published in [4, 10, 11].

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CHAPTER 2

Spectral Theory and Pseudodifferential Operators

2.1. Method of Stationary Phase

We will recall the method of stationary phase, which is used in various places. Sometimes the stationary phase theorem is used directly, and sometimes it is more a guiding principle. We follow roughly the presentation of Hörmander [30, Section 7.7].

The goal is to estimate integrals of the form

$$I(\lambda) = \int e^{i\lambda\phi(x)} a(x) dx, \tag{2.1}$$

where $a \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ real-valued. Denote by K the support of a.

2.1.1. Non-Stationary Phase

The trivial estimate is $|I(\lambda)| \leq |K| \sup |a(x)|$. For large λ the term $e^{i\lambda\phi}$ oscillates rapidly, so we expect to obtain better bounds in the limit $\lambda \to \infty$. First, we observe that if $d\phi \neq 0$ on K, then we have arbitrary decay in λ:

Theorem 2.1.1 (Theorem 7.7.1 in Hörmander [30]). *Let $K \subset \mathbb{R}^d$ be compact, $a \in \mathcal{C}_c^\infty(K)$, and $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ real-valued. For every $k \in \mathbb{N}$,*

$$\lambda^k |I(\lambda)| \lesssim \sum_{|\alpha| \leq k} \sup_{x \in K} |\partial^\alpha a(x)| |d\phi(x)|^{|\alpha| - 2k}, \qquad \lambda > 0.$$

Therefore, using a partition of unity and a suitable choice of coordinates, we may assume that there is an isolated *stationary point*, that is $0 \in K$ and $d\phi = 0$ if and only if $x = 0$. A stationary point of ϕ is called non-degenerate if $\partial^2\phi(0)$ has rank d.

The easiest example of such a phase function is a non-degenerate quadratic form $\phi(x) = \langle Ax, x \rangle$, for a non-degenerate symmetric matrix A.

2.1.2. Quadratic Phase Functions

If A is a non-degenerate $d \times d$ matrix, we can define the operator

$$P = e^{i\langle AD,D\rangle}$$

as a Fourier multiplier. We will calculate an asymptotic expansion of P.

Proposition 2.1.2 (Theorem 7.6.2 in Hörmander [30]). *Assume that $u \in \mathcal{S}(\mathbb{R}^d)$ and $s > d/2$. For all $k \in \mathbb{N}$, we have the estimate*

$$\|e^{i\langle AD,D\rangle}u(x) - \sum_{j=0}^{k-1} \frac{1}{j!}\langle iAD, D\rangle^j u(x)\|_{L^\infty}^2 \lesssim \frac{1}{k!}\sum_{|\alpha|\le s} \|\langle AD, D\rangle^k D^\alpha u\|_{L^2}^2.$$

The idea of the proof is to use the Taylor expansion $e^{ix} = \sum_{k=0}^N \frac{i^k x^k}{k!} + R_{N+1}(x)$ and estimate the remainder in integral form using a Sobolev inequality to obtain L^∞-bounds. This Proposition can now be used to calculate an asymptotic expansion for oscillatory integrals with quadratic phase function $\phi(x) = \langle Qx, x\rangle/2$.

Theorem 2.1.3. *Let Q be a real non-degenerate symmetric matrix. For all positive integers k,*

$$\left|\int e^{i\lambda\langle Qx,x\rangle/2}a(x)dx - e^{\frac{i\pi}{4}\operatorname{sgn} Q}\left(\det\frac{\lambda Q}{2\pi}\right)^{-1/2} T_k(\lambda)a(0)\right| \lesssim \lambda^{-d/2-k}\sum_{|\alpha|\le 2k+d+1}\|\partial^\alpha a\|_{L^\infty},$$

where the differential operators T_k are given by

$$T_k(\lambda)a(x) = \sum_{j=0}^{k-1}\frac{1}{j!}\left(\frac{\langle Q^{-1}D, D\rangle}{2i\lambda}\right)^j a(x).$$

Proof. The Fourier transform of $f(x) = e^{i\lambda\langle Qx,x\rangle/2}$ is given by

$$\hat{f}(\xi) = e^{\frac{i\pi}{4}\operatorname{sgn} Q}\left(\det\frac{\lambda Q}{2\pi}\right)^{-1/2} e^{-i\lambda\langle Q^{-1}\xi,\xi\rangle/2}.$$

Therefore,

$$\int e^{i\lambda\langle Qx,x\rangle/2}a(x)dx = e^{\frac{i\pi}{4}\operatorname{sgn} Q}\left(\det\frac{\lambda Q}{2\pi}\right)^{-1/2} e^{-i\lambda\langle Q^{-1}D,D\rangle/2}u(x)$$

and the claim follows from Proposition 2.1.2. □

2.1.3. Stationary Phase for General Phase Functions

This theorem can be used to prove the general case using Morse lemma, which states that one can always choose coordinates such that, at the stationary point, the phase function is a quadratic form (cf. Grigis–Sjöstrand [17] and Zworski [69]).

Theorem 2.1.4 (Theorem 7.7.5 in Hörmander [30])**.** *Let $\phi \in \mathcal{C}^\infty$ real-valued with non-degenerate stationary point $x = x_0$ and $d\phi \neq 0$ on $K \setminus \{x_0\}$. There are differential operators A_{2j} of order $\leq 2j$ such that for every $k \in \mathbb{N}$,*

$$\left| I(\lambda) - e^{i\lambda\phi(x_0)} \sum_{j=0}^{k-1} \lambda^{-d/2-j} A_{2j}(D_x)a(x_0) \right| \lesssim \lambda^{-d/2-k} \sum_{|\alpha| \leq 2k+d+1} \|\partial^\alpha a\|_{L^\infty}.$$

The zeroth order differential operator is the multiplication operator

$$(A_0 u)(x_0) = (2\pi)^{d/2} e^{i\frac{\pi}{4}\operatorname{sgn}\partial^2\phi(x_0)} \left|\det \partial^2\phi(x_0)\right|^{-1/2} u(x_0).$$

If the second derivatives of the phase vanish, we may still calculate the order of decay, but it becomes difficult to calculate the leading order constant (cf. Stein [60, Proposition 5, p. 342]):

Theorem 2.1.5. *If there is a multiindex $\alpha \in \mathbb{N}^d$ with $|\alpha| > 0$ such that the phase ϕ satisfies*

$$|\partial_x^\alpha \phi(x)| \geq 1$$

on K, then

$$|I(\lambda)| \leq C(\phi)\lambda^{-1/|\alpha|}(\|a\|_{L^\infty} + \|\nabla a\|_{L^1}).$$

2.2. Harmonic Oscillator

The basic idea of quantum mechanics is to take a classical energy (Hamiltonian function) and associate to it a self-adjoint operator on a Hilbert space. The classical energy of a one-particle system is given by $E = E_{\text{kin}} + E_{\text{pot}}$ the sum of the kinetic and potential energy. In classical Hamiltonian mechanics the kinetic energy is assumed to be $E_{\text{kin}} = \frac{1}{2}|\xi|^2$, where we assume the mass of the particle to be normalized to $m = 1$. The potential energy comes from the physical configuration and only depends on the position: $E_{\text{pot}}(x, \xi) = V(x)$. Thus, we are led to consider Hamiltonian functions of the form

$$p(x, \xi) = \frac{1}{2}|\xi|^2 + V(x).$$

The easiest case of a potential is the harmonic oscillator $V(x) = |x|^2/2$.

In this case the quantization just replaces covariables ξ_j by partial derivatives $-i\partial_{x_j}$. Therefore, we want to consider the quantum harmonic oscillator on $\mathbb{R}^d$:

$$H_0 = \frac{1}{2}(\Delta + |x|^2).$$

The eigenvalues of the operator H_0, viewed as a unbounded linear operator on the Hilbert space $L^2(\mathbb{R}^d)$,[1] correspond to the quantized energy levels of the particle and the eigenfunctions to the wavefunction.

2.2.1. Eigenvalues and Eigenfunctions

The eigenvalues of the quantum harmonic oscillator can be explicitly calculated. First, assume that $d = 1$. We define the creation and annihilation operators

$$A_+ = D_x + ix,$$
$$A_- = D_x - ix.$$

We calculate that

$$A_+A_- = 2H_0 - 1,$$
$$A_-A_+ = 2H_0 + 1.$$

One notices that $v_0 = e^{-x^2/2}$ is an eigenfunction of H_0 with eigenvalue $1/2$, because $A_-v_0 = 0$. We set

$$v_n = A_+^n v_0$$

and we show by induction that $H_0 v_n = (n + 1/2)v_n$. The functions v_n are orthogonal and thus $u_n = v_n/\|v_n\|_{L^2}$ is orthonormal system. It remains to show that the set of eigenfunctions $\{u_n\}$ is complete (cf. Zworski [69, Theorem 6.2] for details). The functions $u_n(x)$ can be written as $u_n = e^{-x^2/2}H_n(x)$, where H_n is a polynomial of degree n. The polynomials H_n are called the *Hermite polynomials.*

For arbitrary d, we note that the eigenvalues are given by

$$\lambda_\alpha = |\alpha| + d/2.$$

with eigenfunctions

$$u_\alpha(x) = e^{-|x|^2/2} \prod_{j=1}^{d} H_{\alpha_j}(x_j).$$

[1]The domain of H_0 is the Sobolev space $H^2_{\mathrm{iso}}(\mathbb{R}^d)$, defined in Section 2.4.5.

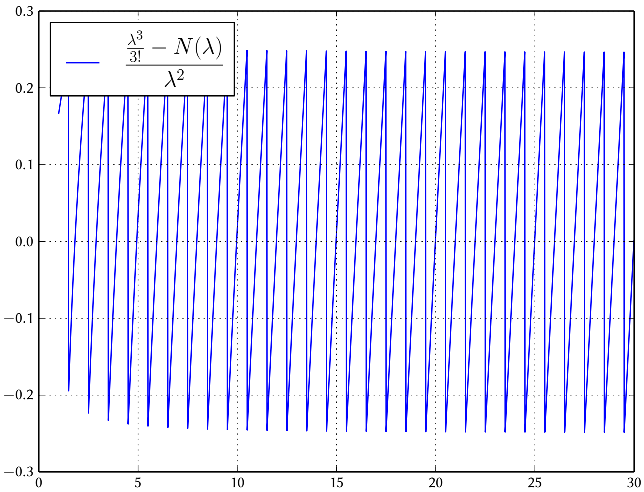

Figure 2.1.: Plot of the rescaled error term $O(\lambda^{d-1})$ of the harmonic oscillator in $d = 3$.

2.2.2. Asymptotic Formula for the Eigenvalues

Using the explicit formula for the eigenvalues, we see that the counting function $N(\lambda)$, which is defined by $N(\lambda) = \#\{j\colon \lambda_j \leq \lambda\}$, is given by

$$N(\lambda) = \#\{\alpha \in \mathbb{N}^d\colon \sum_{j=1}^{d} \alpha_j \leq \lambda - d/2\}.$$

This means that $N(\lambda)$ counts the number of lattice points in the d-simplex with sides $\lambda - d/2$.

We obtain the asymptotic formula

$$N(\lambda) = \frac{\lambda^d}{d!} + O(\lambda^{d-1}),$$

as $\lambda \to \infty$. As it is suggested by Figure 2.1 the error estimate is sharp. In fact, the multiplicity of the eigenspace for $\lambda = k + d/2$ is given by

$$p(k, d) = \#\{\alpha \in \mathbb{N}^d\colon \sum_{j} \alpha_j = k\} = \binom{d+k-1}{k}.$$

Writing

$$p(k, d) = \frac{(d+k-1)!}{(d-1)!k!} = \frac{1}{(d-1)!}(k+d-1)(k+d-2)\cdots(k+1)$$

we see that $p(k,d) \sim \frac{k^{d-1}}{(d-1)!}$, so the jumps are of order k^{d-1} and therefore the asymptotics cannot be further improved.

We can obtain this result without calculating the eigenvalues, by using similar arguments as in Chapter 3, which does not involve any pseudodifferential operators in this case.

2.3. Tauberian Theorems

Since it is rarely possible to explicitly construct a solution to the Schrödinger equation and calculate the inverse Fourier transform of its trace, we have to deal with approximations of the solution operator in a suitable sense and compare this approximated operator to the exact solution operator. This comparision yields an estimate of the counting function via a *Tauberian theorem.*

We consider some essentially self-adjoint operator H on $L^2(\mathbb{R}^d)$ with domain $\mathcal{C}_c^\infty(\mathbb{R}^d)$. The self-adjoint extension is also denoted by H. We assume that it has discrete positive spectrum

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

Let E_λ denote the spectral projector of H onto $(-\infty, \lambda]$. The counting function is given by $\operatorname{Tr} E_\lambda$. There are several different approaches to obtain Weyl-type asymptotics (cf. Hörmander [26]):

1. The Laplace transform

$$e^{-tH} = \int e^{-t\lambda} dE_\lambda, \quad t > 0.$$

   This transform yields the heat kernel and the corresponding Tauberian argument is due to Karamata. This method was used by Minakshisundaram–Pleijel [47].

2. The Mellin transform

$$\zeta_H(s) = \int \lambda^{-s} dE_\lambda,$$

   which yields the Zeta-function. Here, the Tauberian theorem of Ikehara is used (cf. Shubin [59]). This and the method above are robust and give information on the leading asymptotics in various settings, but the error term is not optimal.

3. The Fourier transform

$$e^{-itH} = \int e^{-it\lambda} dE_\lambda.$$

   This transform gives the best results, but the construction of a parametrix is much harder and requires a thorough analysis of the microlocal structure. The Tauberian theorem is much simpler than in the above cases.

Since we are interested in the precise error estimate, we will only consider the last transform.

2.3.1. Hörmander's Tauberian Theorem

Now, we will state the Tauberian theorem and sketch its proof, for details we refer to the second appendix of [56], see also Safarov [55]. The basic idea is to use the Fourier transform and a compactly supported function $\hat{\rho}$ to cut out the high frequencies of a non-smooth function to obtain a smooth function with the same asymptotic behavior, $N(\lambda) \sim \mathcal{F}^{-1}(\hat{\rho}\hat{N})(\lambda)$ in a suitable sense.

We fix a real-valued function $\rho \in \mathcal{S}(\mathbb{R})$ with the following properties

- $\rho(\lambda) > 0$ for all $\lambda \in \mathbb{R}$,
- $\hat{\rho}(0) = 1$,
- $\operatorname{supp} \hat{\rho}$ is compact, and
- ρ is even.

It is proved in [31, Section 17.5] that such a function exists.

We denote by F_+ the set of all real-valued monotone nondecreasing functions N on $\mathbb{R}$ such that $N = 0$ on $(-\infty, 0)$ and N is polynomially bounded. These functions are the natural extension of counting functions. Note that we have no restrictions on the regularity of N.

Theorem 2.3.1 (Hörmander [26]). *Let $n \in \mathbb{R}_+$. If $N \in F_+$ and*

$$(dN * \rho)(\lambda) = O(\lambda^n),$$

then

$$|N(\lambda) - (N * \rho)(\lambda)| = O(\lambda^n).$$

The usual usage of this theorem is to write

$$N(\lambda) = \mathcal{F}^{-1}(\hat{\rho}\hat{N})(\lambda) + O(\lambda^n),$$

where $\widehat{dN}$ is nothing but the trace of the Schrödinger propagator e^{-itH}. Choosing the support of $\hat{\rho}$ small enough shows that the main contribution of the asymptotic comes from the singularity at $t = 0$.

For a complete proof, we refer to [56].

Sketch of Proof. The main step in the proof is to estimate $|N(\lambda+s) - N(\lambda)| \lesssim (1+|s|)^{1+|n|}\lambda^n$ uniformly for all $s \in \mathbb{R}$. This is further reduced to the estimates

$$N(\lambda + s) - N(\lambda) \lesssim s^{1+|n|}\lambda^n,$$
$$N(\lambda) - N(\lambda - s) \lesssim s^{1+|n|}\lambda^n$$

for natural numbers λ, s. By the assumption, we have for some $C > 0$,

$$\begin{aligned} N(\lambda+1) - N(\lambda-1) &= \int_{\lambda-1}^{\lambda+1} dN(\mu) \\ &\leq C \int \rho(\lambda-\mu) dN(\mu) \\ &= C(dN * \rho)(\lambda) = O(\lambda^n). \end{aligned}$$

The claim of Theorem 2.3.1 follows from

$$\begin{aligned} |(N*\rho)(\lambda) - N(\lambda)| &= |\int (N(\lambda-\mu) - N(\lambda))\rho(\mu)d\mu| \\ &\leq \int |N(\lambda-\mu) - N(\lambda)|\rho(\mu)d\mu \\ &\leq C\lambda^n. \end{aligned}$$

□

The extension due to Duistermaat–Guillemin is that under an additional assumption on the singularities at $t \neq 0$, the result can be improved slightly:

Theorem 2.3.2 (Duistermaat–Guillemin [13] and Safarov [54]). *If N satisfies the assumption of Theorem 2.3.1 and if for all $\chi \in \mathcal{S}(\mathbb{R})$ with $\hat{\chi} \in \mathcal{C}_c^\infty(\mathbb{R})$ and $0 \notin \operatorname{supp} \hat{\chi}$ it holds that*

$$(dN * \chi)(\lambda) = o(\lambda^n),$$

then the error is given by

$$|N(\lambda) - (N*\rho)(\lambda)| = o(\lambda^n).$$

If the singularities at $t \neq 0$ are of the same strength as the one at $t = 0$, we can still give an improved remainder estimate, but this becomes more complicated since both vertical and horizontal directions have to be compared:

Theorem 2.3.3 (Safarov [54]). *Let $N_j \in F_+$ with $(dN_j * \rho)(\lambda) = O(\lambda^n)$ for $j = 1, 2$. Assume that*

$$(N_2 * \rho)(\lambda) = (N_1 * \rho)(\lambda) + o(\lambda^n)$$

and

$$(dN_2 * \chi)(\lambda) = (dN_1 * \chi)(\lambda) + o(\lambda^n)$$

for all $\chi \in \mathcal{S}$ such that $\hat{\chi} \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\operatorname{supp} \hat{\chi} \subset (0, \infty)$. Then, there exists a positive function f such that $f \in o(1)$ and

$$N_1(\lambda - f(\lambda)) - \lambda^n f(\lambda) \leq N_2(\lambda) \leq N_1(\lambda + f(\lambda)) + \lambda^n f(\lambda).$$

2.3.2. Mehler's Formula

We want to use Hörmander's Tauberian theorem, Theorem 2.3.1, and thus we are led to consider the solution operator (propagator) of the time-dependent Schrödinger equation:

$$\begin{cases} (i\partial_t - H_0)u(t) = 0 \\ u(0) = u_0. \end{cases} \tag{2.2}$$

The solution operator of the Schrödinger equation is denoted by $U_0(t) = e^{-itH_0}$, that is $u(t) = e^{-itH_0}u_0$ solves (2.2).

The propagator of the quantum harmonic oscillator H_0 can be computed explicitly (cf. Grigis–Sjöstrand [17]).

Proposition 2.3.4. *In the case $d = 1$, the propagator of the quantum harmonic oscillator is given by the kernel*

$$U_0(t, x, y) = (2\pi)^{-1}\cos(t)^{-1/2}\int e^{i(\phi_2(t,x,\eta)-y\eta)}d\eta, \tag{2.3}$$

for $t \in (-\pi/2, \pi/2)$, where $\phi_2(t, x, \eta) = \frac{1}{\cos(t)}\left(x\eta - \frac{1}{2}\sin(t)(x^2 + \eta^2)\right)$.

Furthermore, the propagator satisfies

$$U_0(\pi/2) = \frac{e^{-i\pi/4}}{(2\pi)^{d/2}}\mathcal{F}$$

and therefore, by Fourier inversion formula, we obtain that

$$U_0(2\pi) = -U_0(0),$$

which in turn implies by the group property that $U_0(t + 2\pi) = -U_0(t)$ for all $t \in \mathbb{R}$.

For higher dimensions $d > 1$, the propagator is given by

$$U_0(t, x, y) = (2\pi)^{-d}(-1)^{dk}\cos(t)^{-d/2}\int e^{i(\phi_2(t,x,\eta)-\langle y,\eta\rangle)}d\eta, \tag{2.4}$$

with $\phi_2(t, x, \eta) = \frac{1}{\cos(t)}\left(\langle x, \eta\rangle - \frac{1}{2}\sin(t)(|x|^2 + |\eta|^2)\right)$ for $t \in (2\pi k - \pi/2, 2\pi k + \pi/2)$.

2.3.3. Alternative Proof of the Weyl Asymptotics

Theorem 2.3.5. *The counting function of the eigenvalues of the quantum harmonic oscillator satisfies*

$$N(\lambda) = \frac{\lambda^d}{d!} + \frac{\lambda^{d-1}}{(d-1)!}f(\lambda) + o(\lambda^{d-1}),$$

where f denotes a shifted sawtooth function, $f(\lambda) = \lfloor \lambda + d/2 \rfloor - \lambda - (d-1)/2$.

Proof. First, we calculate the inverse Fourier transform of the Schrödinger trace near $t = 0$. Let $\rho \in \mathcal{S}(\mathbb{R})$ such that $\hat{\rho} \in \mathcal{C}_c^\infty$ and $\operatorname{supp} \hat{\rho} \subset (-\pi/2, \pi/2)$. The kernel of $\hat{\rho}(t) \cdot U_0(t)$ is given by

$$(2\pi)^{-d} \int e^{i(\phi_2(t,x,\eta) - y\eta)} \cos(t)^{-d/2} \hat{\rho}(t)\, d\eta.$$

We have to calculate the inverse Fourier transform of the trace:

$$\begin{aligned}(dN * \rho)(\lambda) &= (2\pi)^{-2d} \int e^{i(\phi_2(t,x,\eta) - x\eta + t\lambda)} \cos(t)^{-d/2} \hat{\rho}(t)\, d\eta\, dx\, dt, \\ &= (2\pi)^{-2d} \lambda^d \int e^{i\lambda(\phi_2(t,x,\eta) - x\eta + t)} \cos(t)^{-d/2} \hat{\rho}(t)\, d\eta\, dx\, dt,\end{aligned}$$

where we have used a rescaling $(x, \eta) \mapsto (\lambda^{1/2} x, \lambda^{1/2} \eta)$. The phase is stationary at the points where

$$\begin{cases} \partial_t \phi_2(t, x, \eta) = -1 \\ \partial_x \phi_2(t, x, \eta) = x \\ \partial_\eta \phi_2(t, x, \eta) = \eta. \end{cases}$$

That is $t = 0$ and $p_2(x, \eta) = 1$ for $p_2(x, \eta) = (1/2)(|x|^2 + |\eta|^2)$.

Changing from (x, ξ) to polar coordinates (r, θ) and applying the Lemma of stationary phase (cf. [11, Proposition 6.1]), we conclude that the leading asymptotic is[2]

$$(dN * \rho)(\lambda) = (2\pi)^{-d} d\lambda^{d-1} \int_{\{p_2 \leq 1\}} dx\, d\eta + o(\lambda^{d-1}).$$

The ball of radius $\sqrt{2}$ in dimension $2d$ has volume

$$\int_{\{|x|^2 + |\xi|^2 \leq 2\}} dx\, d\xi = \frac{(2\pi)^d}{d!}.$$

Using the basic Tauberian theorem (Theorem 2.3.1) this already proves that $N(\lambda) = \lambda^d/d! + O(\lambda^{d-1})$.

For the refined asymptotics, we note that $U_0(t)$ is periodic with period 2π (modulo a sign) and therefore the singularities at $t = 2\pi$ are exactly the same as for $t = 0$. We define the function N_1 by

$$N_1(\lambda) = \frac{\lambda^d}{d!} + \frac{\lambda^{d-1}}{(d-1)!} \sum_{k \in \mathbb{Z} \setminus 0} \frac{(-1)^{dk} e^{2\pi i k \lambda}}{2\pi i k}. \tag{2.5}$$

[2] Note that there is no contribution of order λ^{d-1} because the symbol is homogeneous of degree two.

and note that

$$dN_1(\lambda) = \frac{\lambda^{d-1}}{(d-1)!} \sum_{k \in \mathbb{Z}} (-1)^{dk} e^{2\pi i k \lambda} + o(\lambda^{d-1}).$$

Using that $\sin(x) = (e^{ix} - e^{-ix})/2i$, we obtain

$$\begin{aligned} \sum_{k \in \mathbb{Z} \setminus 0} \frac{e^{2\pi i k \lambda}}{2\pi i k} &= \sum_{k=1}^{\infty} \frac{\sin(2\pi k \lambda)}{\pi k} \\ &= \lfloor \lambda \rfloor - \lambda + 1/2. \end{aligned} \tag{2.6}$$

We note that $(-1)^{dk} e^{2\pi i k \lambda} = e^{2\pi i k(\lambda + d/2)}$, inserting this into (2.6) yields that the series in (2.5) is $f(\lambda) = \lfloor \lambda + d/2 \rfloor - \lambda - (d-1)/2$. Thus,

$$N_1(\lambda) = \gamma_0 \lambda^d + \frac{1}{d \cdot d!} \lambda^{d-1} f(\lambda) + o(\lambda^{d-1}).$$

By the general refined Tauberian theorem 2.3.3 the assertion follows. □

Remark 2.3.6. For $d = 1$ this gives a complete description of the spectrum: $N(\lambda) = \lfloor \lambda + 1/2 \rfloor$ for $\lambda > 0$.

Remark 2.3.7. The fact that $U_0(2\pi) = -U_0(0)$ for odd dimensions caused that there is a shift of $1/2$ for the eigenvalue clusters.

2.4. Isotropic Calculus

Before we turn to the isotropic calculus, we consider the underlying dynamical system of the harmonic oscillator. This shows that the isotropic calculus is the natural calculus associated to the harmonic oscillator.

2.4.1. Hamiltonian Vector Fields

Let $p \in \mathcal{C}^\infty(\mathbb{R}^{2d})$, we define $\mathsf{H}_p := \partial_\xi p(x,\xi)\partial_x - \partial_x p(x,\xi)\partial_\xi$ the Hamiltonian vector field associated to p. The Hamiltonian vector field is related to the Poisson bracket, by

$$\mathsf{H}_p f = \{p, f\}.$$

We denote the flow of H_0 by $\exp(t\mathsf{H}_p)$. One of the most important properties of the Hamiltonian flow is that it preserves the function p:

$$p(\exp(t\mathsf{H}_p)(x,\xi)) = p(x,\xi)$$

for all $t \in \mathbb{R}$.

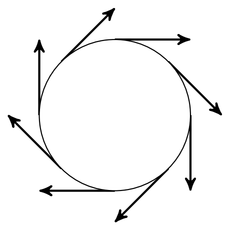

Figure 2.2.: The Hamiltonian vector field H_0 on $\{x^2 + \xi^2 = 1\}$.

For $p_2 = (1/2)(|x|^2 + |\xi|^2)$ the Hamiltonian vector field is $\mathsf{H}_0 = \xi\partial_x - x\partial_\xi$. Its flow is given by $(x(t), \xi(t)) = \exp(t\mathsf{H}_0)(x_0, \xi_0)$ with

$$\begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_0 \\ \xi_0 \end{pmatrix}.$$

Since the flow is homogeneous of degree 1 *jointly* in (x, ξ), we want to define a calculus of pseudodifferential operators with the property that the asymptotic expansion is jointly in (x, ξ). A symbol a composed with the Hamiltonian flow $\exp(t\mathsf{H}_0)$, $a \circ \exp(t\mathsf{H}_0)$, is again in the calculus.

2.4.2. Isotropic Symbol Estimates

The symbol estimates are due to Shubin [59] (cf. also Helffer [22] and Hörmander [28]).

Definition 2.4.1. A function $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ is an isotropic symbol of order $m \in \mathbb{R}$ if for all $\alpha, \beta \in \mathbb{N}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim_{\alpha,\beta} \langle (x,\xi) \rangle^{m-|\alpha|-|\beta|}.$$

We denote the set of all isotropic symbols of order m by Γ^m.

For $\gamma \in \mathbb{N}^{2d}$, we define the corresponding seminorms

$$\|a\|_{\gamma,\Gamma^m} := \sup_{z \in \mathbb{R}^{2d}} |\partial_z^\gamma a(z)| \langle z \rangle^{-m+|\gamma|}.$$

These define a Fréchet topology on Γ^m.

As in the case of the Kohn-Nirenberg symbols, we will be mainly interested in the subclass of classical isotropic symbols. Choose a function $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ such that $\chi \equiv 1$ in the ball of radius $1/4$ centered at the origin and $\chi \equiv 0$ outside of the ball of radius $1/2$ centered at the origin. A symbol $a \in \Gamma^m$ is called *classical*, $a \in \Gamma^m_{\mathrm{cl}}$, if there exist homogeneous functions a_{m-j} in (x, ξ) of order $m - j$ such that

$$a \sim \sum_j a_{m-j},$$

meaning that for all $N \in \mathbb{N}$,

$$a - (1-\chi)\sum_{j=0}^{N-1} a_{m-j} \in \Gamma^N.$$

The space of all symbols will be denoted by $\Gamma = \bigcup_{m\in\mathbb{R}} \Gamma^m$ and the classical symbols are $\Gamma_{\mathrm{cl}} = \bigcup_{m\in\mathbb{R}} \Gamma^m_{\mathrm{cl}}$.

To each symbol $a \in \Gamma$ we can associate a bounded linear operator

$$\mathrm{Op}^w(a) = a^w(x, D) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d),$$

the *Weyl-quantization*:

$$\langle a^w(x,D)u, v\rangle = \int e^{i(x-y)\xi} a((x+y)/2, \xi) u(y) v(x)\, dx\, dy\, đ\xi.$$

The symmetry in x, y implies that the L^2-adjoint is formally $(a^w(x,D))^* = \bar{a}^w(x,D)$. Thus, if a has real-valued symbol, its Weyl-quantized operator is formally self-adjoint. We set

$$G^m = \{a^w(x,D) \colon a \in \Gamma^m\}$$

and

$$G^m_{\mathrm{cl}} = \{a^w(x,D) \colon a \in \Gamma^m_{\mathrm{cl}}\}.$$

It is clear that $G = \bigcup_m G^m$ is a filtered $*$-algebra and $G_{\mathrm{cl}} = \bigcup_m G^m_{\mathrm{cl}}$ is a sub-algebra.

More generally, we can a define the t-quantized operator for $t \in [0,1]$ by

$$a_t(x,D) = \mathrm{Op}_t(a) = \int e^{i(x-y)\xi} a(tx + (1-t)y, \xi)\, đ\xi.$$

If $t = 1/2$, we obtain the Weyl-quantization. The case $t = 0$ is called the left-quantization $a_L(x,D)$ and $t = 1$ is the right-quantization $a_R(x,D)$.

The advantage of the Weyl quantization is that it is metaplectically covariant (cf. Hörmander [31, Theorem 18.5.9]), meaning that for any linear symplectic map

$$\kappa : T^*\mathbb{R}^d \to T^*\mathbb{R}^d,$$

there exists a unique (up to a constant of modulus 1) unitary transformation $U : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that

$$U^{-1} a^w(x,D) U = (a \circ \kappa)^w(x,D).$$

Furthermore, we have for the harmonic oscillator propagator $U_0(t)$ that

$$U_0(t)^{-1} a^w(x,D) U_0(t) = (a \circ \exp(t\mathsf{H}_0))^w(x,D).$$

There is a principal symbol map

$$\sigma^m : G^m \to \Gamma^m/\Gamma^{m-1}$$

such that the following short sequence is exact:

$$0 \to G^{m-1} \to G^m \overset{\sigma^m}{\to} \Gamma^m/\Gamma^{m-1} \to 0.$$

If the operator $A = a^w(x, D)$ is classical with asymptotic expansion $a \sim \sum_j a_{m-j}$, then the principal symbol $\sigma^m(A)$ is the homogeneous function of highest order: $\sigma^m(A) = a_m$.

Example 2.4.2. The most important example of an isotropic pseudodifferential operator is the harmonic oscillator H_0. Its symbol is given by $p_2 = 1/2(|x|^2 + |\xi|^2)$. Complex powers of H_0 (cf. Shubin [59]) and harmonic oscillators with different frequencies are also contained in the calculus. As mentioned above, if $a \in \Gamma^m$, then $a \circ \exp(t\mathsf{H}_0) \in \Gamma^m$ for all $t \in \mathbb{R}$.

The potentials $V(x) \in S^m_{\mathrm{cl}}$ are generally not in Γ^m_{cl} since differentiation in x does not lead to decay in ξ.

Example 2.4.3. More generally, differential operators of the form

$$\sum_{|\alpha|+|\beta|\leq m} a_{\alpha,\beta} x^\alpha D^\beta$$

are in G^m.

A similar calculus was defined by Wunsch [65, 66] to deal with potential and metric perturbations. It was used to study propagation of singularities for the harmonic oscillator and the Schrödinger equation.

2.4.3. Composition

Before we calculate the composition of two pseudodifferential operators, we consider changing the quantization. Let $t, s \in [0, 1]$ and $a \in \mathcal{S}(\mathbb{R}^{2d})$ and set $A = a_t(x, D)$. We want to write $A = b_s(x, D)$ for some $b \in \mathcal{S}(\mathbb{R}^{2d})$.

Proposition 2.4.4 (cf. Theorem 4.13 in Zworski [69]). *The symbol b is given by*

$$b(x, \xi) = e^{i(s-t)\langle D_x, D_\xi \rangle} a(x, \xi).$$

Using that $\mathcal{S}(\mathbb{R}^{2d})$ is dense in $\mathcal{C}^\infty(\mathbb{R}^{2d})$ we may extend this to general symbols. In particular, this show that if $a \in \Gamma^m$, then for any $t \in [0, 1]$, $a_t(x, D) \in G^m$ and the principal symbol $\sigma^m(a^w(x, D))$ is independent of the quantization,

$$\sigma^m \circ \mathrm{Op}_\bullet : a \mapsto [a] \in \Gamma^m/\Gamma^{m-1}.$$

In order to compose two pseudodifferential operators $a^w(x,D)$ and $b^w(x,D)$, we change the quantization of $a^w(x,D)$ to the left-quantization and of $b^w(x,D)$ to the right quantization. We may use that

$$a_L(x,D)b_R(x,D) = \int e^{i(x-y)\xi} a(x,\xi)b(y,\xi)\, đ\xi$$

and change this back to right quantization.

Let $a \in \Gamma^{m_1}$ and $b \in \Gamma^{m_2}$. Define the Moyal product $\#$ by

$$(a\#b)(x,\xi) := e^{i\sigma(D_x,D_\xi,D_y,D_\eta)/2} \left(a(x,\xi)b(y,\eta)\right)\big|_{y=x,\eta=\xi},$$

where the operator $e^{i\sigma(D_x,D_\xi,D_y,D_\eta)/2}$ is defined as a Fourier multiplier and $\sigma(x,\xi,y,\eta) = \langle \xi, y\rangle - \langle x, \eta\rangle$ is the symplectic 2-form.

Proposition 2.4.5. *Let $a \in \Gamma^{m_1}$, $b \in \Gamma^{m_2}$. The product satisfies*

$$a\#b \in \Gamma^{m_1+m_2}$$

and there is an asymptotic expansion

$$a\#b \sim \sum_k \frac{i^k}{k!} \sigma(D_x,D_\xi,D_y,D_\eta)^k a(x,\xi)b(y,\eta)\big|_{y=x,\eta=\xi}.$$

In particular, we have that

- $a\#b = ab + \frac{1}{2i}\{a,b\} + \Gamma^{m_1+m_2-4}$,
- $[a^w(x,D), b^w(x,D)] = -i\{a,b\}^w(x,D) + G^{m_1+m_2-6}$.
- If $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$, then $a\#b \in \Gamma^{-\infty}$.

The crucial property of the isotropic calculus is that the commutator is *two* orders lower, because the Poisson bracket satisfies

$$\{a,b\} \in \Gamma^{m_1+m_2-2}. \tag{2.7}$$

Remark 2.4.6. For the left quantization, we have the product

$$(a\#_L b)(x,\xi) := e^{i\langle D_\xi, D_y\rangle} \left(a(x,\xi)b(y,\eta)\right)\big|_{y=x,\eta=\xi}$$

and for the right quantization

$$(a\#_R b)(x,\xi) := e^{-i\langle D_x, D_\eta\rangle} \left(a(x,\xi)b(y,\eta)\right)\big|_{y=x,\eta=\xi}.$$

Furthermore there are similar expansions as in Proposition 2.4.5.

2.4.4. Ellipticity and Essential Support

We call a symbol $a \in \Gamma^m$ *elliptic* at $(x_0, \xi_0) \in \mathbb{R}^{2d} \setminus \{0\}$ if there is an open conic neighborhood Γ_0 of (x_0, ξ_0) and a constant $C > 0$ such that for all $(x, \xi) \in \Gamma_0$,

$$a(x, \xi) \geq C\langle (x, \xi) \rangle^m.$$

If a is classical the condition is that $a_m(x_0, \xi_0) \neq 0$. The set of all elliptic points is denoted by $\mathrm{ell}(a)$ and its complement is the characteristic set $\Sigma(a)$. Since the principal symbol is invariantly defined, it makes sense to call an operator elliptic at a point (x_0, ξ_0).

A point $z_0 = (x_0, \xi_0) \in \mathbb{R}^{2d} \setminus \{0\}$ is not in the *essential support* of a, $z_0 \notin \text{ess-supp}(a)$, if there exists a symbol $b \in \Gamma^0$, elliptic at z_0 such that $a \cdot b \in \mathcal{S}(\mathbb{R}^{2d})$. While this condition depends on the full symbol a it is invariant under changing the quantization and therefore we also define the essential support of $A \in G^m$ as the the essential support of the symbol for any quantization.

We will denote the essential support of A by $\mathrm{WF}'(A)$, also called the operator wavefront set. Using the conic structure of phase-space, we may view $\mathrm{WF}'(A)$ as a subset of $\mathbb{S}^{2d-1}$. It has the following properties:

1. $\mathrm{WF}'(A^*) = \mathrm{WF}'(A)$,
2. $\mathrm{WF}'(AB) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B)$,
3. $\mathrm{WF}'(A + B) \subset \mathrm{WF}'(A) \cup \mathrm{WF}'(B)$,
4. for any $K \subset \mathbb{S}^{2d-1}$ closed and $U \subset \mathbb{S}^{2d-1}$ open with $K \subset U$, there exists $A \in G^0$ such that $\mathrm{WF}'(A) \subset U$ and $\sigma_0(A) = 1$ on K.
5. For each $A \in G^k$ the following are equivalent:
   - $\mathrm{WF}'(A) = \emptyset$,
   - $A \in G^{-\infty}$,
   - $A : \mathcal{S}' \to \mathcal{S}$.

Elliptic pseudodifferential operators are always invertible up to a regularizing error.

Proposition 2.4.7. *Let $A \in G^m$ be elliptic. There exists a pseudodifferential operator $B \in G^{-m}$ such that*

$$AB - 1 \in G^{-\infty}, \quad BA - 1 \in G^{-\infty}.$$

Such a B is called a parametrix *of A.*

This property can also be microlocalized:

Proposition 2.4.8. *Let $A \in G^m$ be elliptic at $z \in \mathbb{S}^{2d-1}$. There exists a microlocal parametrix $B \in G^{-m}$ meaning that*

$$z \notin \mathrm{WF}'(AB - \mathrm{I}) \cup \mathrm{WF}'(BA - \mathrm{I}).$$

2.4.5. Sobolev Spaces

First, it follows from the Schur test and Hörmander's square-root trick that the isotropic operators of order zero are bounded in L^2 (cf. Nicola–Rodino [49] for a proof in a more general setting).

Proposition 2.4.9. *Let $a \in \Gamma^0$. The operator $A = a^w(x, D)$ satisfies*

$$\|A\|_{L^2 \to L^2} \leq \|a\|_{k,\Gamma^0},$$

for some $k \in \mathbb{N}$.

It also holds that for $a \in \Gamma^{-\epsilon}$ the operator $a^w(x, D) : L^2 \to L^2$ is compact. Now we are able to define a scale of Sobolev spaces adapted to the isotropic calculus.

Definition 2.4.10. Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $s \in \mathbb{R}$. We say that $u \in H^s_{\text{iso}}$ if

$$\|Au\|_{L^2} < \infty$$

for all $A \in G^s$.

In particular, it follows from Proposition 2.4.9 that $H^0_{\text{iso}} = L^2$. Further, the scale of Sobolev spaces is *globally regularizing*, that is

$$\bigcap_{s \in \mathbb{R}} H^s_{\text{iso}} = \mathcal{S}(\mathbb{R}^d), \qquad \bigcup_{s \in \mathbb{R}} H^s_{\text{iso}} = \mathcal{S}'(\mathbb{R}^d).$$

There are several equivalent norms on H^s_{iso} that turn it into a Hilbert space. Let $s \in \mathbb{R}$ be arbitrary. We define the $s/2$-th power of the quantum harmonic oscillator H_0, $\Lambda_s = H_0^{s/2}$ by the spectral theorem. It can be shown that $\Lambda_s \in G^s_{\text{cl}}(\mathbb{R}^d)$ (cf. Nicola–Rodino [49]). The principal symbol of Λ_s is

$$\left(\frac{1}{2}(|\xi|^2 + |x|^2)\right)^{s/2}.$$

Since Λ_s is invertible,

$$\|u\|_{H^s_{\text{iso}}} := \|\Lambda_s u\|_{L^2}$$

is a norm on H^s_{iso}. If $A \in G^s$ is elliptic and $s \geq 0$, then an equivalent norm is given by

$$\|Au\|_{L^2} + \|u\|_{L^2}.$$

This is *not* an equivalent norm if $s < 0$.

By the construction it follows that for all $s, m \in \mathbb{R}$ and all pseudodifferential operators $A \in G^m$,

$$A : H^s_{\text{iso}} \to H^{s-m}_{\text{iso}}$$

is continuous and the operator norm is controlled by a seminorm of the total symbol (in any quantization).

2.4.6. Isotropic Wavefront Sets

Roughly speaking, wavefront sets measure how much a distribution fails to be "regular" in some sense. The classical wavefront set introduced in Section A.4 measures how a distribution fails to be smooth. The isotropic wavefront set detects whether a tempered distribution is not a Schwartz function, so it sees both non-smoothness and non-decay. It was introduced by Hörmander [28].

Definition 2.4.11. Let $u \in \mathcal{S}'(\mathbb{R}^d)$ be a tempered distribution. The *isotropic wavefront set* $\mathrm{WF}_{\mathrm{iso}}(u) \subset \mathbb{R}^{2d} \setminus \{0\}$ is defined as

$$\mathrm{WF}_{\mathrm{iso}}(u) := \bigcap_{\substack{A \in G^0 \\ Au \in \mathcal{S}}} \Sigma(A).$$

We also define the isotropic wavefront set of order $s \in \mathbb{R}$ by

$$\mathrm{WF}^s_{\mathrm{iso}}(u) := \bigcap_{\substack{A \in G^s \\ Au \in L^2}} \Sigma(A).$$

They have the following properties:

- A tempered distribution u is a Schwartz function $u \in \mathcal{S}$ if and only if $\mathrm{WF}_{\mathrm{iso}}(u) = \emptyset$.
- The distribution u is in the isotropic Sobolev space $u \in H^s_{\mathrm{iso}}$ if and only if $\mathrm{WF}^s_{\mathrm{iso}}(u) = \emptyset$.
- If $f \in \mathcal{S}$, then $\mathrm{WF}_{\mathrm{iso}}(fu) \subset \mathrm{WF}_{\mathrm{iso}}(u)$.
- If $A \in G^m$ for any $m \in \mathbb{R}$, then $\mathrm{WF}_{\mathrm{iso}}(Au) \subset \mathrm{WF}_{\mathrm{iso}}(u) \cap \text{ess-supp}(A)$ and $\mathrm{WF}_{\mathrm{iso}}(u) \subset \mathrm{WF}_{\mathrm{iso}}(Au) \cup \Sigma(A)$.
- $\mathrm{WF}_{\mathrm{iso}}(u) = \overline{\bigcup_{s \in \mathbb{R}} \mathrm{WF}^s_{\mathrm{iso}}(u)}$.

Now we will investigate the relationship between smoothness and the isotropic wavefront set. The directions $\{0\} \times \mathbb{R}^d$ measure lack of smoothness. This is illustrated in Figure 2.3. The following Lemma is a special case of Proposition 2.6 by Hörmander [28].

Lemma 2.4.12. *Let $u \in \mathcal{E}'$. The isotropic wavefront set of u is contained in the vertical space,*

$$\mathrm{WF}_{\mathrm{iso}}(u) \subset \{0\} \times \mathbb{R}^d.$$

It is well-known that $\mathrm{WF}_{\mathrm{iso}}(u) \cap \{0\} \times \mathbb{R}^d = \emptyset$ implies that $u \in \mathcal{C}^\infty$ (cf. [11, 28]). We refine this result slightly:[3]

[3]This result was stated by Nakamura [48] for the homogeneous wavefront set.

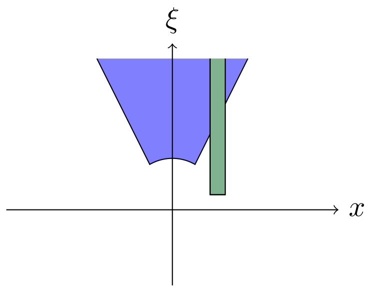

Figure 2.3.: Comparision of the classical and the isotropic wavefront set. The blue area denotes the elliptic set of a and the green area is the support of b. (cf. Schulz [58]).

Lemma 2.4.13 (cf. Doll [10]). *Let* $u \in \mathcal{S}'(\mathbb{R}^d)$ *and* $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ *be an open cone. If* $\mathrm{WF}_{\mathrm{iso}}(u) \cap \{0\} \times \Gamma = \emptyset$, *then* $\mathrm{WF}_{\mathrm{cl}}(u) \cap \mathbb{R}^d \times \Gamma = \emptyset$.

Proof. Let $a \in \Gamma^0_{\mathrm{cl}}$ be such that $a = 1$ in a conic (in (x, ξ)) neighborhood of $\{0\} \times \Gamma$ and $\operatorname{supp} a \cap \mathrm{WF}_{\mathrm{iso}}(u) = \emptyset$. By the properties of the isotropic wavefront set, we obtain

$$\begin{aligned} u &= a(x, D)u + (1 - a(x, D))u \\ &= (1 - a(x, D))u + \mathcal{S}(\mathbb{R}^d) \\ &= (2\pi)^{-d} \int e^{ix\xi}(1 - a(x, \xi))\hat{u}(\xi)d\xi + \mathcal{S}(\mathbb{R}^d). \end{aligned}$$

Choose $b \in S^0_{\mathrm{cl}}$ with $\operatorname{supp} b \subset K \times \Gamma$ for some compact set $K \subset \mathbb{R}^d$. There is an $R > 0$ such that $\{(x, \xi) \in \operatorname{supp} b \colon |\xi| > R\} \subset \operatorname{supp} a$. Therefore, the symbol of the composition $b(x, D)(1 - a(x, D))$ is compactly supported in (x, ξ). This implies that

$$b(x, D)(1 - a(x, D)) : \mathcal{S}' \to \mathcal{S}$$

and thus

$$b(x, D)u = b(x, D)(1 - a(x, D))u + b(x, D)a(x, D)u \in \mathcal{S}(\mathbb{R}^d).$$

□

Note that $u \in \mathcal{C}^\infty$ does not imply that $\mathrm{WF}_{\mathrm{iso}}(u) \cap \{0\} \times \mathbb{R}^d = \emptyset$. The function $u : x \mapsto e^{ix^3/3}$ is smooth, but not rapidly decaying and one can show, using the semiclassical description of isotropic wavefront set, that $\mathrm{WF}(u) \subset \{0\} \times \mathbb{R}^d$.

2.5. Propagation of Singularities

The natural operators in G are of order 2 and, since the commutator of two isotropic operators is *two* orders lower than the sum, we can use a commutator argument similar to the proof of

Lemma 3.2.1 to show that the isotropic wavefront set is rotated according to the Hamiltonian flow of the principal symbol.

Let $P = p^w(x, D) \in G^2_{\mathrm{cl}}(\mathbb{R}^d)$ a self-adjoint elliptic operator and assume that the symbol p admits an asymptotic expansion

$$p \sim \sum_{j=0}^{\infty} p_{2-j}.$$

Recall that H_{p_2} is the Hamiltonian vector field of p_2 satisfying $\mathsf{H}_{p_2} f = \{p_2, f\}$ for all $f \in \mathcal{C}^\infty(\mathbb{R}^{2d})$.

Proposition 2.5.1. *Let $s \in \mathbb{R}$ and $u \in C([0,T], H^s_{\mathrm{iso}}) \cap C^1([0,T], H^{s-1}_{\mathrm{iso}})$ be the solution of the equation*

$$\begin{cases} (i\partial_t - P)u(t,x) = 0 \\ u(0,x) = u_0 \end{cases}$$

for $u_0 \in H^s_{\mathrm{iso}}$. The wavefront set of $u(t)$ is given by

$$\mathrm{WF}_{\mathrm{iso}}(u(t)) = \exp(t\mathsf{H}_{p_2})\,\mathrm{WF}_{\mathrm{iso}}(u_0).$$

Proof. We will use a variant of the method of positive commutators.[4]

Using the substitution

$$u \mapsto \Lambda_{-s} u, \quad P \mapsto \Lambda_{-s} P \Lambda_s$$

we may assume that $u \in C([0,T], H^1_{\mathrm{iso}}) \cap C^1([0,T], L^2(\mathbb{R}^d))$ and $u_0 \in L^2(\mathbb{R}^d)$. We will show by induction that

$$\mathrm{WF}^k_{\mathrm{iso}}(u(t)) = \exp(t\mathsf{H}_{p_2})\,\mathrm{WF}^k_{\mathrm{iso}}(u_0). \tag{2.8}$$

For $k = 0$ this is clear by the assumption that the equation is well-posed in L^2.

We now assume that (2.8) is true for $k-1$. Let $(x_0, \xi_0) \notin \mathrm{WF}^k_{\mathrm{iso}}(u_0)$. In particular, $(x_0, \xi_0) \notin \mathrm{WF}^{k-1}_{\mathrm{iso}}(u_0)$ and by the inductive hypothesis

$$\exp(t\mathsf{H}_{p_2})(x_0, \xi_0) \notin \mathrm{WF}^{k-1}_{\mathrm{iso}}(u(t)), \quad t \in [0,T].$$

Hence, there exists a $B \in \mathcal{C}^\infty([0,T], G^{k-1})$ such that $\exp(t\mathsf{H}_{p_2})(x_0, \xi_0) \in \mathrm{ell}(B(t))$ and $B(t)u(t) \in L^2$ for all $t \in [0,T]$.

Let $a_0 \in \Gamma^k$ such that a_0 is elliptic at (x_0, ξ_0) and $a(t) = a_0 \circ \exp(t\mathsf{H}_{p_2})$ has essential support

$$\operatorname{ess-supp} a(t) \subset \mathrm{ell}(B(t)), \quad t \in [0,T].$$

[4] In our case the commutator is actually zero.

We set $a_\epsilon = a_0 \cdot (1 + \epsilon(|x|^2 + |\xi|^2))^{-1/2}$ and define $a_\epsilon(t) = a_\epsilon \circ \exp(t\mathsf{H}_{p_2})$.

Define the operators $A(t) = a^w(t,x,D)$ and $A_\epsilon(t) = a_\epsilon^w(t,x,D) \in G^{k-1}$ for $\epsilon > 0$ small. The operator $A_\epsilon(t)$ converges to $A(t)$ in the topology of G^{k+1}. By Property 2 of the operator wavefront set, $\mathrm{WF}'(A_\epsilon(t)) \subset \mathrm{ell}(B(t))$ and hence, by microlocal elliptic regularity, $A_\epsilon(t)u(t) \in L^2$. Since P is self-adjoint $\mathrm{Re}\,(iPA_\epsilon(t)u(t), A_\epsilon(t)u(t))_{L^2} = 0$ and we obtain

$$\begin{aligned} 2\|A_\epsilon(t)u(t)\|\partial_t\|A_\epsilon(t)u(t)\| &= \partial_t\|A_\epsilon(t)u(t)\|^2 \\ &= 2\,\mathrm{Re}\,((\partial_t A_\epsilon(t))u(t) + A_\epsilon(t)\partial_t u(t), A_\epsilon(t)u(t))_{L^2} \\ &= 2\,\mathrm{Re}\,([\partial_t + iP, A_\epsilon(t)]u(t), A_\epsilon(t)u(t))_{L^2} \\ &\le 2\,\|[\partial_t + iP, A_\epsilon(t)]u(t)\| \cdot \|A_\epsilon(t)u(t)\|. \end{aligned} \tag{2.9}$$

Note that $[\partial_t, A_\epsilon(t)]u(t) = (\partial_t A_\epsilon(t))u(t)$. Integrating inequality (2.9) yields

$$\|A_\epsilon(t)u(t)\| \le \|A_\epsilon(0)u_0\| + \int_0^t \|[\partial_s + iP, A_\epsilon(s)]u(s)\|\,ds. \tag{2.10}$$

Now, the $k-2$-principal symbol of the commutator is $(\partial_s + \mathsf{H}_{p_2})a_\epsilon(s) = 0$. Thus, the right-hand side of (2.10) is uniformly bounded as $\epsilon \to 0$. We conclude that $A(t)u(t) \in L^2$ and therefore, $\exp(t\mathsf{H}_{p_2})(x_0, \xi_0) \notin \mathrm{WF}^k_{\mathrm{iso}}(u(t))$.

The other inclusion follow from reversing the time $t \mapsto -t$. □

Remark 2.5.2. The positive commutator argument can be extended to more complicated situations, for instance for non-smooth pseudodifferential calculi [29] or in the presence of radial points [42].

In Chapter 4, we calculate the classical wavefront set. From Lemma 2.4.13 and Proposition 2.5.1 it is clear that the singularities reappear only at certain times.

Let $\exp(t\mathsf{H}_{p_2})(y,\eta) = (x(t,y,\eta), \xi(t,y,\eta))$ the Hamiltonian flow of the vector field H_{p_2},

$$\Gamma_t = \left\{\eta \in \mathbb{R}^d \setminus \{0\} \colon\ \exp(t\mathsf{H}_{p_2})(0,\eta) \in \{0\} \times \mathbb{R}^d\right\},$$

and define the function $\Xi_t : \Gamma_t \to \mathbb{R}^d$, which is given by $\Xi_t(\eta) = \xi(t,0,\eta)$. It satisfies

$$\exp(t\mathsf{H}_{p_2})(0,\eta) = (0, \Xi_t(\eta)).$$

Note that Ξ_t is homogeneous of degree one. Set $G_t = \mathrm{supp}\, U_0(t)u \times \Xi_t(\Gamma_t)$ and

$$\mathsf{X}_t f = \int_0^t f \circ \exp(s\mathsf{H}_{p_2})ds.$$

The main theorem of Chapter 4 is the following.

Theorem 2.5.3. *Let $u \in \mathcal{E}' + \mathcal{S}$ and $t \in \mathbb{R}$. The classical wavefront set of $U(t)u$ satisfies*

$$\mathrm{WF}(U(t)u) \subset \{(x, \Xi_t(\eta)) \in G_t \colon \partial_\eta\langle x, \Xi_t(\eta)\rangle - \partial_\eta \mathsf{X}_t p_1(0,\eta) - y \perp \Gamma_t, (y,\eta) \in \mathrm{WF}(u)\}.$$

2.6. Parametrices for Longer Times

For the propagation of the isotropic singularities, we did not have to construct a parametrix of the time-dependent Schrödinger equation

$$\begin{cases} (i\partial_t - H)u(t,x) = 0 \\ u(0,x) = u_0. \end{cases}$$

Here, as before $H \in G^2_{\mathrm{cl}}(\mathbb{R}^d)$ is a self-adjoint elliptic operator and $u_0 \in \mathcal{S}'(\mathbb{R}^d)$.

It was shown by Helffer–Robert [24] that there is always a short-time parametrix $\tilde{U}(t)$ for $U(t) = e^{-itH}$ as an oscillatory integral. For the refined Weyl asymptotics one needs parametrices for arbitrary times.

2.6.1. Helffer–Robert Parametrix

The idea of Helffer–Robert is to construct a parametrix for the operator e^{-itH} as an oscillatory integral of the form

$$\tilde{U}(t,x,y) = \int e^{i(\phi_2(t,x,\eta)+\phi_1(t,x,\eta)-y\eta)} a(t,x,\eta) d\eta,$$

where ϕ_j is homogeneous of degree j and a is an amplitude outside a compact set.

The eikonal equation for $\phi_2 = \phi_2(t,x,\eta)$ is

$$\begin{cases} \partial_t\phi_2 + p_2(x,\partial_x\phi_2) = 0 \\ \phi_2(0,x,\eta) = x\eta. \end{cases} \tag{2.11}$$

The eikonal equation for $\phi_1 = \phi_1(t,x,\eta)$ is more complicated:

$$\begin{cases} \partial_t\phi_1 + p_1(x,\partial_x\phi_2) + \langle \partial_\xi p_2(x,\partial_x\phi_2), \partial_x\phi_1\rangle = 0 \\ \phi_1(0,x,\eta) = 0. \end{cases} \tag{2.12}$$

This type of parametrix is used in Section 3.5 to calculate the contribution to the Schrödinger trace near $t = 0$. For the singularities at times $t = 2\pi k$ this approach is not suited since the eikonal equation degenerates at $t = \pi/2$.

2.6.2. Reduced Parametrix

In Chapter 3 we will construct a parametrix which is better suited if one wants to keep track of the secondary phase function ϕ_1, but at the cost of losing information about the amplitude. Set $H_0 = \mathrm{Op}^w(\tilde{p})$, where $\tilde{p} \in \Gamma^2_{\mathrm{cl}}$ and $\tilde{p} = \sigma^2(H)$ outside a compact set in $\mathbb{R}^{2d}$.

The main idea is to split the propagator into two parts:

$$e^{-itH} = e^{-itH_0} F(t).$$

The operator $U_0(t) = e^{-itH_0}$ is called the "free" propagator and $F(t)$ the reduced propagator. Set $P(t) = U_0(-t)(H - H_0)U_0(t)$. It is straightforward to verify that

$$\begin{cases} (i\partial_t - P(t))F(t) = 0 \\ F(0) = \mathrm{I}\,. \end{cases} \tag{2.13}$$

By Helffer–Robert's Egorov theorem [22], $P(t) \in G^1$. This means that we have lowered the order, but the operator is now time-dependent.

If $H = p^w(x, D)$ with $p \sim p_2 + p_1 + \dots$, then by Egorov's theorem [22] we obtain that

$$\sigma^1(P(t)) = p_1 \circ \exp(t\mathsf{H}_{p_2}), \tag{2.14}$$

In the special case that p_2 is a polynomial, we can apply the exact Egorov theorem (cf. Hörmander [31]) to obtain that

$$P(t) = ((p - p_2) \circ \exp(t\mathsf{H}_{p_2}))^w(x, D),$$

where $p_2 = \sigma^2(H)$.

We write $P(t) = \mathrm{Op}_R(p_R(t))$ for some $p_R \in \mathcal{C}^\infty(\mathbb{R}, \Gamma^1_{\mathrm{cl}}(\mathbb{R}^d))$. Using the ansatz $\tilde{F}(t) = \mathrm{Op}_R(e^{i\phi_1} a)$ to solve (2.13), we arrive at the equation

$$-\partial_t\phi_1(t)e^{i\phi_1(t)}a(t) + ie^{i\phi_1(t)}\partial_t a(t) = p_R(t) \#_R\, e^{i\phi_1(t)}a(t).$$

If we set $\phi_1 \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^{2d}, \mathbb{R})$ such that ϕ_1 is given outside a compact set $K \subset \mathbb{R}^{2d}$ by

$$\phi_1(t, x, \eta) = -\int_0^t p_1(\exp(s\mathsf{H}_{p_2})(x, \eta))ds,$$

then $\phi_1(t)$ solves the eikonal equation

$$\begin{cases} \partial_t\phi_1(t, x, \eta) + (p_1 \circ \exp(t\mathsf{H}_{p_2}))(x, \eta) = 0 \\ \phi_1(0, x, \eta) = 0 \end{cases}$$

outside of $\mathbb{R} \times K$. The system of transport equations can be solved for all t to arbitrary order (cf. Proposition 2.1.2 and Lemma 3.3.1. Repeating the arguments as in the proof of Lemma 3.3.1 we obtain a parametrix of $F(t)$ for arbitrary times t.

Lemma 2.6.1. *There is an operator*

$$\tilde{F}(t) = \mathrm{Op}_R(e^{i\phi_1(t)}a(t))$$

where ϕ_1 as above and $a \in \mathcal{C}^\infty(\mathbb{R}_t, \Gamma^0_{\mathrm{cl}})$, such that

$$\begin{cases} (i\partial_t - P(t))\tilde{F}(t) \in \mathcal{C}^\infty(\mathbb{R}^t, \mathcal{L}(\mathcal{S}', \mathcal{S})) \\ \tilde{F}(0) - \mathrm{I} \in \mathcal{L}(\mathcal{S}', \mathcal{S}). \end{cases}$$

Now, we compose this parametrix with a parametrix for the "free" propagator e^{-itH_0}, which is constructed as in Helffer–Robert. In the case of the quantum harmonic oscillator, we can use the Mehler formula. The composition theorem is proved in Section 3.3.

2.7. Spectral Theory of the Perturbed Harmonic Oscillator

In this section, we want to complement the result of Chapter 3 by the observation that for the quantum harmonic oscillator with at least two rationally related frequencies, we always obtain the improved remainder estimate in the Weyl law.

2.7.1. Second Order Operators

Let $H \in G^2_{\mathrm{cl}}(\mathbb{R}^d)$ be an elliptic self-adjoint operator. We show that H has discrete spectrum and the Schrödinger trace is well-defined as a distribution. Without loss of generality we may assume that H is bounded from below. Thus, the resolvent $(H-\mu)^{-1}$ defined for $\mu \gg 0$ and since $H : H^2_{\mathrm{iso}} \to L^2$, we have

$$(H-\mu)^{-1} : L^2 \to H^2_{\mathrm{iso}}.$$

The embedding $H^2_{\mathrm{iso}} \to L^2$ is compact, therefore $(H-\mu)^{-1}$ is a compact self-adjoint operator on L^2 and has discrete spectrum accumulating at 0. Thus, H has discrete spectrum

$$\lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

To define the trace of e^{-itH}, we first note that we may assume that H is positive. The operator H^{-N} is trace-class for $N \gg 0$. Let $\phi \in \mathcal{S}(\mathbb{R})$ be a test function. We formally calculate

$$\begin{aligned}\langle \phi, \operatorname{Tr} e^{-itH} \rangle &= \langle \phi, \operatorname{Tr} H^{-N} H^N e^{-itH} \rangle \\ &= i^N \langle \phi, \operatorname{Tr} H^{-N} \partial_t^N e^{-itH} \rangle \\ &= \langle (-i\partial_t)^N \phi, \operatorname{Tr} H^{-N} e^{-itH} \rangle.\end{aligned}$$

Now, $H^{-N} e^{-itH}$ is trace-class since e^{-itH} is unitary and therefore the last line is well-defined. We *define* $\operatorname{Tr} e^{-itH}$ by this expression.

2.7.2. Improved Remainder Estimates

Let (M,g) be a compact, connected, and oriented Riemannian manifold and denote by Δ_g the Laplacian and $p(x,\xi) = g_{ij}(x)\xi^i\xi^j$ its principal symbol. The operator $\sqrt{\Delta_g}$ is a first order pseudodifferential operator with principal symbol $\sqrt{p}$.

Hörmander [26] proved that the counting function $N(\lambda)$ of the eigenvalues of $\sqrt{\Delta_g}$ satisfy

$$N(\lambda) = (2\pi)^{-d} \lambda^d \int_{\{p \leq 1\}} dx\, d\xi + O(\lambda^{d-1}).$$

This estimate is sharp. For instance if $M = \mathbb{S}^2$, we can explicitly calculate the eigenvalues and the high multiplicities of the eigenvalues prohibit an improved remainder estimate. The famous Theorem by Duistermaat–Guillemin [13] states that the estimate can be improved if the co-geodesic flow γ_t, which is the Hamiltonian flow for the principal symbol, is not periodic. More precisely, if the set $\{(x,\xi) \in S^*M \colon \exists t > 0, \gamma_t(x,\xi) = (x,\xi)\}$ has measure zero as a subset of S^*M, then one obtains the estimate

$$N(\lambda) = (2\pi)^{-d}\lambda^d \int_{\{p\leq 1\}} dx d\xi + o(\lambda^{d-1}).$$

Note that for arbitrary first order pseudodifferential operators, there is an additional term homogeneous of degree $d-1$.

It was shown by Helffer–Robert, that the general sharp remainder estimate is $O(\lambda^{d-1})$. More precisely they showed:

Theorem 2.7.1 (Helffer–Robert [23]). *Let $H \in G^{2m}_{\mathrm{cl}}(\mathbb{R}^{2d})$ be an elliptic isotropic pseudodifferential operator with real-valued Weyl-quantized symbol $p \sim p_{2m} + p_{2m-1} + \dots$, and $p_{2m}(x,\xi) > 0$ for $(x,\xi) \neq 0$. The counting function $N(\lambda)$ of the eigenvalues of H satisfies*

$$N(\lambda) = \gamma_0 \lambda^{d/m} + \gamma_1 \lambda^{(d-1/2)/m} + O(\lambda^{(d-1)/m}),$$

where

$$\gamma_0 = (2\pi)^{-d} \int_{\{p_{2m}\leq 1\}} dx\, d\xi \quad \text{and} \quad \gamma_1 = -(2\pi)^{-d} \int_{\{p_{2m}=1\}} p_{2m-1} \frac{dS}{|\nabla p_{2m}|}.$$

The goal of Chapter 3 is to show that in the isotropic calculus, even if the Hamiltonian flow of the principal symbol is periodic, under a suitable assumption on the subprincipal symbol, we obtain a similar improvement. In order to see this, set

$$\mathsf{X} f = \int_0^{2\pi} f \circ \exp(t\mathsf{H}_0) dt.$$

This is the average over the flow of the classical harmonic oscillator.

Theorem 2.7.2 (Doll–Gannot–Wunsch [11]). *Let $p \in \Gamma^2_{\mathrm{cl}}(\mathbb{R}^{2d})$ be real-valued with $p_2 = (1/2)(|x|^2 + |\xi|^2)$ and set $H = p^w(x,D)$. Assume that, when restricted to $\mathbb{S}^{2d-1}$, the set where $\nabla \mathsf{X} p_1$ vanishes to infinite order has measure zero. We have the improved Weyl asymptotics*

$$N(\lambda) = (2\pi)^{-d} \int_{\{p_2+p_1\leq\lambda\}} dx d\eta - (2\pi)^{-d} \int_{\{p_2=\lambda\}} p_0(x,\eta) \frac{dS}{|\nabla p_2|} + o(\lambda^{d-1}). \tag{2.15}$$

If we consider the harmonic oscillator with arbitrary frequencies,

$$H_0 = 1/2(\Delta + \sum_j \omega_j^2 x_j^2),$$

for $\omega_j \in (0,\infty)$, then we expect that in the generic case we should obtain a similar improvement. More generally, consider an isotropic symbol $p \in \Gamma^2_{\mathrm{cl}}(\mathbb{R}^d)$ with the following properties:

- $p \sim \sum_{j=0}^{\infty} p_{2-j}$,
- $\sigma^2(p) = p_2 = 1/2(|\xi|^2 + \sum_j \omega_j^2 x_j^2)$, and
- p is real-valued.

As usual set $H = p^w(x, D)$ and H_{p_2} is the Hamiltonian vector field of p_2.

We can use the propagation of isotropic singularities theorem to prove a result on the singular support of the trace of e^{-itH}:

Proposition 2.7.3. *The singularities of the Schrödinger trace satisfy*

$$\operatorname{sing-supp} e^{-itH} \subset \{t \in \mathbb{R} \colon \exists z \in \mathbb{R}^{2d} \setminus \{0\} \text{ such that } \exp(t\mathsf{H}_{p_2})z = z\}.$$

Remark 2.7.4. Note that this result holds for any elliptic self-adjoint pseudodifferential operator $H \in G^2(\mathbb{R}^{2d})$ with positive principal symbol. The proof is almost identical to the proof of Proposition 3.1.1.

The Duistermaat–Guillemin theorem for isotropic pseudodifferential operators was proved by Petkov–Robert [51] in the semiclassical setting. A proof in the non-semiclassical setting follows from similar arguments as in the Chapters 3 and 4. For simplicity, we will assume that the principal symbol comes from an an harmonic oscillator with irrationally related frequencies.

Theorem 2.7.5. *Let p be as above and assume that there are $j_0, j_1 \in \{1, \dots, d\}$ such that $\omega_{j_0}/\omega_{j_1} \in \mathbb{R} \setminus \mathbb{Q}$. Then,*

$$N(\lambda) = (2\pi)^{-d} \int_{\{p_2 + p_1 \leq \lambda\}} dx d\xi - (2\pi)^{-d} \int_{\{p_2 = \lambda\}} p_0(x, \eta) \frac{dS}{|\nabla p_2|} + o(\lambda^{d-1}),$$

where dS is the surface-measure on $\{p_2 = \lambda\}$.

Proof. To calculate the contribution of the trace at $t = 0$, we use the same argument as in the proof of Proposition 3.5.1. Therefore, it only remains to show that for any $\chi \in \mathcal{S}(\mathbb{R})$ with $\hat{\chi} \in \mathcal{C}_c^\infty((0, \infty))$ the trace fulfills

$$(dN * \chi)(\lambda) = o(\lambda^{d-1}).$$

By Proposition 2.7.3 it suffices to consider the case that $\operatorname{supp} \hat{\chi}$ is close to $\omega_{j_0}^{-1} \pi k$ for some fixed j_0 and some $k \in \mathbb{Z}$. By assumption there are two irrationally related frequencies, so there is at least one ω_{j_1} such that $\operatorname{supp} \hat{\chi} \cap \omega_{j_1}^{-1} \pi \mathbb{Z} = \emptyset$ Without loss of generality we may thus assume that there is a $1 \leq k < d$ such that

$$\begin{aligned} \operatorname{supp} \hat{\chi} \cap \frac{\pi}{\omega_j}(1/2 + \mathbb{Z}) = \emptyset, &\quad \text{for all } j \leq k, \\ \operatorname{supp} \hat{\chi} \cap \frac{\pi}{\omega_j}\mathbb{Z} = \emptyset, &\quad \text{for all } j > k. \end{aligned}$$

As the first step, we construct a parametrix of the operator e^{-itH_0} for arbitrary times $t > 0$. Since the propagator is just the product of the 1-dimensional propagators in each variable, we have to only consider the cases $j \leq k$ and $j > k$ and the full propagator is given by the product

$$e^{-itH_0}(x, y) = \prod_{j=1}^{d} U_0(\omega_j t, x_j, y_j).$$

Case 1: $j \leq k$. By Proposition 2.3.4 and the discussion thereafter, it is clear that

$$\hat{\chi}(t) U_0(\omega_j t, x_j, y_j) = \hat{\chi}(t) a_j(t) \int e^{i(\phi_2(\omega_j t, x_j, \eta_j) - y_j \eta_j)} d\eta,$$

where $a_j(t) = i^\nu (2\pi)^{-1/2} \cos(\omega_j t)^{-1/2}$ and $\nu \in \{0, 1, 2, 3\}$ is a Maslov factor.

Case 2: $j > k$. By [17, Exercise 11.1]

$$\chi(t) U_0(\omega_j t, x_j, y_j) = \hat{\chi}(t) a_j(t) e^{i\tilde{\phi}_2(\omega_j t, x_j, y_j)},$$

where $a_j(t) = i^\nu (2\pi)^{-1/2} \sin(\omega_j t)^{-1/2}$ and $\nu \in \{0, 1, 2, 3\}$ is a Maslov factor. The phase function is given by

$$\tilde{\phi}_2(t, x_j, y_j) = \frac{1}{\sin(t)}(-x_j y_j + \frac{1}{2}\cos(t)(x_j^2 + y_j^2)).$$

Combining this yields

$$e^{-itH_0}(x, y) = \int_{\mathbb{R}^k} e^{\phi(t,x,y,\eta)} a(t)\, d\eta,$$

with $a(t) = \prod_{j=1}^{d} a_j(t)$ and

$$\phi(t, x, y, \eta) = \sum_{j=1}^{k} (\phi_2(\omega_j t, x_j, \eta_j) - y_j \eta_j) + \sum_{j=k+1}^{d} \tilde{\phi}_2(\omega_j t, x_j, y_j).$$

We want to construct a parametrix for $U(t) = U_0(t) F(t)$, where the reduced propagator $F(t)$ solves (2.13) with $P(t) = e^{itH_0}(H - H_0) e^{-itH_0}$.

By Lemma 2.6.1 there is a parametrix $\tilde{F}(t)$ of $F(t)$, given by

$$\tilde{F}(t) = \int e^{i(x-y)\xi + i\phi_1(t,y,\xi)} b(t, y, \xi)\, đ\xi,$$

where $\phi_1(t, y, \xi) = -\int_0^t p_1 \circ \exp(t\mathsf{H}_{p_2}) dt$ and $b \in \mathcal{C}^\infty(\mathbb{R}, \Gamma^0_{\mathrm{cl}})$. The composition is almost trivial, because $\tilde{F}(t)$ is right-quantized and the composition is reduced to the identity

$$\delta(x - y) = \int e^{i(x-y)\xi}\, đ\xi.$$

Summing up, we obtain that a parametrix for $U(t)$ is given by

$$\tilde{U}(t,x,y) = \int_{\mathbb{R}^k} e^{i(\phi(t,x,y,\eta)+\phi_1(t,y,\eta,0))} a(t) b(t,y,\eta,0)\, d\eta.$$

Here $(\eta, 0) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$.

Now, calculating the inverse Fourier transform of $\operatorname{Tr} \tilde{U}(t)\hat{\chi}(t)$ in t and changing coordinates $(x,y,\eta) \mapsto \lambda^{1/2}(x,y,\eta)$ yields

$$\begin{aligned} &\mathcal{F}^{-1}_{t\to\lambda}\{\hat{\chi}(t) \operatorname{Tr} \tilde{U}(t,x,x)\}(\lambda) \\ &= \lambda^{(d+k)/2} \int_{\mathbb{R}^{d+k}} e^{i\lambda(\phi(t,x,x,\eta)+\lambda^{-1/2}\phi_1(t,x,\eta,0)+t)} \hat{\chi}(t) a(t) b(t, \lambda^{1/2}x, \lambda^{1/2}\eta, 0)\, d\eta\, dx. \end{aligned}$$

This oscillatory integral satisfies the assumptions 1.–4. from Proposition 3.4.1 and thus, we use the same argument as in the first step of the proof of Proposition 3.4.1 to conclude that

$$(dN * \chi)(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}\{\hat{\chi}(t) \operatorname{Tr} \tilde{U}(t,x,x)\}(\lambda) + O(\lambda^{-\infty}) = O(\lambda^{(d+k)/2-1})$$

and since $k < d$ we have shown that

$$(dN * \chi)(\lambda) = o(\lambda^{d-1}).$$

□

CHAPTER 3

Improved Remainder Estimates for the Weyl Asymptotic

3.1. Introduction

This chapter is taken from the article [11]. The section about the isotropic calculus has been deleted. The results which are needed can be found in Section 2.4.

3.1.1. Main Results

Let $H_0 = \frac{1}{2}(\Delta + |x|^2)$ denote the isotropic harmonic oscillator on $\mathbb{R}^d$, where Δ is the non-negative Laplacian. Thus H_0 is the Weyl quantization $H_0 = \mathrm{Op}^w(p_2)$, where $p_2 = (1/2)(|x|^2 + |\xi|^2)$. Consider a perturbation

$$H = \mathrm{Op}^w(p),$$

where p differs from p_2 by a classical *isotropic* 1-*symbol*. In other words, p admits an asymptotic expansion

$$p \sim p_2 + p_1 + p_0 + \dots, \tag{3.1}$$

where each p_j is homogeneous of degree j jointly in (x, ξ). Furthermore, assume that p is real valued, hence $H^* = H$ by properties of the Weyl calculus.

Since $p_2(x,\xi) > 0$ for $(x,\xi) \neq 0$, the resolvent of H is compact and H has discrete spectrum

$$\lambda_1 \leq \lambda_2 \leq \cdots \to +\infty,$$

where each eigenvalue is listed with multiplicity. Let E_λ denote the corresponding spectral projector onto $(-\infty, \lambda]$, so if $N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$ is the counting function, then $N(\lambda) = \mathrm{Tr}\, E_\lambda$. Moreover, the Fourier transform of the spectral measure satisfies $U(t) = \mathcal{F}_{\lambda\to t} dE_\lambda$, where $U(t)$ is the propagator for the time-dependent Schrödinger equation

$$\begin{cases} (i\partial_t - H)U(t) = 0 \\ U(0) = \mathrm{I}\,. \end{cases}$$

This implies that

$$\mathcal{F}_{\lambda\to t} N'(\lambda) = \mathrm{Tr}\, U(t), \tag{3.2}$$

where the trace of $U(t)$ is defined as a tempered distribution (cf. [13] and Section 3.3.6). It is clear from (3.2) that there is a relationship between the singularities of $\operatorname{Tr} U(t)$ and the growth of $N(\lambda)$ as $\lambda \to \infty$. A proof of the following Poisson relation can be found in [24], but we will give a short and simple proof in the special case of interest here:

Proposition 3.1.1. *Singularities of the Schrödinger trace* $\operatorname{Tr} U(t)$ *satisfy*

$$\text{sing-supp} \operatorname{Tr} U(t) \subset 2\pi\mathbb{Z}.$$

Let H_0 denote the Hamilton vector field of $p_2 = (1/2)(|x|^2 + |\xi|^2)$, whose flow $(x(t), \xi(t)) = \exp(t\mathsf{H}_0)(x_0, \xi_0)$ satisfies

$$\begin{aligned} x(t) &= \cos(t)x_0 + \sin(t)\xi_0, \\ \xi(t) &= \cos(t)\xi_0 - \sin(t)x_0. \end{aligned}$$

Given a function $f \in \mathcal{C}^\infty(\mathbb{R}^{2d-1})$, let $\mathsf{X}f$ denote[1] the average of f over one period of the flow,

$$\mathsf{X}f(x,\xi) = \int_0^{2\pi} f(\exp(t\mathsf{H}_0)(x,\xi))\, dt. \tag{3.3}$$

When restricted to the sphere, $\mathsf{X}f$ can also be viewed as the average of f over the fibers of the complex Hopf fibration $\mathbb{S}^{2d-1} \to \mathbb{CP}^{d-1}$. Indeed, consider the map

$$(x,\xi) \mapsto x + i\xi,$$

which identifies $\mathbb{R}^{2d}$ with $\mathbb{C}^d$. This map intertwines the action of $\exp(t\mathsf{H}_0)$ with complex rotations $z \mapsto e^{-it}z$, and by restriction to $\mathbb{S}^{2d-1}$ the latter action induces the complex Hopf fibration $\mathbb{S}^{2d-1} \to \mathbb{CP}^{d-1}$ with fiber $\mathbb{S}^1$.

The following theorem, which constitutes the main result of this chapter, shows that the singularities of $\operatorname{Tr} U(t)$ at nonzero times, and hence also the remainder term in the Weyl law, depend on properties of $\mathsf{X}p_1$ (recall from (3.1) that p_1 is the subprincipal symbol of H).

Theorem 3.1.2. *Assume that when restricted to* $\mathbb{S}^{2d-1}$*, the set where* $\nabla \mathsf{X}p_1$ *vanishes to infinite order has measure zero. If* $\chi \in \mathcal{C}_c^\infty((-2\pi, 2\pi))$*, then for all* $n \in \mathbb{Z}\backslash\{0\}$*,*

$$\mathcal{F}^{-1}_{t\to\lambda}\chi(t - 2\pi n) \operatorname{Tr} U(t) = o(\lambda^{d-1}). \tag{3.4}$$

If $\mathsf{X}p_1$ *is Morse–Bott on* $\mathbb{S}^{2d-1}$ *with* $k > 0$ *nondegenerate directions, then*

$$\mathcal{F}^{-1}_{t\to\lambda}\chi(t - 2\pi n) \operatorname{Tr} U(t) = O(\lambda^{d-1-k/4}). \tag{3.5}$$

In either of the cases considered above, there holds the Weyl formula

$$N(\lambda) = (2\pi)^{-d} \int_{\{p_2+p_1\le\lambda\}} dx d\eta - (2\pi)^{-d} \int_{\{p_2=\lambda\}} p_0(x,\eta) \frac{dS}{|\nabla p_2|} + o(\lambda^{d-1}). \tag{3.6}$$

[1] This is a kind of X-ray transform, hence the notation.

Observe that $\mathsf{X}p_1$ is never Morse since it is constant along the integral curves of H_0. On the other hand, the pullback of a Morse function on $\mathbb{CP}^{d-1}$ by the complex Hopf fibration yields a function p_1 on $\mathbb{S}^{2d-1}$ such that $\mathsf{X}p_1$ admits $2d-2$ nondegenerate directions. Thus in any dimension $d \geq 2$ there are always examples of p_1 satisfying the Morse–Bott hypothesis of Theorem 3.1.2.

The two-term Weyl asymptotic (3.6) in Theorem 3.1.2 should be viewed as a refinement of the more general asymptotic formula

$$N(\lambda) = (2\pi)^{-d}\lambda^d \int_{\{p_2 \leq 1\}} dx d\eta - (2\pi)^{-d}\lambda^{d-1/2} \int_{\{p_2 = 1\}} p_1 \frac{dS}{|\nabla p_2|} + O(\lambda^{d-1}) \tag{3.7}$$

established earlier by Helffer–Robert [23] for arbitrary 1-symbol perturbations. Indeed, (3.7) is recovered from the leading order term in (3.6) by writing the volume of $\{p_2 + p_1 \leq \lambda\}$ as λ^d times the volume of $\{p_2 + \lambda^{-1/2} p_1 \leq 1\}$ and expanding the latter volume in powers of $\lambda^{-1/2}$.

The necessity of a nondegeneracy hypothesis on p_1 in Theorem 3.1.2 is apparent already from the unperturbed harmonic oscillator H_0. Its eigenfunctions are given by products of Hermite functions, defined for a multiindex $\alpha = (\alpha_1, \ldots, \alpha_d)$ by

$$\psi_\alpha(x) = \pi^{-d/4}(2^{|\alpha|}\alpha!)^{-1/2} H_\alpha(x) e^{-|x|^2/2},$$

with H_j the j'th Hermite polynomial and $H_\alpha = \prod_{j=1}^d H_{\alpha_j}(x_j)$; the corresponding eigenvalues are

$$|\alpha| + \frac{d}{2}.$$

Thus the eigenvalues are $\lambda = j + d/2$ for $j \in \mathbb{N}$, arising with multiplicity

$$p(\lambda - d/2, d),$$

where $p(j,d)$ denotes the the number of ways of writing j as a sum of d nonnegative integers. Since in fact

$$p(j,d) = \binom{d+j-1}{j},$$

and this quantity is bounded below for $j \in \mathbb{N}$ by a multiple of j^{d-1}, the remainder term in the Weyl law for H_0 certainly cannot be $o(\lambda^{d-1})$.

The improvement in the Weyl law is not directly related to the propagation of singularities: If $u \in \mathcal{S}'$, we show that

$$\mathrm{WF}_{\mathrm{cl}}(U(2\pi k)u) = \{(x + k\partial_\xi(\mathsf{X}p_1)(0,\xi), \xi) : (x,\xi) \in \mathrm{WF}_{\mathrm{cl}}(u)\}.$$

If we consider the operator $H = H_0 + \sqrt{H_0}$, for which the symbol of the perturbation is $p_1(x,\xi) = \sqrt{p_2(x,\xi)}$, we see that singularities at time $t = 2\pi k$ are shifted by $2\pi k \partial_\xi |\xi|$. On the other hand there is no improvement in the Weyl law, because the eigenvalues of H are $j + d/2 + \sqrt{j + d/2}$ and the multiplicity remains $p(j,d)$.

3.1.2. Strategy of Proof

As in §3.1.1, denote the free Hamiltonian (namely the exact harmonic oscillator) by $H_0 = \mathrm{Op}^w(p_2)$ and the perturbed one by $H = \mathrm{Op}^w(p)$. Further, let

$$U(t) = e^{-itH}, \quad U_0(t) = e^{-itH_0}, \quad F(t) = U_0(-t)U(t)$$

be the perturbed, free, and "reduced" propagator, respectively. Then, $F(t)$ satisfies the evolution equation

$$\begin{cases} (i\partial_t - P(t))F(t) = 0 \\ \qquad\qquad F(0) = \mathrm{I}, \end{cases} \tag{3.8}$$

where $P(t) = U_0(-t)(H - H_0)U_0(t)$. The main strategy is to show, following the methods of Helffer–Robert in [23], that $F(t)$ has an oscillatory integral parametrix with an explicit phase function. It is then possible to construct a parametrix for $U(t)$ by composing the parametrix for $F(t)$ with the free propagator $U_0(t)$, whose Schwartz kernel is given explicitly by Mehler's formula. Finally, via another more delicate stationary phase computation, we arrive at estimates on the singularities of $\operatorname{Tr} U(t)$. The results on spectral asymptotics then follow via a known Tauberian theorem (see Lemma 3.5.2).

3.1.3. Prior Results

It has been known since the work of Zelditch [68] (see also [64]) that singularities of the propagator for perturbations of the harmonic oscillator by a symbolic potential $V(x) \in S^0(\mathbb{R}^d)$ reconstruct at times $t \in \pi\mathbb{Z}$. Moreover, if the potential is merely bounded with all its derivatives, Zelditch showed that $\operatorname{sing-supp} \operatorname{Tr} U(t) \subset 2\pi\mathbb{Z}$. It was later shown by Kapitanski–Rodnianski–Yajima [34] that the singular support of $\operatorname{Tr} U(t)$ is contained in $2\pi\mathbb{Z}$ supposing only that the perturbation is subquadratic.

More general propagation of singularities for geometric generalizations of the harmonic oscillator to manifolds with large conic ends ("scattering manifolds") was also studied by the Wunsch in [65] and refined by Mao–Nakamura [38], which allows for perturbations in the symbol class $S^{1-\epsilon}(\mathbb{R}^d)$ for any $\epsilon > 0$.

That something dramatic happens for potential perturbations in $S^1(\mathbb{R}^d)$, by contrast, is clear from the results of Doi [9], where the author shows that the location in space of the singularities of the Schrödinger propagator at times $t \in \pi\mathbb{Z}$ is indeed subject to an interesting geometric shift from this type of perturbation.

Helffer–Robert [23] studied the singularity at $t = 0$ of the Schrödinger trace (and, consequently, the Weyl law) for the class of perturbations under consideration here, viz., those that are isotropic operators of order 1. While this class does not include potential perturbations of order 1, hence is perhaps less natural on physical grounds, it is more natural from the point of view of symplectic geometry. The analysis in [23] was limited to the study of the main singularity at $t = 0$, hence did not include the considerations of the global flow studied here.

The parametrix construction of [23] is essential in our work, however, as we extend (a version of) it to long times via composition with the free propagator.

The novelty of our result lies in the delicate perturbation resulting from a one-symbol. This is unlike the case famously considered by Duistermaat–Guillemin in [13] under which a genericity hypothesis on the geodesic flow yields an improvement to the Weyl law remainder for the Laplacian on a compact manifold. For isotropic pseudodifferential operators the analoguous version of the theorem by Duistermaat–Guillemin is also true (see for instance Petkov–Robert [51] in the semiclassical setting). In our case, the most naive version of propagation of singularities, as described by isotropic wavefront set, is unaffected by the perturbation. The perturbative effect can be seen heuristically as a higher-order correction to the motion of Lagrangian subspaces of $T^*\mathbb{R}^n$: at times $t \in 2\pi\mathbb{Z}$, the Lagrangian $N^*\{0\}$, for instance, has evolved under the bicharacteristic flow to another Lagrangian that is asymptotic to $N^*\{0\}$ as $|\xi| \to \infty$, but it is the next-order term in the asymptotics of this Lagrangian that governs the contribution to the Schrödinger trace, and hence to the Weyl law remainder term.

3.2. Singularities of the Trace

3.2.1. Propagation of Isotropic Wavefront Set

Since $P(t) = U_0(-t)(H - H_0)U_0(t)$ and $H - H_0 \in G^1$, it follows from the exact Egorov theorem that

$$P(t) \in G^1, \quad P(t)^* = P(t), \tag{3.9}$$

and $P(t)$ is in fact a smooth family of such operators. Somewhat surprisingly, the evolution generated by $P(t)$ does not move around isotropic wavefront set; this uses essentially the property of the isotropic calculus that errors are two orders lower. The analogous result of course fails for usual wavefront set if $P(t)$ is replaced with an ordinary first order, self-adjoint pseudodifferential operator such as $\sqrt{\Delta}$.

Lemma 3.2.1. *Let $P(t) \in G^1$ be a smooth family of self-adjoint operators, and assume there is a solution $F(t)$ of the equation*

$$\begin{cases} (i\partial_t - P(t))F(t) = 0 \\ F(0) = \mathrm{I} \end{cases}$$

such that $F \in \mathcal{C}^0(\mathbb{R}_t; \mathcal{L}(H^s_{\text{iso}}, H^s_{\text{iso}})) \cap \mathcal{C}^1(\mathbb{R}_t; \mathcal{L}(H^s_{\text{iso}}, H^{s-1}_{\text{iso}}))$ for each $s \in \mathbb{R}$. Then, the isotropic wavefront set satisfies $\mathrm{WF}_{\text{iso}}\, F(t)u = \mathrm{WF}_{\text{iso}}\, u$ for each $u \in \mathcal{S}'$ and $t \in \mathbb{R}$.

Proof. Suppose that $u \in \mathcal{S}'$, hence there exists s_0 such that $u \in H^{s_0}_{\text{iso}}$, and by hypothesis $F(t)u \in H^{s_0}_{\text{iso}}$ for all $t \in \mathbb{R}$. The goal is to show by induction that for every k, the set $\mathrm{WF}^k_{\text{iso}}\, F(t)u$ is invariant; this is trivially true for $k = s_0$, as the wavefront set remains empty.

Suppose that $U \subset \mathbb{S}^{2d-1}$ is open, and $\mathrm{WF}^k_{\text{iso}}\, u \cap U = \emptyset$. The inductive step is completed by showing that

$$\mathrm{WF}^{k-1}_{\text{iso}}\, F(t)u \cap U = \emptyset \Longrightarrow \mathrm{WF}^k_{\text{iso}}\, F(t)u \cap U = \emptyset.$$

Let $A \in G^k$ be fixed independently of t such that $\mathrm{WF}' A \subset U$. Choose a bounded family

$$\{A_\varepsilon \colon \varepsilon \in [0,1)\} \subset G^k$$

such that $A_0 = A$ and $A_\varepsilon \in G^{k-1}$ for each $\varepsilon \in (0,1)$. Furthermore, assume that $\mathrm{WF}' A_\varepsilon \subset U$ for $\varepsilon \in [0,1)$. For instance, let $A_\varepsilon = S_\varepsilon A$, where

$$S_\varepsilon = \mathrm{Op}^w((1+\varepsilon(|x|^2+|\xi|^2))^{-1/2}).$$

Observe in this case that $A_\varepsilon \to A$ in the topology of G^{k+1}. Using (3.8), compute

$$\frac{d}{dt} A_\varepsilon F(t) = -iP(t)(A_\varepsilon F(t)) - i[A_\varepsilon, P(t)]F(t).$$

Since $P(t)$ is self-adjoint and $A_\varepsilon F(t)u \in L^2(\mathbb{R}^d)$ by the inductive hypothesis,

$$\begin{aligned} \frac{d}{dt}\|A_\varepsilon F(t)u\|^2 &= 2\operatorname{Re}\left((d/dt)A_\varepsilon F(t)u, A_\varepsilon F(t)u\right)_{L^2} \\ &= -2\operatorname{Re}\left(i[A_\varepsilon, P(t)]F(t)u, A_\varepsilon F(t)u\right)_{L^2} \\ &\le 2\|A_\varepsilon F(t)u\|\,\|[A_\varepsilon, P(t)]F(t)u\|. \end{aligned} \tag{3.10}$$

On the other hand, since A_ε is bounded in G^k, it (crucially) follows that $[A_\varepsilon, P(t)]$ is bounded in G^{k-1} for $\varepsilon \in [0,1)$. Furthermore, the operator wavefront set of $[A_\varepsilon, P(t)]$ is contained in U. Now integrate to find that

$$\|A_\varepsilon F(t)u\|^2 \le e^t\|A_\varepsilon u\|^2 + e^t\int_0^t e^{-s}\|[A_\varepsilon, P(s)]F(s)u\|^2\,ds$$

for each fixed t, where the right hand side is uniformly bounded as $\varepsilon \to 0$. From the weak compactness of the unit ball in $L^2(\mathbb{R}^d)$, conclude that $A_{\varepsilon_k}F(t)u$ has a weak limit in $L^2(\mathbb{R}^d)$ along a sequence of $\varepsilon_k \to 0$, hence in $\mathcal{S}'(\mathbb{R}^d)$ as well. On the other hand, $A_\varepsilon F(t)u \to AF(t)u$ in $\mathcal{S}'(\mathbb{R}^d)$, since $A_\varepsilon \to A$ in G^{k+1}. It follows that $AF(t)u \in L^2(\mathbb{R}^d)$, and we have shown that for $t > 0$,

$$\mathrm{WF}_{\mathrm{iso}} F(t)u \subset \mathrm{WF}_{\mathrm{iso}} u.$$

To obtain the reverse inclusion, we repeat the argument above, integrating a time-reversed version of (3.10) from t to 0 instead of 0 to t. □

Lemma 3.2.1 can be applied to the evolution equation (3.8): in that case $F(t) = U_0(-t)U(t)$ and both operators in this composition preserve H^s_{iso} for each s; thus $F(t)$ has the requisite mapping properties. The invariance of isotropic wavefront set under $U(t)$ follows directly from Lemma 3.2.1:

Proposition 3.2.2. *For all $u \in \mathcal{S}'$ and $t \in \mathbb{R}$,*

$$\mathrm{WF}_{\mathrm{iso}} U(t)u = \mathrm{WF}_{\mathrm{iso}} U_0(t)u = \exp(t\mathsf{H}_0)\,\mathrm{WF}_{\mathrm{iso}} u.$$

Proof. Since $U_0(t)u = F(-t)U(t)u$, the first equality follows from Lemma 3.2.1, while the second follows from the exact Egorov theorem for $U_0(t)$. □

Equipped with Proposition 3.2.2, there is a simple proof of Proposition 3.1.1 following the strategy of [65].

Proof of Proposition 3.1.1. Pick any small interval I not containing a multiple of 2π. By compactness of the sphere, there exists a partition of unity $\{a_j^2 \colon j \in J\}$ of $\mathbb{S}^{2d-1}$ such that $a_j \cdot (a_j \circ \exp(t\mathsf{H}_0)) = 0$ for all $j \in J$ and $t \in I$. Using an iterative construction in the calculus, it is possible to find $A_j \in G^0$ satisfying $\sigma_0(A_j) = a_j$ and $\mathrm{WF}'(A) \subset \operatorname{supp} a_j$, such that

$$\sum A_j^2 = \mathrm{I} + R,$$

where $R \in G^{-\infty}$ (cf. [65, Corollary 4.7]). Then, computing in the sense of tempered distributions,

$$\begin{aligned} \operatorname{Tr} U(t) &= \operatorname{Tr} \sum A_j^2 U(t) - RU(t) \\ &= \operatorname{Tr} \sum A_j U(t) A_j - RU(t). \end{aligned} \tag{3.11}$$

The term $A_j U(t) A_j$ maps $\mathcal{S}' \to \mathcal{S}$ by propagation of singularities (Proposition 3.2.2), as do all its derivatives, and $RU(t)$ also has this property. Hence the right hand side of (3.11) and all its derivatives are bounded for $t \in I$, so $\operatorname{Tr} U(t) \in \mathcal{C}^\infty(I)$. This completes the proof of Proposition 3.1.1. □

3.3. Parametrix

3.3.1. Oscillatory Integrals

Throughout the rest of the chapter it will be important to consider oscillatory integrals of the form

$$I(a,\psi)(z) = \int e^{i\psi(z,\eta)} a(z,\eta)\, d\eta, \quad (z,\eta) \in \mathbb{R}^k \times \mathbb{R}^m, \tag{3.12}$$

where ψ is a real-valued quadratic form in (z,η). References for this material are [22, Chapter III] and [1]. If ψ satisfies the nondegeneracy hypothesis

$$\operatorname{rank} \begin{pmatrix} \partial^2_{\eta z}\psi & \partial^2_{\eta\eta}\psi \end{pmatrix} = k + m, \tag{3.13}$$

then (3.12) defines a distribution $I(a,\psi) \in \mathcal{S}'(\mathbb{R}^k)$ provided the amplitude $a(z,\eta) \in \mathcal{C}^\infty(\mathbb{R}^{k+m})$ satisfies

$$|\partial^\alpha_{z,\eta} a(z,\eta)| \le C_\alpha \langle z \rangle^M \langle \eta \rangle^M \tag{3.14}$$

for some fixed $M \in \mathbb{R}$ and every α. This also means it is possible to consider phases of the form

$$\psi = \psi_2 + \psi_1,$$

where ψ_2 is a quadratic form satisfying (3.13), and ψ_1 is real-valued satisfying the bounds

$$|\partial_{z,\eta}^{\alpha}\psi_1(z,\eta)| \leq C_\alpha$$

for each $|\alpha| \geq 1$. Indeed, for the purposes of regularization, it suffices to absorb $e^{i\psi_1}$ into the amplitude, since $e^{i\psi_1}a$ satisfies (3.14).

Now suppose that ψ is a real-valued quadratic form in $(x, y, \eta) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m$. If ψ satisfies

$$\det \begin{pmatrix} \partial^2_{xy}\psi & \partial^2_{x\eta}\psi \\ \partial^2_{\eta y}\psi & \partial^2_{\eta\eta}\psi \end{pmatrix} \neq 0 \tag{3.15}$$

and $a(x, y, \eta)$ satisfies (3.14) with $z = (x, y)$, then $I(a, \psi)(x, y)$ is the Schwartz kernel of an operator mapping $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. Furthermore, if $a(x, y, \eta) \in \mathcal{S}(\mathbb{R}^{2d} \times \mathbb{R}^m)$, then the corresponding operator is residual, namely it maps $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

3.3.2. Mehler's Formula

As discussed in Section 3.1.2, the goal is to approximate $U(t)$ by first approximating $F(t)$ by an operator with oscillatory integral kernel of the form

$$\tilde{F}(t)(x, y) = \int e^{i\langle x-y,\eta\rangle + i\phi_1(t,x,\eta)} a(t, x, \eta)\, d\eta,$$

where $\tilde{F}(t) - F(t)$ is regularizing in suitable sense, and ϕ_1 is an explicit phase function which is homogeneous of degree 1 in (x, η). This is useful since $U(t) = U_0(t)F(t)$, and the Schwartz kernel of $U_0(t)$ is explicitly given by Mehler's formula, which is now recalled.

Begin by defining the phase function

$$\phi_2(t, x, \eta) = \sec(t)(\langle x, \eta\rangle - \sin(t)(|x|^2 + |\eta|^2)/2), \tag{3.16}$$

where $(x, \eta) \in \mathbb{R}^{2d}$. This is well defined for any $t \notin 2\pi\mathbb{Z} \pm \pi/2$, and for any such t the quadratic form $\phi_2(t, x, \eta) - \langle y, \eta\rangle$ satisfies (3.15). It is well known that the Schwartz kernel of $U_0(t)$ satisfies

$$U_0(t)(x, y) = (2\pi)^{-d} \frac{(-1)^{dn}}{\cos(t)^{d/2}} \int e^{i\phi_2(t,x,\eta) - i\langle y,\eta\rangle}\, d\eta,$$

where n is such that $t - 2\pi n \in (-\pi/2, \pi/2)$. Thus $U_0(t)(x, y)$ is of the form (3.12), where for each fixed t the amplitude is constant.

3.3.3. Parametrix for the Reduced Propagator

Recall that the reduced propagator $F(t) = U_0(-t)U(t)$ solves the evolution equation

$$\begin{cases} (i\partial_t - P(t))F(t) = 0 \\ F(0) = \mathrm{I}\,. \end{cases} \tag{3.17}$$

Here $P(t) \in G^1_{\text{cl}}$ is a smooth family of classical isotropic operators, and in the notation of (3.1) its total Weyl symbol $\mathsf{p}(t)$ satisfies

$$\mathsf{p}(t) = (p - p_2) \circ \exp(t\mathsf{H}_0)$$

by the exact Egorov theorem. In particular, its homogeneous of degree 1 principal symbol $\mathsf{p}_1(t) = \sigma_1(\mathsf{p}(t))$ is simply $\mathsf{p}_1(t) = p_1 \circ \exp(t\mathsf{H}_0)$. Define

$$\phi_1(t, x, \xi) = -\int_0^t p_1 \circ \exp(s\mathsf{H}_0)(x, \xi)\, ds, \tag{3.18}$$

noting for future reference that $\phi_1(2\pi n, \bullet) = -\mathsf{X}^n p_1 = -n\mathsf{X}p_1$ for each $n \in \mathbb{Z}$, where $\mathsf{X}p_1$ is given by (3.3).

In the following lemma we construct an oscillatory integral parametrix for $F(t)$.

Lemma 3.3.1. *There exists $a \in \mathcal{C}^\infty(\mathbb{R}_t; \Gamma^0_{\text{cl}})$ and an operator $\tilde{F}(t)$ with Schwartz kernel*

$$\tilde{F}(t)(x, y) = \int e^{i\langle x-y,\xi\rangle + i\phi_1(t,x,\xi)} a(t, x, \xi)\, d\xi \tag{3.19}$$

approximately solving (3.17) in the sense that

$$(i\partial_t - P(t))\tilde{F}(t) \in \mathcal{C}^\infty(\mathbb{R}_t; \mathcal{L}(\mathcal{S}', \mathcal{S})), \quad \tilde{F}(0) = \mathrm{I} + K,$$

where $K : \mathcal{S}' \to \mathcal{S}$. Here, the function ϕ_1 is given by (3.18).

Note that unlike the construction of [23] (which we are adapting to our purposes), this holds for arbitrarily long time.

Proof. We seek an approximate solution to (3.17) of the form (3.19). The starting point is the action of an isotropic pseudodifferential operator on oscillatory integral of the form (3.19), as in [23, Section III] or [22, Theorem 2.5.1]. In order to apply these results directly, first write $P(t)$ as a left quantization,

$$P(t) = \mathrm{Op}_L(\widetilde{\mathsf{p}}(t)),$$

where the homogeneous degree 1 part of $\widetilde{\mathsf{p}}(t)$ is still $\mathsf{p}_1(t)$.

Suppose that $a \in \mathcal{C}^\infty(\mathbb{R}_t; \Gamma^0_{\text{cl}})$ and $\phi_1 \in \mathcal{C}^\infty(\mathbb{R}_t \times \mathbb{R}^{2d})$. Let $b(t, x, \xi) = e^{i\phi_1(t,x,\xi)} a(t, x, \xi)$, and then define

$$c(t, x, \xi) = e^{-i\langle x,\xi\rangle} P(t)(e^{i\langle \bullet,\xi\rangle} b(t, \bullet, \xi)).$$

Referring to [23, Section III], it follows that c has an asymptotic expansion

$$c(t, x, \xi) = \sum_{|\alpha|<N} c_\alpha(t, x, y, \xi) + c^{(N)}(t, x, y, \xi),$$

where c_α is given by the formula

$$c_\alpha(t,x,\xi) = (\alpha!)^{-1}\partial_\xi^\alpha \widetilde{\mathsf{p}}(t,x,\xi) D_x^\alpha b(t,z,\xi).$$

Furthermore, given $T > 0$ and $t \in [-T, T]$, the remainder $c^{(N)}$ satisfies the uniform bound

$$|\partial_t^k \partial_x^\beta \partial_\xi^\gamma c^{(N)}(t,x,\xi)| \leq C_{k\beta\gamma} \langle (x,\xi) \rangle^{k+1-N}. \tag{3.20}$$

Disregarding smoothness at $(x,\xi) = 0$ at first, formally apply this result with a symbol having an asymptotic expansion

$$\sum_{k=0}^{\infty} a^{(k)}(t,x,\xi),$$

where each $a^{(k)}(t,\bullet)$ is homogeneous of degree $-k$ outside a compact set, and $\phi(t,\bullet)$ which is homogeneous of degree 1. Recalling that $b = e^{i\phi_1}a$ and separating terms by homogeneity, first obtain from (3.17) the eikonal equation

$$\begin{cases} \partial_t \phi_1 + \mathsf{p}_1(t,x,\xi) = 0 \\ \phi_1(0,x,\xi) = 0. \end{cases}$$

This equation is solved by (3.18), recalling that $\mathsf{p}_1(t) = p_1 \circ \exp(t\mathsf{H}_0)$. Next, obtain a sequence of transport equations, the first of which has the form

$$\begin{cases} \partial_t a^{(0)} = f(t,x,\xi) a^{(0)} \\ a^{(0)}(0,x,\eta) = 1, \end{cases}$$

where $f(t,x,\xi)$ is homogeneous of degree 0. Observe that this equation can be solved for all time since the characteristics are straight lines. There are similar expressions for $a^{(k)}$ (with inhomogeneous term depending on $a^{(0)}, \dots, a^{(k-1)}$ and with vanishing initial value). Let $\widetilde{a} \in \mathcal{C}^\infty(\mathbb{R}_t; \mathbb{R}^{2d} \setminus \{0\})$ be such that

$$\widetilde{a}(t,x,\xi) \sim \sum_{k=0}^{\infty} a^{(k)}(t,x,\xi), \tag{3.21}$$

and then set $a(t,x,\xi) = \zeta(x,\xi)\widetilde{a}(t,x,\xi)$, where $\zeta \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ is such that $\zeta(x,\xi) = 0$ for $|(x,\xi)| \leq 1$ and $\zeta(x,\xi) = 1$ for $|(x,\xi)| \geq 2$. Thus a is everywhere smooth, and ϕ_1 is also smooth on the support of a.

Let $\tilde{F}(t)$ be given by (3.19), and $F_N(t)$ be the corresponding integral when (3.21) is summed from 0 to N. There are two errors when applying $(i\partial_t - H)$ to $F_N(t)$: the first arises since the eikonal and transport equations are only satisfied outside a compact set, hence the corresponding error is residual. The second error arises since the corresponding amplitude a_N is only a finite sum of terms. For this we simply cite [23, Lemma III.6] for mapping properties of the corresponding oscillatory integral with amplitude $c^{(N+1)}(t,x,\xi)$. Since N is arbitrary, the proof is complete. □

Observe that $\tilde{F}(t)(x,y)$ is indeed the distributional kernel of an operator $\mathcal{S}' \to \mathcal{S}'$ as described in Section 3.3.1: clearly the quadratic form $\langle x-y,\xi\rangle$ satisfies the hypotheses (3.13), and as in the proof of Lemma 3.3.1 is may be assumed that ϕ_1 is smooth on the support of a.

3.3.4. Composition

In this section we analyze the composition $\widetilde{U}(t) = U_0(t)\tilde{F}(t)$, which will give a parametrix for $U(t)$. Observe that $\widetilde{U}(t)$ is well defined as an operator between tempered distributions, for example.

Although some information about the composition can be gleaned from the general theory in [22, Chapter 2], a more precise description of the resulting phase is needed here; for this reason the calculations that follow will be explicit. Write

$$\begin{aligned}\tilde{F}(t) &= \int e^{i\langle x-y,\eta\rangle + i\phi_1(t,x,\eta)} b_1(t,x,\eta)\,d\eta,\\ U_0(t) &= \int e^{i\phi_2(t,x,\eta) - i\langle y,\eta\rangle} b_2(t,x,\eta)\,d\eta,\end{aligned}$$

for appropriate amplitudes $b_j \in \mathcal{C}^\infty(\mathbb{R}_t;\Gamma^0)$, where ϕ_2 is given by (3.16), and ϕ_1 is given by (3.18). Of course the formula for $U_0(t)$ only makes sense if $t - 2\pi n \in (-\pi/2,\pi/2)$ for some $n \in \mathbb{Z}$. As remarked at the end of the previous section, it may be assumed that ϕ_1 is smooth on the support of b_1.

Formally then, the composition has Schwartz kernel

$$\widetilde{U}(t)(x,y) = \int e^{i\phi_2(t,x,\eta) - i\langle y,\eta\rangle + i\phi_1(t,y,\eta)} b(t,x,y,\eta)\,d\eta,$$

where the amplitude $b = b(t,x,y,\eta)$ is given by

$$\begin{aligned}b(t,x,y,\eta) &= \int e^{i\langle z-y,\xi-\eta\rangle + i(\phi_1(t,z,\xi)-\phi_1(t,y,\eta))} b_2(t,x,\eta) b_1(t,z,\xi)\,dzd\xi\\ &= \int e^{i\langle z,\xi\rangle} e^{i\phi_1(t,y+z,\eta+\xi) - i\phi_1(t,y,\eta)} b_2(t,x,\eta) b_1(t,y+z,\eta+\xi)\,dzd\xi.\end{aligned} \tag{3.22}$$

In analyzing the latter integral, there is no difficulty in supposing more generally that $b_j \in \mathcal{C}^\infty(\mathbb{R}_t;\Gamma^{m_j})$ for some $m_j \in \mathbb{R}$. Since all the dependence on t henceforth will be smooth and parametric, for notational simplicity the dependence on t will be suppressed. Define

$$a_0(x,y,z,\eta,\xi) = e^{i\phi_1(y+z,\eta+\xi) - i\phi_1(y,\eta)} b_2(x,\eta) b_1(y+z,\eta+\xi).$$

While b_j have improved decay under differentiation for $j = 1,2$, this is not the case for a_0 due to the homogeneous of degree 1 phase factor. Thus

$$|\partial^\alpha a_0| \leq C_\alpha \left\langle (x,\eta)\right\rangle^{m_2} \left\langle (y+z,\eta+\xi)\right\rangle^{m_1}$$

for each α. Now integrate by parts using the operator $L = (1 + |z|^2 + |\xi|^2)^{-1}(1 + \Delta_z + \Delta_\xi)$ to see that

$$|(L^t)^k \, \partial^\alpha_{x,y,\eta} a_0| \leq C_{\alpha k} \left\langle (x,\eta) \right\rangle^{m_2} \left\langle (y,\eta) \right\rangle^{m_1} \left\langle (z,\xi) \right\rangle^{|m_1|-2k}. \tag{3.23}$$

Choosing $k > d + |m_1|/2$ shows that b given by (3.22) is smooth and satisfies

$$|\partial^\alpha b| \leq C_\alpha \left\langle (x,\eta) \right\rangle^{m_2} \left\langle (y,\eta) \right\rangle^{m_1}$$

for each α.

This result must be improved to include symbol bounds when $x = y$; this is important when taking the distributional trace of $\widetilde{U}(t)$.

Lemma 3.3.2. *The pullback of the amplitude b by the map $(t, x, \eta) \mapsto (t, x, x, \eta)$ lies in $\mathcal{C}^\infty(\mathbb{R}_t; \Gamma^{m_1+m_2})$.*

Proof. As in the previous paragraph the smooth dependence on t will follow immediately by differentiating under the integral sign, and so to simplify notation the dependence on t will be again be dropped.

First, observe that it suffices to consider the integral (3.22) over $|(z,\xi)| \leq (1/2)|(x,\eta)|$, since on the complement $b(x,x,\eta)$ is rapidly decaying in (x,η) by (3.23). So now define

$$b_\lambda(x,\eta) = b(\lambda^{1/2}x, \lambda^{1/2}x, \lambda^{1/2}\eta),$$

where $1 \leq |(x,\eta)| \leq 2$. In order to prove the lemma it suffices to show the uniform bounds

$$\left|\partial^\alpha_{x,\eta} b_\lambda(x,\eta)\right| \leq C_\alpha \lambda^{(m_1+m_2)/2} \tag{3.24}$$

as $\lambda \to \infty$. For this, define

$$g_\lambda(z,\xi,x,\eta) = a(\lambda^{1/2}x, \lambda^{1/2}\eta) b(\lambda^{1/2}(x+z), \lambda^{1/2}(\eta+\xi)),$$

noting that

$$|\partial^\alpha g_\lambda(z,\xi,x,\eta)| \leq C_\alpha \lambda^{(m_1+m_2)/2} \tag{3.25}$$

uniformly in $1 \leq |(x,\eta)| \leq 2$ and $|(z,\xi)| \leq 1/2$. A Taylor expansion of $\phi_1(z+x, \xi+\eta)$ at (x,η) yields

$$\begin{aligned}\phi_1(z+x,\xi+\eta) = \phi_1(x,\eta) &+ \langle z, \partial_x \phi_1(x,\eta)\rangle + \langle \xi, \partial_\eta \phi_1(x,\eta)\rangle \\ &+ \sum_{|\alpha|=2} (z,\xi)^\alpha f_\alpha(x,z,\eta,\xi)\end{aligned}$$

for some smooth functions f_α, so if we define

$$\Phi_\mu(x,z,\eta,\xi) = z\xi + \mu\phi_1(z+x,\xi+\eta) - \mu\phi_1(x,\eta)$$

for a parameter $\mu \in \mathbb{R}$, then

$$\Phi_\mu = z\xi + \mu\big(z\partial_y\phi_1(y,\eta) + \xi\partial_\eta\phi_1(y,\eta) + \sum_{|\alpha|=2}(z,\xi)^\alpha f_\alpha(z,\xi,x,\eta)\big).$$

Using homogeneity of the phase, the rescaled amplitude $b_\lambda(x,\eta)$ can be written via a change of variables as

$$b_\lambda(x,\eta) = \lambda^d \int e^{i\lambda\Phi_\mu} g_\lambda(z,\xi,x,\eta)\, dz d\xi \tag{3.26}$$

by setting $\mu = \lambda^{-1/2}$. Let $C_\mu = \{d_{z,\xi}\Phi_\mu = 0\}$ denote the set of stationary points; thus $(z,\xi) \in C_\mu$ if and only if

$$\xi + \mu\partial_z\phi_1(z+x,\xi+\eta) = 0,$$
$$z + \mu\partial_\xi\phi_1(z+x,\xi+\eta) = 0.$$

By the implicit function theorem, we can parametrize (z,ξ) by (μ,x,η) near any fixed (x_0,η_0) for $|\mu|$ sufficiently small, and obtain

$$|z(\mu,x,\eta)| + |\xi(\mu,x,\eta)| \le C|\mu|.$$

In particular these points satisfy $|(z,\xi)| \le 1/2$ for $|\mu|$ sufficiently small and $1 \le |(x,\eta)| \le 2$, hence the derivative bounds (3.25) for g_λ will apply.

We can now estimate the integral (3.26) and its derivatives, initially treating μ as a parameter; assume without loss that $g_\lambda(z,\xi,x,\eta)$ vanishes for $|(z,\xi)| \ge 1/3$. Consider a typical derivative $\partial^\gamma_{x,\eta} g_\lambda$. This is a sum of terms, where those with $\ell \le |\gamma|$ derivatives landing on the exponential factor can be written as

$$\lambda^d(\lambda\mu)^\ell \int e^{i\lambda\Phi_\mu}\left(\partial^{\gamma'}_{x,\eta} g_\lambda\right) \sum_{|\beta|=\ell}(z,\xi)^\beta h_\beta\, dz d\xi \tag{3.27}$$

for some smooth functions $h_\beta = h_\beta(z,\xi,y,\eta,\mu)$ and $|\gamma'| \le |\gamma|$.

Now apply the method of stationary phase, recalling the bounds (3.25). At the critical set C_μ, each term $(z,\xi)^\beta h_\beta(z,\xi,y,\eta)$ in (3.27) gives an additional factor of order $O(|\mu|^{\ell/2})$, since both critical points $z(\mu,y,\eta)$, $\xi(\mu,y,\eta)$ are of order $O(|\mu|)$. When $\mu = \lambda^{-1/2}$ this cancels with the factor of $\lambda^{\ell/2}$ in front of the integral in (3.27). The stationary phase formula eliminates the prefactor of λ^d, showing that

$$|\partial^\alpha b_\lambda(x,\eta)| = O(\lambda^{(m_1+m_2)/2})$$

near (x_0,η_0). Since the set where $1 \le |(x,\eta)| \le 2$ is compact, this implies the symbol estimates (3.24) everywhere on the latter set. ☐

More generally, Lemma 3.3.2 is true whenever ϕ_2 is a quadratic form satisfying (3.13) and ϕ_1 is homogeneous of degree 1.

Corollary 3.3.3. *If $t - 2\pi n \in (-\pi/2, \pi/2)$ for some $n \in \mathbb{Z}$, then the Schwartz kernel of $\widetilde{U}(t)$ is given by an oscillatory integral*

$$\widetilde{U}(t, x, y) = \int e^{i\phi_2(t,x,\eta) - i\langle y, \eta\rangle + i\phi_1(t,x,\eta)} b(t, x, y, \eta)\, d\eta,$$

where ϕ_2 is given by (3.16), and ϕ_1 is given by (3.18). The pullback of $b \in \mathcal{C}^\infty(\mathbb{R}_t \times \mathbb{R}^{3d})$ by the map $(t, x, \eta) \mapsto (t, x, x, \eta)$ lies in $\mathcal{C}^\infty(\mathbb{R}_t; \Gamma^0)$.

Proof. This follows directly from Lemma 3.3.2. □

Let $R(t) = (i\partial_t - P(t))\tilde{F}(t)$. A brief calculation shows that $\widetilde{U}(t)$ satisfies the equation

$$\begin{cases} (i\partial_t - H)\widetilde{U}(t) = U_0(t)R(t) \\ \widetilde{U}(0) = \mathrm{I} + K. \end{cases} \tag{3.28}$$

It follows by Duhamel's principle that

$$\widetilde{U}(t) - U(t) = U(t)K - i\int_0^t U(t-s)U_0(t)R(t)\, ds. \tag{3.29}$$

Recall that $U_0(t)$ and $U(t)$ both preserve the scale of isotropic Sobolev spaces. Since $R(t)$ is a smooth family of residual operators and K is residual, it follows immediately from (3.29) that

$$\widetilde{R}(t) = \widetilde{U}(t) - U(t) \in \mathcal{C}^\infty(\mathbb{R}_t; \mathcal{L}(H^{-N}_{\mathrm{iso}}, H^N_{\mathrm{iso}})) \tag{3.30}$$

for each N.

As in Lemma 3.3.1, there is no loss in assuming that the amplitude $b(t, x, y, \eta)$ in $\widetilde{U}(t)$ is supported away from $(x, y, \eta) = 0$: inserting a cutoff modifies $\widetilde{U}(t)$ by a residual operator which does not affect the error analysis above. In particular, it may be assumed that ϕ_1 is smooth on the support of b.

3.3.5. Propagation of Classical Singularities

Let $u \in \mathcal{E}' + \mathcal{S}$, so in particular $\mathrm{WF}_{\mathrm{iso}}(u) \subset \{(0, \xi) : \xi \in \mathbb{R}^d\}$. We want to calculate the classical wavefront set $\mathrm{WF}_{\mathrm{cl}}(U(t)u)$ of u. By Lemma 2.4.13, if $\{(0, \xi) : \xi \in \mathbb{R}^d\} \cap \mathrm{WF}_{\mathrm{iso}}(v) = \emptyset$ then $v \in \mathcal{C}^\infty$. Applying this to $v = U(t)u$, it follows by Proposition 3.2.2 that $U(t)u \in \mathcal{C}^\infty$ except at times $t \in \pi\mathbb{Z}$, and at those times, it follows by Mehler's formula that

$$\mathrm{WF}_{\mathrm{cl}}(U_0(k\pi)u) = \{(-1)^k(x, \xi) : (x, \xi) \in \mathrm{WF}_{\mathrm{cl}}(u)\}.$$

It remains to calculate how singularities are moved by the reduced propagator $F(t)$.

We now assume more generally that $u \in \mathcal{S}'$. Equation (3.30) (and preceding discussion) implies that the parametrix constructed in Lemma 3.3.1 satisfies $F - \tilde{F} \in \mathcal{C}^\infty(\mathbb{R}_t, \mathcal{L}(\mathcal{S}', \mathcal{S}))$. The classical wavefront set is thus completely determined by the parametrix:

$$\mathrm{WF}_{\mathrm{cl}}(F(t)u) = \mathrm{WF}_{\mathrm{cl}}(\tilde{F}(t)u + (F(t) - \tilde{F}(t))u) = \mathrm{WF}_{\mathrm{cl}}(\tilde{F}(t)u),$$

because $(F(t) - \tilde{F}(t))u \in \mathcal{S} \subset \mathcal{C}^\infty$.

Recall that

$$\begin{aligned}\tilde{F}(t) &= \int e^{i(\langle x-y,\xi\rangle + \phi_1(t,x,\xi))} a(t,x,\xi) d\xi \\ &= \int e^{i\phi(t,x,y,\xi)} \tilde{a}(t,x,\xi) d\xi,\end{aligned}$$

with $\phi = \langle x - y, \xi\rangle + \phi_1(t, 0, \xi)$ and

$$\tilde{a}(t,x,\xi) = e^{i(\phi_1(t,x,\xi) - \phi_1(t,0,\xi))} a(t,x,\xi).$$

Note that ϕ is homogeneous of degree one in ξ and since, locally, $\phi_1(t,x,\xi) - \phi_1(t,0,\xi) \in S^0$ we see that the amplitude is (locally) a Kohn-Nirenberg 0-symbol, $\tilde{a} \in S^0$. Thus, the oscillatory integral $\tilde{F}(t)$ satisfies the assumptions of Theorem 8.1.9 from [30], and we obtain the following:

Proposition 3.3.4. *The wavefront set of the integral kernel of $F(t)$ is given by*

$$\mathrm{WF}_{\mathrm{cl}}(F(t)) \subset \left\{ (x, x + \partial_\xi \phi_1(t,0,\xi), \xi, -\xi) : x \in \mathbb{R}^d, \xi \in \mathbb{R}^d \setminus \{0\} \right\}.$$

If we want to calculate the wavefront set of $F(t)u$ for $u \in \mathcal{S}'$ we have to show that there are no contributions to wavefront set coming from infinity. Fix $t_0 \in \mathbb{R}$ and let $K \subset \mathbb{R}^d$ compact with $\chi_1 \in \mathcal{C}_c^\infty(K)$; set

$$r = \max_{(x,\xi)\in K\times\mathbb{R}^d} |x + \partial_\xi \phi_1(t_0, x, \xi)|.$$

Note that $\partial_\xi \phi_1$ is homogeneous of degree zero in (x,ξ) and therefore $r < \infty$. Let $\chi_2 \in \mathcal{C}^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \chi_2 \cup B_{r+1}(0) = \emptyset$ and homogeneous of degree zero outside of $B_{r+2}(0)$.

It suffices to show that $\chi_1(x)\chi_2(y)\tilde{F}(t_0,x,y) \in \mathcal{S}(\mathbb{R}^{2d})$. Set $\Phi = \langle x - y, \xi\rangle + \phi_1(t_0, x, \xi)$ and define the operator L by

$$Lu = \frac{\langle \partial_\xi \Phi, D_\xi u\rangle}{|\partial_\xi \Phi|^2}.$$

L is well-defined on $\operatorname{supp} \chi_1(x)\chi_2(y)$ and satisfies $Le^{i\Phi} = e^{i\Phi}$ and for all $a \in \Gamma^m$ and $N \in \mathbb{N}$,

$$\begin{aligned}|(L^t)^N a(x,\xi)| &\leq C\langle x - y + \partial_\xi \phi_1(t_0)\rangle^{-N} \langle (x,\xi)\rangle^{m-N} \\ &\leq C\langle y\rangle^{-N} \langle \xi\rangle^{m-N}.\end{aligned}$$

Integration by parts with this operator shows that $\chi_1(x)\chi_2(y)\tilde{F}(t_0,x,y)$ and all its derivatives are rapidly decaying, hence for any $u \in \mathcal{S}'$, we know that $\mathrm{WF}_{\mathrm{cl}} F(t)u \cap \pi^{-1}K$ is determined by the restriction of u to $B_{r+1}(0)$, and is as follows:

Proposition 3.3.5. *For $u \in \mathcal{S}'$,*

$$\mathrm{WF}_{\mathrm{cl}}(F(t)u) = \{(x - \partial_\xi \phi_1(t, 0, \xi), \xi) \colon (x, \xi) \in \mathrm{WF}_{\mathrm{cl}}(u)\}\,.$$

Proof. The usual calculus of wavefront sets, together with Proposition 3.3.4, shows that

$$\mathrm{WF}_{\mathrm{cl}}(F(t)u) \subset \{(x - \partial_\xi \phi_1(t, 0, \xi), \xi) \colon (x, \xi) \in \mathrm{WF}_{\mathrm{cl}}(u)\}\,. \tag{3.31}$$

It remains to upgrade this containment of sets to equality. To do this, we simply observe that by the calculus of wavefront sets and a second use of Proposition 3.3.4,

$$\mathrm{WF}_{\mathrm{cl}} F^*(t)u \subset \{(x, \xi) \colon (x - \partial_\xi \phi_1(t, 0, \xi), \xi) \in \mathrm{WF}_{\mathrm{cl}}(u)\}\,.$$

On the other hand $F(t)^* F(t) = I$, hence the containment in (3.31) must have been equality. □

Corollary 3.3.6. *Let $u \in \mathcal{S}'$ and $k \in \mathbb{Z}$. The wavefront set of the full propagator is given by*

$$\mathrm{WF}_{\mathrm{cl}}(U(\pi k)u) = \left\{(-1)^k \left(x + \int_0^{\pi k} \partial_\xi(\mathsf{p}_1(t, 0, \xi))dt, \xi\right) : (x, \xi) \in \mathrm{WF}_{\mathrm{cl}}(u)\right\}.$$

If $t \notin \pi\mathbb{Z}$ and $u \in \mathcal{E}' + \mathcal{S}$ then $\mathrm{WF}_{\mathrm{cl}}(U(t)u) = \emptyset$.

For $t = 2\pi k$ this becomes

$$\mathrm{WF}_{\mathrm{cl}}(U(2\pi k)u) = \{(x + k\partial_\xi(\mathsf{X}p_1)(0, \xi), \xi) \colon (x, \xi) \in \mathrm{WF}_{\mathrm{cl}}(u)\}.$$

3.3.6. Traces

Recall that $\operatorname{Tr} U(t)$ is well defined as a tempered distribution. More precisely, if $\chi \in \mathcal{S}(\mathbb{R})$, then the Schwartz kernel of

$$\int \chi(t) U(t)\, dt \tag{3.32}$$

lies in $\mathcal{S}(\mathbb{R}^{2d})$, hence the operator is of trace-class. Indeed, if $\{e_j\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ consisting of eigenvectors of H with corresponding eigenvalues λ_j, then (3.32) has Schwartz kernel

$$\sum_{j=0}^{\infty} \hat{\chi}(\lambda_j) e_j(x) e_j(y),$$

which converges in $\mathcal{S}(\mathbb{R}^{2d})$ since $\hat{\chi}$ is rapidly decreasing. In order to obtain results on singularities of $\operatorname{Tr} U(t)$, it suffices to study the trace of $\widetilde{U}(t)$ and its Fourier transform (cf. Lemme (IV.1) of [23]):

Lemma 3.3.7. *If $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$, then $\widetilde{R}(t) = \widetilde{U}(t) - U(t)$ is of trace class, and*

$$\left| \operatorname{Tr} \int e^{it\lambda} \chi(t) \widetilde{R}(t)\, dt \right| \le C_k \langle \lambda \rangle^{-N}$$

for each $\lambda \in \mathbb{R}$ and $N > 0$.

Proof. For $N \gg 0$ operators in $\mathcal{L}(H_{\text{iso}}^{-N}, H_{\text{iso}}^{N})$ are of trace-class (see [31, Lemma 19.3.2]). Using repeated integration by parts, the claim follows from (3.30). □

On the other hand, if $\chi \in \mathcal{C}_c^\infty((-\pi/2, \pi/2))$, then the operator

$$\int \chi(t - 2\pi n)\widetilde{U}(t)\, dt$$

also has its Schwartz kernel in $\mathcal{S}(\mathbb{R}^{2d})$. Replacing χ with $e^{it\lambda}\chi$, it follows that the trace of $\mathcal{F}_{t\to\lambda}^{-1}\chi(t-2\pi n)\widetilde{U}(t)$ is

$$(2\pi)^{-1} \int e^{it\lambda + i\phi_2(t,x,\eta) - i\langle x,\eta\rangle + i\phi_1(t,x,\eta)} \chi(t-2\pi n) b(t,x,x,\eta)\, dt dx d\eta.$$

In the next section we will evaluate this integral as $\lambda \to \infty$.

3.4. Stationary Phase

In this section we apply the method of stationary phase to evaluate an integral of the form

$$I(\lambda) = \int e^{i(t\lambda + \psi_2(t,x,\eta) + \psi_1(t,x,\eta))} \chi(t) a(t,x,\eta)\, dt dx d\eta \tag{3.33}$$

as $\lambda \to \infty$, where $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$. Letting (r, θ) denote polar coordinates on $\mathbb{R}^{2d}$, we will also express various functions of (x, η) in terms of (r, θ). The assumptions are as follows:

1. $\psi_j(t, \bullet)$ is homogeneous of degree j,
2. $a \in \mathcal{C}^\infty(\mathbb{R}_t; \Gamma^0(\mathbb{R}_x^d))$, and ψ_j are smooth on the support of a,
3. there exists a unique $t_0 \in \operatorname{supp} \chi$ such that $\psi_2(t_0, \bullet) = 0$,
4. there exists a unique $r_0 > 0$ such that $\partial_t \psi_2(t_0, r_0, \theta) = -1$ for all $\theta \in \mathbb{S}^{2d-1}$.

Define the set where the restriction of $\nabla\psi_1(t_0, \bullet)$ to $\mathbb{S}^{2d-1}$ vanishes to infinite order,

$$\Pi_{t_0} = \left\{ \theta \in \mathbb{S}^{2d-1} : \partial_\theta^\alpha(\psi_1(t_0, 1, \theta)) = 0 \text{ for all } \alpha \in \mathbb{N}^{2d-1} \setminus 0 \right\}.$$

We can now state our main result on the asymptotics of $I(\lambda)$:

Proposition 3.4.1. *If Π_{t_0} has measure zero, then the integral (3.33) satisfies*

$$I(\lambda) = o(\lambda^{d-1}).$$

If, instead, the restriction of $\psi_1(t_0, \bullet)$ to $\mathbb{S}^{2d-1}$ is Morse-Bott with $k > 0$ non-degenerate directions, then

$$I(\lambda) = O(\lambda^{d-1-k/4}).$$

Proof. To begin, rewrite the integral (3.33) in polar coordinates, and then make the change of variables $r \mapsto \lambda^{1/2} r$. By homogeneity of the phases,

$$I(\lambda) = \lambda^d \int e^{i\lambda(\psi_2(t,r,\theta)+\lambda^{-1/2}\psi_1(t,r,\theta)+t)} \chi(t) a(t, \lambda^{1/2} r, \theta)\, dt\, r^{2d-1} dr d\theta. \tag{3.34}$$

Observe that the exponential term in this integral can be written as $\exp(i\lambda\Psi_\mu)$, where

$$\Psi_\mu(t,r,\theta) = \psi_2(t,r,\theta) + \mu\psi_1(t,r,\theta) + t$$

and $\mu = \lambda^{-1/2}$. The proof proceeds in two steps.

Step 1: Stationary phase in (t,r): First we apply the method of stationary phase to the variables (r,t) for $|\mu|$ small, treating μ and θ as parameters. Let

$$C_\mu = \{(t,r) \colon d_{r,t}\Psi_\mu(t,r,\theta) = 0\}$$

denote the corresponding stationary set. Now $(r\partial_r)\psi_j = j\psi_j$ by homogeneity of the phases, so the stationary points are where

$$\begin{cases} 2\psi_2 + \mu\psi_1 = 0 \\ \partial_t\psi_2 + \mu\partial_t\psi_1 + 1 = 0. \end{cases} \tag{3.35}$$

By hypothesis, if $\theta_0 \in \mathbb{S}^{2d-1}$ is fixed and $\mu = 0$, then these equations are satisfied on the support of the function $(t,r) \mapsto \chi(t)a(t,\lambda^{1/2}r,\theta_0)$ precisely when $t = t_0,\ r = r_0$.

Using the implicit function theorem, parametrize $C_\mu \cap \operatorname{supp}(\chi \cdot a)$ near θ_0 for small $|\mu|$. Indeed, differentiating the equations (3.35) in (t,r) at $\mu = 0,\ r = r_0,\ t = t_0$ yields the invertible Hessian matrix

$$\begin{pmatrix} 0 & -2 \\ -2 & \partial_t^2\psi_2 \end{pmatrix}.$$

Denote by $t = t(\mu,\theta)$ and $r = r(\mu,\theta)$ the corresponding critical points. Furthermore, by the implicit function theorem

$$\begin{pmatrix} \partial_\mu r \\ \partial_\mu t \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \psi_1\partial_t^2\psi_2 + 2\partial_t\psi_1 \\ 2\psi_1 \end{pmatrix}$$

at $\mu = 0,\ r = r_0,\ t = t_0$. Now Taylor expand $\Psi_\mu(t(\mu,\theta), r(\mu,\theta),\theta)$ at $\mu = 0$ to find that

$$\Psi_\mu(t(\mu,\theta), r(\mu,\theta),\theta) = t_0 + \mu\psi_1(t_0,r_0,\theta) + \mu^2\gamma(\mu,\theta)$$

near $\mu = 0,\ \theta = \theta_0$, where $\gamma = \gamma(\mu,\theta)$ is a smooth function of μ and θ.

Next, apply the method of stationary phase to the integral

$$J(\lambda,\mu,\theta) = \lambda^d \int e^{i\lambda\Psi_\mu}\chi(t)a(t,\lambda^{1/2}r,\theta)\, dt\, r^{2d-1} dr,$$

treating $\theta \in \mathbb{S}^{2d-1}$ and μ as parameters. In fact, it may be assumed $a(t, \lambda^{1/2}r, \theta)$ has support on $\{r \leq 3r_0\}$. Indeed, consider the following operator, which is well defined on $\{r \geq 2r_0\} \cap \operatorname{supp} \chi$:

$$L = \lambda^{-1}((\partial_t \phi_2 + 1)^2 + 4\phi_2^2)^{-1} \left((\partial_t \phi_2 + 1)\partial_t + 2\phi_2 \partial_r\right).$$

Due to the symbol bounds on a,

$$|(L^t)^k (e^{i\lambda^{1/2}\psi_1} \chi(t) a(t, \lambda^{1/2}r, \theta) r^{2d-1})| \leq C_k \lambda^{-k/2} r^{2d-1-2k}.$$

Inserting a cutoff to $\{r \geq 2r_0\}$ in the integrand of (3.34) and integrating by parts using L gives a contribution of order $O(\lambda^{-\infty})$. By stationary phase, for any $M \geq 1$,

$$J(\lambda, \mu, \theta) = \lambda^{d-1} e^{i\lambda(t_0 + i\mu\psi_1(t_0, r_0, \theta))} a_M(\lambda^{1/2}, \mu, \theta) + O(\lambda^{d-1-M})$$

uniformly in θ for $|\mu|$ sufficiently small; here, a_M is a function depending smoothly on $(\lambda^{1/2}, \mu, \theta)$. Note that while successive terms in the stationary phase expansion involve differentiation of $a(t, \lambda^{1/2}r, \theta)$ with respect to r, the symbol estimates on a ensure uniform bounds on each a_M as $\lambda \to \infty$.

Step 2: Stationary phase in θ: Recall that $I(\lambda)$ is the integral of $J(\lambda, \lambda^{-1/2}, \theta)$ over $\mathbb{S}^{2d-1}$ with respect to θ. In other words, for each M,

$$I(\lambda) = \lambda^{d-1} e^{i\lambda t_0} \int e^{i\lambda^{1/2}\psi_1(t_0, r_0, \theta)} a_M(\lambda^{1/2}, \lambda^{-1/2}, \theta)\, d\theta + O(\lambda^{d-1-M}). \tag{3.36}$$

We now complete the proof of Proposition 3.4.1. If $(x, \eta) \in \mathbb{S}^{2d-1} \setminus \Pi_{t_0}$, then there exists an $\alpha \in \mathbb{N}^{2d-1}$ such that

$$\partial_\theta^\alpha \psi_1 \neq 0,$$

in a neighborhood of (x, η) within $\mathbb{S}^{2d-1}$. By the weak stationary phase lemma for degenerate stationary points [60, p. 342, Proposition 5] and a covering argument, the contribution of the integral over $\mathbb{S}^{2d-1} \setminus \Pi_{t_0}$ is $o(\lambda^{d-1})$ (cf. [19]). Therefore,

$$I(\lambda) = \lambda^{d-1} e^{i\lambda t_0} \int_{\Pi_{t_0}} e^{i\lambda^{1/2}\psi_1(t_0, r_0, \theta)} b(\lambda^{1/2}, \lambda^{-1/2}, \theta)\, d\theta + o(\lambda^{d-1}).$$

This implies that if Π_{t_0} is of measure zero then

$$I(\lambda) = o(\lambda^{d-1}),$$

which proves the first part of Proposition 3.4.1. For the second part, the condition that $\psi_1(t_0, r_0, \bullet)$ is Morse-Bott with k nondegenerate directions implies that $I(\lambda) = O(\lambda^{d-1-k/4})$ by [30, Theorem 7.7.6], so taking $M \geq k/4$ finishes the proof. $\square$

3.5. Spectral Asymptotics

3.5.1. Singularity at $t = 0$

In this section we calculate the leading order asymptotics of the singularity of $\operatorname{Tr} U(t)$ at $t = 0$. More precisely, we obtain the $\lambda \to \infty$ behavior of its inverse Fourier transform, after a suitable mollification. For this we use a short-time parametrix for $U(t)$ constructed in [23]. This construction actually applies to any self-adjoint classical elliptic isotropic operator of order 2, and for this reason we state Proposition 3.5.1 below quite generally.

Let $p \in \Gamma^2_{\mathrm{cl}}(\mathbb{R}^d)$ be real-valued and elliptic, and then set $P = \operatorname{Op}^w(p)$. Denote by $N(\lambda) = \sum_{\lambda_j \le \lambda} 1$ the counting function for the eigenvalues of P.

Proposition 3.5.1. *Let $\rho \in \mathcal{S}(\mathbb{R})$ be such that $\hat{\rho}$ has compact support in $(-\epsilon, \epsilon)$. If $\epsilon > 0$ is sufficiently small, then*

$$(N * \rho)(\lambda) = (2\pi)^{-d} \int_{\{p_2 + p_1 \le \lambda\}} dx d\eta - (2\pi)^{-d} \int_{\{p_2 = \lambda\}} p_0(x,\eta) \frac{dS}{|\nabla p_2|} \\ + O(\lambda^{d-3/2}).$$

Proof. Let $U(t)$ denote the Schrödinger propagator for P. As remarked above, we will use a parametrix $U_N(t)$ for $U(t)$ taken from [23], which exists on some time interval $(-\epsilon, \epsilon)$ (note that $U_N(t)$ differs from the long time parametrix constructed in Corollary 3.3.3). In the notation of [23],

$$U_N(t,x,y) = (2\pi)^{-d} \int e^{i(S_2(t,x,\eta) - \langle y, \eta\rangle + S_1(t,x,\eta))} a_N(t,x,\eta)\, d\eta.$$

Here S_2, S_1 are appropriate phase functions, and the symbol a_N is a finite sum

$$a_N(t,x,\eta) = \sum_{k=0}^{N} a^{(k)}(t,x,\eta),$$

where each $a^{(k)}(t,\bullet)$ is homogeneous of degree $-k$ outside a compact set and vanishes near $(x,\eta) = 0$. Note, however, that in [23] the operator P is the left quantization of p rather than its Weyl quantization. In order to extract the leading order behavior of these quantities, first write

$$\operatorname{Op}^w(p) = \operatorname{Op}_L(\widetilde{p})$$

with $\widetilde{p} \in \Gamma^2_{\mathrm{cl}}$ and $\widetilde{p}_j = p_j$ for $j = 1, 2$, but

$$\widetilde{p}_0 = p_0 - (i/2) \langle \partial_x, \partial_\xi \rangle p_2. \tag{3.37}$$

Referring to [23, Equations 37-38] for the transport equations satisfied by $a^{(k)}$ and using (3.37), we find that

$$a_N(0,x,\eta) = 1, \quad \partial_t a^{(0)}(0,x,\eta) = -ip_0 - (1/2) \langle \partial_x, \partial_\xi \rangle p_2.$$

Recalling that $\mathcal{F}_{\lambda\to t}N'(\lambda) = \operatorname{Tr} U(t)$, we have $\mathcal{F}_{\lambda\to t}\{N' * \rho\} = \hat\rho(t)\operatorname{Tr} U(t)$. Motivated by this, define the distribution $K(t) = \hat\rho(t)\operatorname{Tr} U_N(t)$, so that

$$K(t) = (2\pi)^{-d}\hat\rho(t)\int e^{i(S_2(t,x,\eta)-\langle x,\eta\rangle+S_1(t,x,\eta))}a_N(t,x,\eta)\,dxd\eta.$$

This makes sense so long as $\hat\rho(t)$ has support on the interval where $U_N(t)$ is well defined.

By [23, Equations 35-36], $S_2(0,x,\eta) = \langle x,\eta\rangle$ and $S_1(0,x,\eta) = 0$, so by Taylor's theorem

$$S_2(t,x,\eta) - \langle x,\eta\rangle + S_1(t,x,\eta) = t\psi(t,x,\eta)$$

with ψ a smooth function. More precisely, ψ is given to leading order in t by

$$\psi(t,\bullet) = -(p_2+p_1) + (t/2)(\langle\partial_\xi p_2,\partial_x p_2\rangle + \langle\partial_\xi p_1,\partial_x p_2\rangle + \langle\partial_\xi p_2,\partial_x p_1\rangle) + t^2 r(t,\bullet).$$

We now follow the argument of [32, Lemma 29.1.3]. First, define

$$A(t,\lambda) = (2\pi)^{-d}\int_{\{-\psi(t)\le\lambda\}} a_N(t,x,\eta)\hat\rho(t)\,dxd\eta.$$

Now for sufficiently small $|t|$, the function $-\psi(t,\bullet)$ is elliptic in Γ^2_{cl}, and as in the aforementioned lemma

$$A(t,\lambda)\in S^d(\mathbb{R}_t;\mathbb{R}_\lambda)$$

is a Kohn–Nirenberg symbol for $|t|$ sufficiently small (see (A.1)). Furthermore, it is an exercise in distribution theory to see that

$$K(t) = \int_{\mathbb{R}} e^{-it\lambda}\partial_\lambda A(t,\lambda)\,d\lambda.$$

Thus $K(t)$ is a conormal distribution, which can be written as the Fourier transform of a symbol by applying [31, Lemma 18.2.1]. If we let $B(\lambda) = e^{iD_tD_\lambda}A(t,\lambda)|_{t=0}$ and recall the definition of $K(t)$, then

$$\mathcal{F}^{-1}_{t\to\lambda}\{\hat\rho(t)\operatorname{Tr} U_N(t)\}(\lambda) = \partial_\lambda B(\lambda).$$

Expand $B(\lambda) = A(0,\lambda) - i\partial_t\partial_\lambda A(0,\lambda) + R(\lambda)$, where $R\in S^{d-2}(\mathbb{R})$. Also let dS denote the induced surface measure on $\{p_2=\lambda\}$. First,

$$A(0,\lambda) = (2\pi)^{-d}\int_{\{p_2+p_1\le\lambda\}} dxd\eta.$$

For the next term in the expansion, recall that $a_N(0,x,\eta) = 1$ and compute

$$\begin{aligned}-i\partial_t A(0,\lambda) &= (2\pi)^{-d}\langle -i\partial_t a, H(\psi+\lambda)\rangle\big|_{t=0} - i(2\pi)^{-d}\langle a\partial_t\psi,\delta(\psi+\lambda)\rangle\big|_{t=0}\\ &= -(2\pi)^{-d}\langle\tilde p_0, H(\lambda-p_2)\rangle - (i/2)(2\pi)^{-d}\langle\langle\partial_x p_2,\partial_\xi p_2\rangle,\delta(\lambda-p_2)\rangle + e(\lambda)\end{aligned}$$

for some $e(\lambda) \in S^{d-1/2}(\mathbb{R})$. Here H denotes the Heaviside function, and the pairings are in the sense of distributions. Integration by parts furthermore yields

$$\langle \langle \partial_x p_2, \partial_\xi p_2 \rangle, \delta(\lambda - p_2) \rangle = \langle \langle \partial_x, \partial_\xi \rangle p_2, H(\lambda - p_2) \rangle.$$

Since the pullback of δ is given by $\delta(\lambda - p_2) = |\nabla p_2|^{-1} dS$, compute from (3.37) that

$$\begin{aligned} -i\partial_\lambda \partial_t A(0, \lambda) &= -(2\pi)^{-d} \langle \tilde{p}_0 + (i/2) \langle \partial_x, \partial_\xi \rangle p_2, \delta(\lambda - p_2) \rangle + O(\lambda^{d-3/2}) \\ &= -(2\pi)^{-d} \int_{\{p_2 = \lambda\}} p_0 |\nabla p_2|^{-1} dS + O(\lambda^{d-3/2}). \end{aligned}$$

Finally, for any k,

$$\begin{aligned} (N' * \rho)(\lambda) &= \mathcal{F}^{-1}_{t \to \lambda} \{\hat{\rho} \operatorname{Tr} U\}(\lambda) \\ &= \mathcal{F}^{-1}_{t \to \lambda} \{\hat{\rho} \operatorname{Tr} U_N\}(\lambda) + O(\lambda^{-k}) \\ &= \partial_\lambda B(\lambda) + O(\lambda^{-k}). \end{aligned}$$

provided $N = N(k)$ is sufficiently large (cf. Lemma IV.1 in [23]). Integrating this equation gives the desired result. □

3.5.2. Proof of Theorem 3.1.2

We now return to the setting of Theorem 3.1.2, so that in Proposition 3.5.1 we take the operator $P = H$. Begin by fixing an appropriate cutoff function in the time domain. Choose a real valued function $\rho \in \mathcal{S}(\mathbb{R})$ with the following properties:

1. $\rho(\lambda) > 0$ for all $\lambda \in \mathbb{R}$,
2. $\hat{\rho}(t) = 1$ on $(-\epsilon, \epsilon)$ for some $\epsilon \in (0, \pi/2)$,
3. $\operatorname{supp} \hat{\rho} \subset (-\pi/2, \pi/2)$,
4. ρ is even.

In order to compare $N(\lambda)$ with $(N * \rho)(\lambda)$, we will need the following Fourier Tauberian theorem, from the appendix of [56]. This result is implicit in [13], and has its roots in [26, 36].

Lemma 3.5.2 (Theorem B.5.1 in [56])**.** *Let ρ be as above, and $\nu \in \mathbb{R}$. If $(N' * \rho)(\lambda) = O(\lambda^\nu)$ and*

$$(N' * \chi)(\lambda) = o(\lambda^\nu)$$

for each function χ satisfying $\hat{\chi} \in C_c^\infty(\mathbb{R})$, $\operatorname{supp} \hat{\chi} \subset (0, +\infty)$, then

$$N(\lambda) = (N * \rho)(\lambda) + o(\lambda^\nu).$$

In order to prove Theorem 3.1.2 it suffices to establish (3.4) and (3.5), since then the Weyl law (3.6) is an immediate corollary of Lemma 3.5.2. Indeed, using Proposition 3.1.1 and a suitable partition of unity, either of the conclusions (3.4) or (3.5) implies that

$$\mathcal{F}^{-1}_{t\to\lambda}\{\chi(t)\operatorname{Tr} U(t)\}(\lambda) = o(\lambda^{d-1}).$$

for any function $\chi \in \mathcal{C}^\infty_c(\mathbb{R})$ with $\operatorname{supp}\chi \subset (0,\infty)$ (here χ is playing the role of $\hat{\chi}$ in Lemma 3.5.2). Now Proposition 3.5.1 in particular shows that

$$(N * \rho)(\lambda) = O(\lambda^d),$$

which together verify the hypotheses of Lemma 3.5.2. This establishes the two term asymptotics (3.6) for $N(\lambda)$.

Thus, we aim to show

$$\mathcal{F}^{-1}_{t\to\lambda}\{\chi(t)\operatorname{Tr} U(t)\}(\lambda) = o(\lambda^{d-1})$$

whenever $\operatorname{supp}\chi \subset (2\pi n - \epsilon, 2\pi n + \epsilon)$, where $n \in \mathbb{N}\setminus 0$ and $\epsilon \in (0, \pi/2)$. By Lemma 3.3.7, for any $N > 0$

$$\mathcal{F}^{-1}_{t\to\lambda}\{\chi(t)\operatorname{Tr} U(t)\}(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}\{\chi(t)\operatorname{Tr} \widetilde{U}(t)\}(\lambda) + O(\lambda^{-N}).$$

Now use Corollary 3.3.3 to see that

$$\mathcal{F}^{-1}_{t\to\lambda}\{\chi(t)\operatorname{Tr} \widetilde{U}(t)\}(\lambda) = \int e^{it\lambda} e^{i(\phi_2(t,x,\eta) - \langle x,\eta\rangle + \phi_1(t,x,\eta))}\chi(t)a(t,x,\eta)\, dtdxd\eta.$$

Apply Proposition 3.4.1 with

$$\psi_2(t,x,\eta) = \phi_2(t,x,\eta) - \langle x,\eta\rangle\,, \quad \psi_1(t,x,\eta) = \phi_1(t,x,\eta).$$

Since $\phi_2(t,x,\eta) = \sec(t)(x\eta - \sin(t)(|x|^2 + |\eta|^2)/2)$ and χ is supported close to $2\pi n$, the hypotheses of Proposition 3.4.1 for the phases ψ_2, ψ_1 and symbol a are satisfied. Indeed, in the notation of the latter proposition, we take

$$t_0 = 2\pi n, \quad r_0 = \sqrt{2}.$$

Now suppose that the restriction of $\nabla \mathsf{X} p_1$ to $\mathbb{S}^{2d-1}$ vanishes to infinite order only on a set of measure zero. Then $\nabla\phi_1(2\pi n, \bullet) = -\nabla \mathsf{X}^n p_1 = -n\nabla\mathsf{X}p_1$, so $\nabla\phi_1(2\pi n,\bullet)$ vanishes to infinite order only on a set of measure zero in $\mathbb{S}^{2d-1}$ as soon as $n \neq 0$. In that case Proposition 3.4.1 shows that

$$\mathcal{F}^{-1}_{t\to\lambda}\{\chi(t)\operatorname{Tr} \widetilde{U}(t)\}(\lambda) = o(\lambda^{d-1}).$$

Similarly, if the restriction of $\mathsf{X}p_1$ to $\mathbb{S}^{2d-1}$ is Morse–Bott with $k > 0$ nondegenerate directions, then $\phi_1(2\pi n, \bullet)$ has the same property for $n \neq 0$. This completes the proof of Theorem 3.1.2. $\square$

CHAPTER 4

Recurrence of Singularities

4.1. Introduction

This chapter is taken from the article [10]. The section about the global pseudodifferential calculi has been shortened, since the isotropic calculus was introduced in Section 2.4.

It is well-known that the harmonic oscillator $H_0 = 1/2(\Delta + |x|^2)$ on $\mathbb{R}^d$ has the property that for compactly supported initial data $u_0 \in \mathcal{E}'(\mathbb{R}^d)$, the solution $u(t) = e^{-itH_0}u_0$ to the dynamical Schrödinger equation is smooth for $t \notin \pi\mathbb{Z}$ and $u(\pi k) = (-iR)^k u$, where $Ru(x) = u(-x)$ is the reflection operator. In particular, we can calculate the wavefront set of $u(t)$:

$$\mathrm{WF}_{\mathrm{cl}}(u(t)) = \begin{cases} (-1)^k \mathrm{WF}_{\mathrm{cl}}(u_0) & t = \pi k, k \in \mathbb{Z}, \\ \emptyset & t \notin \pi\mathbb{Z}. \end{cases} \tag{4.1}$$

Let $p \in \Gamma^2_{\mathrm{cl}}$ be a real-valued classical elliptic isotropic symbol of order 2 and set

$$H = p^w(x, D) \qquad \text{and} \qquad H_0 = p_2^w(x, D),$$ [1]

the pseudodifferential operator and the "free" operator,[2] respectively. We consider the dynamical Schrödinger equation:

$$\begin{cases} (i\partial_t - H)u(t) = 0 \\ \qquad u(0) = u_0. \end{cases} \tag{4.2}$$

We seek to describe the wavefront set of $u(t)$ in terms of the singularities of u_0.

We denote the propagator of the equation by $U(t) = e^{-itH}$, similarly $U_0(t) = e^{-itH_0}$ for the free equation. We proceed in two steps: First, we calculate the wavefront set for the free propagator and then for the reduced propagator $F(t) = U_0(-t)U(t)$.

As usual denote by $\mathsf{H}_0(x, \xi) = \partial_\xi p_2 \partial_x - \partial_x p_2 \partial_\xi$ the Hamiltonian vector field associated to the Hamiltonian function p_2 and $t \mapsto \exp(t\mathsf{H}_0)$ its flow. Let $t > 0$ be arbitrary. We write

$$\exp(t\mathsf{H}_0)(y, \eta) = (x(t, y, \eta), \xi(t, y, \eta)).$$

[1] To be precise, we set $H_0 = \tilde{p}^w(x, D)$, where $\tilde{p} = p_2$ outside a compact set in $\mathbb{R}^{2d}$.

[2] The notion "free" is borrowed from scattering theory.

Let $\Gamma_t = \{\eta \in \mathbb{R}^d \setminus 0\colon \exp(t\mathsf{H}_0)(0,\eta) \in 0 \times \mathbb{R}^d\}$ and define the function $\Xi_t : \Gamma_t \to \mathbb{R}^d$, which is given by $\Xi_t(\eta) = \xi(t,0,\eta)$. It satisfies

$$\exp(t\mathsf{H}_0)(0,\eta) = (0,\Xi_t(\eta)).$$

Note that Ξ_t is homogeneous of degree one. Set $G_t = \operatorname{supp} U_0(t)u \times \Xi_t(\Gamma_t)$.

In the following we assume that the manifolds $\Lambda_t = \{(x,y,\xi,\eta)\colon \exp(t\mathsf{H}_0)(y,\eta) = (x,\xi)\}$ and $0 \times \mathbb{R}^{2d} \setminus \{0\}$ intersect cleanly for all $t \in \mathbb{R}$.

Proposition 4.1.1. *Assume that $u \in \mathcal{S}'$ is a tempered distribution. The wavefront set of $U_0(t)u$ satisfies*

$$\mathrm{WF}_{\mathrm{cl}}(U_0(t)u) \cap G_t \subset \{(x,\Xi_t(\eta)) \in G_t\colon y - \partial_\eta\langle x,\Xi_t(\eta)\rangle \perp \Gamma_t, (y,\eta) \in \mathrm{WF}_{\mathrm{cl}}(u)\}\,.$$

If $u \in \mathcal{E}'$, there cannot appear any other singularities:

$$\mathrm{WF}_{\mathrm{cl}}(U_0(t)u) \subset \{(x,\Xi_t(\eta)) \in G_t\colon y - \partial_\eta\langle x,\Xi_t(\eta)\rangle \perp \Gamma_t, (y,\eta) \in \mathrm{WF}_{\mathrm{cl}}(u)\}\,.$$

Remark 4.1.2. If $p_2(x,\xi) = 1/2(|\xi|^2 + \sum_j \omega_j x_j^2)$ then this proposition follows from Mehler's formula (see Section 4.6).

We follow the notation of [11] and denote the integral over the flow of H_0 by X_t for any $t \in \mathbb{R}$:

$$\mathsf{X}_t f = \int_0^t f \circ \exp(s\mathsf{H}_0) ds.$$

The wavefront set of the reduced propagator can be completely determined:

Proposition 4.1.3. *Let $u \in \mathcal{S}'$ and $t \in \mathbb{R}$. Then*

$$\mathrm{WF}_{\mathrm{cl}}(F(t)u) = \{(x + \partial_\xi \mathsf{X}_t p_1(0,\xi),\xi)\colon (x,\xi) \in \mathrm{WF}_{\mathrm{cl}}(u)\}\,.$$

Combining Proposition 4.1.1 and Proposition 4.1.3 yields

Theorem 4.1.4. *Let $u \in \mathcal{E}' + \mathcal{S}$ and $t \in \mathbb{R}$. The classical wavefront set of $U(t)u$ is a subset of*

$$\{(x,\Xi_t(\eta)) \in G_t\colon \partial_\eta\langle x,\Xi_t(\eta)\rangle - \partial_\eta \mathsf{X}_t p_1(0,\eta) - y \perp \Gamma_t, (y,\eta) \in \mathrm{WF}_{\mathrm{cl}}(u)\}\,.$$

History

The usual setting is the Laplacian on $\mathbb{R}^d$ plus a potential perturbations, that is

$$H = H_0 + V,$$

with H_0 the harmonic oscillator and $V = V(x)$ a potential perturbation. Zelditch [68] proved that for $V \in S^0_{\mathrm{cl}}$ that the singular support of $e^{-itH}u$ is equal to the singular support of

$e^{-itH_0}u$ by calculating the Schwartz-kernel. This was improved by Weinstein [64] to show that the wavefront sets are equal. Further results for zeroth order perturbations were obtained by Kapitanski–Rodnianski–Yajima [34], Mao–Nakamura [38], and Wunsch [65]. In the case $V \in S^1_{\mathrm{cl}}$, Doi [9] and Mao [37] showed for the harmonic oscillator that singularities are shifted. For the anharmonic oscillator there appear weaker singularities even for potentials $V \in \mathcal{C}^\infty_c$ (cf. Doi [9]).

Recurrence of singularities for the harmonic oscillator with perturbation by a pseudodifferential operator in the isotropic calculus was proved in [11].

Outline

The rest of this chapter is structed as follows: In Section 4.2 we recall basic properties about the the SG-calculus. The main part of the chapter is Section 4.3. We construct a parametrix as an oscillatory integral for the free propagator for arbitrary large times t and determine the wavefront set after reducing the oscillatory integral such that the phase is homogeneous of degree one in the fiber-variables. Section 4.4 treats the reduced propagator. There, we use a commutator argument in the SG-calculus to determine the classical wavefront set. Section 4.5 is a refined version of the stationary phase lemma, where the phase is not linear in the large parameter λ. We conclude with two examples, where the leading term is a quadratic form, to illustrate the results.

4.2. SG-Calculus

The SG-calculus is due to Cordes [3], the corresponding wavefront sets at infinity can be found in Coriasco–Maniccia [5]. A self-contained introduction to global pseudodifferential calculi can be found in [49].

The SG-calculus (also called scattering calculus for asymptotically Euclidean manifolds) differs from the isotropic calculus by the fact that taking derivatives in x does not affect the decay in ξ and vice versa. The SG-calculus is in a way the more natural way of defining pseudodifferential operators on $\mathbb{R}^d$, but it is not suited for second order differential operators such as the harmonic oscillator.

Definition 4.2.1. Let m_ψ, m_e be real numbers. The class $\mathrm{SG}^{m_\psi,m_e}(\mathbb{R}^d)$ consists of functions $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ such that for all multiindices $\alpha, \beta \in \mathbb{N}^d$ the is an estimate

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim_{\alpha,\beta} \langle \xi \rangle^{m_\psi - |\beta|} \langle x \rangle^{m_e - |\alpha|}.$$

We define the corresponding class of SG-pseudodifferential operators:

$$\operatorname{Op}\mathrm{SG}^{m_\psi,m_e} = \{a^w(x,D) \colon a \in \mathrm{SG}^{m_\psi,m_e}\}.$$

The SG-calculus enjoys the following properities:

(i) $\operatorname{Op}\mathrm{SG} = \bigcup_{m_\psi,m_e} \operatorname{Op}\mathrm{SG}^{m_\psi,m_e}$ is a $*$-algebra.

(ii) Differential operators of the form

$$\sum_{|\alpha|\le m_e, |\beta|\le m_\psi} a_{\alpha,\beta} x^\alpha D^\beta$$

lie in $\operatorname{Op}\mathrm{SG}^{m_\psi,m_e}$.

(iii) There are two principal symbol maps σ^ψ, σ^e,

$$\sigma^\psi : \operatorname{Op}\mathrm{SG}^{m_\psi,m_e} \to \mathrm{SG}^{m_\psi,m_e} / \mathrm{SG}^{m_\psi-1,m_e},$$
$$\sigma^e : \operatorname{Op}\mathrm{SG}^{m_\psi,m_e} \to \mathrm{SG}^{m_\psi,m_e} / \mathrm{SG}^{m_\psi,m_e-1},$$

such that the following principal symbol sequences are exact:

$$0 \to \operatorname{Op}\mathrm{SG}^{m_\psi-1,m_e} \to \operatorname{Op}\mathrm{SG}^{m_\psi,m_e} \xrightarrow{\sigma^\psi} \mathrm{SG}^{m_\psi,m_e} / \mathrm{SG}^{m_\psi-1,m_e},$$
$$0 \to \operatorname{Op}\mathrm{SG}^{m_\psi,m_e-1} \to \operatorname{Op}\mathrm{SG}^{m_\psi,m_e} \xrightarrow{\sigma^e} \mathrm{SG}^{m_\psi,m_e} / \mathrm{SG}^{m_\psi,m_e-1}.$$

We note that for ellipticity one needs a third principal symbol, $\sigma^{\psi,e}$.

(iv) If $A \in \operatorname{Op}\mathrm{SG}^{m_\psi,m_e}$, $B \in \operatorname{Op}\mathrm{SG}^{m'_\psi,m'_e}$, then

$$[A,B] \in \operatorname{Op}\mathrm{SG}^{m_\psi+m'_\psi-1, m_e+m'_e-1}$$

and satisfies

$$\sigma^\bullet_{m_\bullet+m'_\bullet-1}([A,B]) = \frac{1}{i}\{\sigma^\bullet_{m_\bullet}(A), \sigma^\bullet_{m'_\bullet}(B)\},$$

for $\bullet \in \psi, e$ and with the Poisson bracket indicating the (well-defined) equivalence class of the Poisson bracket of representatives of the equivalence classes of each of the principal symbols.

(v) Every $A \in \operatorname{Op}\mathrm{SG}^{0,0}$ defines a continuous linear map on $L^2(\mathbb{R}^d)$.

(vi) The SG-Sobolev spaces, $H^{s_\psi,s_e}_{\mathrm{SG}}$ are defined for $s_\psi, s_e \in \mathbb{R}$ by

$$f \in H^{s_\psi,s_e}_{\mathrm{SG}} \iff \langle x\rangle^{s_e}\langle D\rangle^{s_\psi} f \in L^2(\mathbb{R}^d).$$

For all $m_\psi, m_e, s_\psi, s_e \in \mathbb{R}$ and all $A \in \operatorname{Op}\mathrm{SG}^{m_\psi,m_e}$,

$$A : H^{s_\psi,s_e}_{\mathrm{SG}} \to H^{s_\psi-m_\psi, s_e-m_e}_{\mathrm{SG}}$$

is continuous.

(vii) The scale of SG-Sobolev spaces satisfies

$$\bigcap_{m_\psi, m_e} H_{\mathrm{SG}}^{m_\psi, m_e} = \mathcal{S}(\mathbb{R}^d), \qquad \bigcup_{m_\psi, m_e} H_{\mathrm{SG}}^{m_\psi, m_e} = \mathcal{S}'(\mathbb{R}^d).$$

(viii) The classical wavefront set of $u \in \mathcal{S}'$ is given by

$$\mathrm{WF}_{\mathrm{cl}} u = \bigcap_{\substack{A \in \mathrm{Op}\,\mathrm{SG}^{0,-\infty} \\ Au \in \mathcal{S}}} \Sigma_\psi(A),$$

where Σ_ψ is set of points $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$ such that $\sigma^\psi(a)(x, \xi) = 0$.

4.3. The Free Propagator

We start with reviewing the construction of a parametrix for the free propagator $U_0(t) = e^{-itH_0}$ in the FIO calculus of Helffer–Robert (cf. [22, Chapter 3]). Let $T > 0$ such that there exists a short-time parametrix $\widetilde{U}_0$ of U_0 until time T and $\widetilde{U}_0$ has the form

$$\widetilde{U}_0(t) = \int e^{i(\phi_2(t,x,\xi) - y\xi)} a(t, x, \xi)\, d\xi,$$

where $a \in \mathcal{C}([0, T], \Gamma_{\mathrm{cl}}^0)$ and $\phi_2 \in \mathcal{C}([0, T], \mathcal{C}^\infty(\mathbb{R}^{2d}))$, with the following properties for $t \in [0, T]$ and $|(x, \xi)| > 1$:

- ϕ_2 is homogeneous of degree 2 in (x, ξ),
- ϕ_2 solves the eikonal equation

$$\begin{cases} \partial_t \phi_2(t) + p_2(x, \partial_x \phi_2) = 0 \\ \phi_2(0) = x\xi, \end{cases}$$

- $\det \partial_x \partial_\xi \phi_2(t) \neq 0$ for $t \in [0, T]$,
- $\exp(-t\mathsf{H}_0)(x, \partial_x \phi_2) = (\partial_\xi \phi_2, \xi)$.

The short-time parametrix is constructed by solving the eikonal equation and transport equations for the homogeneous terms of the amplitude a_j. The time $T > 0$ depends on the eikonal equation and the transport equation for a_0. Using Borel summation and Duhamel's formula (cf. [22, Proposition 3.1.1]), we obtain that

$$\widetilde{U}_0(t) - U_0(t) \in \mathcal{C}^\infty([0, T], \mathcal{L}(\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))).$$

Let $t_0 > 0$ and write $t_0 = \sum_{j=1}^{N} t_j$ such that $t_j \in (0, T)$. Using the group property of U_0, we obtain for $u \in \mathcal{S}'$,

$$\begin{aligned} U_0(t_0) &= U_0(t_1) \cdots U_0(t_N) \\ &\in \widetilde{U}_0(t_1) \cdots \widetilde{U}_0(t_N) + \mathcal{L}(\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)). \end{aligned}$$

The parametrix $\widetilde{U}_0(t_j)$ has kernel

$$\widetilde{U}_0(t_j, z_{j-1}, z_j) = \int e^{i(\phi_2(t, z_{j-1}, \xi_j) - z_j \xi)} a(t, z_{j-1}, \xi_j)\, d\xi_j.$$

We write $x = z_0$ and $y = z_N$, then a parametrix for $t = t_0$ is given by

$$\widetilde{U}_0(t_0) = \int e^{i\phi(x,y,\theta)} a(x, \theta) d\theta,$$

where

$$\begin{aligned} \theta &= (z_1, \dots, z_{N-1}, \xi_1, \dots, \xi_N), \\ \phi(z_0, z_N, \theta) &= \sum_{j=1}^{N} \phi_2(t_j, z_{j-1}, \xi_j) - z_j \xi_j, \\ a(s, z_0, \theta) &= \prod_{j=1}^{N-1} a(t_j, z_{j-1}, \xi_j) \cdot a(t_N, z_{N-1}, \theta). \end{aligned}$$

One advantage of the isotropic calculus is that the new phase function $\phi = \phi(x, y, \theta)$ is homogeneous of degree 2 in all variables for $|(x, y, \theta)|$ large enough.

4.3.1. Classical Flow and Lagrangian Submanifolds

Given an Hamiltonian function p_2 we associate the flow $t \mapsto \exp(t\mathsf{H}_0)$ and define the set

$$\Lambda = \left\{(x, y, \xi, -\eta) \colon \exp(t_0 \mathsf{H}_0)(y, \eta) = (x, \xi)\right\}.$$

We note that Λ is a Lagrangian submanifold of $\mathbb{R}^{4d}$.

In this section, we always assume that $t = t_0$ and omit t from the notation, for instance we write $\exp(-t_0 \mathsf{H}_0)(x, \xi) = (y(x, \xi), \eta(x, \xi))$.

As mentioned in the introduction, we work under the assumption that Λ and $0 \times \mathbb{R}^{2d} \setminus \{0\}$ intersect cleanly, that means that $\Lambda_0 = \Lambda \cap 0 \times \mathbb{R}^{2d} \setminus \{0\}$ is a smooth manifold and

$$T_w \Lambda_0 = T_w \Lambda \cap T_w (0 \times \mathbb{R}^{2d}), \quad w \in \Lambda_0.$$

At a point $w_0 = (0, 0, \xi_0, \eta_0) \in \Lambda_0$ the tangent space is given by

$$T_{w_0}\Lambda_0 = \left\{(0, 0, \xi, \eta) \in \mathbb{R}^{4d} \colon \partial_\xi y(0, \xi_0)\xi = 0, \partial_\xi \eta(0, \xi_0)\xi = \eta\right\}.$$

The dimension of Λ_0 is given by a function $(x, \xi) \mapsto e(\xi, \eta)$, which is a locally constant integer with $e \leq d$. Further,

$$\begin{aligned} e(\xi_0, \eta_0) &= \dim T_{w_0}\Lambda_0 \\ &= d - \operatorname{rk} \partial_\xi y(0, \xi_0) \\ &= \operatorname{rk} \partial_\xi \eta(0, \xi_0). \end{aligned}$$

The critical set of $\phi(x, y, \theta)$ is defined by

$$C_\phi = \{(x, y, \theta) \colon \partial_\theta \phi(x, y, \theta) = 0\},$$

with

$$\partial_\theta \phi = \begin{pmatrix} \partial_z \phi_2(t_j, z_j, \xi_{j+1}) - \xi_j \\ \partial_\xi \phi_2(t_j, z_{j-1}, \xi_j) - z_j \end{pmatrix}.$$

The phase function is non-degenerate and (cf. [12]) as a direct consequence of the regular value theorem, we see that

Lemma 4.3.1. *The set C_ϕ is a manifold of dimension $2d$.*

We have a diffeomorphism

$$\begin{aligned} &\lambda_\phi : C_\phi \to \Lambda, \\ &(x, y, \theta) \mapsto (x, y, \partial_x \phi(x, y, \theta), \partial_y \phi(x, y, \theta)). \end{aligned}$$

Since cleanness is preserved under diffeomorphisms the manifolds C_ϕ and $0 \times \mathbb{R}^N$ intersect cleanly. We denote the intersection by $C_{\phi,0}$. The manifold $C_{\phi,0}$ and its tangential bundle are given by

$$\begin{aligned} C_{\phi,0} &= \{(0, 0, \theta) \colon \partial_\theta \phi(0, 0, \theta) = 0\}, \\ T_{(0,0,\theta_0)} C_{\phi,0} &= \{(0, 0, \delta\theta) \colon \partial_\theta \partial_\theta \phi(0, 0, \theta_0)\delta\theta = 0\}, \end{aligned}$$

so we conclude that for $(0, 0, \theta_0) \in C_{\phi,0}$ and $(0, 0, \xi_0, \eta_0) = \lambda_\phi(0, 0, \theta_0)$,

$$N - \operatorname{rk} \partial_{\theta,\theta} \phi(0, 0, \theta_0) = \operatorname{rk} \partial_\xi \eta(0, \xi_0) = e(\xi_0, \eta_0).$$

The next proposition is implicit in the work of Helffer–Robert:

Proposition 4.3.2. *Let $a \in \Gamma^m$ and $t \in \mathbb{R}$ arbitrary. Then*

$$B = e^{it_0 H_0} a(x, D) e^{-it_0 H_0}$$

is an isotropic pseudodifferential operator, $B \in G^m$ and its principal symbol is given by

$$\sigma^m(B)(y, \eta) = \sigma^m(a)(\exp(t_0 \mathsf{H}_0)(y, \eta)).$$

Proof. By Corollary 2.10.7 from [22] and the parametrix construction, the operator B is an isotropic pseudodifferential operator with principal symbol

$$\sigma^m(B)(y, -\partial_y \phi(x, y, \theta)) = \sigma^m(A)(x, \partial_x \phi(x, y, \theta)),$$

for $(x, y, \theta) \in C_\phi$. Using the diffeomorphism λ_ϕ and the definition of Λ, we see that this is nothing but

$$\sigma^m(B)(y, \eta) = \sigma^m(A)(\exp(t_0 \mathsf{H}_0)(y, \eta))$$

as claimed. □

Proposition 4.3.3. *Let $u \in \mathcal{S}'(\mathbb{R}^d)$. One has*

$$\mathrm{WF}_{\mathrm{iso}}(U_0(t_0)u) = \exp(t_0 \mathsf{H}_0) \, \mathrm{WF}_{\mathrm{iso}}(u).$$

Proof. It suffices to prove that $\mathrm{WF}_{\mathrm{iso}}(U_0(t_0)u) \supset \exp(t_0 \mathsf{H}_0) \, \mathrm{WF}_{\mathrm{iso}}(u)$, equality follows from time-reversal.

Let $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{iso}}(U_0(t_0)u)$. Then there is a $Q \in G^0_{\mathrm{cl}}$ such that $\sigma^0(Q)(x_0, \xi_0) = 1$ and $QU_0(t_0)u \in \mathcal{S}$. This implies by Proposition 4.3.2

$$Pu = U_0(-t_0) Q U_0(t_0) u \in \mathcal{S},$$

and $\sigma^0(P)(y, \eta) = \sigma^0(Q)(\exp(t_0 \mathsf{H}_0)(y, \eta))$. Set $(y_0, \eta_0) = \exp(-t_0 \mathsf{H}_0)(x_0, \xi_0)$. Then the principal symbol at (y_0, η_0) is $\sigma^0(P)(y_0, \eta_0) = 1$ and therefore $(y_0, \eta_0) \notin \mathrm{WF}_{\mathrm{iso}}(u)$. □

4.3.2. Recurrence of Singularities

Now, we investigate the recurrence of classical singularities, which is more delicate. We define the reduced phase function ϕ_{red} by

$$\phi_{\mathrm{red}}(t_0, x, y, \eta) = x \Xi_{t_0}(\eta) - y\eta.$$

Proposition 4.3.4. *The propagator at time $t = t_0$ is locally given by*

$$U_0(t_0, x, y) = \int_\Gamma e^{i\phi_{\mathrm{red}}(t_0, x, y, \eta)} \tilde{a}(t_0, x, y, \eta) d\eta$$

modulo a smoothing operator. Here, $\tilde{a} \in S^0$ is a local Kohn-Nirenberg symbol and $\Gamma = \{\eta \in \mathbb{R}^d \setminus \{0\} \colon (0, 0, \xi, -\eta) \in \Lambda_0$ for some $\xi \in \mathbb{R}^d\}$.

To prove this proposition, we first split the parametrix of $U_0(t_0)$ into a sum of oscillatory integrals supported near connected components of the critical set $C_{\phi,0}$. Then, we reduce the number of fiber-variables similarly as in the case of Fourier integral operators on compact manifolds (cf. Hörmander [32]). In the last step, we show that the resulting amplitude satisfies the Kohn-Nirenberg estimates.

Proof. We write

$$\phi(x,y,\theta) = \phi(0,0,\theta) + x\partial_x\phi(0,0,\theta) + y\partial_y\phi(0,0,\theta) + \sum_{|\alpha|=2}(x,y)^\alpha f_\alpha(x,y,\theta)$$

for some smooth functions f_α and set $f(x,y,\theta) = \sum_{|\alpha|=2}(x,y)^\alpha f_\alpha(x,y,\theta)$. Also, we set $\psi(x,y,\theta) = x\partial_x\phi(0,0,\theta) + y\partial_y\phi(0,0,\theta)$ and $\phi_0(\theta) = \phi(0,0,\theta)$.

By choosing a function $\chi \in \mathcal{C}_c^\infty([0,\infty))$ such that $\chi(r) = 1$ for $r < R$ for some $R > 0$, we may assume that the phase $\phi(0,0,\theta)$ is homogeneous on $\mathbb{R}^N$. In fact, the operator with kernel

$$\int_{\mathbb{R}^N} e^{i\phi(x,y,\theta)}\chi(|\theta|)a(x,\theta)d\theta$$

is regularizing, so may replace in the following $a(x,\theta)$ by $(1-\chi(|\theta|))a(x,\theta)$. The set $C_{\phi,0}$ is conic and we can choose a conic partition of unity $\{\chi_j\}$ such that $C_{\phi,0} \cap \operatorname{supp}\chi_j$ is a connected manifold of dimension $N - e_j$. From now on we restrict our considerations to one χ_j.

After a linear transformation, we may assume $(\theta',\theta'') \in \mathbb{R}^{e_j} \times \mathbb{R}^{N-e_j}$ on $\operatorname{supp}\chi_j$ such that

$$\operatorname{rk}\partial_{\theta''\theta''}\phi_0(\theta) = N - e_j.$$

Homogeneity of $\phi_0(\theta)$ implies that $\phi_0(\theta) = 0$ on $C_{\phi,0}$. Using the implicit function theorem to the equation $\partial_{\theta''}\phi_0(\theta) = 0$, we obtain a smooth map $g : \mathbb{R}^{e_j} \to \mathbb{R}^{N-e_j}$ that is homogeneous of degree 1 outside a compact set such that

$$(\theta',\theta'') \in C_{\phi,0} \text{ if and only if } \theta'' = g(\theta').$$

We introduce new coordinates

$$(\vartheta',\vartheta'') = (\theta', \theta'' - g(\theta'))$$

and the phase function

$$\varphi_0(\vartheta) = \phi_0(\theta).$$

Then $(\vartheta',\vartheta'') \in C_{\phi,0}$ if and only if $\vartheta'' = 0$. So, we have that

$$C_{\varphi_0} = \{\partial_\vartheta\varphi_0 = 0\} = C_{\phi,0}.$$

There exists a quadratic form $Q = Q(\vartheta'')$ with the same signature as $\partial_{\vartheta''\vartheta''}\varphi_0$ such that Q and φ_0 are equivalent in the sense of [22, Definition 2.10.13] by Proposition 2.10.14 in [22]. Since all coordinate transformations are homogeneous of degree 1, the amplitude and the functions ψ and f are of the same form as before.

So, we may assume that $\phi(0,0,\theta)$ depends only on θ'' and $C_{\phi,0} = \mathbb{R}^{e_j} \times 0$. The propagator at time $t = t_0$ becomes

$$\begin{aligned} U_0(t_0) &= \int_{\mathbb{R}^{e_j}} \int_{\mathbb{R}^{N-e_j}} e^{i\varphi(x,y,\vartheta',\vartheta'')} a(x,\vartheta',\vartheta'') d\vartheta' d\theta'' \\ &= \int_{\mathbb{R}^{e_j}} e^{i\psi(x,y,\theta',0)} \tilde{a}(x,y,\theta') d\theta', \end{aligned}$$

with

$$\tilde{a}(x,y,\theta') = \int_{\mathbb{R}^{N-e_j}} e^{i\phi_0(\theta'') + i(\psi(x,y,\theta',\theta'') - \psi(x,y,\theta',0)) + if(x,y,\theta)} a(x,\theta',\theta'') d\theta''.$$

We now show that $\psi(x,y,\theta',0)$ is nothing but ϕ_{red} in local coordinates. Since in our adapted coordinates $C_{\phi,0} = \mathbb{R}^{e_j} \times 0$, we see that

$$\psi(x,y,\theta',0) = \psi(x,y,\theta)|_{C_{\phi,0}}.$$

Using the diffeomorphism $\lambda_\phi : C_\phi \to \Lambda$, we see that

$$\exp(t_0 \mathsf{H}_0)(0, \partial_x \phi(0,0,\theta',0)) = (0, -\partial_y \phi(0,0,\theta')).$$

By the inverse function theorem, there exists a map $\eta \mapsto \theta'$ such that

$$-\partial_y \phi(0,0,\theta'(\eta),0) = \eta.$$

Thus, $\psi(x,y,\theta'(\eta),0) = \phi_{\mathrm{red}}(t_0,x,y,\eta)$.

The map $\eta \mapsto \theta'$ is homogeneous of degree 1. Therefore, it only remains to show that $\tilde{a}$ is a Kohn-Nirenberg symbol. We define the amplitude

$$c(x,y,\theta) = e^{if(x,y,\theta)} a(x,y,\theta)$$

and we write the ψ-phase as

$$\psi(x,y,\theta',\theta'') - \psi(x,y,\theta',0) = \langle \theta'', g(x,y,\theta',\theta'') \rangle,$$

with $g(x,y,\theta',\theta'') = \int_0^1 \partial_{\theta''}\psi(x,y,\theta',t\theta'')dt$. Note that the functions c and g satisfy

$$\begin{aligned} |\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma c(x,y,\theta)| &\lesssim_{\alpha,\beta,\gamma} \langle \theta \rangle^{-|\gamma|} \langle (x,y) \rangle^{|\alpha|+|\beta|+2|\gamma|}, \\ |\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma g(x,y,\theta)| &\lesssim_{\alpha,\beta,\gamma} \langle \theta \rangle^{-|\gamma|} \langle x \rangle^{1-|\alpha|} \langle y \rangle^{1-|\beta|}. \end{aligned}$$

We have to show that $\tilde{a} \in S^0$ on compact sets, in fact we will show that the following estimate holds:

$$|\partial_x^\alpha \partial_y^\beta \partial_{\theta'}^\gamma \tilde{a}(x, y, \theta')| \lesssim_{\alpha,\beta,\gamma} \langle \theta' \rangle^{-|\gamma|} \langle (x, y) \rangle^{|\alpha|+|\beta|+2|\gamma|}.$$

A typical term is of the form

$$I(x, y, \theta') = \int_{\mathbb{R}^{N-e}} e^{i\phi_0(\theta'')} e^{i\langle g, \theta'' \rangle} \prod_{j=1}^k \langle \theta'', \partial_{x,y,\theta'}^{\kappa_j} g \rangle c(x, y, \theta) d\theta''. \tag{4.3}$$

We use the standard Paley-Littlewood decomposition: Choose a $\tilde{\chi} \in \mathcal{C}_c^\infty(\mathbb{R}^{N-e})$ such that $\tilde{\chi} \geq 0$ everywhere, $\tilde{\chi}(x) = 1$ for $|x| \leq 1$, and $\tilde{\chi}(x) = 0$ for $x \geq 2$. Set $\chi_j(x) = \tilde{\chi}(x/2^j) - \tilde{\chi}(x/2^{j-1})$. Then

$$1 = \tilde{\chi}(x) + \sum_{j=1}^\infty \chi_j(x) \qquad \text{for all } x \in \mathbb{R}^{N-e}.$$

For $\lambda = 2^j$, we have

$$I(x, y, \theta') = \sum_{j=1}^\infty \lambda^{N-e} \int_{\mathbb{R}^{N-e}} e^{i\lambda^2 \phi_0(\theta'')} e^{i\lambda \langle g, \theta'' \rangle} \chi_1(\theta'') \prod_{j=1}^k \langle \lambda\theta'', \partial_{x,y,\theta'}^{\kappa_j} g \rangle c(x, y, \theta', \lambda\theta'') d\theta''$$
$$+ \int_{\mathbb{R}^{N-e}} e^{i\phi_0(\theta'')} e^{i\langle g, \theta'' \rangle} \tilde{\chi}(\theta'') \prod_{j=1}^k \langle \theta'', \partial_{x,y,\theta'}^{\kappa_j} g \rangle c(x, y, \theta', \theta'') d\theta''.$$

In order to estimate the sum, we observe that θ''-derivatives of the function

$$e^{i\lambda\langle g, \theta'' \rangle} \chi_1(\theta'') \prod_j \langle \lambda\theta'', \partial_{\theta'}^{\gamma_j} g \rangle c(x, y, \theta', \lambda\theta'')$$

can be estimated by

$$\lambda^{k+|\gamma|} \langle \theta' \rangle^{-|\gamma|} \langle x \rangle^{1-|\alpha|} \langle y \rangle^{1-|\beta|},$$

where $\kappa = (\alpha, \beta, \gamma) \in \mathbb{N}_x^d \times \mathbb{N}_y^d \times \mathbb{N}_{\theta'}^e$ for the multiindex $\kappa = \sum_{j=1}^k \kappa_j$. Using Theorem 7.7.1 from [30], we obtain that for all $M > 0$, each summand can be estimated by

$$\lambda^{N-e-M} \lambda^{k+|\gamma|} \langle \theta' \rangle^{-|\gamma|} \langle x \rangle^{1-|\alpha|} \langle y \rangle^{1-|\beta|}.$$

Choosing $M > N - e + k + |\gamma| + 1$, we can sum the geometric series, which yields the desired bound.

For the last term, we have to use the method of stationary phase. We note that it suffices to show that

$$|\partial_x^\alpha \partial_y^\beta \partial_{\theta'}^\gamma \tilde{a}(x, y, \lambda\theta')| \leq C_{\alpha,\beta,\gamma}$$

as $\lambda \to \infty$ for $C_{\alpha,\beta,\gamma}$ independent of λ. We check that $c(x, y, \lambda\theta)$ and all of its derivatives are bounded by some constant independent of λ. Therefore, Proposition 4.5.1 proves the claim. □

Proof of Proposition 4.1.1. We have constructed a suitable parametrix in Proposition 4.3.4. By Theorem 8.1.9. in [30] the wavefront set of the distribution $U_0(t)$ is given by

$$\mathrm{WF}_{\mathrm{cl}}(U_0(t)) \subset \{(x, y, \Xi_t(\eta), -\eta) \colon \eta \in \Gamma_t, y - \partial_\eta \langle x, \Xi_t(\eta) \rangle \perp \Gamma_t\}\,.$$

By the calculus of wavefront sets, we obtain that for any $(x, \eta) \in \mathbb{R}^{2d}$ such that $\eta \in \Gamma_t$ and $(x, \Xi_t(\eta)) \in \mathrm{WF}_{\mathrm{cl}}(U_0(t)u)$ then $(\partial_\eta \langle x, \Xi(\eta) \rangle, \eta) \in \mathrm{WF}_{\mathrm{cl}}(u)$.

Now, let $u \in \mathcal{E}'$. Assume that there is a $(x_0, \xi_0) \in \mathrm{WF}_{\mathrm{cl}}(U_0(t)u)$ such that $\xi_0 \notin \Xi_t(\Gamma_t)$, that is there exists $(y_0, \eta_0) \in \mathbb{R}^{2d} \setminus \{0\}$ such that $\exp(t\mathsf{H}_0)(0, \xi_0) = (y_0, \eta_0)$ and $y_0 \neq 0$. By Lemma 2.4.13, $(0, \xi_0) \in \mathrm{WF}_{\mathrm{iso}}(U_0(t)u)$. We have seen that the isotropic wavefront set is shifted by the Hamiltonian flow (Proposition 4.3.3) and therefore $(y_0, \eta_0) = \exp(-t\mathsf{H}_0)(0, \xi_0) \in \mathrm{WF}_{\mathrm{iso}}(u)$. By definition of the set Γ_t, $y_0 \neq 0$, but this contradicts the assumption that u was compactly supported using Lemma 2.4.12. □

4.4. The Reduced Equation

The reduced propagator $F(t) = U_0(-t)U(t)$ satisfies

$$\begin{cases} (\partial_t - U_0(-t)(H - H_0)U_0(t))\, F(t) = 0 \\ F(0) = \mathrm{I}\,. \end{cases} \tag{4.4}$$

We define the operator $P(t) = U_0(-t)(H - H_0)U_0(t)$. By Proposition 4.3.2, $P(t) \in G^1_{\mathrm{cl}}$ and the principal symbol is given by

$$\sigma^1(P(t)) = p_1 \circ \exp(t\mathsf{H}_0).$$

Proposition 4.4.1. *For all $u \in \mathcal{S}'$ and $t \in \mathbb{R}$,*

$$\mathrm{WF}_{\mathrm{iso}}(F(t)u) = \mathrm{WF}_{\mathrm{iso}}(u).$$

Proof. This follows from Lemma 3.1 in [11]. □

Proposition 4.4.2. *Let $a \in \mathcal{C}([0, T], \mathrm{SG}^{1,1}_{\mathrm{cl}})$ be real-valued and assume that there is a bounded set $K \subset \mathrm{SG}^{1,1}$ such that for all $|t| \leq T$, $a(t) \in K$ Consider for $u_0 \in \mathcal{S}'$ the equation*

$$\begin{cases} (i\partial_t - a(t, x, D))u(t) = 0 \\ u(0) = u_0. \end{cases} \tag{4.5}$$

Let $u \in C(\mathbb{R}, \mathcal{S}')$ be a solution of (4.5). The wavefront set of $u(t)$ is given by

$$\mathrm{WF}_{\mathrm{cl}}(u(t)) = \Psi_t \mathrm{WF}_{\mathrm{cl}}(u_0),$$

where Ψ_t is the Hamiltonian flow associated to the function $\sigma^\psi(a(t, x, D))$.

Remark 4.4.3. If we exchange the role x and ξ, we can prove that WF_e evolves according to the Hamiltonian flow Ψ_t^e of the symbol $\sigma^e(a(t,x,D))$,

$$\mathrm{WF}_e(u(t)) = \Psi_t^e \, \mathrm{WF}_e(u_0),$$

under the assumption that a admits an asymptotic expansion into terms homogeneous in x.

Using the fact that the SG-estimates are weaker then the isotropic estimates (cf. [49, Section 3.1]) we obtain the propagation of singularities result for $F(t)$:

Proof of Proposition 4.1.3. The symbol $p(t) \in \Gamma^1_{\mathrm{cl}} \subset \mathrm{SG}^{1,1}$ has principal ψ-symbol given by $(p_1 \circ \exp(t\mathsf{H}_0))(0,\xi)$ and Proposition 4.4.2 implies that the wavefront set is

$$\mathrm{WF}_{\mathrm{cl}}(F(t)u) = \{(x + \partial_\xi X_t p_1(0,\xi), \xi) \colon (x,\xi) \in \mathrm{WF}_{\mathrm{cl}}(u)\}\,.$$

□

Proposition 4.4.2 also follows from [5]. We give a self-contained proof using a commutator argument (cf. Hörmander [31, Theorem 23.1.4]).

Proof. Let $(x_0,\xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(u_0)$ then there is a symbol $b \in \mathrm{SG}^{0,-\infty}_{\mathrm{cl}}$ such that $b(x_0,\xi_0) = 1$ and $b(x,D)u_0 \in \mathcal{S}$.

We construct a symbol $q \in \mathcal{C}^\infty([0,T], \mathrm{SG}^{0,-\infty}_{\mathrm{cl}})$ with the following properties

- $[i\partial_t - a(t,x,D), q(t,x,D)] \in \mathcal{C}([0,T], \mathrm{Op}\,\mathrm{SG}^{-\infty,-\infty})$,
- $q(0,x,\xi) = b(x,\xi)$,
- $\sigma^\psi q(t) = \Psi_t \sigma^\psi b$.

If we write $q(t,x,\xi) \sim \sum_j q_{-j}(t,x,\xi)$ where $q_{-j}(t) \in \mathrm{SG}^{-j,-\infty}_{\mathrm{cl}}$ is homogeneous of degree $-j$ in ξ, we can see that the ψ-principal symbol of the commutator is given by

$$\sigma^\psi([i\partial_t - a(t,x,D), q(t,x,D)]) = i\{\tau + a_0(t,x,\xi), q_0(t,x,\xi)\} = i(\partial_t + H_{a_0})q_0,$$

where $H_{a_0} = \partial_\xi a_0 \partial_x - \partial_x a_0 \partial_\xi$ is the Hamiltonian vector field of a_0. The term of order $-j$ is given by

$$i(\partial_t + H_{a_0})q_{-j} + R_j$$

with R_j depending on $q_0, \ldots, q_{-j+1}$. By the assumption that $a(t)$ is contained in a fixed bounded set for all $|t| \leq T$, the equations

$$\begin{cases} (\partial_t y, \partial_t \eta) = (\partial_\xi a_0(t,y,\eta), -\partial_x a_0(t,y,\eta)) \\ (y(0), \eta(0)) = (x,\xi) \end{cases}$$

have a unique solution for time $|t| \le T$. The map $\Psi_t(x,\xi) = (y(t,x,\xi), \eta(t,x,\xi))$ is the Hamiltonian flow of the principal symbol $\sigma^\psi(a(t,x,D))$ and defines a symplectomorphism, which is homogeneous in the second component, $(y(t,x,\lambda\xi), \eta(t,x,\lambda\xi)) = (y(t,x,\xi), \lambda\eta(t,x,\xi))$. If we set $q_0 = b_0(\Psi_t^{-1}(x,\xi))$ then q_0 solves

$$\begin{cases} (\partial_t + H_{a_0}) q_0(t) = 0 \\ q_0(0) = b_0. \end{cases}$$

Similarly, we solve the inhomogeneous equations for q_{-j}, $j > 0$ by

$$q_{-j}(t,x,\xi) = b_{-j}(\Psi_t^{-1}(x,\xi)) + i \int_0^t R_j(\Psi_{t-s}^{-1}(x,\xi)) ds.$$

If we set $q(t,x,\xi) \sim \sum_{j=0}^\infty q_{-j}(t,x,\xi)$ we obtain a symbol with the desired properties. This implies that if $u(t)$ is a solution to (4.5) with initial data u_0 then

$$\begin{cases} (i\partial_t - a(t,x,D)) q(t,x,D) u(t) \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d)) \\ q(0,x,D) u(0) \in \mathcal{S}(\mathbb{R}^d). \end{cases}$$

Using an energy estimate (cf. Hörmander [31, Theorem 23.1.2]) we conclude that $q(t,x,D)u(t) \in H^{s_\psi, s_e}$ for every $s_\psi, s_e \in \mathbb{R}$ and thus $q(t,x,D)u(t) \in \mathcal{S}$. This implies by the construction that $\Psi_t(x_0,\xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(u(t))$.

The whole argument can be carried out if we replace t by $-t$ and therefore we obtain equality of the wavefront sets. $\square$

4.5. Stationary Phase with Inhomogeneous Phase Function

We will derive a formula for calculating stationary phase integrals

$$I(\lambda, y) = \lambda^d \int_{\mathbb{R}^d} e^{i\lambda^2\phi(x) + i\lambda(\psi(x,y) - \psi(0,y))} a(\lambda, x, y) dx.$$

The function a is smooth and satisfies an estimate $|\partial_x^\alpha \partial_y^\beta a(\lambda,x,y)| \le C_{\alpha,\beta}\lambda^m$ and we assume that there is a compact set $K \subset \mathbb{R}^d \times \mathbb{R}^n$ such that for every $\lambda \in \mathbb{R}$, $\operatorname{supp} a(\lambda) \subset K$. The phase functions $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $\psi \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^n)$ satisfy

- ϕ and ψ are real-valued,
- $\phi(0) = \partial_x\phi(0) = 0$,
- $\partial_{xx}\phi(0)$ is non-singular.

Proposition 4.5.1. *For every $\alpha \in \mathbb{N}^n$,*

$$|\partial_y^\alpha I(\lambda, y)| \lesssim_\alpha \lambda^m.$$

Proof. Define the phase function

$$\Phi_\mu(x, y) = \phi(x) + \mu(\psi(x, y) - \psi(0, y)),$$

for $\mu > 0$ small enough the matrix $\partial_{xx}\Phi_\mu$ is invertible and therefore we may apply the regular value theorem to obtain a map $(\mu, y) \mapsto x(\mu, y)$ parametrizing $C_{\Phi_\mu} = \{\partial_x \Phi_\mu = 0\}$. Expanding $x(\mu, y)$ into powers of μ yields

$$x(\mu, y) = \mu\tilde{x}(\mu, y), \qquad \tilde{x} \in \mathcal{C}^\infty(\overline{\mathbb{R}_+} \times \mathbb{R}^n, \mathbb{R}^d).$$

The assumptions on ϕ imply that

$$\begin{aligned} \Phi_\mu|_{C_{\Phi_\mu}} &= \phi(\mu\tilde{x}(\mu, y)) + \mu(\psi(\mu\tilde{x}(\mu, y), y) - \psi(0, y)) \\ &= \phi(0) + \mu\partial_x\phi(0) + O(\mu^2) \\ &= O(\mu^2). \end{aligned}$$

Now, we can estimate $I(\lambda, y)$ and its derivatives. The only case where derivatives could cause problems is when they fall on the exponential, in which case one has terms of the form

$$\lambda^{d+l} \int_{\mathbb{R}^d} e^{i\lambda^2\phi(x) + i\lambda(\psi(x,y) - \psi(0,y))} \prod_{j=1}^{l} \left(\partial_{y_{i_j}}(\psi(x, y) - \psi(0, y))\right) a(x, y) dx.$$

We apply the method of stationary phase and see that each term $\partial_y(\psi(x, y) - \psi(0, y))$ is of order $O(\lambda^{-1})$ since

$$\partial_y(\psi(x, y) - \psi(0, y))|_{C_{\Phi_\mu}} = \mu\partial_{x,y}\psi(0, y) + O(\mu^2)$$

and in our case $\mu = \lambda^{-1}$, further the stationary phase eliminates the prefactor of λ^d and obtain

$$|\partial_y^\alpha I(\lambda, y)| \lesssim \lambda^m$$

as claimed. □

4.6. Examples

We will consider specific cases of $p \in \Gamma^2_{\mathrm{cl}}$ to illustrate the results.

4.6.1. Isotropic Harmonic Oscillator

We consider the free Hamiltonian

$$H_0 = \frac{1}{2}\left(\Delta + |x|^2\right),$$

with principal symbol $p_2 = 1/2(|x|^2 + |\xi|^2)$. It is well-known (cf. Grigis–Sjöstrand [17, Chapter 11]) that the propagator is smoothing for $t \notin \pi\mathbb{Z}$ and compactly supported initial data. For $t \in \pi\mathbb{Z}$, we have

$$e^{-ik\pi H_0} = (-iR)^k,$$

where $Rf(x) = f(-x)$ is the reflection operator. This implies that for $u \in \mathcal{S}'$ such that $\mathrm{WF}_{\mathrm{iso}}(u) \subset 0 \times \mathbb{R}^d$, the wavefront set of e^{-itH_0} is given by

$$\mathrm{WF}_{\mathrm{cl}}(e^{-itH_0}u) = \begin{cases} (-1)^k \mathrm{WF}_{\mathrm{cl}}(u) & t = \pi k, k \in \mathbb{Z}, \\ \emptyset & t \notin \pi\mathbb{Z}. \end{cases}$$

Proposition 4.1.3 then implies

Corollary 4.6.1. *Let $u \in \mathcal{S}'$ such that $\mathrm{WF}_{\mathrm{iso}}(u) \subset 0 \times \mathbb{R}^d$ then*

$$\mathrm{WF}_{\mathrm{cl}}(e^{-i\pi k H}u) = \left\{(-1)^k \left(x + \partial_\xi (\mathsf{X}_{\pi k} p_1)(0,\xi), \xi\right) : (x,\xi) \in \mathrm{WF}_{\mathrm{cl}}(u)\right\}$$

and $\mathrm{WF}_{\mathrm{cl}}(e^{-itH}u) = \emptyset$ if $t \notin \pi\mathbb{Z}$.

This was already proved in [11] using an explicit parametrix of the reduced propagator.

4.6.2. Anisotropic Harmonic Oscillator

Now we take the principal symbol $p_2 \in \Gamma^2_{\mathrm{cl}}(\mathbb{R}^d)$ with

$$p_2 = \frac{1}{2}\left(|\xi|^2 + \sum_{j=1}^d \omega_j^2 x_j^2\right).$$

The Hamiltonian flow of p_2 is given by

$$\begin{aligned} x_j(t) &= \cos(\omega_j t) x_j(0) + \frac{\sin(\omega_j t)}{\omega_j}\xi_j(0), \\ \xi_j(t) &= \cos(\omega_j t)\xi_j(0) + \omega_j \sin(\omega_j t) x_j(0). \end{aligned}$$

Again by Mehler's formula, we have an explicit solution operator:

$$U_0(t) = \int e^{i(\phi(t,x,\eta) - y\eta)} a(t)\, d\eta$$

with

$$\phi(t,x,\eta) = \sum_{j=1}^{d} \frac{1}{\cos(\omega_j t)} \left(x_j\eta_j - 1/2\sin(\omega_j t)(\omega_j x_j^2 + \omega_j^{-1}\eta_j^2) \right)$$

and $a(t) = \prod_{j=1}^d |\cos(\omega_j t)|^{1/2}$.

For simplicity, assume that $d = 2$ and $\omega_j = j$. Then the flow is periodic with minimal period 2π and the propagator is given at the recurrence points by

$$e^{-i\pi\frac{k}{2}H_0} = (-iR)^k \otimes (i^{-1/2}\mathcal{F})^k, \qquad k \in \mathbb{Z}.$$

That means that $e^{-i\pi\frac{k}{2}H_0}u(x,y) = e^{i\pi k/4}(\mathcal{F}_y)^k u((-1)^k x, y)$. Note that we take the unitary Fourier transform

$$\mathcal{F}u(\xi) = (2\pi)^{-d/2} \int e^{-ix\xi} u(x)\, dx.$$

From this, we identify the wavefront set of $e^{-itH_0}u$ for compactly supported initial data $u \in \mathcal{E}'$ as follows:

$$\mathrm{WF_{cl}}(e^{-itH_0}u) = \begin{cases} \{-(x,y,\xi,0) \colon (x,z,\xi,0) \in \mathrm{WF_{cl}}(u) \text{ for some } z \in \mathbb{R}, \\ \qquad\qquad -(x,y) \in \operatorname{supp} e^{-itH_0}u\}, & t \in \pi/2 + \pi\mathbb{Z}, \\ \{(x,(-1)^k y, \xi, (-1)^k\eta) \colon (x,y,\xi,\eta) \in \mathrm{WF_{cl}}(u)\}, & t = \pi k, k \in \mathbb{Z}, \\ \emptyset, & t \notin \frac{\pi}{2}\mathbb{Z}. \end{cases}$$

Let $p \in \Gamma^2_{\mathrm{cl}}$ with principal symbol p_2 as above and set $H = p^w(x,D)$. Using Proposition 4.1.3 we can calculate the wavefront set of $e^{-itH}u$ in terms of the wavefront set of u. This contrasts the case of potential perturbations, where even smooth compactly supported potential can give rise to new singularities (cf. Doi [9] and Zelditch [68]) and we can determine the singularities at time $t = \pi/2$, which was not possible in [9].

CHAPTER 5

Asymptotically Euclidean Manifolds

5.1. Scattering Manifolds

As mentioned in Example 2.4.2, the isotropic calculus is not suited for potential perturbations of the Laplacian. If we change the defining symbol estimates to the one we used in Chapter 4, we can circumvent of this problem:

Definition 5.1.1. Let $m_\psi, m_e \in \mathbb{R}$. A function $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ is in the symbol class $\mathrm{SG}^{m_\psi, m_e}$ if for all $\alpha, \beta \in \mathbb{N}^d$, the estimate

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \lesssim \langle x \rangle^{m_e - |\beta|} \langle \xi \rangle^{m_\psi - |\alpha|}$$

holds.

As in the case of the isotropic symbols, these spaces become Fréchet spaces with the obvious seminorms.

In Chapter 4, we introduced the calculus of SG pseudodifferential operators on $\mathbb{R}^d$. For non-compact manifolds, it is also possible to define the SG calculus, but we have to keep track of the admissible coordinate changes “near infinity” (cf. Schrohe [57]). It is more convenient to *radially compactify* $\mathbb{R}^d$ and define all objects on the compactified space with boundary. The theory will extend to manifolds that asymptotically look like Euclidean space.

Many different applications of the scattering pseudodifferential calculus (or more generally pseudodifferential calculi on non-compact manifolds) have been found in recent years: In general relativity it was shown by Hintz–Vasy [25] that the Kerr–de Sitter solution is nonlinearly stable and Vasy [62] proved meromorphic continuation of the resolvent of the Laplacian for asymptotically hyperbolic manifolds, and there has been a high interest in Hadamard states in quantum field theory (cf. Radzikowski [52] and Vasy–Wrochna [63], and see the examples below).

5.1.1. Manifolds with Corners

Manifolds with corners appear naturally when considering the kernels of operators on manifolds with boundary: The product of two manifolds with boundary is a manifold with corners. Therefore, many questions concerning manifolds with corners already arose for boundary value problems (cf. Grieser [16] and Melrose [40, 41]).

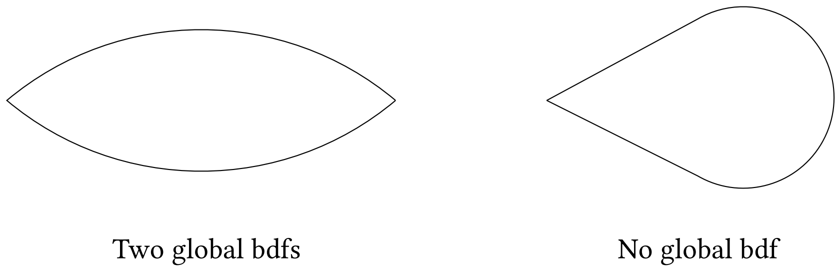

Figure 5.1.: An example and a counterexample of a mwc (cf. Grieser [16]).

There are various definitions of manifolds with corners. Fortunately, we only need products of (two) manifolds with boundaries. The definition of a manifold with boundary causes no difficulties: They are paracompact Hausdorff spaces that are locally homeomorphic to $\mathbb{R}_+ \times \mathbb{R}^{d-1}$ and the transition maps between local charts are smooth functions. If we now naively model d-dimensional manifolds with corners X on $(\mathbb{R}_+)^k \times \mathbb{R}^{d-k}$, we include manifolds, such as the raindrop, that do not have global boundary defining functions (bdf), see Figure 5.1. To avoid this, we will use a definition that asserts that all boundary hypersurfaces are embedded.

Definition 5.1.2 (Joyce [33]). Let X be a paracompact Hausdorff space and $d \geq 1$.

- A d-dimensional chart on X with boundary is a pair (U, ϕ) such that $U \subset \mathbb{R}_+ \times \mathbb{R}^{d-1}$ is open and $\phi : U \to \phi(U) \subset X$ is a homeomorphism.
- A d-dimensional chart on X with corners is a pair (U, ϕ) such that for some $0 \leq k \leq d$ and $U \subset \mathbb{R}_+^k \times \mathbb{R}^{d-k}$ is open and $\phi : U \to \phi(U) \subset X$ is a homeomorphism.

As usual two charts $(U, \phi), (V, \psi)$ are compatible if either $U \cap V = \emptyset$ or $U \cap V \neq \emptyset$ and $\phi^{-1} \circ \psi$ is a diffeomorphism.[1] Furthermore, we define a d-dimensional atlas (with boundary or with corners) to be a family of charts $\{(U_j, \phi_j)\}$ that are pairwise compatible and satisfy $X = \bigcup_j \phi_j(U_j)$.

Definition 5.1.3. A manifold with boundary X is a paracompact Hausdorff space together with a maximal atlas of d-dimensional charts with boundary.

To define the boundary hypersurfaces, we first define the *depth* of a point $p \in X$. For this let X be a topological space and (U, ϕ) a d-dimensional chart with corners such that $\phi(p) = 0 \in \mathbb{R}_+^k \times \mathbb{R}^{d-k}$. We define the depth as the number k, $\mathrm{depth}(p) = k$ and the *depth k stratum*

$$\partial_k X = \{p \in X \,|\, \mathrm{depth}(p) = k\}.$$

Note that this definition of the depth is independent of the choice of chart. A *boundary hypersurface* (bhs) is the closure of a connected component of $\partial_1 X$.

[1]For the half-space (and octants) functions are said to be smooth if they are restrictions of smooth functions.

Definition 5.1.4. Let X be a paracompact Hausdorff space together with a maximal atlas of d-dimensional charts with corners. We say that X is a manifold with corners (mwc) if all boundary hypersurfaces are embedded.

Remark 5.1.5. Joyce calls this notion a (compact) *manifold with embedded corners* (cf. Remark 2.11 in [33]). Melrose [42, 43] calls them just manifold with corners.

Some elementary properties of the depth strata are as follows:

Proposition 5.1.6 (Joyce [33]). *Let X be a manifold with corners. It holds that*

- $X = \bigsqcup_{k=0}^{d} \partial_k X$.
- $\partial_k X$ *is a manifold without boundary.*
- X *is a manifold without boundary if* $\partial_k X = \emptyset$ *for all* $k > 0$.
- X *is a manifold with boundary if* $\partial_k X = \emptyset$ *for all* $k > 1$.

We call $\partial X = \bigsqcup_{k=1}^{d} \partial_k X$ the boundary of X. It is the topological boundary if X is embedded into a manifold without boundary $\tilde{X}$ of the same dimension d. Given a relatively open subset U of a manifold with corner X, we say that U is *interior* if $\overline{U} \cap \partial X = \emptyset$. If, on the other hand, U contains all interior points of the boundary $\overline{U} \cap \partial X$, we call U a *boundary neighborhood.*

In the following, we will always assume that X is compact.

5.1.2. Boundary Defining Functions

As in the case of manifolds without boundary, functions are called smooth if they are smooth in local charts. Let X be a manifold with corners and $K \subset X$ a boundary hypersurface. A smooth function $\rho : X \to \mathbb{R}$ is called a *boundary defining function* (bdf) for K if

- $\rho(x) \geq 0$ if $x \in X$
- $\rho(x) = 0$ if and only if $x \in K$,
- $d\rho(x) \neq 0$ on K.

The assumption that all bhs are embedded implies that there exists a bdf for every bhs. If the boundary hypersurfaces are not embedded, bdfs only exist locally. Thus, we can always assume that there is a finite collection of bdfs $\{\rho_j\}$ such that their differentials are linearly independent and for every bhs there exists exactly one corresponding bdf ρ_j.

For any point $p \in X$ the depth $\operatorname{depth}(p)$ is nothing but the number of independent boundary defining functions vanishing at p.

Let X be a manifold with boundary. By the collar neighborhood theorem (cf. Milnor [46, Corollary 3.5]), there exists a neighborhood U of the boundary ∂X such that U is diffeomorphic to $[0, 1) \times \partial X$. Thus, we can choose coordinates (ρ_X, x) such that ρ_X is the bdf and x are coordinates on the boundary ∂X. Of course we can localize this construction to a small part of the boundary. On manifolds with corners, we may take a small neighborhood U near the boundary such that U is a product of manifolds with boundary. We can apply the theorem for every manifold with boundary and obtain local coordinates $(\rho_1, \dots, \rho_k, x)$, where $k = \max_{x \in U} \operatorname{depth}(x)$.

5.1.3. Compactification

We will now show how to compactify $\mathbb{R}^d$ to the upper hemisphere $\mathbb{S}^d_+$. In Chapter 6 there will be a different compactification and it will be shown that they are equivalent.

Consider the stereographical projection

$$\begin{aligned} \mathrm{SP} : \mathbb{R}^d &\to \mathbb{S}^d_+, \\ x &\mapsto \left(\frac{1}{\langle x \rangle}, \frac{x}{\langle x \rangle} \right) = (\rho, y). \end{aligned}$$

We see that ρ is a boundary defining function on $\mathbb{S}^d_+$ and we have the property that $\rho \cdot x = y$ and we can always choose $d-1$ of these functions y' such that (ρ, y') are coordinates on $\mathbb{S}^d_+$, but mostly we do not need to choose coordinates.

Let $g_{\mathbb{R}^d} = dx^2$ be the standard metric on $\mathbb{R}^d$. In polar coordinates we have that

$$g_{\mathbb{R}^d} = dr^2 + r^2 d\omega_{\mathbb{S}^{d-1}},$$

where $\omega_{\mathbb{S}^{d-1}}$ is the induced metric on the sphere $\mathbb{S}^{d-1}$. Setting $\rho = 1/r$ yields the metric

$$\frac{d\rho^2}{\rho^4} + \frac{\omega_{\mathbb{S}^{d-1}}}{\rho^2}.$$

This is the motivating example of scattering metrics.

5.1.4. Asymptotically Euclidean Manifolds

Now, we want to discuss the geometric structure near the boundary. Let X be a manifold with boundary with fixed bdf ρ_X. Inspired by the example of the compactification of $\mathbb{R}^d$ we call a Riemannian metric g_X on the interior a *scattering metric* if in a neighborhood of the boundary, the metric takes the form

$$g_X(\rho_X, x) = \frac{d\rho_X^2}{\rho_X^4} + \frac{h(\rho_X, x)}{\rho_X^2},$$

where h is a smooth symmetric 2-tensor that is positive-definite if restricted to the boundary ∂X. Locally near the boundary, we may write $h(\rho_X, x) = \sum_{j,k=0}^d h_{ij}(\rho_X, x) dx_i \otimes dx_j$ with the convention that $x_0 = \rho_X$, $x = (x_1, \dots, x_d)$.

The triple (X, ρ_X, g_X) is called a scattering manifold or an asymptotically Euclidean manifold. We will also write (X, g_X), where the choice of bdf ρ_X is implicit. Note that the class of scattering metrics depends on the choice of boundary defining function ρ_X, since a change $\rho_X \mapsto c\rho_X$ adds a factor c^{-2} in the first summand.

5.2. Scattering Calculus

Now we will introduce the calculus of *scattering* pseudodifferential operators. We will mainly follow [42, 43]. Schrohe [57] defined the calculus directly on non-compact manifolds, without using the compactification. A semiclassical version is due to Wunsch–Zworski [67].

5.2.1. Differential Operators

Before defining the algebra of pseudodifferential operators, we define the natural differential operators in a geometric way.

Let $\mathcal{V}_b(X)$ be the set of vector fields that are tangent to the boundary. The scattering vector fields are defined by $\mathcal{V}_{\mathrm{sc}}(X) = \rho_X \mathcal{V}_b(X)$. There is a natural vector bundle over X, the *scattering tangent bundle* ${}^{\mathrm{sc}}TX$, such that the scattering vector fields are the sections of this bundle, $\mathcal{V}_{\mathrm{sc}}(X) = \mathcal{C}^\infty(X, {}^{\mathrm{sc}}TX)$, given by ${}^{\mathrm{sc}}T_pX = \mathcal{V}_{\mathrm{sc}}(X)/I_p(X) \cdot \mathcal{V}_{\mathrm{sc}}(X)$, where $I_p(X)$ is the set of vector fields fV, such that $f \in \mathcal{C}^\infty(X)$, $f(p) = 0$ and $V \in \mathcal{V}_{\mathrm{sc}}(X)$. In local coordinates (ρ_X, x) near the boundary, this vector bundle is spanned by $\{\rho_X^2 \partial_{\rho_X}, \rho_X \partial_{x_1}, \ldots, \rho_X \partial_{x_{d-1}}\}$. The dual bundle to the scattering tangent bundle is denoted by ${}^{\mathrm{sc}}T^*X$ and is called the *scattering cotangent bundle*. Locally, a covector $v \in {}^{\mathrm{sc}}T^*X$ is given by

$$v = a\frac{d\rho_X}{\rho_X^2} + \sum_j b_j \frac{dx_j}{\rho_X},$$

where $a, b_j \in \mathbb{R}$. Here, we view $\frac{d\rho_X}{\rho_X^2}$ and $\frac{dx_j}{\rho_X}$ as the dual covectors to $\rho_X^2 \partial_{\rho_X}$ and $\rho_X \partial_{x_j}$. Hence, they are well-defined up to the boundary $\rho_X = 0$.

As in the case of usual differential operators, we can define the scattering differential operators $\mathrm{Diff}_{\mathrm{sc}}(X)$ as the smallest algebra generated by $\mathcal{V}_{\mathrm{sc}}(X)$ and by multiplication with smooth functions $f \in \mathcal{C}^\infty(X)$. Thus, we have locally

$$A \in \mathrm{Diff}_{\mathrm{sc}}(X) \quad \text{if and only if} \quad A = \sum_{\alpha,k} a_{\alpha,k}(\rho_X, x)(\rho_X^2 \partial_{\rho_X})^k (\rho_X \partial_x)^\alpha,$$

where the sum is locally finite.

Let $P^m({}^{\mathrm{sc}}T^*X)$ the set of smooth functions on ${}^{\mathrm{sc}}T^*X$ that are polynominals of degree m in the fiber and $P^{[m]}({}^{\mathrm{sc}}T^*X)$ the subset of homogeneous polynominal is the fiber.

The set of scattering differential operators is a subset of the differential operators on X and therefore for every $m \in \mathbb{N}$ the symbol map σ restricts to a map

$$\sigma^\psi \colon \operatorname{Diff}^m_{\mathrm{sc}}(X) \to P^{[m]}({}^{\mathrm{sc}}T^*X),$$
$$\sigma^\psi(A) = \sum_{k+|\alpha|=m} a_{\alpha,k}(\rho_X, x)(i\xi_0)^k(i\xi')^\alpha,$$

where $\operatorname{Diff}^m_{\mathrm{sc}} = \operatorname{Diff}_{\mathrm{sc}} \cap \operatorname{Diff}^m$.

There is also another symbol, measuring the decay at infinity:

$$\sigma^e \colon \operatorname{Diff}^m_{\mathrm{sc}}(X) \to P^m({}^{\mathrm{sc}}T^*_{\partial X}X),$$
$$\sigma^e(A) = \sum_{k+|\alpha|\le m} a_{\alpha,k}(0, x)(i\xi_0)^k(i\xi')^\alpha.$$

This definition is independent of the specific quantization, because the commutator of $\rho_X^2\partial_{\rho_X}$ and $\rho_X\partial_x$ is $\rho_X(\rho_X\partial_x)$ and therefore commutators vanish as operators $a\rho_X^2\partial_{\rho_X} + b\rho_X\partial_x$ at $\rho_X = 0$.

The joint symbol $\sigma = (\sigma^\psi, \sigma^e)$ induces an exact sequence

$$0 \to \rho_X \operatorname{Diff}^{m_\psi-1}_{\mathrm{sc}}(X) \to \operatorname{Diff}^{m_\psi}_{\mathrm{sc}}(X) \xrightarrow{\sigma} P^{m_\psi}_{\mathrm{sc}}(X) \to 0,$$

where

$$P^{m_\psi}_{\mathrm{sc}}(X) = \{(f_1, f_2) \in P^{[m_\psi]}({}^{\mathrm{sc}}T^*X) \times P^{m_\psi}({}^{\mathrm{sc}}T^*_{\partial X}X) \colon f_2 - f_1|_{\partial X} \in P^{m_\psi-1}({}^{\mathrm{sc}}T^*_{\partial X}X)\}.$$

We define the full class of scattering differential operators $\operatorname{Diff}^{m_\psi,m_e}_{\mathrm{sc}} = \rho_X^{-m_e}\operatorname{Diff}^{m_\psi}_{\mathrm{sc}}$ and similarly $P^{m_\psi,m_e}_{\mathrm{sc}}$ and obtain the exact sequence

$$0 \to \operatorname{Diff}^{m_\psi-1,m_e-1}_{\mathrm{sc}}(X) \to \operatorname{Diff}^{m_\psi,m_e}_{\mathrm{sc}}(X) \xrightarrow{\sigma} P^{m_\psi,m_e}_{\mathrm{sc}}(X) \to 0. \tag{5.1}$$

The natural space to consider symbols of differential operators is the cotangential bundle. In our case this is a non-compact manifold with boundary, T^*X. Since we do not want to treat non-compactness and boundaries (we introduced the boundary to get rid of the non-compactness in the first place), we will also apply the compactification SP to the fibres of T^*X. This yields the manifold with corners ${}^{\mathrm{sc}}\overline{T}^*X$ with boundary defining functions ρ_X and ρ_Ξ.

The Laplace-Beltrami operator can be defined in the interior of X and extends to a differential operator on $\dot{\mathcal{C}}^\infty(X)$. In fact the Laplacian is the natural example of a scattering differential operator.

For simplicity, we assume that the metric $h(\rho_X, x)$ is of the form

$$h(\rho_X, x) = \sum_{i,j=1}^d h_{ij}(\rho_X, x)dx_i \otimes dx_j.$$

If we set $\eta_0 = \frac{d\rho_X}{\rho_X^2}$ and $\eta = \frac{dy}{\rho_X^2}$, then the metric g is given by

$$g(\rho_X, x, \eta_0, \eta) = \eta_0^2 + \sum_{i,j=1}^d h_{ij}(\rho_X, x)\eta_i\eta_j$$

and the fiberwise compactification in $\eta = (\eta_0, \eta)$, $\mathrm{SP} : \eta \mapsto (\rho_\Xi, \eta \cdot \rho_\Xi)$ yields

$$g(\rho_X, x, \rho_\Xi, \xi) = \frac{1}{\rho_\Xi^2}\left(\xi_0^2 + \sum_{i,j=1}^d h_{ij}(\rho_X, x)\xi_i\xi_j\right).$$

Proposition 5.2.1 (Melrose [40] and Melrose–Zworski [45])**.** *The Laplacian Δ on a scattering manifold (X, g) is a scattering differential operator of order $2, 0$ with principal symbol*

$$g \in \rho_\Xi^{-2}\mathcal{C}^\infty({}^{\mathrm{sc}}\overline{T}^* X)$$

modulo lower order terms $\rho_\Xi^{-1}\mathcal{C}^\infty$ and $\rho_X\mathcal{C}^\infty$.

5.2.2. Pseudodifferential Operators

The space of amplitudes $S_{\mathrm{sc}}^{m_\psi, m_e}$ is given by functions

$$a \in \rho_X^{-m_e}\rho_\Xi^{-m_\psi}\mathcal{C}^\infty({}^{\mathrm{sc}}\overline{T}^* X).$$

More generally, we can define amplitudes on any manifold with corners. This definition does not depend on the choice of boundary defining functions and can be localized in the obvious way.

We will only define scattering pseudodifferential operators on $\mathbb{R}^d$ and use coordinate invariance to define them on arbitrary manifolds. Alternatively, one could define the operators directly by their kernels on $X \times X$.

We denote by

$$\mathrm{SP}_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{S}_+^d \times \mathbb{S}_+^d \tag{5.2}$$

the radial compactification in both components, which is given by $(x, \xi) \mapsto (\mathrm{SP}(x), \mathrm{SP}(\xi))$. The boundary defining function ρ for $\mathbb{S}_+^d$ can be chosen to be given by $\mathrm{SP}^*\,\rho = \langle x \rangle^{-1}$. Let ρ_X such a boundary defining function for the first factor in (5.2) and ρ_Ξ for the second factor.

The classical pseudodifferential operators are defined as follows: The class $\Psi_{\mathrm{sc}}^{m_\psi, m_e}(\mathbb{S}_+^d)$ is given by those operators $A : \mathcal{C}_c^\infty(\mathbb{S}_+^d) \to \dot{\mathcal{C}}^\infty(\mathbb{S}_+^d)$ such that if A' is defined by

$$\mathrm{SP}^*(A\phi) = A'(\mathrm{SP}^*\,\phi)$$

for all $\phi \in \mathcal{C}_c^\infty(\mathbb{S}_+^d)$, then the kernel of A' is given by

$$A'(x, y) = \int e^{i(x-y)\xi}(\mathrm{SP}_2^*\,a)(x, \xi)\,đ\xi,$$

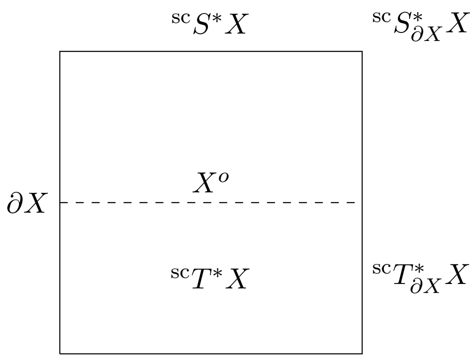

Figure 5.2.: The space ${}^{\mathrm{sc}}\overline{T}^*X$.

where $a \in \rho_X^{-m_e}\rho_\Xi^{-m_\psi}\mathcal{C}^\infty(\mathbb{S}^d_+ \times \mathbb{S}^d_+)$ is an amplitude. Using coordinate-invariance (cf. [42]) we can define $\Psi_{\mathrm{sc}}^{m_\psi,m_e}(X)$ for an arbitrary compact manifold with boundary X.

This gives a calculus of pseudodifferential operators with the following properties (cf. [42]):

- It is a bi-filtered algebra:

$$\Psi_{\mathrm{sc}}^{m_{\psi,1},m_{e,1}}(X) \circ \Psi_{\mathrm{sc}}^{m_{\psi,2},m_{e,2}}(X) \subset \Psi_{\mathrm{sc}}^{m_{\psi,1}+m_{\psi_2},m_{e,1}+m_{e,2}}(X).$$

- It is a superset of the scattering differential operators:

$$\mathrm{Diff}_{\mathrm{sc}}^{m_\psi,m_e}(X) \subset \Psi_{\mathrm{sc}}^{m_\psi,m_e}(X).$$

Remark 5.2.2. We note that the m_e order has a different sign-convention than in [42]. With our convention, the class of residual operators is $\Psi_{\mathrm{sc}}^{-\infty,-\infty}(\mathbb{S}^d_+)$.

5.2.3. Principal Symbol and Wavefront Sets

Set $\mathcal{W} = \partial^{\mathrm{sc}}\overline{T}^*X$.[2] This is a topological manifold, but there is no natural smooth structure. It is the union of two manifolds with boundary $\mathcal{W} = \mathcal{W}^\psi \cup \mathcal{W}^e$, where $\mathcal{W}^\psi = {}^{\mathrm{sc}}S^*X$ and $\mathcal{W}^e = {}^{\mathrm{sc}}\overline{T}^*_{\partial X}X$. We define the smooth functions on $\mathcal{W}$ as follows:

$$\mathcal{C}^\infty(\mathcal{W}) = \{(u_e, u_\psi) \in \mathcal{C}^\infty(\mathcal{W}^e) \times \mathcal{C}^\infty(\mathcal{W}^\psi) \colon u_e|_{\mathcal{W}^{\psi,e}} = u_\psi|_{\mathcal{W}^{\psi,e}}\}.$$

Proposition 5.2.3 (Melrose [42])**.** *There exists a principal symbol map $\sigma = (\sigma^\psi, \sigma^e)$ such that the following sequence is exact:*

$$0 \to \Psi_{\mathrm{sc}}^{m_\psi-1,m_e-1}(X) \to \Psi_{\mathrm{sc}}^{m_\psi,m_e}(X) \xrightarrow{\sigma} \mathcal{C}^\infty(\mathcal{W}) \to 0.$$

[2]Melrose [42] calls this space $C_{\mathrm{sc}}X$.

Consider a scattering pseudodifferential operator A with symbol $a \in \rho_X^{-m_e}\rho_\Xi^{-m_\psi}\mathcal{C}^\infty$. Write $a = \rho_X^{-m_e}\rho_\Xi^{-m_\psi} f$. Near a point $z \in \mathcal{W}$, we define the symbols

$$\begin{aligned}
\sigma^e(a)(\rho_X, x, \rho_\Xi, \xi) &= \rho_X^{-m_e}\rho_\Xi^{-m_\psi} f(0, x, \rho_\Xi, \xi), \quad & z \in {}^{\mathrm{sc}}T^*_{\partial X}X, \\
\sigma^\psi(a)(\rho_X, x, \rho_\Xi, \xi) &= \rho_X^{-m_e}\rho_\Xi^{-m_\psi} f(\rho_X, x, 0, \xi), \quad & z \in {}^{\mathrm{sc}}S^*X, \\
\sigma^{\psi,e}(a)(\rho_X, x, \rho_\Xi, \xi) &= \rho_X^{-m_e}\rho_\Xi^{-m_\psi} f(0, x, 0, \xi), \quad & z \in {}^{\mathrm{sc}}S^*_{\partial X}X.
\end{aligned}$$

The tuple $\sigma(a) = (\sigma^e(a), \sigma^\psi(a))$ can be viewed as a function $\sigma(a) \in \mathcal{C}^\infty(\mathcal{W})$ and $\sigma^{\psi,e}(a)$ is the restriction to the corner.

The symbol a is called *elliptic* at $z \in \mathcal{W}$ if $\sigma(a)(z) \neq 0$ and it is called *characteristic* at z if $\sigma(a)(z) = 0$. The set of all characteristic points of a scattering pseudodifferential operator is denoted by $\Sigma_{m_\psi, m_e}(A)$.

The scattering wavefront set is defined for an arbitrary $u \in \mathcal{C}^{-\infty}(X)$ as

$$\mathrm{WF}_{\mathrm{sc}}(u) = \bigcap_{\substack{A \in \Psi^{0,0}_{\mathrm{sc}}(X) \\ Au \in \dot{\mathcal{C}}^\infty(X)}} \Sigma_{0,0}(A) \subset \mathcal{W}.$$

We may split the wavefront set into three components $\mathrm{WF}_{\mathrm{sc}}(u) = \mathrm{WF}^\psi_{\mathrm{sc}}(u) \cup \mathrm{WF}^e_{\mathrm{sc}}(u) \cup \mathrm{WF}^{\psi,e}_{\mathrm{sc}}(u)$, where $\mathrm{WF}^\bullet_{\mathrm{sc}}(u) = \mathrm{WF}_{\mathrm{sc}}(u) \cap \mathcal{W}^\bullet$.

- The usual wavefront set is contained in the scattering wavefront set:

  $$(x, \xi) \in \mathrm{WF}_{\mathrm{cl}}(u) \quad \text{if and only if} \quad (x, (0, \xi/\langle\xi\rangle)) \in \mathrm{WF}^\psi_{\mathrm{sc}}(u),$$

  for any $x \in X^o$. Note that here we use the compactification to describe the point "at infinity" in the direction of $\xi \neq 0$.

- If $X = \mathbb{S}^d_+$, then the Fourier transform on $\mathbb{R}^d$ can be lifted to the compactification X (cf. Melrose [42]) and changes the variables in the wavefront set:

  $$(x, \xi) \in \mathrm{WF}_{\mathrm{sc}}(u) \Leftrightarrow (\xi, -x) \in \mathrm{WF}_{\mathrm{sc}}(\mathcal{F}u)$$

- Scattering pseudodifferential operators are microlocal (cf. Melrose [42]):

  $$\mathrm{WF}_{\mathrm{sc}}(Au) \subset \mathrm{WF}_{\mathrm{sc}}(u).$$

5.3. Lagrangian Distributions

In Chapter 6 we introduce a class of geometric distributions which are natural in the setting of scattering geometry. There are two main ingredients to Lagrangian distributions: First, we have a definition of amplitudes, which generalize the symbol of a pseudodifferential operator.

Secondly, the underlying symplectic geometry, which describes the structure of the singularities. As in the case of scattering pseudodifferential operators, the relevant manifold is $\mathcal{W}$. Therefore, we define scattering Lagrangian submanifolds as subsets of $\mathcal{W}$: The manifolds $\mathcal{W}^\psi$ and $\mathcal{W}^e$ are contact manifolds with contact forms α^ψ and α^e induced by the symplectic form ω on ${}^{\mathrm{sc}}\overline{T}^*X$.

Definition 5.3.1 (Definition 2.16 in [4]). A closed subset $\Lambda \subset \mathcal{W}$ is called a *sc-Lagrangian* if

- $\Lambda^\psi = \Lambda \cap \mathcal{W}^\psi$ is a smooth Legendrian submanifold with respect to the contact form α^ψ on $\mathcal{W}^\psi$.
- $\Lambda^e = \Lambda \cap \mathcal{W}^e$ is a smooth Legendrian submanifold with respect to the contact form α^e on $\mathcal{W}^e$.
- $\overline{\Lambda^\psi}$ has a boundary if and only if $\overline{\Lambda^e}$ has a boundary and in that case the intersection is clean.

Without assuming Legendrian properties, this gives a natural definition of smooth submanifolds on $\mathcal{W}$. Such submanifolds are given by the restriction of a submanifold $L \subset {}^{\mathrm{sc}}\overline{T}X$ to $\mathcal{W}$. Smooth functions on L are restrictions of smooth functions on $\mathcal{W}$.

We define clean phase functions associated to such Lagrangian submanifold (cf. Definition 2.5 and Definition 2.18). It was shown already by Coriasco–Schulz [7] that there always exists a phase function ϕ locally parametrizing the Lagrangian submanifold. Locally, Lagrangian distributions are given by an oscillatory integral of the form

$$I_\phi(x) = \int_Y e^{i\phi(x,y)} a(x,y)$$

where Y is a manifold with boundary and $a \in \rho_X^{-m_e} \rho_Y^{-m_\psi} \mathcal{C}^\infty(X \times Y, {}^{\mathrm{sc}}\Omega^{1/2}(X) \times {}^{\mathrm{sc}}\Omega^1(Y))$ is an amplitude and ${}^{\mathrm{sc}}\Omega^\bullet$ is a suitably rescaled density bundle $\Omega^\bullet$ (cf. Melrose [42]).

The class of Lagrangian distribution $I^{m_e,m_\psi}(X,\Lambda)$ is now defined as a locally finite sum of such oscillatory integrals with an invariantly defined order, which coincides with the definition of Hörmander [27] for m_ψ and Melrose–Zworski [45] for m_e.

Theorem 5.3.2. *Let $\Lambda \subset \mathcal{W}$ be a* sc*-Lagrangian submanifold. There exists a surjective principal symbol map*

$$j^\Lambda_{m_e,m_\psi} : I^{m_e,m_\psi}(X,\Lambda) \to \mathcal{C}^\infty(\Lambda, M_\Lambda),$$

where M_Λ is the Maslov bundle. Its kernel is $I^{m_e-1,m_\psi-1}(X,\Lambda)$ and therefore we have the identification

$$I^{m_e,m_\psi}(X,\Lambda)/I^{m_e-1,m_\psi-1}(X,\Lambda) \cong \mathcal{C}^\infty(\Lambda, M_\Lambda).$$

5.4. Examples

We will now present two examples of distributions that are contained in our class of Lagrangian distributions. It is obvious by choosing the base manifold X to be compact without boundary that the classical Fourier integral operators defined by Hörmander [27] (cf. Duistermaat–Hörmander [14]) are contained in the class defined in Chapter 6 by choosing the base manifold X to be compact without boundary.

The first example sketches how distributions appear that have wavefront set only in the e-component, which is in a sense dual to the case of Hörmander. The second example illustrates where the classical theory of Duistermaat–Hörmander does not give the best results and one has to take the non-compactness into account.

5.4.1. The Scattering Matrix

Let (X, ρ, g) be a scattering manifold and denote by Δ the Laplace-Beltrami operator on X. Choose an arbitrary smooth function $f \in \mathcal{C}^\infty(\partial X)$. It was shown by Melrose [42] that there is a unique function $u \in \mathcal{C}^\infty(X)$ for every $\lambda \neq 0$ such that

$$(\Delta - \lambda^2)u = 0$$

and u has an asymptotic expansion $u = e^{i\lambda\rho^{-1}}\rho^{(d-1)/2}f_1 + e^{-i\lambda\rho^{-1}}\rho^{(d-1)/2}f_2$, where $f_1, f_2 \in \mathcal{C}^\infty(X)$ and $f_1|_{\partial X} = f$. One may define the *scattering matrix*[3]

$$\begin{aligned} S(\lambda) : \mathcal{C}^\infty(\partial X) &\to \mathcal{C}^\infty(\partial X), \\ f &\mapsto f_2|_{\partial X}. \end{aligned}$$

In [45] it was proved that the scattering matrix $S(\lambda)$ is a Fourier integral operator on the boundary ∂X. The underlying canonical transformation is given by the symplectomorphism

$$\exp(\pi H_{\sqrt{h}}) : T^*\partial X \setminus 0 \to T^*\partial X \setminus 0,$$

where $H_{\sqrt{h}}$ is the Hamiltonian vector field associated to the square root of the metric h on the boundary ∂X.

For the proof the authors introduce a class of geometric distributions on X, which are smooth in the interior of X and oscillatory near the boundary, hence not regular in the sense of Schwartz functions:

The symplectic structure of ${}^{\mathrm{sc}}T^*X$ induces a natural contact structure on ${}^{\mathrm{sc}}T^*_{\partial X}X$, where the contact form ${}^{\mathrm{sc}}\alpha$ is given in canonical coordinates (ρ, x, τ, ξ) by

$${}^{\mathrm{sc}}\alpha = d\tau + \xi \cdot dx.$$

[3] The scattering matrix is usually not a matrix. If $X = \mathbb{R}$ or in the case of hyperbolic manifolds with all ends being cusps it is a matrix.

A submanifold G of ${}^{\mathrm{sc}}T^*_{\partial X}X$ is called *sc-Legendrian* if ${}^{\mathrm{sc}}\alpha$ vanishes on G and $\dim G = \dim X - 1$. As in the case of Lagrangian submanifolds, a non-degenerate phase function $\phi \in \mathcal{C}^\infty(\partial X \times \mathbb{R}^n)$ parametrizes G if

$$(x, \theta) \mapsto (x, -\phi(x, \theta), d_x\phi(x, \theta)) = (x, \tau, \xi)$$

is a local diffeomorphism from $\{d_\theta \phi = 0\}$ to G.

The class of Legendrian distribution is defined as sums of oscillatory integrals of the form

$$u(\rho, x) = \int e^{i\rho^{-1}\phi(x,\theta)} a(\rho, x, \theta)\, d\theta,$$

where $a \in \rho^{m-s/2+d/4}\mathcal{C}^\infty_c(X \times U)$ is supported in a small neighborhood of the boundary and U is an open subset of $\mathbb{R}^s$.

The convention for the order m_e in Section 6.5 is defined such that it coincides with the order of Legendrian distributions.

5.4.2. Quantum Field Theory

Let (M, g) be an orientable complete $d+1$-dimensional pseudo-Riemannian manifold with signature $(-, +, \dots, +)$. We then call (M, g) a space-time. As in the case of Riemannian manifolds, we can define a geometric second order differential operator by

$$\Box u = -|g|^{-1/2}\partial_\mu \left(|g|^{1/2} g^{\mu\nu}\partial_\nu\right) u,$$

which is not elliptic, but hyperbolic and it is the natural generalization of the wave operator $-D_{tt} + \Delta$ on Riemannian manifolds.

At each point $x \in M$, we can define *time-like, space-like* and *null* tangent vectors $v \in T_xM$, by $g_x(v, v) < 0$, $g_x(v, v) > 0$, and $g_x(v, v) = 0$, respectively. A curve is called *causal*, if every tangent vector is either time-like or null.

Let S be a smooth space-like hypersurface meaning that all tangent vectors are space-like. It is called a *Cauchy hypersurface* if every inextendible causal curve intersects S exactly once. The manifold (M, g) is called *globally hyperbolic* if it admits a Cauchy hypersurface S. If this is the case, M is diffeomorphic to $\mathbb{R} \times S$ (cf. Hawking–Ellis [21] for an overview of the causal structure) and the initial value problem for the wave equation

$$\begin{cases} \Box u = 0 \\ u|_S = u(x) \\ \partial_\nu u|_S = v(x) \end{cases}$$

is well-posed for any Cauchy-hypersurface S. Here, ∂_ν denotes the normal derivative with respect to S.

From now on assume that (M, g) is a $d+1$-dimensional globally hyperbolic space-time. Denote by L^*M the co-light cone

$$L^*M = \{(p, \zeta) \in T^*M \setminus 0 \colon g_p^{-1}(\zeta, \zeta) = 0\},$$

where g^{-1} is the dual metric to g on T^*M. We want to consider the inhomogeneous Klein-Gordon equation:

$$(\Box + m^2)u = f,$$

for fixed $m > 0$. The characteristic set of $\Box + m$ is the co-light cone $\Sigma(\Box) = L^*M$. It is clear that this equation has many different solutions, because we have not specified any initial data. Therefore, there are many different fundamental bi-solutions $G_{\mathrm{KG}} \in \mathcal{D}'(M \times M)$ satisfying

$$(\Box_p + m^2)G_{\mathrm{KG}}(p, q) = \delta(p - q),$$
$$(\Box_q + m^2)G_{\mathrm{KG}}(p, q) = \delta(p - q).$$

By Proposition A.4.3, every such fundamental solution has restricted wavefront set

$$\mathrm{WF}_{\mathrm{cl}}(G_{\mathrm{KG}}) \subset N^*\Delta_{M\times M} \cup L^*M \times L^*M,$$

where $\Delta_{M\times M}$ is the diagonal. If $d \geq 2$, then L^*M has two connected components, the forward and the backward light cone. It was shown by Duistermaat–Hörmander [14] that there are four different distinguished fundamental solutions to the Klein-Gordon equation, characterized by its wavefront set, and they constructed parametrices modulo $\mathcal{C}^\infty$. Intuitively speaking, we can choose for every connected component, whether singularities are propagated forward or backward in time (cf. Figure 5.3).

The downside of the distinguished parametrices of Duistermaat–Hörmander is that they differ from the real fundamental solution by a smooth function and that error is in general not compact.

For a *quantum field model*, we consider a distribution $\omega_2 \in \mathcal{D}'(M) \otimes \mathcal{D}'(M)$. We assume that ω_2 has the following properties (cf. [52]):

Positive type If $f \in \mathcal{C}_c^\infty(M)$, then

$$\omega_2(\bar{f} \otimes f) \geq 0.$$

Klein-Gordon equation For all $f, g \in \mathcal{C}_c^\infty(M)$,

$$\omega_2((\Box + m^2)f \otimes g) = \omega_2(f \otimes (\Box + m^2)g) = 0.$$

Commutator For any $f, g \in \mathcal{C}_c^\infty(M)$

$$\omega_2(f \otimes g) - \omega_2(g \otimes f) = i\Delta(f \otimes g),$$

where $\Delta = G_A - G_R$ and G_A, G_R are the advanced and retarded fundamental solutions of the Klein-Gordon equation.

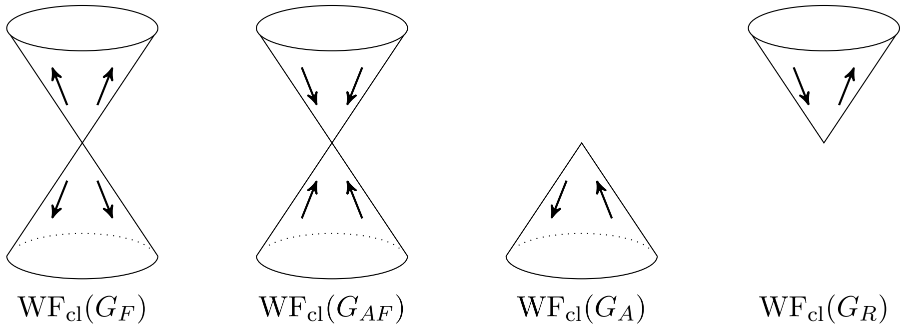

Figure 5.3.: Wave fronts of the distinguished fundamental solutions. From left to right: The Feynman propagator, the anti-Feynman propagator, the advanced fundamental solution, and the retarded fundamental solution.

On Minkowski space $\mathbb{R}^{1+d}_{t,x}$ with metric $g_M = -dt^2 + dx^2$, the two-point function is defined by

$$\omega_2(f \otimes g) = \frac{1}{i}\langle \Delta_+(t-s, x-y), f(t,x)g(s,y)\rangle$$

and Δ_+ is given by the oscillatory integral

$$\Delta_+(t,x) = \frac{i}{2}\int_{\mathbb{R}^d} e^{i\psi(t,x,\xi)}(m^2 + |\xi|^2)^{-1/2}\, đ\xi,$$

where the phase function is $\psi(t,x,\xi) = -t\sqrt{m^2+|\xi|^2} + \langle x, \xi\rangle$ (cf. Reed–Simon [53, Theorem IX.34] and Dang [8]). The phase function ψ is a symbol in $\mathrm{SG}^{1,1}(\mathbb{R}^{d+1} \times \mathbb{R}^d)$ and therefore the oscillatory integral can be interpreted as an oscillatory integral in the scattering calculus (cf. [6]).

Lagrangian Distributions on Asymptotically Euclidean Manifolds

6.1. Introduction

This chapter is taken from [4], where the section about manifolds with corners has been removed and a more general discussion can be found in Section 5.1.1.

Lagrangian distributions were defined by Hörmander [27] as a tool to obtain a global calculus of Fourier integral operators. The latter are widely applied, e.g. in the study of partial differential equations [14], spectral theory [13], index theory [2] and mathematical physics [18]. Motivating examples for the necessity of studying Lagrangian distributions on asymptotically Euclidean spaces include fundamental solutions to the Klein-Gordon equation, which exhibit Lagrangian behavior “at infinity”, see [7], as well as simple or multi-layers which arise when solving partial differential equations along infinite boundaries or Cauchy hypersurfaces, see [3].

In local coordinates, a classical Lagrangian distribution u on a manifold X is given by an oscillatory integral of the form

$$I_{\varphi}(a) = \int_{\mathbb{R}^s} e^{i\varphi} a(x,\theta)\, d\theta, \tag{6.1}$$

for some symbol $a \in S^m(\mathbb{R}^d \times \mathbb{R}^s)$ and a phase function φ on a subset of $\mathbb{R}^d \times \mathbb{R}^s$ bounded in x. A class of oscillatory integrals on Euclidean spaces, the local model for our theory, was studied in [6].

The key feature of the theory of Lagrangian distributions is that each such distribution is globally associated to a Lagrangian submanifold $\Lambda \subset T^*X$ and that its leading order behavior can be invariantly described by its principal symbol which is a section in a line bundle on Λ.

In this chapter, we prove that the situation on asymptotically Euclidean manifolds is similar, but with a more delicate structure “at infinity”. To make this precise, we work within the framework of scattering geometry, developed in [42, 45], see also [20, 67]. In the chapter, we continue the introduction of Chapter 5 to scattering manifolds and add to it a class of naturally arising morphisms, the *scattering maps*. We note that the scattering manifolds may also be seen as Lie manifolds, and in this way our theory complements recent advances in the theory of Lagrangian distributions and Fourier integral operators on such singular spaces (via groupoid techniques), see [35].

The prototype of a scattering geometry is the Euclidean space $\mathbb{R}^d$, identified with a ball under radial compactification. For this setting, a fitting theory of Lagrangian submanifolds on $\mathbb{R}^d$ was developed in [7]. As a first step, we adapt this to general scattering manifolds with boundary $X = X^o \cup \partial X$, the boundary being viewed as infinity. On such manifolds, the environment for microlocalization is then the compactified scattering cotangent bundle ${}^{\mathrm{sc}}\overline{T}^*X$, a manifold with corners of codimension 2 and its boundary $\mathcal{W} = \partial\,{}^{\mathrm{sc}}\overline{T}^*X$. This boundary may be seen as a stratified space, and the two boundary faces of ${}^{\mathrm{sc}}\overline{T}^*X$, which intersect in the corner, inherit a type of contact structure. The geometric objects of study in our theory are then Legendrian submanifolds of the faces $\mathcal{W}$ which intersect in the corner and are the boundary of some Lagrangian submanifold in the interior and smooth (distribution) densities thereon.

The link with Lagrangian distributions is now as follows. We prove that, despite the singular geometry, any Lagrangian submanifold $\Lambda \subset \mathcal{W}$ locally admits a parametrization through some phase function φ, via a generalization of the map

$$\lambda_\varphi : \mathcal{C}_\varphi \to \Lambda_\varphi \quad (x,\theta) \mapsto \big(x, d_x\varphi(x,\theta)\big),$$

where $\mathcal{C}_\varphi = (d_\theta\varphi)^{-1}\{0\}$. For each such a phase function, a Lagrangian distribution can be expressed locally as an oscillatory integral as in (6.1). Up to Maslov factors and some density identifications, the restriction of $a(x,\theta)$ to C_φ yields the (principal) symbol $\sigma(u)$ of u and is interpreted as a (density valued) function on Λ by identification via λ_φ.

Indeed, the main theorem characterizing the principal symbol will be:

Theorem. *Let Λ be a* sc*-Lagrangian on X. Then there exists a surjective principal symbol map*

$$j^{\Lambda}_{m_e,m_\psi} : I^{m_e,m_\psi}(X,\Lambda) \to \mathcal{C}^\infty(\Lambda, M_\Lambda \otimes \Omega^{1/2}),$$

where M_Λ is the Maslov bundle and $\Omega^{1/2}$ denotes the half-density bundle over Λ. Moreover, its null space is $I^{m_e-1,m_\psi-1}(X,\Lambda)$ and we have the short exact sequence

$$0 \longrightarrow I^{m_e-1,m_\psi-1}(X,\Lambda) \longrightarrow I^{m_e,m_\psi}(X,\Lambda) \xrightarrow{j^{\Lambda}_{m_e,m_\psi}} \mathcal{C}^\infty(\Lambda, M_\Lambda \otimes \Omega^{1/2}) \longrightarrow 0.$$

Equivalently,

$$I^{m_e,m_\psi}(X,\Lambda)/I^{m_e-1,m_\psi-1}(X,\Lambda) \simeq \mathcal{C}^\infty(\Lambda, M_\Lambda \otimes \Omega^{1/2}).$$

Summarizing, our results show that the theory of Lagrangian distributions, classically studied either locally or on compact manifolds, may be generalized to a theory of Lagrangian distributions on Euclidean spaces or manifolds with boundaries, hence a much wider class of geometries. It is formulated in a way that makes it easily transferable to other singular geometries as well as manifolds with corners, see [44].

This chapter is organized as follows. In Section 6.2 we give an introduction to scattering geometry. In particular, we discuss the natural class of maps between scattering manifolds, compactification and scattering amplitudes. In Section 6.3 we define the Lagrangian submanifolds and phase functions that arise in our theory. In Section 6.4 we discuss the techniques of classifying phase functions which parametrize the same Lagrangian submanifold. In Section 6.5 we define the Lagrangian distributions in this setting, starting from oscillatory integrals, and study their transformation properties. Finally, in Section 6.6, we define the principal symbol of Lagrangian distributions and prove its invariance.

6.2. Preliminary Definitions

In the following, we will recall some elements of the geometric theory known as "scattering geometry", cf. [42, 43, 45, 67]. To start with, we need to recall some groundwork on the analysis on manifolds with corners, for which we adopt the definition of [41, 44], cf. also [39] and [33] for a discussion on the different notions of manifolds with corners in the literature.

6.2.1. Further Elements of Scattering Geometry

The class of mwc that interest us is that of (products of) fiber bundles where both the base as well as the fiber are allowed to be a compact manifold with boundary (abbreviated "mwb"). The archetype of such a mwc is the product of two mwbs. Indeed, if X and Y are mwbs, $B = X \times Y$ is a mwc. We write $\mathcal{B} = \partial B$ and we have (adopting the notation of [7, 15])

$$\mathcal{B} = \underbrace{(\partial X \times Y^o) \cup (X^o \times \partial Y)}_{=\partial_1 B} \cup \underbrace{(\partial X \times \partial Y)}_{=\partial_2 B} =: \mathcal{B}^e \cup \mathcal{B}^\psi \cup \mathcal{B}^{\psi e}.$$

We present another compactification of $\mathbb{R}^d$, which has the advantage that its image is a subset of $\mathbb{R}^d$ in a natural way.

Definition 6.2.1 (Radial compactification of $\mathbb{R}^d$). Pick any diffeomorphism $\iota : \mathbb{R}^d \to (\mathbb{B}^d)^o$ that, for $|x| > 3$, is given by

$$\iota : x \mapsto \frac{x}{|x|}\left(1 - \frac{1}{|x|}\right).$$

Then its inverse is given, for $|y| \geq \frac{2}{3}$, by

$$\iota^{-1} : y \mapsto \frac{y}{|y|}(1 - |y|)^{-1}.$$

The map ι is called the *radial compactification map*. We may hence view $\mathbb{R}^d$ as the interior of the mwb $\mathbb{B}^d$ and call $\partial\mathbb{B}^d$ "infinity".

Denote by $[x]$ a smooth function $\mathbb{R}^d \to (0, \infty)$ that, for $|x| > 3$, is given by $x \mapsto |x|$. Then $(\iota^{-1})^*[x]^{-1}$ is a boundary defining function on $\mathbb{B}^d$ (and we view $[x]^{-1}$ as a boundary defining function on $\mathbb{R}^d$). Indeed, for $|y| > 2/3$ it is given by $y \mapsto 1 - |y| = \rho_Y$.

Remark 6.2.2. The compactification ι and SP are equivalent, meaning they yield diffeomorphic manifolds. In fact, for $|x| > 3$, we may write

$$\langle x \rangle^{-1} = [x]^{-1} \frac{1}{1 + [x]^{-2}}, \qquad [x]^{-1} = \langle x \rangle^{-1} \frac{1}{\sqrt{1 - \langle x \rangle^{-2}}}.$$

Hence, $\langle x\rangle^{-1}$ and $[x]^{-1}$ yield equivalent boundary defining functions on $\mathbb{R}^d$.

6.2.2. Exterior Derivative

The exterior derivative d lifts to a well-defined scattering differential ${}^{\mathrm{sc}}d$ on the scattering geometric structure. In coordinates, with ρ a local boundary defining function, we write

$$ {}^{\mathrm{sc}}df = \rho^2 \partial_\rho f \, \frac{d\rho}{\rho^2} + \sum_{j=1}^{d-1} \rho \partial_{x_j} f \, \frac{dx_j}{\rho}. \tag{6.2} $$

Note that for $f \in \mathcal{C}^\infty(X)$, this means that as a section of ${}^{\mathrm{sc}}T^*X$, ${}^{\mathrm{sc}}df$ necessarily vanishes on the boundary. In fact, we may extend ${}^{\mathrm{sc}}d$ to the space $\rho^{-1}\mathcal{C}^\infty(X)$ and obtain a map

$$ {}^{\mathrm{sc}}d : \rho^{-1}\mathcal{C}^\infty(X) \longrightarrow {}^{\mathrm{sc}}\Theta(X) = \Gamma({}^{\mathrm{sc}}T^*X). $$

That is, in local coordinates near the boundary,

$$ {}^{\mathrm{sc}}d(\rho^{-1}f) = \rho^{-1}\,{}^{\mathrm{sc}}df - f\frac{d\rho}{\rho^2} = (-f + \rho\partial_\rho f)\,\frac{d\rho}{\rho^2} + \sum_{j=1}^{d-1} \partial_{x_j} f\, \frac{dx_j}{\rho}. $$

Remark 6.2.3. We note that $\rho^{-1}\mathcal{C}^\infty(X)$ and similarly defined spaces are independent of the actual choice of boundary defining function ρ (cf. Remark 5.1.5).

Definition 6.2.4 (Scattering vector fields on product type manifolds). For a product $B = X \times Y$, with (X, ρ_X) and (Y, ρ_Y) mwbs, we may introduce ${}^{\mathrm{sc}}\mathcal{V}(B)$ as $\rho_X\rho_Y({}^b\mathcal{V}(B))$. Near a corner point the resulting bundle ${}^{\mathrm{sc}}T^*B$ is hence generated, if $\mathbf{x} = (\rho_X, x)$ and $\mathbf{y} = (\rho_Y, y)$ are local coordinates on X and Y respectively, by

$$ \rho_X^2\rho_Y\partial_{\rho_X},\ \rho_X\rho_Y\partial_{x_j},\ \rho_X\rho_Y^2\partial_{\rho_Y},\ \rho_X\rho_Y\partial_{y_k}. $$

The space ${}^{\mathrm{sc}}\mathcal{V}(B)$ splits into horizontal and vertical vector fields,[1] ${}^{\mathrm{sc}}\mathcal{V}^X(B)$ and ${}^{\mathrm{sc}}\mathcal{V}^Y(B)$, respectively, and we define ${}^{\mathrm{sc}}\Theta^X(B)$ as the set of (scattering) 1-forms $w \in {}^{\mathrm{sc}}\Theta^1(B)$ such that $w(v) = 0$ for all $v \in {}^{\mathrm{sc}}\mathcal{V}^Y(B)$.

Given complete set of coordinates $\mathbf{x} = (\rho_X, x)$, $\mathbf{y} = (\rho_Y, y)$ on X and Y, respectively, we see that ${}^{\mathrm{sc}}\Theta^X(B)$ is the set of sections generated by

$$ \frac{d\rho_X}{\rho_X^2\rho_Y}, \frac{dx_j}{\rho_X\rho_Y}. $$

[1] Consider the projection $\mathrm{pr}_X : B \to X$. Then $v \in {}^{\mathrm{sc}}\mathcal{V}(B)$ satisfies $v \in {}^{\mathrm{sc}}\mathcal{V}^X(B)$ if $v(\mathrm{pr}_X^* f) = 0$ for all $f \in \mathcal{C}^\infty(X)$. The set ${}^{\mathrm{sc}}\mathcal{V}^Y(B)$ is defined in analogy.

The underlying vector bundle will be denoted by ${}^{\mathrm{sc}}H^X B$. Similarly, we define ${}^{\mathrm{sc}}\Theta^Y(B)$ and ${}^{\mathrm{sc}}H^Y B$. It is important to note that we have the following "rescaling identifications":

$$\begin{aligned} {}^{\mathrm{sc}}\Theta^X(B) \ni \frac{d\rho_X}{\rho_X^2 \rho_Y} &\longleftrightarrow \rho_Y^{-1} \frac{d\rho_X}{\rho_X^2} \in \rho_Y^{-1} \mathcal{C}^\infty(Y, {}^{\mathrm{sc}}\Theta(X)), \\ {}^{\mathrm{sc}}\Theta^X(B) \ni \frac{dx_j}{\rho_X \rho_Y} &\longleftrightarrow \rho_Y^{-1} \frac{dx_j}{\rho_X} \in \rho_Y^{-1} \mathcal{C}^\infty(Y, {}^{\mathrm{sc}}\Theta(X)). \end{aligned} \tag{6.3}$$

Again, we may define the scattering exterior differential ${}^{\mathrm{sc}}d$, induced by the usual exterior differential d, and extend it to a map

$${}^{\mathrm{sc}}d : \rho_X^{-1} \rho_Y^{-1} \mathcal{C}^\infty(B) \longrightarrow {}^{\mathrm{sc}}\Theta(B).$$

In terms of the scattering differentials on X and Y we may decompose ${}^{\mathrm{sc}}d$ as ${}^{\mathrm{sc}}d = {}^{\mathrm{sc}}d_X + {}^{\mathrm{sc}}d_Y$, where

$$\begin{aligned} {}^{\mathrm{sc}}d_X &: \rho_X^{-1} \rho_Y^{-1} \mathcal{C}^\infty(B) \to {}^{\mathrm{sc}}\Theta^X(B), \\ {}^{\mathrm{sc}}d_Y &: \rho_X^{-1} \rho_Y^{-1} \mathcal{C}^\infty(B) \to {}^{\mathrm{sc}}\Theta^Y(B). \end{aligned}$$

6.2.3. Amplitudes

Definition 6.2.5 (Amplitudes of product-type). Let B be a mwc, $\{\rho_j\}_{j=1\ldots k}$ a complete set of bdfs. Then a is called an amplitude of order $m \in \mathbb{R}^k$ if

$$a \in \rho_1^{-m_1} \cdots \rho_k^{-m_k} \mathcal{C}^\infty(B).$$

For an open subset U of X, a *locally defined* amplitude of product type is an element of $\rho_1^{-m_1} \cdots \rho_k^{-m_k} \mathcal{C}^\infty(\overline{U})$. For $p \in \partial X$ we call a *elliptic at* p if $\rho_1^{m_1} \cdots \rho_k^{m_k} a(p) \neq 0$. We write

$$\dot{\mathscr{C}}_0^\infty(X) := \bigcap_{m \in \mathbb{R}^k} \rho_1^{m_1} \cdots \rho_k^{m_k} \mathcal{C}^\infty(B)$$

for the smooth functions vanishing at the boundary of infinite order.

For $p \in \partial B$ we call a *rapidly decaying at* p if there exists a neighbourhood U of p such that a vanishes of infinite order on $U \cap \partial B$, that is $a \in \dot{\mathscr{C}}_0^\infty(\overline{U})$.

We now study the leading boundary behavior of these amplitudes. For simplicity, we only consider $B = X \times Y$ for mwbs X and Y.

Definition 6.2.6. Let $a \in \rho_X^{-m_e} \rho_Y^{-m_\psi} \mathcal{C}^\infty(B)$ and write $a = \rho_X^{-m_e} \rho_Y^{-m_\psi} f$ for some $f \in \mathcal{C}^\infty(B)$. Given a coordinate neighbourhood U of a point $p \in \mathcal{B}^\bullet$, we define symbols $\sigma^\bullet(a)$ of a on U by

$$\begin{cases} \sigma^e(a)(\mathbf{x}, \mathbf{y}) = \rho_X^{-m_e} \rho_Y^{-m_\psi} f(0, x, \mathbf{y}), & p \in \mathcal{B}^e \cup \mathcal{B}^{\psi e} \\ \sigma^\psi(a)(\mathbf{x}, \mathbf{y}) = \rho_X^{-m_e} \rho_Y^{-m_\psi} f(\mathbf{x}, 0, y), & p \in \mathcal{B}^\psi \cup \mathcal{B}^{\psi e} \\ \sigma^{\psi e}(a)(\mathbf{x}, \mathbf{y}) = \rho_X^{-m_e} \rho_Y^{-m_\psi} f(0, x, 0, y) & p \in \mathcal{B}^{\psi e}. \end{cases}$$

The tuple $(\sigma^\psi(a), \sigma^e(a), \sigma^{\psi e}(a))$ is denoted by $\sigma(a)$ and called the *principal symbol.*

Fix $\epsilon > 0$ so small that ρ_X and ρ_Y can be chosen as coordinates on B respectively whenever $\rho_X < \epsilon$ and $\rho_Y < \epsilon$. We choose a cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\chi(t) = 0$ for $t > \epsilon/2$ and $\chi(t) = 1$ for $t < \epsilon/4$.

Definition 6.2.7. For any $a \in \rho_X^{-m_e}\rho_Y^{-m_\psi}\mathcal{C}^\infty(B)$ the amplitude

$$a_p(\mathbf{x}, \mathbf{y}) = \chi(\rho_X)\sigma^e(a)(\mathbf{x}, \mathbf{y}) + \chi(\rho_Y)\sigma^\psi(a)(\mathbf{x}, \mathbf{y}) - \chi(\rho_X)\chi(\rho_Y)\sigma^{\psi e}(a)(\mathbf{x}, \mathbf{y})$$

is called the *principal part* of a.

While a_p does depend on the choice of χ, its leading boundary asymptotic do not. By Taylor expansion of f, we obtain:

Lemma 6.2.8. *The principal part a_p of a satisfies $a - a_p \in \rho_X^{-m_e+1}\rho_Y^{-m_\psi+1}\mathcal{C}^\infty(B)$.*

Example 6.2.9 (Classical SG-symbols). Let $B = \mathbb{B}^d \times \mathbb{B}^s$, where $\mathbb{B}^d$ and $\mathbb{B}^s$ are the radial compactifications of $\mathbb{R}^d$ and $\mathbb{R}^s$. The space of so-called classical SG-symbols, $\mathrm{SG}_{\mathrm{cl}}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^s)$, is that of $a \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^s)$ such that $(\iota^{-1} \times \iota^{-1})^* a \in \rho_X^{-m_e}\rho_Y^{-m_\psi}\mathcal{C}^\infty(B)$. These symbols are then precisely those that satisfy the estimates

$$\left|\partial_x^\alpha \partial_\theta^\beta a(x, \theta)\right| \lesssim \langle x\rangle^{m_e - |\alpha|}\langle\theta\rangle^{m_\psi - |\beta|} \tag{6.4}$$

and admit a polyhomogeneous expansion, see [15, 42, 67] and the principal symbol of a corresponds to its homogeneous coefficients, see [15, Chap. 8.2].

We will need to consider density-valued amplitudes and integrate amplitudes on mwbs. For this, we introduce the space of scattering σ-density bundles, cf. [42], where ${}^{\mathrm{sc}}\Omega^\sigma(X) = \rho^{-\sigma(d+1)}\Omega^\sigma(X)$ in terms of the usual σ-density bundle. Note that ${}^{\mathrm{sc}}\Omega^\sigma$ does not depend on the choice of boundary defining function.

Example 6.2.10. Under the radial compactification, the canonical Lebesgue integration density on $\mathbb{R}^d$, $dx \in \Omega^1(\mathbb{R}^d)$, is mapped to $\iota_* dx \in {}^{\mathrm{sc}}\Omega^1(\mathbb{B}^d)$. In particular, we obtain $\iota_* dx = \rho^{-(d+1)} d\rho\, d\mathbb{S}^d$. More generally, if (X, g) is a scattering manifold, then the metric induces a canonical volume scattering 1-density μ_g.

Since the density bundle is a line bundle, any choice of scattering density provides a section of it and allows for an identification of scattering densities on X and $\mathcal{C}^\infty$-functions.

We denote the set of all smooth sections of the bundle ${}^{\mathrm{sc}}\Omega^\sigma(X)$ by $\mathcal{C}^\infty(X, {}^{\mathrm{sc}}\Omega^\sigma(X))$, and the tempered distribution densities $(\dot{\mathscr{C}}_0^\infty)'(X, {}^{\mathrm{sc}}\Omega^\sigma(X))$ are the continuous linear functionals on $\dot{\mathscr{C}}_0^\infty(X, {}^{\mathrm{sc}}\Omega^{1-\sigma}(X))$.

Lemma 6.2.11. *Let X be a mwb and Y a manifold without boundary. Then, integration over Y induces a map*

$$\int_Y : \mathcal{C}_c^\infty(X \times Y, {}^{\mathrm{sc}}\Omega^1(X \times Y)) \longrightarrow \rho_X^{-\dim Y}\mathcal{C}_c^\infty(X, {}^{\mathrm{sc}}\Omega^1(X)).$$

Remark 6.2.12. More generally, let X, Y be mwbs and Z a manifold without boundary. Consider a differentiable fibration $f : X \to Y$ with typical fiber Z. For every scattering density $\mu \in \mathcal{C}^\infty(X, {}^{\mathrm{sc}}\Omega^1(X))$ the pushforward

$$f_*\mu \in \rho_Y^{-\dim Z}\mathcal{C}_c^\infty(Y, {}^{\mathrm{sc}}\Omega^1(Y))$$

is defined locally by integration along the fiber.

Let (U, ψ) be a trivializing neighborhood of the fiber bundle, that is $U \subset Y$ open, $\psi : X \to U \times Z$ smooth and $f|_{f^{-1}(U)} = \mathrm{pr}_M \circ \psi$. Assume without loss of generality that μ is supported on $f^{-1}(U)$. Then set

$$f_*\mu = \int_Z \mu \circ \psi_j.$$

6.2.4. Scattering Maps

We now introduce and characterize the class of maps whose pull-backs preserve amplitudes of product type. They are a special case of interior b-maps in the sense of [41], and humbly mimicking Melrose's naming conventions we call them sc-maps. We first introduce them on manifolds with boundary and then generalize to manifolds with higher corner degeneracy, such as products of mwcs.

Definition 6.2.13 (sc-maps on mwb). Let Y and Z be mwbs. Suppose $\Psi : Y \to Z$. Then Ψ is called an sc-map if for any $m \in \mathbb{R}$ and $a \in \rho_Z^{-m}\mathcal{C}^\infty(Z)$ it holds that:

1. $\Psi^* a \in \rho_Y^{-m}\mathcal{C}^\infty(Y)$;
2. if $p \in \Psi(Y)$ with $p = \Psi(q)$ and $(\rho_Z^m a)(p) > 0$, then $(\rho_Y^m \Psi^* a)(q) > 0$.

Remark 6.2.14. In particular, Ψ maps the boundary of Y into that of Z. It also follows that $T\Psi$ maps inward pointing vectors at the boundary (meaning vectors with strictly positive ∂_ρ-component) to inward pointing vectors at the corresponding points. Indeed, we see that, at the boundary, $\Psi_* \partial_{\rho_Z} = h^{-1}\partial_{\rho_Y}$.

Remark 6.2.15. It is obvious that the composition of two sc-maps is again a sc-map.

It is straightforward to adapt this definition to that of a local sc-map by replacing Y and Z with open subsets.

Lemma 6.2.16 (sc-maps in coordinates)**.** *Let Y and Z be mwbs, $U \subset Y$ and $V \subset Z$ open subsets. A smooth map $\Psi : U \to V$ is a local sc-map if and only if for the boundary defining functions on Y and Z, ρ_Y and ρ_Z, respectively, we have*

$$\Psi^*\rho_Z = \rho_Y h \text{ for some } h \in \mathcal{C}^\infty(Y) \text{ with } h > 0. \tag{6.5}$$

Hence, any local diffeomorphism of mwbs is a local scattering map. Moreover:

Lemma 6.2.17. *Let X, Z be mwbs. Given any open, bounded set $U \subset \mathbb{R}^d$, define the projection $\mathrm{pr}_Z : Z \times U \to Z, (z, y) \mapsto z$. Then $\mathrm{I}_X \times \mathrm{pr}_Z$ is a* sc*-map.*

We now investigate the action of pull-backs by sc-maps on the objects introduced above. The following assertions can be verified in local coordinates.

Lemma 6.2.18. *Let Y and Z be mwbs, $U \subset Y$ and $V \subset Z$ open subsets. Let $\Psi : U \to V$ be a local* sc*-map. Then, the following properties hold true.*

- *Ψ^* yields a map $\rho_Z^m \,{}^{\mathrm{sc}}\Theta^k(V) \to \rho_Y^m \,{}^{\mathrm{sc}}\Theta^k(U)$ for any $m \in \mathbb{R}$ and $k \in \mathbb{N}$. Moreover, for $\theta \in \rho_Z^m \,{}^{\mathrm{sc}}\Theta^k(V)$, we have ${}^{\mathrm{sc}}d(\Psi^*\theta) = \Psi^*({}^{\mathrm{sc}}d\theta)$.*
- *Ψ^* yields a map ${}^{\mathrm{sc}}\Omega^\sigma(V) \to {}^{\mathrm{sc}}\Omega^\sigma(U)$ for any $\sigma \in [0, 1]$.*
- *The map $T^*\Psi : T^*V \to T^*U$ lifts to a map ${}^{\mathrm{sc}}\overline{T}^*\Psi : {}^{\mathrm{sc}}\overline{T}^*V \to {}^{\mathrm{sc}}\overline{T}^*U$. In local coordinates, away from fiber-infinity, ${}^{\mathrm{sc}}\overline{T}^*\Psi$ is given by*

  $$(\Psi(\mathbf{y}), \boldsymbol{\zeta}) \mapsto \big(\mathbf{y}, \iota({}^t(J\Psi)(\iota^{-1}\boldsymbol{\zeta}))\big),$$

  wherein $J\Psi$ is the Jacobian of Ψ at $\mathbf{y}$. The extension to fiber-infinity is obtained by taking interior limits $|\zeta|^{-1} \to 0$.

We observe that sc-maps provide a natural class of maps between scattering manifolds.

Corollary 6.2.19. *Suppose Y is a mwb, (Z, ρ_Z, g) a scattering manifold, Ψ a* sc*-map $Y \to Z$ which is an immersion. Then $(Y, \Psi^*\rho_Z, \Psi^*g)$ is a scattering manifold.*

Proof. We first observe that $\Psi^*\rho_Z$ is a boundary defining function on Y. Indeed,

$$d\Psi^*\rho_Z = h\, d\rho_Y + \rho_Y dh. \tag{6.6}$$

This implies, at the boundary, $h\, d\rho_Y \neq 0$. The scattering metric on Z pulls back to

$$\Psi^*g = \Psi^*\frac{(d\rho_Z)^{\otimes 2}}{\rho_Z^4} + \Psi^*\frac{g_\partial}{\rho_Z^2} = \frac{(d\Psi^*\rho_Z)^{\otimes 2}}{(\Psi^*\rho_Z)^4} + \frac{\Psi^*g_\partial}{(\Psi^*\rho_Z)^2},$$

which is again a scattering metric. □

Corollary 6.2.20. *Any scattering manifold Y of dimension s is locally isometric to $\mathbb{B}^s$ with some scattering metric. Moreover, any scattering density on Y can locally be written as the pull-back by one on $\mathbb{B}^s$.*

We now extend the notion of sc-map to manifolds with corners.

Definition 6.2.21 (sc-maps on mwc)*.* Let Y and Z be mwcs. Then, a smooth map $\Psi : Y \to Z$ is a local sc-map for some complete sets of local bdfs $\{\rho_{Y_i}\}_{i\in I}$ and $\{\rho_{Z_i}\}_{i\in I}$ if:

For all $i \in I$ we have $\Psi^*\rho_{Z_i} = \rho_{Y_i} h_i$ for some $h_i \in \mathcal{C}^\infty(Y)$ with $h_i > 0$.

Remark 6.2.22. In particular, Ψ maps the boundary of Y into that of Z.

As mentioned before, sc-maps are special cases of b-maps. In fact, they are those *interior* b-maps that are smooth maps in the sense of [33]. The only difference with the smooth maps in [33] is that, therein, $\Psi^*\rho_{Z_i} \equiv 0$ is allowed.

Example 6.2.23. In particular, if $\Psi_1 : Y_1 \to Z_1$ and $\Psi_2 : Y_2 \to Z_2$ are sc-maps on mwb, then $\Psi_1 \times \Psi_2 : Y_1 \times Y_2 \to Z_1 \times Z_2$ is a sc-map on the resulting product mwc.

Remark 6.2.24. Note that we fix the ordering of the boundary defining functions. This is important, in particular, when considering sc-maps between products $X \times Y \to X \times Z$ or of the form $X \times Y \to {}^{\mathrm{sc}}\overline{T}^* X$. Most of the times, the choice of bdfs will be clear from the context.

Note that, on a mwb, it is possible to extend any map $\partial X \mapsto \partial X$ with $x \mapsto x'$ to a scattering map, by setting $(\rho_X, x) \mapsto (\rho_X, x')$ in a collar neighbourhood of ∂X given by $X \cong [0, \epsilon) \times \partial X$. The following proposition grants us the ability to continue scattering maps from a corner into the interior.

Proposition 6.2.25. *Let $B_1 = X_1 \times Y_1$ and $B_2 = X_2 \times Y_2$ be products of mwbs. Let Ψ^e, Ψ^ψ be two (local) scattering maps near a point $p \in \mathcal{B}_1^{\psi e}$,*

$$\Psi^e : \mathcal{B}_1^e \longrightarrow \mathcal{B}_2^e \quad \text{and} \quad \Psi^\psi : \mathcal{B}_1^\psi \longrightarrow \mathcal{B}_2^\psi$$

such that $\Psi^e = \Psi^\psi$ when restricted to $\mathcal{B}_1^{\psi e}$. Then there exists a (local) scattering map Ψ on a neighbourhood $U \subset B_1$ of p with $\Psi^\bullet = \Psi|_{\mathcal{B}^\bullet}$ such that

$$\partial_{\rho_{X_1}} \Psi^* \rho_{Y_2} = \partial_{\rho_{Y_1}} \Psi^* \rho_{X_2} = 0 \quad \text{on } \mathcal{B}_1. \tag{6.7}$$

If Ψ^e and Ψ^ψ are local diffeomorphisms near p (in their respective boundary faces), then Ψ is a local diffeomorphism near p.

Proof. This is Whitney's extension theorem for smooth functions, applied to the system of functions (and their derivatives)

$$\begin{aligned} &(\Psi^e)^* x, (\Psi^e)^* y, (\Psi^e)^* \rho_Y &&\text{on } \mathcal{B}_1^e, \\ &(\Psi^\psi)^* \rho_X, (\Psi^\psi)^* x, (\Psi^\psi)^* y &&\text{on } \mathcal{B}_1^\psi, \end{aligned}$$

together with the conditions (6.7) and

$$\begin{aligned} D_{x,y} \Psi^* \rho_{Y_2} &= 0 \quad \text{on } \mathcal{B}_1^\psi, \\ D_{x,y} \Psi^* \rho_{X_2} &= 0 \quad \text{on } \mathcal{B}_1^e. \end{aligned}$$

Note that, if Ψ^e and Ψ^ψ are local diffeomorphisms at p, the differential of Ψ is an invertible block matrix, and hence Ψ is a local diffeomorphism. $\square$

Lemma 6.2.26. *Consider a* sc*-map* $\Psi : X \times Y \to X \times Y$ *of product form* $\Psi = \Psi_X \times \Psi_Y$*, with* sc*-maps on* X, Y*,* Ψ_X *and* Ψ_Y*, respectively. Assume* $a \in \rho_Y^{-m_\psi} \rho_X^{-m_e} \mathcal{C}^\infty(X \times Y)$*. With the notation of Definition 6.2.6 and 6.2.7, we have:*

$$\begin{aligned} \sigma^\psi(\Psi^* a) - \Psi^*(\sigma^\psi a) &\in \rho_Y^{-m_\psi+1} \rho_X^{-m_e} \mathcal{C}^\infty, \\ \sigma^e(\Psi^* a) - \Psi^*(\sigma^e a) &\in \rho_Y^{-m_\psi} \rho_X^{-m_e+1} \mathcal{C}^\infty, \\ (\Psi^* a)_p - \Psi^*(a_p) &\in \rho_Y^{-m_\psi+1} \rho_X^{-m_e+1} \mathcal{C}^\infty. \end{aligned}$$

Proof. We will only prove the first identity, the others follows by similar arguments. Write $(\Psi^* \rho_X)(\mathbf{x}) = \rho_X h_X(\mathbf{x})$ and $(\Psi^* \rho_Y)(\mathbf{y}) = \rho_Y h_Y(\mathbf{y})$. If $a = \rho_X^{-m_e} \rho_Y^{-m_\psi} f$ then

$$(\Psi^* a)(\mathbf{x}, \mathbf{y}) = \rho_X^{-m_e} \rho_Y^{-m_\psi} h_X^{-m_e}(\mathbf{x}) h_Y^{-m_\psi}(\mathbf{y}) (\Psi^* f)(\mathbf{x}, \mathbf{y}).$$

This implies

$$\begin{aligned} \sigma^\psi(\Psi^* a)(\mathbf{x}, \mathbf{y}) &= \rho_X^{-m_e} \rho_Y^{-m_\psi} h_X^{-m_e}(\mathbf{x}) h_Y^{-m_\psi}(0, y) (\Psi^* f)(\mathbf{x}, 0, y), \\ \Psi^*(\sigma^\psi a)(\mathbf{x}, \mathbf{y}) &= \rho_X^{-m_e} \rho_Y^{-m_\psi} h_X^{-m_e}(\mathbf{x}) h_Y^{-m_\psi}(\mathbf{y}) (\Psi^* f)(\mathbf{x}, 0, y). \end{aligned}$$

Using Taylor's theorem, we obtain that $h_Y^{-m_\psi}(\mathbf{y}) - h_Y^{-m_\psi}(0, y) \in \rho_Y \mathcal{C}^\infty(X \times Y)$, and therefore $\sigma^\psi(\Psi^* a) - \Psi^*(\sigma^\psi a) \in \rho_Y^{-m_\psi+1} \rho_X^{-m_e} \mathcal{C}^\infty(X \times Y)$, as claimed. □

Corollary 6.2.27. *The principal part of* $a \in \rho_Y^{-m_\psi} \rho_X^{-m_e} \mathcal{C}^\infty(X \times Y)$ *is well-defined as an element of*

$$\rho_X^{-m_e} \rho_Y^{-m_\psi} \mathcal{C}^\infty(X \times Y) / \rho_X^{-m_e+1} \rho_Y^{-m_\psi+1} \mathcal{C}^\infty(X \times Y),$$

and does not depend on the choice of boundary-defining functions ρ_X, ρ_Y *on* X, Y*.*

Remark 6.2.28. Note that the space

$$\rho_X^{-m_e} \rho_Y^{-m_\psi} \mathcal{C}^\infty(X \times Y) / \rho_X^{-m_e+1} \rho_Y^{-m_\psi+1} \mathcal{C}^\infty(X \times Y)$$

can be identified with $\mathcal{C}^\infty(\partial(X \times Y))$, which identifies our notion of principal symbol with that of [43, Section 6.4].

The following lemma is one of the main technical tools in this chapter. We have already observed that the local model of a scattering manifold near the boundary is the radial compactification of $\mathbb{R}^d$. We now show that scattering maps arise naturally as the composition of vector-valued amplitudes and radial compactification. Furthermore, we clarify the relation between total derivative and the scattering differential under compactification.

Lemma 6.2.29. *Let Y be a mwb. Let $f \in \rho_Y^{-1}\mathcal{C}^\infty(Y, \mathbb{R}^d)$ with $\rho_Y |f| \neq 0$ on ∂Y.*[2] *Then, $\Psi = \iota \circ f$ extends to a local sc-map $Y \to \mathbb{B}^d$. Moreover, the matrix of coefficients of*

$$^{\mathrm{sc}}df = \begin{pmatrix} ^{\mathrm{sc}}df_1 \\ \vdots \\ ^{\mathrm{sc}}df_d \end{pmatrix}$$

has the same rank as the differential $T\Psi$ of Ψ.

Proof. Since ι is a diffeomorphism, $\iota \circ f$ is a smooth map while $\rho_Y > \varepsilon$ and we may thus restrict our attention to a neighbourhood of ∂Y where $\rho_Y |f|$ is everywhere non-vanishing. As usual, we pick a suitable collar neighbourhood of product type such that locally $Y = [0, \varepsilon) \times \partial Y$, and we write $\dim(Y) = s$ and $\mathbf{y} = (\rho_Y, y)$ for the coordinates. There we need to compute $\Psi^* \rho_Z$. Write $f(\rho_Y, y) = \rho_Y^{-1} h(\rho_Y, y)$ for $h \in \mathcal{C}^\infty(Y, \mathbb{R}^d)$ with $h(0, y) \neq 0$ for all $(0, y) \in \partial Y$. Since ρ_Y is assumed sufficiently small, $|f(\mathbf{y})| = \rho_Y^{-1} |h(\mathbf{y})|$ may be assumed sufficiently large and hence

$$\Psi(\mathbf{y}) = (\iota \circ f)(\mathbf{y}) = \frac{f(\mathbf{y})}{|f(\mathbf{y})|}\left(1 - \frac{1}{|f(\mathbf{y})|}\right) = \frac{h(\mathbf{y})}{|h(\mathbf{y})|}\left(1 - \frac{\rho_Y}{|h(\mathbf{y})|}\right).$$

In this form, Ψ clearly extends up to the boundary. The boundary defining function on $\mathbb{B}^d$ is, in this coordinate patch, $\rho_Z = 1 - |x|$. Thus,

$$\Psi^* \rho_Z = \frac{1}{|f(\mathbf{y})|} = \rho_Y \frac{1}{\rho_Y |f(\mathbf{y})|}.$$

By assumption, $\rho_Y |f(\mathbf{y})| = |h(\mathbf{y})|$ is smooth and non-vanishing, which proves that Ψ is an sc-map.

For the second half of the statement we first observe that, since ι is a diffeomorphism $\mathbb{R}^d \to (\mathbb{B}^d)^o$ and $^{\mathrm{sc}}d$ coincides, up to a rescaling by a non-vanishing factor, with the usual differential in the interior, we may restrict our attention to the boundary ∂Y. Then we compute

$$\begin{aligned} ^{\mathrm{sc}}df(\mathbf{y}) &= \rho_Y^2 \partial_{\rho_Y} f(\mathbf{y}) \frac{d\rho_Y}{\rho_Y^2} + \sum_{j=1}^{s-1} \rho_Y \partial_{y_j} f(\mathbf{y}) \frac{dy_j}{\rho_Y} \\ &= (-h(\mathbf{y}) + \rho_Y \partial_{\rho_Y} h(\mathbf{y})) \frac{d\rho_Y}{\rho_Y^2} + \sum_{j=1}^{s-1} \partial_{y_j} h(\mathbf{y}) \frac{dy_j}{\rho_Y}. \end{aligned}$$

We identify $^{\mathrm{sc}}df$ with its coefficients $(s \times d)$-dimensional block matrix

$$\begin{pmatrix} -h(\mathbf{y}) + \rho_Y \partial_{\rho_Y} h(\mathbf{y}) & (\partial_{y_j} h(\mathbf{y}))_{j=1,\dots,s-1} \end{pmatrix}.$$

[2] This means $\rho_Y f$ is the restriction to Y^o of an element of $g \in \mathcal{C}^\infty(Y, \mathbb{R}^d)$ with $g \neq 0$ on ∂Y.

At the boundary $\rho_Y = 0$ we obtain

$$\begin{pmatrix} -h & (\partial_{y_j} h)_{j=1,\dots,s-1} \end{pmatrix}(0,y). \tag{6.8}$$

We want to compare the rank of (6.8) with that of the differential of Ψ at the point $(0,y) \in \partial Y$. As shown above, the function Ψ is given, at an arbitrary point $\mathbf{y} = (\rho_Y, y)$ close enough to ∂Y, by

$$\frac{h(\mathbf{y})}{|h(\mathbf{y})|}\left(1 - \frac{\rho_Y}{|h(\mathbf{y})|}\right),$$

whose differential at $(0,y)$ is the block matrix

$$T\Psi(0,y) = \begin{pmatrix} -\frac{h}{|h|^2} + \partial_{\rho_Y}\frac{h}{|h|} & \left(\partial_{y_j}\frac{h}{|h|}\right)_{j=1,\dots,s-1} \end{pmatrix}(0,y). \tag{6.9}$$

Now observe that, since they are derivatives of unit vectors, $\partial_{y_j}\frac{h}{|h|}$ and $\partial_{\rho_Y}\frac{h}{|h|}$ are orthogonal to h, which is itself non-zero.[3] Therefore, the rank of $T\Psi(0,y)$ equals that of the block matrix

$$\begin{pmatrix} -h & \left(\partial_{y_j}\frac{h}{|h|}\right)_{j=1,\dots,s-1} \end{pmatrix}(0,y). \tag{6.10}$$

Finally, we have that

$$\partial_{y_j} h = \partial_{y_j}\left(|h|\frac{h}{|h|}\right) = \underbrace{|h|\partial_{y_j}\frac{h}{|h|}}_{\text{collinear to } \partial_{y_j}\frac{h}{|h|}} + \underbrace{\frac{(h\cdot\partial_{y_j}h)}{|h|^2}h}_{\text{collinear to } h}.$$

This means that the null space (and hence the ranks) of (6.8) and (6.10) coincide. □

Example 6.2.30. The simplest example for a map where Lemma 6.2.29 applies is given by the map $f = \iota^{-1} : \mathbb{B}^d \to \mathbb{R}^d$.

Remark 6.2.31. Recall (cf. [31, App. C.3]) that the intersection of two $\mathcal{C}^\infty$-submanifolds Y and Z of a $\mathcal{C}^\infty$-manifold X is *clean* with excess $e \in \mathbb{N}$ if $Y \cap Z$ is a $\mathcal{C}^\infty$-submanifold of X satisfying

$$\begin{aligned} T_x(Y\cap Z) &= T_xY \cap T_xZ, \qquad \forall x \in Y\cap Z,\\ \operatorname{codim}(Y) + \operatorname{codim}(Z) &= \operatorname{codim}(Y\cap Z) + e. \end{aligned}$$

Example 6.2.32. Let X be a mwb and $a \in \rho_X^{-m_e}\rho_{\mathbb{B}^s}^{-m_\psi}\mathcal{C}^\infty(X\times\mathbb{B}^s)$. In this example, we extend a to a local symbol on a suitable subset of $X \times \mathbb{B}^{s+1}$.

We view $\mathbb{B}^{s+1}$ as embedded in $\mathbb{R}^{s+1}$ with coordinates $(y_1,\dots,y_s,\tilde{y})$. Define

$$\jmath : \mathbb{B}^{s+1} \to \mathbb{B}^s \times (-1,1), \qquad (y,\tilde{y}) \mapsto \left(\frac{y}{\sqrt{1-\tilde{y}^2}}, \tilde{y}\right),$$

[3] Recall that, in fact, $|v(t)| = 1 \Leftrightarrow v(t)\cdot v(t) = 1 \Rightarrow 2v(t)\cdot v'(t) = 0 \Leftrightarrow v(t) \perp v'(t)$.

where $y = (y_1, \dots, y_s)$. For every $\varepsilon \in (0,1)$, we obtain coordinates on

$$U = \jmath^{-1}\{\mathbb{B}^s \times (-\varepsilon, \varepsilon)\} = \mathbb{B}^{s+1} \cap \{|\tilde{y}| < \varepsilon\},$$

cf. Figure 6.1. We note that U is a fibration of base $\mathbb{B}^s$ and fiber $(-\varepsilon, \varepsilon)$.

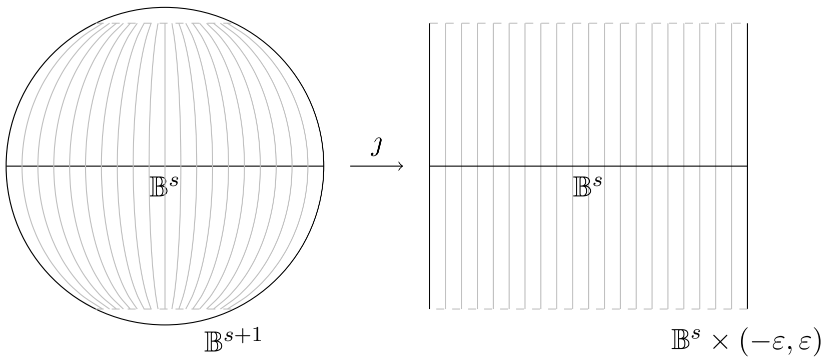

Figure 6.1.: The action of $\jmath$ visualized

We verify that $\jmath$ is a sc-map. For this we now view $\mathbb{B}^s \times (-\varepsilon, \varepsilon)$ as a (non-compact) manifold with boundary[4] with boundary defining function $\rho_Z = 1 - [y]$. Observe that near the boundary we have

$$\begin{aligned}
\jmath^* \rho_Z &= 1 - \frac{[y]}{\sqrt{1 - \tilde{y}^2}} \\
&= (1 - \sqrt{[y]^2 + \tilde{y}^2}) \cdot \frac{1}{\sqrt{1 - \tilde{y}^2}} \cdot \frac{\sqrt{1 - \tilde{y}^2} - [y]}{1 - \sqrt{\tilde{y}^2 + [y]^2}} \\
&= \rho_{\mathbb{B}^{s+1}} h.
\end{aligned}$$

Since $|\tilde{y}| < \epsilon$, h is positive and in $\mathcal{C}^\infty(U)$. Hence $\jmath$ is an sc-map.

As usual, we may perform the same construction fiber-wise on a fiber bundle by considering local product decompositions to obtain a local sc-map. Namely, let X be an arbitrary mwb. Then $\Psi = \mathrm{I}_X \times \jmath$ is again a sc-map on the product $X \times \big(\mathbb{B}^s \times (-\varepsilon, \varepsilon)\big)$. Using Lemma 6.2.17 and Remark 6.2.15, wee see that $\tilde{\Psi} = \Psi \circ (\mathrm{I}_X \times \mathrm{pr}_{\mathbb{B}^s}) : X \times U \to X \times \mathbb{B}^s$ is a sc-map. Hence, $\tilde{\Psi}^* a \in \rho_X^{-m_e} \rho_{\mathbb{B}^{s+1}}^{-m_\psi} C^\infty(X \times U)$.

[4]This means we view $\mathbb{B}^s \times (-\varepsilon, \varepsilon)$ as embedded in the manifold with boundary $\mathbb{B}^s \times \mathbb{S}^1$, which can be embedded in $\mathbb{S}^s \times \mathbb{S}^1$. For higher dimension, we embed $(-\varepsilon, \varepsilon)^r \hookrightarrow \mathbb{T}^r$.

6.3. Phase Functions and Lagrangian Submanifolds

6.3.1. Clean Phase Functions

Definition 6.3.1 (Phase functions). Let X and Y be mwbs, $B = X \times Y$. Let U be an open subset in B. Then, a real valued $\varphi \in \rho_X^{-1}\rho_Y^{-1}\mathcal{C}^\infty(U)$ is a *local* (sc-)phase function if it is the restriction of some $\widetilde{\varphi} \in \rho_X^{-1}\rho_Y^{-1}\mathcal{C}^\infty(B)$ to U such that ${}^{\mathrm{sc}}d\tilde{\varphi}(p) \neq 0$ for all $p \in \overline{\mathcal{B}^\psi} \cap \overline{\partial U}$.

If $U = B$, that is $\varphi \in \rho_X^{-1}\rho_Y^{-1}\mathcal{C}^\infty(B)$ with ${}^{\mathrm{sc}}d\varphi(p)|_{\overline{\mathcal{B}^\psi}} \neq 0$, we call φ a *global* sc-phase function.

Remark 6.3.2. Phrased differently, if U is an interior open set, φ is just a smooth function. In the non-trivial case of U being a boundary neighbourhood, the above definition means that, for every $p \in \partial B$ in the ψ- or ψe-component of the boundary of U, there exists an element $\zeta \in {}^{\mathrm{sc}}\mathcal{V}(B)$ such that $\zeta(\varphi)$ is elliptic at p, meaning $\zeta(\varphi) \in \mathcal{C}^\infty(X \times Y)$ satisfies $\big(\zeta\varphi\big)(p) \neq 0$. It is, by compactness, bounded away from zero at the possible limit points in $\overline{\partial U}$. In the following, we usually do not write $\widetilde{\varphi}$ but simply identify $\widetilde{\varphi}$ and φ at these limit points.

Example 6.3.3 (SG-phase functions). If $B = \mathbb{B}^d \times \mathbb{B}^s$, such φ correspond to so-called (classical) SG-phase functions on $\mathbb{R}^d \times \mathbb{R}^s$, cf. [6, 7], but with a relaxed condition as $\|x\| \to \infty$. Indeed, in light of the SG-estimates (6.4), the previous definition translates to

$$|\langle x\rangle^{-1}\nabla_\theta\varphi|^2 + |\langle\theta\rangle^{-1}\nabla_x\varphi|^2 \geq C \quad \text{for} \quad |\theta| \gg 0. \tag{6.11}$$

The relationship between these and "standard" phase functions which are homogeneous in θ is discussed in [7]. Examples of SG-phase functions are the standard Fourier phase $x \cdot \theta$ on $\mathbb{R}^d_x \times \mathbb{R}^d_\theta$ and $x_0\langle\theta\rangle - x \cdot \theta$ on $\mathbb{R}^{d+1}_{x_0,x} \times \mathbb{R}^d_\theta$.

Definition 6.3.4 (The set of critical points). Let $B = X \times Y$, $\varphi \in \rho_X^{-1}\rho_Y^{-1}\mathcal{C}^\infty(B)$ a (local) phase function. A point $p \in B$ (in the domain of φ) is called a *critical point* of φ if ${}^{\mathrm{sc}}d_Y\varphi(p) = 0$, that is, if $\zeta(\varphi)(p) = 0$ for every $\zeta \in {}^{\mathrm{sc}}\mathcal{V}^Y(B)$. We define

$$C_\varphi = \{p \in B \,|\, {}^{\mathrm{sc}}d_Y\varphi(p) = 0\}. \tag{6.12}$$

We set $\mathcal{C}_\varphi = C_\varphi \cap \mathcal{B}$ and specify

$$\mathcal{C}_\varphi^\bullet = \mathcal{C}_\varphi \cap \mathcal{B}^\bullet \quad \text{for} \quad \bullet \in \{e, \psi, \psi e\}.$$

We now adapt the usual definition of a *clean* phase function from the classical setting to the case with boundary.

Definition 6.3.5 (Clean phase functions). A phase function φ is called *clean* if the following conditions hold:

1.) there exists a neighbourhood $U \subset B$ of ∂B such that $C_\varphi \cap U$ is a manifold with corners with $\partial C_\varphi \subset \partial B$;

2.) the tangent space of T_pC_φ is at every point p given by those vectors in $v \in T_pB$ such that $v(\zeta(\varphi)) = 0$ for all $\zeta \in {}^{\mathrm{sc}}\mathcal{V}^Y$, that is, $T({}^{\mathrm{sc}}d_Y\varphi)v = 0$;

3.) the intersections $\mathcal{C}_\varphi^\bullet = C_\varphi \cap \mathcal{B}^\bullet$ are clean.

The last condition is equivalent to the existence of $w \in T_{\mathcal{C}_\varphi^\bullet}\mathcal{C}_\varphi^\bullet$ such that

$$(T^{\mathrm{sc}}d_Y\varphi)(w + \partial_{\rho_\bullet}) = 0. \tag{6.13}$$

This means that, for some w tangent to $\mathcal{B}^\bullet$, we have $w + \partial_{\rho_\bullet} \in T_{\mathcal{C}_\varphi^\bullet}\mathcal{C}_\varphi$. Here, $\rho_\bullet$ is a bdf of $\mathcal{B}^\bullet$. We now discuss the implications of these conditions.

Lemma 6.3.6. *Let φ be a clean phase function. Then either we are in the "non-corner crossing case" $1a.$) or in the "corner crossing case" $1b.$), namely,*

1. *both $\mathcal{C}_\varphi^e$ and $\mathcal{C}_\varphi^\psi$ are closed manifolds (without boundary) and $\mathcal{C}_\varphi^{\psi e} = \emptyset$;*

2. *$\mathcal{C}_\varphi$ consists of two components, $\overline{\mathcal{C}_\varphi^e}$ and $\overline{\mathcal{C}_\varphi^\psi}$, which are both submanifolds (with boundary), of the same dimension $\dim(C_\varphi) - 1$, with joint boundary $\mathcal{C}_\varphi^{\psi e} = \partial\overline{\mathcal{C}_\varphi^e} = \partial\overline{\mathcal{C}_\varphi^\psi}$ of $\mathcal{B}$. The intersection of $\overline{\mathcal{C}_\varphi^e}$ and $\overline{\mathcal{C}_\varphi^\psi}$ in $\mathcal{C}_\varphi^{\psi e}$ is again clean.*

In both cases, the differential of ${}^{\mathrm{sc}}d_Y\varphi : B \to {}^{\mathrm{sc}}T^*B$, viewed as a map $T({}^{\mathrm{sc}}d_Y\varphi) : TB \to T({}^{\mathrm{sc}}T^*B)$, characterizes $T\mathcal{C}_\varphi^\bullet$: The tangent space of $\overline{\mathcal{C}_\varphi^e}$ and $\overline{\mathcal{C}_\varphi^\psi}$ at each point p is given by those vectors $v \in T\mathcal{B}^\bullet$ such that $v(\zeta(\varphi)) = 0$ for all $\zeta \in {}^{\mathrm{sc}}\mathcal{V}^Y$, that is $T({}^{\mathrm{sc}}d_Y\varphi)v = 0$.

By condition 3.) of Definition 6.3.5, we have $\dim(\ker(T({}^{\mathrm{sc}}d_Y\varphi))) = \dim C_\varphi$. Hence, the restrictions of $T({}^{\mathrm{sc}}d_Y\varphi)$ to the individual boundary components of B on $\mathcal{C}_\varphi$ are of constant rank. Namely,

$$\mathrm{rk}(T({}^{\mathrm{sc}}d_Y\varphi)) = \begin{cases} s - e & \text{on } C_\varphi^o, \\ s - e - 1 & \text{on } \mathcal{C}_\varphi^\psi \text{ and } \mathcal{C}_\varphi^e, \\ s - e - 2 & \text{on } \mathcal{C}_\varphi^{\psi e}, \end{cases}$$

for some fixed number e, called the excess of φ, which is given by

$$e = \dim C_\varphi - d.$$

Remark 6.3.7. Conversely, if the rank of $T({}^{\mathrm{sc}}d_Y\varphi)$ is constant *in a neighborhood* of each critical point of ${}^{\mathrm{sc}}d_Y\varphi$, then φ is clean by the constant rank theorem. In case $e = 0$, φ is called *non-degenerate*, and the two characterizations coincide. The corresponding case of SG-phase functions (on $\mathbb{R}^d$) was studied in [7].

6.3.2. The Associated Lagrangian

In the classical local theory without boundary on subsets of $\mathbb{R}^d \times (\mathbb{R}^s \setminus \{0\})$, see [31, Chapter XXI.2], the set of critical points $\mathcal{C}_\varphi$ is realized as an immersed Lagrangian in $T^*\mathbb{R}^d$ by the map $(x,\theta) \to (x, \varphi'_x(x,\theta))$. In the present setting, the situation is more complicated. Following [7], we define an analogous map λ_φ on the mwc $B = X \times Y$ into ${}^{\mathrm{sc}}\overline{T}^* X$.

For that, we consider the following sequence of maps: Using the "rescaling identifications" (6.3), we may view $(\mathbf{x},\mathbf{y}) \to {}^{\mathrm{sc}}d_X\varphi(\mathbf{x},\mathbf{y})$ as a map in $\rho_Y^{-1}\mathcal{C}^\infty(Y, {}^{\mathrm{sc}}\Theta(X))$. Since ${}^{\mathrm{sc}}\Theta(X)$ are the sections of ${}^{\mathrm{sc}}\overline{T}^* X$, composing with the radial compactification yields, in view of Lemma 6.2.29, a map into the compactified fibers of ${}^{\mathrm{sc}}\overline{T}^* X$.

Definition 6.3.8. The map $\lambda_\varphi : B \to {}^{\mathrm{sc}}\overline{T}^* X$ is defined by

$$(\mathbf{x},\mathbf{y}) \mapsto \big(\mathbf{x}, \iota({}^{\mathrm{sc}}d_X\varphi(\mathbf{x},\mathbf{y}))\big).$$

Lemma 6.3.9. *There is a neighbourhood $U \subset B$ of $\mathcal{C}_\varphi$ such that $\lambda_\varphi : U \to {}^{\mathrm{sc}}\overline{T}^* X$ is a local* sc-*map.*

Proof. We write, $\mathbf{x} = (\rho_X, x)$, $\mathbf{y} = (\rho_Y, y)$ for coordinates in B, $\mathbf{x}$ and $\boldsymbol{\xi} = (\rho_\Xi, \xi)$ for coordinates in ${}^{\mathrm{sc}}\overline{T}^* X$. Since λ_φ is the identity in the first set of variables, we have $\lambda_\varphi^*\mathbf{x} = \mathbf{x}$. In the second set of variables, λ_φ acts as $\iota \circ {}^{\mathrm{sc}}d_X\varphi$, with ${}^{\mathrm{sc}}d_X\varphi \in \rho_Y^{-1}\mathcal{C}^\infty(Y, {}^{\mathrm{sc}}\Theta(X))$. Notice that on $\mathcal{C}_\varphi^\psi \cup \mathcal{C}_\varphi^{\psi e}$, we have ${}^{\mathrm{sc}}d_X\varphi(\mathbf{x},\mathbf{y}) \neq 0$, since ${}^{\mathrm{sc}}d\varphi \neq 0$ on $\mathcal{B}^\psi \cup \mathcal{B}^{\psi e}$ and ${}^{\mathrm{sc}}d_Y\varphi = 0$ on $\mathcal{C}_\varphi$. Hence, due to compactness, we may find a neighbourhood of $\mathcal{C}_\varphi^\psi \cup \mathcal{C}_\varphi^{\psi e}$ on which ${}^{\mathrm{sc}}d_X\varphi(\mathbf{x},\mathbf{y}) \neq 0$. Writing $\varphi = \rho_X^{-1}\rho_Y^{-1} f$ for $f \in \mathcal{C}^\infty(X \times Y)$, this means

$$(-f + \rho_X\partial_{\rho_X} f)\frac{d\rho_X}{\rho_X^2\rho_Y} + \sum_{j=1}^{d-1} \partial_{x_j} f \frac{dx_j}{\rho_X\rho_Y} \neq 0.$$

Rescaling and viewing ${}^{\mathrm{sc}}d_X\varphi$ as a map in $\rho_Y^{-1}\mathcal{C}^\infty(Y, {}^{\mathrm{sc}}\Theta(X))$, we express ${}^{\mathrm{sc}}d_X\varphi$ as

$${}^{\mathrm{sc}}d_X\varphi = \rho_Y^{-1}\left((-f + \rho_X\partial_{\rho_X} f)\frac{d\rho_X}{\rho_X^2} + \sum_{j=1}^{d-1} \partial_{x_j} f \frac{dx_j}{\rho_X}\right). \tag{6.14}$$

Composing with ι, we are therefore in the situation of Lemma 6.2.29, up to additional smooth dependence on the X-variables, and conclude that λ_φ is a local sc-map.

On $\mathcal{C}_\varphi^e$, away from $\mathcal{C}_\varphi^{\psi e}$, we have that $\rho_Y \neq 0$ and correspondingly ${}^{\mathrm{sc}}d_X\varphi(\mathbf{x},\mathbf{y})$ stays bounded. Since ι maps bounded arguments into the interior, we find ${\lambda_\varphi}^*\rho_\Xi \neq 0$. Since λ_φ is smooth, λ_φ is an sc-map. □

In particular, $\iota({}^{\mathrm{sc}}d_X\varphi(\mathbf{x},\mathbf{y}))$ maps boundary points with $\rho_Y = 0$ to boundary points of the fiber, that is to $\mathcal{W}^\psi \cup \mathcal{W}^{\psi e}$.

Definition 6.3.10. We define $L_\varphi = \lambda_\varphi(C_\varphi)$ and $\Lambda_\varphi := \lambda_\varphi(\mathcal{C}_\varphi)$. We further write $\Lambda_\varphi^\bullet$ for $\lambda_\varphi(\mathcal{C}_\varphi^\bullet) \subset \mathcal{W}^\bullet$ for $\bullet \in \{e, \psi, \psi e\}$. We say that φ parametrizes L_φ and Λ_φ.

Theorem 6.3.11. *The map $\lambda_\varphi : \mathcal{C}_\varphi \to {}^{\mathrm{sc}}\overline{T}^* X$ is of constant rank d. Its image L_φ as well as the boundary and corner faces $\Lambda_\varphi^\bullet = \lambda_\varphi(\mathcal{C}_\varphi^\bullet)$ are immersed manifolds of dimension $\dim \Lambda_\varphi^\bullet = \dim \mathcal{C}_\varphi^\bullet - e$. Furthermore, $\lambda_\varphi : \mathcal{C}_\varphi \to \Lambda_\varphi$ is a submersion.*

The proof is inspired by that of Lemma 2.3.2 in [12] (adapted to clean phase functions), but much more involved, due to the presence of the compactification. We treat this new phenomenon by carefully applying Lemma 6.2.29.

Proof. We obtain the rank of $T\lambda_\varphi$ for $\lambda_\varphi : \mathcal{C}_\varphi \to {}^{\mathrm{sc}}\overline{T}^* X$ by computing the dimension of its null space. Let $v = \delta\rho_X \cdot \partial_{\rho_X} + \delta x \cdot \partial_x + \delta\rho_Y \cdot \partial_{\rho_Y} + \delta y \cdot \partial_y$ be a vector at a point $p = (\rho_X, x, \rho_Y, y) \in \mathcal{C}_\varphi$. For the moment, we assume $\rho_Y > 0$. We write $\lambda_\varphi = (\mathrm{I} \times \iota) \circ \ell_\varphi$ with

$$\ell_\varphi : X \times Y^o \to {}^{\mathrm{sc}}T^* X \qquad (x, y) \mapsto (x, {}^{\mathrm{sc}}d_X \varphi(x, y)).$$

Assume that $T\ell_\varphi(p)v = 0$ and $v \in T_p\mathcal{C}_\varphi$. The condition $T\ell_\varphi(p)v = 0$ implies that $\delta\rho_X = 0$ and $\delta x = 0$. Let $\tilde{v} = \delta\rho_Y \cdot \partial_{\rho_Y} + \delta y \cdot \partial_y$. Hence the assumptions are reduced to

$$\begin{aligned} \tilde{v}\,{}^{\mathrm{sc}}d_X \varphi(p) = 0, \\ \tilde{v}\,{}^{\mathrm{sc}}d_Y \varphi(p) = 0, \end{aligned} \tag{6.15}$$

where $\tilde{v}$ is interpreted as acting on the coefficient functions of the differentials.

In coordinates, these coefficient functions are given by

$${}^{\mathrm{sc}}d_X \varphi(p) = \rho_Y^{-1}(-f + \rho_X \partial_{\rho_X} f, \partial_x f)(p), \qquad {}^{\mathrm{sc}}d_Y \varphi(p) = (-f + \rho_Y \partial_{\rho_Y} f, \partial_y f)(p).$$

On $\mathcal{C}_\varphi$, where $-f + \rho_Y \partial_{\rho_Y} f = 0$ and $\partial_y f = 0$ hold true, it is easily seen that (6.15) is equivalent to

$$\begin{pmatrix} \rho_X \rho_Y^{-2}(\rho_Y \partial_{\rho_Y} - 1)\partial_{\rho_X} f & \rho_X \rho_Y^{-1} \partial_{\rho_X} \partial_y f \\ \rho_Y^{-2}(\rho_Y \partial_{\rho_Y} - 1)\partial_x f & \rho_Y^{-1} \partial_x \partial_y f \\ \rho_Y \partial_{\rho_Y} \partial_{\rho_Y} f & \rho_Y \partial_{\rho_Y} \partial_y f \\ \partial_{\rho_Y} \partial_y f & \partial_y \partial_y f \end{pmatrix} \begin{pmatrix} \delta\rho_Y \\ \delta y \end{pmatrix} = 0. \tag{6.16}$$

The cleanness condition translates to the dimension of the nullspace of $T\,{}^{\mathrm{sc}}d_X \varphi$ being constantly e. We identify $T\,{}^{\mathrm{sc}}d_Y \varphi$ with the matrix

$$J = \begin{pmatrix} (\rho_Y \partial_{\rho_Y} - 1)\partial_{\rho_X} f & \partial_y \partial_{\rho_X} f \\ (\rho_Y \partial_{\rho_Y} - 1)\partial_x f & \partial_y \partial_x f \\ \rho_Y \partial_{\rho_Y} \partial_{\rho_Y} f & \partial_y \partial_{\rho_Y} f \\ \rho_Y \partial_{\rho_Y} \partial_y f & \partial_y \partial_y f \end{pmatrix}. \tag{6.17}$$

The matrices appearing in (6.16) and (6.17) are related by

$$J = \begin{pmatrix} \rho_Y \rho_X^{-1} & 0 & 0 & 0 \\ 0 & \rho_Y & 0 & 0 \\ 0 & 0 & \rho_Y^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho_X \rho_Y^{-2}(\rho_Y \partial_{\rho_Y} - 1)\partial_{\rho_X} f & \rho_X \rho_Y^{-1} \partial_{\rho_X} \partial_y f \\ \rho_Y^{-2}(\rho_Y \partial_{\rho_Y} - 1)\partial_x f & \rho_Y^{-1} \partial_x \partial_y f \\ \rho_Y \partial_{\rho_Y} \partial_{\rho_Y} f & \rho_Y \partial_{\rho_Y} \partial_y f \\ \partial_{\rho_Y} \partial_y f & \partial_y \partial_y f \end{pmatrix} \begin{pmatrix} \rho_Y & 0 \\ 0 & 1 \end{pmatrix}.$$

This proves that (6.15) is equivalent to $v \in \ker T^{\mathrm{sc}} d_Y \varphi$ under our assumptions $\rho_Y > 0$ and $\rho_X > 0$, and the rank of ℓ_φ is given by

$$\operatorname{rk} \ell_\varphi = \dim T_p \mathcal{C}_\varphi - \dim \ker T^{\mathrm{sc}} d_Y \varphi = (d + e) - e = d.$$

Now assume that $\rho_X = 0$. We see that the first row of (6.16) vanishes identically, but we have the additional condition (6.13), implying that, at $\rho_X = 0$, the first row of (6.17) depends linearly on the other rows. Therefore, the rank of ℓ_φ is still d at points with $\rho_X = 0$. The composition with $\mathrm{I} \times \iota$ changes nothing for $\rho_Y > 0$, since ι is a diffeomorphism there.

To perform the limit $\rho_Y \to 0$, we have to examine carefully the effect of the presence of the compactification ι, in the spirit of the proof of Lemma 6.2.29. For $v \in T_p \mathcal{C}_\varphi$ such that $T\lambda_\varphi(p) v = 0$, that is, as above, of the form

$$v = \delta\rho_Y \cdot \partial_{\rho_Y} + \delta y \cdot \partial_y,$$

we now obtain the set of equations

$$\begin{aligned} v\big(\iota\, {}^{\mathrm{sc}}d_X \varphi\big)(p) &= 0, \\ v^{\mathrm{sc}} d_Y \varphi(p) &= 0, \end{aligned} \tag{6.18}$$

which are equivalent to the set of equations

$$\begin{pmatrix} \partial_{\rho_Y} \iota^{\mathrm{sc}} d_X \varphi & \partial_y \iota^{\mathrm{sc}} d_X \varphi \\ \partial_{\rho_Y} \partial_y f & \partial_y \partial_y f \end{pmatrix} \begin{pmatrix} \delta\rho_Y \\ \delta y \end{pmatrix} = 0. \tag{6.19}$$

We need to compare the rank of the coefficient matrix in (6.19) with that of $T^{\mathrm{sc}} d_Y \varphi$ at points of the form $(\rho_X, x, 0, y)$. For this purpose, we go through a series of "reductions", along the lines of the proof of Lemma 6.2.29, to simplify the comparison. First, we can identify ${}^{\mathrm{sc}}d_X \varphi$ with

$$\rho_Y^{-1} \begin{pmatrix} -f + \rho_X \partial_{\rho_X} f \\ \partial_x f \end{pmatrix} =: \rho_Y^{-1} h.$$

Note that $h \neq 0$ near $\overline{\mathcal{C}_\varphi^\psi}$, since φ is a phase function. As in the proof of Lemma 6.2.29, the evaluation at $(\rho_X, x, 0, y)$ then gives

$$\begin{pmatrix} \partial_{\rho_Y} \iota^{\mathrm{sc}} d_X \varphi & \partial_y \iota^{\mathrm{sc}} d_X \varphi \\ \partial_{\rho_Y} \partial_y f & \partial_y \partial_y f \end{pmatrix} = \begin{pmatrix} -\frac{h}{|h|^2} + \partial_{\rho_Y} \frac{h}{|h|} & \partial_y \frac{h}{|h|} \\ \partial_{\rho_Y} \partial_y f & \partial_y \partial_y f \end{pmatrix}. \tag{6.20}$$

Since all derivatives of $\frac{h}{|h|}$ are orthogonal to $\frac{h}{|h|}$ and $h \neq 0$, the rank of the matrix (6.20) equals the one of

$$\begin{pmatrix} -\frac{h}{|h|^2} & \partial_y \frac{h}{|h|} \\ 0 & \partial_y \partial_y f \end{pmatrix}. \tag{6.21}$$

In fact, in (6.20), as well as in (6.21), the first column is linearly independent of the others. Now we write

$$\partial_{y_j} \frac{h}{|h|} = \frac{1}{|h|} \partial_{y_j} h - \underbrace{\frac{(h \cdot \partial_{y_j} h)}{|h|^3} h}_{\text{collinear to } h},$$

and remove the collinear summands, which again does not change the rank of the matrix (6.21). Therefore, the rank of (6.20) is the same as the one of

$$\begin{pmatrix} -\frac{h}{|h|^2} & \frac{1}{|h|} \partial_y h \\ 0 & \partial_y \partial_y f \end{pmatrix}. \tag{6.22}$$

Multiplying the first d rows and the first column of (6.22) by the non-vanishing factor $|h|$, again the rank does not change, and we can look at

$$\begin{pmatrix} -h & \partial_y h \\ 0 & \partial_y \partial_y f \end{pmatrix} = \begin{pmatrix} f - \rho_X \partial_{\rho_X} f & -\partial_y f + \rho_X \partial_y \partial_{\rho_X} f \\ -\partial_x f & \partial_y \partial_x f \\ 0 & \partial_y \partial_y f \end{pmatrix}. \tag{6.23}$$

On $\mathcal{C}_\varphi$ at $\rho_Y = 0$ this equals

$$\begin{pmatrix} -\rho_X \partial_{\rho_X} f & \rho_X \partial_y \partial_{\rho_X} f \\ -\partial_x f & \partial_y \partial_x f \\ 0 & \partial_y \partial_y f \end{pmatrix}. \tag{6.24}$$

Finally, we observe that the dimension of the null space of (6.24) is, by cleanness of φ (in particular by (6.13) applied to $\mathcal{C}^{\psi}_{\varphi}$ or $\mathcal{C}^{\psi e}_{\varphi}$), the same as the one of

$$\begin{pmatrix} -\partial_{\rho_X} f & \partial_y \partial_{\rho_X} f \\ -\partial_x f & \partial_y \partial_x f \\ 0 & \partial_y \partial_{\rho_Y} f \\ 0 & \partial_y \partial_y f \end{pmatrix} = T^{\mathrm{sc}} d_Y \varphi|_{\mathcal{C}^{\psi}_{\varphi}}, \tag{6.25}$$

namely e. Therefore, the rank of λ_φ equals $d = (d+e) - e$ near $\mathcal{C}_\varphi$, which concludes the proof. $\square$

Lemma 6.3.12. *The map $\lambda_\varphi : C_\varphi \to L_\varphi$ is a local fibration and the fiber is everywhere a smooth manifold without boundary.*

Proof. Since λ_φ is locally an sc-map, $T\lambda_\varphi$ maps the set of vectors at the boundary that are inwards pointing into itself, see Remark 6.2.14. Therefore λ_φ is a so-called "tame" submersion in the sense of [50, Lemma 1.3]. As such, it is a local fibration and the fiber is a manifold without boundary. □

6.3.3. Symplectic Properties of the Associated Lagrangian

As in the classical theory, L_φ is an immersed Lagrangian submanifold, and its boundary faces $\Lambda^\bullet$ are immersed Legendrian submanifolds. Let us briefly recall these concepts. For more information, the reader is referred to [7, 20, 45].

As a cotangent space, T^*X^o carries a natural symplectic 2-form ω induced by the canonical 1-form $\alpha \in \mathcal{C}^\infty(T^*X^o, T^*(T^*X^o))$ as $\omega = d\alpha$. This 1-form can be recovered from ω by setting $\alpha = \varrho^\psi \lrcorner\, \omega$ for the radial vector field ϱ^ψ on $\mathcal{C}^\infty(T^*X^o)$, which is given by $\varrho^\psi = \xi \cdot \partial_\xi$ in canonical coordinates.
We now write $(\mathbf{x}, \boldsymbol{\xi}) = (\rho_X, x, \rho_\Xi, \xi)$ for the coordinates in the mwc ${}^{\mathrm{sc}}\overline{T}^* X$ which are obtained from the rescaled canonical coordinates under radial compactification in the fiber, cf. [45]. Then ϱ^ψ corresponds to $\rho_\Xi \partial_{\rho_\Xi}$ on $\mathcal{C}^\infty(\overline{T}^* X^o)$. For the purpose of scattering geometry, it is natural to rescale further and define, on $T^*({}^{\mathrm{sc}}\overline{T}^* X)^o$,

$$\alpha^\psi := \rho_\Xi^2 \partial_{\rho_\Xi} \lrcorner\, \omega.$$

There exists another form of interest, namely

$$\alpha^e := \rho_X^2 \partial_{\rho_X} \lrcorner\, \omega.$$

We now extend these forms to $T^*({}^{\mathrm{sc}}\overline{T}^* X)$ and define the boundary restrictions of $\alpha^\bullet$. Observe that, while their explicit form depends on the choice of bdfs, the induced contact structure at the boundary does not, see next Lemma 6.3.13

Lemma 6.3.13. *The forms $\alpha^\bullet$ extend to 1-forms on $\mathcal{W}^\bullet$, denoted by the same letter. The induced contact structures do not depend on the choice of bdfs.*

Example 6.3.14. On $T^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d$, with canonical coordinates (x, ξ), the vector fields ϱ^ψ and ϱ^e correspond to $\varrho^\psi = \xi \cdot \partial_\xi$ and $\varrho^e = x \cdot \partial_x$. The symplectic 2-form is $\sum_j d\xi_j \wedge dx_j$ and hence

$$\varrho^\psi \lrcorner\, \omega = \xi \cdot dx \quad \text{and} \quad \varrho^e \lrcorner\, \omega = -x \cdot d\xi.$$

Obviously, the coefficients of these forms diverge as $[\xi] \to \infty$ and $[x] \to \infty$. The rescaled forms "at the boundary at infinity" then correspond to

$$\alpha^\psi = \frac{\xi}{[\xi]} \cdot dx \quad \text{and} \quad \alpha^e = -\frac{x}{[x]} \cdot d\xi.$$

After a choice of coordinates near the respective boundaries, this is the general local geometric situation.

We are now in the position to formulate the symplectic properties of Λ_φ, cf. [6]. Recall that a submanifold N of a symplectic manifold (M, ω) is Lagrangian if $\omega|_{TN} = 0$ and a submanifold N of a contact manifold (M, α) is Legendrian if $\alpha|_{TN} = 0$.

Proposition 6.3.15. *The immersed manifolds defined in Theorem 6.3.11 satisfy:*

1.) L^o_φ is an immersed Lagrangian submanifold with respect to the 2-form ω on $({}^{\mathrm{sc}}\overline{T}^* X)^o \cong T^*X$;

2.) Λ^ψ_φ is Legendrian with respect to the canonical 1-form α^ψ on $\mathcal{W}^\psi \cong S^*(X^o)$;

3.) Λ^e_φ is Legendrian with respect to the 1-form α^e on $\mathcal{W}^e \cong T^*_{\partial X} X$.

We take this as the definition of an sc-Lagrangian, cf. [7].

Definition 6.3.16 (sc-Lagrangians). Let $\Lambda := \overline{\Lambda^\psi} \cup \overline{\Lambda^e} \subset \mathcal{W}$. Λ is called an sc-Lagrangian if:

1.) $\Lambda^\psi = \Lambda \cap \mathcal{W}^\psi$ is Legendrian with respect to the canonical 1-form α^ψ on $\mathcal{W}^\psi = {}^{\mathrm{sc}}S^*_{X^o} X$;

2.) $\Lambda^e = \Lambda \cap \mathcal{W}^e$ is Legendrian with respect to the 1-form α^e on $\mathcal{W}^e = {}^{\mathrm{sc}}T^*_{\partial X} X$;

3.) $\overline{\Lambda^\psi}$ has a boundary if and only if $\overline{\Lambda^e}$ has a boundary, and, in this case,

$$\Lambda^{\psi e} := \partial\overline{\Lambda^\psi} = \partial\overline{\Lambda^e} = \overline{\Lambda^\psi} \cap \partial\overline{\Lambda^e},$$

with clean intersection.

Figure 6.2, which is taken from [7], summarizes, schematically, the relative positions of Λ^e_φ and Λ^ψ_φ near the corner in W. We may take the analysis one step further in order to stress the

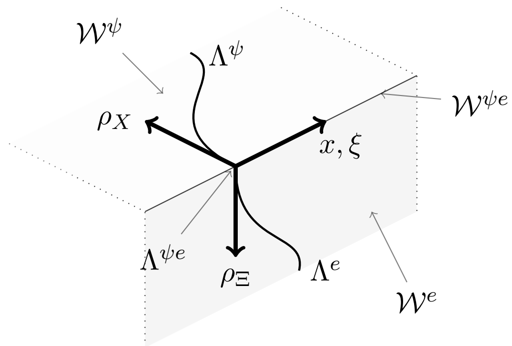

Figure 6.2.: Intersection of $\Lambda^\psi \subset \mathcal{W}^\psi$ and $\Lambda^e \subset \mathcal{W}^e$ at the corner $\mathcal{W}^{\psi e}$

Legendrian character of the boundary components near the corner and to reveal the symplectic properties of $\Lambda^{\psi e}$ by blow-up. For the sake of brevity here, we move this analysis to the appendix, Section 6.7.

We may sum up our previous analysis by stating the next Theorem 6.3.17.

Theorem 6.3.17. *For a clean phase function φ, the image Λ_φ under λ_φ of $\mathcal{C}_\varphi$ is an immersed* sc*-Lagrangian.*

Definition 6.3.18. We say that an sc-Lagrangian Λ is locally parametrized by a phase function φ if, over the domain of definition of φ, we have $\Lambda = \Lambda_\varphi$.

In particular, if Λ is locally parametrized by a phase function, then it is admissible. Conversely, we have the following result, cf. [7].

Proposition 6.3.19. *If Λ is an* sc*-Lagrangian, then it is locally parametrizable by a clean phase function φ, that is $\Lambda^\bullet \cap U^\bullet = \Lambda^\bullet_\varphi \cap U^\bullet$ for some open $U \subset \mathcal{W}^\bullet$. In particular, Λ arises as the boundary of some Lagrangian submanifold L_φ of ${}^{\mathrm{sc}}\overline{T}^* X$.*

Remark 6.3.20. The proof of Proposition 6.3.19 in [7] is based on concrete parametrizations in $\mathbb{R}^d \times \mathbb{R}^d$. It applies here nonetheless, since any d-dimensional manifold with boundary X can be locally modelled by $\mathbb{B}^d$. Hence, ${}^{\mathrm{sc}}\overline{T}^* X$ can be locally modelled by $\mathbb{B}^d \times \mathbb{B}^d$ and thus, under inverse radial compactification (applied to both factors), by $\mathbb{R}^d \times \mathbb{R}^d$. Note that in [7] we imposed additional conditions, namely

$$\Lambda^e \cap (\partial X \times \iota(\{0\})) = \emptyset, \tag{6.26}$$

and that $x \cdot \xi = 0$ in local canonical coordinates on $\Lambda^{\psi e}$, since this is always true for a parametrized Lagrangian (see (6.27) below). However, condition (6.26) is equivalent to the stronger assumption that ${}^{\mathrm{sc}}d\varphi \neq 0$ also on $\mathcal{B}^e$, which we do not impose here. The assumption $x \cdot \xi = 0$, in turn, is superfluous, since it already follows from the symplectic assumptions on $\Lambda^{\psi e}$, as we now show.

Assume that both $\xi \cdot dx \equiv 0$ and $-x \cdot d\xi \equiv 0$ on a bi-conic submanifold L of $\mathbb{R}^d \times \mathbb{R}^d$. Then we must have $d(x \cdot \xi) = 0$. However, when $|x|$ and $|\xi|$ tend to ∞, this blows up unless $x \cdot \xi = 0$. This shows that $x \cdot \xi = 0$ is indeed automatically fulfilled.

This corresponds to the fact that, for the bi-homogenous principal symbol of a phase function $\varphi^{\psi e}$, we have, when $\nabla_\theta \varphi(x, \theta) = 0$, that (cf. [7])

$$\langle x, \nabla_x \varphi(x, \theta)\rangle = \varphi(x, \theta) = \langle \theta, \nabla_\theta \varphi(x, \theta)\rangle = 0, \tag{6.27}$$

where we have used Euler's identity for homogeneous functions twice.

6.3.4. Scattering Conormal Bundles

In this section, we consider the simple example of a scattering conormal bundle. Consider a k-dimensional submanifold $X' \subset X$ which intersects the boundary of X cleanly or not at all (called p-submanifold in [44]). In the following, we assume an intersection with the boundary. Then there exist local coordinates (ρ_X, x', x'') such that X' is locally given by

$$X' = \{(\rho_X, x', x'') \mid \rho_X \geq 0, x' = 0 \in \mathbb{R}^{d-1-k}, x'' \in \mathbb{R}^{k-1}\}.$$

We can now consider the compactified scattering conormal ${}^{\mathrm{sc}}\overline{T}^*X' \subset {}^{\mathrm{sc}}\overline{T}^*_{X'}X$. The boundary faces of ${}^{\mathrm{sc}}\overline{T}^*X'$ constitute a Lagrangian.

In fact, write $X = \iota(\mathbb{R}^d)$, so that X' corresponds to a subspace of $\mathbb{R}^d$ of the form

$$X' = \{(x', x'') \mid x' = 0 \in \mathbb{R}^{d-k}, x'' \in \mathbb{R}^k\}.$$

We can then introduce $Y = \iota(\mathbb{R}^{d-k})$ and $\phi(x,y) = x' \cdot y$ on $\mathbb{R}^d \times \mathbb{R}^{d-k}$, which is an SG-phase function, taking into account (6.11). The true phase function on $X \times Y$ is then $(\iota^{-1} \times \iota^{-1})^*\phi$. We can then compute $C_\varphi = X' \times Y$ and $\Lambda_\varphi = {}^{\mathrm{sc}}\overline{T}^*X'$.

Indeed, in the Euclidean setting, Λ_φ corresponds to the the three conic manifolds

$$\begin{aligned}
\Lambda^e_\varphi &= \{(0, x'', \xi', 0)\} \subset (\mathbb{R} \setminus \{0\}^d) \times \mathbb{R}^d \\
\Lambda^{\psi e}_\varphi &= \{(0, x'', \xi', 0)\} \subset (\mathbb{R} \setminus \{0\}^d) \times (\mathbb{R} \setminus \{0\}^d) \\
\Lambda^\psi_\varphi &= \{(0, x'', \xi', 0)\} \subset \mathbb{R}^d \times (\mathbb{R} \setminus \{0\}^d)
\end{aligned}$$

which have the claimed symplectic properties. Compactification of the $\mathbb{R}^d$-components and projection of the conic $(\mathbb{R} \setminus \{0\}^d)$-component to the corresponding sphere then yields the compactified notions in ${}^{\mathrm{sc}}\overline{T}^*X$.

6.4. Phase Functions which Parametrize the Same Lagrangian

In this section, we adapt the classical techniques for exchanging the phase function locally parametrizing a given Lagrangian, see [61, Chapter 8.1], to the setting with boundary. Since Λ_φ, not L_φ, is our true object of interest, we say that two phase functions φ_i, $i = 1, 2$, locally parametrize the same Lagrangian at $p_0 \in \mathcal{W}$ if $\Lambda_{\varphi_1} = \Lambda_{\varphi_2}$ in a small (relatively) open neighbourhood of p_0 in the respective boundary faces.

Our first observation is the following:

Lemma 6.4.1. *If $\varphi \in \rho_X^{-1}\rho_{\mathbb{B}^s}^{-1}\mathcal{C}^\infty(X \times \mathbb{B}^s)$ is a local phase function and $r \in \mathcal{C}^\infty(X \times \mathbb{B}^s)$, then $\varphi + r$ is still a local phase function and it parametrizes the same Lagrangian as φ.*

Proof. Since $r \in \mathcal{C}^\infty(X \times \mathbb{B}^s)$, ${}^{\mathrm{sc}}dr = 0$ when restricted to the boundary. Therefore, $\varphi + r$ is still a local phase function. By the same reason, $\mathcal{C}_\varphi = \mathcal{C}_{\varphi+r}$. Finally, we have

$$\lambda_{\varphi+r}(\mathbf{x}, by) = (\mathbf{x}, \iota({}^{\mathrm{sc}}d_X(\varphi + r))).$$

Computing ${}^{\mathrm{sc}}d_X(\varphi + r)$ in coordinates, see (6.14),

$${}^{\mathrm{sc}}d_X\varphi = \rho_Y^{-1}\left((-f + \rho_X\partial_{\rho_X} f + \rho_Y\rho_X^2\partial_{\rho_X} r)\frac{d\rho_X}{\rho_X^2} + \sum_{j=1}^{d-1}(\partial_{x_j} f + \rho_Y\rho_X\partial_{x_j} r)\frac{dx_j}{\rho_X}\right),$$

we observe that at $\rho_X = 0$, the contribution from r vanishes. The same is true in the limit of $\rho_Y \to 0$ under application of ι, see also Lemma 6.2.29. ☐

6.4.1. Increasing Fiber Variables

Given a clean phase function $\varphi \in \rho_X^{-1}\rho_{\mathbb{B}^s}^{-1}C^\infty(X \times \mathbb{B}^s)$ with excess e, define the phase function $\widetilde{\psi} \in \rho_X^{-1}\rho_{\mathbb{B}^s}^{-1}C^\infty(X \times \mathbb{B}^s \times (-\varepsilon, \varepsilon))$ as follows:

$$\widetilde{\psi}(\mathbf{x}, \mathbf{y}, \tilde{y}) = \varphi(\mathbf{x}, \mathbf{y}) + \frac{\tilde{y}^2}{\rho_X \rho_{\mathbb{B}^s}}.$$

We see that ${}^{\mathrm{sc}}d\widetilde{\psi} \neq 0$ when ${}^{\mathrm{sc}}d\varphi \neq 0$ and ${}^{\mathrm{sc}}d_{\mathbb{B}^s \times (-\epsilon,\epsilon)}\widetilde{\psi} = 0$ if and only if $\tilde{y} = 0$ and ${}^{\mathrm{sc}}d_{\mathbb{B}^s}\varphi = 0$. Thus,

$$C_{\widetilde{\psi}} = \{(\mathbf{x}, \mathbf{y}, 0) \mid (\mathbf{x}, \mathbf{y}) \in C_\varphi\},$$

which implies that the excess is not changed, and $\Lambda_{\widetilde{\psi}} = \Lambda_\varphi$. Summing up, ψ is a local clean phase function in $s+1$ fiber variables with the same excess e as φ and (locally) parametrizing the same Lagrangian as φ.

This construction may once again be moved to balls, by using Example 6.2.32 and setting $\psi = \Psi^*\widetilde{\psi}$. Then $\psi \in \rho_X^{-1}\rho_{\mathbb{B}^{s+1}}^{-1}C^\infty(X \times U)$. Using the fact that ${}^{\mathrm{sc}}d\psi = \Psi^*\widetilde{\psi}$, we see that ψ is a clean phase function parametrizing Λ_φ with excess e. Again, $X \times \mathbb{B}^s$ can be exchanged by any relatively open subset, hence starting with local phase functions.

6.4.2. Reduction of the Fiber Variables

Starting again from a clean phase function $\varphi \in \rho_X^{-1}\rho_{\mathbb{B}^s}^{-1}C^\infty(X \times \mathbb{B}^s)$ with excess e, we now construct a (local) phase function ψ in the smallest possible number of phase variables (without changing the excess) which (locally) parametrizes the same Lagrangian. The argument is similar to the classical one, but extra attention needs to be paid at to what happens near points with $\rho_Y = 0$, namely, we never seek to get rid of ρ_Y as a parameter.

Remark 6.4.2. In the classical theory, meaning for homogeneous phase functions, it is possible to reduce the number of fiber variables under the assumption that the matrix $\partial^2_{\theta\theta}\varphi(x,\theta)$ has rank $r > 0$ on C_φ. However, since a classical phase function φ is homogeneous in θ, it holds that $\theta \cdot \nabla_\theta \varphi = \varphi$ and hence the second radial derivative is automatically zero on C_φ. Furthermore, the radial variable can always be chosen to parametrize Λ_φ.

We proceed as in the proof of Theorem 6.3.11. We first recall that, for $p_0 \in C_\varphi$, writing $\varphi = \rho_Y^{-1}\rho_X^{-1}f$ with $f \in C^\infty(X \times \mathbb{B}^s)$, we have there

$$0 = {}^{\mathrm{sc}}d_Y\varphi = (-f + \rho_Y\partial_{\rho_Y}f, \partial_{y_k}f). \tag{6.28}$$

We then identify $T_Y{}^{\mathrm{sc}}d_Y\varphi$ in coordinates with the matrix

$$J_Y\varphi = \begin{pmatrix} \rho_Y\partial^2_{\rho_Y}f & -\partial_{y_j}f + \rho_Y\partial_{y_j}\partial_{\rho_Y}f \\ \partial_{\rho_Y}\partial_{y_k}f & \partial_{y_j}\partial_{y_k}f \end{pmatrix}. \tag{6.29}$$

We see, using (6.28), that on $\mathcal{C}_\varphi^\psi \subset \{\rho_Y = 0\}$ this becomes

$$J_Y\varphi\big|_{\mathcal{C}_\varphi^\psi} = \begin{pmatrix} 0 & 0 \\ \partial_{\rho_Y}\partial_{y_k} f & \partial_{y_j}\partial_{y_k} f \end{pmatrix}. \tag{6.30}$$

Therefore, the rank of this matrix is at most $s-1$. Indeed, we observe that, by (6.13), at $\rho_Y = 0$ we have $d\rho_Y \neq 0$ on TC_φ^ψ and hence we can always choose ρ_Y as a parameter to locally describe C_φ^ψ.

Remark 6.4.3. By the same argument, ρ_X can be chosen as a parameter close to $\mathcal{B}^e$, while, close to $\mathcal{B}^{\psi e}$, both ρ_X and ρ_Y can be chosen as parameters to represent C_φ.

We now seek to reduce the remaining set of variables under the assumption that

$$\text{The matrix } \left(\partial_{y_j}\partial_{y_k}\rho_X\rho_Y\varphi\right)_{jk} \text{ has rank } r > 0 \text{ at } p_0 \in \mathcal{C}_\varphi^\psi \cup \mathcal{C}_\varphi^{\psi e}. \tag{6.31}$$

Since at points where $\rho_Y \neq 0$ the variable ρ_Y behaves like all other variables, the same restriction does not hold near a point $p \in \mathcal{C}_\varphi^e$. Here, we simply assume that

$$\text{The matrix } T_Y{}^{\mathrm{sc}}d_Y\varphi \text{ has rank } r > 0 \text{ at } p_0 \in \mathcal{C}_\varphi^e. \tag{6.32}$$

Since up to multiplication by $\rho_Y > 0$ in one row, (6.29) is the Hessian of h (with respect to $\mathbf{y}$), this is equivalent to $\mathrm{rk}(H_Y f) = r > 0$. The two conditions may be summarized into one. Namely, consider the scattering Hessian (with respect to the $\mathbf{y}$-variables) of φ

$$\begin{aligned} {}^{\mathrm{sc}}H_Y\varphi &= \begin{pmatrix} \rho_Y^2\rho_X\partial_{\rho_Y}\rho_Y^2\rho_X\partial_{\rho_Y}\varphi & \rho_Y\rho_X\partial_{y_j}\rho_Y^2\rho_X\partial_{\rho_Y}\varphi \\ \rho_Y^2\rho_X\partial_{\rho_Y}\rho_Y\rho_X\partial_{y_k}\varphi & \rho_Y\rho_X\partial_{y_j}\rho_Y\rho_X\partial_{y_k}\varphi \end{pmatrix} \\ &= \rho_Y\rho_X \begin{pmatrix} \rho_Y^2\partial_{\rho_Y}^2 f & -\partial_{y_j} f + \rho_Y\partial_{y_j}\partial_{\rho_Y} f \\ \rho_Y\partial_{\rho_Y}\partial_{y_k} f & \partial_{y_j}\partial_{y_k} f \end{pmatrix}. \end{aligned} \tag{6.33}$$

Then $\rho_Y^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_Y\varphi$ becomes, at a point in $\mathcal{C}_\varphi$:

$$\rho_Y^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_Y\varphi = \begin{pmatrix} 0 & 0 \\ 0 & \partial_{y_j}\partial_{y_k} f \end{pmatrix}, \quad \text{if } p_0 \in \mathcal{C}_\varphi^\psi \cup \mathcal{C}_\varphi^{\psi e};$$

$$\rho_Y^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_Y\varphi = \begin{pmatrix} \rho_Y^2\partial_{\rho_Y}^2 f & \rho_Y\partial_{y_j}\partial_{\rho_Y} f \\ \rho_Y\partial_{\rho_Y}\partial_{y_k} f & \partial_{y_j}\partial_{y_k} f \end{pmatrix}, \quad \text{if } p_0 \in \mathcal{C}_\varphi^e.$$

Notice that we can factorize these matrices as

$$\begin{pmatrix} \rho_Y & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \partial_{\rho_Y}^2 f & \partial_{y_j}\partial_{\rho_Y} f \\ \partial_{\rho_Y}\partial_{y_k} f & \partial_{y_j}\partial_{y_k} f \end{pmatrix} \begin{pmatrix} \rho_Y & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{6.34}$$

the rank of which therefore is, for $\rho_Y \neq 0$, that of the standard Hessian of f, $H_Y f$. Therefore, our assumption may be expressed as:

$$\text{The matrix } \rho_Y^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_Y\varphi \text{ has rank } r > 0 \text{ at } p_0 \in \mathcal{C}_\varphi. \tag{6.35}$$

We may now proceed as in the standard theory and introduce a splitting of variables $\mathbf{y} = (\mathbf{y}', \mathbf{y}'')$ such that $(\partial_{\mathbf{y}''}\partial_{\mathbf{y}''} f)_{jk}$ is an invertible $r \times r$ matrix. We can then apply the implicit function theorem to

$$0 = {}^{\mathrm{sc}}d_Y \varphi = (-f + \rho_Y \partial_{\rho_Y} f, \partial_{y_k} f)$$

at p_0. We obtain a map from an open neighbourhood of p_0,

$$k : (\mathbf{x}, \mathbf{y}') \mapsto \big(\mathbf{x}, \mathbf{y}', \mathbf{y}''(\mathbf{x}, \mathbf{y}')\big),$$

such that C_φ and the range of k locally coincide. Note that k is a scattering map, since ρ_Y is always one of the $\mathbf{y}'$ near the ψ-face.

Then $\varphi_{\mathrm{red}} = \varphi \circ k$ is a clean local phase function in $d \times (s-r)$ variables with excess e, and k provides a local isomorphism $C_{\varphi_{\mathrm{red}}} \to C_\varphi$. Furthermore, at stationary points p_0 and $k(p_0)$, we have that $\iota({}^{\mathrm{sc}}d_X \varphi_{\mathrm{red}}) = \iota({}^{\mathrm{sc}}d_X \varphi)$, since ${}^{\mathrm{sc}}d_Y \varphi = 0$ there. Hence, φ_{red} locally parametrizes the same Lagrangian as φ.

Remark 6.4.4. Note that, after applying a change of coordinates in the $\mathbf{y}$ variables, φ_{red} may be assumed to be defined on $\mathbb{B}^d \times \mathbb{B}^{s-r}$, see also Lemma 6.4.7 below.

Summing up, we can formulate the next Proposition 6.4.5.

Proposition 6.4.5. *Let $\varphi \in \rho_Y^{-1}\rho_X^{-1}\mathcal{C}^\infty(X \times \mathbb{B}^s)$ be a local clean phase function of excess e. Assume*

$$\rho_Y^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_Y\varphi \textit{ has rank } r > 0 \textit{ at a stationary boundary point } p_0 \in \mathcal{C}_\varphi.$$

We may then define a local phase function $\varphi \in \rho_Y^{-1}\rho_X^{-1}\mathcal{C}^\infty(X \times \mathbb{B}^{s-r})$ of excess e parametrizing the same Lagrangian.

We mention that, locally, the minimal number of fiber variables y that a clean phase function of excess e locally parametrizing L_φ has to possess is

$$s_{\min} = d + e - n,$$

where n is the (local) number of independent x variables on L_φ. This follows from a simple dimension argument: the dimension of L_φ is d, that of C_φ is $d+e$, and the one of the projection to x of C_φ coincides with that of L_φ. Note that, by cleanness of the intersection $\mathcal{C}_\varphi \cap \mathcal{B}^\psi$, near Λ^ψ we have $s_{\min} > 0$.

6.4.3. Increasing the Excess

Given a (local) clean phase function $\varphi \in \rho_X^{-1}\rho_{\mathbb{B}^s}^{-1}C^\infty(X \times \mathbb{B}^s)$ with excess e, define $\psi := \mathrm{pr}^*_{X\times\mathbb{B}^s}\varphi$ on $X \times (\mathbb{B}^s \times (-\varepsilon, \varepsilon))$, viewing $\mathbb{B}^s \times (-\varepsilon, \varepsilon)$ as an open subset of $\mathbb{B}^s \times \mathbb{S}^1$, which is a manifold with boundary whose boundary defining function may be chosen as $\mathrm{pr}^*_{\mathbb{B}^s}\rho_{\mathbb{B}^s}$. In particular we have, with the obvious identifications,

$${}^{\mathrm{sc}}d_{\mathbb{B}^s\times(-\varepsilon,\varepsilon)}\psi = \mathrm{pr}^*_{X\times\mathbb{B}^s}\left({}^{\mathrm{sc}}d_{\mathbb{B}^s}\varphi\right).$$

Then $C_\psi = C_\varphi \times (-\varepsilon, \varepsilon)$ and hence $\dim(C_\psi^\bullet) = \dim(C_\varphi^\bullet) + 1$. Furthermore, $\lambda_\psi = \mathrm{pr}^*_{X\times\mathbb{B}^s}\lambda_\varphi$ and $\Lambda_\varphi = \Lambda_\psi$. Summing up, ψ is a local clean phase function in $s+1$ fiber variables with excess $e+1$, defined and (locally) parametrizing the same Lagrangian as φ.

As before, we may choose to keep working on balls by invoking the construction from Example 6.2.32 and replacing ψ with

$$\Psi^*\psi = \widetilde{\Psi}^*\varphi \in \rho_X^{-1}\rho_{\mathbb{B}^{s+1}}^{-1}\mathcal{C}^\infty(X \times U).$$

In this way, since Ψ is a diffeomorphism, ψ becomes a clean phase function with excess $e+1$ defined on a relatively open subset of $X \times \mathbb{B}^{s+1}$ and similarly we may raise the excess by any natural number.

Example 6.4.6. The standard Fourier phase on $\mathbb{R}\times\mathbb{R}$, $\varphi(x,\xi) = x\cdot\xi$, cannot be seen as an SG-phase on all of $\mathbb{R}\times\mathbb{R}^2$ by setting $\psi(x,\xi,\eta) = x\cdot\xi$. Indeed,

$$\begin{aligned}\langle x\rangle^2|\nabla_x\varphi(x)|^2 + \langle(\xi,\eta)\rangle|\nabla_{\xi,\eta}\varphi|^2 &= (1+x^2)\xi^2 + (1+\xi^2+\eta^2)x^2 \\ &= \langle x\rangle\langle\xi\rangle + x^2\eta^2 - 1\end{aligned} \tag{6.36}$$

For $\xi = 0$ and $x = 0$ and $\eta \to \infty$, this vanishes but should be bounded from below by $c(1+|\eta|)^2$ if ψ were an SG-phase function, given (6.11).

Reviewing Example 6.2.32, the ray $\xi = 0$, $x = 0$ and $\eta \neq 0$ corresponds precisely to the poles in Figure 6.1 which were cut off. Indeed, (6.36) is bounded from below by $\langle x\rangle^2\langle(\xi,\eta)\rangle^2$ in any neighbourhood where $\frac{|\xi|}{|\eta|} > c$ and hence a local phase function in such sets.

6.4.4. Elimination of Excess

Assume now that φ is a phase function on $X \times \mathbb{B}^s$ with excess e and that at some point $p_0 = (\rho_{X,0}, x_0, \rho_{Y,0}, y_0) \in \mathcal{C}_\varphi$ we have $\lambda_\varphi(p_0) = (\rho_{X,0}, x_0, \rho_{\Xi,0}, \xi_0)$. Then, by Lemma 6.3.12, the preimage of $(\rho_{X,0}, x_0, \rho_{\Xi,0}, \xi_0)$ under λ_φ, meaning the fiber in $\mathcal{C}_\varphi$ through p_0, is an e-dimensional smooth submanifold. Locally, since λ_φ is a submersion we may, by [33, Prop. 5.1], reduce to the case of a projection, that is, we may find a splitting $y = (y', y'')$ near p_0 such that λ_φ does not depend on y''. Then,

$$\tilde{\varphi}(\rho_X, x, \rho_Y, y') := \varphi(\rho_X, x, \rho_Y, y', y''_0)$$

defines a phase function without excess (i.e., a non-degenerate phase function) that parametrizes the same Lagrangian as φ. As usual, we may again reduce to the case of a ball and hence replace φ by a phase function on an open subset of $X \times \mathbb{B}^{s-e}$.

6.4.5. Equivalence of Phase Functions

We will now discuss the changes of phase function under a change of coordinates and which phase functions can be considered equivalent. We first check how the stationary points of a phase function transform under changes by local diffeomorphisms.

Lemma 6.4.7. *Let X_1, Y_1, X_2, Y_2 be mwbs, set $B_i = X_i \times Y_i$, $i \in \{1,2\}$, and let $\varphi \in \rho_{X_2}^{-1}\rho_{Y_2}^{-1}C^\infty(B_2)$ be a (local) phase function. Assume $g : X_1 \to X_2$, $h : Y_1 \to Y_2$ to be diffeomorphisms, and set $F = g \times h$. Then, $F^*\varphi \in \rho_{X_1}^{-1}\rho_{Y_1}^{-1}C^\infty(B_1)$ is a (local) phase function with the same excess of φ, and we have*

$$C_{F^*\varphi} = \{(\mathbf{x}_1, \mathbf{y}_1) \in B_1 \,|\, F(\mathbf{x}_1, \mathbf{y}_1) \in C_\varphi\}, \quad L_{F^*\varphi} = ({}^{\mathrm{sc}}\overline{T}^* g)(L_\varphi).$$

Remark 6.4.8. This means that, while the boundary defining function ρ_{Ξ_1} of ${}^{\mathrm{sc}}\overline{T}^* X_1$ does not vanish, $L_{F^*\varphi}$ can then be computed as

$$L_{F^*\varphi} = \{(\mathbf{x}_1,\, \iota({}^t(Jg)\iota^{-1}(\boldsymbol{\xi}_1)) \in {}^{\mathrm{sc}}\overline{T}^* X_1 \mid (g(\mathbf{x}_1), \boldsymbol{\xi}_1) \in L_\varphi\}.$$

As $\rho_\Xi \to 0$, $\Lambda^\psi_{F^*\varphi}$ is obtained by taking interior limits, see also Lemma 6.2.29.

Proof of Lemma 6.4.7. The result for C_φ follows immediately from the first assertion in Lemma 6.2.18. The statement for L_φ then follows by writing

$$\lambda_{F^*\varphi}(\mathbf{x}_1, \mathbf{y}_1) = ({}^{\mathrm{sc}}\overline{T}^* g)(\lambda_\varphi(\mathbf{x}_2, \mathbf{y}_2)) \tag{6.37}$$

near a point $(\mathbf{x}_1, \mathbf{y}_1) \in (C_{F^*\varphi})^o$ such that $(\mathbf{x}_2, \mathbf{y}_2) = (g(\mathbf{x}_1), h(\mathbf{x}_1, \mathbf{y}_1))$. Indeed, at these stationary points, ${}^{\mathrm{sc}}d_X F^*\varphi = F^*({}^{\mathrm{sc}}d_X\varphi)$, since there ${}^{\mathrm{sc}}d_Y\varphi = 0$. Since equality (6.37) holds in the interior, the result at the boundary faces can be obtained as interior limits (see also Lemma 6.3.9). $\square$

Remark 6.4.9. The diffeomorphism $g \times h$ may be replaced by a single diffeomorphism $F : X_1 \times Y_1 \to X_2 \times Y_2$ locally of product type near the boundary faces of $X_2 \times Y_2$, i.e., a (local) diffeomorphism that is a fibered-map at the boundary.

We now define in which sense two phase functions may be considered equivalent.

Definition 6.4.10. Let X, Y_1, Y_2 be mwbs, $B_i = X \times Y_i$. Let $\varphi_i \in \rho_X^{-1}\rho_{Y_i}^{-1}C^\infty(B_i)$. We say that φ_1 and φ_2 are equivalent at a pair of boundary points $(\mathbf{x}^0, \mathbf{y}_1^0) \in \mathcal{B}_1$ and $(\mathbf{x}^0, \mathbf{y}_2^0) \in \mathcal{B}_2$ if there exists a local diffeomorphism $F : X \times Y_2 \to X \times Y_1$ of the form $F = \mathrm{id} \times g$ with $g(\mathbf{x}^0, \mathbf{y}_2^0) = \mathbf{y}_1^0$ such that the following two conditions are met:

$$F^*\varphi_1 - \varphi_2 \text{ is smooth in a neighbourhood } U \text{ of } (\mathbf{x}^0, \mathbf{y}_2^0), \tag{6.38}$$

$$\rho_X\rho_{Y_2}\left(F^*\varphi_1 - \varphi_2\right) \text{ restricted to } \mathcal{C}_{\varphi_2} \cap \partial U \text{ vanishes to second order.} \tag{6.39}$$

Lemma 6.4.11. *Equivalent phase functions parametrize the same Lagrangian, meaning $\Lambda_{F^*\varphi} = \Lambda_\varphi$ and we have $\mathcal{C}_{F^*\varphi_1} = \mathcal{C}_{\varphi_2}$.*

Proof. This follows from Lemmas 6.4.1 and 6.4.7. $\square$

We now associate to any local phase function its *principal phase part*, which corresponds in the SG-case to the leading homogeneous components of φ. From the fact that the principal part of Definition 6.2.7 is obtained from the boundary restrictions of φ, we observe, using $F = \mathrm{I} \times \mathrm{I}$ and Lemma 6.2.8:

Lemma 6.4.12. *A local phase function φ and its principal part φ_p are equivalent.*

Remark 6.4.13. In particular, each phase function is locally equivalent at the e- and ψ-face, respectively, to a homogeneous (w.r.t. ρ_X or ρ_Y) phase function, after a choice of collar decomposition. In general, this is not true near the corner $\mathcal{B}^{\psi e}$.

Since the difference in condition (6.39) is restricted to the boundary, it does not restrict the behavior of $F^*\varphi_1 - \varphi_2$ into the direction transversal to the boundary, e.g. $\partial_{\rho_X} \rho_X \rho_{Y_2}(F^*\varphi_1 - \varphi_2)$ at $\mathcal{C}^e_{\varphi_2}$. The following lemma states the transformation behavior of this directional derivative.

Lemma 6.4.14. *Let X, Y_1, Y_2 be mwbs and let $F : X \times Y_2 \to X \times Y_1$ be a* sc*-map of the form $F = \mathrm{I} \times \Psi$. Set $h = \rho_{Y_2}^{-1} F^* \rho_{Y_1}$. Consider a clean phase function φ on $X \times Y_1$. Write $f = \rho_X \rho_{Y_2} \varphi$. Then we have the following transformation laws:*

$$\begin{aligned} hF^*\partial_{\rho_{Y_1}} \rho_X^{-1} f &= \partial_{\rho_{Y_2}} F^* \rho_X^{-1} f, \quad \text{on } F^*\mathcal{C}^\psi_\varphi, \\ F^*\rho_{Y_1}^{-1} \partial_{\rho_X} f &= \partial_{\rho_X} F^* \rho_{Y_1}^{-1} f, \quad \text{on } F^*\mathcal{C}^e_\varphi. \end{aligned}$$

Proof. On $F^*\mathcal{C}^\psi_\varphi$, we have that

$$\partial_{\rho_{Y_2}} F^* f = hF^*\partial_{\rho_{Y_1}} f + F^*(\partial_{y_1} f)\partial_{\rho_{Y_2}} y_1 = hF^*\partial_{\rho_{Y_1}} f,$$

where we have used $\partial_{y_1} f = 0$ on $F^*\mathcal{C}^\psi_\varphi$. This proves the first equality.

On $F^*\mathcal{C}^e_\varphi$, we compute

$$\begin{aligned} \partial_{\rho_X} F^* \rho_{Y_1}^{-1} f_1 &= F^*\rho_{Y_1}^{-1} \partial_{\rho_X} f_1 + F^*(\partial_{\rho_{Y_1}} \rho_{Y_1}^{-1} f_1)\, \partial_{\rho_X} F^*\rho_{Y_1} + F^*(\rho_{Y_1}^{-1} \partial_{y_1} f_1)\, \partial_{\rho_X} F^* y_1 \\ &= \rho_{Y_2}^{-1} h^{-1} F^* \partial_{\rho_X} f_1. \end{aligned}$$

Therein, we used $\partial_{y_1} f_1 = 0$ and $\partial_{Y_1} \rho_{Y_1}^{-1} f_1 = 0$ on $\mathcal{C}_{\varphi_1}$. ☐

Remark 6.4.15. The previous lemma, combined with Lemma 6.4.12, will imply that, away from the corner, any phase function can be replaced by an equivalent phase function without radial derivative (at $\mathcal{C}_\varphi$) and the vanishing of this derivative at $\mathcal{C}_\varphi$ is preserved under application of scattering maps.

This corresponds to the fact that, in the classical theory, one can always choose a homogeneous phase function. The (non-homogeneous) terms of lower order which arise in transformations can be absorbed into the amplitude.

The rest of this section will be dedicated to establishing a necessary and sufficient criterion for the local equivalence of phase functions.

Lemma 6.4.16. *Let X, Y_1, Y_2 be mwbs such that $\dim(Y_1) = \dim(Y_2)$, and set $B_i = X \times Y_i$, $i \in \{1, 2\}$. Let $\varphi_i \in \rho_X^{-1}\rho_{Y_i}^{-1}C^\infty(B_i)$ be phase functions which have the same excess, and assume that there exist $p_i^0 = (\mathbf{x}^0, \mathbf{y}_i^0) \in \mathcal{C}_{\varphi_i}$, $i \in \{1, 2\}$, such that*

$$\lambda_{\varphi_1}(\mathbf{x}^0, \mathbf{y}_1^0) = \lambda_{\varphi_2}(\mathbf{x}^0, \mathbf{y}_2^0),$$

and, close to $(\mathbf{x}^0, \mathbf{y}_i^0)$, $i \in \{1, 2\}$, both phases parametrize the same Lagrangian Λ, i.e., locally $\Lambda = \Lambda_{\varphi_i}$, $i \in \{1, 2\}$. Then, there exists a local diffeomorphism $F\colon B_2 \to B_1$ of the form $F = \mathrm{I} \times g$ with $F(\mathbf{x}^0, \mathbf{y}_2^0) = (\mathbf{x}^0, \mathbf{y}_1^0)$, such that $F^*\varphi_1 = \rho_X \rho_{Y_2} \widetilde{f}_1$ with $\mathcal{C}_{F^*\varphi_1} = \mathcal{C}_{\varphi_2}$, locally. Moreover, locally near $(\mathbf{x}^0, \mathbf{y}_2^0)$,

$$(f_2 - \widetilde{f}_1)|_{\mathcal{B}_2} \text{ vanishes of second order at any point of } \mathcal{C}_{\varphi_2}. \tag{6.40}$$

Remark 6.4.17. Notice that (6.40) means that the principal part of $F^*\varphi_1$ and φ_2 in Lemma 6.4.16 coincide on $\mathcal{C}_{\varphi_2}$.

Proof of Lemma 6.4.16. Since λ_{φ_i} are local fibrations from $\mathcal{C}_{\varphi_i}$ to Λ_{φ_i}, $i \in \{1, 2\}$, and $\Lambda_{\varphi_1} = \Lambda_{\varphi_2} = \Lambda$, there is a local fibered diffeomorphism $F\colon B_2 \to B_1$ of the form $F = \mathrm{I} \times g$, locally locally near $(\mathbf{x}^0, \mathbf{y}_1^0) = F(\mathbf{x}^0, \mathbf{y}_2^0)$, such that the following diagram is commutative.

$$\begin{tikzcd} & \Lambda & \\ \mathcal{C}_{\varphi_2} \arrow[ur, "\lambda_{\varphi_2}"] \arrow[rr, "\exists F"] & & \mathcal{C}_{\varphi_1} \arrow[ul, "\lambda_{\varphi_1}"'] \end{tikzcd}$$

Note that F is not uniquely determined, not even on $\mathcal{C}_{\varphi_2}$ when the phases are merely clean and not non-degenerate.

After application of F, we may assume that $Y_1 = Y_2 =: Y$, $\mathbf{y}_1^0 = \mathbf{y}_2^0 =: \mathbf{y}^0$ and, locally, $\mathcal{C}_{\varphi_1} = \mathcal{C}_{\varphi_2} =: \mathcal{C}_\varphi$. We now show that the restriction of f_1 and f_2 to a relative neighbourhood of $(\mathbf{x}^0, \mathbf{y}^0)$ in $\mathcal{C}_\varphi$ vanishes of second order. Recall that, since ${}^{\mathrm{sc}}d_Y\varphi_1 = {}^{\mathrm{sc}}d_Y\varphi_2 = 0$, for any $p = (\mathbf{x}, \mathbf{y}) \in \mathcal{C}_\varphi$ we have

$$\big(\rho_Y \partial_{\rho_Y} f_1 - f_1 \quad \partial_{y_k} f_1\big) = \big(\rho_Y \partial_{\rho_Y} f_2 - f_2 \quad \partial_{y_k} f_2\big) = 0 \tag{6.41}$$

Furthermore, since φ_1 and φ_2 parametrize the same Lagrangian, we also have $\lambda_{\varphi_1}(p) = \lambda_{\varphi_2}(p)$, that is, $\iota({}^{\mathrm{sc}}d_X\varphi_1(p)) = \iota({}^{\mathrm{sc}}d_X\varphi_2(p))$. We treat separately the cases $p \in \mathcal{C}_\varphi^e$ and $p \in \mathcal{C}_\varphi^\psi \cup \mathcal{C}_\varphi^{\psi e}$.

If $p \in \mathcal{C}_\varphi^e$, we then find

$$\iota((\rho_Y^{-1}\rho_X\partial_{\rho_X} f_1(p) - f_1(p), \rho_Y^{-1}\partial_{x_k} f_1(p))) = \iota((\rho_Y^{-1}\rho_X\partial_{\rho_X} f_2(p) - f_2(p), \rho_Y^{-1}\partial_{x_k} f_2(p))). \tag{6.42}$$

Since $\rho_Y \neq 0$ on $\mathcal{C}_\varphi^e$, and ι is a diffeomorphism on the interior, this implies

$$f_1(p) = f_2(p), \quad \partial_{x_k} f_1(p) = \partial_{x_k} f_2(p),\ k = 1, \ldots, d-1.$$

Combining this with (6.41), this further implies

$$\partial_{\rho_Y} f_1(p) = \partial_{\rho_Y} f_2(p), \quad \partial_{y_k} f_1(p) = \partial_{y_k} f_2(p),\ k = 1, \dots, s-1.$$

Since $(x, \mathbf{y})$ are a complete set of variables on $\mathcal{B}^e$, we can indeed conclude that $f_1 - f_2$ vanishes of second order along $\mathcal{C}^e_\varphi$.

If $p \in \mathcal{C}^\psi_\varphi$ or $p \in \mathcal{C}^{\psi e}_\varphi$, (6.41) implies that

$$f_1(p) = f_2(p) = 0, \quad \partial_{y_k} f_1(p) = \partial_{y_k} f_2(p),\ k = 1, \dots, s-1.$$

We have to evaluate (6.42) as a limit $\rho_Y \to 0^+$, using, as in Lemma 6.2.29, $\iota(z) = \frac{z}{|z|}(1 - \frac{1}{|z|})$. We obtain that, with

$$v_1 = (\rho_X \partial_{\rho_X} f_1, \partial_{x_k} f_1), \quad v_2 = (\rho_X \partial_{\rho_X} f_2, \partial_{x_k} f_2),$$

$\frac{v_1}{\|v_1\|} = \frac{v_2}{\|v_2\|}$, but not necessarily $v_1 = v_2$, in which case the proof would be complete. We now modify F in order to achieve $v_1 = v_2$. Notice that, since φ_1 and φ_2 are phase functions, we have $v_1 \neq 0$ at $\mathcal{C}_\varphi$. We can therefore scale φ_1 by means of the local diffeomorphism (near $\mathcal{C}_\varphi$)

$$\widetilde{F} \colon (\rho_Y, y) \to (\rho_Y\, r(\rho_X, x, \rho_Y, y), y),$$

where $r(\rho_X, x, \rho_Y, y) = \frac{\|v_2\|}{\|v_1\|}$. Notice that, by our previous computations, $r|_{\mathcal{C}^e_\varphi \cup \mathcal{C}^{\psi e}_\varphi} = 1$, and $\widetilde{F}$ is the identity for $\rho_Y = 0$. Therefore, by Lemma 6.4.7,

$$\mathcal{C}_{\widetilde{F}^* \varphi_1} = \mathcal{C}_{\varphi_1}, \quad \text{and} \quad \Lambda_{\widetilde{F}^* \varphi_1} = \Lambda_{\varphi_1}.$$

By definition, for $\widetilde{F}^* \varphi_1$ we have

$$\widetilde{f}_1 := \rho_X \rho_Y \widetilde{F}^* \varphi_1 = \frac{\|v_2\|}{\|v_1\|} (F^* f_1).$$

Therefore,

$$(\rho_X \partial_{\rho_X} \widetilde{f}_1, \partial_{x_k} \widetilde{f}_1) = \frac{\|v_2\|}{\|v_1\|} \cdot (\rho_X F^*(\partial_{\rho_X} f_1), F^*(\partial_{x_k} \widetilde{f}_1)) =: \widetilde{v}_1,$$

since the derivatives acting on r produce a ρ_Y factor, and then vanish along $\mathcal{C}^\psi_\varphi$. Hence, $\widetilde{v}_1 = v_2$, which completes the proof. $\square$

Remark 6.4.18. The additional computations in the proof of the previous lemma near the face $\mathcal{C}^\psi_\varphi$ correspond to the fact that, classically, $x \cdot \theta$ and $x \cdot (2\theta)$ both parametrize

$$\Lambda = \left\{ (0, \xi) \mid \xi \in \mathbb{R}^d \setminus \{0\} \right\}.$$

In fact, we observe from the same proof that we may choose the norm of $(\rho_X \partial_{\rho_X} f_1, \partial_{x_k} f_1)$ at any point of Λ^ψ_φ without changing Λ_φ.

Theorem 6.4.19 (Equivalence of phase functions)**.** *Let X, Y_1, Y_2 be mwbs such that* $\dim(Y_1) = \dim(Y_2)$*, and set $B_i = X \times Y_i$, $i \in \{1,2\}$. Let $\varphi_i \in \rho_X^{-1}\rho_{Y_i}^{-1}C^\infty(B_i)$, $i \in \{1,2\}$, be phase functions which have the same excess, assume that there exist $(\mathbf{x}^0, \mathbf{y}_i^0) \in \mathcal{C}_{\varphi_i}$, $i \in \{1,2\}$, such that*

$$\lambda_{\varphi_1}(\mathbf{x}^0, \mathbf{y}_1^0) = \lambda_{\varphi_2}(\mathbf{x}^0, \mathbf{y}_2^0),$$

and, close to $(\mathbf{x}^0, \mathbf{y}_i^0)$, $i \in \{1,2\}$, both phase functions parametrize the same Lagrangian Λ, i.e., locally $\Lambda = \Lambda_{\varphi_i}$, $i \in \{1,2\}$. Then, it is necessary and sufficient for φ_1 and φ_2 to be equivalent at $(\mathbf{x}^0, \mathbf{y}_1^0)$ and $(\mathbf{x}^0, \mathbf{y}_2^0)$ that there it holds that

$$\operatorname{sgn}\left(\rho_{Y_1}^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_{Y_1}\varphi_1\right) = \operatorname{sgn}\left(\rho_{Y_2}^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_{Y_2}\varphi_2\right). \tag{6.43}$$

Remark 6.4.20. Before we go into the details of the proof, we recall the expression for the differential in condition (6.43) in coordinates. By (6.34) we have, writing $\varphi = \rho_X^{-1}\rho_Y^{-1}f$,

$$\rho_Y^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_Y\varphi = \begin{pmatrix} \rho_Y & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \partial_{\rho_Y}^2 f & \partial_{y_j}\partial_{\rho_Y} f \\ \partial_{\rho_Y}\partial_{y_k} f & \partial_{y_j}\partial_{y_k} f \end{pmatrix} \begin{pmatrix} \rho_Y & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

Hence, for $\rho_Y \neq 0$, the signature of this matrix is that of $H_Y f$, whereas for $\rho_Y = 0$ it is that of the Hessian of f *restricted to* $\rho_Y = 0$, that is, only with respect to the boundary variables, $\left(\partial_{y_j}\partial_{y_k} f(0,y)\right)_{jk}$.

Proof of Theorem 6.4.19. We first prove that condition (6.43) is necessary. In view of Lemma 6.4.11, we only need to compare ${}^{\mathrm{sc}}H_{Y_1}\varphi_1$ and ${}^{\mathrm{sc}}H_{Y_2}\varphi_2$ by writing

$${}^{\mathrm{sc}}H_{Y_2}\varphi_2 = {}^{\mathrm{sc}}H_{Y_2}F^*\varphi_1 + {}^{\mathrm{sc}}H_{Y_2}(\varphi_2 - F^*\varphi_1). \tag{6.44}$$

We write $r = (\varphi_2 - F^*\varphi_1)$, which, by assumption, satisfies $r \in \mathcal{C}^\infty(X \times Y_2)$. Therefore, $\rho_{Y_2}^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_{Y_2}r$ vanishes at the boundary. Indeed, in local coordinates we have

$$\rho_Y^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_{Y_2}r = \begin{pmatrix} \rho_Y\rho_X\partial_{\rho_Y}\rho_Y^2\partial_{\rho_Y}r & \rho_Y^2\rho_X\partial_{y_j}\partial_{\rho_Y}r \\ \rho_Y\rho_X\partial_{\rho_Y}\rho_Y\partial_{y_k}r & \rho_Y\rho_X\partial_{y_j}\partial_{y_k}r \end{pmatrix}.$$

Thus, we have, at the boundary,

$$\operatorname{sgn}\left(\rho_{Y_2}^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_{Y_2}F^*\varphi_1\right) = \operatorname{sgn}\left(\rho_{Y_2}^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_{Y_2}\varphi_2\right). \tag{6.45}$$

By computing these differentials in coordinates at corresponding stationary points, using (6.34), this implies (6.43).

For the sufficiency of (6.43), we assume familiarity of the reader with the equivalence of phase function theorem in the usual homogeneous setting, see [61, Prop. 4.1.3], [61, Prop. 4.1.3] and sketch briefly that the argument goes through with little modification.

By Lemma 6.4.16 we may assume $Y_1 = Y_2$. Note that equivalence is achieved for $\varphi_i = \rho_X\rho_Y f_i$ if the f_i agree on the boundary. The condition on ${}^{\mathrm{sc}}H_Y\varphi_i$ means precisely that the signatures of the Hessians of the f_i in the tangential derivatives agree in the interior and the signatures of the Hessians of the restriction of the f_i to $\rho_Y = 0$ as well, see Remark 6.4.20. As such, we may use the same techniques as in the classical situation to construct a diffeomorphism on the boundary which transforms the restriction of f_1 into that of f_2, cf. also [7]. This diffeomorphism is then extended by means of Proposition 6.2.25 into the interior. For sake of brevity, we omit the details here.

□

Remark 6.4.21. Note that near $(\mathbf{x}^0, \mathbf{y}^0) \in \mathcal{C}^{\psi}_{\varphi}$, we can also invoke the classical equivalence theorem directly. We need to find a transformation

$$F : (\mathbf{x}, 0, y) \mapsto (\mathbf{x}, 0, \tilde{y}(\mathbf{x}, y))$$

such that $F^*\varphi_1 = \varphi_2$. For $\lambda > 0$ we set $\phi_j(\mathbf{x}, \lambda, y) = \lambda f_j(\mathbf{x}, 0, y)$, $j \in \{1, 2\}$. Then ϕ_j are equivalent *phase functions in the usual homogeneous sense* on $X \times (\mathbb{R}_+ \times Y)$. Indeed, evaluating $d\phi_j$ and ${}^{\mathrm{sc}}d\varphi_j$ in coordinates, we see that $d\phi_j \neq 0$ and ϕ_j is manifestly homogeneous. Furthermore, the signatures of $H_Y\phi_j$ are the same as those of ${}^{\mathrm{sc}}H_Y\varphi_j$. Since the f_j are equal up to second order, the ϕ_j are equivalent in the usual sense and there exists a λ-homogeneous $G : (\mathbf{x}, \lambda, y) \mapsto (\mathbf{x}, \lambda, \tilde{y}(\lambda, \mathbf{x}, y))$ which is homogeneous such that $G^*\phi_1 = \phi_2$. Setting $F = G|_{\lambda=1}$ and possibly applying a scaling, as in the proof of Lemma 6.4.16, concludes the proof for $(\mathbf{x}^0, \mathbf{y}^0) \in \mathcal{C}^{\psi}_{\varphi}$.

6.5. Lagrangian Distributions

In this section, we will address the class of Lagrangian distributions on scattering manifolds. First, we introduce oscillatory integrals associated with a phase function and show that they are well-defined in the usual sense. Then, we define Lagrangian distributions as a locally finite sum of oscillatory integrals, where the phase function parametrizes a Lagrangian submanifold. Using the results from the previous section, we are able to reduce the number of fiber-variables to a minimum and see that the order of the Lagrangian distribution is well-defined independently of the dimension of the fiber.

6.5.1. Oscillatory Integrals Associated with a Phase Function

Definition 6.5.1. Let Y be a mwb. For the remainder of this section, m_ε with $\varepsilon \in (0, 1]$, denotes a family of functions $m_\varepsilon \in \dot{\mathscr{C}}^\infty_0(Y)$ such that for all $k \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^{d-1}$ and $\epsilon > 0$,

$$\left| (\rho_Y^2 \partial_{\rho_Y})^k (\rho_Y \partial_y)^\alpha m_\varepsilon(\mathbf{y}) \right| \leq C_{k,\alpha}\, \rho_Y^{k+|\alpha|}, \tag{6.46}$$

such that, for all $\mathbf{y} \in Y^o$, we have $m_\varepsilon(\mathbf{y}) \to 1$ as $\varepsilon \to 0$.

Remark 6.5.2. We make the observation that (6.46) does not depend on the choice of bdf and is preserved under pullbacks by sc-maps. It is possible to find such a family on any manifold with boundary. In fact, any choice of tubular neighbourhood U of ∂Y such that $U \cong [0, \delta) \times \partial Y$ with coordinates (ρ_Y, y) introduces a dilation in the first variable. Take a function $\chi \in \mathcal{C}_c^\infty[0, \infty)$ such that $\chi(x) = 1$ on $[0, \delta]$. Then set $m_\varepsilon = 1$ on $Y \setminus U$ and

$$m_\varepsilon(\rho_Y, y) = \begin{cases} \chi(\varepsilon\rho_Y^{-1}) & \text{if } \varepsilon\rho_Y^{-1} > \delta/2, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 6.5.3. Consider X, Y mwbs, $U \subset X \times Y$ an open subset, $\varphi \in \rho_X^{-1}\rho_Y^{-1}\mathcal{C}^\infty(U)$ a phase function and $a \in \rho_X^{-m_e}\rho_Y^{-m_\psi}\mathcal{C}^\infty(X \times Y, {}^{\mathrm{sc}}\Omega^{1/2}(X) \times {}^{\mathrm{sc}}\Omega^1(Y))$ an amplitude supported in U. Then $I_\varphi(a) \in (\dot{\mathscr{C}}_0^\infty)'(X, {}^{\mathrm{sc}}\Omega^{1/2}(X))$ is defined as the distributional $1/2$-density acting on $f \in \dot{\mathscr{C}}_0^\infty(X, {}^{\mathrm{sc}}\Omega^{1/2}(X))$ by

$$\langle I_\varphi(a), f\rangle := \lim_{\varepsilon\searrow 0} \iint_{X\times Y} \left(e^{i\varphi} a \cdot (f \otimes m_\varepsilon)\right). \tag{6.47}$$

Remark 6.5.4. If X and Y are equipped with a scattering metric, we have a canonical identification of functions and 1-densities provided by the volume form. Therefore, we can freely choose whether to view functions and distributions as matching (distributional) 1-, 0- or $\frac{1}{2}$-densities.

Remark 6.5.5. When $X = \mathbb{B}^d$ and $Y = \mathbb{B}^s$, these oscillatory integrals correspond, under (inverse) radial compactification, to the tempered oscillatory integrals analyzed in [7, 58].

Lemma 6.5.6. *The expression* (6.47) *yields a well-defined tempered distribution (density) on X. In particular, it is independent of the choice of m_ε.*

Proof. Assume, without loss of generality, that we have a fixed scattering metric and we can identify scattering densities and functions. Let $U \subset X \times Y =: B$ be an open neighborhood of the boundary $\mathcal{B}^\psi$ such that ${}^{\mathrm{sc}}d\varphi \neq 0$ on U.

On $X \times Y \setminus U$, the dominated convergence theorem implies that (6.47) is well-defined. The integrand $u_\varepsilon = e^{i\varphi}a(f \otimes m_\varepsilon)$ converges pointwise and is dominated by $|a \cdot f|$, which is bounded for $\rho_Y > c$.

On U, as in the classical theory, we can define a first order scattering differential $L \in \mathrm{Diff}^1_{\mathrm{sc}}(U)$ which has the property that $Le^{i\varphi} = e^{i\varphi}$. By Proposition 1 from [42], we see that $L^t \in \mathrm{Diff}^1_{\mathrm{sc}}(U)$. Using repeated integration by parts and (6.46), we are able to increase the order in ρ_X and ρ_Y to arbitrary powers, and an application of the dominated convergence theorem then finishes the proof. □

After an arbitrary choice of scattering metrics, we may locally identify (X, g_X) and (Y, g_Y) with subsets of $\mathbb{B}^d$ and $\mathbb{B}^s$, respectively. Then, using some explicit local isomorphism $\Psi =$

$\Psi_X \times \Psi_Y$, we can identify densities with functions using the induced measures μ_X and μ_Y. After use of a partition of unity, we may locally express (6.47) as

$$\langle I_\varphi(a), f\rangle := \lim_{\varepsilon \searrow 0} \iint_{\mathbb{B}^d \times \mathbb{B}^s} \Psi^* \left(e^{i\varphi(\rho_X, x, \rho_Y, y)} a(\rho_X, x, \rho_Y, y) m_\varepsilon(\rho_Y, y) f(\rho_X, x) \right) \tag{6.48}$$

$$= \lim_{\varepsilon \searrow 0} \iint_{\mathbb{B}^d \times \mathbb{B}^s} e^{i\Psi^*\varphi(\rho_X, x, \rho_Y, y)} \widetilde{m}_\varepsilon(\rho_Y, y) \widetilde{a}(\rho_X, x, \rho_Y, y) \widetilde{f}(\rho_X, x) d\mu_{\mathbb{B}^d} d\mu_{\mathbb{B}^s} \tag{6.49}$$

where $\widetilde{f} = \Psi^* f |d\mu_{\mathbb{B}^d}|^{-1/2}$ and $\widetilde{a} \in \rho_{\mathbb{B}^d}^{-m_e} \rho_{\mathbb{B}^s}^{-m_\psi} \mathcal{C}^\infty(\mathbb{B}^d \times \mathbb{B}^s)$ satisfies $\widetilde{a}\widetilde{f} d\mu_{\mathbb{B}^d} d\mu_{\mathbb{B}^s} = af$. Summing up, we may always transform to locally work on $\mathbb{B}^d \times \mathbb{B}^s$ and in local coordinates we work with usual oscillatory integrals.

Since (6.47) does not depend on the choice of m_ε, as it is usual we drop it from the notation and write, *in the sense of oscillatory integrals*,

$$I_\varphi(a) := \int_Y e^{i\varphi} a. \tag{6.50}$$

Singularities of Oscillatory Integrals

Recall that there is a notion of wavefront-set adapted to the pseudo-differential scattering calculus, called the scattering wavefront-set, cf. [3, 5, 42].

Definition 6.5.7. Let $u \in (\dot{\mathscr{C}}_0^\infty)'(X, {}^{\mathrm{sc}}\Omega^{1/2})$. A point $z_0 \in \mathcal{W} = \partial\big({}^{\mathrm{sc}}\overline{T}^* X\big)$ is not in the scattering wavefront-set, and we write $z_0 \notin \mathrm{WF}_{\mathrm{sc}}(u)$, if there exists a scattering pseudo-differential operator A whose symbol is elliptic at z_0 such that $Au \in \dot{\mathscr{C}}_0^\infty(X, {}^{\mathrm{sc}}\Omega^{1/2})$.

Proposition 6.5.8. *For the oscillatory integral in* (6.47)*, we have*

$$\mathrm{WF}_{\mathrm{sc}}(I_\varphi(a)) \subseteq \Lambda_\varphi.$$

Furthermore, if $z \in \Lambda_\varphi$ and a is rapidly decaying near $\lambda_\varphi^{-1}(z)$, then $z \notin \mathrm{WF}_{\mathrm{sc}}(I_\varphi(a))$.

Remark 6.5.9. The (sc-)singular support of u is defined as follows: a point $p_0 \in X$ is contained in $\mathrm{singsupp}_{\mathrm{sc}}(u)$ if and only if for every $f \in \mathcal{C}^\infty(X)$ with $f(p_0) = 1$ we have $fu \notin \dot{\mathscr{C}}_0^\infty(X)$. Similar to the classical wavefront-set and singular support, we have that $\mathrm{pr}_1(\mathrm{WF}_{\mathrm{sc}}(u)) = \mathrm{singsupp}_{\mathrm{sc}}(u)$. Thus, in particular, if a is rapidly decaying near $\mathcal{C}_\varphi$, then $I_\varphi(a) \in \dot{\mathscr{C}}_0^\infty(X)$.

We refer the reader to [6, 58] for the details of this analysis of the wavefront-sets. The proof is carried out as in the classical setting: first, a characterization of $\mathrm{WF}_{\mathrm{sc}}$ in terms of cut-offs and the Fourier transform is achieved, and then one estimates $\mathcal{F} I_\varphi(a)$ in coordinates.

Proposition 6.5.8 gives another insight why we consider Λ_φ as the true object of interest associated with a phase function, not L_φ. In fact, considering (6.47) once more, we see that we may modify phase function and amplitude in the integral by any real valued function $\psi \in \mathcal{C}^\infty(X \times Y)$, writing

$$e^{i\varphi}a = e^{i(\varphi+\psi)}\left(e^{-i\psi}a\right).$$

Then $e^{-i\psi}a \in \rho_X^{-m_e}\rho_Y^{-m_\psi}\mathcal{C}^\infty(X \times Y)$, and hence it is still an amplitude, and $\varphi + \psi$ is a new local phase function. Now, while in general $L_\varphi \neq L_{\varphi+\psi}$, we have $\Lambda_\varphi = \Lambda_{\varphi+\psi}$, by Lemma 6.4.1. This underlines that only Λ_φ and not L_φ can be associated with $I_\varphi(a)$ in an intrinsic way. Nevertheless, it is often convenient to have L_φ available during the proofs.

6.5.2. Definition of Lagrangian Distributions

The class of oscillatory integrals associated with a Lagrangian is – as in the classical theory – not a good distribution space, since in general it is not possible to find a single global phase function to parametrize Λ. Instead, we introduce the following class of Lagrangian distributions. Note that, by our previous findings, we may always reduce an oscillatory integral on $X \times Y$ into a finite sum of oscillatory integrals over $X \times \mathbb{B}^s$ for $s = \dim(Y)$.

Definition 6.5.10 (sc-Lagrangian distributions). Let X be a mwb, $\Lambda \subset \partial\,{}^{\mathrm{sc}}\overline{T}^* X$ a sc-Lagrangian. Then, $I^{m_e,m_\psi}(X,\Lambda)$, $(m_e, m_\psi) \in \mathbb{R}^2$, denotes the space of distributions that can be written as a finite sum of (local) oscillatory integrals as in (6.50), whose phase functions are clean and locally parametrize Λ, plus an element of $\dot{\mathscr{C}}_0^\infty(X)$. More precisely, $u \in I^{m_e,m_\psi}(X,\Lambda)$ if, modulo a remainder in $\dot{\mathscr{C}}_0^\infty(X)$,

$$u = \sum_{j=1}^{N} \int_{Y_j} e^{i\varphi_j} a_j, \tag{6.51}$$

where for $j = 1, \dots, N$:

1.) Y_j is a mwb of dimension s_j;

2.) $\varphi_j \in \rho_{Y_j}^{-1}\rho_X^{-1}\mathcal{C}^\infty(X \times Y_j)$ is a local clean phase function with excess e_j, defined on an open neighbourhood of the support of a_j, which locally parametrizes Λ;

3.) $a_j \in \rho_{Y_j}^{-m_{\psi,j}}\rho_X^{-m_{e,j}}\mathcal{C}^\infty\big(X \times Y_j, {}^{\mathrm{sc}}\Omega^{1/2}(X) \times {}^{\mathrm{sc}}\Omega^1(Y)\big)$ with

$$(m_{\psi,j}, m_{e,j}) = \left(m_\psi + \frac{d}{4} - \frac{s_j}{2} - \frac{e_j}{2}, m_e - \frac{d}{4} + \frac{s_j}{2} - \frac{e_j}{2}\right).$$

We also set

$$I^{-\infty,-\infty}(X,\Lambda) = \bigcap_{(m_\psi,m_e)\in\mathbb{R}^2} I^{m_\psi,m_e}(X,\Lambda),$$
$$I(X,\Lambda) = I^{+\infty,+\infty}(X,\Lambda) = \bigcup_{(m_\psi,m_e)\in\mathbb{R}^2} I^{m_\psi,m_e}(X,\Lambda).$$

Remark 6.5.11. The reason for the choice of the a_j in the scattering amplitude densities spaces of order $(m_{e,j}, m_{\psi,j})$ will be explained in Section 6.5.5.

The next result follows from Proposition 6.5.8.

Proposition 6.5.12. *Let $\Lambda \subset \partial\, {}^{\mathrm{sc}}\overline{T}^* X$ be a* sc*-Lagrangian, and $u \in I(X, \Lambda)$. Then $\mathrm{WF}_{\mathrm{sc}}(u) \subseteq \Lambda$.*

As in the classical case, the class of Lagrangian distributions contains the globally regular functions (cf. Treves [61, Chapter VIII.3.2]):

Lemma 6.5.13. *Let $\Lambda \subset \partial\, {}^{\mathrm{sc}}\overline{T}^* X$ be a* sc*-Lagrangian. Then*

$$\dot{\mathscr{C}}_0^\infty(X, {}^{\mathrm{sc}}\Omega^{1/2}(X)) = I^{-\infty,-\infty}(X, \Lambda). \tag{6.52}$$

Proof. We first prove the inclusion "$\supseteq$". Choose a finite covering of ${}^{\mathrm{sc}}\overline{T}^* X$ with open sets $\{X_j\}_{j=1}^N$ such that there exists a clean phase function φ_j on each X_j parametrizing $\Lambda \cap {}^{\mathrm{sc}}\overline{T}^* X_j$, $j = 1, \ldots, N$. Let $\{g_j\}_{j=1}^N$ be a smooth partition of unity subordinate to such covering. We view X_j as a subset of $X \times \mathbb{B}^d$, $j = 1, \ldots, N$.

Let $\chi \in \dot{\mathscr{C}}_0^\infty(\mathbb{B}^d, {}^{\mathrm{sc}}\Omega^1(\mathbb{B}^d))$ such that $\int \chi = 1$. For any $f \in \dot{\mathscr{C}}_0^\infty(X, {}^{\mathrm{sc}}\Omega^{1/2}(X))$ we set

$$a_j = e^{-i\varphi_j} g_j \cdot (f \otimes \chi), \quad f_j = \int_{\mathbb{B}^d} e^{i\varphi_j} a_j, \quad j = 1, \ldots, N.$$

We see that

$$a_j \in \dot{\mathscr{C}}_0^\infty(X \times \mathbb{B}^d, {}^{\mathrm{sc}}\Omega^{1/2}(X) \times {}^{\mathrm{sc}}\Omega^1(\mathbb{B}^d)), \quad j = 1, \ldots, N,$$

and, summing up,

$$\sum_{j=1}^N f_j(x) = \int_{\mathbb{B}^d} \left(\sum_{j=1}^N g_j(x, y) \right) \cdot (f(x) \otimes \chi(y)) = f(x).$$

The inclusion "$\subseteq$" is achieved by differentiation under the integral sign. ☐

6.5.3. Examples

We have the following examples of (scattering) Lagrangian distributions.

1. Standard Lagrangian distributions of compact support, [27, 32], in particular Lagrangian distributions on compact manifolds X without boundary, are scattering Lagrangian distributions, using the identification

$$\text{Fiber-conic sets in } T^*X \setminus \{0\} \longleftrightarrow \text{Sets in } S^*X \overset{\text{rescaling}}{\longleftrightarrow} \text{Sets in } \mathcal{W}^\psi.$$

2. Legendrian distributions of [45]. Here, the distributions are smooth functions whose singularities at the boundary are of Legendrian type, meaning in $\mathcal{W}^e$.

3. Conormal distributions, meaning the distributions where the Lagrangian, see Section 6.3.4, is $\partial\big({}^{\mathrm{sc}}\overline{T}^* X'\big)$ for a (k-dimensional) p-submanifold $X' \subset Y$. These distributions correspond, under compactification of base and fiber, to the oscillatory integrals given in local (pre-compactified) Euclidean coordinates by
$$u(x', x'') = \int e^{ix'\xi} a(x,\xi)\, d\xi, \qquad a(x,\xi) \in \mathrm{SG}_{\mathrm{cl}}^{m_e,m_\psi}(\mathbb{R}^d \times \mathbb{R}^{d-k}).$$
A prototypical example is given by (derivatives of) $\delta_0(x') \otimes 1$. These arise as (simple or multiple) layers when solving partial differential equations along infinite boundaries or Cauchy surfaces.

4. Examples of scattering Lagrangian distributions which are of none of the previous types arise in the parametrix construction to hyperbolic equations on unbounded spaces, for example the two-point function for the Klein-Gordon equation. For a discussion of this example consider [7].

Remark 6.5.14. Note that, at this stage, the kernels of pseudo-differential operators on $X \times X$ are *not* scattering conormal distributions associated with the diagonal $\Delta \subset X \times X$ when X is a manifold with boundary. In fact, in this case $X \times X$ is a manifold with corners. Furthermore $\Delta \subset X \times X$ does not hit the corner $\partial X \times \partial X$ in a clean way, that is, $\Delta \subset X \times X$ is not a p-submanifold. Similarly, the phase function associated to the SG-phase $(x-y)\xi \in \mathrm{SG}_{\mathrm{cl}}^{1,1}(\mathbb{R}^{2d} \times \mathbb{R}^d)$ is not clean.

However, the formulation of the theory developed in this chapter admits a natural extension to manifolds with corners. The geometric obstruction of $\Delta \subset X \times X$ – or more generally the graphs of (scattering) canonical transformations – not being a p-submanifold can be overcome by lifting the analysis to a blow-up space, see [40, 45]. We postpone this theory of compositions of canonical relations and calculus of scattering Fourier integral operators to a subsequent paper.

6.5.4. Transformations of Oscillatory Integrals

In Section 6.4 we have seen several procedures that allow to switch from one phase function to others that parametrize the same Lagrangian. We will now exploit these to transform oscillatory integrals into "standard form". In the sequel, we will always assume, by a partition of unity, that the support of the amplitude is suitably small.

Transformation Behavior and Equivalent Phase Functions

Now we reconsider (6.48), to express the transformation behavior of the oscillatory integrals under fiber-preserving diffeomorphisms. With the chosen notation and a local phase function

φ_1, we have

$$I_{\varphi_1}(a) = \int_{Y_1} e^{i\varphi_1} a = \int_{Y_2} e^{iF^*\varphi_1} F^* a = I_{F^*\varphi_1}(F^* a) \tag{6.53}$$

for any diffeomorphism $F : X \times Y_2 \to X \times Y_1$ of the form $F = \mathrm{id} \times g$. Assume that φ_2 is equivalent to φ_1 by F, see Definition 6.4.10. After the transformation, we rewrite (6.53) as

$$\int_{Y_2} e^{i\varphi_2} e^{i(F^*\varphi_1 - \varphi_2)} F^* a. \tag{6.54}$$

Now, since $F^*\varphi_1 - \varphi_2$ is smooth up to the boundary, the same holds for $e^{i(F^*\varphi_1-\varphi_2)}$ and this factor can be seen as part of the amplitude. Therefore, we may write

$$I_{\varphi_1}(a) = I_{\varphi_2}\big((F^* a)\, \exp(i(F^*\varphi_1 - \varphi_2))\big). \tag{6.55}$$

In particular, we can express $I_\varphi(a)$, near any boundary point of the domain of definition, using the principal part of φ introduced in Definition 6.2.7, namely

$$I_{\varphi_p}(\widetilde{a}), \text{ with } \widetilde{a} = a \, \exp\big(i(\varphi - \varphi_p)\big). \tag{6.56}$$

By Lemma 6.4.12, $\varphi - \varphi_p \in \mathcal{C}^\infty$ and thus $\widetilde{a} \in \rho_X^{-m_e} \rho_Y^{-m_\psi} \mathcal{C}^\infty(B)$. In the following constructions, we always assume that φ is replaced by its principal part, cf. Remark 6.4.15.

Reduction of the Fiber

We will now analyze the change of boundary behavior under a reduction of fiber variables near $p_0 \in \mathrm{supp}(a) \cap \mathcal{C}_\varphi$. Hence, we assume that

$$\rho_Y^{-1} \rho_X^{-1}\, {}^{\mathrm{sc}}H_Y \varphi \text{ has rank } r > 0 \text{ at } p_0 \in \mathcal{C}_\varphi.$$

We assume, as explained above, that the oscillatory integral is in the form (6.56), namely, φ is replaced by its principal phase part. We observe that, at the boundary point p_0,

$$\mathrm{rk}(\rho_Y^{-1} \rho_X^{-1}\, {}^{\mathrm{sc}}H_Y \varphi) = \mathrm{rk}(\rho_Y^{-1} \rho_X^{-1}\, {}^{\mathrm{sc}}H_Y \sigma(\varphi_p)).$$

By Proposition 6.4.5, we can define a local phase function φ_{red} parametrizing the same Lagrangian as φ. In particular, after a change of coordinates by a scattering map, we can assume $(\mathbf{x}, \mathbf{y}) \in X \times \mathbb{B}^{s-r} \times (-\varepsilon, \varepsilon)^r$, and φ_{red} is given by

$$\varphi_{\mathrm{red}}(\mathbf{x}, \rho_Y, y') = \varphi(\mathbf{x}, \rho_Y, y', 0),$$

where $\rho_Y = \rho_{\mathbb{B}^{s-r}}$ is the boundary defining function on $\mathbb{B}^{s-r}$ and on $\mathbb{B}^{s-r} \times (-\varepsilon, \varepsilon)^r$. We introduce

$$\widetilde{\varphi}(\mathbf{x}, \mathbf{y}) = \varphi_{\mathrm{red}}(\mathbf{x}, \rho_Y, y') + \frac{1}{2} \rho_X^{-1} \rho_Y^{-1} Q(y''), \tag{6.57}$$

where Q is a non-degenerate quadratic form with the same signature as $\partial_{y''}\partial_{y''}f$ at p_0. Then, by Theorem 6.4.19, φ is equivalent to $\widetilde{\varphi}$ by a local diffeomorphism $F = \mathrm{id} \times g$. Note that φ_{red} is equal to its principal part, because we assumed that φ is replaced by φ_p.

We may assume that a is supported in an arbitrarily small neighbourhood of the stationary points of φ. Indeed, we may achieve this for a general amplitude a by applying a cut-off in y'' and writing $a = \phi a + (1-\phi)a$. The oscillatory integral with amplitude $(1-\phi)a$ produces a term in $\dot{\mathscr{C}}_0^\infty(X, \Omega^{1/2}(X))$, by Remark 6.5.9.

Therefore, choosing the support of a small enough, we may perform the change of variables by the local diffeomorphism F as in (6.55). We write, motivated by Lemma 6.2.11 and Example 6.2.32,

$$a_{\mathrm{red}}(\mathbf{x}, \widetilde{\mathbf{y}}) \frac{|d\widetilde{\mathbf{y}}''|}{\rho_{\widetilde{Y}}^r \cdot [h(\mathbf{x}, \widetilde{\mathbf{y}})]^r} = (F^*a)(\mathbf{x}, \widetilde{\mathbf{y}}),$$

which is assumed supported in some compact subset of $(-\epsilon, \epsilon)^r$. Then $I_\varphi(a)$ is transformed into $I_{\varphi_{\mathrm{red}}}(b)$ where

$$b(\mathbf{x}, \rho_Y, y') = \rho_Y^{-r} \int_{(-\varepsilon,\varepsilon)^r} e^{\frac{i}{2}\rho_X^{-1}\rho_Y^{-1}Q(y'')} \Big(e^{i(F^*\varphi(\mathbf{x},\mathbf{y}) - \widetilde{\varphi}(\mathbf{x},\mathbf{y}))}\, a_{\mathrm{red}}(\mathbf{x}, \mathbf{y}) \Big) dy''. \tag{6.58}$$

We claim that $b(\mathbf{x}, \rho_Y, y')$ is again a (density valued) amplitude. First, it is clear that b decays rapidly at $(\mathbf{x}, \rho_Y, y')$ if a decays rapidly at $(\mathbf{x}, \rho_Y, y', 0)$. In particular, b is smooth away from $\mathcal{B}$.

We now we apply the stationary phase lemma [30, Lem. 7.7.3] to (6.58), which yields the asymptotic equivalence, as $\rho_Y\rho_X \to 0$,

$$\begin{aligned} b(\mathbf{x}, \rho_Y, y') = \rho_X^{r/2}\rho_Y^{-r/2} |\det Q|^{-1/2} e^{\frac{i}{4}\pi \operatorname{sgn}(Q)} e^{i(F^*\varphi(\mathbf{x},\rho_Y,y',0) - \widetilde{\varphi}(\mathbf{x},\rho_Y,y',0))} a_{\mathrm{red}}(\mathbf{x}, \rho_Y, y', 0) \\ + \mathcal{O}\big(\rho_Y^{-m_\psi - \frac{r}{2}+1} \rho_X^{-m_e+\frac{r}{2}+1}\big). \end{aligned} \tag{6.59}$$

Similar asymptotics hold for all derivatives of b. We may hence view b as a (density valued) amplitude of the order

$$(m_e', m_\psi') = \Big(m_e - \frac{r}{2}, m_\psi + \frac{r}{2}\Big). \tag{6.60}$$

By Remark 6.4.15 we see that, away from the corner, $F^*\varphi - \widetilde{\varphi}$ vanishes at $\mathcal{C}_\varphi$. Therefore, the principal part of b does not depend on φ. Hence, by comparision of principal parts, cf. Lemma 6.2.8, (6.59) reduces to

$$b(\mathbf{x}, \rho_Y, y') \sim \rho_X^{r/2}\rho_Y^{-r/2} |\det Q|^{-1/2} e^{\frac{i}{4}\pi \operatorname{sgn}(Q)} a_{\mathrm{red}}(\mathbf{x}, \rho_Y, y', 0) \tag{6.61}$$

modulo terms of lower order.

Elimination of Excess

Assume now that φ is a clean phase function of excess $e > 0$. Near some point in $\mathcal{C}_\varphi$, as described in Section 6.4.4, we may make the following geometric assumptions after application of some diffeomorphism F: We assume that $Y = \mathbb{B}^{s-e} \times (-\epsilon, \epsilon)^e$ and that the fibers of $\mathcal{C}_\varphi \to \Lambda_\varphi$ are given by constant $(\mathbf{x}, \rho_Y, y')$ and arbitrary y''. We proceed as in [61] and define

$$\widetilde{\varphi}(\rho_X, x, \rho_Y, y') := \varphi(\rho_X, x, \rho_Y, y', 0). \tag{6.62}$$

We observe that for any fixed y'' the phase function $\phi(y'')$, defined as

$$[\phi(y'')](\mathbf{x}, \rho_Y, y') = \varphi(\mathbf{x}, \rho_Y, y', y''), \tag{6.63}$$

is equivalent to $\widetilde{\varphi}$. Indeed, since $\partial_{y''} {}^{\mathrm{sc}}d_Y \varphi = 0$, the differential ${}^{\mathrm{sc}}H_Y \phi(y'')$ has the same signature as ${}^{\mathrm{sc}}H_{\mathbb{B}^{s-e}} \widetilde{\varphi}$ and both parametrize the same Lagrangian with the same number of phase variables $(s-e)$. Therefore, Theorem 6.4.19 guarantees the existence of a family of diffeomorphisms $G(y'') : (\mathbf{x}, \rho_Y, y') \mapsto (\mathbf{x}, g(\mathbf{x}, \rho_Y, y', y''))$ such that, defining $\widetilde{G} \colon (\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \rho_Y, y', y'') \mapsto (\mathbf{x}, g(\mathbf{x}, \rho_Y, y', y''), y'')$,

$$\widetilde{G}^* \varphi - \widetilde{\varphi} \tag{6.64}$$

is smooth everywhere, and vanishes on $\mathcal{C}_{\widetilde{\varphi}}$ away from the corner by Remark 6.4.15. Then we may express $I_\varphi(a)$ as $I_{\widetilde{\varphi}}(b)$, where

$$b(\mathbf{x}, \rho_Y, y') = \rho_Y^{-e} \int_{(-\varepsilon,\varepsilon)^e} e^{i(\widetilde{G}^*\varphi - \widetilde{\varphi})(\mathbf{x}, \rho_Y, y', y'')} (\widetilde{G}^* a)_{\mathrm{red}}(\mathbf{x}, \rho_Y, y', y'')\, dy'' \tag{6.65}$$

and

$$(\widetilde{G}^* a)_{\mathrm{red}}(\mathbf{x}, \mathbf{y}) \frac{|dy''|}{\rho_{\widetilde{Y}}^e \cdot [h(\mathbf{x}, \mathbf{y})]^e} = (\widetilde{G}^* a)(\mathbf{x}, \mathbf{y}).$$

Since $\widetilde{G}^* \varphi - \widetilde{\varphi}$ is smooth, b is again an amplitude of order

$$(\tilde{m}_e, \tilde{m}_\psi) = (m_e, m_\psi + e)\,. \tag{6.66}$$

Notice that at points in $\mathcal{C}_\varphi$ away from the corner, $\widetilde{G}^* \varphi - \widetilde{\varphi}$ vanishes and hence (6.65) reduces to

$$b(\mathbf{x}, \rho_Y, y') = \rho_Y^{-e} \int_{(-\varepsilon,\varepsilon)^e} (\widetilde{G}^* a)_{\mathrm{red}}(\mathbf{x}, \rho_Y, y', y'')\, dy''. \tag{6.67}$$

6.5.5. The Order of a Lagrangian Distribution

We will now obtain the definition of the order of $I_\varphi(a)$ which is invariant with respect to all the three steps described above.

Lemma 6.5.15. *The numbers $\mu_\psi = m_\psi + s/2 + e/2$ and $\mu_e = m_e - s/2 + e/2$ remain constant under reduction of fiber-variables and elimination of excess.*

Proof. Consider a Lagrangian distribution $A = I_\varphi(a)$ where a has order m_ψ, m_e and $\dim Y = s$ with excess e and r reduceable fiber variables. After the reduction of fiber, we obtain an amplitude a' with order $m'_e = m_e - r/2, m'_\psi = m_\psi + r/2$ (cf. (6.60)), with excess $e' = e$ and number of fiber variables $s' = s - r$. The elimination of excess yields an amplitude $a^\#$ with order $m_e^\# = m_e, m_\psi^\# = m_\psi + e$ (cf. (6.66)), excess $e^\# = 0$ and $s^\# = s - e$. It is now straightforward to check that

$$\begin{aligned} m_\psi + s/2 + e/2 &= m'_\psi + s'/2 + e/2 = m_\psi^\# + s^\#/2 + e^\#/2, \\ m_e - s/2 + e/2 &= m'_e - s'/2 + e/2 \ = m_e^\# - s^\#/2 + e^\#/2. \end{aligned}$$

$\square$

This shows that the tuple (μ_ψ, μ_e) can be used to define the order of a Lagrangian distribution. We still have the freedom to add arbitrary constants to both orders. In order to choose these constants, we compare our class of Lagrangian distributions with Hörmander's Lagrangian distributions and the Legendrian distributions of Melrose–Zworski [45]. First, consider the Delta-distribution δ_0, which is in the Hörmander class $I^{d/4}$ and $\mu_\psi = d/2$. Therefore, we choose $m_\psi = \mu_\psi - d/4$ to obtain the same ψ-order for δ_0. Similarly, the constant function is a Legendrian distribution of order $-d/4$ and $\mu_e = 0$, and therefore we choose $m_e = \mu_e + d/4$. Note that we use the opposite sign convention for the m_e-order then in [45].

6.6. The Principal Symbol of a Lagrangian Distribution

We will now define the principal symbol map $j^\Lambda_{m_e,m_\psi}$ on $I^{m_e,m_\psi}(X, \Lambda)$. Similarly to the classical theory, it takes values in a suitable (density) bundle on Λ. This is coherent with the notion of principal symbol map j_{m_e,m_ψ} for scattering operators, see [42, 43], as well as of principal part for classical SG symbols, see [15, 58], which both provide smooth objects defined on $\mathcal{W} = \partial\, {}^{\mathrm{sc}}\overline{T}^* X \supset \Lambda$. We adapt the construction in [61] (see also [27, 32]), starting from the simplest case of local non-degenerate phase functions parametrizing Λ, up to the general case of local clean functions.

Let $\Lambda \subset \mathcal{W}$ be an sc-Lagrangian, which on $B = X \times Y$ is locally parametrized by a local non-degenerate phase function $\varphi \in \rho_Y^{-1}\rho_X^{-1}\mathcal{C}^\infty(U)$, $U \subset B$. Let $a \in \rho_Y^{-m_\psi}\rho_X^{-m_e}\mathcal{C}^\infty\big(X \times Y, {}^{\mathrm{sc}}\Omega^{1/2}(X) \times {}^{\mathrm{sc}}\Omega^1(Y)\big)$ be supported in U, and let $I_\varphi(a)$ be a (micro-)local representation of $u \in I^{m_e,m_\psi}(X, \Lambda)$ as a single oscillatory integral.

We now fix a 1-density μ_X on X. Any choice of 1 density μ_Y on Y then trivializes the one-dimensional bundle $\mathcal{C}^\infty(X \times Y, {}^{\mathrm{sc}}\Omega^{1/2}(X) \otimes {}^{\mathrm{sc}}\Omega^1(Y))$, and any element is given by a multiple of $\rho_X^{-(d+1)/2}\rho_Y^{-s-1}\sqrt{\mu_X} \otimes \mu_Y$. Any choice of coordinates (ρ_Y, y) in Y allows for us to express μ_Y locally as $\frac{\partial \mu_Y}{\partial(\rho_Y, y)}\, d\rho_Y dy$, meaning as having a smooth density factor with respect to the (local) Lebesgue measure. As such, we rewrite the amplitude $a \in \rho_Y^{-m_\psi}\rho_X^{-m_e}\mathcal{C}^\infty(X \times$

$Y, {}^{\mathrm{sc}}\Omega^{1/2}(X) \otimes {}^{\mathrm{sc}}\Omega^{1}(Y))$ in any choice of local coordinates as

$$\rho_Y^{m_\psi} \rho_X^{m_e} a(\mathbf{x}, \mathbf{y}) = \mathfrak{a}(\mathbf{x}, \mathbf{y})\, \rho_X^{-(d+1)/2} \rho_Y^{-s-1} \sqrt{\mu_X} d\rho_Y dy. \tag{6.68}$$

for $\mathfrak{a} \in \mathcal{C}^\infty(X \times Y)$.

6.6.1. Non-Degenerate Equivalent Phase Functions

As above (cf. (6.14)), when U is a neighbourhood of a point close to the boundary $\mathcal{B}$, we can there identify ${}^{\mathrm{sc}}d_Y\varphi$ with the map,

$$(\mathbf{x}, \mathbf{y}) \mapsto \Phi(\mathbf{x}, \mathbf{y}) = \big(- f(\mathbf{x}, \mathbf{y}) + \rho_Y \partial_{\rho_Y} f(\mathbf{x}, \mathbf{y}) \quad \partial_y f(\mathbf{x}, \mathbf{y}) \big) \in \mathbb{R}^s,$$

locally well-defined on a neighbourhood of C_φ within U.

In view of the non-degeneracy of φ, Φ has a surjective differential, so that we can consider the pullback of distributions $d_\varphi = \Phi^*\delta$, with $\delta = \delta_0 \in \mathcal{D}'(\mathbb{R}^s)$ the Dirac distribution, concentrated at the origin, on $\mathbb{R}^s$ (cf. [30, Ch. VI]). More explicitly, choosing functions $(t_1, \dots, t_d) =: t$, which restrict to a local coordinate system (up to the boundary) on C_φ, the pull-back d_φ can be expressed locally as the density

$$d_\varphi = \left| \det \frac{\partial(t, \Phi)}{\partial(\mathbf{x}, \mathbf{y})} \right|^{-1} dt = \Delta_\varphi(t)\, dt.$$

Consider another local non-degenerate phase function $\widetilde{\varphi}$ parametrizing Λ, defined on an open subset $\widetilde{U} \subset X \times \widetilde{Y}$, such that $\widetilde{\varphi} = F^*\varphi$, with a (local, fibered) diffeomorphism $F = \mathrm{id} \times g\colon X \times \widetilde{Y} \to X \times Y$. Since F is a sc-map, there exists a function $h \in \mathcal{C}^\infty(X \times Y)$ such that $(F^*\rho_Y)(\mathbf{x}, \widetilde{\mathbf{y}}) = \rho_{\widetilde{Y}} \cdot h(\mathbf{x}, \widetilde{\mathbf{y}})$.

As above, we identify ${}^{\mathrm{sc}}d_Y\widetilde{\varphi}$ with the map $\widetilde{\Phi}$ and define $d_{\widetilde{\varphi}}$ and $\Delta_{\widetilde{\varphi}}(\widetilde{t})$ in terms of the functions $\widetilde{t}_j = F^*t_j$, which are local coordinates on $C_{\widetilde{\varphi}}$, provided $\widetilde{U}$ is small enough.

In the sequel, we show how objects defined in these two choices (t, φ) and $(\widetilde{t}, \widetilde{\varphi})$ are related. For that, we implicitly assume all objects evaluated at corresponding points $(\mathbf{x}, \mathbf{y}) \in C_\varphi$ (parametrized by t) and $(\mathbf{x}, \widetilde{\mathbf{y}}) = F(\mathbf{x}, \mathbf{y}) \in C_{\widetilde{\varphi}}$ (parametrized by $\widetilde{t}$).

Lemma 6.6.1. *The functions $\Delta_{\widetilde{\varphi}}(\widetilde{t})$ and $\Delta_\varphi(t)$ are related by*

$$\Delta_{\widetilde{\varphi}}(\widetilde{t}) = h(\mathbf{x}, \mathbf{y})^{s+1} \left| \det \frac{\partial g(\mathbf{x}, \widetilde{\mathbf{y}})}{\partial \widetilde{\mathbf{y}}} \right|^{-2} \Delta_\varphi(t(\widetilde{t})).$$

Proof of Lemma 6.6.1. By direct computation, $\widetilde{\Phi}$ and Φ are related by a matrix $M_{\Phi\widetilde{\Phi}}$ via

$$\widetilde{\Phi}(\mathbf{x}, \widetilde{\mathbf{y}}) = \Phi(F(\mathbf{x}, \widetilde{\mathbf{y}})) \cdot M_{\Phi\widetilde{\Phi}}(\mathbf{x}, \widetilde{\mathbf{y}}), \tag{6.69}$$

where

$$M_{\Phi\widetilde{\Phi}}(\mathbf{x},\widetilde{\mathbf{y}}) = \begin{pmatrix} [h(\mathbf{x},\widetilde{\mathbf{y}})]^{-2}\dfrac{\partial\rho_Y}{\partial\rho_{\widetilde{Y}}}(\mathbf{x},\widetilde{\mathbf{y}}) & [h(\mathbf{x},\widetilde{\mathbf{y}})]^{-2}\rho_{\widetilde{Y}}^{-1}\dfrac{\partial\rho_Y}{\partial\widetilde{y}}(\mathbf{x},\widetilde{\mathbf{y}}) \\ [h(\mathbf{x},\widetilde{\mathbf{y}})]^{-1}\rho_{\widetilde{Y}}\dfrac{\partial y}{\partial\rho_{\widetilde{Y}}}(\mathbf{x},\widetilde{\mathbf{y}}) & [h(\mathbf{x},\widetilde{\mathbf{y}})]^{-1}\dfrac{\partial y}{\partial\widetilde{y}}(\mathbf{x},\widetilde{\mathbf{y}}) \end{pmatrix}$$

and

$$|\det M_{\Phi\widetilde{\Phi}}(\mathbf{x},\widetilde{\mathbf{y}})| = h(\mathbf{x},\widetilde{\mathbf{y}})^{-s-1}\cdot\left|\det\frac{\partial g(\mathbf{x},\widetilde{\mathbf{y}})}{\partial\widetilde{\mathbf{y}}}\right|.$$

Differentiating (6.69), we obtain, using that $\widetilde{\Phi}(\mathbf{x},\mathbf{y}) = \Phi(F(\mathbf{x},\widetilde{\mathbf{y}})) = 0$ on $C_{\widetilde{\varphi}}$,

$$\begin{aligned} \frac{\partial\widetilde{\Phi}}{\partial(\mathbf{x},\widetilde{\mathbf{y}})}(\mathbf{x},\widetilde{\mathbf{y}}) &= {}^tM_{\Phi\widetilde{\Phi}}(\mathbf{x},\widetilde{\mathbf{y}})\cdot\frac{\partial(\Phi(F(\mathbf{x},\widetilde{\mathbf{y}})))}{\partial(\mathbf{x},\widetilde{\mathbf{y}})} \\ &= {}^tM_{\Phi\widetilde{\Phi}}(\mathbf{x},\widetilde{\mathbf{y}})\cdot\left[\frac{\partial\Phi}{\partial(\mathbf{x},\mathbf{y})}(F(\mathbf{x},\widetilde{\mathbf{y}}))\right]\cdot\frac{\partial F}{\partial(\mathbf{x},\widetilde{\mathbf{y}})}(\mathbf{x},\widetilde{\mathbf{y}}). \end{aligned} \tag{6.70}$$

Furthermore, we have

$$\frac{\partial\widetilde{t}}{\partial(\mathbf{x},\widetilde{\mathbf{y}})}(\mathbf{x},\widetilde{\mathbf{y}}) = \left[\frac{\partial t}{\partial(\mathbf{x},\mathbf{y})}(F(\mathbf{x},\widetilde{\mathbf{y}}))\right]\cdot\frac{\partial F}{\partial(\mathbf{x},\widetilde{\mathbf{y}})}(\mathbf{x},\widetilde{\mathbf{y}}).$$

Summing up, we find

$$\frac{\partial(\widetilde{t},\widetilde{\Phi})}{\partial(\mathbf{x},\widetilde{\mathbf{y}})}(\mathbf{x},\widetilde{\mathbf{y}}) = \mathrm{diag}(\mathbb{1}_d,{}^tM_{\Phi\widetilde{\Phi}}(\mathbf{x},\widetilde{\mathbf{y}}))\cdot\left[\frac{\partial(t,\Phi)}{\partial(\mathbf{x},\mathbf{y})}(F(\mathbf{x},\widetilde{\mathbf{y}}))\right]\cdot\frac{\partial F}{\partial(\mathbf{x},\widetilde{\mathbf{y}})}(\mathbf{x},\widetilde{\mathbf{y}}), \tag{6.71}$$

which in turn implies, using $F = \mathrm{I}\times g$,

$$\Delta_{\widetilde{\varphi}}(\widetilde{t}) = \left|\frac{\partial(\widetilde{t},\widetilde{\Phi})}{\partial(\mathbf{x},\widetilde{\mathbf{y}})}(\mathbf{x},\widetilde{\mathbf{y}})\right|^{-1} = [h(\mathbf{x},\widetilde{\mathbf{y}})]^{s+1}\left|\det\frac{\partial g(\mathbf{x},\widetilde{\mathbf{y}})}{\partial\widetilde{\mathbf{y}}}\right|^{-2}\Delta_\varphi(t(\widetilde{t})),$$

as claimed. □

We define

$$w_\varphi = (\rho_X^{-m_e}\rho_Y^{-m_\psi-(s+1)/2}\mathfrak{a})|_{C_\varphi}\cdot\sqrt{|d_\varphi|}, \tag{6.72}$$

with $\mathfrak{a}$ given in (6.68), which is a half-density on (the interior of) C_φ.

To define $w_{\widetilde{\varphi}}$ accordingly, we check that $I_\varphi(a)$ transforms under the action of F as

$$\begin{aligned} \int_Y e^{i\varphi}a &= \int_{\widetilde{Y}} e^{i(F^*\varphi)(\mathbf{x},\widetilde{\mathbf{y}})}F^*\Big[\rho_X^{-m_e}\rho_Y^{-m_\psi}\,\mathfrak{a}\,\rho_X^{-(d+1)/2}\rho_Y^{-s-1}\sqrt{\mu_X}\otimes d\rho_Y\,dy\Big]\,(\mathbf{x},\widetilde{\mathbf{y}}) \\ &= \int_{\widetilde{Y}} e^{i\widetilde{\varphi}(\mathbf{x},\widetilde{\mathbf{y}})}\rho_X^{-m_e}\rho_{\widetilde{Y}}^{-m_\psi}\widetilde{\mathfrak{a}}(\mathbf{x},\widetilde{\mathbf{y}})\,(\rho_X^{-(d+1)/2}\rho_{\widetilde{Y}}^{-s-1}\sqrt{\mu_X}\otimes d\rho_{\widetilde{Y}}d\widetilde{y}), \end{aligned}$$

where

$$\widetilde{\mathfrak{a}}(\mathbf{x},\widetilde{\mathbf{y}}) = \mathfrak{a}(F(\mathbf{x},\widetilde{\mathbf{y}}))h(\mathbf{x},\widetilde{\mathbf{y}})^{-m_\psi-s-1}\left|\det\frac{\partial g(\mathbf{x},\widetilde{\mathbf{y}})}{\partial\widetilde{\mathbf{y}}}\right|. \tag{6.73}$$

We define, coherently with (6.72), $w_{\widetilde{\varphi}} = \rho_X^{-m_e}\rho_{\widetilde{Y}}^{-m_\psi-(s+1)/2}\widetilde{\mathfrak{a}}\sqrt{|d_{\widetilde{\varphi}}|}$.

Lemma 6.6.2. *The half-densities $w_{\widetilde{\varphi}}$ and w_φ are related by*

$$w_{\widetilde{\varphi}} = F^* w_\varphi$$

in (the interior of) $C_{\widetilde{\varphi}}$.

Proof. We obtain from (6.73) and Lemma 6.6.1 that

$$\widetilde{\mathfrak{a}}(\mathbf{x},\widetilde{\mathbf{y}})\left|\Delta_{\widetilde{\varphi}}(\widetilde{t})\right|^{1/2} = \mathfrak{a}(F(\mathbf{x},\widetilde{\mathbf{y}}))h(\mathbf{x},\widetilde{\mathbf{y}})^{-m_\psi-(s+1)/2}\left|\Delta_\varphi(t(\widetilde{t}))\right|^{1/2}.$$

Then, using the local coordinates t and $\widetilde{t} = F^*t$ introduced above, on $C_{\widetilde{\varphi}}$ we find

$$\begin{aligned} w_{\widetilde{\varphi}} &= F^*\Big(\rho_X^{-m_e}\rho_Y^{-m_\psi-(s+1)/2}\mathfrak{a}\Big)\left|\Delta_\varphi(t(\widetilde{t}))\right|^{1/2}\sqrt{|d\widetilde{t}|} \\ &= F^*\Big(\rho_X^{-m_e}\rho_Y^{-m_\psi-(s+1)/2}\mathfrak{a}\,|\Delta_\varphi(t)|^{1/2}\sqrt{|dt|}\Big) = F^*w_\varphi. \end{aligned}$$

□

As a half-density valued amplitude, w_φ is of order $(m_e, m_\psi-(s+1)/2)$, as shown by the computations above. In accordance with the definition of the principal part (cf. Definition 6.2.7), we set

$$\mathfrak{w}_\varphi = \left(\mathfrak{a}\cdot\sqrt{|d_\varphi|}\right)\Big|_{\mathcal{C}_\varphi}.$$

As seen above, $\mathfrak{w}_\varphi$ transforms to $\mathfrak{w}_{\widetilde{\varphi}}$ under the pull-back via F. Since λ_φ is a local diffeomorphism $C_\varphi \to L_\varphi$, we can also consider

$$\alpha_\varphi = (\lambda_\varphi)_*(\mathfrak{w}_\varphi),$$

which yields a local half-density on Λ_φ. The fact that, for the two equivalent phase functions φ and $\widetilde{\varphi}$, we have $\lambda_{\widetilde{\varphi}} = \lambda_\varphi\circ F$, together with the transformation properties of $\mathfrak{w}_\varphi$, shows that

$$\alpha_{\widetilde{\varphi}} = \alpha_\varphi = \alpha,$$

that is, $\alpha_{\widetilde{\varphi}}$ and α_φ are equivalent local representations of a half-density α defined on Λ, in the local parametrizations $\Lambda_{\widetilde{\varphi}}$ and Λ_φ, respectively.

We now prove that the same holds true if $\widetilde{\varphi}$ is merely a non-degenerate phase function equivalent to φ in the sense of Definition 6.4.10. First, if we repeat the construction of $\sqrt{|d_{\widetilde{\varphi}}|}$

described above, all the computations remain valid modulo terms, generated by $\widetilde{\Phi}$, which contain an extra factor $\rho_X \rho_{\widetilde{Y}}$. This is due to

$$\begin{aligned}
&F^*\varphi - \widetilde{\varphi} \in \mathcal{C}^\infty(\widetilde{U}) \\
&\qquad \Leftrightarrow \rho_X^{-1}\rho_{\widetilde{Y}}^{-1}\widetilde{f}(\mathbf{x},\widetilde{\mathbf{y}}) = \rho_X^{-1}\rho_{\widetilde{Y}}^{-1}h(\mathbf{x},\widetilde{\mathbf{y}})^{-1}(F^*f)(\mathbf{x},\widetilde{\mathbf{y}}) + g(\mathbf{x},\widetilde{\mathbf{y}}), g \in \mathcal{C}^\infty(\widetilde{U}) \\
&\qquad \Leftrightarrow \widetilde{f}(\mathbf{x},\widetilde{\mathbf{y}}) = h(\mathbf{x},\widetilde{\mathbf{y}})^{-1}(F^*f)(\mathbf{x},\widetilde{\mathbf{y}}) + \rho_X\rho_{\widetilde{Y}}g(\mathbf{x},\widetilde{\mathbf{y}}), g \in \mathcal{C}^\infty(\widetilde{U}).
\end{aligned}$$

Then, by rescaling $w_{\widetilde{\varphi}}$ through multiplication by $\rho_X^{m_e}\rho_{\widetilde{Y}}^{m_\psi+(s+1)/2}$ and then restricting $\mathfrak{w}_\varphi$ on $\mathcal{C}_{\widetilde{\varphi}}$, such additional terms identically vanish.

Moreover, by Lemma 6.4.12 and Remark 6.4.15, we know that, in a neighbourhood $\widetilde{U}$ of any point in the interior of $\mathcal{C}^e_{\widetilde{\varphi}}$ or $\mathcal{C}^\psi_{\widetilde{\varphi}}$, which does not intersect $\mathcal{C}^{\psi e}_{\widetilde{\varphi}}$, it can be assumed, after passage to the principal parts, that $\widetilde{\varphi} = F^*\varphi$ on $\mathcal{C}_{\widetilde{\varphi}} \cap \partial\widetilde{U}$, see Section 6.5.4. It follows that the factor $\exp(i(F^*\varphi - \widetilde{\varphi}))$, appearing in $\widetilde{\mathfrak{a}}$ (cf. (6.55)) also disappears, away from the corner, when restricting to the faces $\mathcal{C}^e_{\widetilde{\varphi}}$ or $\mathcal{C}^\psi_{\widetilde{\varphi}}$.

Finally, we observe that $\mathfrak{w}_\varphi$ and $\mathfrak{w}_{\widetilde{\varphi}}$ are obtained as restrictions of smooth objects on $X \times Y$ and $X \times \widetilde{Y}$ to their respective boundaries. As such, their transformation behavior extends, by continuity, to the corner as well, producing smooth objects on $\mathcal{C}_\varphi$ and $\mathcal{C}_{\widetilde{\varphi}}$. By push-forward through $\lambda_{\widetilde{\varphi}}$ and λ_φ, we find again that $\alpha_{\widetilde{\varphi}} = \alpha_\varphi = \alpha$ locally on $\Lambda_{\widetilde{\varphi}} = \Lambda_\varphi = \Lambda$.

6.6.2. Non-Degenerate Phase Functions, Reduction of the Fiber

We now consider a φ such that reduction of fiber variables, see Section 6.4.2, is possible. By the argument in Section 6.6.1, we may then write $I_\varphi(a) = I_{\varphi_{\mathrm{red}}}(b)$ with b from (6.58). We now compare α_φ to the analogously defined half-density $\beta_{\varphi_{\mathrm{red}}}$. We can replace the phase function φ by the equivalent phase function given in (6.57), and this does not affect α_φ. Hence we may assume that φ is of the form $\varphi(\mathbf{x},\mathbf{y}) = \varphi_{\mathrm{red}}(\mathbf{x},\mathbf{y}') + \frac{1}{2}\rho_X^{-1}\rho_Y^{-1}\langle Qy'', y''\rangle$.

As such, we assume, in this splitting of coordinates, $C_\varphi \subset \{(\mathbf{x},\mathbf{y}',0)\}$. We find:

Lemma 6.6.3. *Under the identification $C_{\varphi_{\mathrm{red}}} \times \{0\} = C_\varphi$, we have*

$$\sqrt{|d_\varphi|} = |\det Q|^{-\frac{1}{2}}\sqrt{|d_{\varphi_{\mathrm{red}}}|}.$$

Proof. We compute

$$\begin{aligned}
\Phi(\mathbf{x},\mathbf{y}) &= \big(-f_{\mathrm{red}}(\mathbf{x},\mathbf{y}') + \rho_Y\partial_{\rho_Y}f_{\mathrm{red}}(\mathbf{x},\mathbf{y}') \quad \partial_{y'}f_{\mathrm{red}}(\mathbf{x},\mathbf{y}') \quad 0\big) \\
&\quad + \big(-\frac{1}{2}\langle Qy'', y''\rangle \quad 0 \quad \partial_{y''}Q(y'')\big) \\
&=: (\Phi_{\mathrm{red}}(\mathbf{x},\mathbf{y}') \quad 0) + \big(\Psi(y'') \quad Qy''\big) \in \mathbb{R}^{s-r} \times \mathbb{R}^r.
\end{aligned}$$

Therefore,

$$\frac{\partial(t,\Phi)}{\partial(\mathbf{x},\mathbf{y})}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} \dfrac{\partial t}{\partial \mathbf{x}}(\mathbf{x},\mathbf{y}) & \dfrac{\partial t}{\partial \mathbf{y}'}(\mathbf{x},\mathbf{y}) & \dfrac{\partial t}{\partial y''}(\mathbf{x},\mathbf{y}) \\ \dfrac{\partial \Phi_{\mathrm{red}}}{\partial \mathbf{x}}(\mathbf{x},\mathbf{y}') & \dfrac{\partial \Phi_{\mathrm{red}}}{\partial \mathbf{y}'}(\mathbf{x},\mathbf{y}') & -\dfrac{1}{2}\dfrac{\partial \Psi}{\partial y''}(y'') \\ 0 & 0 & Q \end{pmatrix}.$$

Consequently,

$$\begin{aligned} \sqrt{|d_\varphi|} &= \left|\det \frac{\partial(t,\Phi)}{\partial(\mathbf{x},\mathbf{y})}\right|_{C_{\tilde\varphi}}^{-1/2} \sqrt{|dt|} \\ &= \left|\det \frac{\partial(t,\Phi_{\mathrm{red}})}{\partial(\mathbf{x},\mathbf{y}')}\right|_{C_{\varphi_{\mathrm{red}}}}^{-\frac12} \cdot |\det Q|^{-\frac12}\sqrt{|dt|} \\ &= |\det Q|^{-\frac12}\sqrt{|d_{\varphi_{\mathrm{red}}}|}. \end{aligned}$$

□

Notice that[5] $\mathfrak{a} = \mathfrak{a}_{\mathrm{red}}$. We compute, by (6.59), modulo amplitudes of lower order,

$$b(\mathbf{x},\mathbf{y}') \;=\; \rho_X^{-m_e+r/2}\rho_Y^{-m_\psi-r/2}|\det Q|^{-1/2}e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}\mathfrak{a}(\mathbf{x},\mathbf{y}',0)\sqrt{\mu_X}(\rho_Y^{-(s-r+1)/2}|d\mathbf{y}'|). \tag{6.74}$$

We observe that b is an amplitude of order $(m_e - r/2, m_\psi + r/2)$ and find

$$\mathfrak{b}(\mathbf{x},\mathbf{y}') = |\det Q|^{-1/2}e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}\mathfrak{a}(\mathbf{x},\mathbf{y}',0) + \mathcal{O}\big(\rho_X\rho_Y\big),$$

which implies, using Lemma 6.6.3,

$$\begin{aligned} \mathfrak{w}_{\varphi_{\mathrm{red}}} &= \left(\mathfrak{b}(\mathbf{x},\mathbf{y}')\sqrt{|d_{\varphi_{\mathrm{red}}}|}\right)\Big|_{\mathcal{C}_{\varphi_{\mathrm{red}}}} \\ &= e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}\left(\mathfrak{a}(\mathbf{x},\mathbf{y})\sqrt{|d_\varphi|}\right)\Big|_{\mathcal{C}_{\tilde\varphi}} \\ &= e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}\mathfrak{w}_\varphi. \end{aligned}$$

This, in turn, finally gives

$$\beta_{\varphi_{\mathrm{red}}} = (\lambda_{\varphi_{\mathrm{red}}})_*(\mathfrak{w}_{\varphi_{\mathrm{red}}}) = e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}\cdot(\lambda_\varphi)_*(\mathfrak{w}_\varphi) = e^{i\frac{\pi}{4}\mathrm{sgn}(Q)}\cdot\alpha_\varphi.$$

[5]Observe that $\mathfrak{a}_{red}$ is obtained by splitting of the density and weight factors in two steps.

6.6.3. Clean Phase Functions, Elimination of the Excess

We now proceed with the last reduction step, namely, we consider a clean phase function and eliminate its excess. As in Section 6.5.4, we assume $Y = \mathbb{B}^{s-e} \times (-\epsilon, \epsilon)^e$ with the fibers of $\mathcal{C}_\varphi \to \Lambda_\varphi$ given by constant $(\mathbf{x}, \rho_Y, y')$ and arbitrary $y'' \in (-\epsilon, \epsilon)^e$.

Switching to the phase function $\widetilde{\varphi}$ in (6.62), we may write $I_\varphi(a) = I_{\widetilde{\varphi}}(b)$ with b defined in (6.65). We apply the construction of the previous section, and obtain the density $\beta_{\widetilde{\varphi}} = (\lambda_{\widetilde{\varphi}})_* \left(\mathfrak{b} \cdot \sqrt{|d_{\widetilde{\varphi}}|}\right)_{\mathcal{C}_{\widetilde{\varphi}}}$ from the data $(\widetilde{\varphi}, b)$.

Alternatively, we may study the family of oscillatory integrals $I_{\phi(y'')}(a(y''))$ with phase functions $\phi(y'')$ defined in (6.63) and amplitudes

$$a(y'')\colon (\mathbf{x}, \rho_Y, y') \mapsto \rho_Y^{-e}\, a(\mathbf{x}, \rho_Y, y', y'') = \rho_Y^{-e}\, a(\mathbf{x}, \mathbf{y}),$$

with corresponding principal parts $\mathfrak{a}(y'')$. Since $\phi(y'')$ is non-degenerate, we can define the parameter dependent family of half-densities on Λ

$$\alpha_\phi(y'') = (\lambda_{\phi(y'')})_* \left(\mathfrak{a}(y'') \cdot \sqrt{|d_{\phi(y'')}|}\right)_{\mathcal{C}_{\phi(y'')}},$$

and finally set

$$\gamma_{\widetilde{\varphi}} = \int_{(-\varepsilon,\varepsilon)^e} \alpha_\phi(y'')\, dy''. \tag{6.75}$$

Proposition 6.6.4. *The half-densities on $\Lambda_{\widetilde{\varphi}} = \Lambda_\varphi = \Lambda$ given by $\gamma_{\widetilde{\varphi}}$ and $\beta_{\widetilde{\varphi}}$ coincide.*

Proof. We consider the smooth family of diffeomorphisms $G(y'') = \mathrm{id} \times g(y'')$, depending on the parameter y'', involved in $\widetilde{G}$ from (6.64). Assuming the amplitudes $a(y'')$ supported away from the corner points, we can suppose, as above, $G(y'')^* \phi(y'') - \widetilde{\varphi} = 0$. We now compute, using Lemma 6.4.7 and the expression (6.65), together with the transformation properties of $\mathfrak{w}_\varphi$,

$$\begin{aligned}
\left(\mathfrak{b}_{\widetilde{\varphi}} \cdot \sqrt{|d_{\widetilde{\varphi}}|}\right)(\mathbf{x}, \rho_Y, y')|_{\mathcal{C}_{\widetilde{\varphi}}} &= \mathfrak{b}_{\widetilde{\varphi}}(\mathbf{x}, \rho_Y, y')|_{\mathcal{C}_{\widetilde{\varphi}}} \left|\det \frac{\partial(\widetilde{t}, \widetilde{\Phi})}{\partial(\mathbf{x}, \mathbf{y}')}\right|_{\mathcal{C}_{\widetilde{\varphi}}}^{-\frac{1}{2}} \sqrt{|d\widetilde{t}|} \\
\text{((6.73)} \Rightarrow) \qquad &= \int_{(-\varepsilon,\varepsilon)^e} \mathfrak{a}(G(\mathbf{x}, \mathbf{y}))|_{\mathcal{C}_{\widetilde{\varphi}}} \left|\det \frac{\partial g}{\partial \mathbf{y}'}(\mathbf{x}, \mathbf{y})\right|_{\mathcal{C}_{\widetilde{\varphi}}} [h(\mathbf{x}, \mathbf{y})]_{\mathcal{C}_{\widetilde{\varphi}}}^{-m_\psi - s - 1} \times \\
&\qquad \times \left|\det \frac{\partial(\widetilde{t}, \widetilde{\Phi})}{\partial(\mathbf{x}, \mathbf{y}')}\right|_{\mathcal{C}_{\widetilde{\varphi}}}^{-\frac{1}{2}} \sqrt{|d\widetilde{t}|}\, dy'' \\
\text{(Lemma 6.6.1} \Rightarrow) \qquad &= \int_{(-\varepsilon,\varepsilon)^e} G(y'')^* \left[\mathfrak{a}(\mathbf{x}, \mathbf{y})|_{\mathcal{C}_{\phi(y'')}} \left|\det \frac{\partial(t, \Phi(y''))}{\partial(\mathbf{x}, \mathbf{y}')}\right|_{\mathcal{C}_{\phi(y'')}}^{-\frac{1}{2}} \sqrt{|dt|}\right] dy'' \\
\text{(Def. of } d_{\phi(y'')} \Rightarrow) \qquad &= \int_{(-\varepsilon,\varepsilon)^e} G(y'')^* \left[\left(\mathfrak{a}(y'') \cdot \sqrt{|d_{\phi(y'')}|}\right)(\mathbf{x}, \rho_Y, y')\right]_{\mathcal{C}_{\phi(y'')}} dy''.
\end{aligned}$$

Applying $(\lambda_{\widetilde{\varphi}})_*$ to the left-hand side, we obtain $\beta_{\widetilde{\varphi}}$. To apply $(\lambda_{\widetilde{\varphi}})_*$ to the right-hand side, we first recall that $\widetilde{\varphi}$ and $\phi(y'')$ are equivalent by $G(y'')$. Using again Lemma 6.4.7 (see also Lemma 6.4.16), this implies

$$\lambda_{\widetilde{\varphi}} = \lambda_{\phi(y'')} \circ G(y'') \Rightarrow (\lambda_{\widetilde{\varphi}})_* = (\lambda_{\phi(y'')})_* \circ G(y'')_*. \tag{6.76}$$

Since $\lambda_{\widetilde{\varphi}}$ does not depend on y'', we can take it inside the integral and use (6.76), finally obtaining

$$\begin{aligned}
\beta_{\widetilde{\varphi}} = (\lambda_{\widetilde{\varphi}})_* &\left[\int_{(-\varepsilon,\varepsilon)^e} G(y'')^* \Big[\Big(\mathfrak{a}(y'') \cdot \sqrt{|d_{\phi(y'')}|} \Big) \Big]_{\mathcal{C}_{\phi(y'')}} dy'' \right] \\
&= \int_{(-\varepsilon,\varepsilon)^e} (\lambda_{\phi(y'')})_* \circ G(y'')_* \circ G(y'')^* \Big[\Big(\mathfrak{a}(y'') \cdot \sqrt{|d_{\phi(y'')}|} \Big) \Big]_{\mathcal{C}_{\phi(y'')}} dy'' \\
&= \int_{(-\varepsilon,\varepsilon)^e} (\lambda_{\phi(y'')})_* \Big[\Big(\mathfrak{a}(y'') \cdot \sqrt{|d_{\phi(y'')}|} \Big) \Big]_{\mathcal{C}_{\phi(y'')}} dy'' = \int_{(-\varepsilon,\varepsilon)^e} \alpha_\phi(y'')\, dy'' = \gamma_{\widetilde{\varphi}}.
\end{aligned}$$

Extension to the corner points as in the previous subsections proves the claim. ☐

We already showed that the half-density α associated with $I_\varphi(a)$ is invariant under a change of equivalent non-degenerate phase functions. Together with the argument above, this also shows that the half-density γ associated with $I_\varphi(a)$ remains the same under the change of equivalent phase functions which are clean with the same excess.

6.6.4. Principal Symbol and Principal Symbol Map

Let $u \in I^{m_e,m_\psi}(X,\Lambda)$. Consider any local representation of u, as introduced in Definition 6.5.10, with clean phase function φ with excess e associated with Λ and a some local symbol density. The arguments in the previous subsections show how to associate with these data a half-density γ, defined on Λ. We also showed that switching to an equivalent phase function, as well as the elimination of the excess, do not change γ. The reduction of the fiber variables replaces γ with γ' such that

$$\gamma' = e^{i\frac{\pi}{4}\operatorname{sgn}(Q)}\,\gamma,$$

with Q from (6.57). Let $\widetilde{\gamma}$ be the half-density defined by an integral representation $I_{\widetilde{\varphi}}(\widetilde{a})$, with another phase function $\widetilde{\varphi}$ associated with Λ. Then, similarly to [61], in general we have

$$\widetilde{\gamma} = e^{i(\sigma-\widetilde{\sigma})\frac{\pi}{4}}\,\gamma, \tag{6.77}$$

where $\sigma = \operatorname{sgn}\left(\rho_Y^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_Y\varphi\right)$, and $\widetilde{\sigma} = \operatorname{sgn}\left(\rho_{\widetilde{Y}}^{-1}\rho_X^{-1}\,{}^{\mathrm{sc}}H_{\widetilde{Y}}\widetilde{\varphi}\right)$. Denote by $\widetilde{r}$ the number of fiber variable for $\widetilde{\varphi}$, $\widetilde{s}$ the dimension of $\widetilde{Y}$ and $\widetilde{e}$ the excess of $\widetilde{\varphi}$, and define the integer number

$$\kappa = \frac{1}{2}(\sigma - \widetilde{\sigma} - s + \widetilde{s} + e - \widetilde{e}).$$

Then, (6.77) is equivalent to
$$i^{\kappa} e^{i(s-e)\frac{\pi}{4}}\, \gamma = e^{i(\widetilde{s}-\widetilde{e})\frac{\pi}{4}}\, \widetilde{\gamma}. \tag{6.78}$$
We are then led to the following definition of principal symbol map.

Definition 6.6.5. Let $u \in I^{m_e,m_\psi}(X,\Lambda)$. We define $\mathscr{I}(u) = \{(Y_j,\varphi_j)\}$ as the collection of manifolds and associated clean phase functions (Y_j,φ_j) locally parametrizing Λ, giving rise to local representations of u in the form $I_{\varphi_j}(a_j)$. With each pair $(Y,\varphi) \in \mathscr{I}(u)$ we associate the half-density γ, as described in Subsection 6.6.3, in such a manner that, for any other element $(\widetilde{Y},\widetilde{\varphi}) \in \mathscr{I}(u)$, we have the coherence relation (6.78) in $\lambda_\varphi(Y) \cap \lambda_{\widetilde{\varphi}}(\widetilde{Y})$. We call the collection of half-densities $\{\gamma_j\}$, each one associated with $(Y_j,\varphi_j) \in \mathscr{I}(u)$, the *principal symbol of* u, and write $j^{\Lambda}_{m_e,m_\psi}(u) = \{\gamma_j\}$.

By an argument completely similar to the one in [61], we can prove the following result.

Theorem 6.6.6. *Let Λ be a* sc*-Lagrangian on X. Then, the map*
$$j^{\Lambda}_{m_e,m_\psi} : I^{m_e,m_\psi}(X,\Lambda) \ni u \mapsto \{\gamma_j\} \tag{6.79}$$
given in Definition 6.6.5 is surjective. Moreover, the null space of the map (6.79) is given by $I^{m_e-1,m_\psi-1}(X,\Lambda)$, and thus (6.79) defines a bijection
$$\text{classes in } I^{m_e,m_\psi}(X,\Lambda)/I^{m_e-1,m_\psi-1}(X,\Lambda) \mapsto \{\gamma_j\}.$$
The image space of $j^{\Lambda}_{m_e,m_\psi}$ can be seen as $\mathcal{C}^\infty(\Lambda, M_\Lambda \otimes \Omega^{1/2})$, where M_Λ is the Maslov bundle over Λ.

6.7. Resolution of Lagrangian Singularities near the Corner

In this appendix, we show that $\Lambda^{\psi e}$ may be viewed as a Legendre manifold with respect to a (degenerate) contact form, well defined on the blow-up of the corner component $\mathcal{W}^{\psi e}$ of ${}^{\mathrm{sc}}\overline{T}^*X$.

We have already stated that the forms
$$\alpha^\psi := \rho_\Xi^2 \partial_{\rho_\Xi} \lrcorner\, \omega \quad \text{and} \quad \alpha^e := \rho_X^2 \partial_{\rho_X} \lrcorner\, \omega.$$
are well-defined in the interior near the respective boundary face $\mathcal{W}^e$ or $\mathcal{W}^\psi$ and extend to it. The freedom in choosing the boundary defining function has as a consequence that these forms are merely well-defined up to a multiple by a positive function, however their contact structure at the boundary (which is all we need to characterize $\Lambda^\bullet$ as Legendrian) is independent of the choice of bdfs. Neither form extends to the corner component $\mathcal{W}^{\psi e}$. Instead of the rescaled 1-forms, we now consider the non-rescaled forms
$$\begin{aligned} {}^{\mathrm{sc}}\alpha^\psi &:= \rho_\Xi \partial_{\rho_\Xi} \lrcorner\, \omega \\ {}^{\mathrm{sc}}\alpha^e &:= \rho_X \partial_{\rho_X} \lrcorner\, \omega \end{aligned}$$
as sections of ${}^{\mathrm{sc}}T^*({}^{\mathrm{sc}}T^*X^o)$. Then, these extend as *scattering one forms* on ${}^{\mathrm{sc}}\overline{T}^*X$, cf. [45, (2.11)].

Lemma 6.7.1. *The forms* ${}^{\mathrm{sc}}\alpha^{\psi}$ *and* ${}^{\mathrm{sc}}\alpha^{e}$ *extend from* ${}^{\mathrm{sc}}T^*X^o$ *to scattering one-forms on* ${}^{\mathrm{sc}}\overline{T}^*X$*. In a particular choice of coordinates (see [45] and Remark 6.2.2) they are given by*

$$
\begin{aligned}
{}^{\mathrm{sc}}\alpha^{e} &= \frac{d\eta_1}{\rho_X\rho_\Xi} - \frac{\eta_1 d\rho_\Xi}{\rho_X\rho_\Xi^2} + \eta'' \frac{dx}{\rho_X\rho_\Xi},\\
{}^{\mathrm{sc}}\alpha^{\psi} &= \eta_1 \frac{d\rho_X}{\rho_\Xi\rho_X^2} + \eta''\frac{dx}{\rho_X\rho_\Xi}.
\end{aligned}
$$

Here, $\eta = (\eta_1, \eta'')$ *are smooth functions of* (ρ_Ξ, ξ)*,* $d-1$ *of which may be chosen as coordinates.*

Again, the (scattering) contact structures of these forms, when restricted to the respective boundary faces, do not depend on the choice of bdf, since two choices of bdf only differ by positive factors. These forms ${}^{\mathrm{sc}}\alpha^{\bullet}$ will then vanish on $\Lambda^{\bullet}$, $\bullet \in \{e, \psi\}$, since one can identify the kernels of ${}^{\mathrm{sc}}\alpha^{\bullet}$ with that of $\alpha^{\bullet}$ by rescaling there. Furthermore, both ${}^{\mathrm{sc}}\alpha^{\psi}$ as well as ${}^{\mathrm{sc}}\alpha^{e}$ vanish when restricted to $\Lambda^{\psi e}$.

Example 6.7.2. On $T^*\mathbb{R}^d$ with canonical coordinates (x, ξ), this corresponds to both the forms

$$\xi \cdot dx \quad \text{and} \quad -x\cdot d\xi$$

vanishing on the bi-conic (in x and ξ) manifold with base $\Lambda^{\psi e}$, cf. [7].

Hence, $\Lambda^{\psi e}$ is, in some sense, (scattering) isotropic.[6] We note, however, that the $\Lambda^{\psi e}$ is not Lagrangian with respect to any symplectic form on $\mathcal{W}^{\psi e}$, since

$$\dim(\Lambda^{\psi e}) = d-2 \neq d-1 = \frac{\dim(\mathcal{W}^{\psi e})}{2}.$$

However, we may now blow-up the corner $\mathcal{W}^{\psi e}$ in ${}^{\mathrm{sc}}\overline{T}(X)$ and consider the front face $\beta^{-1}(\mathcal{W}^{\psi e})$ in $[{}^{\mathrm{sc}}\overline{T}(X); \mathcal{W}^{\psi e}]$, which is a $2d-1$ dimensional manifold, see Figure 6.3. Here,

$$\beta : [{}^{\mathrm{sc}}\overline{T}(X); \mathcal{W}^{\psi e}] \to {}^{\mathrm{sc}}\overline{T}(X),$$

is the blow-down map.

Proposition 6.7.3. *The lift of the form*

$$\alpha^{\psi e} = \frac{\rho_X\rho_\Xi}{2}({}^{\mathrm{sc}}\alpha^{\psi} + {}^{\mathrm{sc}}\alpha^{e})$$

to the blowup space

$$[{}^{\mathrm{sc}}\overline{T}^*X; \mathcal{W}^{\psi e}] \xrightarrow{\beta} {}^{\mathrm{sc}}\overline{T}^*X$$

restricts to a contact 1-form on the front face $\beta^{-1}\mathcal{W}^{\psi e}$*. Moreover,* $\beta^{-1}(\Lambda^{\psi e})$ *is Legendrian with respect to* $\alpha^{\psi e}$*.*

[6] Not with respect to the standard symplectic form, since it does not extend to the boundary, but to a rescaling of it.

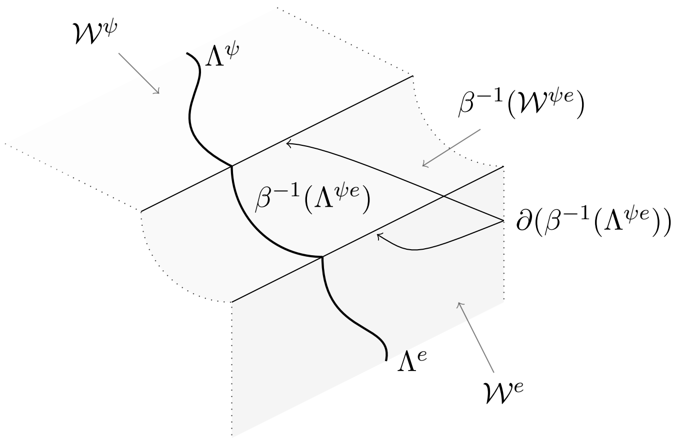

Figure 6.3.: Resolution of Λ^e_φ near the corner

Proof. We note that

$$\alpha^{\psi e} = \rho_X \rho_\Xi \frac{1}{2} (\rho_X \partial_{\rho_X} + \rho_\Xi \partial_{\rho_\Xi}) \lrcorner\, \omega.$$

In the special choice of coordinates of Lemma 6.7.1, we compute

$$\alpha^{\psi e} = \frac{1}{2}\eta_1 \left(\frac{d\rho_X}{\rho_X} - \frac{d\rho_\Xi}{\rho_\Xi} \right) + \frac{1}{2} d\eta_1 + \eta'' dx$$

Now, smooth coordinates on the blow up of ${}^{\mathrm{sc}}\overline{T}^* X$ along $\mathcal{W}^{\psi e} = \{\rho_X = \rho_\Xi = 0\}$ are given by

$$\begin{cases} \rho = \rho_X & \tau = \frac{\rho_\Xi}{\rho_X} \quad (x, \xi) \quad \rho_X > \rho_X \\ \rho = \rho_\Xi & \tau = \frac{\rho_X}{\rho_\Xi} \quad (x, \xi) \quad \rho_\Xi > \rho_X \end{cases} \tag{6.80}$$

In any case, $\beta^* \alpha^{\psi e}$ is of the form

$$\alpha^{\psi e} = \pm \frac{1}{2} \eta_1 \frac{d\tau}{\tau} + \frac{1}{2} d\eta_1 + \eta'' dx$$

Since $\tau = 0$ marks the boundary of the front face $\beta^{-1}\mathcal{W}^{\psi e}$, $\alpha^{\psi e}$ is a 1-form on the interior of $\beta^{-1}\mathcal{W}^{\psi e}$. Finally, $\alpha^{\psi e}$ vanishes on $\beta^{-1}\Lambda^{\psi e}$ since ${}^{\mathrm{sc}}\alpha^\psi$ and ${}^{\mathrm{sc}}\alpha^e$ vanish on $\Lambda^{\psi e}$. $\square$

A

Appendix

A.1. Conventions

We use the following conventions.

- We use the usual notation for the natural numbers $\mathbb{N}$, including 0, the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, and the real numbers $\mathbb{R}$. We will write $\mathbb{R}_+$ for the interval $[0, \infty)$.
- Denote the set of smooth functions by $\mathcal{C}^\infty$, the smooth functions with compact support by $\mathcal{C}_c^\infty$.
- We will use the multiindex notations $x^\alpha = \prod_j x_j^{\alpha_j}$, $\partial_x^\alpha = \prod_j \partial_{x_j}^{\alpha_j}$, $\alpha! = \prod_j \alpha_j!$, and $|\alpha| = \sum_j \alpha_j$ for $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$. Thus, we can write the k-th derivative as

$$f^{(k)}(x; y, \dots, y) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \partial_x^\alpha f(x) y^\alpha.$$

- The L^2-bilinear product will be denoted by

$$\langle u, v \rangle_{L^2} = \int u(x) v(x) dx.$$

  The sesquilinear product and the norm a denoted by

$$\begin{aligned} (u, v)_{L^2} &= \langle u, \bar{v} \rangle_{L^2}, \\ \|u\|_{L^2} &= \langle u, \bar{u} \rangle_{L^2}^{1/2}. \end{aligned}$$

  We will drop the L^2-subscript if it is causes no confusion.
- Estimates are often written as $f \lesssim g$. This means

$$f(x) \lesssim g(x) \text{ if and only if } f(x) \le C g(x),$$

  for some $C > 0$. The constant C might depend on additional parameters.

- We write $n \gg k$ to indicate that n is significant larger than k.
- The big-O and small-O are defined as

$$\begin{aligned} f(\lambda) = O(g(\lambda)) \quad &\text{if and only if} \quad |f(\lambda)| \lesssim |g(\lambda)| \text{ for } \lambda \gg 0, \\ f(\lambda) = o(g(\lambda)) \quad &\text{if and only if} \quad |f(\lambda)|/|g(\lambda)| \to 0 \text{ as } \lambda \to \infty. \end{aligned}$$

- The japanese bracket is defined by

$$\langle x \rangle = (1 + |x|^2)^{1/2},$$

  for $x \in \mathbb{R}^d$, and it satisfies for any $k \in \mathbb{R}$ the *Peetre inequality*

$$\langle x + y \rangle^k \lesssim \langle x \rangle^k \langle y \rangle^{|k|}.$$

- The measure $đx = (2\pi)^{-d}\, dx$ turns the Fourier transform

$$\mathcal{F}u(\xi) = \int e^{-ix\xi} u(x) dx$$

  into a unitary operator $\mathcal{F} : L^2(\mathbb{R}^d, dx) \to L^2(\mathbb{R}^d, đ\xi)$.
- We set $D_{x_j} = -i\partial_{x_j}$ or more generally, $D^\alpha = i^{-|\alpha|}\partial^\alpha$.
- The Laplacian $\Delta = -\sum_j \partial_{x_j}^2$ is non-negative.

A.2. Distributions

Let X be an open subset of $\mathbb{R}^d$. A distribution $u \in \mathcal{D}'(X)$ is a linear form on $\mathcal{C}_c^\infty(X)$ such that for all compact sets $K \subset X$, there exists a $k \in \mathbb{N}$ such that

$$|u(\phi)| \lesssim \sum_{|\alpha| \leq k} \sup_{x \in K} |\partial_x^\alpha \phi(x)|, \quad \phi \in \mathcal{C}_c^\infty(K).$$

If $f \in L^1_{\mathrm{loc}}$ then we can define a distribution u_f by

$$u_f(\phi) = \int_X f(x)\phi(x) dx,$$

and we will identify u_f and f. Therefore, we will also write

$$\langle u, \phi \rangle = u(\phi).$$

We can define multiplication by smooth function and differentiation as follows: For any smooth function $f \in \mathcal{C}^\infty(X)$ and distribution $u \in \mathcal{D}'(X)$ the distribution $f \cdot u$ is given by

$$\langle fu, \phi \rangle = \langle u, f \cdot \phi \rangle$$

and $\partial_x u$ is defined by

$$\langle \partial_x u, \phi \rangle = -\langle u, \partial_x \phi \rangle$$

for all $\phi \in \mathcal{C}_c^\infty(X)$. We say that $x \in \operatorname{supp} u$ if there exists no neighborhood of x such that the restriction of u to U is zero. The set of compactly supported distributions is denoted by

$$\mathcal{E}'(X) := \{u \in \mathcal{D}' \colon \ \operatorname{supp} u \text{ is compact } \}.$$

It is not possible to define the Fourier transform for general distributions. Thus, we consider the set of *Schwartz functions* on $\mathbb{R}^d$. A function $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ is in $\mathcal{S}(\mathbb{R}^d)$ if for all $\alpha, \beta \in \mathbb{N}^d$,

$$|\langle x \rangle^\beta \partial_x^\alpha f(x)| < \infty.$$

These semi-norms turn $\mathcal{S}(\mathbb{R}^d)$ into a Fréchet space. The topological dual space is denoted by $\mathcal{S}'(\mathbb{R}^d)$ and called the set of *tempered distributions*. We have the following inclusions:

$$\mathcal{C}_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d).$$

Example A.2.1. The most important distribution that does not come from a function is the Delta-distribution $\delta_{x_0} \in \mathcal{S}'(\mathbb{R}^d)$ given by

$$\delta_{x_0}(f) = f(x_0),$$

for $x_0 \in \mathbb{R}^d$.

Sometimes we will write for a distributions u, v that $u(x) = v(x)$ this has to be intepreted as $\langle u, f \rangle = \langle v, f \rangle$ for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ or equivalently $u = v$.

A.3. Fourier Transform

Let $u \in \mathcal{S}(\mathbb{R}^d)$ then we define the *Fourier transform* of u by

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} u(x)\, dx.$$

The Fourier transform is an isomorphism $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ with inverse

$$\mathcal{F}^{-1}v(x) = \int_{\mathbb{R}^d} e^{ix\xi} v(\xi)\, đ\xi.$$

Furthermore, the Fourier transform satisfies the Plancharel identity:

$$\langle \mathcal{F}u, v\rangle_{L^2} = \langle u, \mathcal{F}v\rangle_{L^2},$$

and therefore

$$\|\mathcal{F}u\|_{L^2} = (2\pi)^d \|u\|_{L^2}.$$

Thus, we may extend the definition of the Fourier transform to L^2-functions and tempered distributions by duality.

In particular, if $d\mu$ is a measure the Fourier transform is given by

$$\mathcal{F}d\mu(\xi) = \int e^{-ix\xi} d\mu(x).$$

- Derivatives:

$$\mathcal{F}\{\partial_x^\alpha u\}(\xi) = (i\xi)^\alpha \hat{u}(\xi).$$

- Convolution:

$$\mathcal{F}\{u * v\}(\xi) = \hat{u}(\xi)\hat{v}(\xi).$$

- The delta distribution:

$$\mathcal{F}\{\delta_{x_0}\}(\xi) = e^{ix_0\xi}.$$

A.4. Wavefront Sets

Let $u \in \mathcal{D}'(\mathbb{R}^d)$ be a distribution. We want to investigate in which sense u fails to be a smooth function.

Let $x_0 \in \mathbb{R}^d$. We say that x_0 is not in the singular support of u, $x_0 \notin \operatorname{sing-supp} u$ if there exists $\chi_{x_0} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ a cut-off function supported near x_0 such that

$$\chi_{x_0} u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$$

in the sense that there exists a function $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$u(\chi_{x_0}\phi) = \langle f, \chi_{x_0}\phi\rangle.$$

A more refined definition is the *wavefront set*.

Definition A.4.1. Let $(x_0, \xi_0) \in T^*\mathbb{R}^d \setminus 0$. We say that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$ if there exists a smooth cut-off function $\chi_{x_0} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ supported near x_0 such that

$$\mathcal{F}\{\chi_{x_0} u\}(\xi) \lesssim \langle \xi \rangle^{-N}$$

for any $N > 0$ in a small conic neighborhood of ξ_0.

If π_1 denotes the canonical projection $T^*\mathbb{R}^d \to \mathbb{R}^d$ then the singular support is given by

$$\text{sing-supp}\, u = \pi_1 \mathrm{WF}_{\mathrm{cl}}(u).$$

This means that the singular support measures, where the distribution is not smooth and the wavefront set measures also *in which direction.*

Proposition A.4.2. *Let P a differential operator with smooth coefficients and $u \in \mathcal{S}'(\mathbb{R}^d)$ then*

$$\mathrm{WF}_{\mathrm{cl}}(Pu) \subset \mathrm{WF}_{\mathrm{cl}}(u)$$

Proof. This follows from the fact that for smooth functions $a \in \mathcal{C}^\infty$, $\mathrm{WF}_{\mathrm{cl}}(au) \subset \mathrm{WF}_{\mathrm{cl}}(u)$ and $\mathrm{WF}_{\mathrm{cl}}(\partial_x u) \subset \mathrm{WF}_{\mathrm{cl}}(u)$. ☐

Conversely we can characterize the wavefront set of u in terms of the wavefront set of Pu and the set, where P fails to be elliptic: Let $P = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ a differential operator with smooth coefficients and

$$\sigma^m(P) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$$

its principal symbol. Let $\Sigma(P) = \{(x,\xi) \in T^*\mathbb{R}^d \setminus 0 \colon \sigma^m(P)(x,\xi) = 0\}$.

Proposition A.4.3. *If $u \in \mathcal{D}'(\mathbb{R}^d)$ then*

$$\mathrm{WF}_{\mathrm{cl}}(u) \subset \mathrm{WF}_{\mathrm{cl}}(Pu) \cup \Sigma(P).$$

The proof uses a construction of a pseudodifferential parametrix of P away from the characteristic set $\Sigma(P)$ and can be found in [31]. Of course the assumptions on P can be relaxed to P being a pseudodifferential operator of order m.

Example A.4.4. The wavefront set of δ_{x_0} is given by

$$\mathrm{WF}_{\mathrm{cl}}(\delta_{x_0}) = \{x_0\} \times \mathbb{R}^d \setminus \{0\}.$$

The wavefront set of the characteristic function χ_Ω of a smooth domain $\Omega \subset \mathbb{R}^d$ is given by

$$\mathrm{WF}_{\mathrm{cl}}(\chi_\Omega) = N^*\Omega,$$

where $N^*\Omega$ is the conormal bundle of Ω.

A.4.1. Kohn-Nirenberg Symbols

The usual Kohn–Nirenberg symbol class is given by

$$S^m(\mathbb{R}^d;\mathbb{R}^n) = \big\{a(y,\eta) \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^n) \colon |\partial_y^\alpha \partial_\eta^\beta a| \le C_{\alpha\beta}\langle\eta\rangle^{m-|\beta|}\big\}. \tag{A.1}$$

A symbol $a \in S^m$ is elliptic at $(y_0,\eta_0) \in \mathbb{R}^d \times \mathbb{R}^n \setminus \{0\}$ if in a neighborhood of y_0 and in conic neighborhood of η_0 the following estimate holds:

$$|a(y,\eta)| \gtrsim \langle\eta\rangle^m,$$

where the implied constant is independent of η.

The usual class of pseudodifferential operators Ψ^m is defined by using these symbol estimates and then the characterization of the wavefront set is as follows (cf. [31]):

$$\mathrm{WF}_{\mathrm{cl}}(u) = \bigcap_{\substack{A\in\Psi^0 \\ Au\in\mathcal{C}^\infty}} \Sigma(A),$$

where the characteristic set $\Sigma(A)$ is defined as the complement of the elliptic set. Notice that in this case it is conic in η, not in (y,η).

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