

Justification of mean-field coupled modulation equations

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We are interested in reflection symmetric ($x \mapsto -x$) evolution problems on the infinite line. In the systems which we have in mind, a trivial ground state loses stability and bifurcates into a temporally oscillating, spatial periodic pattern. A famous example of such a system is the Taylor–Couette problem in the case of strongly counter-rotating cylinders. In this paper, we consider a system of coupled Kuramoto–Shivashinsky equations as a model problem for such a system. We are interested in solutions which are slow modulations in time and in space of the bifurcating pattern. Multiple scaling analysis is used in the existing literature to derive mean-field coupled Ginzburg–Landau equations as approximation equations for the problem. The aim of this paper is to give exact estimates between the solutions of the coupled Kuramoto–Shivashinsky equations and the associated approximations.

1. Introduction

We consider a system of coupled Kuramoto–Shivashinsky equations:

$$\begin{aligned} \partial_t u &= -(1 + \partial_x^2)u - \alpha_1 \partial_x u - \alpha_2 \partial_x^3 u + \varepsilon^2 u + \partial_x(u^2 + uv + v^2), \\ \partial_t v &= -(1 + \partial_x^2)v + \alpha_1 \partial_x v + \alpha_2 \partial_x^3 v + \varepsilon^2 v + \partial_x(u^2 + uv + v^2), \end{aligned} \tag{1.1}$$

with $t \geq 0$, $x \in \mathbb{R}$, and $\alpha_1, \alpha_2, u, v \in \mathbb{R}$. The variable ε with $1 \gg \varepsilon^2 > 0$ is used as a bifurcation parameter. The linearisation at $(u, v) \equiv (0, 0)$ possesses eigenfunctions of the form $(u, v)(x, t) = (a_k e^{ikx}, b_k e^{ikx})$, $(a_k, b_k \in \mathbb{C}, k \in \mathbb{R})$ with associated eigenvalues $\hat{\lambda}_{1,2}(k, \varepsilon^2) = -(1 - k^2)^2 \mp \alpha_1 ik \pm \alpha_2 ik^3 + \varepsilon^2$. We observe that $\text{Re } \hat{\lambda}_{1,2}(k, \varepsilon^2)$ is positive for wave numbers k close to the values $\pm k_c = \pm 1$.

Centre manifold theory (cf. [9]) allows us to exploit the bifurcation scenario in the weakly unstable case if we assume spatial periodicity with period 2π for the solutions. If we suppress the spatial periodicity assumption, a whole band of Fourier modes with positive growth rates appears. Thus, centre manifold theory is no longer available to describe the dynamics of the bifurcating solutions. We expect that the bifurcating solutions are slow modulations in time and in space of the bifurcating pattern e^{ix} due to the form of $\hat{\lambda}_{1,2}$. Therefore, we make the ansatz.

$$\begin{aligned} u &= \psi_u(t, x, \varepsilon)[A, B] = \varepsilon A(\varepsilon(x - \alpha t), \varepsilon^2 t) e^{i(x - \beta t)} + \varepsilon^2 \sum_{j=0, \pm 2} A_2^j(\varepsilon(x - \alpha t), \varepsilon^2 t) e^{2ix} e^{ij\beta t} \\ &\quad + \varepsilon^2/2 A_0^0(\varepsilon(x - \alpha t), \varepsilon^2 t), \varepsilon^2 t) + c.c., \\ v &= \psi_v(t, x, \varepsilon)[A, B] = \varepsilon B(\varepsilon(x + \alpha t), \varepsilon^2 t) e^{i(x + \beta t)} + \varepsilon^2 \sum_{j=0, \pm 2} B_2^j(\varepsilon(x + \alpha t), \varepsilon^2 t) e^{2ix} e^{ij\beta t} \\ &\quad + \varepsilon^2/2 B_0^0(\varepsilon(x + \alpha t), \varepsilon^2 t) + c.c., \end{aligned} \tag{1.2}$$

with $\alpha, \beta \in \mathbb{R}$. We introduce the spatial translation operator $(\mathcal{F}_y u)(x, t) = u(x + y, t)$

and the spatial scaling operator $(\mathcal{S}_\varepsilon u)(x, t) = u(\varepsilon x, t)$. Inserting the ansatz (1.2) into (1.1) and equating the coefficients in front of $\varepsilon^n e^{imx} e^{ij\beta t}$ to zero gives

$$\alpha = -\alpha_1 + 3\alpha_2, \quad \beta = \alpha_1 - \alpha_2,$$

and the equations for $A = A(X, T)$, $B = B(X, T)$, $A_m^j = A_m^j(X, T)$ and $B_m^j = B_m^j(X, T)$, where $T = \varepsilon^2 t$ and $X = \varepsilon x$.

Elimination of the variables A_m^j and B_m^j for $m = 0, 2$; $j = 0, \pm 2$ by the algebraic equations

$$\begin{aligned} (\mathcal{T}_{-\alpha T/\varepsilon} A_0^0) &= (\mathcal{T}_{\alpha T/\varepsilon} B_0^0) = 0, \\ (9 - 6i\alpha_2)(\mathcal{T}_{-\alpha T/\varepsilon} A_2^{-2}) &= (9 - 4i\alpha_1 + 10\alpha_2)(\mathcal{T}_{\alpha T/\varepsilon} B_2^{-2}) = 2i(\mathcal{T}_{-\alpha T/\varepsilon} A)^2, \\ (9 + 2i\alpha_1 - 8i\alpha_2)(\mathcal{T}_{-\alpha T/\varepsilon} A_2^0) &= (9 - 2i\alpha_1 + 8i\alpha_2)(\mathcal{T}_{\alpha T/\varepsilon} B_2^0) = 2i(\mathcal{T}_{-\alpha T/\varepsilon} A)(\mathcal{T}_{\alpha T/\varepsilon} B), \\ (9 + 4i\alpha_1 - 10i\alpha_2)(\mathcal{T}_{-\alpha T/\varepsilon} A_2^2) &= (9 + 6i\alpha_2)(\mathcal{T}_{\alpha T/\varepsilon} B_2^2) = 2i(\mathcal{T}_{\alpha T/\varepsilon} B)^2, \end{aligned} \tag{1.3}$$

which we have obtained by the above procedure gives finally

$$\begin{aligned} \partial_T A &= A + 4\partial_X^2 A - (\gamma_1 A|A|^2 + a_1 + a_2), \\ \partial_T B &= B + 4\partial_X^2 B - (\gamma_6 B|B|^2 + b_1 + b_2), \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} a_1 &= \gamma_5 A(\mathcal{T}_{2\alpha T/\varepsilon} \bar{B})(\mathcal{T}_{2\alpha T/\varepsilon} B), \\ a_2 &= \gamma_4 A(\mathcal{T}_{2\alpha T/\varepsilon} \bar{B})Ae^{-2i\beta T/\varepsilon^2} + \gamma_2 A\bar{A}(\mathcal{T}_{2\alpha T/\varepsilon} B)e^{2i\beta T/\varepsilon^2} \\ &\quad + \gamma_3(\mathcal{T}_{2\alpha T/\varepsilon} B)\bar{A}(\mathcal{T}_{2\alpha T/\varepsilon} B)e^{4i\beta T/\varepsilon^2} + \gamma_6(\mathcal{T}_{2\alpha T/\varepsilon} B)(\mathcal{T}_{2\alpha T/\varepsilon} \bar{B})(\mathcal{T}_{2\alpha T/\varepsilon} B)e^{2i\beta T/\varepsilon^2}, \\ b_1 &= \gamma_2 B(\mathcal{T}_{-2\alpha T/\varepsilon} \bar{A})(\mathcal{T}_{-2\alpha T/\varepsilon} A), \\ b_2 &= \gamma_5(\mathcal{T}_{-2\alpha T/\varepsilon} A)\bar{B}Be^{-2i\beta T/\varepsilon^2} + \gamma_4(\mathcal{T}_{-2\alpha T/\varepsilon} A)(\mathcal{T}_{-2\alpha T/\varepsilon} A)\bar{B}e^{-4i\beta T/\varepsilon^2} \\ &\quad + \gamma_3 B(\mathcal{T}_{-2\alpha T/\varepsilon} \bar{A})Be^{2i\beta T/\varepsilon^2} + \gamma_1(\mathcal{T}_{-2\alpha T/\varepsilon} A)(\mathcal{T}_{-2\alpha T/\varepsilon} \bar{A})(\mathcal{T}_{-2\alpha T/\varepsilon} A)e^{-2i\beta T/\varepsilon^2}. \end{aligned}$$

As usual, \bar{F} denotes the complex conjugate of F . For problem (1.1) we have the coefficients

$$\begin{aligned} \gamma_1 &= \left(\frac{4}{9 - 6i\alpha_2} + \frac{2}{9 - 4i\alpha_1 + 10i\alpha_2} \right), & \gamma_2 &= \left(\frac{4}{9 + 2i\alpha_1 - 8i\alpha_2} + \frac{2}{9 - 2i\alpha_1 + 8i\alpha_2} \right), \\ \gamma_3 &= \left(\frac{4}{9 + 4i\alpha_1 - 10i\alpha_2} + \frac{2}{9 + 6i\alpha_2} \right), & \gamma_4 &= \left(\frac{2}{9 - 6i\alpha_2} + \frac{4}{9 - 4i\alpha_1 + 10i\alpha_2} \right), \\ \gamma_5 &= \left(\frac{2}{9 + 2i\alpha_1 - 8i\alpha_2} + \frac{4}{9 - 2i\alpha_1 + 8i\alpha_2} \right), & \gamma_6 &= \left(\frac{2}{9 + 4i\alpha_1 - 10i\alpha_2} + \frac{4}{9 + 6i\alpha_2} \right). \end{aligned}$$

Throughout the paper, we assume that α_1 and α_2 are given in such a way that α and β are nonvanishing real constants which are independent of the bifurcation parameter ε . Under this assumption the equations (1.4) still depend strongly on the small bifurcation parameter ε . The aim of the existing literature (see below) is to obtain equations which are independent of the small parameter ε . We follow the arguments of these papers and neglect the terms a_2 and b_2 since they are highly

oscillating in time for $\varepsilon \rightarrow 0$. Secondly, the terms $\bar{B}B$, and $\bar{A}A$ depend on the fast drift terms $X \pm 2\alpha T/\varepsilon$ in a_1 and b_1 . Thus, it is widely expected that the dynamics of A and B is only influenced by the mean value of the terms $\bar{B}B$ and $\bar{A}A$ over a certain space region. Thus, a so-called mean-field coupled system of Ginzburg-Landau equations is considered:

$$\begin{aligned} \partial_T A &= A + 4\partial_X^2 A - (\gamma_1 A|A|^2 + \gamma_5 A \langle |\mathcal{F}_{2\alpha T/\varepsilon} B|^2 \rangle_L), \\ \partial_T B &= B + 4\partial_X^2 B - (\gamma_6 B|B|^2 + \gamma_2 B \langle |\mathcal{F}_{-2\alpha T/\varepsilon} A|^2 \rangle_L), \end{aligned} \tag{1.5}$$

where

$$\langle F \rangle_L(Y) = \frac{1}{2L} \int_{-L}^L F(Y + X) dX$$

for a fixed $L > 0$. These equations are independent of the bifurcation parameter ε if we assume $A(X) = A(X + q)$, $B(X) = B(X + q)$ and $L = q$ for a $q > 0$ fixed. In this case, the shift operator $\mathcal{F}_{2\alpha T/\varepsilon}$ acts trivially on the terms $\langle |\mathcal{F}_{-2\alpha T/\varepsilon} A|^2 \rangle_L$ and $\langle |\mathcal{F}_{2\alpha T/\varepsilon} B|^2 \rangle_L$ since they are constants in space. It is the aim of this paper to show exact estimates between the solutions of (1.1) and the approximations which are obtained by the solutions of (1.5). In the last case our result, which is formulated in Theorem 1.2, simplifies as follows.

THEOREM 1.1. *Let $\|u\|_{H_{1,u}^1} = \sup_{m \in \mathbb{R}} \|u\|_{H^{(m,m+1)}}$. For arbitrary, but fixed, $q > 0$ the following holds: let $(A, B) \in C([0, T_0], (H_{1,u}^8)^2)$ be solutions of (1.5) to $L = q$ which fulfil $A(X) = A(X + q)$, $B(X) = B(X + q)$, and $\sup_{T \in [0, T_0]} \|(A, B)\|_{(H_{1,u}^8)^2} \leq C_1 < \infty$, for a constant C_1 . Then there exists $\varepsilon_0, C > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions (u, v) of (1.1) with*

$$\sup_{0 \leq t \leq T_0/\varepsilon^2} \|(u, v) - (\psi_u[A, B], \psi_v[A, B])\|_{(H_{1,u}^4)^2} \leq C\varepsilon^2.$$

The error of order $\mathcal{O}(\varepsilon^2)$ is small compared with the order $\mathcal{O}(\varepsilon)$ of the solution and of the approximation.

The question arises as to how large the length L of the spatial region can be chosen such that there are really solutions of (1.1) which can be described by (1.5). We will show that solutions of the original system (1.1) exist which can be approximated by the solutions of (1.5), if L is not too large.

THEOREM 1.2. *For arbitrary, but fixed, $q > 0$ and $v \in [0, 1)$ the following holds: let $\varepsilon \in (0, 1) \mapsto (A, B) \in C([0, T_0], (H_{1,u}^8)^2)$ be a family of solutions of (1.5) to $L = q/\varepsilon^v$ which fulfils $\sup_{\varepsilon \in (0, 1)} \sup_{T \in [0, T_0]} \|(A, B)\|_{(H_{1,u}^8)^2} \leq C_1 < \infty$, for a constant C_1 . Then there exist $\varepsilon_0, C > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions (u, v) of (1.1) with*

$$\sup_{0 \leq t \leq T_0/\varepsilon^2} \|(u, v) - (\psi_u[A, B], \psi_v[A, B])\|_{(H_{1,u}^4)^2} \leq C\varepsilon^{2-v}.$$

COROLLARY 1.3. *Assume a spatial period $L = q/\varepsilon^v$ for (1.5) with $v \in [0, 1)$. Then $\langle \bar{A}A \rangle_L$ and $\langle \bar{B}B \rangle_L$ are constants in space and depend only on time. This yields to a spatial period q/ε^{v+1} for the original system (1.1). In this case, the equations (1.5) are independent of ε . Only the underlying interval depends on ε .*

The theory of modulation equations for dissipative systems started with the famous paper of Newell and Whitehead [16] in 1969, in which they derived a single

Ginzburg–Landau equation for Bérnard’s problem. In the last few years, approximation properties for single Ginzburg–Landau equations in the sense of Theorem 1.2 have been shown, for instance, in [1, 4, 10, 21, 25].

Mean-field coupled modulation equations are derived, for instance, in [5, 6, 11–14, 17, 18] for several problems. There are two justification results in the sense of Theorem 1.2. In [19] a system of coupled nonlinear Schrödinger equations is justified as an approximation system for the Sine–Gordon equation. For this problem, highly oscillating terms as in a_2 and b_2 do not appear. In [26] systems similar to (1.4) and (1.5) are compared. In both papers the case $\nu = 0$ is considered. In contrast to these papers, our system (1.1) contains quadratic terms which are responsible for the difficulties in the justification of the Ginzburg–Landau equation. See, for instance, [20].

Here we consider a model problem, but we expect that it is possible to transfer our analysis to more general problems. System (1.1) stands as a model for a translationally invariant dissipative system in a cylindrical domain with one unbounded space direction close to the threshold of instability. At criticality, a temporally oscillating, spatial periodic pattern with period $2\pi/k_c$ bifurcates from the trivial branch. Mean-field coupled Ginzburg–Landau equations can be derived if the system possesses a certain symmetry under reflections $x \mapsto -x$, the so-called reversibility. For such systems, all eigenvalues which occur for the wave number k must also appear for the wave number $-k$. The eigenfunctions of such systems are of the form $e^{ikx}f_k(z)$, with $k \in \mathbb{R}$ and $f_k(z)$ a Banach-space-valued function which is defined on the bounded cross-section alone. For fixed wave number k , an ordered set of eigenvalues $(\tilde{\lambda}_j(k))_{j \in \mathbb{N}}$ with $\operatorname{Re} \tilde{\lambda}_j \geq \operatorname{Re} \tilde{\lambda}_{j+1}$ occurs. The curves $\tilde{\lambda}_{1,2}(k, \varepsilon^2)$ of the most unstable eigenvalues correspond to the curves $\hat{\lambda}_{1,2}(k, \varepsilon^2)$ in our model problem. In the general case, $\operatorname{Im} \tilde{\lambda}'_1(k_c, 0)$ takes the role of α and $-\operatorname{Im} \tilde{\lambda}'_1(k_c, 0)$ the role of β . A famous example of such a situation is the Taylor–Couette problem in the case of strongly counter-rotating cylinders [3]. In our model problem, the nonlinear terms do not respect the reflection symmetry described above.

We finish this introduction with the interesting observation that $\|\langle \bar{A}A \rangle_{\varepsilon^{-1/2}}\|_{H_{1,u}^n} = \mathcal{O}(\varepsilon^{1/2})$ for ε -independent functions A which vanish for $|x| \rightarrow \infty$. Thus, the equations (1.5) decouple for such A s and B s in lowest order. As a consequence, solutions A and B of (1.5) to ε -independent compactly supported initial conditions do not influence each other on timescales of order $\mathcal{O}(1/\varepsilon^2)$. There exists a wide class of other initial conditions on the infinite line which lead to such a decoupling and to a nondependence of (1.5) on the small bifurcation parameter ε in the leading order. Therefore, it is advantageous to include the case $\nu > 0$ in our analysis and not to restrict ourselves to $\nu = 0$. This observation is of great importance for hyperbolic problems, since the modulation equations which occur in this case (NLS-, KdV-equations) possess solitons of homoclinic form. For instance, in [24] it is shown that the Boussinesq equation decouples into two independent KdV-equations for initial conditions which decay sufficiently fast for $|x| \rightarrow \infty$. See also [19], where this decoupling is shown for the Sine–Gordon equation and the nonlinear Schrödinger equation as a modulation equation.

This paper is organised as follows. As a first step, we show that the solutions of system (1.1) can be approximated by the solutions of system (1.4). This part follows strongly along the lines of [20]. Therefore, we recommend the reading of [20]

together with this paper. In a second step, we show that the solutions of (1.4) can be approximated by the solutions of (1.5). Moreover, we explain why the attractivity of solutions in the Ginzburg–Landau form (1.2) (cf. [7, 15, 23]) also holds for our system (1.1). The local existence and uniqueness for solutions of (1.4) and (1.5) is handled in Lemma 2.5.

Throughout this paper, constants are uniformly denoted by C . Moreover, we assume $0 < \varepsilon^2 \ll 1$.

2. Transfer of the classical results

In this section, we show that solutions of (1.1) can be approximated by the solutions of (1.4). In order to do so, we follow the lines of the existing literature about the validity of the Ginzburg–Landau equation. We start this section by introducing the functional analytic set-up, followed by a remark on the attractivity of the set of solutions which are in Ginzburg–Landau form (1.2).

Functional analytic set-up for the infinite line

We recommend the reading of [15] to those interested in the details of the following functional analytic set-up. Here we only cite definitions and theorems which are necessary for what follows. We define

$$\tilde{H}_{l,u}^n = \left\{ u \in L^2_{loc} \mid \partial_x^j u \in L^2_{loc}, (j = 0, \dots, n), \|u\|_{\tilde{H}_{l,u}^n} = \sup_{y \in \mathbb{R}} \|u\|_{\tilde{H}^n(y, y+1)} < \infty \right\},$$

the space of uniform locally square-integrable Sobolev functions. Moreover, we define the Banach space

$$H_{l,u}^n = \{u \in \tilde{H}_{l,u}^n \mid \|\mathcal{T}_y u - u\|_{H_{l,u}^n} \rightarrow 0 \text{ for } y \rightarrow 0\},$$

where $(\mathcal{T}_y u)(x) = u(x + y)$ again denotes the translation operator. This space is equipped with the norm of $\tilde{H}_{l,u}^n$. We write $L^2_{l,u}$ for $H_{l,u}^0$. The space $H_{l,u}^n$ is a dense subspace of $\tilde{H}_{l,u}^n$ for $m < n$, in contrast to $H_{l,u}^n \subset \tilde{H}_{l,u}^m$. As a consequence of this density, the theory of analytic semigroups can be applied in the spaces $H_{l,u}^n$.

In these spaces so-called multipliers can be defined. Multipliers m are operators which are defined by multiplying the Fourier transform $\hat{u} = \mathcal{F}u$ of a function u with an $\hat{m} \in L^\infty(\mathbb{R}, \mathbb{C})$ in Fourier space. In [21] it is shown that it is possible to define operators in the physical space in this way.

LEMMA 2.1. *Let $q, s \geq 0$ and $w_{s-q}(k) = (1 + k^2)^{(s-q)/2} \hat{m}(k) \in C_b^2(\mathbb{R}, \mathbb{C})$. Then $M : H_{l,u}^q \rightarrow H_{l,u}^s; u \mapsto \mathcal{F}^{-1}(\hat{m}\mathcal{F}u)$ is well defined with the estimate*

$$\|Mu\|_{H_{l,u}^s} \leq C(q, s) \|w_{s-q}\|_{C_b^2(\mathbb{R}, \mathbb{C})} \|u\|_{H_{l,u}^q}, \tag{2.1}$$

where $C(q, s)$ does not depend on \hat{m} .

REMARK 2.2. Assume $q \geq 1$ and $w_{s-q}(k) = (1 + k^2)^{(s-q)/2} \hat{m}(k)k \in C_b^2(\mathbb{R}, \mathbb{C})$. Then $M : H_{l,u}^q \rightarrow H_{l,u}^s; u \mapsto \mathcal{F}^{-1}(\hat{m}\mathcal{F}\partial_x u)$ is well defined with the estimate (2.1). This is demonstrated similarly to the proof of the above Lemma 2.1 in [21].

Note that the translation operator \mathcal{T}_y , all differential operators, analytic semigroups and many more operators can be represented by multipliers. The estimate in Lemma 2.1 does not give optimal estimates for many operators, in particular

not for the translation operator. We will denote multipliers in Fourier space with a $\hat{\cdot}$, in physical space without a $\hat{\cdot}$.

The attractivity

It is known (cf. [2, 7, 8, 15, 23]) that all small solutions of a single Kuramoto–Shivashinsky equation develop in such a way that after a time $\mathcal{O}(1/\varepsilon^2)$ they can be approximated by the solutions of the associated Ginzburg–Landau equation. In order to do so, it has to be shown that after this time the solutions can be represented in the form (2.2). It is obvious if we look at the proofs in [2, 7, 23] that the attractivity result also holds for the coupled system (1.1). Nowhere in the proofs is it required that a scalar problem be considered or that the problem possess real eigenvalues. So we state without proof:

THEOREM 2.3. *For fixed $C > 0$, $N \in \mathbb{N}$, and all initial conditions $(u(0), v(0))$ of (1.1) with $\|(u(0), v(0))\|_{(H^4_{i,u})^2} \leq C\varepsilon$ the following is true: there exist $\varepsilon_0(C, N)$, $T_0(C, N) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we obtain a representation of the solution (u, v) at a time T_0/ε^2 through*

$$\begin{aligned} u(T_0/\varepsilon^2, x, \varepsilon) &= \sum_{m=-N}^N \varepsilon^{\gamma(m)} A_m(\varepsilon x) e^{imx} + \mathcal{O}(\varepsilon^{(N+3)/2}), \\ v(T_0/\varepsilon^2, x, \varepsilon) &= \sum_{m=-N}^N \varepsilon^{\gamma(m)} B_m(\varepsilon x) e^{imx} + \mathcal{O}(\varepsilon^{(N+3)/2}). \end{aligned} \tag{2.2}$$

The functions $(A_m, B_m) \in (H^N_{i,u})^2$ fulfil $\|(A_m, B_m)\|_{(H^N_{i,u})^2} \leq (C_0(C, N))$ for a constant $C_0(C, N)$ independent of ε . The function γ is defined by $\gamma(\pm 2) = \gamma(0) = 2$, $\gamma(\pm 1) = 1$ and $\gamma(m) = 1 + |m|/2$ for $|m| \geq 3$.

The approximation property

In this sub-section, we show that solutions of (1.1) can be approximated by the solutions of (1.4). This part follows strongly the lines of the existing literature about the validity of the Ginzburg–Landau equation. Therefore, we recommend the reading of [20] together with this sub-section.

THEOREM 2.4. *Let $\varepsilon \in (0, 1) \mapsto (A, B) \in C([0, T_0], (H^6_{i,u})^2)$ be a family of solutions of (1.4) which fulfils $\sup_{\varepsilon \in (0,1)} \sup_{T \in [0, T_0]} \|(A, B)\|_{(H^6_{i,u})^2} \leq C_1 < \infty$, for a constant C_1 . Then there exists $\varepsilon_0, C > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions $(u, v)(t, x, \varepsilon)$ of (1.1) with*

$$\sup_{0 \leq t \leq T_0/\varepsilon^2} \|(u, v) - (\psi_u[A, B], \psi_v[A, B])\|_{(H^4_{i,u})^2} < C\varepsilon^2.$$

Proof. We use Lemma 2.1 to define so-called model filters. These operators allow us to extract certain intervals of Fourier modes from a given function. Let φ be a C^∞_0 -function with the following properties: $\varphi(k) \in [0, 1]$, $\varphi(k) = 1$ for $k \in [-7/6, -5/6] \cup [5/6, 7/6]$, and $\varphi(k) = 0$ for $k \notin [-5/4, -3/4] \cup [3/4, 5/4]$. According to Lemma 2.1, we denote by E_c the operator in physical space which is associated with the multiplier φ in Fourier space. The operator E_c enables us to extract the unstable Fourier modes from a function. The stable Fourier modes can be extracted with $E_s u = u - E_c u$.

The residual contains all terms which remain after inserting the ansatz ψ_u, ψ_v into

(1.1). It is defined by

$$\begin{aligned} \text{Res}_1(u, v) &= -\partial_t u - (1 + \partial_x^2)^2 u - \alpha_1 \partial_x u - \alpha_2 \partial_x^3 u + \varepsilon^2 u + \partial_x(u^2 + uv + v^2), \\ \text{Res}_2(u, v) &= -\partial_t v - (1 + \partial_x^2)^2 v + \alpha_1 \partial_x v + \alpha_2 \partial_x^3 u + \varepsilon^2 v + \partial_x(u^2 + uv + v^2). \end{aligned} \tag{2.3}$$

Looking at the validity proofs for the Ginzburg–Landau equation in [21, 25], or especially in [20], it is obvious that Theorem 2.4 holds if the residual is sufficiently small, more precisely:

$$\begin{aligned} \sup_{t \in [0, T_0/\varepsilon^2]} \sum_{j=1}^2 \|E_c \text{Res}_j(\psi_u, \psi_v)\|_{H^1_{t,u}} &\leq C\varepsilon^4, \\ \sup_{t \in [0, T_0/\varepsilon^2]} \sum_{j=1}^2 \|E_s \text{Res}_j(\psi_u, \psi_v)\|_{H^1_{t,u}} &\leq C\varepsilon^3. \end{aligned} \tag{2.4}$$

It is obvious that due to the construction of the approximations ψ_u and ψ_v the inequality (2.4) holds, if we can estimate the terms $\partial_t A^j_m$ and $\partial_t B^j_m$ to be of order $\mathcal{O}(\varepsilon)$ for $m = 0, 2; j = 0, \pm 2$. We have the same error bounds ε^4 and ε^3 as in [20], respectively, where the residual is estimated in C^1_b and the solution of the Ginzburg–Landau equation is assumed to be in C^5_b , since $C^1_b \subset H^1_{t,u}$ and $H^6_b \subset C^5_b$.

The terms $\partial_t A^j_m$ and $\partial_t B^j_m$ can be expressed in terms of $\partial_t A$ and $\partial_t B$ by differentiating the algebraic equations (1.3), for instance $(9 + 2i\alpha_1 - 8i\alpha_2)\partial_t A^0_2 = 2i(\mathcal{T}_{2\alpha\varepsilon} B)\partial_t A + A(\mathcal{T}_{2\alpha\varepsilon}(\partial_t B) + 2\alpha\varepsilon\mathcal{T}_{2\alpha\varepsilon}(\partial_x B))$. If we can make use of $\partial_t A = \varepsilon^2\partial_T A$ and $\partial_t B = \varepsilon^2\partial_T B$, respectively, we are done. In order to do so, we need the following lemma.

LEMMA 2.5. *Let $(\tilde{A}, \tilde{B}) \in (H^n_{t,u})^2$ for some $n \geq 2$. Then there exist $C_1, T_0 > 0$ such that for all $\varepsilon \in (0, 1)$ there are solutions $(A, B) \in C([0, T_0], (H^n_{t,u})^2) \cap C^1([0, T_0], (H^{n-2}_{t,u})^2)$ of (1.4) with $(A(0), B(0)) = (\tilde{A}, \tilde{B})$. We have $\sup_{\varepsilon \in (0,1)} \sup_{T \in [0, T_0]} \|(A, B)\|_{(H^n_{t,u})^2} \leq C_1 < \infty$, where C_1 depends only on $\|(\tilde{A}, \tilde{B})\|_{(H^n_{t,u})^2}$. The same result holds for (1.5) for fixed $q > 0, v \in [0, 1)$, and $L = q/\varepsilon^v$.*

Proof. From Lemma 2.1, it follows that $(1 + 4\partial_x^2, 1 + 4\partial_x^2)$ is the generator of an analytic semigroup \mathcal{G}_T in $(L^2_{t,u})^2$ with $\|\mathcal{G}_T\|_{\mathcal{L}((H^n_{t,u})^2, (H^n_{t,u})^2)} \leq C_{m-n} \max(1, T^{-(m-n)/2})$. The nonlinear problem is solved by using the variation of constant formula, i.e.

$$\begin{pmatrix} A(T) \\ B(T) \end{pmatrix} = \mathcal{G}_T \begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} + \int_0^T \mathcal{G}_{T-s} \begin{pmatrix} -(\gamma_1 A|A|^2 + a_1 + a_2)[S] \\ -(\gamma_6 B|B|^2 + b_1 + b_2)[S] \end{pmatrix} \tag{2.5}$$

For a sufficiently small T_0 , the right-hand side defines a contraction in a ball in $C([0, T_0], (H^n_{t,u})^2)$ centred in $\mathcal{G}_T(\tilde{A}, \tilde{B})$. With the usual fixed-point argument, we find the required solution. The existence interval $[0, T_0]$ of the solutions can be chosen independently of $\varepsilon \in (0, 1)$, since the Lipschitz-constant of the contraction is independent of ε . Its length only depends on $\|(\tilde{A}, \tilde{B})\|_{(H^n_{t,u})^2}$. The time derivative $(\partial_T A, \partial_T B)$ is estimated by the right-hand side of (1.4), which is of order $\mathcal{O}(1)$ in $H^{n-2}_{t,u}$. In the same way the result follows for (1.5). \square

Now we can complete the proof of Theorem 2.4. In the following, we use the abbreviations $Z = (H^4_{t,u})^2$ and $Y = (H^1_{t,u})^2$. We write (1.1) as

$$\partial_t \mathcal{U} = \Lambda \mathcal{U} + B(\mathcal{U}, \mathcal{U}), \tag{2.6}$$

where $\mathcal{U} = (u, v)$, $\Lambda = -(1 + \partial_x^2)^2 + \varepsilon^2 - \alpha_1 \partial_x - \alpha_2 \partial_x^3$, $-(1 + \partial_x^2)^2 + \varepsilon^2 + \alpha_2 \partial_x + \alpha_2 \partial_x^3$. The nonlinear terms are written as a symmetric bilinear mapping $B(\mathcal{U}, \mathcal{U})$ which fulfils $\|B(\mathcal{U}, \mathcal{V})\|_Y \leq C \|\mathcal{U}\|_Z \|\mathcal{V}\|_Z$. The modefilters are denoted by $\mathcal{E}_c = (E_c, E_c)$ and $\mathcal{E}_s = (E_s, E_s)$, and the approximation is denoted by $\Psi = (\psi_u, \psi_v)$. Moreover, we define the critical part $\varepsilon \Psi_c = \mathcal{E}_c \Psi$ and the noncritical part $\varepsilon^2 \Psi_s = \mathcal{E}_s \Psi$ of the approximation. We have the estimate

$$\sup_{t \in [0, T_0/\varepsilon^2]} (\|\Psi_c\|_Z + \|\Psi_s\|_Z) \leq C_\psi = \mathcal{O}(1)$$

for $\varepsilon \rightarrow 0$. The operator Λ generates an analytic semigroup $e^{\Lambda t}$ which can be estimated by $\|e^{\Lambda t} \mathcal{E}_c\|_{\mathcal{L}(Y, Z)} \leq C e^{C\varepsilon^2 t}$ and by $\|e^{\Lambda t} \mathcal{E}_s\|_{\mathcal{L}(Y, Z)} \leq C \max(1, t^{-3/4}) e^{-\sigma t}$ for a $\sigma = \mathcal{O}(1) > 0$. We make the ansatz $\mathcal{U} = \varepsilon \Psi_c + \varepsilon^2 \Psi_s + \varepsilon^2 \mathcal{R}_c + \varepsilon^3 \mathcal{R}_s$ and define the error functions \mathcal{R}_c and \mathcal{R}_s to be the solutions of

$$\begin{aligned} \partial_t \mathcal{R}_c &= \Lambda \mathcal{R}_c + \varepsilon^2 L_c(\mathcal{R}) + \varepsilon^3 N_c(\mathcal{R}) + \varepsilon^{-2} \mathcal{E}_c \text{Res}(\Psi), \\ \partial_t \mathcal{R}_s &= \Lambda \mathcal{R}_s + L_s(\mathcal{R}_c) + \varepsilon N_s(\mathcal{R}) + \varepsilon^{-3} \mathcal{E}_s \text{Res}(\Psi). \end{aligned} \tag{2.7}$$

We have used the abbreviations

$$\begin{aligned} L_c(\mathcal{R}) &= \mathcal{E}_c [2B(\mathcal{R}_c, \Psi_s) + 2B(\mathcal{R}_s, \Psi_c)], \\ L_s(\mathcal{R}_c) &= \mathcal{E}_s [2B(\mathcal{R}_c, \Psi_c)], \\ N_c(\mathcal{R}) &= \mathcal{E}_c [B(\mathcal{R}_s, 2\mathcal{R}_c + 2\Psi_s + \varepsilon \mathcal{R}_s)], \\ N_s(\mathcal{R}) &= \mathcal{E}_s [2B(\mathcal{R}_c, \Psi_s) + 2B(\mathcal{R}_s, \Psi_c) + B(\mathcal{R}_c, \mathcal{R}_c) + \varepsilon B(\mathcal{R}_s, 2\mathcal{R}_c + 2\Psi_s + \varepsilon \mathcal{R}_s)]. \end{aligned}$$

Now system (2.7) has exactly the same properties as the corresponding systems [20, (17)] and [22, (12)]. Following the remaining arguments of these papers, we obtain

$$\sup_{t \in [0, T_0/\varepsilon^2]} (\|\mathcal{R}_c\|_Z + \|\mathcal{R}_s\|_Z) \leq C = \mathcal{O}(1) < \infty$$

for $\varepsilon \rightarrow 0$. This concludes the proof of Theorem 2.4. \square

3. Mean-field coupled description

In this section, we prove the validity of Theorem 1.2. In order to do so, it remains to show that the solutions of (1.4) can be approximated by the solutions of (1.5).

THEOREM 3.1. *For arbitrary, but fixed, $q > 0$ and $v \in [0, 1)$ the following holds: let $\varepsilon \in (0, 1) \mapsto (A, B) \in C([0, T_0], (H_{1,u}^8)^2)$ be a family of solutions of (1.5) to $L = q/\varepsilon^v$ which fulfils $\sup_{s \in (0, 1)} \sup_{T \in [0, T_0]} \|(A, B)\|_{(H_{1,u}^8)^2} \leq C_1 < \infty$, for a constant C_1 . Then there exist $\varepsilon_0, C > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions (\hat{A}, \hat{B}) of (1.4) with*

$$\sup_{0 \leq t \leq T_0} \|(\hat{A}, \hat{B}) - (A, B)\|_{(H_{1,u}^6)^2} \leq C\varepsilon^{1-v}.$$

Proof of Theorem 1.2. Let $Z = (H_{1,u}^4)^2$. Using Theorem 2.4 and Theorem 3.1, we obtain

$$\begin{aligned} \|(u, v) - (\psi_u[A, B], \psi_v[A, B])\|_Z &\leq \|(u, v) - (\psi_u[\hat{A}, \hat{B}], \psi_v[\hat{A}, \hat{B}])\|_Z \\ &\quad + \|(\psi_u[\hat{A}, \hat{B}], \psi_v[\hat{A}, \hat{B}]) - (\psi_u[A, B], \psi_v[A, B])\|_Z \\ &\leq C\varepsilon^2 + C\varepsilon^{2-v}. \quad \square \end{aligned}$$

Proof of Theorem 3.1. Now let $Z = (H_{l,\mu}^6)^2$. The nonlinearity in (1.4) is written as a symmetric trilinear mapping \mathcal{N} . We have the estimate $\|\mathcal{N}(u, v, w)\|_Z \leq C\|u\|_Z\|v\|_Z\|w\|_Z$ for a $C > 0$. The terms which remain after inserting a solution $\mathcal{A} = (A, B)$ of (1.5) into (1.4) define again the residual $\text{RES}(\mathcal{A})$ by

$$\begin{pmatrix} \text{RES}_1(\mathcal{A}) \\ \text{RES}_2(\mathcal{A}) \end{pmatrix} = \begin{pmatrix} -\partial_T A + A + 4\partial_X^2 A - (\gamma_1 A|A|^2 + a_1 + a_2) \\ -\partial_T B + B + 4\partial_X^2 B - (\gamma_6 B|B|^2 + b_1 + b_2) \end{pmatrix}.$$

A solution (\hat{A}, \hat{B}) of (1.4) is a sum of an approximation $\mathcal{A} = (A, B)$ and an error $\varepsilon^\mu \mathcal{R}$ with $\mu + \nu = 1$. The error fulfils the differential equation

$$\partial_T \mathcal{R} = \mathcal{L}\mathcal{R} + 3\mathcal{N}(\mathcal{A}, \mathcal{A}, \mathcal{R}) + 3\varepsilon^\mu \mathcal{N}(\mathcal{A}, \mathcal{R}, \mathcal{R}) + \varepsilon^{2\mu} \mathcal{N}(\mathcal{R}, \mathcal{R}, \mathcal{R}) + \varepsilon^{-\mu} \text{RES}(\mathcal{A})$$

with zero-initial conditions and $\mathcal{L} = (1 + 4\partial_X^2, 1 + 4\partial_X^2)$. The local existence and uniqueness of solutions in Z follows immediately from Lemma 2.5. In order to show the assertion of Theorem 3.1, it remains to demonstrate the $\mathcal{O}(1)$ -boundedness of \mathcal{R} on the time interval $[0, T_0]$. The semigroup $\mathcal{G}_T = (S_T, S_T)$ generated by \mathcal{L} can be estimated with the help of Lemma 2.1. This gives the existence of a $C_2 > 0$ such that $\|\mathcal{G}_T\|_{\mathcal{L}(Z,Z)} \leq C_2$ for all $T \in [0, T_0]$. The application of \mathcal{G}_T on (1.4) gives

$$\begin{aligned} \mathcal{R}(T) &= \int_0^T \mathcal{G}_{T-\tau} (3\mathcal{N}(\mathcal{A}, \mathcal{A}, \mathcal{R}) + 3\varepsilon^\mu \mathcal{N}(\mathcal{A}, \mathcal{R}, \mathcal{R}) + \varepsilon^{2\mu} \mathcal{N}(\mathcal{R}, \mathcal{R}, \mathcal{R}))(\tau) d\tau \\ &\quad + \int_0^T \varepsilon^{-\mu} \mathcal{G}_{T-\tau} \text{RES}(\mathcal{A})(\tau) d\tau. \end{aligned}$$

By assumption there exists a $C_1 > 0$ such that $\sup_{T \in [0, T_0]} \|\mathcal{A}\|_Z \leq C_1$. Moreover, we have:

LEMMA 3.2. *There exist $\varepsilon_0, C_{\text{RES}} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$:*

$$\sup_{T \in [0, T_0]} \left\| \int_0^T \varepsilon^{-\mu} \mathcal{G}_{T-\tau} \text{RES}(\mathcal{A})(\tau) d\tau \right\|_Z \leq C_{\text{RES}}.$$

For the proof, see below. We show that there exist $\varepsilon_0, D > 0$ with $\sup_{T \in [0, T_0]} \|\mathcal{R}(T)\|_Z \leq D$ for all $\varepsilon \in (0, \varepsilon_0)$. Assume now the existence of such a D ; then

$$\|\mathcal{R}(T)\|_Z \leq \left(\int_0^T C_2 C C_1^2 \|\mathcal{R}(\tau)\|_Z d\tau \right) + T_0 C_2 C (\varepsilon^\mu C_1 D^2 + \varepsilon^{2\mu} D^3) + C_{\text{RES}}.$$

Suppose now that

$$(\varepsilon^\mu C_1 D^2 + \varepsilon^{2\mu} D^3) \leq 1. \tag{3.1}$$

Then Gronwall’s inequality yields

$$\|\mathcal{R}(T)\|_Z \leq (T_0 C_2 C + C_{\text{RES}}) e^{T_0 C_2 C C_1^2} =: D$$

for $T \in [0, T_0]$. This defines D . We are done, since we can choose $\varepsilon_0 > 0$ such that the assumption (3.1) is fulfilled for all $\varepsilon \in (0, \varepsilon_0)$. \square

Proof of Lemma 3.2. Obviously we can restrict the analysis to the first component of the residual $(\text{RES}_1(\mathcal{A}))_1 = s_1 + a_2$, where $s_1 = a_1 - \gamma_5 A \langle |\mathcal{F}_{2\alpha T/\varepsilon} B|^2 \rangle_L$. The terms

in a_2 are highly oscillating for $\varepsilon \rightarrow 0$. They can be estimated by partial integration in the following way:

$$\begin{aligned} & \sup_{T \in [0, T_0]} \left\| \int_0^T \varepsilon^{-\mu} S_{T-\tau} (A(\mathcal{F}_{2\alpha\tau/\varepsilon} \bar{B})A)(\tau) e^{-2i\beta\tau/\varepsilon^2} d\tau \right\|_{H_{l,u}^6} \\ & \leq \sup_{T \in [0, T_0]} \left\| \int_0^T \varepsilon^{-\mu} \partial_\tau [S_{T-\tau} (A(\mathcal{F}_{2\alpha\tau/\varepsilon} \bar{B})A)] e^{-2i\beta\tau/\varepsilon^2} \varepsilon^2 / (-2i\beta) d\tau \right\|_{H_{l,u}^6} \\ & \quad + \sup_{T \in [0, T_0]} \left\| \varepsilon^{-\mu} [S_{T-\tau} (A(\mathcal{F}_{2\alpha\tau/\varepsilon} \bar{B})A)] e^{-2i\beta\tau/\varepsilon^2} \varepsilon^2 / (-2i\beta) \right\|_{\tau=0}^T \|_{H_{l,u}^6} \\ & \leq \varepsilon^{1-\mu} C \|A\|_{H_{l,u}^8}^2 \|\bar{B}\|_{H_{l,u}^8} + \varepsilon^{2-\mu} C \|A_1\|_{H_{l,u}^8}^2 \|\bar{B}\|_{H_{l,u}^8} = \mathcal{O}(\varepsilon^{1-\mu}). \end{aligned}$$

We have used that $\partial_\tau(\mathcal{F}_{2\alpha\tau/\varepsilon} \bar{B}) = \mathcal{O}(1/\varepsilon)$, $\partial_\tau S_{T-\tau} = -S_{T-\tau}(1 + 4\partial_X^2)$ and that $\partial_\tau A$ can be expressed by the right-hand side of (1.5). All other terms in a_2 can be estimated in the same way.

It remains to bound the term

$$\sup_{T \in [0, T_0]} \left\| \int_0^T \varepsilon^{-\mu} S_{T-\tau} [A(\mathcal{F}_{2\alpha\tau/\varepsilon} \bar{B})(\mathcal{F}_{2\alpha\tau/\varepsilon} B) - A \langle |\mathcal{F}_{2\alpha\tau/\varepsilon} B|^2 \rangle_L] d\tau \right\|_{H_{l,u}^6}, \quad (3.2)$$

for $L = q\varepsilon^{-\nu}$. According to Lemma 2.1 the translation operator $m_\tau = \mathcal{F}_{2\alpha\tau/\varepsilon}$ can be represented by the multiplier $\hat{m}_\tau = e^{2i\alpha k\tau/\varepsilon}$ in Fourier space. We use partial integration to estimate the term (3.2). Integrating the multiplier m_τ with respect to τ gives

$$\begin{aligned} & \sup_{T \in [0, T_0]} \left\| \int_0^T \varepsilon^{-\mu} S_{T-\tau} (A(m_\tau [\bar{B}B - \langle |B|^2 \rangle_L])) d\tau \right\|_{H_{l,u}^6} \\ & \leq \sup_{T \in [0, T_0]} \left\| \int_0^T \varepsilon^{-\mu} S_{T-\tau} (1 + 4\partial_X^2) (A(m_I m_\tau [\bar{B}B - \langle |B|^2 \rangle_L])) d\tau \right\|_{H_{l,u}^6} \\ & \quad + \sup_{T \in [0, T_0]} \left\| \int_0^T \varepsilon^{-\mu} S_{T-\tau} ([\partial_\tau A](m_I m_\tau [\bar{B}B - \langle |B|^2 \rangle_L])) d\tau \right\|_{H_{l,u}^6} \\ & \quad + \sup_{T \in [0, T_0]} \left\| \int_0^T \varepsilon^{-\mu} S_{T-\tau} (A(m_I m_\tau \partial_\tau [\bar{B}B - \langle |B|^2 \rangle_L])) d\tau \right\|_{H_{l,u}^6} \\ & \quad + \sup_{T \in [0, T_0]} \left\| \varepsilon^{-\mu} S_{T-\tau} (A(m_I m_\tau [\bar{B}B - \langle |B|^2 \rangle_L])) \right\|_{\tau=0}^T \|_{H_{l,u}^6}, \quad (3.3) \end{aligned}$$

with the new multiplier $\hat{m}_I = \varepsilon/(2i\alpha k)$. This partial integration is well defined due to Remark 2.2 if m_τ is applied to a gradient of a bounded function. Since

$$f(X) = \bar{B}(X)B(X) - \langle |B(X)|^2 \rangle_L \in H_{l,u}^8 \subset C_b^1$$

possesses a zero in $[X - L, X + L]$, there exists a function $F \in H_{l,u}^9$ with $f = \partial_X F(X)$ and $\sup_{T \in [0, T_0]} \|F\|_{H_{l,u}^9} \leq C\varepsilon^{-\nu}$. We also have $\sup_{T \in [0, T_0]} \|\partial_T F\|_{H_{l,u}^7} \leq C\varepsilon^{-\nu}$ since the same is true for $\partial_T f \in H^6$.

The differential operator ∂_X can be represented as a multiplier which is given in Fourier space by ik . Thus, according to Remark 2.2, we have $\|m_I \partial_X\|_{\mathcal{L}(H_{l,u}^8, H_{l,u}^8)} = \mathcal{O}(\varepsilon)$. We obtain $\|m_\tau\|_{\mathcal{L}(H_{l,u}^8, H_{l,u}^8)} = 1$ since m_τ is a translation

operator. Using that all scalar multipliers commute, we finally come to

$$\|m_\tau m_I [\bar{B}B - \langle |B|^2 \rangle_L]\|_{H_{i,u}^n} \leq \|m_\tau\|_{\mathcal{L}(H_{i,u}^n, H_{i,u}^n)} \|m_I \partial_X\|_{\mathcal{L}(H_{i,u}^n, H_{i,u}^n)} \|F\|_{H_{i,u}^0} \leq \mathcal{O}(e^{1-\nu})$$

for all $n \in \{0, \dots, 9\}$. From $\nu + \mu = 1$ it follows that the last four lines of (3.3) can be estimated by $T_0 C \|A\|_{H_{i,u}^s} \|F\|_{H_{i,u}^0} = \mathcal{O}(1)$. Therefore the proof is complete. \square

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