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On Vertex-Edge-Critically n-Connected Graphs

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For Paul Erdős on his 80th birthday

All digraphs are determined that have the property that when any vertex and any edge that are not adjacent are deleted, the connectivity number decreases by two.

1. Introduction and notation

Whereas the characterization of all graphs having the property that the deletion of any two edges decreases the connectivity number by two is rather easy, and well known [6] (see Section 2), the characterization of all graphs with the analogous property for the deletion of two vertices instead of two edges seems to be hopeless. So the following idea suggests itself. A graph or digraph G is called *vertex-edge-critically n-connected* (abbreviated to *n-ve-critical*), if the deletion of any vertex v and any edge e not incident to v decreases the connectivity number n of G by two (and such v and e exist). If we do not want to specify the connectivity number, we write *vertex-edge-critical* or *ve-critical.* When I determined the minimum number of 1-factors of a $(2k)$-connected graph containing a 1-factor, the ve-critical graphs played an important role and all ve-critical undirected graphs were characterized there [2]. It was shown in [2] that every ve-critical undirected graph is obtained in the following way. For an integer $m \geq 1$, take vertex-disjoint circuits of length $m+2$ and vertex-disjoint copies of $\overline{K}_m$ (the complementary graph of the complete graph K_m on m vertices) and take all edges between these vertex-disjoint graphs. We will give an easier proof of this characterization in Section 3 by using the characterization of all minimally n-connected graphs with exactly $n+1$ vertices of degree n, given in [3]. The main result of the paper is the characterization of all ve-critical digraphs in Sections 4 and 5: *every vertex-edge-critical digraph arises from a vertex-edge-critical undirected graph by replacing every edge with a pair of oppositely directed edges.*

First we will put together our notation and definitions. A (directed) multigraph $G = (V, E)$ consisting of the vertex set $V(G) = V$ and the edge set $E(G) = E$ may have multiple edges, but no loops. A multidigraph is a directed multigraph. The set of edges between the vertices x and y (in the directed case, from x to y) in G is denoted by $[x,y]_G$, and, for X, $Y \subseteq V(G)$, $[X,Y]_G := \bigcup_{x\in X, y\in Y}[x,y]_G$. Distinct edges from $[x,y]_G$ are distinguished by an upper index, for instance $[x,y]^i$. If G is directed and $e \in [x,y]_G$, then x is the *tail* and y is the *head* of e. The set of edges in G with tail in x (head in x) is denoted by $E^+(x;G)$ ($E^-(x;G)$). A *graph* has no multiple edges and is undirected and a *directed graph* or *digraph* has no multiple edges of the same direction. For emphasis, we sometimes say undirected (multi-)graph for (multi-)graph. In a graph or digraph we write $[x,y]$ for the edge from x to y. An edge $[x,y]$ of a digraph D is *symmetric*, if $[y,x] \in E(D)$ also, and *asymmetric*, otherwise. If every edge of a digraph D is symmetric, we call D *symmetric*. In a drawing of a digraph, a pair of symmetric edges is displayed as a line without an arrow-head. Edges $e \in [x,y]_G$ and $e' \in [x',y']_G$ of a directed multigraph G are *consecutive* if $y = x'$ or $y' = x$ holds. For a multigraph G, the directed multigraph $\overleftrightarrow{G}$ arises from G by replacing every edge of G with a pair of oppositely directed edges. For a directed or undirected multigraph G and a positive integer n, G^n is constructed from G by replacing every edge of G with n edges. The *dual* of a digraph D arises from D by reversing the direction of every edge of D. The vertex number and the edge number of G are denoted by $|G|$ and $||G||$, respectively. For a vertex set A, we define $A \cap G := A \cap V(G)$, and $x \in G$ means $x \in V(G)$. For $A \subseteq V(G)$, the submultigraph of G spanned by A is $G(A) := G-(V(G)-A)$. For undirected G and $x \in G$, we use $d(x;G)$ to denote the degree of x in G, and $N(x;G)$ is the set of neighbours of x in G. For directed G and $x \in G$, we use $d^+(x;G)$ ($d^-(x;G)$) to denote the outdegree (indegree) of x in G, and $N^+(x;G)$ ($N^-(x;G)$) is the set of outneighbours (inneighbours) of x in G. For a digraph D and $x \in D$, we define $N_s(x;D) := N^+(x;D) \cap N^-(x;D)$, $N_a^\epsilon(x;D) := N^\epsilon(x;D) - N_s(x;D)$ and $d_a^\epsilon(x;D) := |N_a^\epsilon(x;D)|$ for $\epsilon \in \{+,-\}$, and $\Delta_a(D) := \max\{d_a^\epsilon(x;D) : x \in D \text{ and } \epsilon \in \{+,-\}\}$. A directed multigraph D is called *n-regular* if $d^+(x;D) = d^-(x;D) = n$ for every $x \in D$. If there is no doubt which graph is meant, we drop it in the above notation. $\mathbb{N}$ denotes the set of positive integers, n is always from $\mathbb{N}$, and $\mathbb{N}_m := \{n \in \mathbb{N} : n \leq m\}$ for $m \geq 0$.

A *path* and a *circuit* in G pass through every vertex of G at most once. If G is directed, they are *continuously directed*. For $x,y \in G$, an x,y-path P is a path from x to y, and for $u,v \in P$ such that u is before v on P in the directed case, $P[u,v]$ is the u,v-path contained in P and $P[u,v) := P[u,v] - \{v\} =: P[u,v] - v$. We consider paths and circuits as subgraphs, but write them as a sequence of their vertices in the order passed through (for multidigraphs, in the direction of the path or the circuit). We say that the paths $P_1,\ldots,P_n$ in G cover G if $\bigcup_{i\in\mathbb{N}_n} V(P_i) = V(G)$ holds. In a directed or undirected multigraph G, x,y-paths P,Q are *openly disjoint* if they are distinct and $V(P) \cap V(Q) = \{x,y\}$ holds. The maximum number of pairwise openly disjoint x,y-paths in G is denoted by $\kappa(x,y;G)$. The *connectivity number* $\kappa(G)$ of G is defined by $\kappa(G) := \min_{x\neq y}\kappa(x,y;G)$ for $|G| \geq 2$ and $\kappa(G) := |G|-1$ for $|G| \leq 1$. In an analogous manner, the edge-connectivity number $\lambda(G)$ is defined by edge-disjoint paths. A directed or undirected multigraph G is *k-minimally n-(edge-)connected*, for $k \in \mathbb{N}$, iff $||G|| \geq k$ and, for all $E' \subseteq E(G)$ with $|E'| \leq k$, we have $\kappa(G-E') = n-|E'|$ ($\lambda(G-E') = n-|E'|$). For 1-minimally n-connected we say *minimally n-connected*. Let us make precise the definition of ve-criticality: a (di-)graph G is *vertex-edge-critically n-connected* iff $\kappa(G) = n \geq 2$, and for every $v \in V(G)$ and $e \in E(G-v)$, $\kappa(G-v-e) = n-2$ holds.

A sequence $v_1, [v_1, \overline{v_1}], \overline{v_1}, [v_2, \overline{v_1}], v_2, [v_2, \overline{v_2}], \ldots, [v_n, \overline{v_n}], \overline{v_n}, [v_1, \overline{v_n}]$ of vertices $v_1, \overline{v_1}, \ldots, \overline{v_n}$ and distinct edges of a digraph D is called an *alternating cycle* in D. Normally, we omit the edges in the notation and write $v_1^+, \overline{v_1}, v_2^+, \ldots, \overline{v_n}$ for an alternating cycle, where the upper index $+$ at v_i means that the edges (cyclically) on either side of v_i have their tails in v_i. Sometimes we consider an alternating cycle as a subdigraph of D. If $G_1, \ldots, G_n$ are graphs (digraphs) with $V(G_i) \cap V(G_j) = \emptyset$ for $i \neq j$, the graph (digraph) $\sum_{i=1}^n G_i = G_1 + G_2 + \cdots + G_n$ is defined as

$$\left(\bigcup_{i=1}^n V(G_i), \bigcup_{i=1}^n E(G_i) \cup \bigcup_{i=1}^n \left\{ [x,y] : x \in G_i \text{ and } y \in \bigcup_{j \in \mathbb{N}_n - \{i\}} V(G_j) \right\} \right).$$

If all G_i are isomorphic, we write $nG_1 := \sum_{i=1}^n G_i$. If $G_1, \ldots, G_n$ are not vertex-disjoint, we define $\sum_{i=1}^n G_i$ by vertex-disjoint copies of $G_1, \ldots, G_n$. If $G = H_1 + H_2$ and $E(H_1) = \emptyset$, we also write $G = V(H_1) + H_2$. For an integer $m \geq 3$, C_m denotes an undirected circuit of length m. For integers $m \geq 3$, $k \geq 0$, $l \geq 0$, the multidigraph $D = C_m^{k,l}$ is defined by $V(D) := \mathbb{N}_m$ and $|[i, i+1]_D| = k$, $|[i+1, i]_D| = l$ for i modulo m.

2. 2-minimally n(-edge)-connected graphs

B. Maurer and P. Slater determined in [6] all 2-minimally n-connected graphs and all 2-minimally n-edge-connected multigraphs. We give a simpler proof of the latter result, and show that, in the proof of the former, it is not necessary to use the fact from [1] that every minimally n-connected graph has at least $n+1$ vertices of degree n.

Let $G = (V, E)$ be a 2-minimally n-connected multigraph. Consider any $x \in V$. There are an $e \in E$ incident to x, say, $e \in [x, y]_G$ and a system of n openly disjoint x,y-paths $P_1, \ldots, P_n$. From $\kappa(G-e) < n$, it easily follows that $\kappa(x, y; G) = n$ by Menger's Theorem. Hence $e \in \bigcup_{i=1}^n E(P_i)$. For every $e' \in E - \{e\}$, $\kappa(G - \{e, e'\}) = n-2$ holds and implies $\kappa(x, y; G - \{e, e'\}) = n-2$ by Menger's Theorem. Hence $e' \in \bigcup_{i=1}^n E(P_i)$ and thus $E = \bigcup_{i=1}^n E(P_i)$ and $d(x) = n$ follow. Hence G is finite and n-regular. If $|G| \geq 3$, there is a $z \in V - \{x, y\}$, and z is on exactly one of the paths $P_1, \ldots, P_n$, since $E = \bigcup_{i=1}^n E(P_i)$ holds and $P_1, \ldots, P_n$ are openly disjoint. Hence $E = \bigcup_{i=1}^n E(P_i)$ implies $d(z) = 2$, and G is 2-regular. So we have somewhat generalized a result from Maurer and Slater.

Theorem 1. [6] *The only 2-minimally n-connected multigraphs are K_2^n and, for $n = 2$, the circuits C_m.*

Let $G = (V, E)$ now be a 2-minimally n-edge-connected multigraph, and choose $x \in V$ and $e \in [x, y]_G$ as above. Now there are edge-disjoint x,y-paths $P_1, \ldots, P_n$. As above, $E = \bigcup_{i=1}^n E(P_i)$ and $d(x) = n$ follow. Again, G is finite and n-regular. Let us assume $|G| \geq 3$ and consider $z \in V - \{x, y\}$. Then every edge incident to z belongs to exactly one of $P_1, \ldots, P_n$. Hence n is even and exactly $n/2$ of $P_1, \ldots, P_n$ pass through z. Suppose $N(x) = \{y, y_1, \ldots, y_k\}$. Then $|G| \geq 3$ and $E = \bigcup_{i=1}^n E(P_i)$ imply $\{y_1, \ldots, y_k\} \neq \emptyset$. We define a directed multigraph D on the vertex set $\{y_1, \ldots, y_k\}$. Every path P_j of length at least 2 generates the following edges of D (and there are no further edges in D):

if z is the first vertex of P_j after x, then $z \in \{y_1, \dots, y_k\}$ and we add the edges $[z,u]^j$ for all $u \in (P_j - z) \cap \{y_1, \dots, y_k\}$. We prove that D has no circuit. Suppose there is a circuit in D, and this may have the edges $[z_1,u_1]^{j_1}, \dots, [z_m,u_m]^{j_m}$ in this cyclic order (hence $u_i = z_{i+1}$). By definition of D, $j_1, \dots, j_m$ are distinct, since $z_1, \dots, z_m$ are. If we replace P_{j_i} with the x,y-path $P'_{j_i} := P_{j_i}[x,z_i] \cup P_{j_{i-1}}[u_{i-1},y]$ for $i = 1, \dots, m$ (i modulo m), we get from $P_1, \dots, P_n$ a system $P'_1, \dots, P'_n$ of edge-disjoint x,y-paths in G. But $\bigcup_{i=1}^n E(P'_i) \subsetneq \bigcup_{i=1}^n E(P_i)$ holds, contradicting the remarks above. Hence D is acyclic and there is a $z \in V(D)$ with $d^-(z;D) = 0$. This means that all the $n/2$ paths P_i containing z have $E(P_i) \cap [x,z]_G \neq \emptyset$. But this implies $|[x,z]_G| = n/2$. Considering an $e' \in [x,z]_G$ instead of e, we get, in the same way, a vertex $z' \neq z$ with $|[x,z']_G| = n/2$. Since G is n-regular and finite, we get $G \cong C_{|G|}^{n/2}$. The following theorem summarizes what we have proved.

Theorem 2. [6] *The only 2-minimally n-edge-connected multigraphs are K_2^n and, for even $n \geq 4$, also $C_m^{n/2}$.*

Obviously, the deletion of two consecutive edges of a digraph cannot decrease the connectivity number or edge-connectivity number by two (*cf.* [6]). So it is natural to consider only the deletion of non-consecutive edges in a digraph. Let us call a multidigraph D weakly 2-minimally n-connected (weakly 2-minimally n-edge-connected), if $\kappa(D) = n \geq 2$ ($\lambda(D) = n \geq 2$), but for all non-consecutive $e_1 \neq e_2$ from $E(D)$, we have $\kappa(D - \{e_1, e_2\}) = n - 2$ ($\lambda(D - \{e_1, e_2\}) = n - 2$).

Let $D = (V,E)$ be a weakly 2-minimally n-connected multigraph. Choosing $[x,y]_D \neq \emptyset$ and openly disjoint x,y-paths $P_1, \dots, P_n$, we conclude, as above, $E - (E^-(x) \cup E^+(y)) \subseteq \bigcup_{i=1}^n E(P_i)$. Hence D is n-regular and finite. Let us assume $|D| \geq 3$. Then D has no multiple edges, since D is n-regular and $\kappa(D) = n \geq 2$. Since only one edge of $\bigcup_{i=1}^n E(P_i)$ has its head in $z \in V - \{x,y\}$, we get $n = 2$ and $[y,z] \in E$ for all $z \in V - \{x,y\}$. Now $D \cong \overleftrightarrow{K}_3$ follows easily.

Theorem 1d. *The only weakly 2-minimally n-connected multidigraphs are*

$$\overleftrightarrow{K}_2^n \text{ and } \overleftrightarrow{K}_3 \text{ for } n = 2.$$

Let us now consider a weakly 2-minimally n-edge-connected multidigraph $D = (V,E)$. If $[x,y]_D \neq \emptyset$ and $P_1, \dots, P_n$ are edge-disjoint x,y-paths, $E - (E^-(x) \cup E^+(y)) \subseteq \bigcup_{i=1}^n E(P_i)$ follows as above. Hence D is n-regular and finite. Put $m := \max_{x,y \in V} |[x,y]_D|$ and choose $x,y \in V$ such that $m = |[x,y]_D|$ holds. If $N^+(x) = \{y\}$ holds, then $m = n$ and $E - (E^-(x) \cup E^+(x)) \subseteq [x,y]_D$. But this implies $D \cong \overleftrightarrow{K}_2^n$ or $D \cong C_3^{n,0}$. So we assume $|N^+(x)| \geq 2$. Let $P_1, \dots, P_n$ be edge-disjoint x,y-paths. As in the proof of Theorem 2, we find a $z \in N^+(x) - \{y\}$ such that $z \in P_i$ implies $[x,z]_D \cap E(P_i) \neq \emptyset$. Set $k := |\{i \in \mathbb{N}_n : z \in P_i\}|$. Since $z \in N^+(x)$ and $d^+(x) = n$, we have $k = |[x,z]_D| \geq 1$. Since $E - (E^-(x) \cup E^+(y)) \subseteq \bigcup_{i=1}^n E(P_i)$ and $d^-(z) = n$, we conclude $|[y,z]_D| = n - k$. Since $m + k \leq d^+(x) = n$ and $n - k \leq m$ by choice of m, it follows that $n - k = m$. Since $E - (E^-(x) \cup E^+(z))$ is contained in the n-edge-disjoint x,z-paths of $D(\{x,y,z\})$, we conclude $E^-(y) - [x,y]_D = [z,y]_D$ and $E^+(y) - [y,z]_D = [y,x]_D$, hence $|[z,y]_D| = k = |[y,x]_D|$. Furthermore, $|D| = 3$ follows, since D is n-regular and $k \geq 1$. So $D \cong C_3^{k,n-k}$, and we have proved the following theorem.

Theorem 2d. *The only weakly 2-minimally n-edge-connected multidigraphs are* $\overleftrightarrow{K}_2^n$ *and* $C_3^{k,n-k}$ *for* $k \in \mathbb{N}_n$.

3. Vertex-edge-critical undirected graphs

First we will deduce some common properties of undirected and directed ve-critical graphs. Subsequently, we will determine all ve-critical undirected graphs.

Let $G = (V, E)$ be an n-ve-critical graph or digraph. Consider an edge $e = [x, y] \in E$ and openly disjoint x, y-paths $P_1, \dots, P_n$ in G. For $v \in V - \{x, y\}$, $\kappa(G - v - e) = n - 2$ by assumption, hence $\kappa(x, y; G - v - e) = n - 2$ follows easily from Menger's Theorem. But this implies $v \in \bigcup_{i=1}^n V(P_i)$, hence $V = \bigcup_{i=1}^n V(P_i)$ and G is finite. On the other hand, if for every edge $[x, y]$ of a graph or digraph G with $\kappa(G) = n \geq 2$, every system of n openly disjoint x, y-paths covers G, obviously G is n-ve-critical. We state this equivalence formally.

Lemma 1. *A graph or digraph* G *with* $\kappa(G) = n \geq 2$ *is n-ve-critical, iff for every edge* $[x, y]$ *of* G*, every system of openly disjoint* x, y*-paths* $P_1, \dots, P_n$ *covers* G.

From this, the following property is easily deduced.

Lemma 2. *Every n-ve-critical graph or digraph is finite and n-regular.*

Proof. It remains to show that an n-ve-critical G is n-regular. Consider an edge $[x, y]$ of G and openly disjoint x, y-paths $P_1, \dots, P_n$ in G. Suppose there is an edge $[x, z]$ in G that is not on any P_i. Since $P_1, \dots, P_n$ cover G by Lemma 1, there is a P_i containing z, say, $z \in P_n$. Then the openly disjoint x, y-paths $P_1, \dots, P_{n-1}, P_n'$, where $P_n' := x, P_n[z, y]$, do not cover G, contradicting Lemma 1. Hence $d(x) = n$ or $d^+(x) = n$, respectively, and G is n-regular. □

If we delete a vertex v from an n-ve-critical graph or digraph G, then $G - v$ is minimally $(n-1)$-connected. So one can apply the results on minimally n-connected graphs and digraphs. By Lemma 2, $G - v$ has exactly n vertices of degree $n - 1$ or n vertices of outdegree $n - 1$ and n vertices of indegree $n - 1$, respectively. On the other hand, it is well known [1] that a minimally $(n-1)$-connected graph has at least n vertices of degree $n - 1$, and in [3] even a characterization of all minimally $(n - 1)$-connected graphs containing exactly n vertices of degree $n - 1$ was obtained. This permits a straightforward proof of the characterization theorem on n-ve-critical undirected graphs, which was first proved in [2] using the fact known from [1] that every circuit in a minimally n-connected graph contains a vertex of degree n. First, we state the above mentioned result for minimally n-connected graphs.

Theorem A. [3] *For* $n \geq 2$*, all minimally n-connected graphs containing exactly* $n+1$ *vertices of degree n are obtained in the following way.*

(a) *For an integer* $m \in \mathbb{N}_n \cup \{0\}$*, let* H *be an* $(n - m)$*-regular,* $(n - m)$*-connected graph on* $n + 1$ *vertices. Then* $\overline{K}_m + H$ *is minimally n-connected, containing exactly* $n + 1$ *vertices of degree n.*

(b) *For an integer m with $4 \leq m \leq n$, let H be an $(n-m)$-regular, $(n-m)$-connected graph on $n-1$ vertices, and let P be a path with $|P| = m$. Then $P + H$ is minimally n-connected, containing exactly $n+1$ vertices of degree n.*

For characterizing all ve-critical graphs, we need a further lemma.

Lemma 3. *If $G + H$ is a non-complete, ve-critical, undirected or directed graph, H is ve-critical or $||H|| = 0$.*

Proof. Set $n := \kappa(G+H)$ and $m := n - |G|$. Using Lemma 2, we see that H is m-regular and m-connected. We assume $||H|| > 0$. Then $\kappa(H) = m > 0$ holds. Suppose $m = 1$ and consider an edge $[x, y] \in E(H) \neq \emptyset$. There are n openly disjoint x, y-paths in $G+H(\{x, y\})$. This implies $|G + H| = n + 1$ by Lemma 1, hence $G + H$ is complete. This contradiction shows $\kappa(H) \geq 2$. Since for $e \in E(H)$, every separating vertex set S of $(G + H) - e$ with $|S| = n - 1$ must contain $V(G)$, it is easy to see that H is ve-critical, since $G + H$ is. □

Without difficulty, we now get the following result.

Theorem 3. [2] *The vertex-edge-critical graphs are exactly the graphs $G_{m,k,l} := k\overline{K}_m + lC_{m+2}$, where $m \geq 1, k, l$ are non-negative integers such that $\kappa(G_{m,k,l}) \geq 2$ holds.*

Proof. Suppose G is a ve-critical graph of the form $\sum_{i=1}^{k} \overline{K}_{m_i} + \sum_{j=1}^{l} C_{n_j}$ with $m_i \in \mathbb{N}$. Since G is regular by Lemma 2, we get immediately that $m_1 = m_2 = \cdots = m_k$ and $n_1 = n_2 \cdots = n_l$ and $n_1 = m_1 + 2$, if $k > 0$ and $l > 0$. This implies $G \cong G_{m_1,k,l}$. On the other hand, it is easy to check that the graphs $G_{m,k,l}$ with $\kappa(G_{m,k,l}) \geq 2$ are ve-critical. So it remains to show that every ve-critical graph has the form $\sum \overline{K}_{m_i} + \sum C_{n_j}$. We will prove this by induction on the connectivity number.

Let G be an $(n + 1)$-ve-critical graph. If $n = 1$, then G is a circuit by Lemma 2. So suppose $n \geq 2$ and choose $v \in V(G)$. Then $G - v$ is minimally n-connected and has exactly $n + 1$ vertices of degree n, namely $N(v; G)$. So $G - v$ has the structure described in Theorem A. If $G - v = \overline{K}_m + H$, as in case (a) of Theorem A, then $G = \overline{K}_{m+1} + H$, where $V(\overline{K}_{m+1}) = V(\overline{K}_m) \cup \{v\}$. If G is complete or $||H|| = 0$, then G has the form wanted. Otherwise, Lemma 3 implies that H is ve-critical, and hence, by the induction hypothesis, H has the form $\sum \overline{K}_{m_i} + \sum C_{n_j}$, and hence G does as well. If $G - v = P + H$, as in case (b) of Theorem A, then $G = C + H$, where C is a circuit containing v with the property $C - v = P$. By an application of Lemma 3 and the induction hypothesis as in case (a), the proof is complete. □

4. Vertex-edge-critical directed graphs: introduction and preliminaries.

Of course, every n-ve-critical graph G provides an n-ve-critical digraph $\overleftrightarrow{G}$. The aim of this paper is to show that we get every ve-critical digraph in this way, *i.e.*, that every ve-critical digraph is symmetric. With regard to Theorem 3, we will then have proved the following theorem.

Theorem 4. *The vertex-edge-critical digraphs are exactly the digraphs $\overleftrightarrow{G}_{m,k,l}$ with $\kappa(G_{m,k,l}) \geq 2$.*

The proof of this theorem cannot be based on an analogue of Theorem A: only recently [5], I have shown that every minimally n-connected digraph has at least $n+1$ vertices of outdegree n (that there are at least n such vertices was known before from [4]), but at the moment there is no hope to characterize all minimally n-connected digraphs containing exactly $n+1$ vertices of outdegree n and $n+1$ vertices of indegree n. However, there is an analogue in the directed case to the fact that a minimally n-connected graph does not contain a circuit consisting only of vertices of degree exceeding n, which we will state now.

Let $D = (V, E)$ be a minimally n-connected digraph. Let D_0 be the subdigraph of D given by $V(D_0) := \{v \in V : d^+(v;D) > n \text{ or } d^-(v;D) > n\}$ and $E(D_0) := \{[x,y] \in E : d^+(x;D) > n \text{ and } d^-(y;D) > n\}$. It was proved in [4] that D_0 has no alternating cycle.

Theorem B. [4] *For every minimally n-connected digraph D, D_0 does not contain an alternating cycle.*

To every digraph D, we let correspond a bipartite undirected graph $\overline{D}$ as follows: take vertices $x' \neq x''$ for every $x \in V(D)$ so that $\{x', x''\} \cap \{y', y''\} = \emptyset$ holds for $x \neq y$, and define $\overline{D}$ by $V(\overline{D}) := \bigcup_{x \in D}\{x', x''\}$ and $E(\overline{D}) := \{[x', y''] : [x,y] \in E(D)\}$. The following equivalence is easily seen and was shown in [4].

Lemma C. [4] *A digraph D does not have an alternating cycle iff $\overline{D}$ is a forest.*

In the following, $D = (V, E)$ always denotes an n-ve-critical digraph containing an asymmetric edge that has a minimum number of vertices. Our aim is to show that such a digraph cannot exist. By Lemma 2, D is finite and n-regular and $|D| \geq n+2$ holds. Since the dual digraph of a ve-critical digraph is ve-critical again, for every result on D, there is a dual one, which we will use, but, in general, not state explicitly. For $x \in V$, $H := D - x$ is minimally $(n-1)$-connected. So H_0 and $\overline{H}_0$ are defined, and we set $D_x := H_0$ and $F_x := \overline{H}_0 - (\{y'' : y \in N_a^+(x)\} \cup \{y' : y \in N_a^-(x)\})$. Defining $R(x) := V - (N^+(x) \cup N^-(x) \cup \{x\})$, we observe $D_x = (V - (N_s(x) \cup \{x\}), [N_a^+(x) \cup R(x), R(x) \cup N_a^-(x)]_D)$, since D is n-regular. Furthermore, F_x has the partition $F'_x := \{y' : y \in N_a^+(x) \cup R(x)\}$, $F''_x := \{y'' : y \in R(x) \cup N_a^-(x)\}$ into independent vertex sets. Since D is n-regular by Lemma 2, we have $d_a^+(x) = d_a^-(x) =: d_a(x)$, and hence $|F'_x| = |F''_x|$. By Theorem B and Lemma C, F_x is a forest. Theorem B implies the following important properties of D.

Lemma 4.

(a) *If $a_1^+, \overline{a}_1, a_2^+, \ldots, \overline{a}_k$ is an alternating cycle of D, then for every $x \in V - \{a_1, \ldots, \overline{a}_k\}$, there is an $i \in \mathbb{N}_k$ such that $[a_i, x] \in E$ or $[x, \overline{a}_i] \in E$ holds.*

(b) *If $z \notin N^+(x) \cup N^+(y)$ for distinct $x, y, z \in V$, then $|N^+(z) \cap N^+(x) \cap N^+(y)| \geq |N^+(x) \cap N^+(y)| - 1$.*

Proof. (a) If $[a_i, x] \notin E$ and $[x, \overline{a}_i] \notin E$ holds for all $i \in \mathbb{N}_k$, then $a_1^+, \overline{a}_1, \ldots, \overline{a}_k$ is an alternating cycle in D_x, contradicting Theorem B.

(b) For $u \neq v$ in $N^+(x) \cap N^+(y)$, x^+, u, y^+, v is an alternating cycle in D. If $z \notin N^+(x) \cup N^+(y) \cup \{x, y\}$, we get $[z, u] \in E$ or $[z, v] \in E$ by (a). This implies $|N^+(z) \cap N^+(x) \cap N^+(y)| \geq |N^+(x) \cap N^+(y)| - 1$. □

Lemma 5. *For all vertices $x \neq y$ of D, the following statements are true:*

(a) *if $[x,y] \in E$, then $|N^+(x) \cap N^-(y)| \leq n-2$;*
(b) *if $[x,y] \in E$, then $|N^+(x) \cap N^+(y)| \leq n-2$;*
(c) *$N^+(x) \neq N^+(y)$.*

Proof. (a) Suppose $[x,y] \in E$ and $|N^+(x) \cap N^-(y)| \geq n-1$. Then there are n openly disjoint x,y-paths in $D(N^+(x) \cup \{x\})$. These paths cover D by Lemma 1, which implies the contradiction $|D| = n+1$.
(b) Suppose $[x,y] \in E$ and $|N^+(x) \cap N^+(y)| \geq n-1$. Let z be the element of $N^+(y) - N^+(x)$. Suppose $z \neq x$, and consider a system of n openly disjoint y,z-paths in D. Obviously, these paths cannot contain x. So Lemma 1 implies $z = x$. But then $S := N^+(x) - \{y\} = N^+(y) - \{x\}$ with $|S| = n-1$ is separating, since $|D| \geq n+2$ holds. This contradiction proves (b).
(c) We suppose there are vertices $x \neq y$ in D with $N^+(x) = N^+(y) =: N$. For $z \in V - (N \cup \{x,y\})$, we get $|N^+(z) \cap N| \geq n-1$ by Lemma 4 (b). Hence (b) implies $[z,x] \notin E$ and $[z,y] \notin E$ and, therefore, $N^-(x) = N = N^-(y)$ holds. Suppose there is an edge $[z,\bar{z}] \in E(D - (N \cup \{x,y\}))$ and consider n openly disjoint $z,\bar{z}$-paths in D. Since $|N^+(z) \cap N| \geq n-1$ and $N^+(x) = N^+(y) = N$, these paths cannot contain both the vertices x and y. So Lemma 1 implies $||D - (N \cup \{x,y\})|| = 0$. In particular, for $z \in V - (N \cup \{x,y\})$, we get $N^+(z) = N$ and so also $N^-(z) = N$. Altogether, we have shown $D = (V-N) + D(N)$. Hence $||D(N)|| = 0$ holds or $D(N)$ is ve-critical by Lemma 3. If $||D(N)|| = 0$ holds, D is symmetric, contrary to our assumption. So $D(N)$ is ve-critical. But then $D(N)$ is symmetric by choice of D as a minimal counterexample, hence D is symmetric as well. This contradiction proves (c). □

Lemma 5 (a) and (b) mean that for every $x \in V$, the maximum outdegree and maximum indegree of $D(N^+(x))$ are at most $n-2$, and, dually, the same holds for $D(N^-(x))$. We now deduce some properties of F_x from Lemma 5.

Lemma 6.

(a) *For every $v \in F_x$, $d(v;F_x) \geq 1$ and for every $v \in F_x - \bigcup_{y \in R(x)} \{y', y''\}$, $d(v;F_x) \geq 2$ holds.*
(b) *For $F = F'_x$ and $F = F''_x$, $|\{v \in F : d(v;F_x) = 1\}| = c + \sum_{\substack{v \in F \\ d(v;F_x) \geq 3}} (d(v;F_x) - 2)$ holds, where c denotes the number of components of F_x.*

Proof. (a) By duality, it suffices to consider a $v \in F'_x$. Then there is a $z \in N_a^+(x) \cup R(x)$ such that $z' = v$ holds. Since $[z,x] \notin E$, we have $d(z';F_x) = n - |N^+(z;D) \cap N^+(x;D)|$. So Lemma 5 (c) implies $d(z';F_x) \geq 1$. Assume $z \in N_a^+(x)$. Then $d(z';F_x) \geq 2$ by Lemma 5 (b).
(b) Since F'_x, F''_x is a bipartition of the forest F_x into independent sets F'_x and F''_x of the same cardinality, we get

$$\sum_{v \in F'_x} d(v;F_x) = ||F_x|| = |F_x| - c = 2|F'_x| - c = \Big(\sum_{v \in F'_x} 2\Big) - c.$$

This proves assertion (b), since there are no isolated vertices in F_x by (a) and the case $F = F''_x$ is dual. $\square$

Lemma 6 (b) provides at least one vertex $z \in N_a^+(x) \cup R(x) \neq \emptyset$ with $d(z'; F_x) = 1$, and by Lemma 6 (a), every such vertex is in $R(x)$. So there is a vertex $z \in R(x)$ with $|N^+(z) \cap N^+(x)| = n-1$, and we define $R^+(x) := \{z \in R(x) : |N^+(z) \cap N^+(x)| = n-1\}$. $R^-(x)$ is defined dually as $\{z \in R(x) : |N^-(z) \cap N^-(x)| = n-1\}$. We emphasize once again that $R^+(x) \neq \emptyset$ and $R^-(x) \neq \emptyset$.

We need a series of preliminary lemmas. Herein, x_0 always denotes any vertex of D.

Lemma 7.

(a) *If $x \in N_a^+(x_0)$ such that $R^-(x_0) \not\subseteq N^+(x)$ holds, then $|N^-(x) \cap N^-(x_0)| = n-2$ and $|N^-(x) \cap (N_a^+(x_0) \cup R(x_0))| = 1$.*

(b) *For all $x \in N_a^+(x_0)$ but at most one, $|N^-(x) \cap (N_a^+(x_0) \cup R(x_0))| = 1$ holds.*

Proof. (a) Suppose there is $z \in R^-(x_0) - N^+(x)$. Since $[x, x_0] \notin E$, we can apply the dual of Lemma 4 (b) for x_0, z, x, and get $|N^-(x) \cap N^-(x_0) \cap N^-(z)| \geq |N^-(x_0) \cap N^-(z)| - 1 = n-2$, by definition of $R^-(x_0)$. This implies $|N^-(x) \cap N^-(x_0)| = n-2$ by the dual of Lemma 5(b), so $|N^-(x) \cap (N_a^+(x_0) \cup R(x_0))| = n - |N^-(x) \cap N^-(x_0)| - 1 = 1$ follows.

Since for every $z \in R^-(x_0)$, the definition of $R^-(x_0)$ implies that $|N^-(z) \cap N_a^+(x_0)| \leq 1$ holds, (b) follows from (a), since $R^-(x_0) \neq \emptyset$. $\square$

Lemma 8.

(a) *If $v \in R^+(x_0)$, $x \in N^+(x_0) - N^+(v)$, and $y \in N^-(x) \cap (N_a^-(x_0) \cup R(x_0))$, then $[v, y] \in E$ or $y \in R^+(x_0)$ holds.*

(b) *If $x \in N^+(x_0)$ such that $|R^+(x_0) - N^-(x)| \geq 2$ holds, then $N^-(x) \cap N_a^-(x_0) = \emptyset$.*

Proof. (a) Suppose $[v, y] \notin E$. Since $[x_0, y] \notin E$ also, we can apply Lemma 4 (b) to x_0, v, y and get $|N^+(y) \cap N^+(x_0) \cap N^+(v)| \geq |N^+(x_0) \cap N^+(v)| - 1 = n-2$. This implies $|N^+(y) \cap N^+(x_0)| \geq n-1$, since $x \in N^+(y) - N^+(v)$ holds. Hence, by Lemma 5 (b), $y \notin N^-(x_0)$ holds, so $y \in R(x_0)$ and even $y \in R^+(x_0)$ by Lemma 5 (c).

(b) Suppose there are $v_1 \neq v_2$ in $R^+(x_0) - N^-(x)$ for an $x \in N^+(x_0)$, and there is a $y \in N^-(x) \cap N_a^-(x_0)$. Then (a) implies $[v_i, y] \in E$ for $i = 1, 2$. Since $N^+(v_i) \cap N^+(x_0) = N^+(x_0) - \{x\}$ for $i = 1, 2$, we get $N^+(v_1) = N^+(v_2)$, which contradicts Lemma 5 (c). $\square$

For every $[x, y] \in E$, we have $|N^+(x) \cap N^-(y)| \leq n-2$ by Lemma 5 (a). Let us assume equality holds. Then $(D - [x, y]) - (N^+(x) \cap N^-(y))$ has exactly one x, y-path, since every such path does contain $V - (N^+(x) \cap N^-(y))$ by Lemma 1. This path has length at least 3.

Lemma 9. *Let $[x, y] \in E$ be such that $S := N^+(x) \cap N^-(y)$ has exactly $n-2$ vertices, and let $P : x, x_1, \ldots, x_k, x_{k+1}$ be the x, y-path in $(D - [x, y]) - S$. Then the following statements are true.*

(a) $N^+(x_1) = \{x, x_2\} \cup S$ *and* $N^-(x_k) = \{x_{k-1}, y\} \cup S$*;*

(b) $S \subseteq N^-(x) \rightarrow [y, x] \in E$.

Proof. Since, by Lemma 1, $N^+(x_1) \cap \{x_3, \ldots, x_{k+1}\} = \emptyset$ holds, we must have $N^+(x_1) \subseteq V - \{x_3, \ldots, x_{k+1}\}$, hence $N^+(x_1) = \{x, x_2\} \cup S$. The other claim in (a) follows by duality. Now suppose $S \subseteq N^-(x)$. Then there are $n-1$ openly disjoint x_1, x-paths in $D(\{x_1, x\} \cup S)$ by (a). There is a $z \in N^-(x) \cap \{x_2, \ldots, x_{k+1}\}$, and by Lemma 1, $P[x_1, z], x$ does contain $\{x_2, \ldots, x_{k+1}\}$, which implies $z = x_{k+1} = y$, and hence (b). □

If we assume that $[x, y] \in E$ is asymmetric, and that $\Delta_a(D) = 1$ holds, then $S := N^+(x) \cap N^-(y)$ is a subset of $N^-(x)$, and Lemma 9 (b) implies $|S| \leq n - 3$. It is possible to improve this result.

Lemma 10. *Let* $[x, y] \in E$ *be asymmetric and assume* $\Delta_a(D) = 1$. *Then*

(a) $|N^+(x) \cap N^-(y)| \leq n - 4$, *and*

(b) $|N^+(x) \cap N^+(y)| \leq n - 3$ *hold.*

Proof. (a) We suppose $S := N^+(x) \cap N^-(y)$ has at least $n - 3$ elements. Since $\Delta_a(D) = 1$ holds, $[x, y]$ is the only asymmetric edge with tail in x. Hence, there is exactly one asymmetric edge in D with head in x, say $[y', x]$. In particular, we see $S \subseteq N^-(x)$ and, dually, $S \subseteq N^+(y)$. Then Lemma 9 (b) implies $|S| = n-3$, since $[x, y]$ is asymmetric. Hence, there are two openly disjoint x, y-paths $P_i : x, x_1^i, \ldots, x_{k_i+1}^i$ $(i = 1, 2)$ in $(D - [x, y]) - S$. Furthermore, $k_i \geq 2$ holds, and P_1, P_2 cover $D - S$ by Lemma 1, in particular, $y' \in P_1[x_2^1, y) \cup P_2[x_2^2, y)$. First, we prove a few properties.

(1) $S \cup \{x_1^{i+1}\} \nsubseteq N^+(x_1^i)$ for $i = 1, 2 \pmod 2$.

Suppose $S \cup \{x_1^2\} \subseteq N^+(x_1^1)$. Then $S \cup \{x_1^2\} \subseteq N^+(x_1^1) \cap N^-(x)$ holds. Applying Lemma 9 to $[x_1^1, x] \in E$, we get from the second equality in Lemma 9 (a) the contradiction that $[y', x]$ is symmetric.

(2) For $i = 1, 2 \pmod 2$, $[x_1^i, x_2^{i+1}] \in E$ and $|N^+(x_1^i) \cap (S \cup \{x_1^{i+1}\})| = n - 3$ hold.

By (1), there is at least one edge from x_1^i to $P_i[x_3^i, y] \cup P_{i+1}[x_2^{i+1}, y]$ for $i = 1, 2$. By Lemma 1, this can be only the edge $[x_1^i, x_2^{i+1}]$, since $k_i \geq 2$. Hence (2) follows.

Dually, we get $[x_{k_i-1}^i, x_{k_{i+1}}^{i+1}] \in E$ for $i = 1, 2$ (see Figure 1).

(3) $S \subseteq N^+(x_1^i)$ for $i = 1, 2$.

Suppose, for instance, $S \nsubseteq N^+(x_1^1)$, say $s \in S - N^+(x_1^1)$. Then $S' := (S - \{s\}) \cup \{x_1^2\} \subseteq N^+(x_1^1)$ holds by (2). Set $D' := (D - [x_1^1, x]) - S'$. If $y' \notin \{x_{k_1}^1, x_{k_2}^2\}$, we can easily find openly disjoint x_1^1, x-paths Q_1 with $y' \in Q_1$ and Q_2 with $[y, s] \in E(Q_2)$ in D' such that $Q_1 \cap \{x_{k_1}^1, x_{k_2}^2\} = \emptyset$ and $|Q_2 \cap \{x_{k_1}^1, x_{k_2}^2\}| = 1$, contradicting Lemma 1. Hence, $y' \in \{x_{k_1}^1, x_{k_2}^2\}$ holds. Suppose there is a $z \neq y'$ in $N^-(s) \cap (P_1[x_2^1, y) \cup P_2[x_2^2, y))$. If $\{y', z\} \nsubseteq V(P_1)$ and $\{y', z\} \nsubseteq V(P_2)$, using $[x_1^1, x_2^2] \in E$, we get, obviously, openly disjoint x_1^1, x-paths Q_1, Q_2 in D' with $y \notin V(Q_1) \cup V(Q_2)$, contradicting Lemma 1. So $\{y', z\} \subseteq V(P_1)$ or $\{y', z\} \subseteq V(P_2)$ holds. Then we find, again, two openly disjoint x_1^1, x-paths in $D' - y$, namely in the former case (then $x_{k_1}^1 = y'$) the paths $x_1^1, P_2[x_2^2, x_{k_2-1}^2], y', x$ and $P_1[x_1^1, z], s, x$, and in the latter case (then $x_{k_2}^2 = y'$) the paths $x_1^1, P_2[x_2^2, z], s, x$ and $P_1[x_1^1, x_{k_1-1}^1], y', x$. (Note that $k_2 \geq 3$ in the former case, since in this case $k_1 \geq 3$, hence $x_2^1 \neq x_{k_1}^1$ holds, but there is only the edge $[x_1^2, x_2^1]$ from x_1^2 to $P_1[x_2^1, y]$ by (2)). This contradiction with Lemma 1 shows

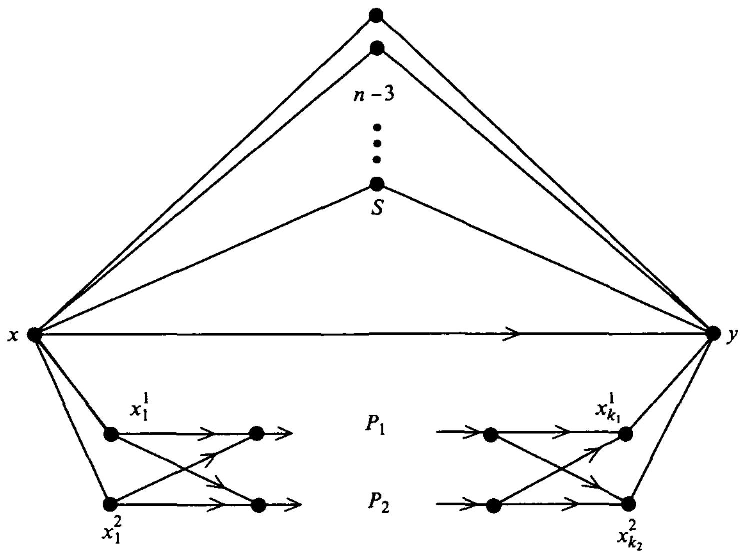

Figure 1

$N^-(s) = (S - \{s\}) \cup \{x, y, y', x_1^2\}$. Now it is easy to find in D n openly disjoint x,s-paths not containing $\{x_{k_1}^1, x_{k_2}^2\} - \{y'\}$. This contradiction to Lemma 1 proves (3).

We may assume $y' \in P_2$, hence $y' \in P_2[x_2^2, y)$. If $[x_2^1, x_1^1] \in E$ holds, by using (3) and (2), it is easy to find n openly disjoint x_1^2,x-paths in $D - y$, contradicting Lemma 1. So $[x_1^1, x_2^1] \in E$ is asymmetric. Hence $[x_1^2, x_2^1] \in E$ is symmetric, since $\Delta_a(D) = 1$, that is $[x_2^1, x_1^2] \in E$ holds. By using (3) and (2) again, we now easily find n openly disjoint x_1^1,x-paths in $D - y$. This contradiction to Lemma 1 proves (a).

(b) If $|N^+(x) \cap N^+(y)| \geq n-2$ holds, then $|N^+(x) \cap N^-(y)| \geq n-3$ follows, since $\Delta_a(D) = 1$, thus at most one of the edges $[y, z]$ for $z \in N^+(x) \cap N^+(y)$ is asymmetric. Hence (b) follows from (a). □

Lemma 11. $|D| \geq n + 4$ *holds.*

Proof. If $\Delta_a(D) \geq 2$, we choose an x_0 with $d_a(x_0) \geq 2$ and get $|N^+(x_0) \cup N^-(x_0)| \geq n + 2$, hence $|D| \geq n + 4$, since $R(x_0) \neq \emptyset$. If $\Delta_a(D) = 1$, we choose an asymmetric edge $[x, y] \in E$ and get $|N^+(x) \cup N^+(y)| \geq n + 3$ by Lemma 10 (b), hence $|D| \geq n + 4$. □

Lemma 12. *If* $d_a(x_0) > 0$, *then*

(a) $\bigcup_{v \in R^+(x_0)} N^+(v) \supseteq N^+(x_0)$ *and*

(b) $|R^+(x_0)| \geq 2$ *hold.*

Proof. Since $|N^+(v) \cap N^+(x_0)| = n-1$ for $v \in R^+(x_0)$, it suffices to prove (a). Suppose there is an $x \in N^+(x_0) - \bigcup_{v \in R^+(x_0)} N^+(v)$. There is a $v \in R^+(x_0)$, and $N^+(v) - N^+(x_0)$ has exactly one element, say, z. Consider any $y \in V - (N^+(x_0) \cup \{x_0, v, z\})$ and suppose $[y, x] \in E$. Then Lemma 8 (a) implies $y \in R^+(x_0)$, contradicting the choice of x. This contradiction shows

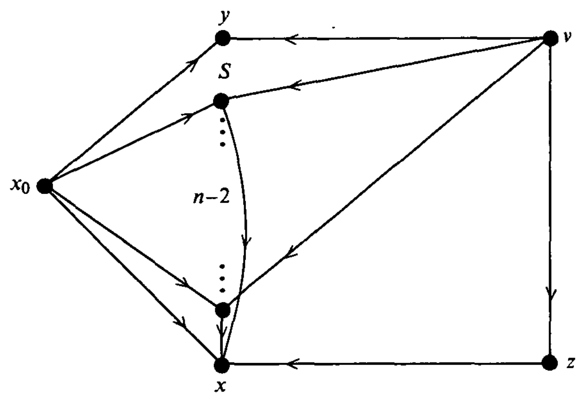

Figure 2

$N^-(x) \subseteq N^+(x_0) \cup \{x_0, z\}$, hence $|N^-(x) \cap N^+(x_0)| = n-2$ and $[z, x] \in E$ by Lemma 5 (a). Set $S := N^-(x) \cap N^+(x_0)$, and let y be the vertex of $N^+(x_0) - (S \cup \{x\})$ (see Figure 2). We apply Lemma 9 (a) to $[x_0, x] \in E$ and to the only x_0, x-path $P : x_0, y, \dots, z, x$ in $(D - [x_0, x]) - S$, and get $S \subseteq N^-(z)$ and $[y, x_0] \in E, [x, z] \in E$. Since $[v, z] \in E$ and $S \subseteq N^+(v) \cap N^-(z)$, the path v, y, x_0, x, z must cover $D - S$ by Lemma 1. But this implies $|D| = n + 3$, contradicting Lemma 11. □

5. Proof of Theorem 4

As in section 3, let $D = (V, E)$ be a minimal counterexample to Theorem 4, and $n := \kappa(D)$. We will show that D cannot have an asymmetric edge. First we prove that D must contain many symmetric edges.

(1) $\Delta_a(D) \leq 2$.

Suppose there is an $x_0 \in V$ with $d_a(x_0) \geq 3$. Since $|R^+(x_0)| \geq 2$ by Lemma 12 (b), Lemma 7 (b) implies $d_a(x_0) = 3$ and $|R^+(x_0)| = 2$, since $|N^+(v) \cap N^+(u) \cap N_a^+(x_0)| \geq d_a(x_0) - 2$ for $u \neq v$ in $R^+(x_0)$. Set $N_a^+ = \{x_1, x_2, x_3\}$ and choose $v \in R^-(x_0) \neq \emptyset$. Then $|N^-(v) \cap N_a^+(x_0)| \leq 1$, say, $[x_i, v] \notin E$ for $i = 1, 2$. Hence, Lemma 7 (a) implies $|N^-(x_i) \cap N^-(x_0)| = n-2$ and thus $|N^-(x_i) \cap R^+(x_0)| = 1$ by Lemma 7 (b) (or 12 (a)) for $i = 1, 2$. Therefore, we have $d^-(x_i; D(N_a^+(x_0))) = 0$ for $i = 1, 2$. Furthermore, $|N^-(x_i) \cap R^+(x_0)| = 1$ for $i = 1, 2$ implies $R^+(x_0) \subseteq N^-(x_3)$. Since $[x_3, x_1] \notin E$, and $[x_3, x_0] \notin E$, we get from the dual of Lemma 4 (b) that $|N^-(x_3) \cap N^-(x_0) \cap N^-(x_1)| \geq |N^-(x_0) \cap N^-(x_1)| - 1 = n - 3$, so $N^-(x_3) \subseteq N^-(x_0) \cup \{x_0\} \cup R^+(x_0)$. Together, we have seen $||D(N_a^+(x_0))|| = 0$ and so $d(x_i'; F_{x_0}) \geq 3$ for $i \in \mathbb{N}_3$. But this implies $|R^+(x_0)| \geq 4$ by Lemma 6 (b) and (a). This contradiction proves (1). □

In the next step we show the following.

(2) $\Delta_a(D) = 1$.

Suppose $\Delta_a(D) \geq 2$, hence $\Delta_a(D) = 2$ by (1). Let $x_0 \in V$ with $d_a(x_0) = 2$, say,

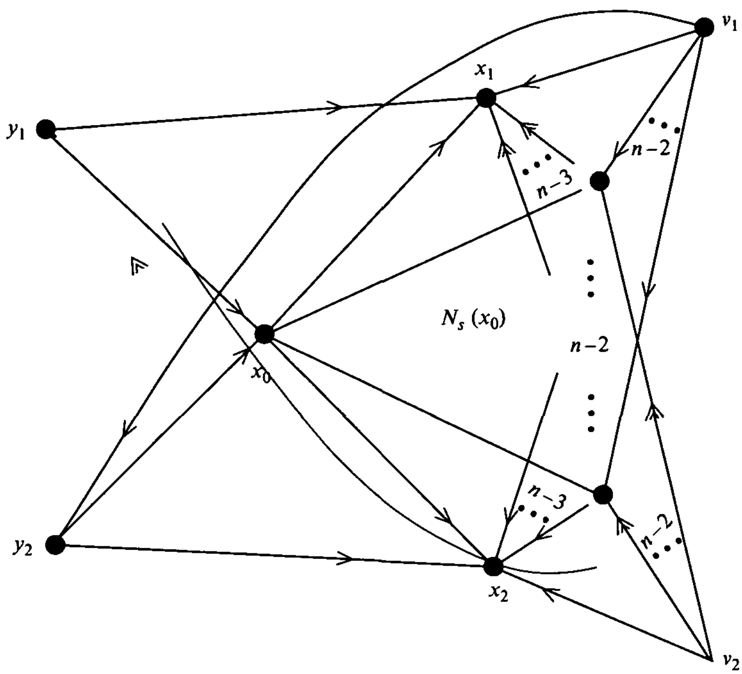

Figure 3

$N_a^+(x_0) = \{x_,x_2\}$ and $N_a^-(x_0) = \{y_1, y_2\}$. By Lemma 12 (b), $|R^+(x_0)| \geq 2$ holds. Consider any $v \in R^-(x_0) \neq \emptyset$. Since $|N^-(v) \cap N_a^+(x_0)| \leq 1$, say, $[x_2, v] \notin E$. Then $|N^-(x_2) \cap N^-(x_0)| \geq n-2$ by Lemma 4 (b). Since $|N^-(x_2) \cap N_s(x_0)| \geq n-2$ is not possible by Lemma 9 (b), $N^-(x_2) \cap N_a^-(x_0) \neq \emptyset$ follows, say, $[y_2, x_2] \in E$. Since $|N^-(x_2) \cap (N^-(x_0) \cup \{x_0\})| \geq n-1$ and $y_2 \in N^-(x_2)$, Lemma 8 (b) implies $|R^+(x_0)| = 2$, say, $R^+(x_0) = \{v_1, v_2\}$ and $|N^-(x_2) \cap R^+(x_0)| = 1$, say, $v_2 \in N^-(x_2)$. Hence $[v_1, x_2] \notin E$, so $N^+(v_1) \cap N^+(x_0) = N_s(x_0) \cup \{x_1\}$, and since $y_2 \in N^-(x_2)$, we get $[v_1, y_2] \in E$ from Lemma 8 (a). Hence $[v_1, y_1] \notin E$, and Lemma 8 (a) implies $[y_1, x_2] \notin E$, hence $|N^-(x_2) \cap N_s(x_0)| = n-3$. Furthermore, $[v_1, y_1] \notin E$ implies $|N^+(y_1) \cap N^+(v_1) \cap N^+(x_0)| \geq n-2$ by Lemma 4 (b). Since $N_s(x_0) \subseteq N^+(y_1)$ is not possible by the dual of Lemma 9 (b), the last inequality implies $[y_1, x_1] \in E$. (So far, we have got the edges without or with one arrow-head in Figure 3.) Since $[x_1, x_2] \notin E$ and $|N^-(x_2) \cap N^-(x_0)| = n-2$, we get $|N^-(x_1) \cap N^-(x_2) \cap N^-(x_0)| \geq n-3$ from Lemma 4 (b). Since $\{x_0, v_1, y_1\} \subseteq N^-(x_1)$, but $\{x_0, v_1, y_1\} \cap N^-(x_2) \cap N^-(x_0) = \emptyset$, we conclude $[v_2, x_1] \notin E$, hence $N_s(x_0) \subseteq N^+(v_2)$. So Lemma 8 (a) implies, as above, that $[v_2, y_1] \in E$ and $[y_2, x_1] \notin E$, hence $|N^-(x_1) \cap N_s(x_0)| = n-3$.

Since $|R^-(x_0)| = 2$ holds by duality, there are only two $z \in F''_{x_0}$ with $d(z; F_{x_0}) = 1$ by Lemma 6 (a), and so Lemma 6 (b) implies $d(y_i''; F_{x_0}) \leq 2$ for $i = 1$ or $i = 2$. Suppose $d(y_1''; F_{x_0}) \leq 2$. This means $|N^-(y_1) \cap N^-(x_0)| \geq n-2$. Now we will point out n openly disjoint v_1, x_1-paths that do not cover D. If z denotes the vertex of $N_s(x_0) - N^-(x_1)$, then $\kappa(v_1, x_1; D(\{v_1, x_1\} \cup (N_s(x_0) - \{z\}))) = n-2$ and $\kappa(v_1, x_1; D(\{v_1, x_1, z, y_2, x_0, y_1\}) - [v_1, x_1])) = 2$ hold, since $N^-(y_1) \cap \{z, y_2\} \neq \emptyset$. So we get n openly disjoint v_1, x_1-paths that do not contain x_2 (and v_2). This contradiction to Lemma 1 proves $\triangle_a(D) \leq 1$, since the case $d(y_2''; F_{x_0}) \leq 2$ is analogous. Since D is not symmetric, $\triangle_a(D) = 1$ follows.

By (2), there is an $x_0 \in V$ with $d_a(x_0) = 1$, say, $N_a^+(x_0) = \{x\}$ and $N_a^-(x_0) = \{y\}$. For such an x_0, we now prove the following.

(3) $R^+(x_0) = R^-(x_0) = R(x_0)$, $||D(R(x_0))|| = 0$, and $R(x_0) \subseteq N^+(x) \cap N^-(y)$ hold.

Choose $v \in R^+(x_0)$. If $[v, y] \notin E$, then $|N^+(y) \cap N^+(x_0)| \geq n - 2$ by Lemma 4 (b), which contradicts Lemma 10 (b). Hence $y \in N^+(v)$ and $N^+(v) \subseteq N^+(x_0) \cup N^-(x_0)$ follows. Consider $z \in R(x_0) - \{v\}$. Since $z \notin N^+(v)$, Lemma 4 (b) implies

$$|N^+(z) \cap N^+(x_0) \cap N^+(v)| \geq n - 2. \tag{α}$$

Let us suppose $[z, v] \in E$. Then $[z, v]$ is an asymmetric edge with $|N^+(z) \cap N^+(v)| \geq n-2$, contradicting Lemma 10 (b). Hence $N^-(v) \subseteq N^+(x_0) \cup N^-(x_0)$ follows. This implies $v \in R^-(x_0)$ and $[x, v] \in E$ by Lemma 5 (c). So we have shown $R^+(x_0) \subseteq R^-(x_0)$. Since the other inclusion is dual, we get $R^+(x_0) = R^-(x_0)$. Furthermore, we have seen

$$N^+(v) \cup N^-(v) \subseteq N^+(x_0) \cup N^-(x_0) \text{ for all } v \in R^+(x_0), \tag{β}$$

hence $||D(R^+(x_0))|| = 0$, and $R^+(x_0) \subseteq N^+(x) \cap N^-(y)$.

So it only remains to prove $R(x_0) = R^+(x_0)$. Suppose $\overline{R} := R(x_0) - R^+(x_0) \neq \emptyset$. First we show

$$N^+(z_1) \cap N^+(x_0) = N^+(z_2) \cap N^+(x_0) \text{ for all } z_1, z_2 \in \overline{R} \tag{γ}$$

We can choose $v_1 \neq v_2$ from $R^+(x_0)$, since $|R^+(x_0)| \geq 2$ by Lemma 12 (b). Since $y \in N^+(v_1) \cap N^+(v_2)$, Lemma 5 (c) implies $N^+(v_1) \cap N^+(x_0) \neq N^+(v_2) \cap N^+(x_0)$. Using ($\alpha$) for v_1 and v_2, we conclude $N^+(z) \cap N^+(x_0) = N^+(v_1) \cap N^+(v_2) \cap N^+(x_0)$ for every $z \in \overline{R}$, since $|N^+(z) \cap N^+(x_0)| \leq n - 2$. This implies ($\gamma$).

Let us now consider $\overline{D} := D_{x_0} - R^+(x_0)$ and $\overline{F} := F_{x_0} - \bigcup_{v \in R^+(x_0)} \{v', v''\}$. Since $|N^+(z) \cap N^+(x_0)| \leq n - 2$ and $|N^-(z) \cap N^-(x_0)| \leq n - 2$ for $z \in \overline{R} = R(x_0) - R^-(x_0)$, using ($\beta$), we see $d(z; \overline{F}) \geq 2$ for all $z \in V(\overline{F}) - \{x', y''\}$. Since $\overline{F}$ is a forest with $|\overline{F}| \geq 4$, it must be an x', y''-path. There are $z_1, z_2 \in \overline{R}$, such that $[x', z_1''] \in E(\overline{F})$ and $[z_2', y''] \in E(\overline{F})$ hold. Using (α) and (γ), Lemma 10 (b) implies, that $D_{x_0}(\overline{R})$ is symmetric. This implies $z_1 = z_2$, since $d^+(z; \overline{D}) = d^-(z; \overline{D}) (= 2)$ holds for every $z \in \overline{R}$. Then the undirected graph G with the property $\overleftrightarrow{G} = D_{x_0}(\overline{R})$ has one vertex of degree 1, namely z_1, and all the other vertices of degree 2. This contradiction shows $\overline{R} = \emptyset$, and (3) is proved.

By Lemma 12 (b) and (3), $r := |R(x_0)| = |R^+(x_0)| \geq 2$ holds. Define $S := N^-(y) \cap N^+(x_0)$ and $T := N^+(x_0) - S$. Since $R(x_0) \subseteq N^-(y)$ by (3), $N^-(y) = R(x_0) \cup S$, hence $|S| = n - r$ and $|T| = r$ hold. Since $d_a(y) = 1$, and thus $N^+(y) - N^-(y) = \{x_0\}$, we conclude $R(y) = T$. Since $||D(T)|| = 0$ by (3), using Lemma 5 (c), we immediately have the following.

(4) For every $t \in T$, $|N^+(t) \cap (S \cup R(x_0))| \geq n - 1$ and for all $t_1 \neq t_2$ from T, $S \cup R(x_0) \subseteq N^+(t_1) \cup N^+(t_2)$ holds.

Now we will complete the proof of Theorem 4 by constructing for an edge $[u, v]$ in D n openly disjoint u, v-paths, that do not cover D. This contradicts Lemma 1.

First, we assume there is a $z_0 \in R(x_0)$ with $N^+(z_0) \not\supseteq S$. Then $[z_0, y] \in E$ by (3). Since $z_0 \in R^+(x_0)$ by (3), there is only one vertex in $S - N^+(z_0)$, say s_0, and $T \subseteq N^+(z_0)$ holds. Then $\kappa(z_0, y; D(\{z_0, y\} \cup (S - \{s_0\}))) = n - r$ holds. By (4), it is easy to find r disjoint edges in $[T, (R(x_0) - \{z_0\}) \cup \{s_0\}]_D$, since $r \geq 2$ holds. This implies $\kappa(z_0, y; D(\{y, s_0\} \cup R(x_0) \cup T) - [z_0, y]) = r$. Together, this gives n openly disjoint z_0, y-paths not containing x_0. Since the asymmetric edge $[y, x_0]$ was arbitrary, this contradiction establishes our next claim.

(5) For every asymmetric edge $[u, v] \in E$ and every $t \in R(v)$, $N^-(u) \cap N^+(v) \subseteq N^+(t)$ holds.

For every $z \in R(x_0)$, therefore, $S \subseteq N^+(z)$ and thus $\kappa(z, y; D(\{z, y\} \cup S)) = n - r + 1$ holds. Choose $z_0 \in R(x_0)$ and define $T' := N^+(z_0) \cap T$. Then $|T'| = r - 1$ holds. If $r \geq 3$, we get, as above, $r - 1$ disjoint edges in $[T', R(x_0) - \{z_0\}]_D$, hence $\kappa(z_0, y; D(\{y\} \cup R(x_0) \cup T') - [z_0, y]) = r - 1$, so there are n openly disjoint z_0, y-paths not containing x_0. This contradiction shows $r = 2$.

So we have $|S| = n - 2$, and Lemma 10 (b) shows that $N^+(y) \supseteq S$ is impossible. Hence there is an asymmetric edge $[s_0, y] \in [S, \{y\}]_D$. Since $d_a(y) = 1$, all the edges $[R(x_0), \{y\}]_D$ are symmetric, hence $R(x_0) \subseteq N^+(y)$ holds by (3). So we have $R(x_0) \subseteq N^-(s_0) \cap N^+(y)$, and (5) implies $R(x_0) \subseteq N^+(t)$ for every $t \in R(y) = T$. So we see $\kappa(z_0, y; D - x_0) = n$, since $\kappa(z_0, y; D(\{y\} \cup R(x_0) \cup T')) = 2$. This contradiction to Lemma 1 completes the proof of Theorem 4.

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