Equivariant Indices of Vector Fields and 1-Forms on Varieties

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Kurzzusammenfassung

Zu einer gegebenen G-invarianten, holomorphen 1-Form auf dem Keim einer komplex-analytischen G-Varietät, die einen isolierten singulären Punkt aufweist, lassen sich der äquivariante homologische und der (reduzierte) äquivariante radiale Index als Elemente im Ring der komplexen Darstellungen der (endlichen) Gruppe G definieren. Wir zeigen, dass diese Indizes auf einer glatten, komplex-analytischen G-Varietät übereinstimmen. Das ermöglicht uns, die Differenz zwischen diesen beiden Indizes als eine Version der äquivarianten Milnorzahl eines Keims einer G-Varietät mit isoliertem singulären Punkt zu betrachten. Für zyklische Gruppen der Ordnung zwei und drei geben wir außerdem einen alternativen Ansatz, um die Gleichheit der beiden Indizes zu zeigen. Dieser Ansatz liefert uns dann eine Klassifikation singulärer Punkte, die man nicht ausschließen kann, indem man Deformationen von 1-Formen, die invariant bezüglich der Wirkung einer zyklischen Gruppe der Ordnung 3 sind, betrachtet.

Stichworte: endliche Gruppenwirkungen, Invariante Vektorfelder, Invariante 1-Formen, Indizes, äquivariante Deformationen

Abstract

Given a G-invariant holomorphic 1-form with an isolated singular point on a germ of a complex analytic G-variety with an isolated singular point, its equivariant homological index and (reduced) equivariant radial index are defined as elements of the ring of complex representations of the finite group G. We show that these indices coincide on a germ of a smooth complex analytic G-variety. This makes it possible to consider the difference between them as a version of the equivariant Milnor number of a germ of a G-variety with an isolated singular point. For cyclic groups of order two and three we additionally describe another approach to prove the coincidence. This gives us a classification of singular points which cannot be excluded by deformations of 1-forms invariant with respect to an action of a cyclic group of order 3.

Key words: finite group actions, invariant vector fields, invariant 1-forms, indices, equivariant deformations.

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Contents

to my mother

Chapter 1

Introduction

An isolated singular point of a (continuous) vector field or a 1-form on a smooth manifold has a well-known integer invariant, the index. It has been an important topological invariant since the classical works of Poincaré [29] and Hopf [24] on the result, which is now known as Poincaré-Hopf theorem and which states that the sum of indices of isolated singular points of a vector field on a smooth compact manifold is equal to the Euler characteristic of the manifold.

For more than half a century there is a lively interest in generalization of the fundamental results for smooth (real or complex) manifolds to the case of singular varieties. In particular, there are several types of indices of isolated singular points of vector fields or 1-forms on singular (real or complex) analytic varieties [3, 10]. One of these generalizations is radial index. The idea of this index was inspired by the Poincaré-Hopf theorem. The first definition of the radial index was introduced for so-called radial vector fields by M.-H. Schwartz [30, 31, 32]. Later it was defined for vector fields or 1-forms on arbitrary real or complex analytic varieties by W. Ebeling and S. M. Gusein-Zade in [13, 14].

In [17] X. Goméz-Mont introduced the so-called homological index for a germ of a complex analytic variety with an isolated singular point at the origin and a holomorphic vector field on it. It was used to obtain a generalization of the algebraic formula for the index of a vector field [26, 28] to the case of a hypersurface with an isolated singularity. The homological index was defined for 1-forms in [15]. For a germ of a variety with an isolated singular point and a holomorphic 1-form on it the authors suggested to consider the difference between the homological and the radial indices as a version of the Milnor number of the singular point of the variety. It was motivated by the fact that the difference of these two indices does not depend on the 1-form and is equal to zero in the non singular case, as both indices are equal to the classical index.

For a finite group G acting on a variety, a G-variety, there are several versions of the Euler characteristic, called the equivariant Euler characteristic. For instance, in [34] it was introduced as an element of the representation ring $R(G)$ of the group G, and in [5, 8] as an element of the Burnside ring $A(G)$. As the sum of the radial indices of a vector field or 1-form on a compact variety with isolated singularities is equal to its Euler characteristic [13, 14], it is natural to try to obtain the same result for the case of a G-variety. That was done in [8]. The equivariant radial index of a G-invariant vector field or 1-form there was defined as an element of the Burnside ring $A(G)$ of the group. There is a natural homomorphism from the Burnside ring $A(G)$ to the ring $R(G)$ of (complex) representations of the group G. It sends the equivariant Euler characteristic defined in [34] to the equivariant Euler characteristic from [5, 8] and gives a version of the equivariant radial index, the reduced equivariant radial index, with values in the ring $R(G)$.

There are very natural generalizations of the notions of the homological index of a vector field or of a 1-form to the equivariant setting (see Section 3.4). These generalizations take values in the ring $R(G)$ of representations. For a holomorphic vector field on a germ of a smooth complex analytic manifold, the equivariant homological index and the equivariant "usual" (radial) index with values in the ring $R(G)$ of representations coincide (see Section 5.1). It follows from the fact that there exists a G-invariant deformation of a G-invariant vector field with only non degenerate singular points. On the other hand, a similar argument does not work for 1-forms. G-invariant deformations of a G-invariant holomorphic 1-form on $(\mathbb{C}^n, 0)$ have, as a rule, complicated singular points.

Aiming to prove that the equivariant homological index and the equivariant radial index of a holomorphic 1-form coincide, we might try to describe all singular points which can appear in generic G-invariant deformations and to compare these indices for them. This approach was used in the current work for 1-forms invariant under actions of groups of order two and three (Chapter 4). However, in the general setting this seems to be a rather involved task and it is not clear to which extent this program can be implemented.

Using another approach, we prove that the equivariant homological index and the reduced equivariant radial index of a singular point of a holomorphic 1-form on a smooth complex-analytic manifold coincide. The proof is by induction on the dimension of the manifold and on the order of a (cyclic) group. This statement allows us to consider the difference between these indices as a version of the equivariant Milnor number of a germ of a G-variety with an isolated singular point.

Structure of the Thesis

In Chapter 2 we recall the definitions of the classical index of a vector field or 1-form on a smooth manifold and of its generalizations to the case of vector fields and 1-forms on singular varieties. We discuss the main results for these notions and mention several advantages of 1-forms over vector fields.

In Chapter 3 we give the necessary definitions of the Burnside ring and the ring of representations, discuss the equivariant analogues of the indices defined in the previous chapter and the equivariant Euler characteristic. In Section 3.4 we introduce the notion of the equivariant homological index and prove the law of conservation for it.

In Chapter 4 for the case of cyclic groups of order two and three we prove the equivalence of the reduced equivariant radial and homological indices and classify simplest zeros of 1-forms which cannot be excluded by deformations with respect to these group actions.

Chapter 5 is mostly based on the article [22] and dedicated to the proof of the coincidence of equivariant homological and reduced equivariant radial indices for G-invariant 1-forms (Theorem 5.3). Section 5.1 contains the same result for vector fields.

In Chapter 6 we discuss the equivariant version of the generalized Milnor number and obtain an algebraic formula for the radial index of a 1-form on a quotient singularity as a corollary of Theorem 5.3.

Chapter 2

Indices of Vector Fields and 1-Forms

2.1 The Classical Poincaré-Hopf Index

Let ξ be a vector field on a smooth manifold $M^n \subset \mathbb{R}^N$, i.e. a section of its tangent bundle TM^n. A neighborhood of a point on M^n may be identified with a neighborhood of the origin in $\mathbb{R}^n$. We can therefore present the vector field using local coordinates in the form: $\xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}$. The vector field is continuous, smooth, analytic, etc. if the components $\{\xi_1, \ldots, \xi_n\}$ are continuous, smooth, analytic, etc.

Definition: A point $p \in M^n$ is called *singular* (or *zero*) for the vector field ξ if all of its components $\{\xi_1, \ldots, \xi_n\}$ vanish simultaneously at p. A singular point p is called *isolated* if there are no other singular points of ξ in a neighborhood of p.

Let B^n_ε be the ball of a small radius ε centered at the origin in $\mathbb{R}^n$ such that the vector field ξ has no other singular points in B^n_ε except the origin itself. Let $S^{n-1}_\varepsilon := \partial B^n_\varepsilon$ be the $(n-1)$-dimensional sphere corresponding to B^n_ε.

Definition: The *Poincaré-Hopf index* (or just the *index)* of the vector field ξ at the point $p \in M^n$ is the degree of the map

$$\frac{\xi}{||\xi||} : S^{n-1}_\varepsilon \to S^{n-1}_1 \tag{2.1}$$

from the $(n-1)$-dimensional sphere of radius ε centered at the origin in $\mathbb{R}^n$ (which corresponds to $p \in M^n$) to the unit sphere S^{n-1}_1.

As one may see, it is independent of the choice of coordinates.

Definition: A singular point p of the vector field ξ is called *non-degenerate* if the Jacobi matrix of the map $(\xi_1, \ldots, \xi_n)$ is non-degenerate at p:

$$J_{\xi,p} = \det\left(\frac{\partial \xi_i}{\partial x_j}(p)\right) \neq 0.$$

Then the (Poincaré-Hopf) index is equal to the sign of the Jacobian, i.e., $\operatorname{ind}(\xi, M; p) = 1$, if $J_{\xi,p} > 0$, and $\operatorname{ind}(\xi, M; p) = -1$, if $J_{\xi,p} < 0$.

The index satisfies the law of conservation of number.

Proposition 2.1 *Let p be an isolated singular point of the vector field ξ on a manifold M. If $\tilde{\xi}$ is a small perturbation of ξ with isolated singular points, then the sum of indices* *of the singular points of $\tilde{\xi}$ near p is equal to the index of ξ at p:*

$$\operatorname{ind}(\xi; M, p) = \sum_{\tilde{p} \in \operatorname{Sing} \tilde{\xi} \cap U(p)} \operatorname{ind}(\tilde{\xi}; M, \tilde{p}),$$

where $U(p)$ is a small neighborhood of p in M.

From the law of conservation it follows that the index of an arbitrary singular point p is equal to the number of non-degenerate singular points of a generic deformation of the vector field ξ in a neighborhood of the point p counted with the corresponding signs of the determinant of the Jacobi matrix of ξ.

The following statement is one of the main properties of the index.

Theorem 2.1 (Poincaré-Hopf theorem) *Let M be a closed manifold (i.e. compact without boundary) and ξ is a vector field with finitely many singular points on it. The sum of indices of singular points of the vector field ξ is equal to the Euler characteristic of the manifold M:*

$$\sum_{p \in \operatorname{Sing} \xi} \operatorname{ind}(\xi; M, p) = \chi(M).$$

In case of a holomorphic vector field there is an algebraic formula for the index of an isolated singular point, which was first proved by V. Palamodov in 1967. Namely, in local coordinates centered at the singular point p let $\xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}$. Denote by $\mathcal{O}_{(\mathbb{C}^n,0)}$ the ring of germs of holomorphic functions of n variables.

Theorem 2.2 *[28]*

$$\operatorname{ind}(\xi; \mathbb{C}^n, p) = \dim_{\mathbb{C}} \mathcal{O}_{(\mathbb{C}^n,0)}/(\xi_1, \ldots, \xi_n),$$

where $(\xi_1, \ldots, \xi_n)$ is the ideal generated by $\xi_1, \ldots, \xi_n$

2.2 Advantages of 1-Forms

In [1] Arnold suggested to consider indices of 1-forms instead of indices of vector fields.

For the case of smooth manifolds there is no difference between indices of vector fields and indices of 1-forms. Vector fields and 1-forms on smooth manifolds can be identified via a Riemannian metric. Therefore, one can define the Poincaré-Hopf index of a 1-form ω as the degree of the map

$$\frac{\omega}{||\omega||} : S_{\varepsilon}^{n-1} \to S_1^{n-1}$$

between the spheres in the dual space.

In the local complex-analytic case the vector field $\sum_{i=1}^{n} \xi_i \frac{\partial}{\partial z_i}$ corresponds to the 1-form $\sum_{i=1}^{n} \xi_i dz_i$, where $(z_1, \ldots, z_n)$ are local coordinates. The Poincaré-Hopf theorem in this case is formulated in a slightly different way: the sum of indices of singular points of a 1-form on a complex manifold M is equal to $(-1)^n \chi(M^n)$, where $n := \dim M$. (See the explanation of the sign in the next section.) Also, in the case of odd-dimensional smooth complex-analytic manifolds non-trivial holomorphic vector fields can exist only if the Euler characteristic of the manifold is non-negative. Whereas, non-trivial holomorphic 1-forms exist only on manifolds with non-positive Euler characteristic.

In the case of singular varieties the difference between vector fields and 1-forms becomes more essential. For example, complex-analytic vector fields with isolated zeros exist on germs of complex-analytic varieties with an isolated singularity, but not in general (see [10, Example 1]). One can say that the tangency condition for vector fields on singular varieties is a very restrictive one. We can see it directly for the GSV-indices (see Section 2.4). The GSV-index of an isolated singular point of a holomorphic 1-form on a complex isolated complete intersection singularity (i.e. a complex variety V given by simultaneous vanishing of $\operatorname{codim} V$ holomorphic functions) may be expressed as the dimension of an appropriate local algebra [9]. On the other hand, the algebraic formula for the GSV-index of a vector field in the same case becomes more complicated [20].

2.3 The Radial Index

Let V be a (real or complex) analytic variety of dimension n in some manifold M.

Definition:[3] An *analytic stratification* of V is a locally finite family $(V_i)_{i\in I}$ of non-singular analytic subspaces of V such that:

(1) V_i's are pairwise disjoint and $\bigcup_{i\in I} V_i$ covers V.

(2) For each V_i, the closures in V of both V_i and $\overline{V_i} \setminus V_i$ are analytic in V.

(3) For each pair (V_i, V_j) such that $V_i \cap \overline{V_j} \neq \emptyset$ one has $V_i \subset \overline{V_j}$.

Definition:[3] A stratification $(V_i)_{i\in I}$ of V is called *Whitney stratification* if it further satisfies Whitney (a) and (b) conditions for every pair (V_i, V_j) such that $V_i \subset \overline{V_j}$.

Let $\{x_\alpha\} \in V_j$ be an arbitrary sequence converging to some point $y \in V_i$ and $\{y_\alpha\} \in V_i$ a sequence that also converges to $y \in V_i$. Suppose these sequences are such that the sequence of secant lines $l_\alpha = \overline{x_\alpha y_\alpha}$ also converges to some limiting line l, and the tangent planes $T_{x_\alpha} V_j$ converge to some limiting plane τ. The Whitney conditions (a) and (b) are the following:

(a) The limit space τ contains the tangent space of the stratum V_i at y: $T_y V_i \subset \tau$.

(b) The limit space contains all the limits of secants: $l \subset \tau$.

Definition:[19] Let $Z \subset M$ be a Whitney stratified subspace of a manifold M. Denote by $D(p)$ a small disk centered at a point $p \in Z$ in M transverse to the stratum $S \subset Z$ containing p, such that:

1) $D(p) \cap S = p$,

2) $\dim S + \dim D(p) = \dim M$.

The *normal slice* to S at p is the intersection of $D(p)$ with Z:

$$N(p) := D(p) \cap Z.$$

Its boundary $L(p) = \partial D(p) \cap Z$ is called the *link* of the stratum S.

Let $(X, 0)$ be a germ of a closed real analytic variety in $(\mathbb{R}^N, 0)$ and $X = \cup_{i=1}^q X_i$ be an analytic Whitney stratification of X.

Definition:[3] A *stratified vector field* on $X = \bigcup_{i=0}^q X_i$ is a vector field such that at each point $x \in X$ it is tangent to the stratum containing x.

Definition:[3] A vector field on $(X, 0)$ is called *radial* if it is transverse (outwards-pointing) to all the spheres of small enough radii around the origin.

Let us prove explicitly the following statement from [8] for a stratified vector field ξ on $(X, 0)$ with an isolated singular point at the origin.

Proposition 2.2 *There exists a (continuous) stratified vector field $\widetilde{\xi}$ on X with the following properties:*

1) the vector field $\widetilde{\xi}$ coincides with ξ on a neighborhood of the intersection of X with the sphere $S_\varepsilon = \partial B_\varepsilon$ of a small radius ε around the origin;

2) the vector field $\widetilde{\xi}$ has a finite number of singular points (zeros);

3) In a neighborhood of each singular point $x_0 \in B_\varepsilon \setminus \{0\}$ there exists a (local) analytic diffeomorphism h which identifies $\widetilde{\xi}$ with the vector field of the form

$$\widetilde{\xi}_{X_i} + \widetilde{\xi}_{\rm rad},$$

where $\widetilde{\xi}_{X_i}$ is a germ of a vector field on X_i with an isolated singular point at the origin, $\widetilde{\xi}_{\rm rad}$ is a radial vector field on the normal slice N_i of X to the stratum X_i at the point x_0.

Proof. Such a vector field may be constructed by induction over the dimensions of the strata X_i, $i = 1, \ldots, q$. If there is a zero-dimensional stratum (which is the origin itself), we take the radial vector field $y_1 \frac{\partial}{\partial y_1} + \ldots + y_N \frac{\partial}{\partial y_N}$ on it ($y_1, \ldots, y_N$ are the local coordinates around the origin in $\mathbb{R}^N$). If the smallest possible dimension of the strata is $m > 0$, we first perturb the restriction of ξ to the stratum of dimension m in such a way that it has only non-degenerate zeros and then add the radial vector field $y_1 \frac{\partial}{\partial y_1} + \ldots + y_{N-m} \frac{\partial}{\partial y_{N-m}}$. We extend the new field to B_ε in such a way that near the neighborhood ∂B_ε it coincides with ξ. Wherein, generally speaking, we obtain new singular points (zeros) in $B_\varepsilon \setminus \{0\}$.

The step of induction is the following. Assume that we already have a stratified vector field ξ' which satisfies the conditions 1) and 2), and the condition 3) holds for all the singular points on the union of all the strata of dimension less than k.

At each of the singular points on the union X^k of strata of dimension k we take the vector field

$$\xi'_{|X^k} + y_1 \frac{\partial}{\partial y_1} + \ldots + y_{N-k} \frac{\partial}{\partial y_{N-k}},$$

where $\xi'_{|_{X^k}}$ is the restriction of the vector field ξ' to X^k, $(y_1, \ldots, y_{N-k})$ are local coordinates on the complement to X^k. We extend it to the whole ball B_ε in such a way that it coincides with ξ near ∂B_ε. Therein, we obtain a vector field which satisfies the properties 1) and 2), and the property 3) now holds for the points on the union of the strata of dimensions less or equal to k. $\square$

Definition:[10] The *radial index* of the vector field ξ on X at the origin is defined as the following sum:

$$\mathrm{ind}_{\mathrm{rad}}(\xi; X, 0) := \sum_{p \in \mathrm{Sing}\, \tilde{\xi}} \mathrm{ind}\,(\tilde{\xi}_{|X_{(p)}}; X_{(p)}, p),$$

where $\mathrm{ind}\,(\tilde{\xi}_{|X_{(p)}}; X_{(p)}, p)$ is the classical index of the restriction of the vector field $\tilde{\xi}$ to the smooth manifold $X_{(p)}$.

Well-definition (i.e. independence of the choice of the vector field $\tilde{\xi}$) of the radial index follows directly from [8, Proposition 1].

Let now ω be a continuous 1-form on the germ $(X, 0) \subset (\mathbb{R}^N, 0)$.

Definition: A *singular point* of a 1-form ω on $(X, 0)$ is a singular point (zero) of its restriction to a stratum of the Whitney stratification of $(X, 0)$. If the stratum is zero-dimensional its point is assumed to be singular.

Definition: A 1-form ω is called *radial* on $(X, 0)$ if, for an arbitrary nontrivial analytic arc $\varphi : (\mathbb{R}, 0) \to (X, 0)$ on $(X, 0)$, the value of the 1-form ω on the tangent vector $\dot{\varphi}(t)$ is positive for positive t (small enough).

Let $\varepsilon > 0$ be small enough so that in the closed ball B_ε of radius ε centered at the origin in $\mathbb{R}^N$ the 1-form ω has no singular points on $X \setminus \{0\}$. As in the case for vector fields one can show that on a neighborhood of B_ε there exists a 1-form $\widetilde{\omega}$ possessing the following properties.

1) The 1-form $\widetilde{\omega}$ coincides with ω on a neighborhood of the sphere $S_\varepsilon = \partial B_\varepsilon$.

2) The 1-form $\widetilde{\omega}$ is radial on $(X, 0)$ at the origin.

3) In a neighborhood of each singular point $x_0 \in (X \cap B_\varepsilon) \setminus \{0\}$, $x_0 \in X_i$, $\dim X_i = k$, the 1-form $\widetilde{\omega}$ looks as follows. There exists a (local) analytic diffeomorphism $h : (\mathbb{R}^N, \mathbb{R}^k, 0) \to (\mathbb{R}^N, X_i, x_0)$ such that $h^*\widetilde{\omega} = \pi_1^*\widetilde{\omega}_1 + \pi_2^*\widetilde{\omega}_2$, where π_1 and π_2 are the natural projections $\pi_1 : \mathbb{R}^N \to \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^N \to \mathbb{R}^{N-k}$ respectively, $\widetilde{\omega}_1$ is the germ of a 1-form on $(\mathbb{R}^k, 0)$ with an isolated singular point at the origin, and $\widetilde{\omega}_2$ is a radial 1-form on $(\mathbb{R}^{N-k}, 0)$.

We therefore can define the radial index of a 1-form in the same way as we did for vector fields.

Definition:[10] The *radial index* of the 1-form ω is the following sum

$$\mathrm{ind}_{\mathrm{rad}}(\omega; X, 0) := \sum_{p \in \mathrm{Sing}\, \tilde{\omega}} \mathrm{ind}\,(\tilde{\omega}_{|X_{(p)}}; X_{(p)}, p).$$

Remark. According to its definition the radial index of a vector field or a 1-form on a smooth manifold is equal to the classical Poincaré-Hopf index.

In the case of complex-analytic varieties the radial index of a (complex valued) vector field is equal to the radial index of its real part, whereas for (complex valued) 1-forms the situation is slightly different. The classical index of a continuous complex valued 1-form ω on a smooth complex-analytic manifold of pure dimension n is $(-1)^n$ times the index of its real part. For example, one can see it if one considers the complex 1-form $\omega = \sum_{j=1}^{n} z_j dz_j$ with $(z_1, \ldots, z_n)$ being local coordinates in a neighborhood of the origin in $\mathbb{C}^n$. In this case the Poincaré-Hopf index of ω is equal to 1, whereas for $\mathrm{Re}(\omega) = \sum_{j=1}^{n}(x_j dx_j - y_j dy_j)$, where $z_j = x_j + iy_j$, $j = 1, \ldots, n$, it is equal to $(-1)^n$. Therefore, we have the following definition:

Definition:[10] The complex radial index of a complex 1-form ω on an n-dimensional variety X at the origin is $(-1)^n$ times the index of the real 1-form $\mathrm{Re}(\omega)$ on X:

$$\mathrm{ind}_{\mathrm{rad}}(\omega; X, 0) = (-1)^n \mathrm{ind}_{\mathrm{rad}}(\mathrm{Re}(\omega); X, 0).$$

From the definition of the radial index one obtains directly the law of conservation of number:

Proposition 2.3 *Let ξ (or ω) be a stratified vector field (or 1-form) with an isolated singular point at the origin on a germ of a variety $(X, 0) \subset (\mathbb{R}^N, 0)$ (with Whitney stratification $X = \bigcup_{i=0}^{q} X_i$) and let ξ' (or ω') be a stratified vector field (or a 1-form) close to ξ (or ω) with only isolated singular points. Then*

$$\mathrm{ind}_{\mathrm{rad}}(\xi; X, 0) = \sum_{p' \in \mathrm{Sing}\,\xi'} \mathrm{ind}_{\mathrm{rad}}(\xi'; X, p')$$

$$(\mathrm{ind}_{\mathrm{rad}}(\omega; X, 0) = \sum_{p' \in \mathrm{Sing}\,\omega'} \mathrm{ind}_{\mathrm{rad}}(\omega'; X, p')).$$

Proof. We prove the statement for vector fields. For 1-forms it is proved in a similar way.

Let $\tilde{\xi}_{p'}$ be vector fields satisfying the properties 1) - 3) from Proposition 2.2 in neighborhoods of each of the singular points p' of ξ'. Denote by $\tilde{\xi}'$ the vector field on $(X, 0)$ that coincides with $\tilde{\xi}_{p'}$ inside each closed ball $B_{\varepsilon'}(p')$ and with ξ' near the boundaries $\partial B_{\varepsilon'}(p')$ and outside of them. Note that it also satisfies the properties 2) and 3) from Proposition 2.2. Let $\tilde{\xi}$ be a vector field satisfying the properties 1) - 3) for the vector field ξ. We can always chose ε (the radius of the closed ball $B_\varepsilon(0)$ around the origin) in such a way that there are no singular points of $\tilde{\xi}'$ in a neighborhood of the boundary $\partial B_\varepsilon(0)$ of the ball $B_\varepsilon(0)$ and outside of it.

Fix a stratum X_i of X. Denote by Y_i the manifold that is obtained by an appropriate smoothing of the intersection with the boundary $\partial B_\varepsilon(0)$ of the complement in X_i of a small tubular neighborhood of ∂X_i. On the tubular neighborhood of ∂X_i, $\tilde{\xi}$ and $\tilde{\xi}'$ are homotopy equivalent, in the class of non-vanishing vector fields on the boundary, to vector fields pointing inside Y_i. It suffices to add the normal vector of a corresponding (large) length, multiplied by a coefficient varying from 0 to 1. On the other part of the boundary, which is the intersection $Y_i \cap \partial B_\varepsilon$, $\tilde{\xi}$ and $\tilde{\xi}'$ are equal to ξ and ξ' correspondingly, and are also homotopy equivalent (as fields close to each other) in the class of non-vanishing fields on the boundary (due to the choice of ε). Therefore from the classical Poincaré-Hopf index theory for manifolds with boundary (see [27]) one has the equality:

$$\sum_{\tilde{p}' \in (\operatorname{Sing} \tilde{\xi}' \cap X_i)} \operatorname{ind}(\tilde{\xi}'_{|X_i}; X_i, \tilde{p}') = \sum_{\tilde{p} \in (\operatorname{Sing} \tilde{\xi} \cap X_i)} \operatorname{ind}(\tilde{\xi}_{|X_i}; X_i, \tilde{p})$$

of sums of indices of two homotopic vector fields on a manifold with boundary.

Therefore:

$$\begin{aligned} \sum_{p' \in \operatorname{Sing} \xi'} \operatorname{ind}_{\mathrm{rad}}(\xi'; X, p') &= \sum_{i=1}^{q} \sum_{\tilde{p}' \in (\operatorname{Sing} \tilde{\xi}' \cap X_i)} \operatorname{ind}(\tilde{\xi}'_{p'|X_i}; X_i, \tilde{p}') \\ &= \sum_{i=1}^{q} \sum_{\tilde{p} \in (\operatorname{Sing} \tilde{\xi} \cap X_i)} \operatorname{ind}(\tilde{\xi}_{p|X_i}; X_i, \tilde{p}) = \operatorname{ind}_{\mathrm{rad}}(\xi; X, 0). \end{aligned}$$

□

2.4 The GSV-Index

The notion of the GSV-index (named after X. Gómez-Mont, J. Seade and A. Verjovsky) was first introduced for vector fields on hypersurfaces with isolated singularities in [18], then generalized to vector fields on isolated complete intersection singularities (ICIS) in [33], and to 1-forms in [9, 12].

Let $(X, 0) \subset (\mathbb{C}^{n+k}, 0)$ be a germ of an isolated complete intersection singularity (ICIS) of dimension n, given as the common zero set of k (germs of) holomorphic functions:

$$X = \{f_1 = \ldots = f_k = 0\}.$$

Let ξ be a continuous vector field tangent to $X \setminus \{0\}$ with an isolated zero at the origin. Let ε be small enough, so that the sphere S_ε of radius ε centered at the origin intersects X transversally and ξ has no other singular points except the origin inside the ball B_ε. Consider the intersection $K_\varepsilon = X \cap S_\varepsilon$. It is a real smooth $(2n-1)$-dimensional oriented manifold, called the *link of the ICIS X*.

Definition: The *GSV-index* $\operatorname{ind}_{\mathrm{GSV}}(\xi; X, 0)$ of the vector field ξ at the origin is the degree of the map:

$$\Psi = (\xi, \operatorname{grad} f_1, \ldots, \operatorname{grad} f_k) : K \to W_{n+k,k+1}$$

from the link K to the Stiefel manifold $W_{n+k,k+1}$ of collections of $k+1$ linearly independent vector fields. By $\operatorname{grad} f_i$ we denote the gradient vector field $\left(\overline{\frac{\partial f_i}{\partial z_1}}, \ldots, \overline{\frac{\partial f_i}{\partial z_{n+k}}}\right)$, whose j-th component is the complex conjugate of $\partial f_i / \partial z_j$.

There is an alternative way to define the GSV-index. First, let us introduce an important notion. Consider the map $f = (f_1, \ldots, f_k) : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}, 0)$. Denote by $\Delta \subset (\mathbb{C}^k, 0)$ the discriminant of f. For $0 < \varepsilon \ll \delta$ small enough the restriction of f to the closed ball of radius δ around the origin in $\mathbb{C}^{n+k}$ is a locally trivial fibration over $B_\varepsilon^{2k} \setminus \Delta$, where B_ε^{2k} is the closed ball of radius ε around the origin in $\mathbb{C}^k$.

Definition: The fiber $X_{\overline{\varepsilon}} = f^{-1}(\overline{\varepsilon}) \cap B_\delta^{2(n+k)}$, where $\overline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k) \in B_\varepsilon^{2k} \setminus \Delta$, is a smooth complex manifold with boundary, called the *Milnor fiber* of the ICIS $(X, 0)$.

One can construct a vector field on a neighborhood of the boundary of the Milnor fiber $X_{\overline{\varepsilon}}$ approximating the vector field ξ on a neighborhood of $X \cap \partial B_\delta^{2(n+k)}$: see in [3]. We extend this vector field to a vector field $\tilde{\xi}$ on the entire fiber $X_{\overline{\varepsilon}}$ with only isolated zeros.

Definition: The *GSV-index* of the vector field ξ at the origin is the following sum:

$$\mathrm{ind}_{\mathrm{GSV}}(\xi; X, 0) = \sum_{\tilde{p} \in \mathrm{Sing}\,\tilde{\xi}} \mathrm{ind}\,(\tilde{\xi}; X_{\overline{\varepsilon}}, \tilde{p}). \tag{2.2}$$

Let now X be a compact variety with only local ICIS's as singularities and let ξ be a vector field on X with isolated singular points. (The set of *singular points* of a vector field ξ (or of a 1-form ω) on a variety X with only isolated singularities is the union of the set of singular points S of X and the set of singular points (i.e. zeros) of ξ (or of ω) on $X \setminus S$.)

Proposition 2.4 *The sum of the GSV-indices of a vector field on the compact variety X is equal to the Euler characteristic of a smoothing $\tilde{X}$ of X.*

$$\sum_{P \in \mathrm{Sing}\,\xi} \mathrm{ind}_{\mathrm{GSV}}(\xi; X, P) = \chi(\tilde{X}).$$

Let ω be a germ of a continuous 1-form on $(\mathbb{C}^{n+k}, 0)$ such that, when considered on an ICIS $(X, 0)$, it has at most an isolated zero at the origin.

Definition:[9] The *GSV-index* of the 1-form ω on the ICIS X at the origin is the degree of the map:

$$\Psi = (\omega, df_1, \ldots, df_k) : K \to W_{n+k,k+1}.$$

The GSV-index of a 1-form on an ICIS X is defined in terms of the Milnor fiber of X just in the same way, as it was done for vector fields in the formula (2.2). One has also an analogue of Proposition 2.4.

2.5 The Homological Index

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a germ of a complex-analytic variety of pure dimension n with an isolated singular point at the origin. Let ξ be a holomorphic vector field tangent to $(X, 0)$ with an isolated zero at the origin. Consider the complex $(\Omega^\bullet_{X,0}, \xi)$:

$$0 \to \Omega^n_{X,0} \to \Omega^{n-1}_{X,0} \to \ldots \to \Omega^0_{X,0} \to 0\,, \tag{2.3}$$

where $\Omega^i_{X,0}$ are the modules of germs of differential i-forms on $(X, 0)$ ($\Omega^0_{X,0} = \mathcal{O}_{X,0}$) and the arrows are given by the contraction of germs of differential forms on $(X, 0)$ with the vector field ξ. This complex has finite-dimensional cohomology groups $H^i(\Omega^\bullet_{X,0}, \xi)$. (This follows from the fact that the corresponding complex of sheaves consists of coherent sheaves and its cohomologies are concentrated at the origin.)

Definition:[17] The *homological index* of the vector field ξ on $(X, 0)$ is the Euler characteristic of the complex $(\Omega^\bullet_{X,0}, \xi)$:

$$\mathrm{ind}_{\mathrm{hom}}(\xi; X, 0) = \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H_j(\Omega^\bullet_{X,0}, \xi).$$

From Lemma 1.1 in [17] it follows that in the case of smooth varieties , i.e. when the germ $(X, 0)$ is equivalent to $(\mathbb{C}^N, 0)$, for the complex (2.3) there is only one non-trivial cohomology group, which is in the dimension 0. Therefore:

$$\mathrm{ind}_{\mathrm{hom}}(\xi; \mathbb{C}^N, 0) = H^0(\Omega^\bullet_{\mathbb{C}^N,0}, \xi) = \mathcal{O}_{\mathbb{C}^N,0}/(\xi_1, \dots, \xi_N),$$

where $\xi = \sum_{i=1}^N \xi_i \frac{\partial}{\partial z_i}$, and $(z_1, \dots, z_N)$ are local coordinates in $(\mathbb{C}^N, 0)$.

Hence, if X is smooth, then, using the algebraic formula for the classical index (Theorem 2.2), one obtains, that the homological index of a vector field with an isolated zero at the origin on $(X, 0)$ is equal to the classical Poincaré-Hopf index:

$$\mathrm{ind}_{\mathrm{hom}}(\xi; X, 0) = \mathrm{ind}\,(\xi; X, 0).$$

The homological index satisfies the following law of conservation of number.

Proposition 2.5 *[17] Let ξ have only one isolated singular point at the origin on $(X, 0)$, and ξ' be a holomorphic vector field on X close to ξ, then:*

$$\mathrm{ind}_{\mathrm{hom}}(\xi; X, 0) = \mathrm{ind}_{\mathrm{hom}}(\xi'; X, 0) + \sum_{p \in \mathrm{Sing}\, \xi' \cap (X \setminus \{0\})} \mathrm{ind}_{\mathrm{hom}}(\xi'; X, p),$$

where the sum is taken over all the singular points (zeros) of ξ' in a small punctured neighborhood of the origin in X.

In [17] it was shown that the homological index of a zero point of a vector field on a hypersurface singularity is equal to the GSV-index. The result was generalized to isolated complete intersection singularities in [20].

The notion of homological index also exists for holomorphic 1-forms: [15]. Let ω be a holomorphic 1-form with an isolated singular point at the origin on $(X, 0)$. One can consider the complex $(\Omega^\bullet_{X,0}, \wedge\omega)$ dual to (2.3):

$$0 \to \Omega^0_{X,0} \to \Omega^1_{X,0} \to \dots \to \Omega^n_{X,0} \to 0\,, \tag{2.4}$$

where the arrows denote the exterior multiplication $\wedge\omega$ by the 1-form ω.

Definition:[15] The *homological index* of the 1-form ω on $(X,0)$ is $(-1)^n$ times the Euler characteristic of the complex $(\Omega^\bullet_{X,0}, \wedge\omega)$:

$$\mathrm{ind}_{\mathrm{hom}}(\omega; X, 0) = \sum_{j=0}^{n} (-1)^{j+n} \dim_{\mathbb{C}} H_j(\Omega^\bullet_{X,0}, \wedge\omega).$$

It was shown in [15] that in the case of smooth varieties the homological index of a 1-form is equal to the classical Poincaré-Hopf index, and in the case of isolated complete intersection singularity it coincides with the GSV-index. The authors have also pointed out that the law of conservation of number, which is formulated in the same way as it was done for vector fields, is a particular case of the main theorem in [16].

Chapter 3

Equivariant Indices

In this chapter we discuss the main points in the theory of equivariant indices, recall the definition of the equivariant radial index and introduce the equivariant homological index.

3.1 The Representation Ring and the Burnside Ring

Most of the current section is based on extracts from the book [25].

Definition: For a finite group G and a vector space V a *(linear) representation of* G *on* V is a homomorphism $\rho : G \to \operatorname{Aut} V$. V is called a (left) G-*module* and ρ is said to give *an action of* G *on* V.

Definition: A *map of* G-*modules* $f : (V_1, \rho_1) \to (V_2, \rho_2)$ is a linear map of vector spaces $f : V_1 \to V_2$ satisfying, for any $v \in V$ and $g \in G$, $f(gv) = gf(v)$. It is an *isomorphism* if there is a map of G-modules $f' : (V_2, \rho_2) \to (V_1, \rho_1)$ so that the compositions ff' and $f'f$ are the identity maps on V_2 and V_1. (V_1, ρ_1) and (V_2, ρ_2) are *isomorphic* if there exists such an isomorphism.

Definition: The *sum* $V \oplus W$ of the two representations is constructed by taking the vector space sum $V \oplus W$ and letting G act by, for $g \in G$,

$$g(v, w) = (gv, gw).$$

The *product* $V \cdot W$ of the two representations V and W is the vector space $V \otimes W$ on which $g \in G$ acts by

$$g(v \otimes w) = gv \otimes gw.$$

Proposition 3.1 *[25] Let U, V and W be G-modules. Then there are isomorphisms:*

i) $(U \oplus V) \oplus W \cong U \oplus (V \oplus W)$

ii) $U \oplus V \cong V \oplus U$

iii) $0 \oplus V \cong V \oplus 0 \cong V$

iv) $(U \cdot V) \cdot W \cong U \cdot (V \cdot W)$

v) $U \cdot V \cong V \cdot U$

vi) $U \cdot 1 \cong 1 \cdot U \cong U$

vii) $U \cdot (V \oplus W) \cong U \cdot V \oplus U \cdot W$,

where 0 is the zero representation of G and 1 the trivial one-dimensional representation.

Consider the set of all formal sums $\sum_i n_i[V_i]$, where V_i are representations of G and n_i are integers. This set is a group under the operation

$$\sum_i n_i[V_i] + \sum_i m_i[V_i] = \sum_i (n_i + m_i)[V_i].$$

Require that

a) If V and W are isomorphic representations, $[V] = [W]$.

b) For all V and W, $[V] + [W] = [V \oplus W]$.

Denote the resulting set $R(G)$. Parts i), ii), iii) of the above proposition imply that $R(G)$ is an abelian group under the operation +. Define the multiplication in $R(G)$ as

$$[V] \cdot [W] := [V \cdot W].$$

From iv), v), vi), vii) of the same proposition we can then obtain that $R(G)$ is a commutative ring. We call it the *representation ring of the group G*.

For a subgroup $H \subset G$ there are natural maps: the *restriction map* $R_H^G : R(G) \to R(H)$ and the *induction map* $I_H^G : R(H) \to R(G)$. The restriction map sends each representation $\rho : G \to \operatorname{Aut} V$ to its restriction $R_H^G(\rho) : H \to \operatorname{Aut} V$. The induction map sends each representation $\rho : H \to \operatorname{Aut} V$ to the *induced representation* $I_H^G(\rho)$ of H (we will also use the equivalent notation: $I_H^G(V)$), which is defined as follows: H acts on G as a set of permutations by $h(g) = gh$. More explicitly, let $\mathbb{C}G$ be the vector space with basis the elements of G. This action makes $\mathbb{C}G$ a right H-module. $I_H^G(V)$ is then defined as the vector space

$$I_H^G(V) := (\mathbb{C}G \otimes V) \Big/ \langle gh \otimes v - g \otimes hv \rangle ,$$

where $\langle gh \otimes v - g \otimes hv \rangle$ is the subspace generated by all the elements of the form $gh \otimes v - g \otimes hv$, $g \in G$, $h \in H$. G acts on $I_H^G(V)$ by $g_1(g \otimes v) = (g_1 g) \otimes v$ for $g_1 \in G$.

Definition: A representation V of a group G is called *reducible* if there is a subspace $W \subset V$ ($W \neq \{0\}, V$), with $gw \in W$ for all $w \in W$, $g \in G$. Otherwise V is irreducible. V is called *decomposable* if V is isomorphic to the sum of two representations of G: $V \cong V_1 \oplus V_2$.

From Maschke's Theorem about decomposition of reducible representations it follows directly that any representation V of G can be written as a finite sum of irreducible representations. Therefore, the representation ring $R(G)$ can be considered as a free abelian group of isomorphism classes of irreducible representations.

Definition: The *group algebra* $\mathbb{C}[G]$ of the group G is a vector space over $\mathbb{C}$ with the basis $\{e_g\}$ corresponding to the elements of G and multiplication induced by the group operation: $e_{g_1} \cdot e_{g_2} = e_{g_1 g_2}$.

Definition: An *action of a finite group G on a finite set S* is a homomorphism of G to the group of permutations of S. S is called a *finite G-set*. If S and T are two G-sets then a *G-map* is a map of sets $f : S \to T$ such that $f(g(s)) = g(f(s))$, for all $g \in G$, $s \in S$. A G-map f is an *isomorphism* if it is an isomorphism as map of sets.

Let S and T be G-sets. Then the disjoint union $S \sqcup T$ is a G-set (under the obvious action: G acts on each component), called the *sum* of S and T. G also acts on the Cartesian product $S \times T$ by $g(s,t) = (g(s), g(t))$, giving the *product* $S \times T$.

The *Burnside ring*, $A(G)$, of a group G, consists of all finite formal sums,

$$\sum_i n_i[S_i],$$

where $n_i \in \mathbb{Z}$, $[S_i]$ are the isomorphism classes of the G-sets S_i, modulo the relation:

$$[S_1] + [S_2] = [S_1 + S_2].$$

The product, defined above, makes $A(G)$ a ring.

Definition: An *orbit* of the action of G on S is any subset of S of the form $\{gs \mid g \in G\}$ for some $s \in S$. The action is called *transitive* if all of S is one orbit. S is then called *irreducible.*

It is not difficult to see, that any finite G-set can be written (uniquely up to order) as a finite sum of irreducible G-sets (or G-orbits). On the other hand, every irreducible G-set S is of the form G/H for some subgroup $H \subset G$ (One can take the subgroup $H = \{g \in G \mid g(s) = s, \text{for all } s \in S\}$ and the G-isomorphism $\varphi : G/H \to S$, $gH \mapsto gs$.) Isomorphism classes of irreducible G-sets are in one-to-one correspondence with the elements of the set $\operatorname{Conjsub} G$ of conjugacy classes of subgroups of G: If G/U and G/V are isomorphic as G-sets, then U and V are conjugate subgroups of G. Therefore one obtains the following statement:

Proposition 3.2 *As an abelian group, the Burnside ring is free of rank k, where k is the number of conjugacy classes of subgroups of G, i.e. each element of $A(G)$ can be written in a unique way as*

$$\sum_{[H] \in \operatorname{Conjsub} G} a_{[H]}[G/H]$$

with $a_{[H]} \in \mathbb{Z}$.

Definition:[4] A *G-set* (not necessarily finite) is a set with an action of the group G, i.e. a set X together with a continuous map $G \times X \to X$, $(g,x) \mapsto gx$, such that $(gh)x = g(hx)$, $ex = x$.

This definition agrees with the definition of a finite G-set. Moreover, the definition of a G-map for any two (not necessarily finite) G-sets is the same as before.

For a point x in a G-set X let $G_x = \{g \in G \mid gx = x\}$ be the isotropy subgroup (i.e. stabilizer) of the point x. For a subgroup $H \subset G$ let

$$X^H = \{x \in X \mid Hx = x\}$$

be the H-fixed point set of X and let

$$X^{(H)} = \{x \in X \mid G_x = H\}$$

be the set of points with the isotropy group H. For a conjugacy class $h := [H] \in \operatorname{Conjsub} G$, let $X^h = \bigcup_{K \in h} X^K$ and $X^{(h)} = \bigcup_{K \in h} X^{(K)}$.

As for the representation rings, there are restriction and induction maps for Burnside rings. For a subgroup $H \subset G$ the *restriction map* $R_H^G : A(G) \to A(H)$ sends a G-set X to the same set considered with H-action. The *induction map* I_H^G sends an H-set X to the product $G \times X$ factorized by the natural equivalence: $(g_1, x_1) \equiv (g_2, x_2)$ if there exists $g \in H$ such that $g_2 = g_1 g$, $x_2 = g^{-1} x_1$ with the natural (left) action of G.

To each permutation representation of G (i.e. to each action of G on a finite set S) one can associate a linear representation of G by considering the corresponding permutations of the basis $\{e_s \mid s \in S\}$ of the $\mathbb{C}$-vector space $\mathbb{C}S$. This gives us a natural homomorphism r_G ("reduction") from the Burnside ring $A(G)$ to the representation ring $R(G)$ of the group G.

Another way to see the same homomorphism is by sending a G-set to the vector space of complex valued functions on it with the induced representation of the group G.

3.2 The Equivariant Euler Characteristics

There are different versions of equivariant Euler characteristic. In this section we will discuss only two of them.

Definition: Let X be a "sufficiently good" topological space (e.g. a real or complex analytic variety). Then its *Euler characteristic* is defined as the following sum

$$\chi(X) := \sum_k (-1)^k \dim H_c^k(X; \mathbb{R}),$$

where $H_c^k(X; \mathbb{R})$ are the cohomology groups of X with compact support.

Unlike the Euler characteristic, which is defined as alternating sum of dimensions of the cohomology groups of X, this Euler characteristic is additive, i.e.

$$\chi(X \sqcup Y) = \chi(X) + \chi(Y)$$

for a disjoint union of two "sufficienty good" spaces.

The construction of the equivariant Euler characteristic as an element of the representation ring follows the lines of [34].

Let X be a finite simplicial complex and the finite group G acts on it in such a way, that if $g \in G$ leaves any simplex σ of X invariant, then it fixes it pointwise. The action of G on the chain complex of X,

$$0 \to C_n(X) \to C_{n-1}(X) \to \ldots \to C_1(X) \to C_0(X) \to 0,$$

with complex coefficients, is induced by the permutation of the simplices. Therefore, the chain complex can be regarded as a sequence of $\mathbb{C}[G]$-modules.

Definition:[34] The *equivariant Euler characteristic* is the alternating sum

$$\chi_G(X) := \sum_{i=0}^{n} (-1)^i [C_i(X)] \in R(G).$$

Using the standard arguments one can rewrite the definition:

$$\chi_G(X) = \sum_{i=0}^{n} (-1)^i [H_i(X; \mathbb{C})] \in R(G)$$

where the action of G on $H_i(X; \mathbb{C})$ is induced by the action on X.

On the other hand, we have the following

Definition: The *equivariant Euler characteristic* is the element of the Burnside ring of G defined by (see, e.g., [5])

$$\chi^G(X) = \sum_{h \in \operatorname{Conjsub} G} \chi(X^{(h)}/G)[G/H] \in A(G),$$

where $H \in h$ is a representative of the class h. The *reduced equivariant Euler characteristic* is

$$\overline{\chi}^G(X) = \chi^G(X) - [G/G] \in A(G).$$

Remark.[8] The natural homomorphism from the Burnside ring to the representation ring described above sends the equivariant Euler characteristic $\chi^G(X) \in A(G)$ to the equivariant Euler characteristic $\chi_G(X) \in R(G)$.

Remark.[8] There is a natural homomorphism $|\cdot| : B(G) \to \mathbb{Z}$ sending a (virtual) G-set A to the number of elements of A. This homomorphism sends the equivariant Euler characteristic $\chi^G(X)$ to the usual Euler characteristic $\chi(X)$.

3.3 The Equivariant Radial Index

We recall the notion of the equivariant radial index of a G-invariant vector field or 1-form on a (real or complex) analytic variety from [8].

Let the space $(\mathbb{R}^N, 0)$ be endowed with a smooth action of a finite group G. Without loss of generality we may assume that the action is linear. Let $(X, 0) \subset (\mathbb{R}^N, 0)$ be a germ of a G-invariant real analytic variety at the origin. There exists an analytic Whitney stratification $X = \bigcup_{i=0}^q X_i$ of the germ $(X, 0)$ such that each X_i is G-invariant, the isotropy subgroups $\operatorname{St}(x) = \{g \in G : gx = x\}$ of all points x of X_i are conjugate to each other and X_i/G is connected. The stratification with these properties is *called G-invariant Whitney stratification*. One can obtain it if one considers an analytic Whitney stratification of $(X, 0)$ and then takes intersections/unions of the strata and their transforms by the elements of G. If the stratum is zero-dimensional, its points are assumed to be singular.

Let ξ be a (continuous) G-invariant stratified vector field on $(X, 0)$ with an isolated singular point (zero) at the origin. The following statement from [8] is the G-invariant version of the Proposition 2.2.

Proposition 3.3 *There exists a (continuous) G-invariant stratified vector field $\widetilde{\xi}$ on X with the following properties:*

1) the vector field $\widetilde{\xi}$ coincides with ξ on a neighbourhood of the intersection of X with the sphere $S_\varepsilon = \partial B_\varepsilon$ of a small radius ε around the origin;

2) the vector field $\widetilde{\xi}$ has a finite number of singular points (zeros);

3) In a neighbourhood of each singular point $x_0 \in B_\varepsilon \setminus \{0\}$ there exists a (local) analytic diffeomorphism h which identifies $\widetilde{\xi}$ with the vector field of the form

$$\widetilde{\xi}_{X_i} + \widetilde{\xi}_{\text{rad}},$$

where $\widetilde{\xi}_{X_i}$ is a germ of a vector field on X_i with an isolated singular point at the origin, $\widetilde{\xi}_{\text{rad}}$ is a radial vector field on the normal slice N_i of X to the stratum X_i at the point x_0.

Proof. The statement is proved in exactly the same way as Proposition 2.2. However, the induction is conducted over the dimensions of the strata, corresponding to the classes $h \in \operatorname{Conjsub} G$ of conjugate subgroups of G. Another difference is that, in order to obtain a G-invariant vector field, we take the mean of the vector field over the group action. □

Definition: [8] The *equivariant radial index* $\operatorname{ind}^G_{\text{rad}}(\xi; X, 0)$ of the vector field ξ on X at the origin is the element of the Burnside ring which is equal to the following sum

$$\operatorname{ind}^G_{\text{rad}}(\xi, X, 0) = \sum_{\bar{p} \in (\operatorname{Sing} \tilde{\xi})/G} \operatorname{ind}\,(\tilde{\xi}_{X_{(p)}}; X_{(p)}, p)[Gp],$$

where p is a point of the preimage of $\bar{p}$ under the canonical projection, $X_{(p)}$ is the stratum containing p and $Gp \cong G/\operatorname{St}(p)$ is the G-set corresponding to the orbit $\bar{p}$.

Now let the space $(\mathbb{C}^N, 0)$ be endowed with an analytic action of the group G and $(X, 0) \subset (\mathbb{C}^N, 0)$ be a germ of a G-invariant analytic variety of pure dimension n, and let ξ be a (continuous, complex-valued) G-invariant vector field on $(X, 0)$.

Definition: The *equivariant radial index* $\operatorname{ind}^G_{\text{rad}}(\xi; X, 0)$ of the complex vector field on the variety X at the origin is the equivariant radial index of the real part of ξ:

$$\operatorname{ind}^G_{\text{rad}}(\xi; X, 0) := \operatorname{ind}^G_{\text{rad}}(Re(\xi); X, 0).$$

Definition: The *reduced equivariant radial index* $\operatorname{rind}^G_{\text{rad}}(\xi; X, 0)$ of a (real or complex) vector field ξ on a (real or complex) analytic variety $(X, 0)$ is the image of the equivariant radial index of ξ under the reduction map

$$\operatorname{rind}^G_{\text{rad}}(\xi; X, 0) = r_G\left(\operatorname{ind}^G_{\text{rad}}(\xi; X, 0)\right) \in R(G)\,.$$

Consider again a germ $(X, 0) \subset (\mathbb{R}^N, 0)$ of a G-invariant real analytic variety at the origin, and let ω be a (continuous) G-invariant 1-form on $(\mathbb{R}^N, 0)$. We assume that ω has an isolated singular point at the origin on $(X, 0)$.

Let $\varepsilon > 0$ be small enough so that in the closed ball B_ε of radius ε centered at the origin in $\mathbb{R}^N$ the 1-form ω has no singular points on $X \setminus \{0\}$. As in the case for vector fields one can show that there exists a G-invariant 1-form $\widetilde{\omega}$ on a neighborhood of B_ε possessing the properties 1) - 3) from the Section 2.3.

The usual index $\operatorname{ind}(\widetilde{\omega}_{|X_i}; X_i, p)$ of the restriction of the 1-form $\widetilde{\omega}$ to the corresponding stratum (a smooth manifold) will be called the *multiplicity* of the 1-form $\widetilde{\omega}$ at the point x_0. If the origin is a stratum of the stratification itself (the zero-dimensional one), the multiplicity of $\widetilde{\omega}$ at the origin is assumed to be equal to 1.

Definition: [8] The *equivariant radial index* $\operatorname{ind}^G_{\rm rad}(\omega; X, 0)$ of the 1-form ω on the variety X at the origin is the element of the Burnside ring $A(G)$ of the group G represented by the set of singular points of the 1-form $\widetilde{\omega}$ regarded with the multiplicities.

Remark. It is possible to assume that the restrictions of the 1-form $\widetilde{\omega}$ to the strata have only non-degenerate singular points. In this case all the multiplicities are equal to ± 1.

Let the space $(\mathbb{C}^N, 0)$ be endowed with an (analytic) action of a finite group G. (Without loss of generality we may assume that the action is linear.) Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a germ of a G-invariant complex analytic variety of pure dimension n and let ω be a (continuous, complex-valued) G-invariant 1-form on $(\mathbb{C}^N, 0)$.

Definition: The *equivariant radial index* $\operatorname{ind}^G_{\rm rad}(\omega; X, 0)$ of the complex 1-form ω on the variety X at the origin is defined by the equation

$$\operatorname{ind}^G_{\rm rad}(\omega; X, 0) = (-1)^n \operatorname{ind}^G_{\rm rad}(\operatorname{Re}\omega; X, 0) \in A(G)\,,$$

where $\operatorname{Re}\omega$ is the real part of the 1-form ω (see the explanation of the sign in Section 2.3).

Definition: The *reduced equivariant radial index* $\operatorname{rind}^G_{\rm rad}(\omega; X, 0)$ of a (real or complex) 1-form ω on a (real or complex) analytic variety $(X, 0)$ is the image under the reduction map of the equivariant radial index

$$\operatorname{rind}^G_{\rm rad}(\omega; X, 0) = r_G\left(\operatorname{ind}^G_{\rm rad}(\omega; X, 0)\right) \in R(G)\,.$$

The equivariant radial index of a vector field (or of a 1-form) on a germ of a G-variety $(X, 0)$ satisfies the law of conservation of number in the following sense:

Proposition 3.4 *[8] For a G-invariant deformation $\tilde{\xi}$ (or $\tilde{\omega}$) of a vector field ξ (or of a 1-form ω) in a neighborhood of the origin we have the following equality:*

$$\begin{gathered} \operatorname{ind}^G_{\rm rad}(\xi; X, 0) = \sum_{\bar{q} \in (\operatorname{Sing}\tilde{\xi})/G} I^G_{\operatorname{St}(p)}(\operatorname{ind}^{\operatorname{St}(q)}_{\rm rad}(\tilde{\xi}; X, q)) \\ (\text{or } \operatorname{ind}^G_{\rm rad}(\omega; X, 0) = \sum_{\bar{q} \in (\operatorname{Sing}\tilde{\omega})/G} I^G_{\operatorname{St}(p)}(\operatorname{ind}^{\operatorname{St}(q)}_{\rm rad}(\tilde{\omega}; X, q))), \end{gathered} \tag{3.1}$$

where q is a point of the preimage of the orbit $\bar{q}$ under the canonical projection, $I^G_{\operatorname{St}(p)} : A(\operatorname{St}(q)) \to A(G)$ is the induction map described before.

3.4 The Equivariant Homological Index

As in the previous section, let the space $(\mathbb{C}^N, 0)$ be endowed with a linear action of a finite group G and let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a germ of a G-invariant complex analytic variety of pure dimension n. Let us assume that X has an isolated singular point at the origin. Let ξ be a G-invariant holomorphic vector field tangent to $(X, 0)$ without singular points (zeroes) outside of the origin. All the spaces $\Omega^i_{X,0}$ in the complex $(\Omega^\bullet_{X,0}, \xi)$ (2.3) and thus the cohomology groups $H^i(\Omega^\bullet_{X,0}, \xi)$ carry natural representations of the group G. The definition of the "usual" (non-equivariant) homological index of a vector field (see Section 2.5) inspires the following definition.

Definition: The *equivariant homological index* $\mathrm{ind}^G_{\mathrm{hom}}(\xi; X, 0)$ of the vector field ξ on $(X, 0)$ is defined by the relation

$$\mathrm{ind}^G_{\mathrm{hom}}(\xi; X, 0) = \sum_{i=0}^{n} (-1)^i [H^i(\Omega^\bullet_{X,0}, \xi)] \in R(G)\,, \tag{3.2}$$

where $[H^i(\Omega^\bullet_{X,0}, \xi)]$ is the class of the (finite-dimensional) G-module $H^i(\Omega^\bullet_{X,0}, \xi)$ in the ring $R(G)$ of complex representations of the group G.

Let ω be a G-invariant holomorphic 1-form on $(X, 0)$ (that is the restriction to $(X, 0)$ of a (G-invariant) holomorphic 1-form on $(\mathbb{C}^N, 0)$) with an isolated zero at the origin. We have the following definition.

Definition: The *equivariant homological index* $\mathrm{ind}^G_{\mathrm{hom}}(\omega; X, 0)$ of the 1-form ω on $(X, 0)$ is defined by the equation

$$\mathrm{ind}^G_{\mathrm{hom}}(\omega; X, 0) = \sum_{i=0}^{n} (-1)^{n-i} [H^i(\Omega^\bullet_{X,0}, \wedge\omega)] \in R(G)\,, \tag{3.3}$$

where $[H^i(\Omega^\bullet_{X,0}, \wedge\omega)]$ is the class of the G-module $H^i(\Omega^\bullet_{X,0}, \wedge\omega)$ in the ring $R(G)$.

One has a law of conservation for the equivariant homological index as well. We will formulate and prove it for vector fields. The statement and the proof for 1-forms is exactly the same with obvious changes.

Let ξ' be a small G-invariant holomorphic deformation of the vector field ξ. For a singular point x of the vector field ξ' in a punctured neighbourhood of the origin 0 in X, denote by $\mathrm{ind}^{\mathrm{St}(x)}_{\mathrm{hom}}(\xi'; X, x) \in R(G_x)$ the equivariant homological index of the vector field ξ' at the point x.

Proposition 3.5

$$\mathrm{ind}^G_{\mathrm{hom}}(\xi; X, 0) = \mathrm{ind}^G_{\mathrm{hom}}(\xi'; X, 0) + \sum_{[x]\in(\mathrm{Sing}(\xi'\setminus\{0\}))/G} I^G_{\mathrm{St}(x)}\left(\mathrm{ind}^{\mathrm{St}(x)}_{\mathrm{hom}}(\xi'; X, x)\right),$$

where the sum on the right hand side is over all orbits $[x]$ of singular points of the vector field ξ' in a small punctured neighbourhood of the origin 0 in X, x is a representative of the orbit $[x]$.

Proof. The proof of this proposition can be obtained from the proof of a more general statement in [16] by considering all sheaves and modules with the corresponding actions (representations) of the group G. In this case all modules are in fact $\mathbb{C}[G]$-modules, and the projective resolution from the proof of one of the main statements [16, Lemma 1] is the resolution of sheaves of $\mathbb{C}[G]$-modules. Therefore, all isomorphisms that are considered in the article commute with the group action. In particular, the isomorphism in homology induced from the map β_* in [16, Proposition 4] is the isomorphism of $\mathbb{C}[G]$-modules, that provides us with the complex whose homology groups are isomorphic as $\mathbb{C}[G]$-modules to the homologies of our complex. Moreover, the alternating sum of the representations on the homologies does not depend on the parameter of the deformation of the vector field, what proves the law of conservation in our case. □

Remark. If X is non-singular (i.e. $(X,0) \cong (\mathbb{C}^n,0)$), the only non-trivial cohomology group of the complex $(\Omega^\bullet_{X,0},\xi)$ is in the dimension 0. If the vector field ξ on $(\mathbb{C}^n,0)$ is equal to $\sum_{i=1}^{n} f_i \frac{\partial}{\partial z_i}$, where $z_1,\ldots,z_n$ are local coordinates near the origin in $\mathbb{C}^n$, one has

$$\mathrm{ind}^G_{\mathrm{hom}}(\xi;\mathbb{C}^n,0) = [\mathcal{O}_{\mathbb{C}^n,0}/\langle f_1,\ldots,f_n\rangle] \in R(G). \tag{3.4}$$

In this case the statement can be reduced to an equivariant version of the law of conservation of number for the multiplicity $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/\langle f_1,\ldots,f_n\rangle)$ of the map $F = (f_1,\ldots,f_n) : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$.

Let $\xi_\lambda = \sum_{j=0}^{n} f_{i\lambda} \frac{\partial}{\partial z_i}$, where $\lambda \in [0,1] \subset \mathbb{C}^1$, be a G-invariant holomorphic deformation of ξ, such that $\xi_0 = \xi$, $\xi_1 = \xi'$, and let $Z := \{f_{1\lambda} = 0,\ldots,f_{n\lambda} = 0\}$, and $\iota : Z \hookrightarrow \mathbb{C}^n \times [0,1]$ be the natural inclusion. Denote by φ the composition of ι and the projection to the second factor. Consider the function

$$\nu(\lambda) := \sum_{z\in\varphi^{-1}(\lambda)} \dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathcal{O}_{(\mathbb{C}^1,\lambda)}} \mathcal{O}_{(Z,z)}).$$

Note that $\nu(0)$ is equal to the multiplicity $\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/\langle f_1,\ldots,f_n\rangle)$. According to the law of conservation [2, Theorem 5.10] $\nu(\lambda)$ is locally constant. Therefore, using the results discussed in [6] we obtain that Z is flat over $\mathbb{C}$ and, therefore, $\varphi_*\mathcal{O}_Z$ is a locally free sheaf, which corresponds to some vector bundle $\mathcal{E}$, whose fibers are the $\mathbb{C}[G]$-modules $\bigoplus_{z\in\varphi^{-1}(\lambda)} \mathbb{C} \otimes_{\mathcal{O}_{(\mathbb{C}^1,\lambda)}} \mathcal{O}_{(Z,z)}$. The group acts fiber-wise on the bundle, hence the representations of the group action on the fibers are equivalent.

3.5 The Equivariant GSV-index

In this section we discuss the equivariant version of the GSV-index for G-invariant 1-forms on G-invariant isolated complete intersection singularities. Analogous definitions and statements exist for G-invariant vector fields as well. However, in the current work we focus on equivariant GSV-index of 1-forms only.

Let $(X,0) = \{f_1 = \ldots = f_k = 0\} \subset (\mathbb{C}^{n+k},0)$ be a G-invariant ICIS defined by relations with G-invariant left-hand sides $f_1,\ldots,f_k$.The notion of the equivariant GSV-index of a G-invariant 1-form ω on $(X,0)$ was introduced by Ebeling and Gusein-Zade in [8]. It was defined as an element of the Burnside ring $A(G)$ of the group G. One way to define it is the following. Let us take a G-invariant representative of the 1-form

ω defined in a neighborhood of the origin in $\mathbb{C}^{n+k}$. We will denote it by ω as well. Let $X_{\overline{\varepsilon}} = \overline{f}^{-1}(\overline{\varepsilon}) \cap B_\delta^{2(n+k)}(0)$ be the Milnor fiber of the ICIS $(X, 0)$ ($\overline{f} = (f_1, \ldots, f_k)$, $\overline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k)$, $0 < \|\overline{\varepsilon}\| \ll \delta$, δ is small enough). One may assume that the set $\operatorname{Sing}\omega$ of the singular points of the restriction of the 1-form ω to $X_{\overline{\varepsilon}}$ is finite (i.e., this restriction has only isolated singular points (zeroes)).

Definition:[8] The *equivariant GSV-index* of the 1-form ω at the origin is defined as the following sum:

$$\operatorname{ind}^G_{\mathrm{GSV}}(\omega; X, 0) := \sum_{[p]\in \operatorname{Sing}\omega/G} I^G_{\mathrm{St}(p}(\operatorname{ind}^{\mathrm{St}(p)}_{\mathrm{rad}}(\omega; X_{\overline{\varepsilon}}, p))\,,$$

where p is a representative of the G-orbit $[p]$.

Let X be a compact complex analytic G-variety with only local ICIS's as singular points and ξ be a G-invariant vector field on V with isolated singular points. If $p \in X$ is a singular point of ξ, then the whole G-orbit $[p]$ of the point p consists of singular points of ξ. One defines (see in [8]) the equivariant GSV-index of the orbit $[p]$ as the image under the induction map $I^G_{\mathrm{St}(p)}$ of the $\mathrm{St}(p)$-equivariant GSV-index of the point p:

$$\operatorname{ind}^G_{\mathrm{GSV}}(\xi; X, [p]) := I^G_{\mathrm{St}(p)}(\operatorname{ind}^{\mathrm{St}(p)}_{\mathrm{GSV}}(\xi; X, p)). \tag{3.5}$$

The equivariant analogue of the Proposition 2.4 follows directly from (3.5).

Proposition 3.6 *[8] The sum of the equivariant GSV-indices of the singular orbits of a G-invariant vector field ξ on the compact G-variety X is equal to the equivariant Euler characteristic of a G-smoothing $\tilde{X}$ of X.*

$$\sum_{[p]\in \operatorname{Sing}\xi/G} \operatorname{ind}^G_{\mathrm{GSV}}(\xi; X, [p]) = \chi^G(\tilde{X}) \in A(G).$$

Chapter 4

Simplest zeros of 1-forms invariant under actions of $\mathbb{Z}_2$ and $\mathbb{Z}_3$

As it was mentioned in the Introduction (Chapter 1), G-invariant deformations of a G-invariant holomorphic 1-form on $(\mathbb{C}^n, 0)$ have, as a rule, complicated singular points (at least for a group G with more than two elements). In order to prove the equivalence of the homological index and the reduction of the equivariant radial index it is natural to try to describe the simplest singularities of G-invariant deformations of generic G-invariant 1-forms and compare the indices for them. This approach seems to be rather complicated for the general case. In this chapter we realize this program for the cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

Remark. All the germs of 1-forms considered in this chapter are germs of holomorphic 1-forms with only isolated singularities.

Definition: A *deformation* of a (germ of a) 1-form ω on $(\mathbb{C}^n, 0)$ is a 1-form Ω on $(\mathbb{C}^n \times \mathbb{C}_\lambda, 0)$ such, that $\Omega\left(\frac{\partial}{\partial\lambda}\right) = 0$ (i. e. the 1-form Ω can be considered as a family ω_λ of 1-forms on a neighborhood of the origin in $\mathbb{C}^n$ for small enough λ) and $\Omega_{|(\mathbb{C}^n,0)} = \omega$.

Let $\mathbb{C}^n$ be endowed with an action of the group $\mathbb{Z}_2$ of order 2. Without loss of generality we can assume that:

$$\sigma \cdot (z_1, \dots, z_n) = (z_1, \dots, z_{s_0}, -z_{s_0+1}, \dots, -z_n),$$

where σ represents the generator of $\mathbb{Z}_2$, $s_0 \in \mathbb{Z}_{\geq 0}$.

It is easy to show that every $\mathbb{Z}_2$-invariant holomorphic 1-form on $\mathbb{C}^n$ with an isolated zero at the origin has a deformation with only non-degenerate zeros. The set of germs of 1-forms with non-degenerate zero at the origin is connected and contains the 1-form

$$\omega_0 = z_1 dz_1 + \ldots + z_n dz_n.$$

Computations of the equivariant radial index and of the equivariant homological index of the 1-form ω_0 give the same result. This implies the following statement.

Theorem 4.1 *For a $\mathbb{Z}_2$-invariant 1-form on $(\mathbb{C}^n, 0)$ one has:*

$$\mathrm{ind}_{\mathrm{hom}}^{\mathbb{Z}_2}(\omega, 0) = \mathrm{rind}_{\mathrm{rad}}^{\mathbb{Z}_2}(\omega, 0).$$

Now let $\mathbb{C}^n$ be endowed with an action of the group $\mathbb{Z}_3$ of order 3. There exist coordinates on $\mathbb{C}^n$ such that the action may be presented in the following way:

$$\sigma(z_1, \dots, z_n) = (z_1, \dots, z_{s_0}, e z_{s_0+1}, \dots, e z_{s_0+s_1}, e^2 z_{s_0+s_1+1}, \dots, e^2 z_{s_0+s_1+s_2}).$$

where σ is the generator of $\mathbb{Z}_3$, $e := \exp(2\pi i/3)$, $s_0, s_1, s_2 \in \mathbb{Z}_{\geq 0}$ and $s_0+s_1+s_2 = n$. For our convenience we assume that $s_2 \geq s_1$. (It could be done by changing the generator of the group.)

Proposition 4.1 *For every $\mathbb{Z}_3$-invariant holomorphic 1-form ω on $(\mathbb{C}^n, 0)$ there exists a $\mathbb{Z}_3$-invariant holomorphic deformation $\tilde{\omega}$ such that it has only non-degenerate zeros outside the invariant subspace $\langle z_1, \dots, z_{s_0}\rangle$, and for each of the zeros $p \in \langle z_1, \dots, z_{s_0}\rangle$ we have the following picture: in coordinates (which we denote by the same variables) centered at point p it is of the form:*

$$\begin{aligned}
&\sum_{i=1}^{s_0}(l_i(z_1,\dots,z_{s_0}) + R_i)dz_i + \sum_{j=1}^{s_1}(l_{s_0+j}(z_{s_0+1},\dots,z_{s_0+s_1}) + R_{s_0+j})dz_{s_0+s_1+j}\\
&\sum_{k=1}^{s_1}(l_{s_0+s_1+k}(z_{s_0+s_1+1},\dots,z_n) + R_{s_0+s_1+k})dz_{s_0+k}+ \qquad (4.1)\\
&\sum_{l=1}^{s_2-s_1}(p_l(z_{s_0+2s_1+1},\dots,z_n) + R_{s_0+2s_1+l})dz_{s_0+2s_1+l},
\end{aligned}$$

where $l_1, \dots, l_{s_0+2s_1}$ are linear functions, $p_1, \dots, p_{n-s_0-2s_1}$ are homogeneous polynomials of degree 2, $R_1, \dots, R_n$ are the terms of higher order, and the system:

$$\begin{cases} l_1 = 0,\\ \dots\\ l_{s_0+2s_1} = 0,\\ p_1 = 0,\\ \dots\\ p_{s_2-s_1} = 0 \end{cases}$$

has only one solution: $(0, \dots, 0)$.

Proof. First we perturb the 1-form in the following way: $\omega_{t_1} = \omega + t_1\omega_1$, where

$$\omega_1 = \sum_{i=1}^{s_0} z_i dz_i + \sum_{j=1}^{s_1}(z_{s_0+j}dz_{s_0+s_1+j} + z_{s_0+s_1+j}dz_{s_0+j}) + \sum_{l=1}^{s_2-s_1} z^2_{s_0+2s_1+l}dz_{s_0+2s_1+l},$$

choosing t_1 and changing the coordinates in such a way that the perturbation is of the form (4.1) at the origin. The new 1-form ω_{t_1} may have zero points outside the origin.

For each new zero $(a_1, \dots, a_{s_0}, 0, \dots, 0)$ on the invariant subspace $\langle z_1, \dots, z_{s_0}\rangle$ we perform similar perturbations: $\omega'_{t_1} = \omega + t_1\omega'$, where

$$\omega' = \sum_{i=1}^{s_0}(z_i - a_i)dz_i + \sum_{j=1}^{s_1}(z_{s_0+j}dz_{s_0+s_1+j} + z_{s_0+s_1+j}dz_{s_0+j}) + \sum_{l=1}^{s_2-s_1} z^2_{s_0+2s_1+l}dz_{s_0+2s_1+l},$$

choosing again the parameters and changing the coordinates in such a way that the perturbations are of the form (4.1) at each of the zeros including the origin. We repeat the same procedure for each of the new occurring zero points on $\langle z_1, \dots, z_{s_0}\rangle$. The process will stop after a finite amount of iterations. At the end we obtain the 1-form $\tilde{\omega}$ which is of the form (4.1) at each of the zeros on the invariant subspace.

Let $(b_1, \ldots, b_n)$ be a representative of one of the orbits of zeros of $\tilde{\omega}$ outside the invariant subspace. Consider the 1-form ω_2' such that $\omega_2' = ((z_1-b_1)+R_1)dz_1+\ldots+((z_n-b_n)+R_n)dz_n$, where R_i, $i = 1, \ldots, n$, are terms of higher order, at $(b_1, \ldots, b_n)$, and its components do not contain the first order terms (z_i-b_i), $i = 1, \ldots, n$, at the other two points of the orbit. After taking the mean over the group action we obtain a $\mathbb{Z}_3$-invariant 1-form ω_2 with non-degenerate zeros at $(b_1, \ldots, b_n)$, $(b_1, \ldots, b_{s_0}, \sigma b_{s_0+1}, \ldots, \sigma b_{s_0+s_1}, \sigma^2 b_{s_0+s_1+1}, \ldots, \sigma^2 b_n)$ and $(b_1, \ldots, b_{s_0}, \sigma^2 b_{s_0+1}, \ldots, \sigma^2 b_{s_0+s_1}, \sigma b_{s_0+s_1+1}, \ldots, \sigma b_n)$. Now we can construct the new $\mathbb{Z}_3$-invariant perturbation $\omega_{t_2} = \tilde{\omega} + t_2\omega_2$ with only non-degenerate zeros, where t_2 and the coordinates are chosen in such a way that ω_{t_2} is of the form (4.1) at the origin.

Applying the same procedure to the other orbits of zeros (in each case choosing the perturbing parameter and the coordinates in such a way, that the perturbation has the form (4.1) at the zero points on $\langle z_1, \ldots, z_{s_0}\rangle$) we obtain a G-invariant holomorphic deformation, which has only non-degenerate zeros outside the space $\langle z_1, \ldots, z_{s_0}\rangle$, and is of the form (4.1) at the zero points on $\langle z_1, \ldots, z_{s_0}\rangle$. □

Proposition 4.2 *In a neighbourhood of the origin the 1-form (4.1) and the 1-form*

$$\sum_{i=1}^{s_0} z_i dz_i + \sum_{j=1}^{s_1} (z_{s_0+j} dz_{s_0+s_1+j} + z_{s_0+s_1+j} dz_{s_0+j}) + \sum_{l=1}^{s_2-s_1} z_{s_0+2s_1+l}^2 dz_{s_0+2s_1+l}. \tag{4.2}$$

can be connected by a continuous family of holomorphic 1-forms that turn to zero only at the origin.

Proof. First let us construct a continuous family between the 1-form (4.1) and the 1-form:

$$\begin{aligned} &\sum_{i=1}^{s_0} z_i dz_i + \sum_{j=1}^{s_1} (z_{s_0+j} dz_{s_0+s_1+j} + z_{s_0+s_1+j} dz_{s_0+j}) + \\ &\quad + \sum_{l=1}^{s_2-s_1} (p_l(z_{s_0+2s_1+1}, \ldots, z_n) + R_{s_0+2s_1+l}) dz_{s_0+2s_1+l}. \end{aligned} \tag{4.3}$$

Let $S^{2n}_{\varepsilon,0}$ be a sphere of a small radius ε centered at the origin. For the vector of linear components $l_1, \ldots, l_{s_0+2s_1}$ of the 1-form we have the following inequality:

$$||(l_1, \ldots, l_{s_0+2s_1})_{|_{S^{2n}_{\varepsilon,0}}}|| \geq \varepsilon c$$

for some $c > 0$. On the other hand, for the vector of higher order terms we have $||(R_1, \ldots, R_{s_0+2s_1})_{|_{S^{2n}_{\varepsilon,0}}}|| = O(\varepsilon^2)$. Therefore

$$\omega_\delta = \sum_{i=1}^{s_0+2s_1} (l_i + (1-\delta)R_i) dz_i + \sum_{j=1}^{s_2-s_1} (p_j + R_{s_0+2s_1+j}) dz_{s_0+2s_1+j}, \tag{4.4}$$

where l_i, $i = 1, \ldots, s_0 + 2s_1$, and p_j, $j = 1, \ldots, s_2 - s_1$ are the same as in (4.2), is the continuous family of holomorphic 1-forms we need.

Applying the idea above but for the vector $(p_1, \dots, p_{s_2-s_1})$ we can also eliminate the higher order terms of the last $s_2 - s_1$ components of the 1-form.

Using proposition 4.3 we finally obtain the continuous family of holomorphic 1-forms between the 1-form (4.3) and the 1-form (4.2). □

Proposition 4.3 *In the (vector) space V of all the n-tuples $(p_1, \dots, p_n)$ of homogeneous polynomials of degree 2 the set M of all the n-tuples vanishing simultaneously only at the origin is connected.*

Proof. The complement $V \setminus M$ of M is a proper complex-analytic subspace of V. Therefore its complement M is connected. □

Theorem 4.2 *For a $\mathbb{Z}_3$-invariant 1-form on $(\mathbb{C}^n, 0)$ one has:*

$$\mathrm{ind}^{\mathbb{Z}_3}_{\mathrm{hom}}(\omega, 0) = \mathrm{rind}^{\mathbb{Z}_3}_{\mathrm{rad}}(\omega, 0). \tag{4.5}$$

Proof. According to the Propositions 4.1 and 4.2 (and to the fact, that for the indices in both parts of (4.5) the sum of indices of singular points of a deformation is equal to the index of the 1-form ω at the origin) it is sufficient to check the equality (4.5) for the 1-form (4.2). For the equivariant homological index we have:

$$\begin{aligned}
\mathrm{ind}^{\mathbb{Z}_3}_{\mathrm{hom}}(\omega, 0) =& [\Omega^n / \Omega^{n-1} \wedge \omega] \\
=& [\langle z_{s_0+2s_1+1}^{a_1} \cdot \dots \cdot z_n^{a_{s_2-s_1}} | (a_1, \dots, a_{s_2-s_1}) \in \mathbb{Z}_2^{s_2-s_1} \rangle dz_1 \wedge \dots \wedge dz_n] \\
=& ([1] + \underbrace{[\sigma^2] + \dots + [\sigma^2]}_{s_2-s_1} + \underbrace{[\sigma^4] + \dots + [\sigma^4]}_{\binom{s_2-s_1}{2}} + \underbrace{[\sigma^6] + \dots + [\sigma^6]}_{\binom{s_2-s_1}{3}} + \dots \\
&+ [\sigma^{2(s_2-s_1)}])[\sigma^{s_1+2s_2}] \\
=& (A_{s_2-s_1}[1] + C_{s_2-s_1}[\sigma] + B_{s_2-s_1}[\sigma^2])[\sigma^{s_1+2s_2}],
\end{aligned} \tag{4.6}$$

where by $[\sigma^k]$ we denote the corresponding class in the representation ring of the group, $A_{s_2-s_1}$, $B_{s_2-s_1}$ and $C_{s_2-s_1}$ are integer numbers that we calculate below.

Note that the class $[1]$ in (4.6) is obtained from classes of those degrees of σ^2 that are divisible by 3. Therefore,

$$A_{s_2-s_1} = \binom{s_2-s_1}{0} + \binom{s_2-s_1}{3} + \binom{s_2-s_1}{6} + \dots . + \binom{s_2-s_1}{3 \cdot \lfloor \frac{s_2-s_1}{3} \rfloor}.$$

The class $[\sigma]$ in (4.6) is obtained from classes of those degrees of σ^2 that are congruent to 1 modulo 3. Therefore:

$$B_{s_2-s_1} = \binom{s_2-s_1}{1} + \binom{s_2-s_1}{4} + \binom{s_2-s_1}{7} + \dots + \binom{s_2-s_1}{b}.$$

where $b \equiv 1 \mod 3$ and $0 \le s_2 - s_1 - b \le 2$.

The class $[\sigma^2]$ in (4.6) is obtained from classes of those degrees of σ^2 that are congruent to 2 modulo 3. Therefore:

$$C_{s_2-s_1} = \binom{s_2-s_1}{2} + \binom{s_2-s_1}{5} + \binom{s_2-s_1}{8} + \dots + \binom{s_2-s_1}{c},$$

where $c \equiv 2 \mod 3$ and $0 \leq s_2 - s_1 - c \leq 2$.

Let $f(x) := (1+x)^m$. Using binomial expansion and the fact that the sum of roots of unity is equal to 0 we have:

$$A_m = \frac{f(1)+f(\sigma)+f(\sigma^2)}{3} = \frac{(1+1)^m + (1+\sigma)^m + (1+\sigma^2)^m}{3}$$
$$= \frac{2^m + (-\sigma^2)^m + (-\sigma)^m}{3} = \begin{cases} \frac{2^m+(-1)^m\cdot 2}{3}, & \text{if } m \equiv 0 \mod 3, \\ \frac{2^m-(-1)^m}{3}, & \text{if } m \equiv 1 \mod 3, \\ \frac{2^m-(-1)^m}{3}, & \text{if } m \equiv 2 \mod 3. \end{cases}$$

On the other hand:

$$A_m + B_m + C_m = \sum_{0\leq k\leq m} \binom{m}{k} = (1+1)^m = 2^m. \tag{4.7}$$

Using Pascal identity for each of the summands in the sums for A_m, B_m and C_m together with (4.7) we get the following:

$$\begin{aligned} A_m &= A_{m-1} + C_{m-1} = 2^m - B_{m-1}, \\ B_m &= B_{m-1} + A_{m-1} = 2^m - C_{m-1}, \\ C_m &= C_{m-1} + B_{m-1} = 2^m - A_{m-1}. \end{aligned} \tag{4.8}$$

And finally:

$$B_m = \begin{cases} \frac{2^m-(-1)^m}{3}, & \text{if } m \equiv 0 \mod 3, \\ \frac{2^m-(-1)^m}{3}, & \text{if } m \equiv 1 \mod 3, \\ \frac{2^m+(-1)^m\cdot 2}{3}, & \text{if } m \equiv 2 \mod 3, \end{cases}$$
$$C_m = \begin{cases} \frac{2^m-(-1)^m}{3}, & \text{if } m \equiv 0 \mod 3, \\ \frac{2^m+(-1)^m\cdot 2}{3}, & \text{if } m \equiv 1 \mod 3, \\ \frac{2^m-(-1)^m}{3}, & \text{if } m \equiv 2 \mod 3. \end{cases}$$

Note that:

$$\sigma^{s_1+2s_2} = \begin{cases} [1], & \text{if } s_2 - s_1 \equiv 0 \mod 3, \\ [\sigma^2], & \text{if } s_2 - s_1 \equiv 1 \mod 3, \\ [\sigma], & \text{if } s_2 - s_1 \equiv 2 \mod 3, \end{cases} \tag{4.9}$$

Therefore:

$$\begin{aligned} \mathrm{ind}^{\mathbb{Z}_3}_{\mathrm{hom}} &= \begin{cases} A_{s_2-s_1}[1] + C_{s_2-s_1}[\sigma] + B_{s_2-s_1}[\sigma^2], & \text{if } s_2 - s_1 \equiv 0 \mod 3, \\ C_{s_2-s_1}[1] + B_{s_2-s_1}[\sigma] + A_{s_2-s_1}[\sigma^2], & \text{if } s_2 - s_1 \equiv 1 \mod 3, \\ B_{s_2-s_1}[1] + A_{s_2-s_1}[\sigma] + C_{s_2-s_1}[\sigma^2], & \text{if } s_2 - s_1 \equiv 0 \mod 3, \end{cases} \\ &= \frac{2^{s_2-s_1} + (-1)^{n-s_0}\cdot 2}{3}[1] + \frac{2^{s_2-s_1} - (-1)^{n-s_0}}{3}([\sigma] + [\sigma^2]). \end{aligned} \tag{4.10}$$

In the last equality we replaced the signs due to the following:

$$(-1)^{s_2-s_1} = (-1)^{s_2+s_1-2s_1} = (-1)^{s_2+s_1} = (-1)^{n-s_0}.$$

On the other hand, we have the equivariant radial index:

$$\mathrm{ind}^G_{\mathrm{rad}}(\mathrm{Re}(\omega), 0) = \alpha[\mathbb{Z}_3/\mathbb{Z}_3] + \beta[\mathbb{Z}_3/\{e\}],$$

where $\alpha + 3\beta = (-1)^n 2^{s_2-s_1}$, $\alpha = (-1)^{s_0}$ (as the index of the 1-form $\mathrm{Re}(z_1 dz_1 + \ldots + z_{s_0} dz_{s_0})$). Therefore $\beta = \frac{1}{3}(-1)^n(2^{s_2-s_1} - (-1)^{s_0-n})$. And, finally:

$$\begin{aligned} \mathrm{rind}^G_{\mathrm{rad}}(\omega, 0) &= r((-1)^n \mathrm{ind}^G_{\mathrm{rad}}(\mathrm{Re}(\omega), 0)) \\ &= (-1)^{s_0+n}[1] + \frac{(-1)^{2n}(2^{s_2-s_1} - (-1)^{s_0-n})}{3}([1] + [\sigma] + [\sigma^2]) \\ &= \frac{2^{s_2-s_1} + 2(-1)^{s_0-n}}{3}[1] + \frac{2^{s_2-s_1} - (-1)^{s_0-n}}{3}([\sigma] + [\sigma^2]) \end{aligned}$$

what coincides with (4.10). □

Remark. Propositions 4.1 - 4.3 imply that for $G = \mathbb{Z}_2$ or $\mathbb{Z}_3$ the set of germs of G-equivariant 1-forms with the minimal multiplicity is connected and contains the differential of a G-invariant function. One can conjecture that this holds for an arbitrary finite group.

Chapter 5

Coincidence of Equivariant Radial and Homological Indices

5.1 Coincidence for Vector Fields

As we mentioned before a neighbourhood of a point on a smooth complex analytic variety can be identified with a neighbourhood of the origin in $\mathbb{C}^n$. Therefore we may consider $(\mathbb{C}^n, 0)$ instead of a germ of a smooth complex analytic variety. Let a finite group G act on $(\mathbb{C}^n, 0)$.

Lemma 5.1 *For every G-invariant holomorphic vector field ξ on $(\mathbb{C}^n, 0)$ there exists a G-invariant holomorphic deformation $\tilde{\xi}$ with only non-degenerate zeros.*

Proof. We construct such a deformation using the same method as in Proposition 4.1. First we perturb the vector field in the following way:

$$\xi_{t_1} = \xi + t_1 \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}$$

choosing t_1 such that at the origin ξ_{t_1} is of the form

$$A \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + \text{terms of higher order}, \tag{5.1}$$

where A is a non-degenerate constant matrix. For the new occurring singular points we perform similar deformations choosing parameters in the same way as we chose t_1. The process will stop after a finite amount of iterations.

We eliminate the higher order terms for each of the singular points by the same method as in Proposition 4.2. At the end we obtain a vector field with only non-degenerate zeros. □

Theorem 5.1 *For a G-invariant holomorphic vector field ξ on $(\mathbb{C}^n, 0)$*

$$\operatorname{rind}^G_{\text{rad}}(\xi; \mathbb{C}^n, 0) = \operatorname{ind}^G_{\text{hom}}(\xi; \mathbb{C}^n, 0).$$

Proof. Let $\tilde{\xi}$ be the holomorphic deformation of ξ from Lemma 5.1. By the law of conservation (Proposition 3.4) for the equivariant radial index

$$\operatorname{ind}^G_{\mathrm{rad}}(\xi; \mathbb{C}^n, 0) = \sum_{\bar{p}\in \operatorname{Sing}\tilde{\xi}/G} I^G_{\mathrm{St}(p)}(\operatorname{ind}^{\mathrm{St}(p)}_{\mathrm{rad}}(\tilde{\xi}; \mathbb{C}^n, p)). \tag{5.2}$$

On the other hand, from the definition of the equivariant radial index one obtains that:

$$\begin{aligned}\operatorname{ind}^{\mathrm{St}(p)}_{\mathrm{rad}}(\tilde{\xi}; \mathbb{C}^n, p) &= \sum_{\overline{p'}\in \operatorname{Sing}\tilde{\xi}/\operatorname{St}(p)} \operatorname{ind}(\tilde{\xi}_{|X_{(p')}}; X, p')\cdot[\operatorname{St}(p)/\operatorname{St}(p')] \\ &= \operatorname{ind}(\tilde{\xi}_{|X_{(p)}}; X_{(p)}, p)\cdot[\operatorname{St}(p)/\operatorname{St}(p)] = [\operatorname{St}(p)/\operatorname{St}(p)] = [1] \in A(\operatorname{St}(p)),\end{aligned}$$

as p is a non-degenerate singular point of the vector field $\tilde{\xi}$.

We can therefore rewrite (5.2) for the reduced equivariant radial index:

$$\operatorname{rind}^G_{\mathrm{rad}}(\xi; \mathbb{C}^n, 0) = \sum_{\bar{p}\in (\operatorname{Sing}\tilde{\xi})/G} I^G_{\mathrm{St}(p)}([1]_{\mathrm{St}(p)}) \in R(G), \tag{5.3}$$

where by $[1]_{\mathrm{St}(p)}$ we denote the class of trivial representation in the representation ring $R(\mathrm{St}(p))$.

By the law of conservation for the equivariant homological index (Proposition 3.5):

$$\operatorname{ind}^G_{\mathrm{hom}}(\xi; \mathbb{C}^n, 0) = \sum_{\bar{p}\in (\operatorname{Sing}\tilde{\xi})/G} I^G_{\mathrm{St}(p)}(\operatorname{ind}^{\mathrm{St}(p)}_{\mathrm{hom}}(\tilde{\xi}; \mathbb{C}^n, p)). \tag{5.4}$$

For $(\mathbb{C}^n, 0)$ the only non trivial cohomology group of the complex $(\Omega^\bullet_{X,0}, \xi)$ (2.3) is in the dimension 0. Therefore for any singular point p of $\tilde{\xi}$

$$\operatorname{ind}^{\mathrm{St}(p)}_{\mathrm{hom}}(\tilde{\xi}; \mathbb{C}^n, p) = [\mathcal{O}_{(\mathbb{C}^n,0)}/(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)]_{\mathrm{St}(p)} \in R(\mathrm{St}(p)), \tag{5.5}$$

where $\tilde{\xi}_1, \ldots, \tilde{\xi}_n$ are the components of $\tilde{\xi}$.

On the other hand, for a non-degenerate singular point p of the vector field $\tilde{\xi}$ there is a $\mathbb{C}$-algebra isomorphism

$$\mathcal{O}_{(\mathbb{C}^n,p)}/(\tilde{\xi}_1, \ldots, \tilde{\xi}_n) \cong \mathcal{O}_{(\mathbb{C}^n,p)}/(z_1, \ldots, z_n), \tag{5.6}$$

where $z_1, \ldots, z_n$ are local coordinates centered in p. Therefore:

$$\begin{aligned}\operatorname{ind}^G_{\mathrm{hom}}(\xi; \mathbb{C}^n, 0) &= \sum_{\bar{p}\in (\operatorname{Sing}\tilde{\xi})/G} I^G_{\mathrm{St}(p)}([\mathcal{O}_{(\mathbb{C}^n,p)}/(z_1, \ldots, z_n)]) \\ &= \sum_{\bar{p}\in (\operatorname{Sing}\tilde{\xi})/G} I^G_{\mathrm{St}(p)}([1]_{\mathrm{St}(p)}) \in R(G)\end{aligned} \tag{5.7}$$

From (5.3) and (5.7) we obtain the statement of the theorem. □

5.2 Equivariant Radial and Homological Indices in the One-dimensional Case.

A finite group G acting faithfully on the line $(\mathbb{C}, 0)$ is cyclic; let it be $\mathbb{Z}_m$, and let σ be a generator of $\mathbb{Z}_m$. Without loss of generality we can assume that σ acts on $\mathbb{C}$ as multiplication by $\sigma := \exp(2\pi i/m)$. (The coincidence of notations for a generator of $\mathbb{Z}_m$ and for $\exp(2\pi i/m)$ here and below does not lead to a confusion. Moreover, we shall use the same notation for the representation of the group $\mathbb{Z}_m$ on $\mathbb{C}$, which we describe) A (non-trivial) $\mathbb{Z}_m$-invariant 1-form on $(\mathbb{C}, 0)$ is right-equivalent to $z^{sm-1}dz$ (i.e., can be reduced to the latter by a change of variable on $\mathbb{C}$).

Proposition 5.1 *The reduced equivariant radial index and the equivariant homological index of the 1-form $\omega_s = z^{sm-1}dz$ (as elements of the ring $R(\mathbb{Z}_m)$) are equal to $s(1 + \sigma + \sigma^2 + \ldots + \sigma^{m-1}) - 1$.*

Proof. The usual (non-equivariant) index of this (complex) 1-form is equal to $sm - 1$. Therefore, the index of its real part is equal to $1 - sm$. The G-equivariant 1-form $\widetilde{\omega}_s$ in the definition of the radial index of the 1-form $\mathrm{Re}\,\omega_s$ is radial at the origin of $\mathbb{C} \cong \mathbb{R}^2$, and the orbits of its singular points outside of the origin are free. Therefore,

$$\mathrm{rind}^G_{\mathrm{rad}}(\mathrm{Re}\,\omega_s; \mathbb{R}^2, 0) = 1 - sI^{\mathbb{Z}_m}_{(e)}(1) = 1 - s(1 + \sigma + \sigma^2 + \ldots + \sigma^{m-1})\,.$$

Thus $\mathrm{rind}^G_{\mathrm{rad}}(\omega_s; \mathbb{C}, 0) = s(1 + \sigma + \sigma^2 + \ldots + \sigma^{m-1}) - 1$.

A basis of $\Omega^1_{\mathbb{C},0}/\omega_s \wedge \Omega^0_{\mathbb{C},0}$ consists of the (monomial) 1-forms dz, zdz, …, $z^{sm-2}dz$. On the element $z^j dz$ the generator σ acts by the representation σ^{j+1}. This gives $\mathrm{ind}^G_{\mathrm{hom}}(\omega_s; \mathbb{C}, 0) = s(1 + \sigma + \sigma^2 + \ldots + \sigma^{m-1}) - 1$. □

Proposition 5.2 *The reduced radial and the homological equivariant index of the 1-form $\omega_s = z^{sm-1}dz$, i. e., $\mathrm{ind} = s(1 + \sigma + \sigma^2 + \ldots + \sigma^{m-1}) - 1$, is not a divisor of zero in $R(\mathbb{Z}_m)$.*

Proof. The table of multiplication of the basis elements σ^i, $i = 0, 1, \ldots, m-1$, by the element ind is given by the $m \times m$-matrix $sI - E$, where I is the matrix all of whose entries are equal to 1, E is the identity matrix. This matrix is non-degenerate (since its eigenvalues are $s(m-1)$ and -1, the latter one with multiplicity $m-1$). □

5.3 A Sebastiani–Thom Formula for the Equivariant Indices.

Let $\mathbb{C}^n$ and $\mathbb{C}^m$ be spaces with actions (representations) of the group G, and let ω and η be G-invariant 1-forms on $(\mathbb{C}^n, 0)$ and $(\mathbb{C}^m, 0)$, respectively, with isolated singular points at the origin. The Sebastiani–Thom sum (the direct sum) $\omega \oplus \eta$ of the 1-forms ω and η (a 1-form on $(\mathbb{C}^n \oplus \mathbb{C}^m, 0) \cong (\mathbb{C}^{n+m}, 0)$) is defined by $(\omega \oplus \eta)_{(x,y)}(u, v) = \omega_x(u) + \eta_y(v)$ ($x \in \mathbb{C}^n$, $y \in \mathbb{C}^m$, $u \in T_x\mathbb{C}^n \cong \mathbb{C}^n$, $v \in T_y\mathbb{C}^m \cong \mathbb{C}^m$).

Theorem 5.2 *(a version of the Sebastiani–Thom theorem) The following relation holds:*

$$\begin{aligned}\mathrm{ind}^G_{\mathrm{rad}}(\omega\oplus\eta;\mathbb{C}^{n+m},0) &= \mathrm{ind}^G_{\mathrm{rad}}(\omega;\mathbb{C}^{n},0)\cdot\mathrm{ind}^G_{\mathrm{rad}}(\eta;\mathbb{C}^{m},0)\in A(G)\,,\\ \mathrm{ind}^G_{\mathrm{hom}}(\omega\oplus\eta;\mathbb{C}^{n+m},0) &= \mathrm{ind}^G_{\mathrm{hom}}(\omega;\mathbb{C}^{n},0)\cdot\mathrm{ind}^G_{\mathrm{hom}}(\eta;\mathbb{C}^{m},0)\in R(G)\,.\end{aligned}$$

Proof. For the radial index, this is a consequence the following construction. Let $\widetilde{\omega}$ and $\widetilde{\eta}$ be the 1-forms in the definition of the equivariant radial index (which correspond to the 1-forms $\omega' := \mathrm{Re}\,\omega$ and $\eta' := \mathrm{Re}\,\eta$, respectively). Without loss of generality we may assume that $\widetilde{\omega}$ and $\widetilde{\eta}$ of the same radius ε (centered at the origin) are defined on the balls B^{2n}_ε and B^{2m}_ε in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, and that they coincide with ω' and η', respectively, outside the balls of radius $\varepsilon/4$. Let $\psi(r)$ be a (continuous) function on $[0,\varepsilon]$ such that $0\le\psi(r)\le 1$, $\psi(r)\equiv 1$ for $r\le\varepsilon/2$, $\psi(r)\equiv 0$ for $r\ge 3\varepsilon/4$. Consider the 1-form $\widetilde{\omega\oplus\eta}$ on $B^{2(n+m)}_\varepsilon\subset\mathbb{C}^{n+m}$ defined by

$$\widetilde{\omega\oplus\eta}_{(x,y)} = (1-\psi(r))\omega'_x\oplus\eta'_y + \psi(r)\widetilde{\omega}_x\oplus\widetilde{\eta}_y\,,$$

where $x\in\mathbb{C}^n$, $y\in\mathbb{C}^m$, and $r:=\sqrt{\|x\|^2+\|y\|^2}$. One can see that the 1-form $\widetilde{\omega\oplus\eta}$ considered on the ball $B^{2(n+m)}_\varepsilon\subset\mathbb{C}^{n+m}$ is suited for the definition of the equivariant radial index of the 1-form $\mathrm{Re}\,(\omega\oplus\eta)=\omega'\oplus\eta'$ (i. e., satisfies the conditions 1)–3) from Section 2.3). Moreover, the set of its singular points (considered as a G-set) is the direct product of the sets of singular points of the 1-forms $\widetilde{\omega}$ and $\widetilde{\eta}$. This follows from the fact that, for a point (x,y) outside $B^{2n}_{\varepsilon/2}\times B^{2m}_{\varepsilon/2}$ either $\widetilde{\omega}_x=\omega'_x$ or $\widetilde{\eta}_y=\eta'_y$ and, therefore, the 1-form $\widetilde{\omega\oplus\eta}$ does not vanish.

For the homological index, this follows from the fact that, for a 1-form on a non-singular manifold, the only non-trivial cohomology group of the complex (2.4) is in the highest dimension, we have

$$\Omega^{n+m}_{\mathbb{C}^{n+m},0}/(\omega\oplus\eta)\wedge\Omega^{n+m-1}_{\mathbb{C}^{n+m},0} = (\Omega^{n}_{\mathbb{C}^{n},0}/\omega\wedge\Omega^{n-1}_{\mathbb{C}^{n},0})\otimes(\Omega^{m}_{\mathbb{C}^{m},0}/\eta\wedge\Omega^{m-1}_{\mathbb{C}^{m},0})$$

(as spaces with G-representations). □

Remark. For the radial index, the same relation holds for two 1-forms on (singular) varieties and for the corresponding 1-form (the direct sum) on the product of these varieties. For the homological index defined here, the corresponding relation does not make sense. Here the homological index is defined for a 1-form on a variety with an isolated singular point, while the product of two varieties with isolated singular points has non-isolated singular points.

Corollary. The following relation holds

$$\mathrm{rind}^G_{\mathrm{rad}}(\omega\oplus\eta;\mathbb{C}^{n+m},0)=\mathrm{rind}^G_{\mathrm{rad}}(\omega;\mathbb{C}^{n},0)\cdot\mathrm{rind}^G_{\mathrm{rad}}(\eta;\mathbb{C}^{m},0)\in R(G)\,.$$

5.4 Destabilization of Singular Points

Let $\mathbb{C}^m=\mathbb{C}^{m-k}\oplus\mathbb{C}^k$ be a G-invariant decomposition of the space $\mathbb{C}^m$ with a representation of the group G such that the kth exterior power of the action of G on $\mathbb{C}^k$ (i.e., the action of G on the space of k-forms on $\mathbb{C}^k$) is trivial. Let ω be a G-invariant holomorphic 1-form on $(C^m, 0)$ such that its restriction to $(\mathbb{C}^k, 0)$ is non-degenerate.

Proposition 5.3 *There exists a G-invariant complex analytic 1-form η on $(\mathbb{C}^{m-k}, 0)$ such that* $\operatorname{ind}^G_{\rm hom}(\omega; \mathbb{C}^m, 0) = \operatorname{ind}^G_{\rm hom}(\eta; \mathbb{C}^{m-k}, 0)$, $\operatorname{ind}^G_{\rm rad}(\omega; \mathbb{C}^m, 0) = \operatorname{ind}^G_{\rm rad}(\eta; \mathbb{C}^{m-k}, 0)$, *and therefore* $\operatorname{rind}^G_{\rm rad}(\omega; \mathbb{C}^m, 0) = \operatorname{rind}^G_{\rm rad}(\eta; \mathbb{C}^{m-k}, 0)$.

Proof. For small $x \in \mathbb{C}^{m-k}$, the restriction of the 1-form ω to the affine subspace $\{x\}\times\mathbb{C}^k$ has one non-degenerate zero $\{x\}\times f(x)$ in a neighborhood of $\{x\}\times\{0\}$, where f is a G-equivariant analytic map from $(\mathbb{C}^{m-k}, 0)$ to $(\mathbb{C}^k, 0)$. Let $H : \mathbb{C}^{m-k}\oplus\mathbb{C}^k \to \mathbb{C}^{m-k}\oplus\mathbb{C}^k$ be defined by $H(x, y) = (x, y + f(x))$. This map is a local G-equivariant holomorphic automorphism of $\mathbb{C}^{m-k} \oplus \mathbb{C}^k$. The 1-form $H^*\omega$ has the same equivariant radial and homological indices as ω. Moreover, for any $x \in (\mathbb{C}^{m-k}, 0)$, the restriction of $H^*\omega$ to $\{x\} \times \mathbb{C}^k$ has a non-degenerate singular point at the origin $\{x\} \times \{0\}$. If $\varphi_i(\overline{z})$, $i = 1, \ldots, m$, are the components of the 1-form $H^*\omega$ ($H^*\omega = \sum_{i=1}^{m} \varphi_i(\overline{z})dz_i$, $\overline{z} = (z_1, \ldots, z_m)$), then the ideal in $\mathcal{O}_{\mathbb{C}_m,0}$ generated by the elements $\varphi_{m-k+1}(\overline{z})$, ..., $\varphi_m(\overline{z})$ coincides with the ideal $\langle z_{m-k+1}, \ldots, z_m\rangle$. Therefore,

$$\mathcal{O}_{\mathbb{C}^m,0}/\langle\varphi_1, \ldots, \varphi_m\rangle = \mathcal{O}_{\mathbb{C}^{m-k},0}/\langle\varphi_{1|(\mathbb{C}^{m-k},0)}, \ldots, \varphi_{m-k|(\mathbb{C}^{m-k},0)}\rangle\,.$$

This implies that $\operatorname{ind}^G_{\rm hom}(\omega; \mathbb{C}^m, 0) = \operatorname{ind}^G_{\rm hom}((H^*\omega)_{|(\mathbb{C}^{m-k},0)}; \mathbb{C}^{m-k}, 0)$.

Let π_1 and π_2 be the natural projections of $T_p\mathbb{C}^m \cong \mathbb{C}^m$ to $T_p\mathbb{C}^{m-k} \cong \mathbb{C}^{m-k}$ and to $T_p\mathbb{C}^k \cong \mathbb{C}^k$, respectively, ($p \in (\mathbb{C}_m, 0)$). Let $\omega_i = \pi_i^* H^*\omega$, $i = 1, 2$. One has $H^*\omega = \omega_1 + \omega_2$. (Note that π_1 and π_2 are not maps from $(\mathbb{C}^m, 0)$ to $(\mathbb{C}^{m-k}, 0)$ and to $(\mathbb{C}^k, 0)$.) Let $\varepsilon > 0$ be small enough, and let $\psi(r)$ be the function described in the proof of Theorem 5.2. Let $\widehat{\omega}$ be the 1-form defined by $\widehat{\omega}_{|(\overline{z}',\overline{z}'')} = \omega_{1|(\overline{z}',0)} + \omega_{2|(0,\overline{z}'')}$, where $\overline{z}' \in (\mathbb{C}^{m-k}, 0)$, $\overline{z}'' \in (\mathbb{C}^k, 0)$. One can see that the 1-form $\psi(r)\widehat{\omega} + (1 - \psi(r))H^*\omega$ has no zeros in the ball of radius ε outside of the origin, coincides with $H^*\omega$ in a neighborhood of the boundary of this ball, and coincides with $(H^*\omega)_{(\mathbb{C}^{m-k},0)} \oplus (H^*\omega)_{(\mathbb{C}^k,0)}$ in the ball of radius $\varepsilon/2$. According to Theorem 5.2, this implies that

$$\begin{aligned}\operatorname{ind}^G_{\rm rad}(\omega; \mathbb{C}^m, 0) &= \operatorname{ind}^G_{\rm rad}((H^*\omega)_{(\mathbb{C}^{m-k},0)}; \mathbb{C}^{m-k}, 0) \cdot \operatorname{ind}^G_{\rm rad}((H^*\omega)_{(\mathbb{C}^k,0)}; \mathbb{C}^k, 0)\\ &= \operatorname{ind}^G_{\rm rad}((H^*\omega)_{(\mathbb{C}^{m-k},0)}; \mathbb{C}^{m-k}, 0).\end{aligned}$$

□

5.5 Coincidence for 1-Forms

Theorem 5.3 *For a G-invariant holomorphic 1-form ω on $(\mathbb{C}^n, 0)$*

$$\operatorname{rind}^G_{\rm rad}(\omega; \mathbb{C}^n, 0) = \operatorname{ind}^G_{\rm hom}(\omega; \mathbb{C}^n, 0)\,.$$

Proof. For a subgroup H of the group G, the indices $\operatorname{rind}^H_{\rm rad}(\omega; \mathbb{C}^n, 0)$ and $\operatorname{ind}^H_{\rm hom}(\omega; \mathbb{C}^n, 0)$ are the images of the indices $\operatorname{rind}^G_{\rm rad}(\omega; \mathbb{C}^n, 0)$ and $\operatorname{ind}^G_{\rm hom}(\omega; \mathbb{C}^n, 0)$ under the reduction homomorphism R^G_H. A representation of a finite group is determined by its character: the trace of the corresponding operator as a function on the group. Each element of a finite group is contained in a cyclic subgroup. Therefore it is sufficient to prove the statement for G being a cyclic group $\mathbb{Z}_d$.

The proof is by the induction both on the dimension n of the space and on the number d of elements of the group $G = \mathbb{Z}_d$. For $n = 1$ (the 1-dimensional case), the statement is proved in Section 5.4. For the trivial group G (i. e., in the non-equivariant setting), the statement is well known (see, e. g., [15]). Assume first that the representation of the group G on $\mathbb{C}^n$ has a non-trivial summand $\mathbb{C}^k$ with the trivial representation of G and $\mathbb{C}^n = \mathbb{C}^{n-k} \oplus \mathbb{C}^k$ is a decomposition of the representation space. There exists a G-invariant holomorphic deformation $\widetilde{\omega}$ of the 1-form ω such that, at each singular point (zero) $p \in \{0\} \times \mathbb{C}^k$ (that is, $p = (0, y_0)$) of the 1-form $\widetilde{\omega}$, the restriction of this form to the (affine) subspace $\{0\} \times \mathbb{C}^k$ has a non-degenerate zero. Proposition 5.3 implies that there exists a G-invariant 1-form ω' on $(\mathbb{C}^{n-k}, 0)$ such that

$$\begin{aligned} \mathrm{rind}^G_{\mathrm{rad}}(\widetilde{\omega}; \mathbb{C}^n, p) &= \mathrm{rind}^G_{\mathrm{rad}}(\omega'; \mathbb{C}^{n-k}, 0), \\ \mathrm{ind}^G_{\mathrm{hom}}(\widetilde{\omega}; \mathbb{C}^n, p) &= \mathrm{ind}^G_{\mathrm{hom}}(\omega'; \mathbb{C}^{n-k}, 0). \end{aligned}$$

According to the induction hypothesis, one has $\mathrm{rind}^G_{\mathrm{rad}}(\omega'; \mathbb{C}^{n-k}, 0) = \mathrm{ind}^G_{\mathrm{hom}}(\omega'; \mathbb{C}^{n-k}, 0)$ and, therefore,

$$\mathrm{rind}^G_{\mathrm{rad}}(\widetilde{\omega}; \mathbb{C}^n, p) = \mathrm{ind}^G_{\mathrm{hom}}(\widetilde{\omega}; \mathbb{C}^n, p).$$

For a singular point p of the 1-form $\widetilde{\omega}$ outside $\{0\} \times \mathbb{C}^k$, one has $G_p \subsetneq G$. The induction hypothesis gives $\mathrm{rind}^{G_p}_{\mathrm{rad}}(\widetilde{\omega}; \mathbb{C}^n, p) = \mathrm{ind}^{G_p}_{\mathrm{hom}}(\widetilde{\omega}; \mathbb{C}^n, p)$ and, therefore,

$$I^G_{G_p} \mathrm{rind}^{G_p}_{\mathrm{rad}}(\widetilde{\omega}; \mathbb{C}^n, p) = I^G_{G_p} \mathrm{ind}^{G_p}_{\mathrm{hom}}(\widetilde{\omega}; \mathbb{C}^n, p).$$

The laws of conservation of number for the equivariant radial and equivariant homological indices imply $\mathrm{rind}^G_{\mathrm{rad}}(\omega; \mathbb{C}^n, 0) = \mathrm{ind}^G_{\mathrm{hom}}(\omega; \mathbb{C}^n, 0)$. Therefore, we can assume that the representation of G on the space $\mathbb{C}^n$ has no trivial summands.

Let σ be a generator of the group $G = \mathbb{Z}_d$, and let σ act on $\mathbb{C}^n$ by $\sigma \star (z_1, \ldots, z_n) = (\sigma^{k_1} z_1, \ldots, \sigma^{k_n} z_n)$, where (on the right-hand side) $\sigma = \exp \frac{2\pi i}{d}$ and $0 < k_i < d$ for $i = 1, \ldots, n$. Let the space $\mathbb{C}^{n+1} = \mathbb{C}^n \oplus \mathbb{C}^1$ be endowed with the representation

$$\sigma \star (z_1, \ldots, z_n, z_{n+1}) = (\sigma^{k_1} z_1, \ldots, \sigma^{k_n} z_n, \sigma^{-k_n} z_{n+1})$$

of the group $\mathbb{Z}_d$, and let $\widehat{\omega} = \omega \oplus z_{n+1}^{d-1} dz_{n+1}$. One has

$$\begin{aligned} \mathrm{rind}^G_{\mathrm{rad}}(\widehat{\omega}; \mathbb{C}^{n+1}, 0) &= \mathrm{rind}^G_{\mathrm{rad}}(\omega; \mathbb{C}^n, 0) \cdot \mathrm{rind}^G_{\mathrm{rad}}(z_{n+1}^{d-1} dz_{n+1}; \mathbb{C}^1, 0), \\ \mathrm{ind}^G_{\mathrm{hom}}(\widehat{\omega}; \mathbb{C}^{n+1}, 0) &= \mathrm{ind}^G_{\mathrm{hom}}(\omega; \mathbb{C}^n, 0) \cdot \mathrm{ind}^G_{\mathrm{hom}}(z_{n+1}^{d-1} dz_{n+1}; \mathbb{C}^1, 0) \end{aligned}$$

(by Theorem 5.2). Since $\mathrm{rind}^G_{\mathrm{rad}}(z_{n+1}^{d-1} dz_{n+1}; \mathbb{C}^1, 0) = \mathrm{ind}^G_{\mathrm{hom}}(z_{n+1}^{d-1} dz_{n+1}; \mathbb{C}^1, 0)$ is not a divisor of zero (see Section 5.2), it is sufficient to show that $\mathrm{rind}^G_{\mathrm{rad}}(\widehat{\omega}; \mathbb{C}^{n+1}, 0) = \mathrm{ind}^G_{\mathrm{hom}}(\widehat{\omega}; \mathbb{C}^{n+1}, 0)$. Let $\widehat{\omega}' := \widehat{\omega} + \lambda(z_{n+1} dz_n + z_n dz_{n+1})$ be a deformation of the 1-form $\widehat{\omega}$ (where λ is small enough). The restriction of the 1-form $\widehat{\omega}'$ to the subspace $\mathbb{C}^2$ corresponding to the last two coordinates has a non-degenerate singular point at the origin. By Proposition 5.3 there exists a holomorphic 1-form η on $(\mathbb{C}^{n-1}, 0)$ such that

$$\operatorname{ind}^G_{\rm hom}(\widehat{\omega}'; \mathbb{C}^{n+1}, 0) = \operatorname{ind}^G_{\rm hom}(\eta; \mathbb{C}^{n-1}, 0),$$
$$\operatorname{rind}^G_{\rm rad}(\widehat{\omega}'; \mathbb{C}^{n+1}, 0) = \operatorname{rind}^G_{\rm rad}(\eta; \mathbb{C}^{n-1}, 0).$$

According to the induction hypothesis, we have

$$\operatorname{rind}^G_{\rm rad}(\eta; \mathbb{C}^{n-1}, 0) = \operatorname{ind}^G_{\rm hom}(\eta; \mathbb{C}^{n-1}, 0).$$

For a singular point p of the 1-form $\widehat{\omega}'$ outside the origin, we have $G_p \subsetneq G$. The induction hypothesis gives $\operatorname{rind}^{G_p}_{\rm rad}(\widehat{\omega}'; \mathbb{C}^{n+1}, p) = \operatorname{ind}^{G_p}_{\rm hom}(\widehat{\omega}'; \mathbb{C}^{n+1}, p)$ and, therefore,

$$I^G_{G_p} \operatorname{rind}^{G_p}_{\rm rad}(\widehat{\omega}'; \mathbb{C}^{n+1}, p) = I^G_{G_p} \operatorname{ind}^{G_p}_{\rm hom}(\widehat{\omega}'; \mathbb{C}^{n+1}, p).$$

The laws of conservation of number for the equivariant radial and equivariant homological indices imply

$$\operatorname{rind}^G_{\rm rad}(\widehat{\omega}; \mathbb{C}^{n+1}, 0) = \operatorname{ind}^G_{\rm hom}(\widehat{\omega}; \mathbb{C}^{n+1}, 0).$$

□

Chapter 6

Applications

6.1 An Equivariant Version of the Milnor Number

The Milnor number is a well-known topological invariant, defined in [26] as the multiplicity of an isolated zero of a complex map between complex spaces of the same dimensions. In the same work it was showed, that the Milnor fiber of an isolated hypersurface singularity, given by the equation $f = 0$ (where $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a holomorphic function germ), has the homotopy type of a bouquet of μ spheres, where μ is the Milnor number (multiplicity) of the zero point of the map $(\partial f/\partial z_1, \ldots, \partial f/\partial f_{n+1})$ (or the singular point of the hypersurface $\{f = 0\}$). In [23] Hamm extended this result to isolated complete intersection singularities. The Milnor number for a germ $(V, 0)$ of an isolated hypersurface singularity at the origin may also be expressed in algebraic terms (see in [28]) as the dimension of the local algebra of the singular point of the function germ f, that defines the hypersurface at 0 (cf. Theorem 2.2):

$$\mu(V, 0) = \dim_{\mathbb{C}} \mathcal{O}_{(\mathbb{C}^{n+1},0)}/I,$$

where I is the ideal in $\mathcal{O}_{(\mathbb{C}^{n+1},0)}$ generated by the partial derivatives of f.

For a 1-form ω on an isolated complete intersection singularity $(X, 0)$ with an isolated zero at the origin it was proved in [15], that the difference between its GSV-index and radial index does not depend on the 1-form ω and is equal to the Milnor number of the ICIS:

$$\mathrm{ind}_{\mathrm{GSV}}(\omega; X, 0) - \mathrm{ind}_{\mathrm{rad}}(\omega; X, 0) = \mu(X, 0).$$

Also, in [15] was shown that this difference may be considered as a generalized Milnor number of an isolated singularity of a germ of a complex analytic space. Here we give an equivariant analogue of these results.

In [9] an algebraic formula for the GSV-index of a holomorphic 1-form was given. (The proof given in [9] contained a minor mistake corrected in [11, Theorem 4].) In [15] it was shown that in this case the GSV-index coincides with the homological one. In fact, this follows directly from the algebraic formula for the GSV-index given in [9] and the fact that for a holomorphic 1-form ω with an isolated singular point on an n-dimensional ICIS $(X, 0)$ the only non-trivial (co)homology group of the complex (2.3) is the one in dimension n: [21]. Strictly speaking, in [21] this was proved for $\omega = df$, where f is a holomorphic function on $(X, 0)$; however, G.-M. Greuel explained that, in the general case, the same result holds.

Denote by

$$\mathrm{rind}_{\mathrm{GSV}}(\omega; X, 0) := r\left(\mathrm{ind}_{\mathrm{GSV}}(\omega; X, 0)\right) \in R(G)$$

the reduction of the equivariant GSV-index to the ring $R(G)$ of representations of G.

An argument in [11, Theorem 4] (together with the fact that the only non-trivial (co)homology group of the complex (2.4) is the one in the dimension n) proves the following statement.

Proposition 6.1 *For a holomorphic G-invariant 1-form ω on the G-invariant ICIS $(X, 0)$, the equivariant homological index is equal to the reduction of the equivariant GSV-index:*

$$\mathrm{ind}^G_{\mathrm{hom}}(\omega, X, 0) = \mathrm{rind}^G_{\mathrm{GSV}}(\omega; X, 0).$$

Let $\chi^G(X_{\overline{\varepsilon}}) \in A(G)$ be the equivariant Euler characteristic of the Milnor fiber $X_{\overline{\varepsilon}}$ and let $\overline{\chi}^G(X_{\overline{\varepsilon}}) := \chi^G(X_{\overline{\varepsilon}}) - 1$ be the *reduced equivariant Euler characteristic* of this fiber. For a G-invariant radial (real) 1-form ω_{rad} on the ICIS $(X, 0)$, the equivariant GSV-index $\mathrm{ind}^G_{\mathrm{GSV}}(\omega_{\mathrm{rad}}; X, 0)$ is equal to $\chi^G(X_{\overline{\varepsilon}})$. This implies the following statement (an equivariant analogue of Proposition 5.3 in [8] for 1-forms).

Proposition 6.2 *For a G-invariant real 1-form ω on the ICIS $(X, 0)$,*

$$\mathrm{ind}^G_{\mathrm{GSV}}(\omega; X, 0) - \mathrm{ind}^G_{\mathrm{rad}}(\omega; X, 0) = \overline{\chi}^G(X_{\overline{\varepsilon}})\,.$$

For a G-invariant complex 1-form ω on the ICIS $(X, 0)$,

$$\mathrm{ind}^G_{\mathrm{GSV}}(\omega; X, 0) - \mathrm{ind}^G_{\mathrm{rad}}(\omega; X, 0) = (-1)^n\overline{\chi}^G(X_{\overline{\varepsilon}})\,.$$

The reduction $r_G((-1)^n\overline{\chi}^G(X_{\overline{\varepsilon}})) \in R(G)$ is the equivariant Milnor number of the ICIS $(X, 0)$ in the sense of [34], i.e., it is equal to the class in $R(G)$ of the G-module $H^n(X_{\overline{\varepsilon}})$.

Let $(X, 0)$ be a complex analytic G-variety of pure dimension n with an isolated singular point at the origin. The laws of conservation of number for the equivariant radial and for the equivariant homological indices, together with the fact that they coincide on a smooth manifold, imply the following statement.

Proposition 6.3 *For a G-invariant holomorphic 1-form ω on $(X, 0)$ with an isolated singular point at the origin, the difference*

$$\mathrm{ind}^G_{\mathrm{hom}}(\omega; X, 0) - \mathrm{rind}^G_{\mathrm{rad}}(\omega; X, 0) \in R(G)$$

does not depend on the 1-form ω.

As shown above, for a G-invariant ICIS, this difference is its equivariant Milnor number. This allows us to regard

$$\mathrm{ind}^G_{\mathrm{hom}}(\omega; X, 0) - \mathrm{rind}^G_{\mathrm{rad}}(\omega; X, 0)$$

as a version of the equivariant Milnor number of a germ of the G-variety $(X, 0)$ with an isolated singular point.

6.2 Index of a 1-Form on a Quotient Singularity

Let $\mathbb{C}^n$ be endowed with a linear action of a finite group G. Consider a germ of a G-invariant holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. It induces an analytic function $\check{f} : (\mathbb{C}^n/G, 0) \to (\mathbb{C}, 0)$ on the quotient $\mathbb{C}^n/G$, such that $f = \check{f} \circ \pi$, where $\pi : \mathbb{C}^n \to \mathbb{C}^n/G$. Let $\Omega_f^{\mathbb{C}} := \Omega^n_{(\mathbb{C}^n,0)}/df \wedge \Omega^{n-1}_{(\mathbb{C}^n,0)}$, where $\Omega^j_{(\mathbb{C}^n,0)}$ is the module of germs of differential j-forms on $(\mathbb{C}^n, 0)$.

Theorem 6.1 *In the setting described above the following relation holds:*

$$\mathrm{ind}_{\mathrm{rad}}(d\check{f}; \mathbb{C}^n/G, 0) = \dim(\Omega_f^{\mathbb{C}})^G,$$

where $(\Omega_f^{\mathbb{C}})^G$ *is the* G*-invariant part of* $\Omega_f^{\mathbb{C}}$.

Proof. Consider the map $r^{(0)} : A(G) \to \mathbb{Z}$, that sends a G-set $\sum a_{[H]}[G/H] \in A(G)$ (where, as before, $a_{[H]} \in \mathbb{Z}$, $[G/H]$ is the class of the G-set G/H in $A(G)$, and the sum is over all the classes of conjugate subgroups of G) to the integer $\sum a_{[H]}$. From Proposition 1 of [7] one has:

$$\mathrm{ind}_{\mathrm{rad}}(d\check{f}; \mathbb{C}^n/G, 0) = r^{(0)}\mathrm{ind}^G_{\mathrm{rad}}(df; \mathbb{C}^n, 0). \tag{6.1}$$

On the other hand, by Theorem 5.3:

$$\mathrm{rind}^G_{\mathrm{rad}}(df; \mathbb{C}^n, 0) = \mathrm{ind}^G_{\mathrm{hom}}(df; \mathbb{C}^n, 0) \in R(G).$$

In Section 3.4 we have already discussed, that, in the case of $(\mathbb{C}^n, 0)$, the only non trivial cohomology group of the complex (2.4) is the one in the dimension n. Therefore,

$$\mathrm{ind}^G_{\mathrm{hom}}(df; \mathbb{C}^n, 0) = [\Omega_f^{\mathbb{C}}] \in R(G).$$

Recall, that, $\mathrm{rind}^G_{\mathrm{rad}}(df; \mathbb{C}^n, 0) = r_G(\mathrm{ind}^G_{\mathrm{rad}}(df; \mathbb{C}^n, 0)$, where $r_G : A(G) \to R(G)$ is the reduction map. Note, that the image of any class $[G/H] \in A(G)$ under r_G may be considered as the permutation representation of the set G/H. On the other hand, the trivial representation is a direct summand of each permutation representation, and the sum $\sum a_{[H]}$ is, in fact, the multiplicity of the trivial representation in $r_G(\sum a_{[H]}[G/H])$. Therefore, $r^{(0)}\mathrm{ind}^G_{\mathrm{rad}}(df; \mathbb{C}, 0)$ is the multiplicity of the trivial representation in $[\Omega_f^{\mathbb{C}}]$, which is exactly the dimension of the G-invariant part of the space $\Omega_f^{\mathbb{C}}$. $\square$

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