On the combinatorics of Tits arrangements

Von der Fakultät für Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover

zur Erlangung des akademischen Grades
Doktor der Naturwissenschaften
Dr. rer. nat.

genehmigte Dissertation von

M. Sc. David Geis

2018

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Tag der Promotion: 31.5.2018

Deutschsprachige Kurzzusammenfassung

Die vorliegende Dissertationsschrift befasst sich mit simplizialen Arrangements von Hyperebenen. Ein Arrangement von Hyperebenen in $\mathbb{R}^r$ heißt simplizial, falls jede Komponente von $\mathbb{R}^r \setminus \left(\bigcup_{H\in\mathcal{A}} H\right)$ ein offener simplizialer Kegel ist. In Kapitel 1 geben wir eine kurze Einleitung. Genaue Definitionen, etwas mehr Hintergrund und wichtige bekannte Resultate folgen in Kapitel 2, wo wir auch das allgemeinere Konzept des *Tits Arrangements* definieren.

In Kapitel 3 werden einige Resultate für die obigen Tits Arrangements bewiesen. Es wird eine kombinatorische Charakterisierung von Paaren von Tits Arrangements gegeben, die sich lediglich um eine Hyperebene unterscheiden. Hierzu nutzen wir ungerichtete Varianten klassischer Dynkin Diagramme (sog. *schwache Dynkin Diagramme*). Weiterhin betrachten wir *endliche Spiegelungs Gruppoide*; diese sind Varianten der sogenannten Weyl Gruppoide, lassen sich aber nur aus gewissen Tits Arrangements konstruieren. Schließlich klassifizieren wir kristallographische Tits Arrangements, deren Rang mindestens sieben beträgt und die eine Kammer mit gewissen vorgegebenen Dynkin Diagrammen enthalten.

Kapitel 4 enthält eine Klassifikation affiner Tits Arrangements des Rangs drei, deren Normalenvektoren auf einer projektiven kubischen Kurve liegen.

In Kapitel 5 studieren wir simpliziale Arrangements von Geraden in $\mathbb{P}^2(\mathbb{R})$. Selbst in dieser einfachsten Situation liegt keine vollständige Klassifikation vor. Grünbaum hat jedoch eine Liste gegeben, welche bis auf vier weitere Arrangements (gefunden von Michael Cuntz) alle zur Zeit bekannten Beispiele enthält. Man vermutet, dass die aktuell vorhande Liste bis auf endlich viele weitere Korrekturen vollständig ist. Wir bestätigen diese Vermutung für gewisse Klassen von simplizialen Arrangements mit speziellen Eigenschaften. Beispielsweise wird gezeigt, dass ein *freies* simpliziales Arrangement $\mathcal{A}$ mit $t_i^{\mathcal{A}} = 0$ für $i > 5$ aus höchstens vierzig Geraden besteht. Weiterhin zeigen wir, dass es bis auf Isomorphie nur endlich viele simpliziale Arrangements in $\mathbb{P}^2(\mathbb{R})$ gibt, deren Normalenvektoren auf einer irreduziblen projektiven Kurve beschränkten Grades liegen.

In Kapitel 6 untersuchen wir ein interessantes Phänomen: ist ein simpliziales Arrangement $\mathcal{A}$ in $V := \mathbb{P}^{r-1}(\mathbb{R})$ gegeben, dann lassen sich aus $\mathcal{A}$ in natürlicher Weise gewisse duale Arrangements in V^* konstruieren. Diese dualen Arrangements sind in vielen Fällen ebenfalls simplizial und wir geben einige Klassifikations- und Endlichkeitssätze in diesem Kontext. Weiterhin zeigen wir, wie gewisse *sporadische* Arrangements aus Spiegelungsgruppen konstruiert werden können.

Abstract

In this thesis, we study simplicial arrangements of hyperplanes. Classically, a simplicial arrangement $\mathcal{A}$ in $\mathbb{R}^r$ is a finite set of linear hyperplanes such that every component of $\mathbb{R}^r \setminus \left(\bigcup_{H\in\mathcal{A}} H\right)$ is an open simplicial cone. A short introduction is given in Chapter 1.

In Chapter 2, we review the precise definitions and collect some important known results; we also recall the more general concept of a *Tits arrangement* and the corresponding notions.

In Chapter 3, we establish some results for these Tits arrangement. In particular, we give a combinatorial characterization of pairs of Tits arrangements differing by one hyperplane. For this, we introduce *weak Dynkin diagrams*, which generalize the classical Dynkin diagrams in the crystallographic case. Moreover, we show how one may associate a so called *finite reflection groupoid* to certain Tits arrangements, generalizing the concept of the Weyl groupoid in the crystallographic case. Furthermore, we classify crystallographic Tits arrangements of rank at least seven containing a chamber whose associated Dynkin diagram is of a certain prescribed type.

Chapter 4 contains a classification of affine rank three Tits arrangements whose associated root vectors lie on a projective cubic curve.

In Chapter 5, we focus on the classical subject of simplicial line arrangements in $\mathbb{P}^2(\mathbb{R})$. A classification even in this case is still an open problem. However, there exists a catalogue published by Grünbaum listing almost all currently known examples; only four additional arrangements have been discovered by Michael Cuntz. One of the main conjectures in the field is that the current catalogue is complete up to finitely many additions. We prove this conjecture for various special kinds of simplicial arrangements. For instance, we prove that a *free* simplicial arrangement $\mathcal{A}$ such that $t_i^{\mathcal{A}} = 0$ for $i \geq 6$ consists of at most forty lines. We also show that there are only finitely many (combinatorial isomorphism classes of) simplicial line arrangements whose associated root vectors lie on an irreducible projective algebraic curve of fixed (or at least bounded) degree.

In the last Chapter 6, we study an interesting phenomenon: given a simplicial arrangement $\mathcal{A}$ in $V := \mathbb{P}^{r-1}(\mathbb{R})$, one may associate in a natural way certain arrangements in the dual space V^*. It turns out that these dual arrangements are also simplicial in many cases and we give finiteness, classification as well as some experimental results in this setup. For instance, we show how to construct certain *sporadic* arrangements via reflection groups.

Keywords: combinatorics, simplicial hyperplane arrangements, discrete geometry

Contents

List of abbreviations

i.e. . . . . id est

Chapter 1

Introduction

An arrangement of hyperplanes is a finite set of hyperplanes in a finite dimensional vector space. Arrangements form a classical field of study in both pure and applied mathematics. Arguably more than in most other fields of study, there is a beautiful interaction between geometry, algebra, combinatorics, topology and probably even more. Classical references are the books by Orlik and Terao (see [27]) and Grünbaum (see [18]). The latter focuses more on combinatorial aspects of the theory and its relations to polytope theory, while the first also deals with more algebraic and group theoretic aspects of the subject. A more recent monograph has been published by Dimca in 2017 (see [13]), focusing more on the algebraic geometric part of the story.

One of the most fascinating aspects of the subject lies in the fact that many problems concerning arrangements may be formulated on a very elementary level, which makes the subject accessible for a broad variety of people. However, solutions to such seemingly easy problems often turn out to be very difficult and thus require new ideas, which in turn stimulate the progress of mathematics itself.

This way, new fields of study arise and connections between seemingly unrelated subjects manifest themselves. For instance, it turns out that certain properties or invariants, which seem to be algebraic or topological, can be shown to be combinatorial in nature: this is exemplified by the fact that, given a complex hyperplane arrangement $\mathcal{A}$ in $V := \mathbb{C}^r$, the Betti numbers of the space obtained by removing all hyperplanes in $\mathcal{A}$ from V are determined entirely by the so called *intersection lattice* of $\mathcal{A}$, which is a purely combinatorial invariant of the arrangement.

In this thesis, we will be interested in *simplicial* arrangements of hyperplanes in a *real* vector space. Classically, simplicial arrangements are defined very elementary: the arrangement $\mathcal{A}$ in $V := \mathbb{R}^r$ is called simplicial, if all hyperplanes in $\mathcal{A}$ pass through the origin and if every component of

$V \setminus \left(\bigcup_{H\in\mathcal{A}} H\right)$ is bounded by precisely r hyperplanes of $\mathcal{A}$ (we give more precise definitions and some more background in Chapter 2). It is an immediate observation that in $\mathbb{R}^2$ the second condition is automatic, once the first is satisfied. Thus geometrically, the first case of actual interest is $r = 3$. Natural examples are for instance the arrangements consisting of the reflecting hyperplanes of finite real reflection groups.

In the paper [12], the notion of a simplicial arrangement is generalized to the more flexible notion of so called *Tits arrangements*. In Chapter 2, Sections 2-3, we recall their definition and associated notions.

In Chapter 3, we then provide some interesting results for these Tits arrangements. In the first section, we prove a theorem which characterizes Tits arrangements differing by only one hyperplane. It turns out that a hyperplane may be removed from a Tits arrangement if a certain combinatorial condition involving so called *weak Dynkin diagrams* is satisfied. These diagrams are graphs associated with the chambers of an arrangement which contain geometric and combinatorial data. They already proved to be very useful in [8].

In the second section, we follow ideas of Michael Cuntz and introduce *finite reflection groupoids* associated to Tits arrangements. These are algebraic structures which may be associated with certain (but not all) Tits arrangements. In some sene, they are non-integral generalizations of so called *Weyl groupoids*, which correspond to *crystallographic* Tits arrangements (see Chapter 2, Section 3).

We give a classification result for highly symmetric finite Tits arrangements in $\mathbb{R}^3$ which can be coordinatized over $\mathbb{Q}$ and which permit such a groupoid structure. Moreover, we can prove a finiteness result for spherical Tits arrangements which are realizable over some algebraic number field $\mathbb{K}$ and which admit a groupoid structure as above.

In the last section of Chapter 3, we are interested in *locally spherical crystallographic* Tits arrangements of rank at least seven. We show that, up to isomorphism, there exists only one locally spherical crystallographic Tits arrangement containing a chamber whose associated Dynkin diagram is of type $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ respectively (we use the notation in [24, Chapter 4]). The same is true for arrangements containing a chamber whose diagram is of type $A_{r-1}^{(1)}$ or $D_r^{(2)}$ for $r \geq 8$. Similarly, we can prove that there are, again up to isomorphism, exactly two Tits arrangements as above containing a chamber whose associated Dynkin diagram is of type E_8.

In Chapter 4, we give a classification of *affine* rank three Tits arrangements whose normal vectors are contained in a cubic curve. In the process, we discover an affine Tits arrangement $\mathcal{A}$ having a vertex v such that infinitely many lines of $\mathcal{A}$ are incident with v. We remark that to our knowledge, no classical affine simplicial arrangement with this property already appeared in the literature. Moreover, we observe the interesting geometric fact that the normal vectors of $\mathcal{A}$ are contained in the union of an irreducible conic σ with a line ℓ touching σ in the real projective plane. This in turn implies that the incidence structure of $\mathcal{A}$ is similar to the incidence structure of arrangements belonging to the infinite series $\mathcal{R}(1)$: these are *spherical* rank three Tits arrangements which are closely related to the symmetry structure of the regular n-gon for $n \geq 3$ (see [4, Section 3] or Chapter 2, Section 4 for precise definitions and background). The similarity of the incidence structures originates from the fact that the normal vectors of arrangements belonging to $\mathcal{R}(1)$ also lie on the union of a projective conic σ' and a projective line ℓ'; however, in the spherical case the curves σ' and ℓ' do not meet in the real projective plane.

Clearly, simplicial arrangements can also be defined naturally in projective space (using the canonical map $\pi : \mathbb{R}^r \setminus \{0\} \longrightarrow \mathbb{P}^{r-1}(\mathbb{R})$). Doing so, we may visualize a simplicial arrangement in $\mathbb{R}^3$ as a triangulated projective plane. Thus, at least for $r = 3$ simplicial arrangements in $\mathbb{R}^r$ *appear* to be elementary objects which ought to be easily understood. However, the reality is different: it is still one the great open problems to give a complete classification of such triangulations, even in the case for arrangements which may be defined over $\mathbb{Q}$ (see [19] and [6]). Only under some very strong conditions, one can give bounds for the size or even classifications of simplicial arrangements in $\mathbb{R}^r$. So far, only the crystallographic and supersolvable arrangements are well understood and completely classified (see Section 2.4 for additional results and references).

In Chapter 5, we will have a special focus on the case $r = 3$ and we will present certain conditions which permit finiteness and in some cases even classification results. The conditions we present will be combinatorial, geometric and in some sense algebraic. For instance, we will give a classification of *free* simplicial line arrangements having the property that every point of the plane is incident with at most four lines of the arrangement. By a similar argument, we show that any (not necessarily simplicial) free arrangement $\mathcal{A}$, having the property that every point of the plane is incident with at most five lines of $\mathcal{A}$, consists of at most 185 lines.

Motivated by the observation that the root vectors of all arrangements from the infinite series $\mathcal{R}(1)$ (see Section 2.4 for definitions) lie on a cubic curve, we also study simplicial line arrangements whose associated root vectors are contained in the locus of some homogeneous polynomial of bounded degree d. We show that for fixed d, there are only finitely many combinatorial isomorphism classes of arrangements having this property.

Finally, in the last Chapter 6 we present certain natural constructions suggested by Michael Cuntz which associate with a simplicial arrangement in $V := \mathbb{P}^{r-1}(\mathbb{R})$ certain *dual* arrangements in V^* which turn out to be simplicial in many cases.

Basically, the idea is the following: given an arrangement $\mathcal{A}$ in V, one considers the set $V_{\mathcal{A}}$ of vertices determined by $\mathcal{A}$. Clearly, by projective duality, any subset of $V_{\mathcal{A}}$ defines an arrangement in V^*. The crucial observation is that for certain *well chosen* subsets, the corresponding dual arrangements tend to be simplicial as well.

We note that the above construction works particularly well if we take subsets of vertices with fixed or bounded multiplicity. Doing so, we are able to construct many *sporadic* simplicial arrangements from the root sets of certain finite reflection groups. Moreover, there are arrangements which are invariant under the above construction, i.e. there exist arrangements $\mathcal{A}$ with the property that a certain dual arrangement of $\mathcal{A}$ is isomorphic to $\mathcal{A}$ itself. For instance, this holds for all arrangements corresponding to exceptional reflection groups of type F_4, H_4, E_6, E_7, E_8. We note that for $r \geq 6$, the arrangements of type C_r, D_r have this property as well.

We also give certain classification and finiteness results related to these phenomena in the case $V = \mathbb{P}^2(\mathbb{R})$.

At the end of Chapters $3 - 6$, we present some possibly interesting related open problems and questions which we have not been able to resolve yet, but whose solutions would mean -in our opinion- considerable progress in the field of simplicial arrangements.

Acknowledgement. *I am very grateful to my advisor Michael Cuntz for his steady support and for many helpful and stimulating discussions. I also wish to thank Bernhard Mühlherr for some inspiring conversations on the subject. Moreover, I want to thank Piotr Pokora for his interest in the subject and for bringing some interesting papers to my attention. Thanks are also due to my family and to everybody who supported me during the process of writing this dissertation. Last but not least, I thank both the Deutsche Forschungsgemeinschaft (DFG) for their financial support and the Institut für Algebra, Zahlentheorie und Diskrete Mathematik at Leibniz Universität Hannover for providing a place to work with the right atmosphere.*

Chapter 2

Background, definitions and known results

2.1 Simplicial arrangements of hyperplanes in $\mathbb{R}^r$ and $\mathbb{P}^{r-1}(\mathbb{R})$, the classical case

The aim of this section is to introduce arrangements of hyperplanes in a finite dimensional real vector space. Arrangements in projective space will be defined as well. We introduce useful associated notions and report on some already known results, which will turn out to be relevant for our upcoming work.

We start with the following most basic definition:

Definition 2.1. i) Let $\mathcal{A}$ be a finite set of linear hyperplanes in $V := \mathbb{R}^r$ for some positive $r \in \mathbb{N}$. Then $\mathcal{A}$ is called an *arrangement of hyperplanes.* If $\bigcap_{H\in\mathcal{A}} H = \{0\}$, then $\mathcal{A}$ is called *essential.* The connected components of $V \setminus \left(\bigcup_{H\in\mathcal{A}} H\right)$ are called the *chambers* of $\mathcal{A}$. We call $\mathcal{A}$ *simplicial* if every chamber of $\mathcal{A}$ is an open simplicial cone. Equivalently, $\mathcal{A}$ is simplicial if every chamber is bounded by precisely r hyperplanes of $\mathcal{A}$.
Let $L_{\mathcal{A}}$ be the collection of all subspaces of V of the form $\bigcap_{H\in\mathcal{B}} H$ for some subset $\mathcal{B} \subset \mathcal{A}$. Ordering $L_{\mathcal{A}}$ by reverse inclusion makes it into a lattice, the so called *intersection lattice* of $\mathcal{A}$. We denote the Möbius function of $L_{\mathcal{A}}$ by μ and define the *characteristic polynomial* of $\mathcal{A}$ by the formula $\chi(\mathcal{A}, t) := \sum_{X\in L_{\mathcal{A}}} \mu(X)t^{\dim(X)}$.
Any arrangement of hyperplanes in V induces a cell-decomposition on the unit sphere $\mathbb{S}_{r-1}$ and two arrangements $\mathcal{A}_1, \mathcal{A}_2$ are called *combinatorially isomorphic* if the associated cell-decompositions are isomorphic.
ii) Let $\pi : \mathbb{R}^r \setminus \{0\} \longrightarrow \mathbb{P}^{r-1}(\mathbb{R})$ be the natural map and let $\mathcal{A}$ be an essential arrangement of hyperplanes in $\mathbb{R}^r$. Then by taking the image of each hyperplane of $\mathcal{A}$ under π, we obtain an arrangement $\mathbb{P}(\mathcal{A})$ in $\mathbb{P}^{r-1}(\mathbb{R})$.
Now if $\mathcal{B} = \mathbb{P}(\mathcal{A})$ for some essential hyperplane arrangement $\mathcal{A}$, then $\mathcal{B}$ is

called a *projective arrangement of hyperplanes*. The arrangement $\mathcal{B}$ induces a cell-decomposition of $\mathbb{P}^{r-1}(\mathbb{R})$ which is simplicial if and only if the corresponding cell-decomposition induced by $\mathcal{A}$ on the unit sphere is simplicial. Consequently, the arrangement $\mathcal{B}$ is called *simplicial*, if $\mathcal{A}$ is simplicial. We write $f_i^{\mathcal{B}}, 0 \leq i \leq r-1$, for the number of cells having (projective) dimension i. The vector $f^{\mathcal{B}}$ defined by $(f^{\mathcal{B}})_i := f_i^{\mathcal{B}}$ is called the *f-vector* of $\mathcal{B}$. In case $r = 3$, we call $\mathcal{B}$ a *(projective) line arrangement.* Two projective arrangements $\mathbb{P}(\mathcal{A}), \mathbb{P}(\mathcal{A}')$ are called *combinatorially isomorphic* if the arrangements $\mathcal{A}, \mathcal{A}'$ are combinatorially isomorphic.

Remark 2.1. Assume that $\mathcal{A}$ is not essential. Then $X := \bigcap_{H \in \mathcal{A}} H$ has positive dimension. By passing to the quotient space $\overline{V} := V/X$, we obtain an arrangement $\overline{\mathcal{A}}$ such that $\bigcap_{\overline{H} \in \overline{\mathcal{A}}} \overline{H} = \{0\}$. Thus $\overline{\mathcal{A}}$ is essential. Here $\overline{H}$ denotes the image of H under the natural map $V \longrightarrow \overline{V}$.

Next we introduce a simple (but very useful) combinatorial invariant associated to an arbitrary projective arrangement of hyperplanes.

Definition 2.2. Let $\mathcal{A}'$ be an essential arrangement of hyperplanes in $\mathbb{R}^r$ and let $\mathcal{A} := \mathbb{P}(\mathcal{A}')$ be the associated projective arrangement in $\mathbb{P}^{r-1}(\mathbb{R})$. If $v \in L_{\mathcal{A}'}$ such that $\dim(v) = 1$, then v is called a *vertex of* $\mathcal{A}$.
We define $m(v) := |\{H \in \mathcal{A}' \mid v \subset H\}|$ and call this number the *weight* or *multiplicity* of v. Denote the set of vertices of $\mathcal{A}$ by $\mathcal{V}$. Then we define the *multiplicity* of $\mathcal{A}$ to be $m(\mathcal{A}) := \max_{v \in \mathcal{V}} m(v)$.
Writing $t_i^{\mathcal{A}} := |\{v \in \mathcal{V} \mid m(v) = i\}|$ we define the *t-vector* of $\mathcal{A}$ by $(t^{\mathcal{A}})_i := t_i^{\mathcal{A}}$ for $r-1 \leq i \leq m(\mathcal{A})$. By definition we have $t_i^{\mathcal{A}} = 0$ for $i > m(\mathcal{A})$.

We remark that in case $r = 3$, the t-vector of an arrangement $\mathcal{A}$ captures a lot of information. In particular, the number of chambers can be read off directly from $t^{\mathcal{A}}$:

Remark 2.2. Let $\mathcal{A}$ be a projective line arrangement with associated f-vector $f^{\mathcal{A}} = (f_0^{\mathcal{A}}, f_1^{\mathcal{A}}, f_2^{\mathcal{A}})$. Then we have $f_0^{\mathcal{A}} = \sum_{i \geq 2} t_i^{\mathcal{A}}$ by definition. Moreover, we have $f_1^{\mathcal{A}} = \sum_{i \geq 2} i t_i^{\mathcal{A}}$, because the vertices partition the lines of $\mathcal{A}$ into segments, called the *edges* of $\mathcal{A}$. Finally, as $\mathbb{P}^2(\mathbb{R})$ has Euler characteristic equal to one, we conclude that $f_2^{\mathcal{A}} = 1 + \sum_{i \geq 2}(i-1)t_i^{\mathcal{A}}$.

Moreover, we have the following useful relations which are satisfied by the t-vector of any real projective line arrangement:

Theorem 2.1. *Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$. Then the following statements hold:*

$$\binom{n}{2} = \sum_{i \geq 2} \binom{i}{2} t_i^{\mathcal{A}}, \tag{2.1}$$

$$t_2^{\mathcal{A}} \geq 3 + \sum_{i \geq 4} (i-3) t_i^{\mathcal{A}}. \tag{2.2}$$

If the multiplicity of $\mathcal{A}$ is at most $n-3$, then the following inequality holds:

$$t_2^{\mathcal{A}} + \frac{3}{2}t_3^{\mathcal{A}} \geq 8 + \sum_{i\geq 4} \frac{4i-15}{2} t_i^{\mathcal{A}}. \tag{2.3}$$

If the multiplicity of $\mathcal{A}$ is at most $\frac{2n}{3}$, then we also have the following estimates:

$$\sum_{i\geq 2} it_i^{\mathcal{A}} \geq \frac{n^2+3n}{3}, \tag{2.4}$$

$$\sum_{i\geq 2} i^2 t_i^{\mathcal{A}} \geq \frac{4}{3}n^2. \tag{2.5}$$

Proof. Equation (2.1) follows from counting pairs of lines in two different ways. Inequality (2.2) is known as Melchior's inequality and a proof can be found in [26]. The estimate (2.3) is more recent, see [32] for a proof. Finally, inequalities (2.4), (2.5) are proved in [25]. □

Remark 2.3. i) Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$. Note that by definition we always have $t_n^{\mathcal{A}} = 0$, as the hyperplane arrangement in $\mathbb{R}^3$ corresponding to $\mathcal{A}$ is required to be essential. Usually, one has to require this additional condition in order to have (2.2). Moreover, we observe that $\mathcal{A}$ is simplicial if and only if we have equality in (2.2).
ii) We remark that inequality (2.3) also holds true for *arrangements of pseudolines*. See [2] for more background on this topic. We shall make use of this observation in Chapter 5.

We define *Coxeter diagrams*, a combinatorial invariant attached to the chambers of a hyperplane arrangement:

Definition 2.3. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^{r-1}(\mathbb{R})$ and let K be a chamber of $\mathcal{A}$. Then we associate to K a weighted undirected graph Γ_C^K in the following way: the vertices of Γ_C^K are given by the s hyperplanes $H_1, ..., H_s$ bounding K. Two vertices H_i, H_j are connected by an edge in Γ_C^K if and only if there are at least three hyperplanes of $\mathcal{A}$ containing the subspace $H_i \cap H_j$. The weight of this edge is then given by the precise number of hyperplanes of $\mathcal{A}$ containing $H_i \cap H_j$. The graph Γ_C^K is referred to as the *Coxeter diagram at* K. If $\mathcal{A}$ is simplicial then Γ_C^K has precisely r vertices for every chamber K. A simplicial projective arrangement $\mathcal{A}$ is called *irreducible* if Γ_C^K is connected for any chamber K of $\mathcal{A}$. Otherwise $\mathcal{A}$ is called *reducible*. We observe that up to isomorphism the graph Γ_C^K depends only on the chamber K.

Remark 2.4. Let $\mathcal{A}$ be a simplicial projective arrangement of hyperplanes. Assume that $\mathcal{A}$ contains a chamber K whose corresponding Coxeter diagram Γ_C^K is not connected. So by definition, the arrangement $\mathcal{A}$ is reducible.

Moreover, it is not hard to see that in this case $\mathcal{A}$ may be realized as product arrangement: if Γ_C^K has m components, then there are arrangements $\mathcal{A}_1, ..., \mathcal{A}_m$ such that $\mathcal{A} = \mathcal{A}_1 \times ... \times \mathcal{A}_m$ (see [27] for more details on this construction). This observation justifies the introduced terminology.

The following result shows that an irreducible simplicial arrangement of n lines has multiplicity at most $\frac{n}{2}$. For the sake of completeness, we include a proof.

Proposition 2.1. *Let $\mathcal{A}$ be an irreducible simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$ with $m := \frac{|\mathcal{A}|}{2}$. Then we have $t_i^{\mathcal{A}} = 0$ for all $i > m$. Moreover, we always have $t_m^{\mathcal{A}} \leq 1$.*

Proof. Suppose that there was some $i > m$ such that $t_i^{\mathcal{A}} > 0$. Pick a vertex v of weight i and denote the set of lines passing through v by L_v. Then there are $2i$ chambers $K_1, ..., K_{2i}$ having v as a vertex. Each of these chambers has precisely one wall supported by a line not contained in L_v. As $|\mathcal{A} \setminus L_v| < m$, we conclude that there must be some $\ell \in \mathcal{A} \setminus L_v$ such that ℓ is a wall in three neighbouring chambers $K_{j_1}, K_{j_2}, K_{j_3}$. But then ℓ contains a segment bounded by two vertices of weight two. It follows that $\mathcal{A}$ is not irreducible, contradicting our initial assumption. This proves the first claim. Next we show that $t_m \leq 1$. Suppose that $t_m > 1$. Then clearly $t_m = 2$ and we denote the two vertices of weight m by v_1 and v_2. Then v_1 and v_2 may or may not be connected by a line of $\mathcal{A}$ and one checks that any line of $\mathcal{A}$ not passing through both v_1 and v_2 contains a segment bounded by two vertices of weight two. Therefore $\mathcal{A}$ is not irreducible. This completes the proof. □

Remark 2.5. Let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines. Then the above result shows that the multiplicity of $\mathcal{A}$ is less than $\frac{2n}{3}$. Therefore, the relations (2.4) and (2.5) are always true for irreducible simplicial line arrangements (see Theorem 2.1). This allows for instance to give lower bounds on the number of chambers for such arrangements.

Next we introduce *free* hyperplane arrangements. These will occur in Chapter 5, where we establish some finiteness and classification results concerning free simplicial line arrangements.

Definition 2.4. Let $\mathbb{K}$ be a field and let $\mathcal{A} = \mathbb{P}(\mathcal{A}')$ be a projective arrangement of hyperplanes in $\mathbb{P}^{r-1}(\mathbb{K})$. By choosing coordinates we identify the symmetric algebra $\mathrm{Sym}((\mathbb{K}^r)^*)$ with the polynomial ring $S := \mathbb{K}[x_1, ..., x_r]$. For every $H \in \mathcal{A}'$ choose a homogeneous linear form $l_H \in S$ such that $H = \ker(l_H)$ and let $\mathcal{D}$ be the free S-module of derivations of S over $\mathbb{K}$. We define a submodule

$$\mathcal{D}_{\mathcal{A}} := \left\{ \theta \in \mathcal{D} \mid \theta(l_H) \in (l_H), H \in \mathcal{A}' \right\} \subset \mathcal{D}.$$

Then $\mathcal{A}$ is called *free* if $\mathcal{D}_{\mathcal{A}}$ is a free S-module.

We will mainly need the following property of free arrangements, which is known as Terao's factorization theorem. A proof can be found for instance in [27].

Theorem 2.2. *Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{P}^{r-1}(\mathbb{R})$. If $\mathcal{A}$ is free, then for the characteristic polynomial $\chi(\mathcal{A},t)$ we have the following formula:*

$$\chi(\mathcal{A},t) = \prod_{1 \leq i \leq r} (t - d_i).$$

The numbers $d_1, ..., d_r$ are the degrees of homogeneous generators in a basis for $\mathcal{D}_{\mathcal{A}}$. In particular, these degrees are combinatorially determined.

Remark 2.6. i) Theorem 2.2 is also true for hyperplane arrangements defined over arbitrary fields.
ii) The theorem shows that the degrees of the generators in a homogeneous basis for $\mathcal{D}_{\mathcal{A}}$ are uniquely determined by the combinatorics, if $\mathcal{A}$ is free. However, it is not clear whether or not the property of being free is determined by the combinatorics as well: given two hyperplane arrangements $\mathcal{A}_1 = \mathbb{P}(\mathcal{A}'_1), \mathcal{A}_2 = \mathbb{P}(\mathcal{A}'_2)$ such that $\mathcal{A}_1$ is free and such that $L_{\mathcal{A}'_1} \cong L_{\mathcal{A}'_2}$, is it true that $\mathcal{A}_2$ is free as well?

This problem is known as Terao's Conjecture and is still open even in the case of line arrangements. We point out that the conjecture is false if one does not fix the characteristic of the base field (see [34]).

2.2 A generalization: Tits arrangements in $\mathbb{R}^r$

In this section we define so called *Tits arrangements.* These are a generalization of classical simplicial arrangements of hyperplanes defined in the last section. Tits arrangements have been first introduced in [12]. The main difference to the classical objects lies in the fact that we are not necessarily taking complements with respect to the entire ambient space V, but only with respect to a certain open convex cone $T \subset V$. We give the following definition:

Definition 2.5. Let $\mathcal{A}$ be a set of linear hyperplanes in $V := \mathbb{R}^r$ and let T be an open convex cone in V. We say that $\mathcal{A}$ *is locally finite in T* if for every $x \in T$ there exists a neighbourhood $U_x \subset T$ of x, such that $\{H \in \mathcal{A} \mid H \cap U_x \neq \emptyset\}$ is a finite set. A *hyperplane arrangement (of rank r)* is a pair $(\mathcal{A}, T)$, where T is a convex open cone in V, and $\mathcal{A}$ is a set of linear hyperplanes such that the following holds:

- $H \cap T \neq \emptyset$ for all $H \in \mathcal{A}$,
- $\mathcal{A}$ is locally finite in T.

Usually we omit the reference to T, since it should always be clear from the context. Hence often times we just speak of the set $\mathcal{A}$ as hyperplane arrangement. Denote by $\overline{T}$ the topological closure of T with respect to the standard topology of V. If $X \subset \overline{T}$ then the *localization at X (in $\mathcal{A}$)* is defined as

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}.$$

If $X = \{x\}$ happens to be a singleton, then we write $\mathcal{A}_x$ instead of $A_{\{x\}}$. We call $(\mathcal{A}_x, T)$ the *parabolic subarrangement at x*. The pair $(\mathcal{A}', T)$ is called a *parabolic subarrangement of* $(\mathcal{A}, T)$ if there exists $x \in \overline{T}$ such that $\mathcal{A}' = \mathcal{A}_x$. Similarly, we define the *restriction of $\mathcal{A}$ with respect to X* by

$$\mathcal{A}^X := \{H \cap X \mid X \not\subset H\}.$$

The connected components of $T \setminus \bigcup_{H \in \mathcal{A}} H$ are called the *chambers* of $\mathcal{A}$. If K is a chamber then its *walls* are given by the hyperplanes contained in the set

$$W^K := \{H \leq V \mid \dim(H) = r - 1, \langle H \cap \overline{K} \rangle_{\mathbb{R}} = H, H \cap K = \emptyset\}.$$

The arrangement $(\mathcal{A}, T)$ is called *thin* if $W^K \subset \mathcal{A}$ for each chamber K. A *simplicial hyperplane arrangement (of rank r)* is an arrangement $(\mathcal{A}, T)$ such that each chamber K is an open simplicial cone. T is called the *Tits cone* of the arrangement. A simplicial arrangement is called a *Tits arrangement* if it is also thin. Let the pair $(\mathcal{A}, T)$ be a Tits arrangement and denote the set of chambers by $\mathcal{K}$. Then we have the following thin chamber complex

$$\mathcal{S}(\mathcal{A}, T) := \left\{ \overline{K} \cap \bigcap_{H \in X} H \mid K \in \mathcal{K}, X \subset W^K \right\},$$

whose poset-structure is given by set-wise inclusion. If $U \leq V$ such that $\dim(U) = 1$ and $\overline{K} \cap U \in \mathcal{S}(\mathcal{A}, T)$ for some chamber K, then $v := \overline{K} \cap U$ is called a *vertex* of $\mathcal{A}$. The number $m(v) := |\{H \in \mathcal{A} \mid v \subset H\}|$ is called the *weight* or *multiplicity* of v. We call the Tits arrangement $(\mathcal{A}, T)$ *locally spherical* if all vertices of $\mathcal{A}$ meet T. Denote by $\mathcal{V}$ the set of vertices of $\mathcal{A}$ and define the multiplicity of $\mathcal{A}$ by $m(\mathcal{A}) := \sup_{v \in \mathcal{V}} m(v)$. The arrangement $(\mathcal{A}, T)$ is called *spherical* if $T = V$. If $\partial T = \ker(\alpha)$ for some $\alpha \in V^*$, then the Tits arrangement $(\mathcal{A}, T)$ is called *affine*.

Remark 2.7. i) We observe that classical simplicial hyperplane arrangements in $V := \mathbb{R}^r$ may be identified with spherical Tits arrangements in V.
ii) Let $(\mathcal{A}, T)$ be a Tits arrangement in $\mathbb{R}^r$ and let $x \in T$. We write $\overline{\mathcal{A}_x}$ for the essential arrangement of rank $r - 1$ corresponding to the parabolic subarrangement $\mathcal{A}_x$. Then by [12, Proposition 4.4] we know that $\overline{\mathcal{A}_x}$ is again a Tits arrangement. Thus by abuse of notation, we may say that every

parabolic subarrangement of a Tits arrangement is again a Tits arrangement.
iii) Note that for a simplicial arrangement $(\mathcal{A},T)$, the closure of T can be reconstructed from the chambers of $\mathcal{A}$: we have $\overline{T} = \overline{\bigcup_{K\in\mathcal{K}(\mathcal{A})} K}$. In particular, the Tits cone T is determined by $\mathcal{A}$. For details on this, see [12, Lemma 3.24].

Next we recall the so called *chamber graph* of $\mathcal{A}$ and state some of its elementary properties.

Definition 2.6. i) Let $(\mathcal{A},T)$ be a Tits arrangement with set of chambers $\mathcal{K}$. We say that two chambers $K,L\in\mathcal{K}$ are *(H-)adjacent*, if $H := K\cap L$ is a hyperplane. Define the *chamber graph of* $\mathcal{A}$ to be the simplicial graph $G := G(\mathcal{A},T)$ whose vertex set is given by $\mathcal{K}$ such that $K,L\in\mathcal{K}$ are connected by an edge if and only if they are adjacent. A path in G connecting two chambers $K,L\in\mathcal{K}$ is called a *gallery from K to L*. The gallery is called *minimal* if it is of length $d_G(K,L)$, where d_G is the natural distance function of the metric space $\mathcal{K}$.
ii) Let $x\in\overline{T}$ and denote by $\mathcal{K}_x\subset\mathcal{K}$ the set of chambers whose closure contains x. Define the graph G_x to be the graph with vertex set $\mathcal{K}_x$ such that $K,L\in\mathcal{K}_x$ are connected by an edge if and only if they are adjacent. Then G_x is a subgraph of G, called the *parabolic subgraph at x*. A subgraph of G is called *parabolic* if it is of the form G_x for some $x\in\overline{T}$.
iii) Let (M,d) be an arbitrary connected metric space. For $a,b\in M$ we denote by $\sigma(a,b)$ the segment between a and b. Now let $A\subset M$ and $x\in M$. Then $y\in A$ is called a *gate from x to A*, if $y\in\sigma(x,z)$ for any $z\in A$. This gate is uniquely determined, existence provided. The set A is called *gated* if every $x\in M$ has a gate to A.

We then have the following basic results for the graph $G(\mathcal{A},T)$ (see [12] for proofs of these statements).

Proposition 2.2. *Let $(\mathcal{A},T)$ be a Tits arrangement with chamber graph G.*
i) The graph G and all its parabolic subgraphs are connected.
ii) For $x\in\overline{T}$, the set $\mathcal{K}_x$ is a gated subset of $(\mathcal{K},d_G)$.

Next we define the notion of *isomorphism* for Tits arrangements and we introduce *crystallographic* Tits arrangements.

Definition 2.7. a) Let $V := \mathbb{R}^r$. A *root system* is a set $R\subset V^*$ such that

- $0\notin R$,
- if $\alpha\in R$ then $-\alpha\in R$,
- there exists a Tits arrangement $(\mathcal{A},T)$ such that $\mathcal{A} = \{\ker(\alpha)\mid\alpha\in R\}$.

If R is a root system and $(\mathcal{A},T)$ is as above, then we say that the Tits arrangement $\mathcal{A}$ is *associated to R*. A root system R is called *reduced* if

for all $\alpha \in R$ we have $R \cap \langle\alpha\rangle_{\mathbb{R}} = \{\pm\alpha\}$. If $\mathcal{A}$ is associated to R, then we sometimes write $(\mathcal{A}, T, R)$ instead of just $\mathcal{A}$ or $(\mathcal{A}, T)$. The image of R under the natural map $V^* \longrightarrow \mathbb{P}(V^*)$ is called the *dual point set corresponding to* $\mathcal{A}$ and is denoted by $\mathcal{A}^*$.
b) Let $(\mathcal{A}, T, R), (\mathcal{A}', T', R')$ be two Tits arrangements. Then these are called *isomorphic* if there exists $g \in \mathrm{GL}(V)$ such that $g\mathcal{A} = \mathcal{A}'$ and $g * R = R'$, where $g * \alpha = \alpha \circ g^{-1}$ for $\alpha \in V^*$. If one can choose coordinates such that $R \subset \mathbb{K}^r$ for some subring $\mathbb{K} \subset \mathbb{R}$, then we say that $\mathcal{A}$ *is realizable over* $\mathbb{K}$.
c) Let $(\mathcal{A}, T)$ be a Tits arrangement associated to R and let K be a chamber. Then the *root basis of* K is the set

$$B^K := \{\alpha \in R \mid \ker(\alpha) \in W^K, \forall x \in K : \alpha(x) > 0\}.$$

We call the Tits arrangement $(\mathcal{A}, T)$ *crystallographic (with respect to R)* if

$$R \subset \pm \sum_{\alpha \in B^K} \mathbb{N}_0 \alpha$$

for each chamber K.

Remark 2.8. As pointed out already, we may identify spherical Tits arrangements of rank r with classical simplicial arrangements of hyperplanes in $\mathbb{R}^r$. Thus, we now have two notions of isomorphism for such arrangements. Namely, *combinatorial isomorphism* in the sense of Definition 2.1 and *isomorphism* in the sense of Definition 2.7. We note that the latter implies the first.

2.3 Crystallographic Tits arrangements and Cartan graphs

In this section we recall the correspondence between crystallographic Tits arrangements and connected simply connected Cartan graphs permitting a root system described in [33]. This generalizes the respective correspondence in the classical case obtained in [3].

First we recall the notion of a *generalized Cartan matrix*. For this, we use the same terminology as introduced in [24]. In the following definition, all relations of the form $X > 0, X \geq 0$ are understood componentwise.

Definition 2.8. Let $r \in \mathbb{N}$, $I := \{1, ..., r\}$ and consider a matrix $C \in \mathbb{Z}^{I \times I}$.
i) The matrix $C = (c_{i,j})_{i,j \in I}$ is called a *generalized Cartan matrix* if the following holds:

- $c_{i,i} = 2$ for all $i \in I$.
- If $i \neq j \in I$ then $c_{i,j} \leq 0$.

- For all $i \neq j \in I$ the inequality $c_{i,j} < 0$ implies that $c_{j,i} < 0$.

ii) We say that a generalized Cartan matrix C is of *finite type*, if the following conditions hold:

- $\det(C) \neq 0$
- there exists $u > 0$ such that $Cu > 0$
- $Cv \geq 0$ implies $v > 0$ or $v = 0$

iii) We say that a generalized Cartan matrix C is of *affine type*, if the following conditions hold:

- $\operatorname{corank}(C) = 1$
- there exists $u > 0$ such that $Cu = 0$
- $Cv \geq 0$ implies $Cv = 0$

Now we are ready to define *Cartan graphs* and their associated *Weyl groupoids*.

Definition 2.9. i) Let $r \in \mathbb{N}$ and set $I := \{j \in \mathbb{N} \mid 1 \leq j \leq r\}$. For $i \in I$ we denote by α_i the i-th standard basis vector in $\mathbb{Z}^I$. Sometimes we refer to α_i as a *simple root*.
ii) Let $r \in \mathbb{N}$ and let A be a non-empty set. A *Cartan graph* $\mathcal{C}$ consists of the following data:

- For each $i \in I$ there is an involutive bijection $\rho_i : A \longrightarrow A$.
- For each $a \in A$ there is a generalized Cartan matrix $C^a = (c^a_{i,j})_{i,j \in I} \in \mathbb{Z}^{I \times I}$ such that $c^a_{i,j} = c^{\rho_i(a)}_{i,j}$ for all $i, j \in I$.

We write $\mathcal{C} = \big(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A}\big)$. The integer r is called the *rank* of $\mathcal{C}$ and A is referred to as the set of *objects* of $\mathcal{C}$.
iii) For each $i \in I$ and each $a \in A$ the generalized Cartan matrix C^a defines involutive linear maps $\sigma^a_i : \mathbb{Z}^I \longrightarrow \mathbb{Z}^I$ in the following way:

$$\sigma^a_i(\alpha_j) := \alpha_j - c^a_{i,j}\alpha_i.$$

For fixed $a \in A$ the maps $\sigma^a_i, i \in I$, are called the *simple reflections at* a. The *Weyl groupoid* $W(\mathcal{C})$ associated to $\mathcal{C} = \big(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A}\big)$ is the category whose objects are given by the elements of the set A and whose morphisms are given by compositions of the maps σ^a_i, viewed as elements of $\operatorname{Aut}(\mathbb{Z}^I)$. Here the morphism σ^a_i is considered as an element in $\operatorname{Hom}(a, \rho_i(a))$. The category $W(\mathcal{C})$ is a groupoid because all morphisms are actually isomorphisms.
iv) The Cartan graph $\mathcal{C}$ is called *connected* if for each $a, b \in A$ there is a

morphism $\omega \in \mathrm{Hom}(a,b)$. It is called *simply connected* if for all $a \in A$ the set $\mathrm{Hom}(a,a)$ consists only of the identity map at a.
v) The Cartan graph $\mathcal{C}$ is called *standard* if there exists a generalized Cartan matrix C such that for all $a \in A$ we have $C^a = C$.
vi) Let $\mathcal{C} = \big(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A}\big), \mathcal{C}' = \big(I', A', (\rho')_{i\in I}, (C'^a)_{a\in A'}\big)$ be two Cartan graphs. Then $\mathcal{C}$ and $\mathcal{C}'$ are called *equivalent* if there are bijections $\phi_0 : I \longrightarrow I', \phi_1 : A \longrightarrow A'$ such that we have

$$\phi_1(\rho_i(a)) = \rho'_{\phi_0(i)}(\phi_1(a)),$$
$$C^{\phi_1(a)}_{\phi_0(i),\phi_0(j)} = C^a_{i,j}$$

for all $i, j \in I, a \in A$.

Remark 2.9. If $\mathcal{C}$ is standard then the corresponding Weyl groupoid $W(\mathcal{C})$ is actually a Weyl group. It is obtained as a subgroup of $\mathrm{Aut}(\mathbb{Z}^I)$ generated by the simple reflections associated with the unique Cartan matrix appearing in $\mathcal{C}$.

We continue by introducing root systems of type $\mathcal{C}$, where $\mathcal{C}$ is a Cartan graph.

Definition 2.10. Let $\mathcal{C} := \big(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A}\big)$ be a Cartan graph of rank r. For every $a \in A$, let $R^a \subset \mathbb{Z}^I$ be a non-empty subset. Then for all $a \in A$ and each $i, j \in I$ we define $m^a_{i,j} := |R^a \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$. We say that

$$\mathcal{R} := \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following conditions:

$(R1)$ $R^a = R^a_+ \cup -R^a_+$, where $R^a_+ = R^a \cap \mathbb{N}_0^r$ for all $a \in A$.
$(R2)$ $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$, for all $i \in I, a \in A$.
$(R3)$ $\sigma^a_i(R^a) = R^{\rho_i(a)}$, for all $i \in I, a \in A$.
$(R4)$ If $i \neq j \in I$ and $a \in A$ such that $m^a_{i,j}$ is finite, then $(\rho_i\rho_j)^{m^a_{i,j}}(a) = a$.

The root system $\mathcal{R}$ is called *finite* if for all $a \in A$ the set R^a is finite. In this case the corresponding Cartan graph is called *finite*, if it is connected. The root system $\mathcal{R}$ is called *irreducible* if all Cartan matrices in the corresponding Cartan graph are indecomposable. Finally, $\mathcal{C}$ is called *locally spherical* if it possesses a root system and if $|R^a \cap U| < \infty$ for all $a \in A$ and every proper subspace $U \subset \mathbb{Q}^I$, generated by simple roots at a.

Remark 2.10. i) Let $\mathcal{C}$ be a Cartan graph. For each object $a \in A$ we define the *real roots at* a by

$$(R^{re})^a := \{\omega(\alpha_j) \mid \omega \in \mathrm{Hom}(b,a), b \in A, j \in I\} \subset \mathbb{Z}^I.$$

We remark that if the Cartan graph $\mathcal{C}$ possesses a root system of type $\mathcal{C}$ then the collection $\big((R^{re})^a\big)_{a\in A}$ forms a root system of type $\mathcal{C}$ as well.
ii) If $\mathcal{C}$ is simply connected then condition (R4) implies that for each $a \in A$ and arbitrary $i \neq j \in I$ we have the following identity:

$$(\sigma_i\sigma_j)^{m^a_{i,j}-1}\sigma_i\sigma^a_j = (\sigma_j\sigma_i)^{m^a_{i,j}-1}\sigma_j\sigma^a_i = \mathrm{Id}_r,$$

where Id_r denotes the identity matrix in $\mathbb{Z}^{I\times I}$.

Next we introduce Dynkin diagrams. These are graph theoretical encodings of generalized Cartan matrices. In some sense, these are finer invariants than Coxeter diagrams. They will be useful in Chapter 3.

Definition 2.11. Let $\mathcal{C} := \big(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A}\big)$ be a Cartan graph of rank r. Then the *Dynkin diagram* Γ^a at a is a labelled directed graph given by the Cartan matrix C^a in the following way: the vertices of Γ^a are indexed by the simple roots α_i, where $1 \leq i \leq r$. Two different vertices $\alpha_i \neq \alpha_j$ are connected by an arrow pointing towards α_i with label $-c^a_{i,j}$ if and only if $c^a_{i,j} \neq 0$. For an object $a \in A$ we shall say that a is of "*finite type* X" if Γ^a is the diagram associated to some finite Weyl group X. Similarly, the diagram Γ^a is said to be of *weakly finite type*, if Γ^a occurs in a finite Cartan graph of rank at least eight. Moreover, Γ^a is said to be of *affine type*, if the corresponding generalized Cartan matrix is of affine type.

In [24], Kac gives a complete classification of generalized Cartan matrices of finite and affine type and we will use the same notation for the corresponding Dynkin diagrams in Chapter 3.

Remark 2.11. Let $R \subset \mathbb{N}_0^r$ be such that $|R \cap (\mathbb{Z}\alpha_i + \mathbb{Z}\alpha_j)| < \infty$ for all $1 \leq i, j \leq r$. Then we may associate with R a generalized Cartan matrix $C = (c_{i,j})_{1\leq i,j\leq r} \in \mathbb{Z}^{r\times r}$ in the following way: for $i \neq j$ as above we set $c_{i,j} := -\max\{k \mid k \in \mathbb{N}, k \cdot \alpha_i + \alpha_j \in R\}$ while $c_{i,i} := 2$. So by Definition 2.11 there is a Dynkin diagram Γ associated with R.

Now we can formulate the correspondence between connected simply connected Cartan graphs permitting a root system and crystallographic Tits arrangements. We give the following theorem (see [33, Corollary 2.6.24]):

Theorem 2.3. *Let $\mathfrak{A}$ be the set of all crystallographic Tits arrangements with reduced root systems, let $\mathfrak{C}$ be the set of all connected, simply connected Cartan graphs permitting a root system. Let $\cong$ denote isomorphism on $\mathfrak{A}$ as well as equivalence on $\mathfrak{C}$. Then there is a one to one correspondence*

$$\Lambda : \mathfrak{C}/_{\cong} \longrightarrow \mathfrak{A}/_{\cong}.$$

Remark 2.12. i) We remark that a Tits arrangement is locally spherical in the sense of Definition 2.5 if and only if the corresponding Cartan graph is locally spherical in the sense of Definition 2.10.

ii) If $\mathcal{C}$ is locally spherical with associated root system $\mathcal{R}$ and corresponding hyperplane arrangement (A, T), then by [12, Lemma 3.24] for the Tits cone T it holds

$$T = \bigcup_{K \in \mathcal{K}(A)} \overline{K},$$

showing the analogy to the Tits cone associated with a Coxeter group (see [23]).

2.4 Important problems and results in the classical case

In this section we present some open problems as well as some important results on simplicial arrangements in the classical case, i.e. problems and results on spherical Tits arrangements in some $\mathbb{R}^r$.

We will be mainly (but not exclusively) interested in classification and finiteness results (and problems). These results will turn out to be relevant in Chapter 5 and Chapter 6, where we try to make some advances concerning some of the mentioned problems.

We start with the following positive result, which asserts that at least our knowledge of spherical crystallographic Tits arrangements appears to be complete:

Theorem 2.4. *We have a complete list (up to combinatorial isomorphism) of spherical crystallographic Tits arrangements in any rank $r \geq 2$.*

This result has been established in the papers [9], [7], [8]. For $r = 2$ one has a bijective correspondence to triangulations of convex n-gons by non-intersecting diagonals while for $r \geq 3$ the classification is obtained via machine computations by exploiting several strong conditions induced by the crystallographic property. For instance, one can prove that in a spherical crystallographic Tits arrangement every root is either simple or a sum of two positive roots, which is a crucial fact for the classification algorithm. Clearly, this need not be true for a general simplicial arrangement. Therefore, the techniques used in the classification do not seem to generalize to other classes of simplicial arrangements.

However, using the famous correspondence between (combinatorial isomorphism classes of) pseudoline arrangements and *wiring diagrams* (see [2] or [6] for details on this), one can prove the following result for simplicial line arrangements. These may of course be identified with spherical rank three Tits arrangements.

Theorem 2.5. *We have a complete list (up to combinatorial isomorphism) of simplicial line arrangements in $\mathbb{P}^2(\mathbb{R})$ consisting of at most 27 lines.*

This result is again obtained by machine computations. This time, one has to use clever ideas to recognize whether or not a given *wiring-fragment* may be completed to a simplicial (pseudo)line arrangement. A proof and a detailed description of the algorithm can be found in [6]. It turns out that the problem of determining whether or not a given arrangement of pseudo-lines has a realization as arrangement of straight lines is not the crucial point of the classification. An algorithm that is capable of determining whether or not this is the case (at least in every instance occurring in the classification) is described in [4].

We now come to a more recent result. For this we recall that an arrangement of hyperplanes $\mathcal{A}$ in $\mathbb{R}^r$ (in the sense of Definition 2.1) is called *supersolvable*, if the intersection lattice $L_{\mathcal{A}}$ admits a maximal chain of modular elements (see [27] for more details). Moreover, we define the following three families of simplicial line arrangements (we use the same notation as in [19]):

Definition 2.12. Let $n \geq 3$ be a natural number and for $1 \leq k \leq n$ let

$$P_k := \left(\cos\left(\frac{2k\pi}{n}\right) : \sin\left(\frac{2k\pi}{n}\right) : 1\right) \in \mathbb{P}^2(\mathbb{R})$$

be the vertices of a regular n-gon centred at the origin of the affine $z = 1$ plane. We denote by $\mathcal{A}_n$ the arrangement in $\mathbb{P}^2(\mathbb{R})$ obtained by taking the lines containing the n sides of the n-gon together with its n lines of mirror symmetry. Then $\mathcal{A}_n$ is a simplicial arrangement consisting of $2n$ lines whose isomorphism class is denoted $A(2n, 1)$ in [19]. We define the set

$$\mathcal{R}(1) := \{\mathcal{A}_n \mid n \in \mathbb{N}_{\geq 3}\}.$$

If n is even, then we obtain another simplicial arrangement $\mathcal{A}'_n$ from $\mathcal{A}_n$ by adding the line at infinity (which is given by the equation $z = 0$). The corresponding isomorphism class is denoted by $A(2n + 1, 1)$. We define

$$\mathcal{R}(2) := \left\{\mathcal{A}'_n \mid n = 2k, k \in \mathbb{N}_{\geq 2}\right\}.$$

Finally, we call a projective line arrangement $\mathcal{A}$ a *near pencil arrangement* if there exists $\ell \in \mathcal{A}$ such that all lines in $\mathcal{A} \setminus \{\ell\}$ pass through a single point p with $p \notin \ell$. We define

$$\mathcal{R}(0) := \{\mathcal{A} \mid \mathcal{A} \text{ is a near pencil arrangement}\}.$$

These are referred to as the *infinite series of simplicial line arrangements*, the third one $\mathcal{R}(0)$ being considered trivial.

Remark 2.13. i) We remark that the arrangement $\mathcal{A}_3$ is combinatorially isomorphic to the reflection arrangement $\mathcal{A}(A_3)$. Moreover, the arrangement $\mathcal{A}'_4$ is combinatorially isomorphic to the reflection arrangement $\mathcal{A}(B_3)$.
ii) For $n \geq 3$ let $\mathcal{A}_n$ be defined as above. Then the dual point set of $\mathcal{A}_n$ is located on a reducible projective cubic curve consisting of a conic σ and a line ℓ such that σ and ℓ do not meet in the real projective plane. Moreover, we note that the arrangements belonging to $\mathcal{R}(1) \cup \mathcal{R}(2)$ are *projectively unique*. For details on this, see for instance [4].
iii) The arrangements $\mathcal{A}_n$ are relevant also in the context of the so called *Dirac-Motzkin conjecture*, which asserts that up to finitely many exceptions, every arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$ determines at least $\frac{n}{2}$ vertices of weight two. Indeed, a quick computation shows that the only non-zero values of the t-vector of the arrangement $\mathcal{A}_n$ are $t_2^{\mathcal{A}_n} = n, t_3^{\mathcal{A}_n} = \binom{n}{2}, t_n^{\mathcal{A}_n} = 1$. Moreover, in the paper [17] it is shown that the Dirac-Motzkin conjecture is actually a theorem (at least for sufficiently large arrangements). Thus, for sufficiently large n, the arrangements $\mathcal{A}_n$ may be regarded as extremisers with respect to said theorem.
iv) We note that for $n \geq 3$ one always has $t_3^{\mathcal{A}_n} \geq t_2^{\mathcal{A}_n}$. Let $x \in \mathbb{N}$ be a natural number. It is an interesting observation that for any simplicial arrangement $\mathcal{A}$ in $\mathbb{P}^2(\mathbb{R})$ with $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$ such that $t_i^{\mathcal{A}} = 0$ for $i \notin \{2, 3, x\}$ one always has $x \leq 8$: by assumption $\mathcal{A}$ is simplicial and therefore $t_2^{\mathcal{A}} = 3 + (x-3)t_x^{\mathcal{A}}$. By Proposition 2.1 and [28, Theorem 2.2] this yields the following chain of inequalities:

$$\frac{7t_2^{\mathcal{A}}}{4} \geq t_2^{\mathcal{A}} + \frac{3t_3^{\mathcal{A}}}{4} \geq n + \frac{x(x-4)}{4}t_x^{\mathcal{A}} = n + \frac{x(x-4)(t_2^{\mathcal{A}} - 3)}{4(x-3)}.$$

If $x \geq 9$ this implies $0 \leq \left(\frac{x(x-4)}{4(x-3)} - \frac{7}{4}\right) t_2^{\mathcal{A}} \leq \frac{3x(x-4)}{4(x-3)} - n$. From this we may deduce that $n \leq \frac{3x(x-4)}{4(x-3)} < \frac{3x}{4} < \frac{3n}{4} < n$, which is absurd.

Using the notation introduced above, we can now formulate the following classification result whose proof can be found in [11].

Theorem 2.6. *We have a complete list (up to combinatorial isomorphism) of irreducible supersolvable simplicial arrangements in any rank $r \geq 3$. If $r = 3$, then the arrangements in question are exactly those combinatorially isomorphic to one of the infinite series $\mathcal{R}(1), \mathcal{R}(2)$. Moreover, if $r \geq 4$, then the only examples are combinatorially isomorphic to the reflection arrangements $\mathcal{A}(A_r), \mathcal{A}(B_r)$ or to an arrangement obtained from $\mathcal{A}(B_r)$ by removing a suitable hyperplane.*

The proof proceeds by induction on the rank by a very careful examination of Coxeter diagrams together with several other geometric and combinatorial observations, most of which rely on the fact that, in a simplicial arrangement, one may use reflections to pass from a given chamber to an adjacent chamber.

We now turn to unsolved problems concerning spherical Tits arrangements. In the rank three case, there is an impressive catalogue published by Grünbaum (see [19]) listing almost all currently known examples: only four additional arrangements have been discovered in the paper [6]. The most challenging problem in rank three is therefore probably the following one:

Conjecture 1. *Up to finitely many corrections, the present catalogue of (combinatorial isomorphism classes of) simplicial arrangements in $\mathbb{P}^2(\mathbb{R})$ is complete.*

In Chapter 5 and Chapter 6 we will provide proofs for this conjecture for some special classes of arrangements. In particular, we prove that our list of free simplicial arrangements in $\mathbb{P}^2(\mathbb{R})$ having multiplicity at most four is complete (see Theorem 5.5). Moreover, we show that a free simplicial line arrangement with multiplicity at most five consists of at most forty lines (see Theorem 5.8). We can also prove that a free simplicial line arrangement $\mathcal{A}$ whose multiplicity is bounded by six, and which has the property $t_3^{\mathcal{A}} \geq \frac{3}{4} t_2^{\mathcal{A}}$, consists of at most 1480 lines (see Theorem 5.10). We can also prove finiteness results for some classes of simplicial arrangements exhibiting some sort of duality (see for instance Corollary 6.1, Theorem 6.2 and Theorem 6.3).

Of course we can formulate a similar conjecture in arbitrary rank. We remark that there is also a lesser known publication by Grünbaum and Shephard (see [21]), listing some examples in rank four. However, for ranks greater than four, in the general case almost nothing is known. Nonetheless, we formulate the following generalization of Conjecture 1.

Conjecture 2. *There are only finitely many combinatorial isomorphism classes of simplicial arrangements in $\mathbb{P}^{r-1}(\mathbb{R})$ for every $r \geq 3$.*

We close this chapter with the following extreme relaxation of Conjecture 2, which appears to be still unproved and which shows how limited our current understanding of simplicial arrangements still is (see [21], p.99).

Conjecture 3. *Let $r \geq 3$ be a natural number. Then there exists a hyperplane arrangement in $\mathbb{R}^r$ which does not occur as a subarrangement of a simplicial arrangement in $\mathbb{R}^r$.*

Chapter 3

Some results on Tits arrangements

In this chapter we establish some results on Tits arrangements. The first section contains a characterization of pairs of Tits arrangements $(\mathcal{A}, \mathcal{A}')$ such that $\mathcal{A}'$ is obtained from $\mathcal{A}$ by removing one hyperplane (see Theorem 3.1). This is achieved by generalizing some of the concepts presented in Section 2.3 to the non-crystallographic case.

In the second section, we follow ideas of Michael Cuntz to study locally spherical Tits arrangements which admit a groupoid structure similar to the *Weyl groupoid* associated to a crystallographic arrangement. Again, we give a characterization of those Tits arrangements admitting such a structure (see Theorem 3.2). Moreover, we give a classification result for highly symmetric rational arrangements in the spherical rank three case (see Theorem 3.3).

The third section contains some results on locally spherical crystallographic Tits arrangements. More precisely, we show that the appearance of a certain Dynkin diagram Γ determines the arrangement in question in many cases. Using the notation in [24, Chapter 4], we prove this for $\Gamma \in \{A_{r-1}^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, D_r^{(2)} \mid r \geq 8\}$. If Γ is of type E_8, then we can prove that there are two possible arrangements containing a chamber whose Dynkin diagram is given by Γ (see Theorems 3.5 - 3.9).

3.1 Pairs of Tits arrangements differing by one hyperplane

The goal of this section is to give a characterization of pairs of Tits arrangements $((\mathcal{A}, T), (\mathcal{A}', T))$ in $\mathbb{R}^r$ such that $|\mathcal{A} \setminus \mathcal{A}'| = 1$. The characterization will be in terms of so called *weak Dynkin diagrams* (see Definition 3.3). The goal is achieved by proving Theorem 3.1.

To get started, we give the following definition.

Definition 3.1. Let $(\mathcal{A}, T)$ be a Tits arrangement with set of chambers $\mathcal{K}$. Let $K \in \mathcal{K}$ and $B := (v_1, ..., v_r)$ be a basis for $\mathbb{R}^r$. Then B is called a *basis for* K, if $K = \{\sum_{i=1}^{r} \lambda_i v_i \mid \lambda_i > 0 \text{ for } 1 \leq i \leq r\}$. In this case, we write $K = (v_1, ..., v_r)_{>0}$.

We begin with the following well known observation which will be central for most ideas in this chapter.

Lemma 3.1. *Let $(\mathcal{A}, T)$ be a Tits arrangement in $\mathbb{R}^r$ and assume that K and K' are two adjacent chambers of $\mathcal{A}$. Choose a basis $B := (v_1, ..., v_r)$ for K and a unit vector w such that $K' = (v_1, ..., v_{i-1}, w, v_{i+1}, ..., v_r)_{>0}$. Then there exists a unique reflection $\sigma_i^{K_B}$ fixing all vertices v_j for $j \neq i$ and mapping v_i to $t \cdot w$ for some $t > 0$ such that the following holds: if $(\alpha_1, ..., \alpha_r)$ and $(\beta_1, ..., \beta_r)$ are dual bases for $(v_1, ..., v_r), (v_1, ..., v_{i-1}, t \cdot w, v_{i+1}, ..., v_r)$ respectively, then $\beta_i = -\alpha_i$.*

Proof. We may assume $i = 1$. Set $B := (v_1, ..., v_r)$ and let $(\gamma_1, ..., \gamma_r)$ be the dual basis for $(w, v_2, ..., v_r)$. Now choose real numbers λ_i for $1 \leq i \leq r$ such that $w = \sum_{j=1}^{r} \lambda_j v_j$. Observe that since K and K' are adjacent simplicial chambers, we necessarily have $\lambda_1 < 0$ while $\lambda_j \geq 0$ for all $j \neq 1$. Further it holds that $(t^{-1} \cdot \gamma_1, \gamma_2, ..., \gamma_r)$ is a dual basis for the basis $(t \cdot w, v_2, ..., v_r)$:

$$\begin{aligned}
&t^{-1} \cdot \gamma_1(t \cdot w) = \frac{t}{t} = 1, \\
&t^{-1} \cdot \gamma_1(v_i) = 0 \text{ for all } i > 1, \\
&\gamma_j(t \cdot w) = t \cdot \gamma_j(w) = 0 \text{ and } \gamma_j(v_j) = 1 \text{ for all j} > 1, \\
&\gamma_j(v_i) = 0 \text{ for all } j \neq i > 1.
\end{aligned}$$

Clearly, the condition $t^{-1} \cdot \gamma_1 = -\alpha_1$ holds precisely when $t = -\lambda_1^{-1} > 0$. Set $c_{1,i}^{K_B} := \frac{\lambda_i}{\lambda_1}$ for $2 \leq i \leq r$. Then by specifying

$$\begin{aligned}
\sigma_1^{K_B}(v_1) &= -v_1 - \sum_{i \geq 2} c_{1,i}^{K_B} v_i, \\
\sigma_1^{K_B}(v_i) &= v_i
\end{aligned}$$

we obtain a linear map $\sigma_1^{K_B} : \mathbb{R}^r \longrightarrow \mathbb{R}^r$ whose corresponding matrix with respect to $(v_1, ..., v_r)$ is given by $\begin{pmatrix} -1 & 0 & ... & 0 \\ -c_{1,2}^{K_B} & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -c_{1,r}^{K_B} & 0 & \cdots & 1 \end{pmatrix}$. In particular, $\sigma_1^{K_B}$ is a reflection. $\square$

Remark 3.1. The reflections $\sigma_i^{K_B}$ also induce maps $\sigma_i^{K_{B^*}}$ on the dual of $\mathbb{R}^r$: if $B = (v_1, ..., v_r)$ is a basis for K with corresponding dual basis $B^* = (\alpha_1, ..., \alpha_r)$, then with respect to B^* the map $\sigma_i^{K_{B^*}}$ is given by the transposed matrix of the matrix representing $\sigma_i^{K_B}$ with respect to B.

In the next lemma we want to clarify in which way the matrix for the reflection $\sigma_i^{K_B}$ changes if we rescale the elements in a basis B chosen for the chamber K. To this end we introduce some notation.

Definition 3.2. Let $(\mathcal{A}, T)$ be a Tits arrangement with set of chambers $\mathcal{K}$. If $K \in \mathcal{K}$ with basis $B := (v_1^K, ..., v_r^K)$, then we associate with the pair (K, B) a matrix $C^{K_B} := (c_{i,j}^{K_B})_{i,j \in I}$, where for $i \neq j$ the entry $c_{i,j}^{K_B}$ is defined as in the proof of Lemma 3.1 and $c_{i,i}^{K_B} := 2$ for all $1 \leq i \leq r$. The matrix C^{K_B} is called the *weak Cartan matrix with respect to* (K, B).

If $B^* = (\alpha_1^K, ..., \alpha_r^K)$ is dual to B, then by the above definition we have

$$\sigma_i^{K_{B^*}}(\alpha_j^K) = \alpha_j^K - c_{i,j}^{K_B} \alpha_i^K,$$

for the maps induced on the dual space by the reflections constructed in Lemma 3.1.

Now we can conveniently describe what happens with the reflection constructed in Lemma 3.1 if we rescale the basis chosen for our chamber K.

Lemma 3.2. *Let B, K, K' be as in the previous lemma and suppose that $\mu_1, ..., \mu_r$ are positive real numbers such that $B' := (\mu_1 v_1, ..., \mu_r v_r)$ is another basis for the chamber K. Then we have $c_{i,j}^{K_{B'}} = \frac{\mu_i}{\mu_j} c_{i,j}^{K_B}$. In particular, we have $c_{i,j}^{K_B} \cdot c_{j,i}^{K_B} = c_{i,j}^{K_{B'}} \cdot c_{j,i}^{K_{B'}}$ for all $i \neq j$.*

Proof. We may assume that $i = 1$. So take two bases $B := (v_1, ..., v_r)$, $B' := (\mu_1 v_1, ..., \mu_r v_r)$ for the chamber K. Let w be the same unit vector as in Lemma 3.1 not lying on the hyperplane determined by $v_2, ..., v_r$. So we have $K' = (w, \mu_2 v_2, ..., \mu_r v_r)_{>0}$. Choose real numbers $\nu_1, ..., \nu_r, \lambda_1, ..., \lambda_r$ such that $w = \sum_{j=1}^r \nu_j \mu_j v_j$ as well as $w = \sum_{j=1}^r \lambda_j v_j$. Then it follows that $\nu_j \mu_j = \lambda_j$, in particular we have $\nu_1 \mu_1 = \lambda_1$. By the construction in Lemma 3.1 we find $\sigma_1^{K_{B'}}(\mu_1 v_1) = -{\nu_1}^{-1} w = -\frac{\mu_1}{\lambda_1} w = -\frac{\mu_1}{\lambda_1} \sum_{j=1}^r \lambda_j v_j = \sum_{j=1}^r \frac{\mu_1}{\mu_j}(-\frac{\lambda_j}{\lambda_1}) \mu_j v_j = -\mu_1 v_1 + \sum_{j=1}^r (-\frac{\mu_1}{\mu_j} c_{1,j}^{K_B}) \mu_j v_j$. This proves the claim. $\square$

Remark 3.2. The proof of the last lemma shows that rescaling the elements in B does not change the reflection $\sigma_i^{K_B}$. Consider the case $i = 1$: $\mu_1 \sigma_1^{K_{B'}}(v_1) = \sigma_1^{K_{B'}}(\mu_1 v_1) = -{\nu_1}^{-1} w = -\frac{\mu_1}{\lambda_1} w = \mu_1 \sigma_1^{K_B}(v_1)$ and therefore $\sigma_1^{K_{B'}}(v_1) = \sigma_1^{K_B}(v_1)$. Since $\sigma_1^{K_{B'}}$ fixes $\mu_j v_j$ we clearly also have $\sigma_1^{K_{B'}}(v_j) = v_j = \sigma_1^{K_B}(v_j)$ for each $j > 1$. Altogether it follows that $\sigma_1^{K_{B'}} = \sigma_1^{K_B}$. However, the matrix representing the reflection $\sigma_1^{K_B}$ with respect to B' will of course be conjugate to the matrix representing the same reflection with respect to B.

Using the last two results we may associate with each chamber of a simplicial hyperplane arrangement an undirected weighted graph as described in the following definition.

Definition 3.3. Let $(\mathcal{A}, T)$ be a Tits arrangement in $\mathbb{R}^r$ and denote by $\mathcal{K}$ its set of chambers. Then we may associate with each $K \in \mathcal{K}$ an undirected weighted graph $\tilde{\Gamma}^K$ in the following way: we choose a basis $B = (v_1^K, ..., v_r^K)$ for K and we denote the vertices of $\tilde{\Gamma}^K$ by the symbols $v_1^K, ..., v_r^K$.

We agree that v_i^K, v_j^K are joined by an edge with weight $\tilde{\Gamma}_{ij}^K := c_{i,j}^{K_B} \cdot c_{j,i}^{K_B}$ if and only if this quantity is nonzero. We call $\tilde{\Gamma}^K$ the *weak Dynkin diagram at* K. Note that by Lemma 3.2 the graph $\tilde{\Gamma}^K$ does not change if we rescale the elements in B. Thus, up to isomorphism, the graph $\tilde{\Gamma}^K$ depends only on K. Compare also Definition 2.11.

We continue with some technical statements which will turn out to be useful in the rest of this chapter.

Lemma 3.3. *Let $(\mathcal{A}, T)$ be a Tits arrangement in $\mathbb{R}^r$ and let K be a chamber of $\mathcal{A}$ with basis $B = (v_1^K, ..., v_r^K)$. Suppose that in $\tilde{\Gamma}^K$ the vertices v_i^K, v_j^K are not adjacent. Then it follows that $c_{i,j}^{K_B} = 0 = c_{j,i}^{K_B}$.*

Proof. Let B be as above and denote its dual basis by B^*. We write $B^* = (\alpha_1^K, ..., \alpha_r^K)$ and we denote the chamber adjacent to K via the root α_i^K by K'. We take $\sigma_i^{K_B}(B)$ as basis for K' with corresponding dual basis $(\alpha_1^{K'}, ..., \alpha_r^{K'})$. Hence $\alpha_j^{K'} = \sigma_i^{K_B*}(\alpha_j^K) = \alpha_j^K - c_{i,j}^{K_B}\alpha_i^K$ for all $i \neq j$ and $\sigma_i^{K_B*}(\alpha_i^K) = -\alpha_i^K$. Now by assumption at least one of $c_{i,j}^{K_B}, c_{j,i}^{K_B}$ must be zero because α_i, α_j are not connected in $\tilde{\Gamma}^K$. So Suppose that $c_{i,j}^{K_B} = 0$ and assume $c_{j,i}^{K_B} = t < 0$. We consider the parabolic subarrangement corresponding to α_i^K, α_j^K and denote it by $\mathcal{B}$. Then $c_{i,j}^{K_B} = 0$ implies that $-\alpha_i^K$ and α_j^K constitute the walls of a certain chamber κ of $\mathcal{B}$. Now write $\sigma_j^{K_B*}(\alpha_i^K) = \alpha_i^K - c_{j,i}^{K_B}\alpha_j^K = \alpha_i^K - t\alpha_j^K$ with respect to the basis $(-\alpha_i^K, \alpha_j^K)$. We obtain $\sigma_j^{K_B*}(\alpha_i^K) = -(-\alpha_i^K) - t\alpha_j^K$. But as $t < 0$, this implies that the hyperplane corresponding to the root $\sigma_j^{K_B*}(\alpha_i^K)$ has non-empty intersection with the chamber κ. This contradiction shows that $c_{i,j}^{K_B} = 0$ whenever $c_{j,i}^{K_B} = 0$. The claim follows. □

By a similar argument we can prove the following lemma. However, as we do not need this result for our further investigations, we omit the proof.

Lemma 3.4. *Let $(\mathcal{A}, T)$ be a Tits arrangement in $\mathbb{R}^r$ and let K be a chamber of $\mathcal{A}$ with basis $B = (v_1^K, ..., v_r^K)$. If the vertices v_i^K, v_j^K are adjacent in $\tilde{\Gamma}^K$ then the weight of the connecting edge is at least* 1.

Now we are in a position to prove the announced theorem which characterizes pairs of Tits arrangements which differ by only one hyperplane.

Theorem 3.1. *Let $(\mathcal{A}, T)$ be a Tits arrangement in $\mathbb{R}^r$ and denote by $\mathcal{K}$ its set of chambers. Fix some $H \in \mathcal{A}$ and denote the set of chambers containing H as a wall by $\mathcal{K}_H$. Then $(\mathcal{A} \setminus \{H\}, T)$ defines a Tits arrangement if and only if for each $K \in \mathcal{K}_H$ it holds that the vertex in $\tilde{\Gamma}^K$ corresponding to H is a leaf.*

Proof. We need to check that each chamber of the hyperplane arrangement defined by $\mathcal{A} \setminus \{H\}$ is a simplicial cone. Clearly, the only chambers affected by the removal of H are those contained in $\mathcal{K}_H$. So let $K \in \mathcal{K}_H$ and choose a basis B for K with corresponding dual basis $B^* = (\alpha_1^K, ..., \alpha_i^K, ..., \alpha_r^K)$, where i is the label corresponding to H: thus $W^K = \left\{\ker(\alpha_j^K) \mid 1 \leq j \leq r\right\}$.

Consider the chamber $K_i \in \mathcal{K}_H$ which is adjacent to K via the wall corresponding to $\ker(\alpha_i^K)$, so $K_i = \sigma_i^{K_B}(K)$. Now take $B_i := \sigma_i^{K_B}(B)$ as basis for K_i with corresponding dual basis $B_i^* := (\alpha_1^{K_i}, ..., \alpha_r^{K_i})$. By Lemma 3.1 we have $\alpha_j^{K_i} = \sigma_i^{K_{B^*}}(\alpha_j^K)$ for $j \in I$. It follows that

$$W^{K_i} = \left\{\ker(\sigma_i^{K_{B^*}}(\alpha_j^K)) \mid 1 \leq j \leq r\right\}.$$

If we remove the hyperplane H, then the chambers K, K_i will merge to a single chamber K_0 of $\mathcal{A} \setminus \{H\}$ and the walls of K_0 are clearly given by

$$W^{K_0} = \left(W^K \cup W^{K_i}\right) \setminus \left\{\ker(\alpha_i^K)\right\}.$$

On the other hand, the chamber K_0 is a simplicial cone if and only if $|W^{K_0}| = r$. The last equation holds precisely when there exists exactly one $l \in \{1, ..., i-1, i+1, ..., r\}$ such that $\alpha_l^{K_i} = \sigma_i^{K_{B^*}}(\alpha_l^K)$ and α_l^K are non-collinear. But by Lemma 3.3, the last condition is easily seen to be equivalent to the condition that there is precisely one vertex in $\tilde{\Gamma}^K$ which is connected with α_i by an edge. This finishes the proof of the theorem. $\square$

3.2 Reflection groupoids associated to Tits arrangements

In this section we want to use the reflections defined in Lemma 3.1 of the previous section to associate with certain locally spherical Tits arrangements a so called *finite reflection groupoid* with the property that for each pair of objects a, b there is precisely one morphism mapping a to b. It turns out that not every locally spherical Tits arrangement admits such a structure. Hence our goal is to give a characterization of those locally spherical Tits arrangements which do admit such a groupoid structure. Following ideas proposed by Michael Cuntz, we begin with the following definition.

Definition 3.4. Let $r \geq 2$ and let $\mathcal{A}$ be a locally spherical Tits arrangement in $\mathbb{R}^r$ with set of chambers $\mathcal{K}$. Fix a chamber K_0 and let B_0 be a basis for

K_0. Assume that each product of reflections $\omega := \sigma_{i_k}...\sigma_{i_1}^{K_0 B_0}$ (as constructed in Lemma 3.1) such that $\omega(K_0) = K_0$ is given by the identity matrix with respect to B_0. Then by composition of the reflections constructed in Lemma 3.1, our choice of B_0 determines bases for all other chambers in a natural way: given a chamber K we choose a shortest word $\omega_K := \sigma_{j_l}...\sigma_{j_1}^{K_0 B_0}$ mapping K_0 to K and we choose as a basis for K the image of B_0 under ω. We denote by B_K the basis associated to the chamber K in this way and we say that $\mathcal{A}$ *admits a finite reflection groupoid (with respect to B_0).*

We denote this groupoid by $\mathcal{G}_{\mathcal{A}}^{B_0}$. The objects of $\mathcal{G}_{\mathcal{A}}^{B_0}$ are the chambers of the arrangement $\mathcal{A}$ and its morphisms are given by compositions of the simple reflections $\sigma_i^{K_{B_K}}$, where $\sigma_i^{K_{B_K}}$ is identified with its matrix with respect to B_K and the composition takes place in $\mathrm{GL}(\mathbb{R}^r)$. The reflection $\sigma_i^{K_{B_K}}$ is considered as an element of $\mathrm{Hom}(K, \sigma_i^{K_{B_K}}(K))$ and products of simple reflections are interpreted accordingly. Moreover, we write $\omega_{B_0}^{L,K}$ for the unique morphism from L to K (with respect to B_0). By construction, each such morphism is represented by a unique matrix in $\mathrm{GL}(\mathbb{R}^r)$.

Remark 3.3. i) In the above definition we have omitted most of the upper indices in the equations $\omega := \sigma_{i_k}...\sigma_{i_1}^{K_0 B_0}, \omega_K := \sigma_{j_l}...\sigma_{j_1}^{K_0 B_0}$. However, this is no problem because all other upper indices are uniquely determined from the one given. Most of the time we will follow this convention.
ii) The condition stated in the above definition does not depend on the chosen chamber K_0, i.e. if there is a chamber K_0 such that the condition is satisfied for K_0, then it is also satisfied for every other chamber K: indeed, suppose the condition of the definition is satisfied for a chamber K_0 and let K be a different chamber. Suppose that $\gamma = \sigma_{i_n}...\sigma_{i_1}^{K_{B_K}}$ is a product of reflections mapping K to itself. Now choose a product of reflections $\omega = \sigma_{j_m}...\sigma_{j_1}^{K_0 B_0}$ mapping K_0 to K. Then $\omega^{-1}\gamma\omega$ is a product of reflections mapping K_0 to itself and therefore it is given by the identity matrix with respect to B_0. But then it is given by the identity matrix with respect to B_K as well. This implies that also γ is given by the identity matrix with respect to B_K.
iii) Let $\mathcal{A}, \mathcal{A}'$ be Tits arrangements such that $\mathcal{A}' = g\mathcal{A}$ for some $g \in \mathrm{GL}(\mathbb{R}^r)$. Then $\mathcal{A}$ admits a finite reflection groupoid if and only if $\mathcal{A}'$ does so.

Example 3.1. *i) Let $\mathcal{A}$ be the spherical Tits arrangement associated to some finite real reflection group G. Then $\mathcal{A}$ admits a finite reflection groupoid. In this case, the associated groupoid is actually a group and it may be identified with the group G.*
ii) Let $\mathcal{A}$ be a locally spherical crystallographic Tits arrangement. Then by Theorem 2.3 we know that $\mathcal{A}$ admits a finite reflection groupoid. In this case, the associated groupoid can be identified with the corresponding Weyl groupoid.

iii) Let $\mathcal{A}$ be a line arrangement belonging to one of the infinite series $\mathcal{R}(1), \mathcal{R}(2)$. Then $\mathcal{A}$ admits a finite reflection groupoid (see Corollary 3.1).
iv) Let $\mathcal{A}$ be the Tits arrangement in $\mathbb{R}^2$ whose lines are spanned by the following vectors: $(1,0),(0,1),(-1,2),(-3,1)$. Then $\mathcal{A}$ does not admit a finite reflection groupoid. Indeed, let ω be the morphism corresponding to a full turn starting at the chamber $K := ((1,0),(0,1))_{>0}$. Let $r,s \in \mathbb{R}_{>0}$ be positive real numbers. Then one checks that with respect to the basis $B_K^{r,s} := ((r,0),(0,s))$, the morphism ω is given by the matrix

$$\omega_{r,s} := \begin{pmatrix} 25 & 0 \\ 0 & \frac{1}{25} \end{pmatrix}.$$

Our strategy in order to characterize those locally spherical Tits arrangements that do admit a reflection groupoid as described above will be as follows: we show that $\mathcal{A}$ yields such a groupoid structure if and only if every parabolic subarrangement of $\mathcal{A}$ yields such a structure. Since our arrangements are locally spherical every parabolic subarrangement is finite. Hence it makes sense to start with the following proposition, which clarifies the situation for spherical rank two Tits arrangements. It turns out that there exists a connection to so called 1-*quiddity cycles* associated to *frieze patterns* (see [10]). For the proof, we use ideas taken from a manuscript provided by Michael Cuntz.

Proposition 3.1. *Let $\mathcal{A}$ be a spherical Tits arrangement in $\mathbb{R}^2$ containing the chamber $K_0 := (e_1, e_2)_{>0}$ where $B_0 := (e_1, e_2)$ is the standard basis of $\mathbb{R}^2$. Let $|\mathcal{A}| = n$ and label the chambers counterclockwise $K_0, ..., K_{2n-1}, K_{2n}$ where $K_{2n} = K_0$. Consider the reflections $\sigma_1, \sigma_2, ..., \sigma_{2n}$ with the property that $\sigma_i : K_{i-1} \longrightarrow K_i$ as constructed in Lemma 3.1. Denote by c_i the off-diagonal entry of the matrix representing σ_i with respect to $B_{i-1} := \sigma_{i-1}(B_{i-2})$. For $1 \leq i \leq 2n$ we define $\eta(c_i) := \begin{pmatrix} c_i & 1 \\ -1 & 0 \end{pmatrix}$.*
Then $\mathcal{A}$ admits a finite reflection groupoid if and only if we have:

$$\eta(c_{2n})...\eta(c_2)\eta(c_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.1}$$

Proof. We define $\tau := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then for any $1 \leq i \leq 2n$ we have

$$\tau \begin{pmatrix} -1 & 0 \\ c_i & 1 \end{pmatrix} = \eta(c_i) = \begin{pmatrix} 1 & c_i \\ 0 & -1 \end{pmatrix} \tau.$$

Thus by Lemma 3.1 and the above identity we conclude that

$$\sigma_{2n}...\sigma_1 = \sigma_{2n}\tau^2\sigma_{2n-1}\sigma_{2n-2}\tau^2...\tau^2\sigma_3\sigma_2\tau^2\sigma_1 = \eta(c_{2n})...\eta(c_1).$$

Hence (3.1) is equivalent to the equation $\sigma_{2n}...\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which holds if and only if $\mathcal{A}$ admits a finite reflection groupoid. This completes the proof. $\square$

Remark 3.4. Proposition 3.1 shows that up to a change of basis every spherical rank two Tits arrangement admitting a finite reflection groupoid is determined by a sequence $c_m, ..., c_1$ such that

$$\eta(c_m)...\eta(c_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

As mentioned above, these sequences are called 1-*quiddity cycles* (see [10]). Let $\mathcal{A}$ be a locally spherical Tits arrangement admitting a finite reflection groupoid. Fix a chamber K_0 of $\mathcal{A}$ and a corresponding basis B_0. Observe that the entries of the 1-quiddity cycles corresponding to rank two localizations of $\mathcal{A}$ are entries of suitable weak Cartan matrices (with respect to (K_0, B_0)) appearing in $\mathcal{A}$. If $(c_m, ..., c_1)_{B_0}$ is such a 1-quiddity cycle, then we say that $(c_m, ..., c_1)_{B_0}$ is *associated* to $\mathcal{A}$.

In the next theorem we are going to reduce the problem of determining whether or not $\mathcal{A}$ admits the desired groupoid structure to parabolic subarrangements. Together with Proposition 3.1, this will yield the promised characterization of locally spherical Tits arrangements admitting a finite reflection groupoid. To do this, we give a little lemma.

Lemma 3.5. *Suppose that $(\mathcal{A}, T)$ is a Tits arrangement in $\mathbb{R}^r$. Let K_1, K_2 be two adjacent chambers and let L be another chamber. Then the distance between K_1 and L in the chamber graph corresponding to $(\mathcal{A}, T)$ is different from the distance between K_2 and L.*

Proof. Let $H \in \mathcal{A}$ be the unique hyperplane that separates K_1 and K_2. Then L lies either on the same side as K_1 with respect to H or it lies on the same side as K_2 with respect to H. In the first case we obtain $\text{dist}(K_2, L) = \text{dist}(K_1, L) + 1$, because every hyperplane that separates K_1 and L also separates K_2 and L. In the latter case a similar argument gives $\text{dist}(K_1, L) = \text{dist}(K_2, L) + 1$. $\square$

Theorem 3.2. *Assume $r > 2$ and let $(\mathcal{A}, T)$ be a locally spherical Tits arrangement in $\mathbb{R}^r$. Then $\mathcal{A}$ admits a finite reflection groupoid if and only if every parabolic subarrangement of $\mathcal{A}$ admits a finite reflection groupoid.*

Proof. If $\mathcal{A}$ admits a finite reflection groupoid, then clearly the same is true for every parabolic subarrangement. So it suffices to show that if every parabolic subarrangement of $\mathcal{A}$ admits a finite reflection groupoid, then $\mathcal{A}$ does so as well.

So fix a chamber K_0 and a basis B_{K_0} for K_0. Let ω be a product of reflections (constructed as in Lemma 3.1) mapping K_0 to itself. The strategy is now to prove the statement by induction on the maximal occurring distance between K_0 and chambers along the path of ω.

If the maximal occurring distance is one, then there is nothing to prove, since reflections are involutive. So assume that the maximal occurring distance to K_0 along ω is greater than one. Let K be a chamber along the path ω whose distance to K_0 is maximal. From all vertices x of $\mathcal{A}$ incident with K we choose an x_0 with the property that the gate G from K_0 to $\mathcal{A}_{x_0}$ has minimal distance to K_0. Now let K_1 be the chamber of $\mathcal{A}_{x_0}$ that comes right before the chamber K along the path of ω; similarly let K_2 be the chamber of $\mathcal{A}_{x_0}$ that comes right after the chamber K along the path of ω.

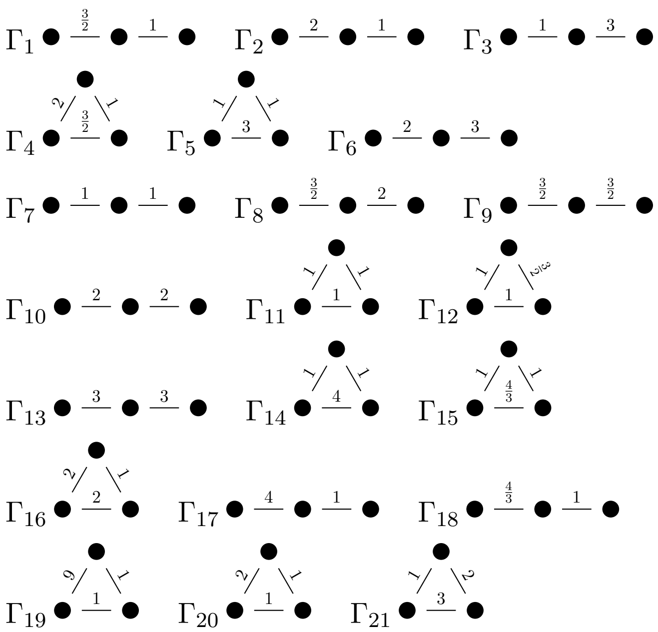
We observe that the distance between K_1 and K_0 as well as the distance between K_2 and K_0 is less than the distance between K_0 and K: by Lemma 3.5 we have $\mathrm{dist}(K_1, K_0) \neq \mathrm{dist}(K, K_0), \mathrm{dist}(K_2, K_0) \neq \mathrm{dist}(K, K_0)$. Thus, by choice of K it follows $\max\{\mathrm{dist}(K_1, K_0), \mathrm{dist}(K_2, K_0)\} < \mathrm{dist}(K, K_0)$. Since all parabolic subarrangements admit finite reflection groupoids, we may replace the subpath of ω connecting K_1 and K_2 by a path δ connecting K_1 and K_2 and visiting only chambers whose distance to K_0 is less than the distance between K and K_0. We can do this for every chamber along the path of ω whose distance to K_0 is maximal. Therefore we may replace ω by ω' such that the maximal occurring distance between K_0 and chambers along the path of ω' is smaller than the maximal distance between K_0 and chambers along the path of ω. By induction the claim follows. $\square$

In the following example we will give an application of Theorem 3.2. Namely, we are going to determine all combinatorial isomorphism classes of irreducible spherical rank three Tits arrangements that admit a finite reflection groupoid and which consist of at most 27 hyperplanes. In the process, we collect all appearing weak Dynkin diagrams. By part ii) of Example 3.1 it remains to consider non-crystallographic arrangements, as the crystallographic arrangements always correspond to Weyl groupoids. We note that such a non-crystallographic arrangement consists of at least 10 hyperplanes. Thus, we may restrict attention to arrangements satisfying $10 \leq |\mathcal{A}| \leq 27$. In the following, we will use the results and the same notation as in [6].

Example 3.2. *We consider non-crystallographic spherical Tits arrangements $\mathcal{A}$ in $\mathbb{R}^3$ with $10 \leq |\mathcal{A}| \leq 27$. Up to combinatorial isomorphism, these have been classified in [6]. Note also that a posteriori, all obtained arrangements are projectively unique: if $\mathcal{A}'$ is a Tits arrangement which is combinatorially isomorphic to an arrangement $\mathcal{A}$ as above, then there exists an element $g \in \mathrm{GL}(\mathbb{R}^3)$ such that for any $H \in \mathcal{A}$ we have $gH \in \mathcal{A}'$. It is conjectured that this is true for every spherical Tits arrangement in $\mathbb{R}^3$. It seems that so far, no proof is known. We start by giving an enumeration (up to labelling) of all weak Dynkin diagrams appearing in arrangements as* *above. For this, let ρ be the real zero of the polynomial $P := X^3 - 3X - 25$ and define the following constants, which will appear as edge weights of the corresponding weak Dynkin diagrams:*

$$
\begin{aligned}
\eta_1 &:= 4\sin\left(\frac{\pi}{14}\right) - 2\sin\left(\frac{3\pi}{14}\right) + 4\cos\left(\frac{\pi}{7}\right),\\
\eta_2 &:= 2 - 2\sin\left(\frac{\pi}{18}\right) + 2\cos\left(\frac{\pi}{9}\right),\\
\eta_3 &:= 4\sin\left(\frac{\pi}{22}\right) - 4\sin\left(\frac{3\pi}{22}\right) + 4\sin\left(\frac{5\pi}{22}\right) + 4\cos\left(\frac{\pi}{11}\right) - 2\cos\left(\frac{2\pi}{11}\right),\\
\eta_4 &:= -4\sin\left(\frac{\pi}{26}\right) + 4\sin\left(\frac{3\pi}{26}\right) - 4\sin\left(\frac{5\pi}{26}\right)\\
&\quad + 4\cos\left(\frac{\pi}{13}\right) - 2\cos\left(\frac{2\pi}{13}\right) + 4\cos\left(\frac{3\pi}{13}\right),\\
\eta_5 &:= \frac{1}{9}\rho^2 + \frac{1}{9}\rho + \frac{7}{9},\\
\eta_6 &:= \frac{1}{9}\rho^2 + \frac{4}{9}\rho + \frac{13}{9},\\
\eta_7 &:= \frac{1}{3}\rho + \frac{2}{3},\\
\eta_8 &:= \frac{1}{9}\rho^2 + \frac{1}{9}\rho - \frac{2}{9}.
\end{aligned}
$$

Now we are ready to give the enumeration of the weak Dynkin diagrams:



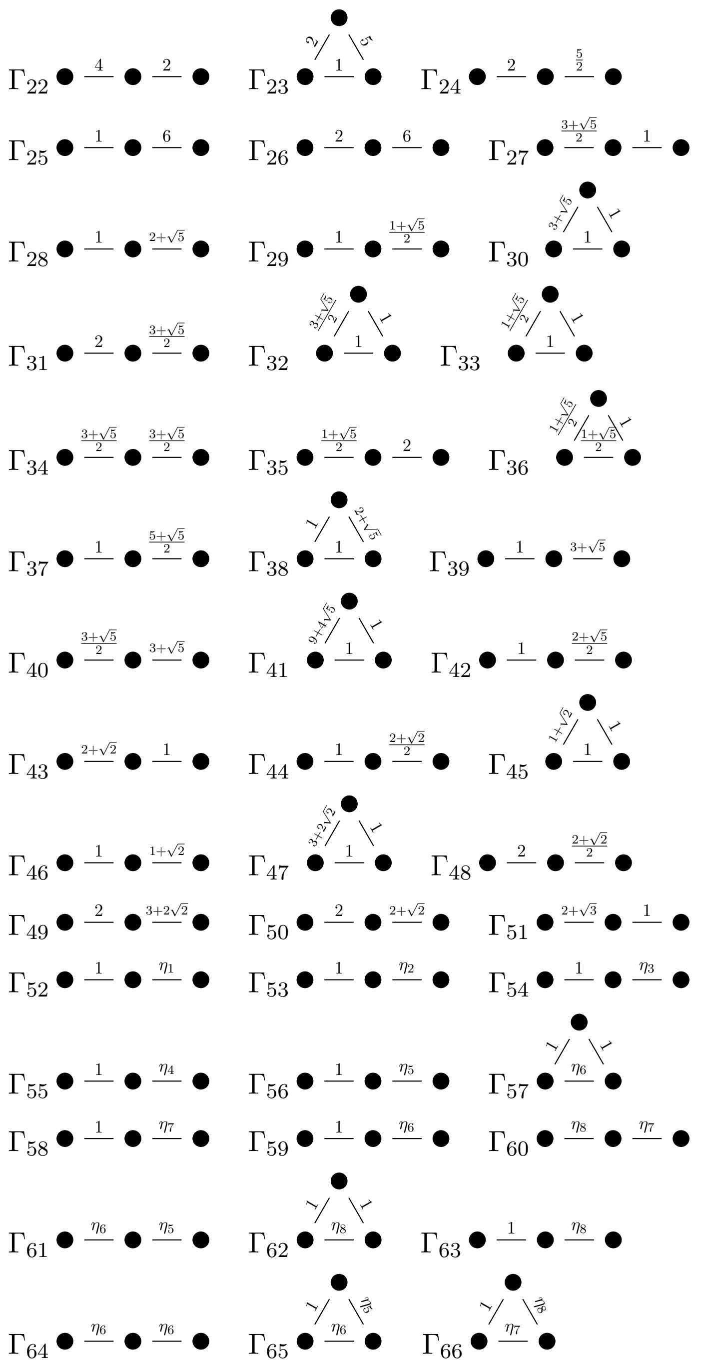


Figure 3.1: Weak Dynkin diagrams of some non-crystallographic spherical Tits arrangements

Next we give a table in which we associate the weak Dynkin diagrams to the Tits arrangements in which they arise. We also determine all Tits arrangements of the considered type admitting a finite reflection groupoid.

label	weak Dynkin diagrams	finite reflection groupoid
$(10, 1)$	7,11,27	yes
$(12, 1)$	3,7,11	yes
$(13, 1)$	2,3,7,20	yes
$(13, 4)$	2,7,11,20	yes
$(14, 1)$	7,11,52	yes
$(14, 3)$	2,7,11,20,27,28,29,30,31,32,33,35	no
$(14, 4)$	7,11,27,28,29,32,34,36	no
$(15, 1)$	27	yes
$(15, 3)$	2,7,10,14,15,16,17,18	no
$(15, 4)$	2,7,11,20,27,28,29,31,32,33,35	no
$(15, 5)$	7,11,56,57,58,59	yes
$(16, 1)$	7,11,43	yes
$(16, 4)$	7,27,32	yes
$(16, 5)$	2,7,27,29,31,32,33,34	no
$(16, 6)$	1,2,3,4,5,6	no
$(16, 7)$	1,2,3,5,6,7,8,9,10,11,12,13	no
$(17, 1)$	2,7,11,20,43	yes
$(17, 5)$	2,7,27,31,32	yes
$(17, 6)$	1,2,3,4,5,6,7,8,9,12,13,21	no
$(17, 7)$	1,2,3,5,7,8,11,12,20,21	no
$(17, 8)$	2,7,11,20,43,44,48,49,50	yes
$(18, 1)$	7,11,53	yes
$(18, 2)$	1,2,3,17,19,20	yes
$(18, 4)$	2,7,27,31,32	yes
$(18, 5)$	2,7,27,31,32	yes
$(18, 6)$	1,2,3,4,5,6,7,8,9,10,12,13,21	no
$(18, 8)$	1,2,3,5,7,8,10,11,12,16,17,20,21,22	no
$(19, 2)$	2,7,27,31,32	yes
$(19, 7)$	1,2,3,5,6,7,8,9,10,12,13	no
$(20, 1)$	7,11,37	yes
$(20, 2)$	2,7,27,31,32	yes
$(21, 1)$	2,7,11,20,37	yes
$(21, 2)$	2,31	yes
$(21, 7)$	7,11,56,57,58,59,60,61,62,63,64,65,66	no
$(22, 1)$	7,11,54	yes
$(22, 2)$	1,2,3,5,7,8,11,12,21,23,24,25	no
$(22, 3)$	2,7,11,27,28,29,31,32,33,38,39,40	yes

label	weak Dynkin diagrams	finite reflection groupoid
$(22,5)$	7,11,41,42	yes
$(23,1)$	1,2,3,5,6,7,8,10,11,12,17,20,21,22,23,24,25,26	no
$(23,2)$	2,7,11,20,27,39,41,42	yes
$(24,1)$	7,11,51	yes
$(24,2)$	2,7,43,44,45,46,47	yes
$(24,3)$	1,2,3,5,6,7,8,10,11,12,17,20,21,22,23,24,25,26	no
$(24,4)$	2,7,11,20,27,39,41,42	yes
$(25,1)$	2,7,11,20,51	yes
$(25,3)$	2,20,27	yes
$(25,5)$	2,7,43,45,46	yes
$(25,8)$	2,7,11,20,27,39	yes
$(26,1)$	7,11,55	yes
$(26,2)$	2,20,27,28,29,30	yes
$(27,1)$	2,20,27,28,29,30	yes

Table 3.1: Non-crystallographic spherical Tits arrangements in $\mathbb{R}^3$ with at most 27 hyperplanes

We remark that the list of appearing weak Dynkin diagrams determines the Tits arrangement in question in many cases: the 51 *different arrangements yield* 44 *different lists of weak Dynkin diagrams. Hence these diagrams appear to be rather strong invariants (at least in the spherical case).*
We also observe that Tits arrangements having the same list of weak Dynkin diagrams are pretty "close" in the Hasse diagram corresponding to the set of all finite Tits arrangements in rank three; for this consider [6, Figure 6*].*
Finally, we see that from all the arrangements in consideration there are sixteen that do not admit a finite reflection groupoid. From those there are eleven arrangements defined over the rational numbers, four arrangements defined over the quadratic field $\mathbb{Q}(\sqrt{5})$ *and one arrangement defined over the cubic extension* $\mathbb{Q}(\rho)$*, where* ρ *is the real number defined in the beginning of this example. Moreover, the computed data yields that certain weak Dynkin diagrams only appear in arrangements admitting a finite reflection groupoid while others only appear in arrangements not having this property. This agrees with our theoretical results, see also Remark 3.4.*

The computed data shows that all considered arrangements belonging to one of the infinite series $\mathcal{R}(1), \mathcal{R}(2)$ admit finite reflection groupoids. Using symmetries of the regular n-gon, it is another immediate consequence of Theorem 3.2 that this is true in general. We obtain the following result:

Corollary 3.1. *Let* $\mathcal{A}$ *be an arrangement from one of the infinite series* $\mathcal{R}(1), \mathcal{R}(2)$*. Then* $\mathcal{A}$ *admits a finite reflection groupoid.*

We continue by giving a result on highly symmetric Tits arrangements in rank three which admit a finite reflection groupoid. To state the result, we need the following definition, which generalizes the concept of "real roots" introduced in Chapter 2 to non-crystallographic situations.

Definition 3.5. Let $\mathcal{A}$ be a locally spherical Tits arrangement in $V := \mathbb{R}^r$ with set of chambers $\mathcal{K}$. Assume that $\mathcal{A}$ admits a finite reflection groupoid. Fix a chamber K_0 and a corresponding basis B_0 for K_0.

Consider two chambers $K, L \in \mathcal{K}$ and let $L = K_1, K_2, ..., K_{m-1}, K_m = K$ be a minimal gallery from L to K. The basis for K_i determined by B_0 is denoted by B_i for $1 \leq i \leq m$ and the corresponding dual bases are denoted by B_i^*. Thus we have $\omega_{B_0}^{L,K} = \sigma_{i_{m-1}}^{K_{m-1}B_{m-1}} ... \sigma_{i_1}^{K_1 B_1}$ for the unique morphism taking L to K.

We identify the dual morphism $\sigma_{i_j}^{K_j B_j^*}$ with its matrix with respect to B_j^* and write

$$\tilde{\omega}_{B_0}^{L,K} := \sigma_{i_{m-1}}^{K_{m-1}B_{m-1}^*} ... \sigma_{i_1}^{K_1 B_1^*} \in \mathrm{GL}(\mathbb{R}^r)$$

for the matrix corresponding to the morphism dual to $\omega_{B_0}^{L,K}$ (see Definition 3.2).

Now if K is any chamber of $\mathcal{A}$ then we define

$$R_{B_0}^K := \left\{ \tilde{\omega}_{B_0}^{L,K}(\alpha_i) \mid 1 \leq i \leq r, L \in \mathcal{K} \right\} \subset \pm \left(\mathbb{R}_{\geq 0}\right)^r,$$

where $\alpha_1, ..., \alpha_r$ are the standard basis vectors in $\mathbb{R}^r$.

We call $R_{B_0}^K$ the *root set at* K (with respect to B_0). Two chambers K, L are called *equivalent*, if $R_{B_0}^K = R_{B_0}^L$. Clearly, this defines an equivalence relation on $\mathcal{K}$.

For fixed K, we define $G_{B_0}^K := \left\{ \tilde{\omega}_{B_0}^{K,L} \mid R_{B_0}^L = R_{B_0}^K \right\} \subset \mathrm{GL}(\mathbb{R}^r)$ and call it the *automorphism group of* K (with respect to B_0). Moreover, we define $\mathcal{R}(\mathcal{A})_{B_0} := \left\{ R_{B_0}^K \mid K \in \mathcal{K} \right\}$ to be the set of different root sets occurring in $\mathcal{A}$ (with respect to B_0).

Remark 3.5. Let $\mathcal{A}$ be as in the above definition.
i) The set $G_{B_0}^K$ is a group because for two chambers L, L' with $R_{B_0}^L = R_{B_0}^{L'}$, the matrices $\sigma_i^{L B_L^*}, \sigma_i^{L' B_{L'}^*}$ are the same for every $1 \leq i \leq r$. Furthermore, it is clear that we have $\omega\left(R_{B_0}^K\right) = R_{B_0}^K$ for any $\omega \in G_{B_0}^K$.
ii) If K, L are two chambers of $\mathcal{A}$, then the automorphism groups $G_{B_0}^K, G_{B_0}^L$ are conjugate: let $\omega \in G_{B_0}^L$ and define $\omega' := \tilde{\omega}_{B_0}^{K,L}$. Then the matrix $\omega'^{-1}\omega\omega'$ is an element of $G_{B_0}^K$. In particular, all automorphism groups are isomorphic.
iii) If $\mathcal{A}$ is spherical, then we have $|\mathcal{K}| = |G_{B_0}^K||\mathcal{R}(\mathcal{A})_{B_0}|$ for any chamber $K \in \mathcal{K}$. Moreover, the number $|\mathcal{R}(\mathcal{A})_{B_0}|$ serves as a measure for the degree of symmetry exhibited by the arrangement $\mathcal{A}$. In particular, if $|\mathcal{R}(\mathcal{A})_{B_0}| = 1$, then $\mathcal{A}$ is a reflection arrangement.

iv) If $\mathcal{A}$ is not crystallographic, then the root set $R^K_{B_0}$ need not be reduced. This happens for instance for the (non-crystallographic) arrangements belonging to the infinite series $\mathcal{R}(1)$. Thus, these arrangements are naturally *multiarrangements* via their groupoid structure.

Now we are ready to give the following result:

Theorem 3.3. *Let $\mathcal{A}$ be a spherical rank three Tits arrangement with set of chambers $\mathcal{K}$. Assume that $\mathcal{A}$ admits a finite reflection groupoid. Fix some $K_0 \in \mathcal{K}$ and a corresponding basis B_0. If $\mathcal{A}$ is realizable over $\mathbb{Q}$ and if $|\mathcal{R}(\mathcal{A})_{B_0}| \leq 8$, then $|\mathcal{A}| \leq 27$. In particular, we have a complete list (up to combinatorial isomorphism) of such arrangements.*

Proof. For a simplicial arrangement $\mathcal{A}$ in $\mathbb{R}^3$ with set of chambers $\mathcal{K}$ it holds

$$|\mathcal{K}| = \frac{4}{3}\sum_{i\geq 2} it_i^{\mathcal{A}}.$$

On the other hand, as $\mathcal{A}$ is realizable over $\mathbb{Q}$ we have $|G^{K_0}_{B_0}| \leq 48$ (see [16]). Thus, using part iii) of Remark 3.5 and relation (2.4) in Theorem 2.1, we obtain the following inequality:

$$\frac{4}{9}(|\mathcal{A}|^2 + 3|\mathcal{A}|) \leq |\mathcal{K}| \leq |\mathcal{R}(\mathcal{A})_{B_0}||G^{K_0}_{B_0}| \leq 8 \cdot 48.$$

For this, recall that by Proposition 2.1, the multiplicity of an irreducible simplicial arrangement in $\mathbb{R}^3$ is at most $\frac{|\mathcal{A}|}{2}$; so relation (2.4) does indeed apply. We conclude that $|\mathcal{A}| \leq 27$. The classification result now follows from Theorem 2.5. □

If we are only interested in finiteness results, then the idea behind the above proof may be generalized to arrangements realizable over algebraic number fields in arbitrary dimension. We give the following theorem, which concludes this section.

Theorem 3.4. *Let $r \geq 3$ and let $\mathcal{A}$ be a spherical Tits arrangements in $\mathbb{R}^r$ which has a realization over some algebraic number field $\mathbb{K}$. Assume that $\mathcal{A}$ admits a finite reflection groupoid. Fix a chamber K_0 and a corresponding basis B_0 and suppose that $|\mathcal{R}(\mathcal{A})_{B_0}| \leq m|\mathcal{A}|^{1-\epsilon}$ for some $m, \epsilon > 0$. Then there exists $M := M(r, \mathbb{K}, m, \epsilon)$ such that $|\mathcal{A}| \leq M$.*

Proof. Let $S(r, \mathbb{K})$ denote the Schur bound: the order of every finite subgroup G in $\mathrm{GL}(r, \mathbb{K})$ divides $S(r, \mathbb{K})$ (see [22, Theorem 14]). In particular, we have $|G| \leq S(r, \mathbb{K})$ for every finite subgroup G in $\mathrm{GL}(r, \mathbb{K})$. Thus, using similar arguments as in the proof of Theorem 3.3 and applying Shannon's Theorem (see [30]), we obtain

$$2|\mathcal{A}| \leq |\mathcal{K}| \leq m|\mathcal{A}|^{1-\epsilon} S(r, \mathbb{K}).$$

Hence, we may take $M := \left(\frac{mS(r,\mathbb{K})}{2}\right)^{1/\epsilon}$. This completes the proof. □

3.3 Crystallographic Tits arrangements of rank at least seven

In this section, we are interested in locally spherical crystallographic Tits arrangements of rank at least seven. We show that the appearance of certain Dynkin diagrams determines the arrangement in quite a few cases. More precisely, for $r \geq 8$ we classify locally spherical crystallographic Tits arrangements containing an object of Dynkin type $E_8, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, A_{r-1}^{(1)}, D_r^{(2)}$ respectively (we use the notation in [24, Chapter 4]). Our methods rely heavily on the classification of spherical crystallographic Tits arrangements obtained in [8].

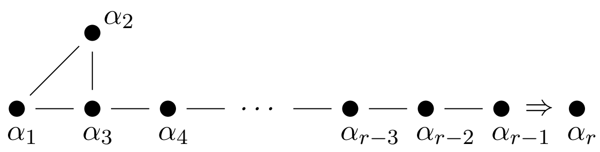
As in the mentioned paper [8], instead of dealing with Tits arrangements themselves, we choose the language of Cartan graphs to formulate and prove our results. This approach is justified by the correspondence described in Theorem 2.3.

In the following, $\mathcal{C} := \big(I, A, (\rho_i)_{i\in I}, (C^b)_{b\in A}\big)$ will always denote a connected simply connected locally spherical Cartan graph of some rank $r := |I|$. By Theorem 2.3, the Cartan graph $\mathcal{C}$ then corresponds to a locally spherical crystallographic Tits arrangement $\mathcal{A}$ in $\mathbb{R}^r$. Recall that a Dynkin diagram Γ is said to be of *weakly finite type*, if Γ appears as Dynkin diagram in some finite Cartan graph of rank at least eight (see Definition 2.11); similarly, Γ is said to be of affine type, if the corresponding Cartan matrix is of affine type.

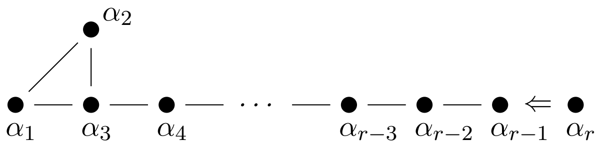
We start with the following proposition, which determines the possible Dynkin diagrams that may appear in a rank r Cartan graph $\mathcal{C}$ as above for $r \geq 8$.

Proposition 3.2. *Let $\mathcal{C} = (I, A, (\rho_i)_{i\in I}, (C^b)_{b\in A})$ be as above with $|I| \geq 8$. Then at each object $a \in A$ the corresponding Dynkin diagram Γ^a is either of weakly finite type or it is of affine type or it is one of the following (up to obvious symmetries and labelling of the vertices):*

$D'B_r$:



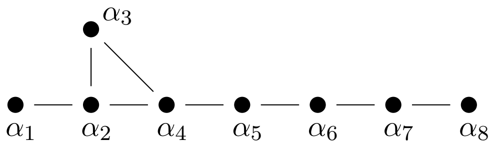
$D'C_r$:



$D'D_r$:

$D'D'_r$:

$\tilde{D}'_8$:

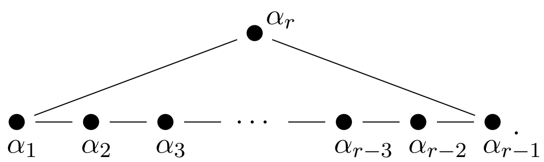


Proof. Let $a \in A$ and let Γ^a be the Dynkin diagram at a. As $\mathcal{C}$ is locally spherical, we conclude that each connected subgraph Γ' of Γ^a on $r-1$ vertices is one of those listed in [8]. So it remains to examine in how many ways one can attach a further vertex to Γ' such that each connected subgraph (of the obtained graph) on $r-1$ vertices is still one of those listed in [8]. This yields the above possibilities for Γ^a, proving the claim. □

The last result already puts us in a position where we can prove our first theorem. It asserts that for $r \geq 8$, there exists precisely one locally spherical Cartan graph containing an object with a Dynkin diagram of type $A^{(1)}_{r-1}, D^{(2)}_r$ respectively.

Theorem 3.5. *Suppose that $r \geq 8$ and consider the Cartan graph $\mathcal{C} = (I, A, (\rho_i)_{i \in I}, (C^b)_{b \in A})$. Assume that there is an object $a \in A$ such that the associated Dynkin diagram Γ^a is of type $A^{(1)}_{r-1}$ or $D^{(2)}_r$. Then $\mathcal{C}$ is standard.*

Proof. i) Consider the case where Γ^a is of type $A^{(1)}_{r-1}$. We may assume that Γ^a is labelled in the following way:



As $\mathcal{C}$ is connected it is enough to show that $\Gamma^{\rho_i(a)} = \Gamma^a$ for all $i \in I$. Consider $\Gamma^{\rho_r(a)}$ first. By [5, Lemma 4.5] and the definition of Cartan graphs, we see that $c^a_{r,j} = c^{\rho_r(a)}_{r,j}$ for all $j \in I$ different from r; also $c^a_{i,j} = c^{\rho_r(a)}_{i,j}$ for all $i \in I$ such that $c^a_{r,i} = 0$ and arbitrary $j \in J$. Thus, by inspecting Γ^a it follows that each entry of $C^{\rho_r(a)}$ is determined except for the entries

$c^{\rho_r(a)}_{1,r}, c^{\rho_r(a)}_{1,2}, c^{\rho_r(a)}_{r-1,r}, c^{\rho_r(a)}_{r-1,r-2}$. But Proposition 3.2 shows that all these entries must be equal to -1 and thus equal to $c^a_{1,r}, c^a_{1,2}, c^a_{r-1,r-2}, c^a_{r-1,r}$ respectively. This proves that $\Gamma^{\rho_r(a)} = \Gamma^a$.

By symmetry the other cases are dealt with in the same way. We conclude that $\Gamma^{\rho_i(a)} = \Gamma^a$ for all $i \in I$. Hence $\mathcal{C}$ is standard of type $A^{(1)}_{r-1}$.

ii) Now assume that Γ^a is of type $D^{(2)}_r$. We assume that Γ^a is labelled in the following way:

$$\begin{array}{ccccccccccccccccc} \bullet & \Leftarrow & \bullet & — & \bullet & — & \bullet & — & \cdots & — & \bullet & — & \bullet & — & \bullet & \Rightarrow & \bullet \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & & & \alpha_{r-3} & & \alpha_{r-2} & & \alpha_{r-1} & & \alpha_r \end{array}.$$

Since $\mathcal{C}$ is connected, it is enough to show that $\Gamma^{\rho_i(a)} = \Gamma^a$ for each $i \in I$. Because $\mathcal{C}$ is locally spherical of rank $r \geq 8$, the results in [8] show that the parabolic Cartan graphs $\mathcal{C}_{J_1}$, $\mathcal{C}_{J_2}$ corresponding to the index sets $J_1 := I \setminus \{r\}$, $J_2 := I \setminus \{r-1, r\}$ are standard of type B_{r-1}, B_{r-2} respectively.

Hence, the parabolic Dynkin diagram $\Gamma^{\rho_i(a)}_{J_1}$ corresponding to J_1 is of type B_{r-1} for all $1 \leq i \leq r-1$. On the other hand, by inspection of Γ^a we conclude that $\sigma_r|_{\sum_{1\leq i\leq r-2}\mathbb{Z}\alpha_i}$ is the identity map . Hence the parabolic diagram $\Gamma^{\rho_r(a)}_{J_2}$ corresponding to J_2 is of type B_{r-2}.

Defining $J_3 := I \setminus \{1\}$, $J_4 := I \setminus \{1, 2\}$, the same argument shows that for $2 \leq i \leq r$ the parabolic diagram $\Gamma^{\rho_i(a)}_{J_3}$ is of type B_{r-1} while the parabolic diagram $\Gamma^{\rho_1(a)}_{J_4}$ is of type B_{r-2}.

It follows that $\Gamma^{\rho_i(a)}$ is of type $D^{(2)}_r$ for all $i \in I$ and therefore $\mathcal{C}$ is standard of type $D^{(2)}_r$.

□

We continue with the following result for the diagram of type $E^{(1)}_8$.

Theorem 3.6. *Suppose that $\mathcal{C} = (I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ contains an object b such that the Dynkin diagram Γ^b is given as follows:*

$$\Gamma^b = \begin{array}{ccccccccccccccc} & & & & & & & & & & \bullet & & & & \\ & & & & & & & & & & |\ \alpha_9 & & & & \\ \bullet & — & \bullet & — & \bullet & — & \bullet & — & \bullet & — & \bullet & — & \bullet & — & \bullet \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 & & \alpha_8 \end{array}.$$

Then for each object $a \in A$, it holds that $\Gamma^a = \Gamma^b$. Thus, the Cartan graph $\mathcal{C}$ is standard of type $E^{(1)}_8$.

Proof. Let a be an arbitrary object of $\mathcal{C}$. As $\mathcal{C}$ is connected and simply connected, there exists a unique morphism ω_a which takes b to a. We will prove the claim by induction on the length of ω_a.

- Assume that $\omega_a = \sigma^b_i$ for some $i \in I$. As $\mathcal{C}$ is locally spherical, we conclude that the parabolic root set R^b_J corresponding to $J := I \setminus \{1\}$

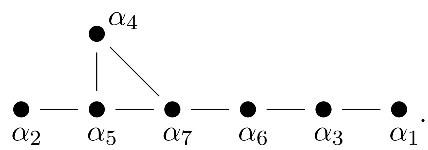
is standard of type E_8 (this follows from the classification results given in [8]). Recall that for $k,l \in I$ we have $c^b_{k,l} = c^{\rho_k(b)}_{k,l}$; further for $j,k,l \in I$ such that $c^b_{j,k} = 0$ we have $c^b_{k,l} = c^{\rho_j(b)}_{k,l}$. Using this together with Proposition 3.2, we may conclude that $\Gamma^{\rho_i(b)} = \Gamma^b$ for $i \neq 1$. We observe that the same rules combined with Proposition 3.2 also yield $\Gamma^{\rho_1(b)} = \Gamma^b$.

- Let $n \geq 2$ and assume that the claim holds for all morphisms of length less than n. Assume that $\omega_a = \sigma_j \sigma_{i_{n-1}} ... \sigma^b_{i_1}$. We define $b' := \rho_{i_{n-1}} ... \rho_{i_1}(b)$ and observe that by induction $\Gamma^{b'} = \Gamma^b$. By the same reasoning as above, we conclude that $\Gamma^a = \Gamma^{\rho_j(b')} = \Gamma^{b'} = \Gamma^b$.

□

In order to deal with Cartan graphs containing an object with Dynkin diagram of type $E_7^{(1)}$, we need the following lemma, which describes the behaviour of occurring parabolic subarrangements. The proof is immediate via the results obtained in [8]

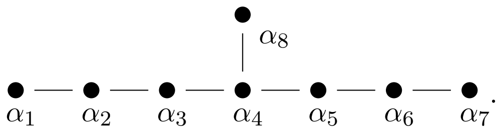
Lemma 3.6. *Let $\mathcal{C} = (I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ denote the uniquely determined finite Cartan graph containing an object b with the following Dynkin diagram:*

$\Gamma^b =$ 

Then $\mathcal{C}$ contains eight different root sets $R_1, ..., R_8$ with associated Dynkin diagrams $\Gamma_1, ..., \Gamma_8$. We may order the root sets in such a way that the following holds:
a) $\Gamma_i = E_7$ for $1 \leq i \leq 4$, $\Gamma_5 = \Gamma^b$, $\Gamma_6 = \Gamma_7 = \Gamma_{D_7}$ and $\Gamma_8 = \Gamma_{A_7}$.
b) $\sigma_1(R_1) = R_2$ and $\sigma_i(R_1) = R_1$ for $i \neq 1$.
c) $\sigma_1(R_2) = R_1$ and $\sigma_3(R_2) = R_3$, $\sigma_i(R_2) = R_2$ for $i \neq 1,3$.
d) $\sigma_3(R_3) = R_2$ and $\sigma_6(R_3) = R_4$, $\sigma_i(R_3) = R_3$ for $i \neq 3,6$.
e) $\sigma_6(R_4) = R_3$ and $\sigma_7(R_4) = R_5$, $\sigma_i(R_4) = R_4$ for $i \neq 6,7$.
f) $\sigma_7(R_5) = R_4$, $\sigma_5(R_5) = R_6$ and $\sigma_4(R_5) = R_8$, $\sigma_i(R_5) = R_5$ for $i \neq 4,5,7$.
g) $\sigma_5(R_6) = R_5$ and $\sigma_2(R_6) = R_7$, $\sigma_i(R_6) = R_6$ for $i \neq 2,5$.
h) $\sigma_2(R_7) = R_6$, $\sigma_i(R_7) = R_7$ for $i \neq 2$.
i) $\sigma_4(R_8) = R_5$, $\sigma_i(R_8) = R_8$ for $i \neq 4$.

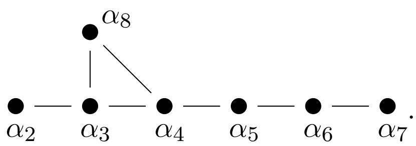
Now we can prove the following theorem which takes care of Cartan graphs containing an object whose Dynkin diagram is of type $E_7^{(1)}$:

Theorem 3.7. *Let $\mathcal{C} = (I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan graph containing an object $b \in A$ such that Γ^b is of type $E_7^{(1)}$ with the labelling given as follows:*

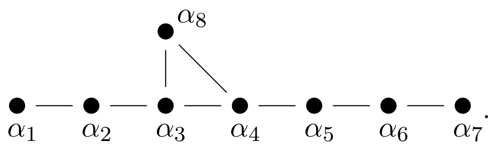
$\Gamma^b =$ 

Then $\Gamma^a = \Gamma^b$ for every $a \in A$. Thus, $\mathcal{C}$ is standard of type $E_7^{(1)}$.

Proof. Let a be an object of $\mathcal{C}$. We first show that $\Gamma^b = \Gamma^a$ automatically implies that the parabolic root set R^a_J corresponding to $J := I \setminus \{1\}$ is standard of type E_7. To see this, observe that the classification results in the finite case imply that there are two possibilities for R^a_J: either it is standard of type E_7 or it is given by some root set whose associated finite Cartan graph contains an object with the following Dynkin diagram:



For this we refer to the results in [8]. But then by Lemma 3.6, $\mathcal{C}$ itself would contain an object with the following Dynkin diagram:



However, Proposition 3.2 tell that there is no such diagram appearing in a Cartan graph as above. Hence R^a_J must be standard of type E_7. Using this and the fact that $\mathcal{C}$ is connected and simply connected, we will prove the claim by induction on the length of the unique morphism ω_a which takes b to a.

- Assume that $\omega_a = \sigma^b_j$ for some $j \in I$. We use that for $k, l \in I$ we have $c^b_{k,l} = c^{\rho_k(b)}_{k,l}$; further for $j, k, l \in I$ such that $c^b_{j,k} = 0$ we have $c^b_{k,l} = c^{\rho_j(b)}_{k,l}$. These rules show that $\Gamma^{\rho_1(b)} = \Gamma^b$. For $j \neq 1$ we may also use that the parabolic root set R^b_J corresponding to $J = I \setminus \{1\}$ is standard of type E_7. We obtain $\Gamma^{\rho_j(b)} = \Gamma^b$ for all $j \in I$.

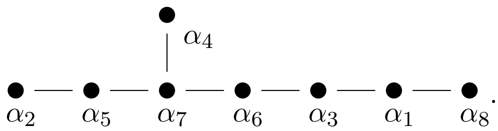
- Let $n \geq 2$ and assume that the claim holds for all morphisms of length less than n. Consider the morphism $\omega_a := \sigma_j \sigma_{i_{n-1}} ... \sigma^b_{i_1}$ and define $b' := \rho_{i_{n-1}} ... \rho_{i_1}(b)$. Then by induction we have $\Gamma^{b'} = \Gamma^b$. Thus, by the argument given at the beginning of the proof, we know that the parabolic root set $R_J^{b'}$ corresponding to $J = I \setminus \{1\}$ is standard of type E_7. As above, we may conclude that $\Gamma^a = \Gamma^{\rho_j(b')} = \Gamma^{b'} = \Gamma^b$ for all $j \in I$. This completes the proof.

□

We continue by classifying those locally spherical Cartan graphs containing an object whose Dynkin diagram is of type E_8. For this, we need the following lemma.

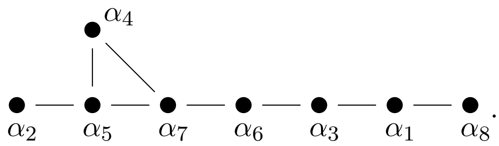
Lemma 3.7. *Let $\mathcal{C} = (I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a non-standard Cartan graph containing an object b of Dynkin type E_8 with labelling given as follows:*

$\Gamma^b =$



Then $\mathcal{C}$ also contains an object a such that Γ^a is of Dynkin type $\tilde{D}'_8$ with labelling given as follows:

$\Gamma^a =$



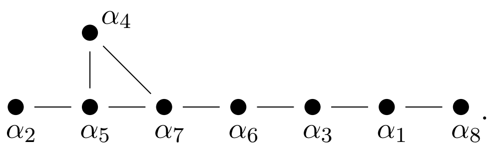
Proof. Let c be an object such that $\Gamma^c = \Gamma^b$. Then the parabolic root set at c corresponding to $J := I \setminus \{8\}$ will be either standard of type E_7 or it will be given by one of the root sets R_1, R_2, R_3, R_4 introduced in Lemma 3.6. If we are in the latter case then the claim follows by the same lemma. Now suppose that for any object c such that $\Gamma^c = \Gamma^b$ the parabolic root set R_J^c is standard of type E_7. Then $\mathcal{C}$ itself is standard of type E_8. But by assumption $\mathcal{C}$ is a non-standard Cartan graph. This contradiction proves the claim. □

Next we prove that there is at most one locally spherical Cartan graph containing an object of Dynkin type $\tilde{D}'_8$. Together with the fact that there is a restriction of $E_8^{(1)}$ containing such an object, this will yield the desired result.

Proposition 3.3. *Let $\mathcal{C} = (I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan graph. If $\mathcal{C}$ contains an object of Dynkin type $\tilde{D}'_8$, then $\mathcal{C}$ is uniquely determined.*

Proof. Suppose that a is an object of $\mathcal{C}$ with Dynkin diagram

$\Gamma^a =$



We set $J := I \setminus \{8\}$ and we use the root sets $R_1, ..., R_8$ as introduced in Lemma 3.6. It is enough to prove the following three statements:
i) If $i > 1$ and if b is an object such that $R_J^b = R_i$, then $R_J^{\rho_8(b)} = R_i$.
ii) If $R_J^b = R_1$, then $R_J^{\rho_8(b)} = E_7$.
iii) If $R_J^b = E_7$, then $R_J^{\rho_8(b)} = R_1$.
We start with statement i): for $i = 5$ the claim is an immediate consequence of Lemma 3.6. Now consider $i = 4$. By Lemma 3.6 we have $\sigma_8\sigma_7^b = \sigma_7\sigma_8^b$ and therefore $\sigma_8^b(R_J^b) = \sigma_7\sigma_8\sigma_7^b(R_4) = \sigma_7\sigma_8(R_5) = \sigma_7(R_5) = R_4$. Using a similar argument one reduces the claim for $i = 3$ to the already established case for $i = 4$. And similarly the claim for $i = 2$ reduces to the claim for $i = 3$. Now consider $i = 6$. Again it holds that $\sigma_8\sigma_5^b = \sigma_5\sigma_8^b$ and therefore $\sigma_8^b(R_J^b) = \sigma_5\sigma_8\sigma_5^b(R_6) = \sigma_5\sigma_8(R_5) = \sigma_5(R_5) = R_6$. As above the claim for $i = 7$ reduces to the claim for $i = 6$. Hence in order to establish claim i) it remains to consider $i = 8$. Again $\sigma_8\sigma_4^b = \sigma_4\sigma_8^b$ and therefore $\sigma_8^b(R_J^b) = \sigma_4\sigma_8\sigma_4^b(R_8) = \sigma_4\sigma_8(R_5) = \sigma_4(R_5) = R_8$. Next we turn to statement ii): we need to show that $R_J^b = R_1$ implies $R_J^{\rho_8(b)} = E_7$. We have $\sigma_1\sigma_8\sigma_1^b = \sigma_8\sigma_1\sigma_8^b$. Hence $\sigma_1\sigma_8\sigma_1^b(R_1) = \sigma_1\sigma_8(R_2) = \sigma_1(R_2) = R_1$ and so $R_1 = \sigma_8\sigma_1\sigma_8^b(R_1)$. Now suppose that $\sigma_8^b(R_1) = R_j$ for some $j \in \{1, 2, 3, 4\}$. If $j > 1$ then by Lemma 3.6 we conclude $\sigma_8\sigma_1\sigma_8^b(R_1) = \sigma_8\sigma_1(R_j) = \sigma_8(R_j) = R_j$, contradicting the equation $R_1 = \sigma_8\sigma_1\sigma_8^b(R_1)$. On the other hand $j = 1$ implies that $R_1 = \sigma_8\sigma_1\sigma_8^b(R_1) = \sigma_8\sigma_1(R_1) = \sigma_8(R_2) = R_2$, a contradiction. This shows that $\sigma_8^b(R_1)$ must be standard of type E_7. Statement iii) is proven similarly. Hence the proof is finished. $\square$

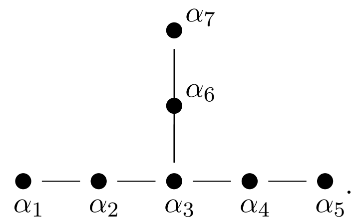
With the last proposition we can prove the following result:

Theorem 3.8. *Let $\mathcal{C} = (I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan graph. Assume that there is an object b such that Γ^b is of finite type E_8. Then $\mathcal{C}$ is either standard of type E_8 or it corresponds to the restriction of the affine reflection arrangement $E_8^{(1)}$ to a hyperplane.*

Proof. We may assume that $\mathcal{C}$ is a non-standard Cartan graph. Let $\mathcal{A}'$ be the crystallographic arrangement obtained as the restriction of the affine reflection arrangement $E_8^{(1)}$ to a hyperplane and denote by $\mathcal{C}'$ the corresponding Cartan graph. By Lemma 3.7 and Proposition 3.3, we conclude that $\mathcal{C}$ is uniquely determined by the fact that it is non-standard Cartan graph containing an object a such that Γ^a is of type E_8. Moreover, one checks that $\mathcal{C}'$ is an example of such a non-standard Cartan graph. This proves the claim. $\square$

We close this section with the following result, which asserts that there is only one locally spherical Cartan graph containing an object whose Dynkin diagram is of type $E_6^{(1)}$.

Theorem 3.9. *Let $\mathcal{C} = (I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan graph. Assume that there exists an object b such that Γ^b is given as follows:*

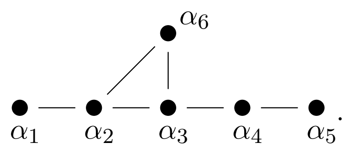
$\Gamma^b =$  (3.2)

Then we have $\Gamma^a = \Gamma^b$ for every object a. Hence, $\mathcal{C}$ is standard of type $E_6^{(1)}$.

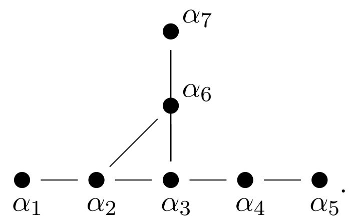
Proof. Using similar arguments as in the proofs before, we observe that it is enough to show that for every object a such that Γ^a is as in (3.2), the parabolic root set R^a_J corresponding to $J := I \setminus \{7\}$ is standard of type E_6. By the results in [8], we have the following possibilities for R^a_J and the theorem is proved once we have excluded the last three of them:

- Case 1: R^a_J is the standard root set of type E_6.
- Case 2: R^a_J is one of the root sets appearing in the Cartan graph of rank six with Nr.2 given in [8].
- Case 3: R^a_J is one of the root sets appearing in the Cartan graph of rank six with Nr.3 given in [8].
- Case 4: R^a_J is one of the root sets appearing in the Cartan graph of rank six with Nr.4 given in [8].

Case 2: Denote the Cartan graph corresponding to R^a_J by $\mathcal{C}$. Then $\mathcal{C}$ contains seven different root sets $R_1, ..., R_7$ with corresponding Dynkin diagrams Γ_i. The results in [8] show that we may label the R_i in such a way that the following holds:
i) $\Gamma_1, \Gamma_2, \Gamma_3$ are the only diagrams of type E_6. These are given by the graph in (3.2) with the vertex α_7 removed.

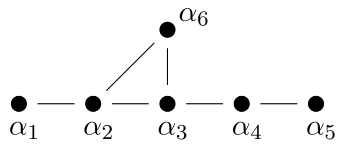
ii) The Dynkin diagram Γ_4 is given as follows: 

iii) $\sigma_5(R_1) = R_2; \sigma_4(R_2) = R_3; \sigma_3(R_3) = R_4$.
By the above we conclude that there exists a morphism ω mapping b to some object a with the following diagram:

$\Gamma^a =$  (3.3)

However, Γ^a contains a forbidden subgraph (remove the vertex α_5). For this, remember that $\mathcal{C}$ is locally spherical and examine the list of possible Dynkin diagrams in the finite case given in [8]. Thus, Case 2 cannot occur.

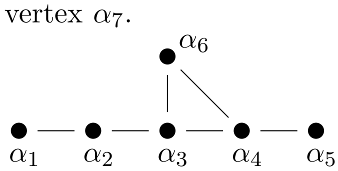
Case 3: The argument is similar to Case 2. Denote the Cartan graph corresponding to R^a_J by $\mathcal{C}$ and observe that there are 14 different root sets $R_1, ..., R_{14}$ with corresponding Dynkin diagrams $\Gamma_1, ..., \Gamma_{14}$ occurring in $\mathcal{C}$. The work in [8] shows that we may label them in such a way that the following statements hold:
i) $\Gamma_1, \Gamma_2, \Gamma_3$ are the only diagrams of type E_6. These are given by the graph in (3.2) with the vertex α_7 removed.

ii) The Dynkin diagram Γ_4 is given as follows: 

.

iii) $\sigma_5(R_1) = R_2; \sigma_4(R_2) = R_3; \sigma_3(R_3) = R_4$.
As in Case 2, this yields an object a whose diagram is the same as in (3.3). Again, this is not possible.

Case 4: Again, the argument is similar to the preceding cases. We denote the Cartan graph corresponding to R^a_J by $\mathcal{C}$ and observe that in this case, we have 21 different root sets $R_1, ..., R_{21}$ and associated Dynkin diagrams $\Gamma_1, ..., \Gamma_{21}$ which appear in $\mathcal{C}$. This time, we label them such that the following holds:
i) $\Gamma_4, \Gamma_5, \Gamma_6$ are the only diagrams of type E_6 that appear in $\mathcal{C}$. These are obtained from the graph (3.2) by removing the vertex α_7.

ii) The Dynkin diagram Γ_3 is given as follows: 

.

iii) $\sigma_1(R_6) = R_4; \sigma_2(R_5) = R_4; \sigma_3(R_4) = R_3$. As above, this yields a forbidden diagram. Hence, Case 4 cannot occur as well. $\square$

3.4 Open problems and related questions

In this section, we collect two interesting open problems related to the results obtained in this chapter.

Motivated by Theorem 3.8, we start with the following one, which asks to classify locally spherical crystallographic Tits arrangements containing a chamber whose associated Dynkin diagram is of type E_6 or E_7. We state the problem in terms of Cartan graphs.

Problem 1. *We ask to classify all locally spherical connected simply connected Cartan graphs $\mathcal{C}$ containing an object b whose Dynkin diagram Γ^b is* *given by one of the following:*

$$\Gamma = \begin{array}{ccccccccc} & & & & \bullet & & & & \\ & & & & \vert\ \alpha_6 & & & & \\ \underset{\alpha_1}{\bullet} & - & \underset{\alpha_2}{\bullet} & - & \underset{\alpha_3}{\bullet} & - & \underset{\alpha_4}{\bullet} & - & \underset{\alpha_5}{\bullet} \end{array},$$

$$\Gamma' = \begin{array}{ccccccccccc} & & & & \bullet & & & & & & \\ & & & & \vert\ \alpha_7 & & & & & & \\ \underset{\alpha_1}{\bullet} & - & \underset{\alpha_2}{\bullet} & - & \underset{\alpha_3}{\bullet} & - & \underset{\alpha_4}{\bullet} & - & \underset{\alpha_5}{\bullet} & - & \underset{\alpha_6}{\bullet} \end{array}.$$

Compared to the situation of Theorem 3.8, the problem appears to be much more difficult. This comes from the fact that there are a lot more possible parabolic subarrangements in ranks five and six (see the results in [8]). Thus, a potential solution has to address many different cases.

We finish the chapter with the following problem, which asks for bounds on the number of different root sets that may occur in a finite reflection groupoid associated to some spherical Tits arrangement.

Problem 2. *Let $r \geq 3$ and let $\mathcal{A}$ be a spherical Tits arrangements in $\mathbb{R}^r$ which has a realization over some (fixed) algebraic number field $\mathbb{K}$. Assume that $\mathcal{A}$ admits a finite reflection groupoid and let K_0 be a chamber of $\mathcal{A}$ together with a corresponding basis B_0. Do there exist positive real numbers m_0, ϵ_0 such that $|\mathcal{R}(\mathcal{A})_{B_0}| \leq m_0|\mathcal{A}|^{1-\epsilon_0}$ for any arrangement $\mathcal{A}$ as above?*

If this was true, then by Theorem 3.4 we could conclude that there are only finitely many combinatorial isomorphism classes of spherical Tits arrangements satisfying the conditions in Problem 2.

Chapter 4

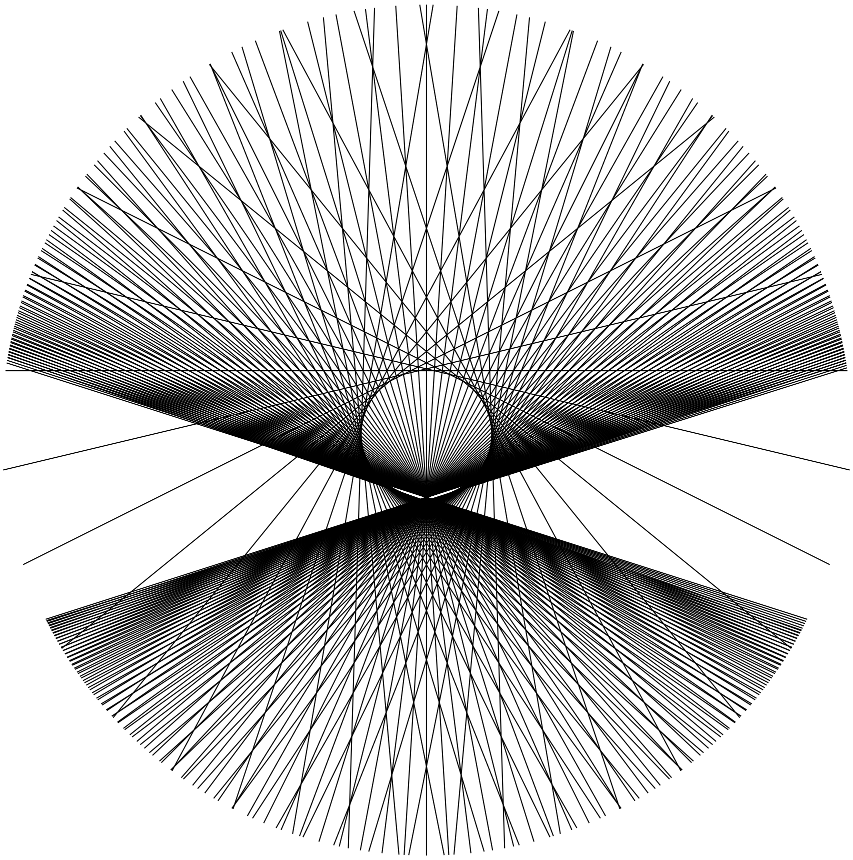
Affine Tits arrangements on cubic curves

It is an important observation that after choosing suitable coordinates, the dual point sets corresponding to simplicial arrangements from the infinite series $\mathcal{R}(1)$ are all contained in the locus of a (reducible) homogeneous cubic polynomial. More precisely, each point set is contained in the union of an irreducible conic σ and some line not meeting σ.
Motivated by this, we ask for possible *infinite* simplicial arrangements whose dual point sets are contained in a cubic curve. More precisely, we shall be interested in *affine rank three Tits arrangements* having this property.

In this chapter, we give a classification of such arrangements. Our strategy for the classification builds upon the results obtained in [12] and on elementary results from the geometry of the projective plane, like Bézout's theorem and the fact that the irreducible conic in $\mathbb{P}^2(\mathbb{R})$ is a self-dual curve. Indeed, if one chooses coordinates such that σ is given by the equation $x^2 + y^2 - z^2 = 0$, then the corresponding dual curve σ' is given by the same equation. Here, by the *dual curve* of a curve C in $\mathbb{P}^2(\mathbb{R})$ we mean the curve C' in $(\mathbb{P}^2(\mathbb{R}))^*$ consisting of all points which are dual to tangent lines of C.

We find that there are only two classes of irreducible affine Tits arrangements satisfying the above property: namely the arrangement of type $A_2^{(1)}$ whose corresponding dual point set is contained in the union of three projective lines, and a class of arrangements which we call $\tilde{A}_2^0$ (see Figure 4.1). The dual point set of $\tilde{A}_2^0$ is contained in the union of a projective conic σ and a projective line $\mathfrak{l}$ *touching* σ. It turns out that the arrangement $\tilde{A}_2^0$ is an example of an irreducible affine Tits arrangement which is not locally spherical. More precisely, we have the following main theorem:

Theorem. *Let the pair $(\mathcal{A}, T)$ be an affine rank three Tits arrangement and assume that the projective root vectors of $\mathcal{A}$ are contained in the locus of a homogeneous cubic polynomial. Then $\mathcal{A}$ is either a near pencil, an arrangement of type $A_2^{(1)}$, or it is an arrangement of type $\tilde{A}_2^0$.*



A subset of the arrangement of type $\tilde{A}_2^0$

Figure 4.1

This result is established by proving Theorem 4.2 in Section 4.1. In Section 4.2 we discuss some related open questions.

4.1 The classification

In order to prove the main theorem of this chapter, we use the following notation:

Notation. Let $(\mathcal{A}, T)$ be a Tits arrangement of rank three. By abuse of notation we denote the set of projective lines $\{\mathfrak{g} \mid \exists H \in \mathcal{A} : \mathfrak{g} = \pi(H)\}$ by $\mathcal{A}$ as well; here $\pi : \{U \leq \mathbb{R}^3 \mid \dim U \geq 1\} \longrightarrow \{U \leq \mathbb{P}^2(\mathbb{R})\}$ is the natural projection. If $p \in (\mathbb{P}^2(\mathbb{R}))^*$ then we denote the corresponding dual line by $p^* \subset \mathbb{P}^2(\mathbb{R})$. Likewise, if $\mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$ is a projective line, then its corresponding dual point is denoted by $\mathfrak{l}^* \in \mathbb{P}^2(\mathbb{R})$. Similarly, if $\mathcal{A}$ is a set of projective lines in $\mathbb{P}^2(\mathbb{R})$, we write $\mathcal{A}^* \subset (\mathbb{P}^2(\mathbb{R}))^*$ for the corresponding set of dual projective points (and vice versa).

The main strategy of the proof can be summarized as follows: according to the possible factorizations of a homogeneous cubic polynomial P, there are naturally three cases to consider. Namely, P may factor as a product of three linear polynomials, or it may factor as a product of an irreducible quadratic polynomial and a linear polynomial, or P may be irreducible. We examine all three cases and collect all (up to projectivity) affine Tits arrangements $\mathcal{A}$ such that $\mathcal{A}^* \subset V(P)$.

We start with the following lemma which will be used extensively to rule out the possibility of existence of certain Tits arrangements. It basically says that near pencils are the only rank three Tits arrangements containing a segment bounded by two vertices of weight two.

Lemma 4.1. *Let $\mathcal{A}$ be an Tits arrangement of rank three. Suppose there is a line $\mathfrak{g} \in \mathcal{A}$ containing two vertices v_1, v_2 of weight two such that there is no other vertex contained in the bounded segment between v_1 and v_2 on $\mathfrak{g}$. Then $\mathcal{A}$ is a near pencil.*

Proof. Denote by $\mathfrak{g}_1, \mathfrak{g}_2$ the two lines meeting $\mathfrak{g}$ in v_1 respectively v_2 and set $v := \mathfrak{g}_1 \cap \mathfrak{g}_2$. Using $w(v_1) = w(v_2) = 2$ it follows that there are two chambers with vertices v_1, v_2, v and it is easy to see that every line $\mathfrak{g}' \in \mathcal{A} \setminus \{\mathfrak{g}\}$ needs to pass through v. □

We state two further lemmas, which will turn out to be useful and may be interesting in their own right.

Lemma 4.2. *Let $\mathcal{A}$ be an affine Tits arrangement of rank three. Then there is at most one vertex of $\mathcal{A}$ contained in ∂T.*

Proof. Suppose there were two vertices $v \neq w \in \partial T$. Then there is a chamber K having v as a vertex. As $\mathcal{A}$ is thin, it follows that K has to be contained in the cone C generated by two neighbouring lines passing through v. As the lines passing through w accumulate at ∂T we conclude that there are infinitely many lines passing through w and intersecting K, a contradiction. □

Lemma 4.3. *Let $\mathcal{A}$ be a Tits arrangement of rank three. Suppose there is a vertex v of weight two which is surrounded by neighbouring vertices v_1, v_2, v_3, v_4 of weight three. Then $\mathcal{A}$ is spherical and $|\mathcal{A}| \in \{6, 7\}$.*

Proof. We denote the lines intersecting in v by $\mathfrak{g}_1, \mathfrak{g}_2$ and we agree that $v_1, v_3 \in \mathfrak{g}_1$ while $v_2, v_4 \in \mathfrak{g}_2$. It is clear that there are no further vertices lying in the bounded segment between v_1 and v_4 and the same is true for the bounded segments between v_1 and v_2, v_2 and v_3, v_3 and v_4. Denote the line passing through v_i and v_j by $\mathfrak{g}_{i,j}$ and observe that the spherical arrangement $\mathcal{B} \subset \mathcal{A}$ defined by $\mathcal{B} := \{\mathfrak{g}_{i,j} \mid 1 \leq i, j \leq 4\}$ is simplicial. Now by the above there are cells $K_{i,j}$ of $\mathcal{A}$ containing the vertices v_i and v_j for $\{i,j\} \in \{\{1,4\},\{1,2\},\{2,3\},\{3,4\}\}$ and these cells are triangles. Suppose there was a line in $\mathcal{A}$ not contained in $\mathcal{B}$ supporting an edge of such a cell $K_{i,j}$. This edge needs to pass through either v_i or v_j. But then the weight of either v_i or v_j needs to be strictly greater than three, contradicting our assumption. This shows that the only line one may add to $\mathcal{B}$ in such a way that the obtained arrangement is simplicial with the vertices v_1, v_2, v_3, v_4 having weight three is the line passing through the points $\mathfrak{g}_{1,2} \cap \mathfrak{g}_{3,4}$ and $\mathfrak{g}_{1,4} \cap \mathfrak{g}_{2,3}$. □

The next proposition and the following theorem are preliminary results which will be used to simplify the proof of Proposition 4.2.

Proposition 4.1. *Let $\mathcal{A}$ be a Tits arrangement of rank three. Assume that $\mathcal{A}^*$ is contained in the union of two lines. Then $\mathcal{A}$ is a near pencil.*

Proof. Suppose $\mathcal{A}^* \subset \mathfrak{l}_1 \cup \mathfrak{l}_2$. Then after dualizing the lines $\mathfrak{l}_1, \mathfrak{l}_2 \subset (\mathbb{P}^2(\mathbb{R}))^*$ become two points $v_1, v_2 \in \mathbb{P}^2(\mathbb{R})$. Suppose that $w(v_1) = |\mathcal{A}_{v_1}| \geq 3$ and pick a line $\mathfrak{g}$ of $\mathcal{A}$ such that $v_1 \notin \mathfrak{g}$ and $v_2 \in \mathfrak{g}$. Observe that there is at most one line in $\mathcal{A}_{v_1}$ meeting $\mathfrak{g}$ in a vertex of weight greater than two. Further, different lines in $\mathcal{A}_{v_1}$ produce different intersections with $\mathfrak{g}$. Since $\mathfrak{g}$ contains only one vertex with weight possibly bigger than two, we may use Lemma 4.1 to conclude that $\mathcal{A}$ is a near pencil. If on the other hand $w(v_1) = 2$ then we choose $\mathfrak{g} \in \mathcal{A}$ passing through v_1 but not through v_2. Then $\mathfrak{g}$ contains a segment bounded by two vertices of weight two. Hence by Lemma 4.1 it follows that $\mathcal{A}$ is a near pencil. □

Remark 4.1. We observe that in the situation of the last proposition there is a unique $p \in \mathcal{A}^*$ such that $\mathcal{A}^* \setminus \{p\}$ is contained in one of the two lines $\mathfrak{l}_1, \mathfrak{l}_2$ while p is contained in the other one.

Theorem 4.1. *The near pencil is the only Tits arrangement $\mathcal{A}$ of rank three such that $\mathcal{A}^*$ lies on a conic.*

Proof. Let $P \in \mathbb{R}[x,y,z]$ be a homogeneous polynomial of degree two and set $\sigma := V(P)$. Suppose that $\mathcal{A}^* \subset \sigma \subset (\mathbb{P}^2(\mathbb{R}))^*$ for some rank three Tits arrangement $\mathcal{A}$. First, assume that P is the product of two distinct linear polynomials. Then by Proposition 4.1 the only Tits arrangements lying on σ are near pencils. If P splits as a square of a linear polynomial, then every $p \in \mathcal{A}^*$ lies on a single line which means that all lines of $\mathcal{A}$ pass through a single point. Hence $\mathcal{A}$ is not simplicial. Now, finally suppose that P is irreducible. By Bézout's theorem every line meets σ in at most two points. Hence the weight of any vertex of $\mathcal{A}$ is bounded by two. But this implies that $\mathcal{A}$ is a near pencil consisting of three lines. □

The next proposition is a first step towards our main theorem.

Proposition 4.2. *Let $\mathcal{A}$ be an affine Tits arrangement of rank three. Suppose that $\mathcal{A}^*$ is contained in the union of at most three lines. Then $\mathcal{A}$ is either a near pencil or it is an arrangement of type $A_2^{(1)}$.*

Proof. Taking into account Theorem 4.1 it is enough to consider the case where $\mathcal{A}^*$ is contained in the union of exactly three lines: $\mathcal{A}^* \subset \mathfrak{l}_1 \cup \mathfrak{l}_2 \cup \mathfrak{l}_3 \subset (\mathbb{P}^2(\mathbb{R}))^*$. We define $v_1 := \mathfrak{l}_1^*, v_2 := \mathfrak{l}_2^*, v_3 := \mathfrak{l}_3^* \in \mathbb{P}^2(\mathbb{R})$ and consider two cases:
a) Suppose that $\mathfrak{l}_1 \cap \mathfrak{l}_2 \cap \mathfrak{l}_3 = w \in (\mathbb{P}^2(\mathbb{R}))^*$. Then the corresponding points v_1, v_2, v_3 all lie on the line $w^* \subset \mathbb{P}^2(\mathbb{R})$. If $|\mathcal{A}_{v_i}| = |\mathcal{A}_{v_j}| = \infty$ for two different values i, j, then we have $w^* = \partial T$ because $\mathcal{A}$ is locally finite in T.

Let $k \neq i, j$ and assume that $|\mathcal{A}_{v_k}| < \infty$. Then it is easy to see that $\mathcal{A}$ contains a segment bounded by two vertices of weight two. By Lemma 4.1 we may assume that $|\mathcal{A}_{v_k}| = \infty$. Observe that all vertices in T have weight bounded by three. But since $\mathcal{A}$ is not spherical, Lemma 4.3 shows that every vertex in T has weight exactly three. From this it is easy to see that the arrangement $\mathcal{A}$ is of type $A_2^{(1)}$.

Now suppose that there is precisely one i such that $|\mathcal{A}_{v_i}| = \infty$ and pick a line $\mathfrak{g} \in \mathcal{A}$ such that $v_j \in \mathfrak{g}, v_i \notin \mathfrak{g}$ for some $j \neq i$. Then it is easy to see that $\mathfrak{g}$ contains a segment bounded by two vertices of weight two. Hence by Lemma 4.1 the arrangement $\mathcal{A}$ is a near pencil.
b) Assume $\mathfrak{l}_1 \cap \mathfrak{l}_2 \cap \mathfrak{l}_3 = \emptyset$. Then the three points v_1, v_2, v_3 are not collinear. Hence it is impossible to have $|\mathcal{A}_{v_i}| = \infty$ for all $1 \leq i \leq 3$. But then we may assume that $|\mathcal{A}_{v_1}| = \infty$ and $|\mathcal{A}_{v_3}| < \infty$. Now if $|\mathcal{A}_{v_2}| < \infty$ as well, then we may argue as in case a) to show that $\mathcal{A}$ is a near pencil. So assume that $|\mathcal{A}_{v_1}| = |\mathcal{A}_{v_2}| = \infty$ and $|\mathcal{A}_{v_3}| < \infty$. Again, we may argue as in case a) to conclude that $\mathcal{A}$ is a near pencil. $\square$

Remark 4.2. If we drop the condition on $\mathcal{A}$ to be affine, then we find some more possible (spherical) arrangements such that A^* is contained in the union of three lines: for instance the arrangement of type $A(10, 3)$ (as denoted in [19]) and some of its subarrangements.

Our next goal is to show that there is no affine Tits arrangement $\mathcal{A}$ such that $\mathcal{A}^*$ is contained in the locus of an irreducible homogeneous cubic polynomial. This may be deduced from Lemma 4.3 and the following result:

Lemma 4.4. *Let $\mathcal{A}$ be an affine Tits arrangement of rank three. Assume that every vertex of $\mathcal{A}$ has weight three and suppose that $\mathcal{A}^* \subset V(F)$ for some homogeneous cubic polynomial F. Then F is not irreducible.*

Proof. Consider the affine space $\mathbb{E} := \mathbb{P}^2(\mathbb{R}) \setminus \{\partial T\}$ and look at the arrangement induced by $\mathcal{A}$ on $\mathbb{E}$; by abuse of notation we denote this arrangement by $\mathcal{A}$ as well.

Fix a line $\mathfrak{g}$ of the arrangement $\mathcal{A}$. Denote by $\mathcal{A}_\mathfrak{g}$ the set of all lines in $\mathcal{A}$ which are not parallel to $\mathfrak{g}$ and set $\mathcal{A}' := \mathcal{A}_\mathfrak{g} \cup \{\mathfrak{g}\} \subset \mathcal{A}$. Observe that if

$\mathfrak{g}'$ is a line of $\mathcal{A}$ parallel to $\mathfrak{g}$, then $\mathcal{A}_\mathfrak{g} = \mathcal{A}_{\mathfrak{g}'}$. Assume that there is a vertex v of $\mathcal{A}'$ of weight three not lying on $\mathfrak{g}$. We may choose v in such a way that there is a line $\mathfrak{g}_v \in \mathcal{A}'$ passing through v such that the bounded segment on $\mathfrak{g}_v$ reaching from v to $\mathfrak{g} \cap \mathfrak{g}_v$ does not contain any other vertex of $\mathcal{A}'$ of weight three. We say that v has *distance* k to $\mathfrak{g}$ if k is minimal with the property that there is a line $\mathfrak{g}' \in \mathcal{A}'$ such that the interior of the bounded segment on $\mathfrak{g}'$ reaching from v to $\mathfrak{g}' \cap \mathfrak{g}$ contains exactly k vertices of $\mathcal{A}'$ all of which have weight two.

Let us first consider the case where v has distance zero to $\mathfrak{g}$. We will show that then there must be a vertex of $\mathcal{A}$ of weight three, contradicting our assumption on $\mathcal{A}$. There are two possibilities: either there are two lines $\mathfrak{g}_1, \mathfrak{g}_2$ passing through v such that there is no vertex of weight two of $\mathcal{A}'$ contained in the bounded segments reaching from v to $\mathfrak{g} \cap \mathfrak{g}_1, \mathfrak{g} \cap \mathfrak{g}_2$ respectively, or there is only one such line. Consider the first possibility. Let $\mathfrak{g}_1, \mathfrak{g}_2$ be as above and denote by $\mathfrak{g}_3$ the third line passing through v. Similarly, denote by $\mathfrak{g}_4$ the third line passing through $\mathfrak{g} \cap \mathfrak{g}_2$ and assume that the bounded segment s on $\mathfrak{g}_3$ reaching from v to $\mathfrak{g} \cap \mathfrak{g}_3$ contains the vertex $\mathfrak{g}_3 \cap \mathfrak{g}_4$. Using the fact that there can be only finitely many lines of $\mathcal{A}'$ passing through the bounded segment on $\mathfrak{g}$ reaching from $\mathfrak{g} \cap \mathfrak{g}_3$ to $\mathfrak{g} \cap \mathfrak{g}_2$, we see that $\mathcal{A}'$ has a vertex w of weight two contained in s. As by assumption every vertex of $\mathcal{A}$ has weight three, there must be a line $\mathfrak{g}_0 \in \mathcal{A}$ passing through w which is parallel to $\mathfrak{g}$. But then there is a vertex of weight two of $\mathcal{A}$ contained in the line $\mathfrak{g}_2$, a contradiction. Now we deal with the second possibility. Denote the three lines passing through v again by $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$. We may assume that the bounded segment on $\mathfrak{g}_1$ reaching from v to $\mathfrak{g} \cap \mathfrak{g}_1$ does not contain any vertices of $\mathcal{A}'$. Moreover, we may assume that $\mathfrak{g} \cap \mathfrak{g}_1$ is not contained in the bounded segment on $\mathfrak{g}$ reaching from $\mathfrak{g} \cap \mathfrak{g}_2$ to $\mathfrak{g} \cap \mathfrak{g}_3$ and that $\mathfrak{g} \cap \mathfrak{g}_3$ is not contained in the bounded segment on $\mathfrak{g}$ reaching from $\mathfrak{g} \cap \mathfrak{g}_1$ to $\mathfrak{g} \cap \mathfrak{g}_2$. Again, using the fact that there are only finitely many lines of $\mathcal{A}'$ passing through the bounded segment on $\mathfrak{g}$ reaching from $\mathfrak{g} \cap \mathfrak{g}_1$ to $\mathfrak{g} \cap \mathfrak{g}_2$, we conclude that $\mathcal{A}'$ must have a vertex w' of weight two contained in the bounded segment on line $\mathfrak{g}_2$ reaching from v to $\mathfrak{g} \cap \mathfrak{g}_2$. Thus, there must be a line in $\mathcal{A}$ parallel to $\mathfrak{g}$ and passing through w'. But then $\mathcal{A}$ must have a vertex of weight two, contradicting our assumption.

Now assume that the distance from v to $\mathfrak{g}$ is greater than zero and call it k. Let $\mathfrak{g}' \in \mathcal{A}'$ be a line passing through v and containing exactly k vertices $v_0, ..., v_{k-1}$ of weight two of $\mathcal{A}'$ between v and $\mathfrak{g} \cap \mathfrak{g}'$. Without loss of generality we may assume that v_{k-1} is closest to v. As the arrangement $\mathcal{A}$ has only vertices of weight three we conclude that there must be a line $\mathfrak{g}''$ parallel to $\mathfrak{g}$ and passing through v_{k-1}. Now consider the arrangement $\mathcal{A}'' := \mathcal{A}_\mathfrak{g} \cup \{\mathfrak{g}''\} \subset \mathcal{A}$. We observe that $\mathcal{A}'$ and $\mathcal{A}''$ differ only by one line not belonging to $\mathcal{A}_\mathfrak{g}$, hence v is also a vertex of $\mathcal{A}''$ and its distance to $\mathfrak{g}''$ is zero. As above this implies that $\mathcal{A}$ has a vertex of weight two, which is impossible by assumption. Hence $\mathcal{A}'$ has no vertex v as above: every vertex of $\mathcal{A}'$ of weight three must lie on $\mathfrak{g}$. This shows that $\mathcal{A}$ must contain infinitely many lines which are parallel to $\mathfrak{g}$. But then F cannot be irreducible as it must contain a linear factor corresponding to the infinitely many lines of $\mathcal{A}$ parallel to $\mathfrak{g}$. □

Corollary 4.1. *There is no affine Tits arrangement $\mathcal{A}$ of rank three such that $\mathcal{A}^*$ is contained in the locus of an irreducible homogeneous polynomial of degree three.*

Proof. Consider an arrangement $\mathcal{A}$ of lines in the real projective plane such that $\mathcal{A}^* \subset V(P) \subset (\mathbb{P}^2(\mathbb{R}))^*$ for some irreducible $P \in \mathbb{R}[x,y,z]$ with $\deg(P) = 3$. Let $v \in \mathbb{P}^2(\mathbb{R})$ be an arbitrary vertex of $\mathcal{A}$. Then in the dual setting v^* is given by a line and the weight of v is bounded by $|v^* \cap V(P)|$. Bézout's theorem gives $|v^* \cap V(P)| \leq \deg(v^*) \cdot \deg(P) = 3$. Now if $\mathcal{A}$ is simplicial and affine then by Lemma 4.3 each vertex of $\mathcal{A}$ has weight exactly three. But then by Lemma 4.4 it follows that P cannot be irreducible. □

Remark 4.3. i) Let P be an irreducible cubic polynomial. If one drops the assumption on $\mathcal{A}$ to be affine in Corollary 4.1, then the proof above shows that there are possible candidates for (spherical) Tits arrangements $\mathcal{A}$ such that $\mathcal{A}^* \subset V(P)$: namely all spherical arrangements having only vertices of weight two or three. Since these are precisely the arrangements $A(6,1), A(7,1)$ and the near pencils with at most four lines, we will not investigate this further.
ii) If $\mathcal{A}^* \subset V(P)$ for some possibly reducible polynomial P, we may still apply Bézout's theorem to conclude the following: suppose that P is a product of three linear factors. Then $\mathcal{A}$ has at most three vertices of weight possibly bigger than three and all other vertices have weight bounded by three.

If on the other hand P is the product of an irreducible quadratic factor and a linear factor, then $\mathcal{A}$ has at most one vertex of weight possibly bigger than three while all other vertices have weight bounded by three.

It remains to consider the possibility that $\mathcal{A}^*$ is contained in the locus of a cubic homogeneous polynomial having an irreducible quadratic factor. As preparation, we introduce some more notation.

Definition 4.1. a) Let σ be an irreducible conic in $\mathbb{P}^2(\mathbb{R})$ and consider a subset $M \subset \sigma$. There exists a projectivity Ψ such that $\Psi(\sigma)$ is given by the polynomial $P := x^2 + y^2 - z^2$ and is thus contained entirely in the affine $z = 1$ chart of $\mathbb{P}^2(\mathbb{R})$. We say that $p_1, ..., p_k \in M$ are *consecutive with respect to* Ψ, if for any $1 \leq i \leq k-1$ it is true that one of the segments on $\Psi(\sigma)$ bounded by $\Psi(p_i), \Psi(p_{i+1})$ contains no other point of $\Psi(M)$.
b) Consider the map $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ sending $v_1, v_2 \in \mathbb{R}^3$ to their vector product $v_1 \times v_2$. This induces a map $\psi : \big(\mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})\big) \setminus \Delta \longrightarrow (\mathbb{P}^2(\mathbb{R}))^*$, where $\Delta := \{(x,x) \mid x \in \mathbb{P}^2(\mathbb{R})\}$. By a slight abuse of notation, we write

$\psi(v_1, v_2) = v_1 \times v_2 \in (\mathbb{P}^2(\mathbb{R}))^*$ for two different projective points $v_1, v_2 \in \mathbb{P}^2(\mathbb{R})$. Observe that for $p, q \in (\mathbb{P}^2(\mathbb{R}))^*$ the vector product $p \times q$ gives the vertex in $\mathbb{P}^2(\mathbb{R})$ obtained as the intersection of the dual lines p^*, q^*. Similarly, if v, v' are two points in $\mathbb{P}^2(\mathbb{R})$, then the vector product $v \times v'$ gives the point in $(\mathbb{P}^2(\mathbb{R}))^*$ which is dual to the line passing through v and v'.

Now we can prove the following statement (compare also [4, Thm. 3.6], where case c) of the following proposition is examined for spherical Tits arrangements).

Proposition 4.3. *Suppose that $\mathcal{A}$ is an affine rank three Tits arrangement and assume that $\mathcal{A}^* \subset \sigma \cup \mathfrak{l}$ for some irreducible conic $\sigma \subset (\mathbb{P}^2(\mathbb{R}))^*$ and an arbitrary line $\mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$. Then the following statements hold:*

- *a) $|\mathcal{A}^* \cap \sigma| = \infty$, unless $\mathcal{A}$ is a near pencil.*
- *b) $|\mathcal{A}^* \cap \mathfrak{l}| = \infty$ and $(\partial T)^* \in \mathfrak{l}$.*
- *c) If $|\sigma \cap \mathfrak{l}| = 0$ then $\mathcal{A}$ is a near pencil.*
- *d) If $|\sigma \cap \mathfrak{l}| = 1$ then $\sigma \cap \mathfrak{l} = (\partial T)^*$, unless $\mathcal{A}$ is a near pencil.*
- *e) If $|\sigma \cap \mathfrak{l}| = 2$ then $\mathcal{A}$ is a near pencil.*

Proof. a) Define $\mathcal{B} := \mathcal{A} \cap \sigma^*$ and suppose that $|\mathcal{B}| < \infty$. Since $\mathcal{A}$ is affine and hence necessarily infinite, we set $L := \mathcal{A} \cap \mathfrak{l}^*$ and conclude that $|L| = \infty$. So we have $\mathcal{A} = \mathcal{B} \cup L$ and it is easy to see that we find a line in $\mathcal{B}$ containing a segment bounded by two vertices of weight two. By Lemma 4.1 we conclude that $\mathcal{A}$ is a near pencil.
b) If $\mathcal{A}$ is a near pencil then both statements are easily seen to be true. So we may assume that $\mathcal{A}$ is not a near pencil. We show that the second statement is a consequence of the first. So suppose that $|\mathcal{A}^* \cap \mathfrak{l}| = \infty$ and assume that $(\partial T)^* \notin \mathfrak{l}$. Dualizing we obtain that the point $\mathfrak{l}^*$ does not lie on the line ∂T. Hence $\mathfrak{l}^*$ lies in T and there are infinitely many lines of $\mathcal{A}$ passing through $\mathfrak{l}^*$. But since $\mathcal{A}$ is locally finite in T this is impossible. So it suffices to prove that $|\mathcal{A}^* \cap \mathfrak{l}| = \infty$. We show that $|\mathcal{A}^* \cap \mathfrak{l}| < \infty$ gives a contradiction: fix some $q \in \sigma \cap \mathcal{A}^*$ and consider the pencil $\mathcal{P}_q$ of lines $\mathfrak{l}_{q,q'} \subset (\mathbb{P}^2(\mathbb{R}))^*$ passing through q and $q' \in (\sigma \cap \mathcal{A}^*) \setminus \{q\}$. By part a) it follows that $|\mathcal{A}^* \cap \sigma| = \infty$, since by assumption $\mathcal{A}$ is not a near pencil. In particular, we have $|\mathcal{P}_q| = \infty$. Hence there must be a pair of neighbouring lines $\mathfrak{l}_{q,q'}, \mathfrak{l}_{q,q''} \in \mathcal{P}_q$ whose intersections with $\mathfrak{l}$ are both not contained in $\mathcal{A}^*$. This is true because by assumption there are only finitely many points in $\mathcal{A}^* \cap \mathfrak{l}$. But this means that the line $q^* \in \mathcal{A}$ must contain a segment bounded by two vertices of weight two, which by Lemma 4.1 implies that $\mathcal{A}$ is a near pencil. This is the desired contradiction.
c) Since $\sigma \cap \mathfrak{l} = \emptyset$ we may use part b) to conclude that $(\partial T)^* \notin \sigma$. But then it follows that $|\mathcal{A}^* \cap \sigma| < \infty$, since points of $\mathcal{A}^*$ may accumulate only in a

neighbourhood of $(\partial T)^*$ (because $\mathcal{A}$ is locally finite in T). Now by part a) it follows that $\mathcal{A}$ is a near pencil.
d) By part b) we already know that $(\partial T)^* \in \mathfrak{l}$. Assume that $(\partial T)^* \notin \sigma$. Then it follows that $|\mathcal{A}^* \cap \sigma| < \infty$, because points of $\mathcal{A}^*$ may accumulate only in a neighbourhood of $(\partial T)^*$. Hence we may use part a) to conclude that $\mathcal{A}$ must be a near pencil.
e) After applying a projectivity as in part a) of Definition 4.1, we may assume that $\sigma = V(P)$ where $P := x^2 + y^2 - z^2$. So σ is contained entirely in the affine $z = 1$ patch of $(\mathbb{P}^2(\mathbb{R}))^*$. We write σ' for the conic in $\mathbb{P}^2(\mathbb{R})$ defined by the same polynomial.

Suppose that $\mathcal{A}$ is not a near pencil. As points of $\mathcal{A}^*$ may accumulate only in a neighbourhood of $(\partial T)^*$, we have $(\partial T)^* \in \sigma \cap \mathfrak{l}$. Observe that for $p = (a : b : 1) \in \sigma \cap \mathcal{A}^* \subset (\mathbb{P}^2(\mathbb{R}))^*$ the corresponding dual line p^* is the tangent to σ' at the point $(-a : -b : 1) \in \mathbb{P}^2(\mathbb{R})$. In particular, if $(\partial T)^* = (x : y : 1)$, this implies that there is a sequence of tangent lines to σ' converging towards the tangent line at the point $(-x : -y : 1)$, and this tangent line is precisely ∂T. It remains to identify the dual lines q^* corresponding to $q \in \mathfrak{l} \cap \mathcal{A}^*$. We may assume without loss of generality that in the $z = 1$ patch of $(\mathbb{P}^2(\mathbb{R}))^*$ the line $\mathfrak{l}$ is given by the equation $y = \lambda$ for some $0 \leq \lambda < 1$. Hence any $q \in \mathfrak{l}$ will have homogeneous coordinates $q = (x_0 : \lambda : 1)$. So if $\lambda > 0$, the equation of the dual line q^* in the $z = 1$ patch of $\mathbb{P}^2(\mathbb{R})$ will be $y = -\frac{x_0 \cdot x}{\lambda} - \frac{1}{\lambda}$; if on the other hand $\lambda = 0$, then the equation of q^* will be $x = -\frac{1}{x_0}$. Hence if $\lambda > 0$, then all lines pass through the point $(0 : -\frac{1}{\lambda} : 1)$ which implies that $\mathfrak{l}^* = (0 : -\frac{1}{\lambda} : 1)$; if $\lambda = 0$, then all lines pass through $\mathfrak{l}^* = (0 : 1 : 0)$. This shows that $\mathfrak{l}^* \notin \sigma'$. Since $(\partial T)^* \in \mathfrak{l}$ we conclude that $\mathfrak{l}^* \in \partial T$. Now we take ∂T as line at infinity. Doing so, we obtain $\mathcal{A}$ as union of tangent lines to a parabola together with infinitely many parallel lines each of which being non-parallel to the symmetry axis of the parabola. But then $\mathcal{A}$ is not simplicial. $\square$

The following lemma will be the key to proving the main theorem of this chapter.

Lemma 4.5. *Let σ be an irreducible conic together with a projectivity Ψ as in part a) of Definition 4.1. Assume that $\mathfrak{l}$ is a line touching σ. If $\mathcal{A}$ is an irreducible affine rank three Tits arrangement such that $\mathcal{A}^* \subset \sigma \cup \mathfrak{l}$, then $\mathcal{A}$ is determined by specifying four points on σ which are consecutive with respect to Ψ. More precisely, if $p_{-1}, p_0, p_1, p_2, p_3, p_4 \in \mathcal{A}^* \cap \sigma$ are six consecutive points (with respect to Ψ), then we have the following formulas for p_{-1} and p_4 in terms of $p_0, ..., p_3$:*

$$p_4 = (p_0 \times (\mathfrak{l}^* \times (p_1 \times p_3))) \times (p_1 \times (\mathfrak{l}^* \times (p_2 \times p_3))), \tag{4.1}$$
$$p_{-1} = (p_2 \times (\mathfrak{l}^* \times (p_0 \times p_1))) \times (p_3 \times (\mathfrak{l}^* \times (p_0 \times p_2))). \tag{4.2}$$

Proof. Denote by $L_1, L_2 \subset \mathcal{A}$ the set of lines corresponding to elements in $\mathcal{A}^* \cap \sigma, \mathcal{A}^* \cap \mathfrak{l}$ respectively. Observe that every $\mathfrak{h} \in L_2$ passes through the point $\mathfrak{l}^*$ while no line belonging to L_1 passes through $\mathfrak{l}^*$: if $\mathfrak{l}^* \in \mathfrak{g}$ and $\mathfrak{g}^* \in \sigma$ for some $\mathfrak{g}$, then $\mathfrak{g}^* \in \mathfrak{l} \cap \sigma = \{(\partial T)^*\}$, by part e) of Proposition 4.3. As $\mathcal{A}$ is thin by definition, we conclude that $\mathfrak{g} \notin \mathcal{A}$.

Note also that every vertex of weight two of $\mathcal{A}$ must lie on a line belonging to L_2. Indeed, assume there was a vertex v of weight two such that $v = \mathfrak{g} \cap \mathfrak{g}'$ for some $\mathfrak{g}, \mathfrak{g}' \in L_1$. As $\mathcal{A}^* \subset \sigma \cup \mathfrak{l}$ and because no line belonging to L_1 passes through $\mathfrak{l}^*$, we may use part ii) of Remark 4.3 to conclude that every neighbour of v has weight bounded by three. But then by Lemma 4.1 every neighbour of v has weight precisely three, because $\mathcal{A}$ was assumed to be irreducible. By Lemma 4.3 we obtain that $\mathcal{A}$ is spherical, a contradiction. In particular, it follows that for every vertex v' obtained as intersection of elements in L_1 there is a line $\mathfrak{h} \in L_2$ passing through v'. Also, every vertex of weight two is a neighbour of $\mathfrak{l}^*$.

These conditions already suffice to prove the claim. Let $p_0, p_1, p_2, p_3 \in \mathcal{A}^* \cap \sigma$ be four consecutive points (with respect to Ψ). We need to construct the points $p_{-1}, p_4 \in \mathcal{A}^* \cap \sigma$ such that both p_{-1}, p_0, p_1, p_2 and p_1, p_2, p_3, p_4 are consecutive (with respect to Ψ). By symmetry, it suffices to construct p_4. For this, denote the line corresponding to p_i by $\mathfrak{g}_i$ and let $\mathfrak{h}$ be the line passing through the vertices $\mathfrak{l}^*, \mathfrak{g}_1 \cap \mathfrak{g}_3$. Similarly, denote by $\mathfrak{h}'$ the line passing through the vertices $\mathfrak{l}^*, \mathfrak{g}_2 \cap \mathfrak{g}_3$. Then $\mathfrak{g}_4$ is the line passing through the vertices $\mathfrak{g}_0 \cap \mathfrak{h}, \mathfrak{g}_1 \cap \mathfrak{h}'$. From this, one reads off that (4.1) holds. This completes the proof. $\square$

Remark 4.4. Let P be a homogeneous cubic polynomial having an irreducible quadratic factor. If $\mathcal{A}$ is an irreducible spherical Tits arrangement such that $\mathcal{A}^* \subset V(P)$, then one may use part ii) of Remark 4.3 to conclude that there are two possibilities for $\mathcal{A}$: either $\mathcal{A}$ is the arrangement $A(7,1)$ or $\mathcal{A}$ belongs to the infinite family $\mathcal{R}(1)$.

Now we can construct the arrangement of type $\tilde{A}_2^0$ and prove that up to projectivity it is the only irreducible affine rank three Tits arrangement whose dual point set is contained in the locus of a cubic polynomial having an irreducible quadratic factor:

Proposition 4.4. *Up to projectivity, there is only one irreducible affine rank three Tits arrangement $\mathcal{A}$ such that $\mathcal{A}^*$ is contained in the locus of a cubic polynomial P having an irreducible quadratic factor. The arrangement $\mathcal{A}$ may be defined by the following set of dual points:*

$$\mathcal{A}^* = \left\{ \left(k : \frac{k(k-1)}{2} : 1\right), \left(1 : \frac{k}{2} : 0\right) \mid k \in \mathbb{Z} \right\}.$$

Proof. Let $\mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$ be the line corresponding to the linear factor of P and let $\sigma \subset (\mathbb{P}^2(\mathbb{R}))^*$ be the irreducible conic corresponding to the quadratic factor of P. We then have $\mathcal{A}^* \subset \sigma \cup \mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$ and by Proposition 4.3 we may assume that $\mathfrak{l}$ touches σ at the point $(\partial T)^*$.

Let $p_1, p_2, p_3, p_4 \in \mathcal{A}^* \cap \sigma$ be four consecutive points (with respect to some projectivity Ψ). After a change of coordinates we may assume that

$$\begin{aligned}(\partial T)^* = (0:1:0)\,, p_2 = (1:0:1)\,,\\ p_3 = (2:1:1)\,, p_4 = (3:3:1)\,.\end{aligned}$$

We then have $p_1 = (x:y:z)$ for some $x, y, z \in \mathbb{R}$. Now consider the vertices $v := p_2 \times p_3, v' := p_1 \times p_4 \in \mathbb{P}^2(\mathbb{R})$ and let $\mathfrak{g} \subset \mathbb{P}^2(\mathbb{R})$ be the line passing through v and v'. Then by (the proof of) Lemma 4.5 we know that $\mathfrak{g} \in \mathcal{A}$ and that $\mathfrak{g}$ passes through the vertex $\mathfrak{l}^*$. As $\mathfrak{l}^* \in \partial T$, we may write $\mathfrak{l}^* = (a:0:b)$ for certain $a, b \in \mathbb{R}$. In order to prove the statement we will distinguish four cases.

Case 1. Assume that $x = y = 0$. This implies that $p_1 = (0:0:1)$. We claim that $\mathfrak{l}^* = (0:0:1)$. To see this write $\mathfrak{l}^* = (a:0:b)$ for some $a, b \in \mathbb{R}$ as above. The fact that $\mathfrak{g}$ passes through $\mathfrak{l}^*$ implies that $a = 0$ and therefore we have $\mathfrak{l}^* = (0:0:1)$.

Now consider the projectivity $\Phi : (\mathbb{P}^2(\mathbb{R}))^* \longrightarrow (\mathbb{P}^2(\mathbb{R}))^*$ taking the point p_i to p_{i+1} for $1 \leq i \leq 4$. We obtain $\mathcal{A}^* \cap \sigma = \big\{\Phi^k(p_1) \mid k \in \mathbb{Z}\big\} = \Big\{\Big(k : \frac{k(k-1)}{2} : 1\Big) \mid k \in \mathbb{Z}\Big\}$, using Lemma 4.5 and induction. Observe that the lines of $\mathcal{A}$ corresponding to points in $\mathcal{A}^* \cap \mathfrak{l}$ are exactly the lines passing through $\mathfrak{l}^*$ and a vertex of the form $p \times p'$ for $p, p' \in \mathcal{A}^* \cap \sigma$ (see the proof of Lemma 4.5). We conclude that $\mathcal{A}^* \cap \mathfrak{l} = \big\{\big(1 : \frac{k}{2} : 0\big) \mid k \in \mathbb{Z}\big\}$. It is now easy to check that $\mathcal{A}^* = \Big\{\Big(k : \frac{k(k-1)}{2} : 1\Big), \big(1 : \frac{k}{2} : 0\big) \mid k \in \mathbb{Z}\Big\}$ defines an irreducible affine Tits arrangement.

Case 2. Assume that $x \neq 0$ and $y = 0$. Then we may assume that $p_1 = (1:0:z)$. Write $\mathfrak{l}^* = (a:0:b)$ for $a, b \in \mathbb{R}$. The fact that $\mathfrak{g}$ passes through $\mathfrak{l}^*$ implies that $a \neq 0$. Thus, we may assume that $\mathfrak{l}^* = (1:0:b)$. It follows that $z = \frac{b+4}{3}$ and therefore $p_1 = \big(1 : 0 : \frac{b+4}{3}\big)$. Observe that the five given points $(\partial T)^*, p_1, p_2, p_3, p_4$ on σ determine its equation. Using this together with Lemma 4.5, the condition $p_5 \in \sigma$ implies that $b \in \{-1, -\frac{3}{2}, -\frac{7}{3}, -3\}$. As $p_0, p_5 \neq p_i$ for $1 \leq i \leq 4$, we conclude that $b \in \{-1, -\frac{3}{2}, -3\}$ is impossible. In the remaining case $b = -\frac{7}{3}$, we observe that the conic σ may be defined by the polynomial $f = -\frac{10}{3}X^2 + 2XY + \frac{28}{3}XZ - \frac{10}{3}YZ - 6Z^2$. By assumption, we know that the line $\mathfrak{l}$ touches σ at the point $(\partial T)^*$. Thus, as $\mathfrak{l}^* = \big(1 : 0 : -\frac{7}{3}\big)$, there exists $0 \neq \lambda \in \mathbb{R}$ such that the following equations are satisfied:

$$\begin{aligned}1 &= \lambda\,\frac{\partial f}{\partial_X}\Big|_{(\partial T^*)},\\ 0 &= \lambda\,\frac{\partial f}{\partial_Y}\Big|_{(\partial T^*)},\\ -\frac{7}{3} &= \lambda\,\frac{\partial f}{\partial_Z}\Big|_{(\partial T^*)}.\end{aligned}$$

The first equation gives $\lambda = \frac{1}{2}$. But then the third equation reads $-\frac{7}{3} = -\frac{5}{3}$. This contradiction shows that Case 2 cannot occur.

Case 3. Assume that $x = 0$ and $y \neq 0$. Then without loss of generality, we may assume that $p_1 = (0 : 1 : z)$. Again, we write $\mathfrak{l}^* = (a : 0 : b)$ for suitable $a, b \in \mathbb{R}$ and as $\mathfrak{g}$ passes through $\mathfrak{l}^*$, we obtain $a \neq 0$. Thus, we may assume that $\mathfrak{l}^* = (1 : 0 : b)$, leading to $z = -\frac{b+3}{3}$. We conclude that $p_1 = (0 : 1 : -\frac{b+3}{3})$. The relation $p_5 \in \sigma$ gives $b \in \{-3, -1\}$. As $p_5 \neq p_i$ for $1 \leq i \leq 4$, we conclude that this is impossible.

Case 4. Assume that both $x \neq 0$ and $y \neq 0$. Then we may suppose that $p_1 = (1 : y : z)$. Write $\mathfrak{l}^* = (a : 0 : b)$ for suitable $a, b \in \mathbb{R}$. As before, by considering the line $\mathfrak{g}$, we conclude that $-3za - 3ay - by + 4a + b = 0$. Suppose that $a = 0$. Then without loss of generality $b = 1$ and we have $y = 1$, in particular $p_1 = (1 : 1 : z)$. As $p_5 \in \sigma$, we conclude that $z \in \{\frac{1}{3}, \frac{1}{2}\}$. Again, this is not possible because $p_0, p_5 \neq p_i$ for $1 \leq i \leq 4$.

Hence, we may assume that $a = 1$. In particular, we have $z = \frac{4}{3} - \frac{b(y-1)}{3} - y$ and $p_1 = (1 : y : \frac{4}{3} - \frac{b(y-1)}{3} - y)$.

Suppose that $b \neq -3$. Using the condition $p_5 \in \sigma$, we compute that $y \in \left\{1, \frac{-3b^2-10b-7}{2(b+3)}, \frac{2b^2+5b+3}{2(b^2+3b+3)}\right\}$. As $p_1 \neq p_4$, we can exclude the case $y = 1$. Assume that $y = \frac{2b^2+5b+3}{2(b^2+3b+3)}$. Then we obtain $p_1 = p_5$, a contradiction. So we necessarily have $y = \frac{-3b^2-10b-7}{2(b+3)}$. In particular, this implies that $p_1 = \left(1 : \frac{-3b^2-10b-7}{2(b+3)} : \frac{b^2+4b+5}{2}\right)$. Therefore, the conic σ may be defined by the polynomial $f := (b-1)X^2 + 2XY - (b-7)XZ - 2(b+4)YZ - 6Z^2$. To see this, one only has to check that $f(p_i) = 0$ for $1 \leq i \leq 5$. The line $\mathfrak{l}$ touches σ at the point $(\partial T)^* = (0 : 1 : 0)$. Therefore, as $\mathfrak{l}^* = (1 : 0 : b)$, we know that there exists $0 \neq \lambda \in \mathbb{R}$ such that the following equations hold:

$$1 = \lambda \, \frac{\partial f}{\partial_X}\Big|_{(\partial T^*)},$$
$$0 = \lambda \, \frac{\partial f}{\partial_Y}\Big|_{(\partial T^*)},$$
$$b = \lambda \, \frac{\partial f}{\partial_Z}\Big|_{(\partial T^*)}.$$

The first equation gives $\lambda = \frac{1}{2}$. Thus, the third equation yields $b = -2$ and we obtain $p_1 = \left(1 : \frac{1}{2} : \frac{1}{2}\right) = (2 : 1 : 1) = p_3$, a contradiction.

It remains to consider the case $b = -3$. Then we have $\mathfrak{l}^* = (1 : 0 - 3)$ and $p_1 = \left(1 : y : \frac{1}{3}\right)$. Clearly, we have $y \neq 1$ because $p_1 \neq p_4$. Then Lemma 4.5 yields $p_5 = (3 : 3 : 1) = p_4$, another contradiction. This completes the proof. $\square$

We obtain the following Corollary:

Corollary 4.2. *There are irreducible affine Tits arrangements which are not locally spherical.*

Proof. This follows from Proposition 4.4. The arrangement constructed there is such an example: the vertex $\mathfrak{l}^*$ is incident with infinitely many lines of $\mathcal{A}$. $\square$

Finally, using Proposition 4.2, Corollary 4.1, Proposition 4.3, and Proposition 4.4, we obtain the promised main theorem:

Theorem 4.2. *Let $\mathcal{A}$ be an affine rank three Tits arrangement such that $\mathcal{A}^*$ is contained in the locus of a homogeneous polynomial of degree three. Then, up to projectivity, $\mathcal{A}$ is either a near pencil, an arrangement of type $A_2^{(1)}$, or it is an arrangement of type $\tilde{A}_2^0$.*

4.2 Open problems and related questions

In this section we want to point out some possibly interesting related problems. First, we ask if there exists an affine rank three Tits arrangement $\mathcal{A}$ (viewed as arrangement of lines in the real projective plane) such that $\mathcal{A}^*$ is contained entirely in the locus of an irreducible homogeneous polynomial:

Problem 3. *Does there exist some irreducible homogeneous polynomial $P \in \mathbb{R}[x, y, z]$ such that $\mathcal{A}^* \subset V(P)$ for a suitable irreducible affine rank three Tits arrangement $\mathcal{A}$?*

Observe that given a Tits arrangement $\mathcal{A}$ and an irreducible homogeneous polynomial P of degree d such that $\mathcal{A}^* \subset V(P)$, it follows immediately that $\mathcal{A}$ is locally spherical. Indeed, suppose there was a vertex v of $\mathcal{A}$ such that infinitely many lines of $\mathcal{A}$ pass through v. Then after dualizing it follows that infinitely many points of $\mathcal{A}^*$ lie on the line v^*. But since by assumption $\mathcal{A}^* \subset V(P)$, it follows that infinitely many points lie on the intersection $V(P) \cap v^*$. But Bézout's theorem tells that $|V(P) \cap v^*| \leq d \cdot 1 = d < \infty$, because P was assumed to be irreducible and hence v^* cannot be a component of $V(P)$. This contradiction shows that $\mathcal{A}$ must be locally spherical.

This leads to the next problem. Are there other examples of irreducible affine rank three Tits arrangements which are not locally spherical?

Problem 4. *Classify (up to projectivities) all irreducible affine rank three Tits arrangements $\mathcal{A}$ which are not locally spherical.*

Observe that if $\mathcal{A}$ is not locally spherical, then by Lemma 4.2 there is precisely one vertex v on the boundary of the Tits cone T. In particular, it follows that for every line $\mathfrak{l} \neq v^*$ we have $|\mathcal{A}^* \cap \mathfrak{l}| < \infty$. If in addition we know that $\mathcal{A}^* \subset V(P)$ for some homogeneous polynomial P of degree d, then by Bézout's theorem the last inequality can be strengthened to

$$|\mathcal{A}^* \cap \mathfrak{l}| \leq |V(P) \cap \mathfrak{l}| \leq d$$

for every line $\mathfrak{l} \neq v^*$ which is not a component of $V(P)$.

We close this section by proposing the following final problem which is probably the most difficult:

Problem 5. *Classify (up to projectivities) all affine rank three Tits arrangements $\mathcal{A}$ such that $\mathcal{A}^* \subset V(P)$ for some homogeneous polynomial $P \in \mathbb{R}[x, y, z]$.*

A solution to the last problem seems to be an important step towards a classification of all affine rank three Tits arrangements. Indeed, if $\mathcal{A}$ is such an arrangement and if $\mathcal{A} = \bigcup_{i \in I} L_i$ for some finite index set I and sets of mutually parallel lines $L_i, i \in I$, then $\mathcal{A}^*$ is contained in the locus of a polynomial P of degree $|I|$: the polynomial P is a product of linear factors corresponding to the sets $L_i, i \in I$. For example, affine Tits arrangements coming from Nichols algebras of diagonal type are always of this type.

Even if we enlarge $\mathcal{A}$ by finitely many countable subsets of tangent lines to certain conics, we still find a polynomial P' such that the enlarged arrangement is contained in the locus of P'. The polynomial P' may be taken as the product of P together with the irreducible quadratic polynomials defining the (dual) conics in question. This gives the impression that the class of affine rank three Tits arrangements whose dual points sets are contained in the locus of some polynomial is rather large, as demonstrated by the fact that only usage of at most quadratic polynomials already leads to nontrivial considerations. It may even be conjectured that for every irreducible affine rank three Tits arrangement $\mathcal{B}$ there is a certain polynomial Q such that $\mathcal{B}^* \subset V(Q)$. If this is true, then clearly a solution to Problem 5 amounts to a complete classification of affine rank three Tits arrangements.

Chapter 5

Combinatorics of free and simplicial line arrangements

In this chapter we are interested in spherical rank three Tits arrangements. These may be identified with simplicial arrangements of lines in the real projective plane. Despite some considerable progress (see for instance the papers [6], [7], [8]), a complete classification of simplicial line arrangements in $\mathbb{P}^2(\mathbb{R})$ still remains an open problem. However, there is a catalogue published by Grünbaum (see [19]), listing all currently known combinatorial isomorphism classes of simplicial line arrangements except for four arrangements discovered in [6].
The current belief is that -up to finitely many further corrections- the given catalogue is complete.

In this chapter we collect some more evidence for this belief. In particular, we show that Grünbaum's catalogue contains all free simplicial line arrangements whose vertices have weight bounded by four (see Theorem 5.5). Similarly, we prove that a free simplicial line arrangement whose vertices have weight bounded by five consists of at most forty lines (see Theorem 5.8). This implies that there is only a (rather small) finite number of such arrangements possibly missing in Grünbaum's catalogue. See also Theorem 5.10 for a similar (but somewhat weaker) statement concerning free simplicial arrangements having only vertices of weight bounded by six.

Motivated by the observation that the dual point set of a simplicial arrangement belonging to the infinite series $\mathcal{R}(1)$ is contained in a cubic curve, we also study simplicial arrangements whose dual point sets are contained in some projective curve of bounded degree (see Theorem 5.13).

Moreover, we prove a combinatorial analogue of the fact that hyperplane arrangements in $\mathbb{R}^3$ having isometric chambers are Coxeter arrangements (see Theorem 5.3 and the paper [14]).

Our techniques also allow us to prove finiteness results concerning line arrangements which are only free but not necessarily simplicial. In particular, we prove that there are only finitely many combinatorial isomorphism classes of free line arrangements in $\mathbb{P}^2(\mathbb{R})$ having only vertices of weight bounded by five (see Theorem 5.11).

Most given arguments are purely combinatorial and focus on the t-vector of an arrangement (see Chapter 2). Therefore, one obtains corresponding statements for arrangements of pseudolines in many cases.

5.1 Combinatorics of simplicial line arrangements

In this section we study the combinatorics of simplicial line arrangements. The main goal is the classification up to combinatorial isomorphism of all simplicial line arrangements $\mathcal{A}$ having multiplicity at most four such that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$. It turns out that such an arrangement is automatically crystallographic. Moreover, we prove that there are only finitely many combinatorial isomorphism classes of simplicial arrangements whose multiplicities are at most five and whose characteristic polynomials have only real roots. We are able to give classification results in this situation if we impose some further restriction. Furthermore, we prove a combinatorial analogue of the theorem which says that every hyperplane arrangement in $\mathbb{R}^3$ having isometric chambers is a Coxeter arrangement. All goals (except Theorem 5.2) are achieved by using purely combinatorial methods.

5.1.1 Basic relations involving $t^{\mathcal{A}}$ and bounds for $t_2^{\mathcal{A}}, t_3^{\mathcal{A}}$

Let $\mathcal{A}$ be a simplicial projective line arrangement. In this subsection we first collect some known results on $t^{\mathcal{A}}$. We then proceed to derive upper and lower bounds for the numbers $t_2^{\mathcal{A}}, t_3^{\mathcal{A}}$. These results will then be used in the following two subsections to derive some interesting results on arrangements having low multiplicity.

We start with the following basic but very useful lemma.

Lemma 5.1. *Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$. Then for the t-vector $t^{\mathcal{A}}$ the following relations hold:*

$$\sum_{i\geq 2} \binom{i}{2} t_i^{\mathcal{A}} = \binom{n}{2}, \tag{5.1}$$

$$1 + \sum_{i\geq 2} (i-1) t_i^{\mathcal{A}} = f_2^{\mathcal{A}}, \tag{5.2}$$

$$\sum_{i\geq 2} t_i^{\mathcal{A}} = f_0^{\mathcal{A}}, \tag{5.3}$$

$$\sum_{i\geq 2} i t_i^{\mathcal{A}} = f_1^{\mathcal{A}}, \tag{5.4}$$

$$3 + \sum_{i \geq 4} (i-3) t_i^{\mathcal{A}} \leq t_2^{\mathcal{A}}. \tag{5.5}$$

Moreover, we have equality in (5.5) if and only if $\mathcal{A}$ is simplicial. In this case, we also have $2(f_0^{\mathcal{A}} - 1) = f_2^{\mathcal{A}} = \frac{2}{3} f_1^{\mathcal{A}}$.

Proof. This follows from Remark 2.2 and Theorem 2.1. □

The next two lemmas are easy but very important results (see also [6, Lemma 3.2] and Lemma 4.1 from the previous chapter).

Lemma 5.2. *a) Let $\mathcal{A}$ be a simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$. Suppose there is a line $\ell \in \mathcal{A}$ containing an edge bounded by two vertices of weight two. Then $\mathcal{A}$ is not irreducible.*
b) Assume that $\mathcal{A}$ is an irreducible simplicial line arrangement. Then we have the estimate $4t_2^{\mathcal{A}} \leq f_2^{\mathcal{A}}$. Equality holds if and only if every chamber contains a vertex of weight two.
c) Assume that $\mathcal{A}$ is an irreducible simplicial line arrangement. Then we have the following analogue of Melchior's inequality involving the number of triple points:

$$t_3^{\mathcal{A}} \geq 4 + \sum_{i \geq 5} (i-4) t_i^{\mathcal{A}}.$$

Proof. a) By definition $\mathcal{A}$ is not irreducible if there exists a chamber K such that the Coxeter diagram Γ_C^K is not connected. If $\ell \in \mathcal{A}$ contains an edge e bounded by two vertices of weight two, then the Coxeter diagrams $\Gamma_C^{K_i}$ corresponding to the chambers K_1, K_2 containing e are not connected.
b) By part a) every chamber has at most one vertex of weight two. Denote the set of chambers of $\mathcal{A}$ by $\mathcal{K}$ and denote by $\mathcal{V}_2$ the set of vertices of $\mathcal{A}$ which have weight two. Then clearly we have $\sum_{K \in \mathcal{K}} \overline{K} \cap \mathcal{V}_2 \leq f_2^{\mathcal{A}}$. On the other hand, every $v \in \mathcal{V}_2$ is contained in exactly four chambers. Hence $4t_2^{\mathcal{A}} = \sum_{K \in \mathcal{K}} \overline{K} \cap \mathcal{V}_2 \leq f_2^{\mathcal{A}}$. The statement about equality is now obvious.
c) By part b) and Lemma 5.1 we have $-2 + 2\sum_{i \geq 2} t_i^{\mathcal{A}} = f_2^{\mathcal{A}} \geq 4t_2^{\mathcal{A}}$. After rearranging terms, this yields $t_3^{\mathcal{A}} \geq 4 + \sum_{i \geq 5} (i-4) t_i^{\mathcal{A}}$. □

Remark 5.1. If $\mathcal{A}$ is simplicial but not irreducible, then $\mathcal{A}$ is a near pencil arrangement with $t_2^{\mathcal{A}} = |\mathcal{A}| - 1$ and $f_2^{\mathcal{A}} = 2(|\mathcal{A}| - 1)$. So in this case one has $f_2^{\mathcal{A}} < 4t_2^{\mathcal{A}}$.

Lemma 5.3. *Let $\mathcal{A}$ be a simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$. Assume that $\mathcal{A}$ has some vertex v of weight two such that every neighbour of v has weight three. Then $|\mathcal{A}| \in \{6, 7\}$.*

Proof. This follows from Lemma 4.3. □

The following corollary has no further applications in this chapter. Still, it seems to be interesting in its own right.

Corollary 5.1. *Let $\mathcal{A}$ be an irreducible simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$ and assume that every chamber of $\mathcal{A}$ contains at least two vertices of weight three. Then $\mathcal{A}$ is combinatorially isomorphic to one of the arrangements $A(6,1), A(7,1)$ (as denoted in [19]).*

Proof. Suppose that every chamber of $\mathcal{A}$ has at least two vertices of weight three. By relation (5.5) in Lemma 5.1 we know that there exists a chamber containing a vertex of weight two. Hence, by assumption we have a vertex of weight two whose neighbours all have weight three. By the last lemma we conclude that $6 \leq |\mathcal{A}| \leq 7$, and the claim follows from Theorem 2.5. $\square$

We proceed to give a little lemma which tells us about the combinatorial consequences for an arrangement $\mathcal{A}$ if $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$.

Lemma 5.4. *Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$ and consider the number $m := (n+1)^2 - 4f_2^{\mathcal{A}}$. Then for the characteristic polynomial $\chi(\mathcal{A}, t)$ we have the following formula:*

$$\chi(\mathcal{A}, t) = t^3 - nt^2 + (f_2^{\mathcal{A}} - 1)t + n - f_2^{\mathcal{A}}.$$

In particular, the roots of $\chi(\mathcal{A}, t)$ are given by $1, \frac{n-1+\sqrt{m}}{2}, \frac{n-1-\sqrt{m}}{2}$. Thus, $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$ if and only if $m \geq 0$. In this case we have $f_2^{\mathcal{A}} \leq \frac{(n+1)^2}{4}$.

Proof. Let $\tilde{\mathcal{A}}$ be the hyperplane arrangement associated to $\mathcal{A}$ and let L denote its intersection lattice. By definition $\chi(\mathcal{A}, t) = \sum_{X \in L} \mu(X) t^{\dim(X)}$.

If $X \in L$ has codimension two, then $\mu(X) = |\mathcal{A}_X| - 1$ and if $X \in L$ has codimension one, then $\mu(X) = -1$. Moreover, we have $\mu(V) = 1$ as well as the identity $\mu(\{0\}) = -\sum_{\{0\} \neq Y \in L} \mu(Y)$.

Writing L_2 for the subset of L consisting of all elements of codimension two, the claim then follows from the identity $\sum_{X \in L_2} |A_X| - 1 = \sum_{i \geq 2} (i-1) t_i$. For this, observe that $\sum_{i \geq 2} (i-1) t_i = f_2^{\mathcal{A}} - 1$, as the Euler characteristic of $\mathbb{P}^2(\mathbb{R})$ equals one. $\square$

Remark 5.2. It is known that every crystallographic simplicial line arrangement is inductively free and therefore the corresponding characteristic polynomial splits over $\mathbb{R}$ (see [1] for a definition of inductive freeness and for a proof of the mentioned result). Moreover, among the 51 known examples of combinatorial isomorphism classes of simplicial arrangements with up to 27 lines which are non-crystallographic, there are 29 such that their characteristic polynomial has only real roots.

In order to obtain an upper bound for the number of double points in an irreducible simplicial line arrangement, it suffices by part b) of Lemma 5.2 to bound the number of chambers. This is done in the following result.

Proposition 5.1. *a) Let $\mathcal{A}$ be an irreducible simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$. Then we have the estimate $t_2^{\mathcal{A}} \leq \frac{\binom{n}{2}+6}{7}$, where $n := |\mathcal{A}|$. Moreover, this bound is tight.*
b) If additionally $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$ then we have $t_2^{\mathcal{A}} \leq \frac{(n+1)^2}{16}$, which is stronger then the bound in a) for sufficiently large n.

Proof. a) By part b) of Lemma 5.2 we have $4t_2^{\mathcal{A}} \leq f_2^{\mathcal{A}}$. Further, using the relations given in Lemma 5.1, one can check that

$$f_2^{\mathcal{A}} = \binom{n}{2} + 1 - \sum_{i \geq 3} \binom{i-1}{2} t_i^{\mathcal{A}} \leq \binom{n}{2} + 1 - t_3^{\mathcal{A}} - 3t_2^{\mathcal{A}} + 9.$$

Using $t_3^{\mathcal{A}} \geq 4$, we may conclude that $4t_2^{\mathcal{A}} \leq f_2^{\mathcal{A}} \leq \frac{n^2-n+12}{2} - 3t_2^{\mathcal{A}}$, proving the first claim. The reflection arrangement of type B_3 is an example for which the given bound is tight.
b) By Lemma 5.4 the characteristic polynomial $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$ if and only if $\frac{(n+1)^2}{4} \geq f_2^{\mathcal{A}}$. Using part b) of Lemma 5.2 it follows that $\frac{(n+1)^2}{4} \geq f_2^{\mathcal{A}} \geq 4t_2^{\mathcal{A}}$. This proves the claim. $\square$

We can determine all simplicial line arrangements in $\mathbb{P}^2(\mathbb{R})$ whose characteristic polynomials split over $\mathbb{R}$ and for which the bound in part a) of Proposition 5.1 is tight.

Theorem 5.1. *Let $\mathcal{A}$ be an irreducible spherical Tits arrangement in $\mathbb{R}^3$ and write $n := |\mathcal{A}|$. Suppose that $t_2^{\mathcal{A}} = \frac{\binom{n}{2}+6}{7}$ and assume that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$. Then $\mathcal{A}$ is combinatorially isomorphic to one of the arrangements $A(6,1), A(9,1), A(13,2)$ (as denoted in [19]).*

Proof. By part b) of Proposition 5.1 we obtain the inequality

$$\frac{n^2-n}{14} + \frac{6}{7} = \frac{\binom{n}{2}+6}{7} = t_2^{\mathcal{A}} \leq \frac{n^2}{16} + \frac{n}{8} + \frac{1}{16}.$$

This in turn leads to the inequality $\frac{n^2}{112} - \frac{11n}{56} + \frac{89}{112} \leq 0$, which clearly only holds for $6 \leq n \leq 16$. Now we may use Theorem 2.5 to verify that the given examples are the only ones in the corresponding range for n. $\square$

Remark 5.3. Let $\mathcal{A}$ be a simplicial line arrangement with $|\mathcal{A}| = n$. If $t_2^{\mathcal{A}} = \frac{\binom{n}{2}+6}{7}$ then we have equality in the chain of inequalities $4t_2^{\mathcal{A}} \leq f_2^{\mathcal{A}} \leq \frac{n^2-n}{3} - \frac{2}{3}t_2^{\mathcal{A}} + 4$. This in turn implies that $\mathcal{A}$ has multiplicity at most four. Hence, Theorem 5.1 may also be deduced as a consequence of Theorem 5.5 from Subsection 5.1.3. Moreover, we observe that for any simplicial line arrangement $\mathcal{A}$ we have $t_3^{\mathcal{A}} = 4 \Leftrightarrow t_4^{\mathcal{A}} = \frac{\binom{n}{2}-15}{7} \Leftrightarrow t_2^{\mathcal{A}} = \frac{\binom{n}{2}+6}{7}$.

In the following, we want to establish an upper bound for $\min(t_2^{\mathcal{A}}, t_3^{\mathcal{A}})$ and a lower bound for $\max(t_2^{\mathcal{A}}, t_3^{\mathcal{A}})$. In order to do this we use a little lemma which may be interesting in its own right.

Lemma 5.5. *Let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$. Then the following statements are true:*
a) We have $2t_2^{\mathcal{A}} \leq \frac{f_2^{\mathcal{A}}}{2} < 2t_2^{\mathcal{A}} + t_3^{\mathcal{A}} \leq 2t_2^{\mathcal{A}} + t_3^{\mathcal{A}} + \frac{t_4^{\mathcal{A}}}{3} \leq \frac{\binom{n}{2}}{3} + 5$.
b) If $t_i^{\mathcal{A}} = 0$ for $i > 6$ then we have $2t_2^{\mathcal{A}} + t_3^{\mathcal{A}} + \frac{t_4^{\mathcal{A}}}{3} = \frac{\binom{n}{2}}{3} + 5$. In particular, we cannot have $t_4^{\mathcal{A}} \equiv 2 \pmod 3$ for such an arrangement.

Proof. a) Equation (5.1) from Lemma 5.1 gives $3t_3^{\mathcal{A}} = \binom{n}{2} - t_2^{\mathcal{A}} - \sum_{i\geq 4}\binom{i}{2}t_i^{\mathcal{A}}$. Moreover, for $i \geq 5$ we always have $\binom{i}{2} \geq 5(i-3)$ and so we conclude that

$$6t_4^{\mathcal{A}} + \sum_{i\geq 5}\binom{i}{2}t_i^{\mathcal{A}} \geq t_4^{\mathcal{A}} + 5\sum_{i\geq 4}(i-3)t_i^{\mathcal{A}} = 5t_2^{\mathcal{A}} + t_4^{\mathcal{A}} - 15.$$

It follows $3t_3^{\mathcal{A}} \leq \binom{n}{2} - 6t_2^{\mathcal{A}} - t_4^{\mathcal{A}} + 15$. This proves the upper bound for $2t_2^{\mathcal{A}} + t_3^{\mathcal{A}} + \frac{t_4^{\mathcal{A}}}{3}$. Now suppose that $2t_2^{\mathcal{A}} + t_3^{\mathcal{A}} \leq \frac{f_2^{\mathcal{A}}}{2}$. From this we deduce that

$$3 + \sum_{i\geq 4}(i-3)t_i^{\mathcal{A}} = t_2^{\mathcal{A}} \leq \sum_{i\geq 4} t_i^{\mathcal{A}} - 1,$$

which leads to $4 \leq 0$. This contradiction shows that $2t_2^{\mathcal{A}} + t_3^{\mathcal{A}} > \frac{f_2^{\mathcal{A}}}{2}$. Finally, the inequality $2t_2^{\mathcal{A}} \leq \frac{f_2^{\mathcal{A}}}{2}$ follows from Lemma 5.2, part b).
b) If $t_i^{\mathcal{A}} = 0$ for $i > 6$ then $t_2^{\mathcal{A}} + 3t_3^{\mathcal{A}} + 6t_4^{\mathcal{A}} + 10t_5^{\mathcal{A}} + 15t_6^{\mathcal{A}} = \binom{n}{2}$, using equation (5.1) from Lemma 5.1. By simpliciality of $\mathcal{A}$ we have $t_2^{\mathcal{A}} = 3 + t_4^{\mathcal{A}} + 2t_5^{\mathcal{A}} + 3t_6^{\mathcal{A}}$ (see again Lemma 5.1). We conclude that $6t_2^{\mathcal{A}} + 3t_3^{\mathcal{A}} + t_4^{\mathcal{A}} - 15 = \binom{n}{2}$.

It follows $n^2 - n - 2t_4^{\mathcal{A}} \equiv 0 \pmod 3$. As the polynomial $X^2 - X + 2$ is irreducible over the finite field $\mathbb{F}_3$, it follows that $t_4^{\mathcal{A}} \not\equiv 2 \pmod 3$. This completes the proof. □

Corollary 5.2. *Let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$. Then we have $\min(t_2^{\mathcal{A}}, t_3^{\mathcal{A}}) \leq \frac{n^2-n+30}{18}$ and $\max(t_2^{\mathcal{A}}, t_3^{\mathcal{A}}) > \frac{f_2^{\mathcal{A}}}{6}$.*

Proof. Assume that $\max_{i\geq 2} t_i^{\mathcal{A}} = t_3^{\mathcal{A}}$. Then by Lemma 5.5 we have

$$3t_2^{\mathcal{A}} \leq 2t_2^{\mathcal{A}} + t_3^{\mathcal{A}} \leq \frac{\binom{n}{2}}{3} + 5,$$
$$\frac{f_2^{\mathcal{A}}}{2} < 2t_2^{\mathcal{A}} + t_3^{\mathcal{A}} \leq 3t_3^{\mathcal{A}}.$$

This proves the claim in case $\max_{i\geq 2} t_i^{\mathcal{A}} = t_3^{\mathcal{A}}$. The case $\max_{i\geq 2} t_i^{\mathcal{A}} = t_2^{\mathcal{A}}$ is dealt with similarly. □

We close this subsection with a theorem which gives a quadratic lower bound for $\max(t_2^{\mathcal{A}}, t_3^{\mathcal{A}})$ if $\mathcal{A}$ is an irreducible simplicial line arrangement:

Theorem 5.2. *Let $\mathcal{A}$ be an irreducible simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$. Then we have the inequality*

$$f_2^{\mathcal{A}} \geq \left\lceil \frac{2(|\mathcal{A}|^2 + 3|\mathcal{A}|)}{9} \right\rceil.$$

Thus, by Corollary 5.2 we obtain:

$$\max(t_2^{\mathcal{A}}, t_3^{\mathcal{A}}) > \left\lceil \frac{|\mathcal{A}|^2 + 3|\mathcal{A}|}{27} \right\rceil.$$

Proof. Observe that by Proposition 2.1 we know that $t_i^{\mathcal{A}} = 0$ for $i > \frac{|\mathcal{A}|}{2}$. Thus, we may apply inequality (2.4) from Theorem 2.1 to obtain the following estimate:

$$f_1^{\mathcal{A}} = \sum_{i \geq 2} i t_i^{\mathcal{A}} \geq \left\lceil \frac{|\mathcal{A}|^2 + 3|\mathcal{A}|}{3} \right\rceil.$$

As $\mathcal{A}$ is simplicial we have $3f_2^{\mathcal{A}} = 2f_1^{\mathcal{A}}$ (see Lemma 5.1), which proves the lower bound for $f_2^{\mathcal{A}}$. Hence, using Corollary 5.2 we obtain the inequality

$$\max(t_2^{\mathcal{A}}, t_3^{\mathcal{A}}) > \frac{f_2^{\mathcal{A}}}{6} \geq \left\lceil \frac{|\mathcal{A}|^2 + 3|\mathcal{A}|}{27} \right\rceil,$$

finishing the proof of the theorem. □

Remark 5.4. If $\mathcal{A}$ has multiplicity at most five, then the estimate given in Theorem 5.2 can be improved by using [31, Theorem 1]: this result says that $f_2^{\mathcal{A}} \geq 2\frac{n^2-n+2t}{t+3}$, where t denotes the multiplicity of $\mathcal{A}$.

5.1.2 A combinatorial characterization of finite rank three Coxeter arrangements

In this subsection we prove that, up to combinatorial isomorphism, spherical rank three Coxeter arrangements are characterized as those arrangements $\mathcal{A}$ having the property that $\Gamma_C^K \cong \Gamma_C$ for any chamber K of $\mathcal{A}$ and a suitable fixed connected Coxeter diagram Γ_C. This may be regarded as a combinatorial analogue of the theorem which asserts that spherical Coxeter arrangements may be characterized as those arrangements having isometric chambers (see [14]).

Lemma 5.6. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2(\mathbb{R})$. Assume that there is a connected Coxeter diagram Γ_C such that $\Gamma_C^K \cong \Gamma_C$ for every chamber K of $\mathcal{A}$. Then $\mathcal{A}$ is simplicial and there exists $x \in \mathbb{N}$ such that*

$$\Gamma_C = \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\!\!\overset{x}{-\!\!\!-\!\!\!-}\!\!\bullet.$$

Proof. Write $n := |\mathcal{A}|$ and observe that $\mathcal{A}$ is not a near pencil arrangement. Then by Shannon's theorem (see [30]) we know that $\mathcal{A}$ contains at least n chambers which are triangles. By assumption, this implies that every chamber of $\mathcal{A}$ must be a triangle and hence the arrangement $\mathcal{A}$ is necessarily simplicial. We have $t_2^{\mathcal{A}} \geq 3$ by inequality (5.5) from Lemma 5.1. Moreover, we have $t_3^{\mathcal{A}} \geq 4$ by part c) of Lemma 5.2. This proves the claim. $\square$

Proposition 5.2. *Fix $x \in \mathbb{N}$ and let $\mathcal{A}$ be a simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$ such that $\Gamma_C^K \cong \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\!\!\overset{x}{-\!\!\!-\!\!\!-}\!\!\bullet$ for every chamber K of $\mathcal{A}$. Then $x \in \{3,4,5\}$. Moreover, if $x = 3$ then $\mathcal{A}$ is combinatorially isomorphic to the reflection arrangement of type A_3, if $x = 4$ then $\mathcal{A}$ is combinatorially isomorphic to the reflection arrangement of type B_3 and if $x = 5$ then $\mathcal{A}$ is combinatorially isomorphic to the reflection arrangement of type H_3.*

Proof. Set $n := |\mathcal{A}|$ and suppose that $x = 3$, so $\mathcal{A}$ has only vertices of weight two or three. Then $\mathcal{A}$ is of type $A(6,1)$ or $A(7,1)$ (as denoted in [19]). Since the arrangement $A(7,1)$ contains a chamber having only vertices of weight three, it follows that $\mathcal{A}$ is of type $A(6,1)$.

Now assume that $x > 3$. As $\mathcal{A}$ is simplicial we have equality in relation (5.5) of Lemma 5.1:

$$t_2^{\mathcal{A}} - (x-3)t_x^{\mathcal{A}} - 3 = 0. \tag{5.6}$$

On the other hand, Lemma 5.1 also yields the following identities:

$$t_2^{\mathcal{A}} + \binom{3}{2} t_3^{\mathcal{A}} + \binom{x}{2} t_x^{\mathcal{A}} - \binom{n}{2} = 0, \tag{5.7}$$

$$2t_2^{\mathcal{A}} - 3t_3^{\mathcal{A}} = 0, \tag{5.8}$$

$$3t_3^{\mathcal{A}} - x t_x^{\mathcal{A}} = 0. \tag{5.9}$$

Interpret x as an indeterminate and consider the function field $\mathbb{K} := \mathbb{Q}(x)$. Think of $n, t_2^{\mathcal{A}}, t_3^{\mathcal{A}}, t_x^{\mathcal{A}}$ as variables in a polynomial ring $R := \mathbb{K}[n, t_2^{\mathcal{A}}, t_3^{\mathcal{A}}, t_x^{\mathcal{A}}]$ and consider the ideal I generated by the relations (5.6), (5.7), (5.8), (5.9). We compute the following Gröbner basis for I:

$$I = \left(2\binom{n}{2} + \frac{12x + 6x^2}{x-6}, t_2^{\mathcal{A}} + \frac{3x}{x-6}, t_3^{\mathcal{A}} + \frac{2x}{x-6}, t_x^{\mathcal{A}} + \frac{6}{x-6} \right).$$

As $t_2^{\mathcal{A}} > 0$ and $x \geq 4$ we infer that $x - 6 < 0$, hence $4 \leq x \leq 5$. For $x = 4$ we obtain $n = 9$ and $t^{\mathcal{A}} = (6,4,3)$. If $x = 5$ then $n = 15$ and $t^{\mathcal{A}} = (15,10,0,6)$. Now we may use Theorem 2.5 to obtain the full statement. $\square$

Lemma 5.6 and Proposition 5.2 now immediately give us the desired theorem:

Theorem 5.3. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2(\mathbb{R})$ with associated set of chambers $\mathcal{K}$. Then the following statements are equivalent:*
i) There exists a connected Coxeter diagram Γ_C such that $\Gamma_C^K \cong \Gamma_C$ for every chamber $K \in \mathcal{K}$.
ii) $\mathcal{A}$ is combinatorially isomorphic to a spherical Coxeter arrangement.

5.1.3 Simplicial line arrangements having low multiplicities

In this subsection we prove that there are only finitely many (combinatorial isomorphism classes of) irreducible simplicial line arrangements $\mathcal{A}$ whose characteristic polynomials split over $\mathbb{R}$ and which have multiplicity bounded by five. Among these arrangements, we are able to give a classification of those having multiplicity bounded by four. If $\mathcal{A}$ has multiplicity bounded by six and if $t_3^{\mathcal{A}} \geq \frac{3}{4} t_2^{\mathcal{A}}$, then we can also prove that there are only finitely many possibilities for the isomorphism class of $\mathcal{A}$, again provided that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$. Finally, we show that the validity of an old conjecture stated in [15] is related to Theorem 5.4, which gives estimates for the values of $t_2^{\mathcal{A}}, t_3^{\mathcal{A}}, t_4^{\mathcal{A}}, t_5^{\mathcal{A}}, t_6^{\mathcal{A}}$ and which is considered the main result of this subsection.

We start with the mentioned main result:

Theorem 5.4. *Let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$. Suppose that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$ and assume that $\mathcal{A}$ has multiplicity at most six. Then we have:*

$$\frac{n^2 - 46n + 273}{16} \leq t_2^{\mathcal{A}} \leq \frac{(n+1)^2}{16}, \tag{5.10}$$

$$\frac{n^2 - 130n + 797}{24} \leq t_3^{\mathcal{A}} \leq \frac{n^2 + 134n - 699}{24}, \tag{5.11}$$

$$t_4^{\mathcal{A}} + t_5^{\mathcal{A}} \leq \frac{3}{2} n - \frac{17}{2}, \tag{5.12}$$

$$\frac{n^2 - 46n + 225}{48} \leq t_6^{\mathcal{A}} \leq \frac{n^2 + 2n - 47}{48}. \tag{5.13}$$

In particular, all estimates hold true if $\mathcal{A}$ is a free arrangement having multiplicity bounded by six.

Proof. By Lemma 5.1 and part b) of Proposition 5.1 we have $t_2^{\mathcal{A}} = 3 + t_4^{\mathcal{A}} + 2t_5^{\mathcal{A}} + 3t_6^{\mathcal{A}} \leq \frac{(n+1)^2}{16}$. We conclude $t_6^{\mathcal{A}} \leq \frac{n^2}{48} + \frac{n}{24} - \frac{t_4^{\mathcal{A}}}{3} - \frac{2t_5^{\mathcal{A}}}{3} - \frac{47}{48}$.

As $3t_3^{\mathcal{A}} = \binom{n}{2} - t_2^{\mathcal{A}} - \sum_{i \geq 4} \binom{i}{2} t_i^{\mathcal{A}}$ and $f_2^{\mathcal{A}} = 2\left(f_0^{\mathcal{A}} - 1\right)$, we may use Lemma 5.4 to rewrite the condition that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$ as $\frac{n^2}{48} - \frac{5n}{24} +$

$\frac{7}{16} - \frac{t_5^{\mathcal{A}}}{2} - \frac{t_4^{\mathcal{A}}}{6} \leq t_6^{\mathcal{A}}$. Combining the last two inequalities we obtain:

$$\frac{n^2}{48} - \frac{5n}{24} + \frac{7}{16} - \frac{t_5^{\mathcal{A}}}{2} - \frac{t_4^{\mathcal{A}}}{6} \leq t_6^{\mathcal{A}} \leq \frac{n^2}{48} + \frac{n}{24} - \frac{t_4^{\mathcal{A}}}{3} - \frac{2t_5^{\mathcal{A}}}{3} - \frac{47}{48}. \tag{5.14}$$

This implies inequality (5.12). Inequality (5.13) now follows from (5.14) using (5.12): we have $\frac{n^2}{48} - \frac{5n}{24} + \frac{7}{16} - \frac{t_4^{\mathcal{A}}+t_5^{\mathcal{A}}}{2} \leq \frac{n^2}{48} - \frac{5n}{24} + \frac{7}{16} - \frac{t_5^{\mathcal{A}}}{2} - \frac{t_4^{\mathcal{A}}}{6}$ and as $t_4^{\mathcal{A}} + t_5^{\mathcal{A}} \leq \frac{3n}{2} - \frac{17}{2}$ we conclude $\frac{n^2-46n+225}{48} = \frac{n^2}{48} - \frac{5n}{24} + \frac{7}{16} - \frac{3n}{4} + \frac{17}{4} \leq t_6^{\mathcal{A}}$. Moreover, $t_6^{\mathcal{A}} \leq \frac{n^2}{48} + \frac{n}{24} - \frac{t_4^{\mathcal{A}}}{3} - \frac{2t_5^{\mathcal{A}}}{3} - \frac{47}{48} \leq \frac{n^2}{48} + \frac{n}{24} - \frac{47}{48}$ because $t_i^{\mathcal{A}} \geq 0$ for all $i \geq 2$.

Finally, $t_2^{\mathcal{A}} \geq 3 + 3t_6^{\mathcal{A}} \geq 3\left(1 + \frac{n^2-46n+225}{48}\right) = \frac{n^2-46n+273}{16}$, which establishes inequality (5.10). Inequality (5.11) now follows from (5.10),(5.12) and (5.13) using equation (5.1) of Lemma 5.1. This completes the proof. □

We now draw some conclusions from Theorem 5.4.

Theorem 5.5. *Let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$ having multiplicity at most four. Assume that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$. Then $n \leq 16$. In particular, we have a complete list (up to combinatorial isomorphism) of such arrangements.*

Proof. We have $t_i^{\mathcal{A}} = 0$ for $i > 4$ and as $\mathcal{A}$ is simplicial, we have equality in relation (5.5) of Lemma 5.1. It follows $t_2^{\mathcal{A}} = 3 + t_4^{\mathcal{A}}$. Consequently, by Theorem 5.4 we obtain the upper bound $t_2^{\mathcal{A}} \leq 3 + \frac{3}{2}n - \frac{17}{2}$.

Similarly, equation (5.1) of Lemma 5.1 gives $3t_3^{\mathcal{A}} = \binom{n}{2} - 7t_2^{\mathcal{A}} + 18$. Using this, the condition $(n+1)^2 \geq 4f_2^{\mathcal{A}}$ translates to $t_2^{\mathcal{A}} \geq \frac{(n-10)n+45}{8}$. It follows $\frac{(n-10)n+45}{8} \leq t_2^{\mathcal{A}} \leq \frac{3}{2}n - \frac{11}{2}$ which implies $1 \leq n \leq 16$. This proves the first claim. The second claim follows from Theorem 2.5. □

Corollary 5.3. *Let $\mathcal{A}$ be a line arrangement satisfying the conditions of Theorem 5.5. Then $\mathcal{A}$ admits a crystallographic rootset. In particular, every free simplicial line arrangement having multiplicity bounded by four admits a crystallographic rootset.*

Proof. Both claims can be verified using Theorem 2.5. For the second claim, observe that the roots of the characteristic polynomial of any free arrangement are given by its exponents. Thus, these roots are integral and in particular real. □

Remark 5.5. a) As every crystallographic arrangement is inductively free (see [1]), the last corollary shows that for a simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$ whose multiplicity is at most four, the notions of being free and being inductively free coincide.
b) We observe that every line arrangement satisfying the conditions of Theorem 5.5 is a subarrangement of the arrangement $A(13, 2)$. This can be

verified by inspecting the Hasse diagram given in [6]. One should also observe that the latter arrangement is obtained as a restriction of the reflection arrangement of type F_4. Thus, all free simplicial line arrangements having multiplicity bounded by four originate from a reflection group.

We can prove another classification result concerning simplicial arrangements in $\mathbb{P}^2(\mathbb{R})$ having multiplicity bounded by four. In this case we do not need any assumption on $\chi(\mathcal{A}, t)$, thus the following theorem is not a direct consequence of Theorem 5.4.

Theorem 5.6. *Let $\mathcal{A}$ be an irreducible simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$ having multiplicity at most four. Suppose that every chamber of $\mathcal{A}$ has at most one vertex of weight four and assume that every vertex of weight three has at least two neighbours of weight two. Then $\mathcal{A}$ is combinatorially isomorphic to one of the arrangements $A(6,1), A(7,1), A(8,1), A(9,1), A(10,2)$ (as denoted in [19]).*

Proof. We may assume that $n := |\mathcal{A}| \geq 8$. If every chamber has at most one vertex of weight four, then it follows that $8t_2^{\mathcal{A}} - 24 = 8t_4^{\mathcal{A}} \leq f_2^{\mathcal{A}} = \frac{n^2-n}{3} + 4 - \frac{2}{3}t_2^{\mathcal{A}}$. Thus, the number of vertices of weight two is bounded from above by $t_2^{\mathcal{A}} \leq \frac{n^2-n+84}{26}$. Write $\mathcal{V}_2(\mathcal{A})$ for the set of vertices of weight two of the arrangement $\mathcal{A}$. For $v \in \mathcal{V}_2(\mathcal{A})$, denote by e_v the number of vertices of weight three which are connected to v by an edge. Then by assumption and Lemma 5.3 we have the estimate $2t_3^{\mathcal{A}} \leq \sum_{v\in\mathcal{V}_2} e_v \leq 3t_2^{\mathcal{A}}$. Using the identity $2t_2^{\mathcal{A}} + t_3^{\mathcal{A}} + \frac{1}{3}t_4^{\mathcal{A}} = \frac{n^2-n}{6} + 5$, we obtain the lower bound $t_2^{\mathcal{A}} \geq \frac{n^2-n+36}{23}$. Hence, we have established the inequality $\frac{n^2-n+36}{23} \leq t_2^{\mathcal{A}} \leq \frac{n^2-n+84}{26}$. This implies $n \leq 18$, so the statement is obtained by Theorem 2.5. □

Remark 5.6. The conditions on the vertices of weight three in the last theorem can be relaxed. If we assume that only $\frac{7}{10}$ of the vertices of weight three have at least two neighbours of weight two, while the remaining vertices of weight three have at least one neighbour of weight two, then it follows that $n \leq 28$. But for $n = 28$ we have $32 < \frac{612}{19} \leq t_2^{\mathcal{A}} \leq \frac{420}{13} < 33$, contradicting the fact that $t_2^{\mathcal{A}} \in \mathbb{Z}$. However, we obtain the same arrangements as in Theorem 5.6.

We can also relax the condition on the vertices of weight four in the last theorem. However, we only obtain a finiteness result in this case.

Theorem 5.7. *Let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$ having multiplicity at most four. Write $\mathcal{K}_{4,i}$ for the number of chambers containing i vertices of weight four. If $\mathcal{K}_{4,3} \leq n, \mathcal{K}_{4,2} \leq \frac{1}{8}f_2^{\mathcal{A}} + n$ and if every vertex of weight three has at least two neighbours of weight two, then $n \leq 923$.*

Proof. The proof is similar to that of Theorem 5.6. Writing $\mathcal{V}_4$ for the set of vertices of weight four and using the conditions on $\mathcal{K}_{4,2}$ and $\mathcal{K}_{4,3}$, we obtain the estimate $8t_2^{\mathcal{A}} - 24 = 8t_4^{\mathcal{A}} = \sum_{K\in\mathcal{K}} |\overline{K} \cap \mathcal{V}_4| \leq \frac{2}{8} f_2^{\mathcal{A}} + \frac{7}{8} f_2^{\mathcal{A}} + 2n + 3n = \frac{9}{8} f_2^{\mathcal{A}} + 5n = \frac{9}{8}\left(\frac{n^2-n}{3} + 4 - \frac{2}{3}t_2^{\mathcal{A}}\right) + 5n$. We derive the upper bound $t_2^{\mathcal{A}} \leq \frac{3n^2+37n+228}{70}$. By the same argument as in the proof of Theorem 5.6 we obtain the lower bound $t_2^{\mathcal{A}} \geq \frac{n^2-n+36}{23}$, leading to the inequality $\frac{n^2-n+36}{23} \leq \frac{3n^2+37n+228}{70}$. It follows $n \leq 923$. □

The next theorem yields the announced finiteness result for simplicial arrangements whose multiplicity is bounded by five and whose characteristic polynomial splits over $\mathbb{R}$.

Theorem 5.8. *Let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$ having multiplicity at most five. Assume that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$. Then it follows that $n \leq 40$.*

Proof. By assumption and Theorem 5.4 we have $\frac{n^2-46n+225}{48} \leq t_6^{\mathcal{A}} = 0$. This implies $n \leq 40$. □

If in the situation of the last theorem we restrict attention to those arrangements satisfying $t_3^{\mathcal{A}} \geq \frac{13}{16} t_2^{\mathcal{A}}$, then we can give the following classification result:

Theorem 5.9. *Let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$ having multiplicity at most five. If $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$ and if $t_3^{\mathcal{A}} \geq \frac{13}{16} t_2^{\mathcal{A}}$, then $n \leq 27$. In particular, we have a complete list (up to combinatorial isomorphism) of such arrangements.*

Proof. First, we observe that $\binom{n}{2} = t_2^{\mathcal{A}} + 3t_3^{\mathcal{A}} + 6t_4^{\mathcal{A}} + 10t_5^{\mathcal{A}}$. Moreover, as $\mathcal{A}$ is simplicial we may write $5t_4^{\mathcal{A}} + 10t_5^{\mathcal{A}} = 5(t_2^{\mathcal{A}} - 3)$. Using this, we conclude that $\binom{n}{2} = 6t_2^{\mathcal{A}} + 3t_3^{\mathcal{A}} + t_4^{\mathcal{A}} - 15$, which implies $t_3^{\mathcal{A}} = \frac{n^2-n}{6} - 2t_2^{\mathcal{A}} - \frac{t_4^{\mathcal{A}}}{3} + 5$. Since $f_2^{\mathcal{A}} = \binom{n}{2} + 1 - \sum_{j\geq 3}\binom{j-1}{2} t_j^{\mathcal{A}}$, the splitting of $\chi(\mathcal{A}, t)$ yields the estimate $t_3^{\mathcal{A}} + 3t_2^{\mathcal{A}} - 9 = t_3^{\mathcal{A}} + 3t_4^{\mathcal{A}} + 6t_5^{\mathcal{A}} \geq \binom{n}{2} + 1 - \frac{(n+1)^2}{4} = \frac{n^2-4n+3}{4}$. Plugging in the expression for $t_3^{\mathcal{A}}$ obtained above and remembering that $t_4^{\mathcal{A}} \geq 0$, we conclude that $t_2^{\mathcal{A}} \geq \frac{n^2-4n+3}{4} - \frac{n^2-n}{6} + \frac{t_4^{\mathcal{A}}}{3} + 4 \geq \frac{n^2-10n+57}{12}$. Finally, because $\frac{13}{16} t_2^{\mathcal{A}} \leq t_3^{\mathcal{A}}$, we may invoke Lemma 5.5 to arrive at the inequality $\frac{n^2-10n+57}{12} \leq t_2^{\mathcal{A}} \leq \frac{8(n^2-n+30)}{135}$. It is now easy to see that we necessarily must have $n \leq 27$. □

Next we show that there are only finitely many combinatorial isomorphism classes of simplicial line arrangements $\mathcal{A}$ such that $\frac{3}{4} t_2^{\mathcal{A}} \leq t_3^{\mathcal{A}}$, provided $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$ and $\mathcal{A}$ has multiplicity at most six.

Theorem 5.10. *Let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$ having multiplicity bounded by six. Assume that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$. If $\frac{3}{4} t_2^{\mathcal{A}} \leq t_3^{\mathcal{A}}$ then $n \leq 1480$.*

Proof. By Theorem 5.4 we have $\frac{n^2-46n+273}{16} \leq t_2^{\mathcal{A}}$ and $t_3^{\mathcal{A}} \leq \frac{n^2+134n-699}{24}$. By assumption we obtain $\frac{n^2+134n-699}{24} \geq \frac{3(n^2-46n+273)}{64}$. This is possible only when $n \leq 1480$. $\square$

Unfortunately, we are not able to prove that the set M of combinatorial isomorphism classes of arrangements considered in Theorem 5.4 is finite: the case $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$ remains open in general. However, we can relate the finiteness of this set to the following conjecture stated in [15]:

Conjecture 4. *Let $5 \leq k \in \mathbb{N}$ be a natural number and let $\mathfrak{A}_k$ denote the set of all line arrangements in $\mathbb{P}^2(\mathbb{R})$ having multiplicity at most k. Then for any sequence of arrangements $(\mathcal{A}_\nu)_{\nu\in\mathbb{N}}$ such that $\mathcal{A}_\nu \in \mathfrak{A}_k$ and $\lim_{\nu\to\infty} |\mathcal{A}_\nu| = \infty$ we have*

$$\lim_{\nu\to\infty} \frac{t_k^{\mathcal{A}_\nu}}{|\mathcal{A}_\nu|^2} = 0.$$

Then Theorem 5.4 yields the following corollary, which closes this section.

Corollary 5.4. *If $|M| = \infty$ then Conjecture 4 is false for $k = 6$. Equivalently, if Conjecture 4 is true for $k = 6$, then $|M| < \infty$.*

Proof. By Theorem 5.4 we have $\frac{|\mathcal{A}|^2-46|\mathcal{A}|+225}{48} \leq t_6^{\mathcal{A}} \leq \frac{|\mathcal{A}|^2+2|\mathcal{A}|-47}{48}$, if $\mathcal{A} \in M$. So if $|M| = \infty$ we find a sequence $(\mathcal{A}_\nu)_{\nu\in\mathbb{N}}$ such that $\lim_{\nu\to\infty} |\mathcal{A}_\nu| = \infty$ and $\mathcal{A}_\nu \in M$ for every $\nu \in \mathbb{N}$. But then $\lim_{\nu\to\infty} \frac{t_6^{\mathcal{A}_\nu}}{|\mathcal{A}_\nu|^2} = \frac{1}{48} > 0$ and Conjecture 4 is false for $k = 6$. $\square$

5.1.4 Free line arrangements having only few vertices of high multiplicity

In this subsection we investigate the combinatorics of projective line arrangements whose characteristic polynomials have only real roots. Our first main result is Theorem 5.11, which implies that a free line arrangement having only vertices of weight bounded by five consists of at most 185 lines. In particular, there are only finitely many combinatorial isomorphism classes of such arrangements. Moreover, it turns out that if the number of vertices of $\mathcal{A}$ having high multiplicity is not too large, then the number of lines of $\mathcal{A}$ may be bounded from above. This is made precise in our second main result, Theorem 5.12.

We start with the following lemma which is similar to Theorem 5.4 of the last subsection.

Lemma 5.7. *Let $\mathcal{A}$ be an arrangement of $n \geq 8$ lines in $\mathbb{P}^2(\mathbb{R})$ having multiplicity at most five. Assume that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$. Then the following estimates hold:*

$$\frac{t_4^{\mathcal{A}}}{3} + t_5^{\mathcal{A}} \geq \frac{n^2 - 10n + 21}{24}, \tag{5.15}$$

$$t_2^{\mathcal{A}} \geq \frac{n^2 - 46n + 233}{8} + 2t_4^{\mathcal{A}}, \tag{5.16}$$

$$\max(t_4^{\mathcal{A}}, t_5^{\mathcal{A}}) \geq \frac{n^2 - 10n + 21}{32}. \tag{5.17}$$

Proof. By Theorem 2.1, relation (2.3) we have $t_2^{\mathcal{A}} + \frac{3t_3^{\mathcal{A}}}{2} \geq 8 + \frac{t_4^{\mathcal{A}}}{2} + \frac{5t_5^{\mathcal{A}}}{2}$. As $3t_3^{\mathcal{A}} = \binom{n}{2} - t_2^{\mathcal{A}} - 6t_4^{\mathcal{A}} - 10t_5^{\mathcal{A}}$ we may rewrite this as $\frac{t_2^{\mathcal{A}}}{2} + \frac{n^2-n}{4} - 3t_4^{\mathcal{A}} - 5t_5^{\mathcal{A}} \geq 8 + \frac{t_4^{\mathcal{A}}}{2} + \frac{5t_5^{\mathcal{A}}}{2}$. It follows $t_5^{\mathcal{A}} \leq \frac{n^2-n}{30} + \frac{t_2^{\mathcal{A}}}{15} - \frac{7t_4^{\mathcal{A}}}{15} - \frac{16}{15}$. By Lemma 5.4 the splitting of $\chi(\mathcal{A}, t)$ over $\mathbb{R}$ translates to $(n+1)^2 \geq 4f_2^{\mathcal{A}}$. Equation (5.2) in Lemma 5.1 yields $f_2^{\mathcal{A}} = 1 + f_1^{\mathcal{A}} - f_0^{\mathcal{A}} = 1 + t_2^{\mathcal{A}} + 2t_3^{\mathcal{A}} + 3t_4^{\mathcal{A}} + 4t_5^{\mathcal{A}} = \frac{n^2-n+3}{3} + \frac{t_2^{\mathcal{A}}}{3} - t_4^{\mathcal{A}} - \frac{8t_5^{\mathcal{A}}}{3}$. Now we use inequality (5.5) from Lemma 5.1 to conclude that the estimate

$$\frac{n^2}{4} + \frac{n}{2} + \frac{1}{4} \geq f_2^{\mathcal{A}} \geq \frac{n^2 - n + 6}{3} - \frac{2t_4^{\mathcal{A}}}{3} - 2t_5^{\mathcal{A}}$$

holds. From this we deduce that $t_5^{\mathcal{A}} \geq \frac{n^2}{24} - \frac{5n}{12} + \frac{7}{8} - \frac{t_4^{\mathcal{A}}}{3}$, proving (5.15). We have thus established the following chain of inequalities:

$$\frac{n^2 - 10n}{24} + \frac{7}{8} - \frac{t_4^{\mathcal{A}}}{3} \leq t_5^{\mathcal{A}} \leq \frac{n^2 - n}{30} + \frac{t_2^{\mathcal{A}} - 16}{15} - \frac{7t_4^{\mathcal{A}}}{15}.$$

This implies (5.16). In order to prove (5.17) we consider two cases. First assume that $t_4^{\mathcal{A}} \leq t_5^{\mathcal{A}}$. Then by the above we know that $t_5^{\mathcal{A}} \geq \frac{n^2-10n}{24} + \frac{7}{8} - \frac{t_4^{\mathcal{A}}}{3} \geq \frac{n^2-10n}{24} + \frac{7}{8} - \frac{t_5^{\mathcal{A}}}{3}$. From this we conclude that $t_5^{\mathcal{A}} \geq \frac{n^2-10n+21}{32}$. The case $t_4^{\mathcal{A}} \geq t_5^{\mathcal{A}}$ is dealt with similarly. This finishes the proof. $\square$

With the last lemma we are ready to prove our first result for this section.

Theorem 5.11. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2(\mathbb{R})$ having multiplicity bounded by five. Assume that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$. Then $|\mathcal{A}| \leq 185$.*

Proof. We may assume that $n := |\mathcal{A}| > 7$. As $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$ we have $\frac{(n+1)^2}{4} \geq f_2^{\mathcal{A}}$. Together with the first estimate in Lemma 5.7 this yields

$$\frac{(n+1)^2}{4} \geq f_2^{\mathcal{A}} \geq 1 + t_2^{\mathcal{A}} + 2t_3^{\mathcal{A}} + 4\left(\frac{t_4^{\mathcal{A}}}{3} + t_5^{\mathcal{A}}\right) \geq 1 + t_2^{\mathcal{A}} + \frac{n^2 - 10n + 21}{6}.$$

We conclude $\frac{n^2+26n-51}{12} \geq t_2^{\mathcal{A}}$. But now the second estimate in Lemma 5.7 gives us the following chain of inequalities: $\frac{n^2+26n-51}{12} \geq t_2^{\mathcal{A}} \geq \frac{n^2-46n+233}{8}$. This implies $n \leq 185$, finishing the proof. $\square$

Remark 5.7. If one additionally requires $t_5^{\mathcal{A}} = 0$ in the above theorem, then we can conclude $n \leq 19$. This follows immediately from [31, Theorem 1].

We observe that Lemma 5.7 also yields the following classification result in the simplicial case:

Corollary 5.5. *Let $\mathcal{A}$ be a simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$ such that $\chi(\mathcal{A}, t)$ has only real roots. If $\mathcal{A}$ has only vertices of weight two, three or five and if every chamber of $\mathcal{A}$ contains at most one vertex of weight five, then $|\mathcal{A}| \leq 26$. In particular, we have a complete list (up to combinatorial isomorphism) of such arrangements.*

Proof. By Lemma 5.7 we have $t_5^{\mathcal{A}} \geq \frac{n^2-10n+21}{24}$. On the other hand, as every chamber contains at most one vertex of weight five and because $\chi(\mathcal{A}, t)$ has only real roots, it follows that $10t_5^{\mathcal{A}} \leq f_2^{\mathcal{A}} \leq \frac{(n+1)^2}{4}$. Thus, n must satisfy the inequality $\frac{n^2-10n+21}{24} \leq \frac{(n+1)^2}{40}$. This implies $n \leq 26$. □

Next we show that Theorem 5.11 is actually a special case of a more general statement. Observe that contrary to the preceding results, in the following we do not require all vertices of the considered arrangements to have weight bounded by five.

Moreover, in the proof of the next theorem we use the concept of *pseudoline arrangements*. For precise definitions and background on this, see [2]. For us it will be enough to know the following: an arrangement of pseudolines is a finite set $\mathcal{B}$ of $n \geq 3$ smooth closed curves in $\mathbb{P}^2(\mathbb{R})$ such that the following conditions are satisfied:

- Curves in $\mathcal{B}$ do not intersect themselves.
- Different curves in $\mathcal{B}$ intersect transversally at precisely one point.
- Not all curves in $\mathcal{B}$ pass through the same point.

Observe that one may define a t-vector $t^{\mathcal{B}}$ for a pseudoline arrangement $\mathcal{B}$ in the same way as for straight line arrangements. Moreover, the values $f_i^{\mathcal{B}}$ for $0 \leq i \leq 2$ are defined in the same way as well. Using these definitions, all results (except Theorem 5.2) from Subsection 3.1 as well as Lemma 5.7 hold true for arrangements of pseudolines. Using this, we are ready to prove our second main theorem.

Theorem 5.12. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2(\mathbb{R})$ such that $\chi(\mathcal{A}, t)$ has only real roots . Write $\Delta_i := \sum_{j=4}^{i-2} j$ and assume $0 \leq t_i^{\mathcal{A}} \leq \alpha_i$ for $i \geq 6$ and arbitrary real numbers α_i. Then we have $|\mathcal{A}| \leq 95+2\sqrt{2056 + 63\sum_{i\geq 6} \Delta_i\alpha_i}$. In particular, if $\alpha_i = 0$ for all $i \geq 6$ then $|\mathcal{A}| \leq 185$. If additionally $t_5^{\mathcal{A}} = 0$ then $|\mathcal{A}| \leq 19$.*

Proof. We construct a pseudoline arrangement $\mathcal{A}'$ with $n := |\mathcal{A}| = |\mathcal{A}'|$ in the following way: by a perturbation we transform any vertex v of $\mathcal{A}$ having weight $x \geq 6$ into one vertex v' of $\mathcal{A}'$ of weight five and $\lambda(x)$ vertices $w'_1, ..., w'_{\lambda(x)}$ of $\mathcal{A}'$ of weight two. All vertices of $\mathcal{A}$ of weight at most five remain unchanged. One checks that $\lambda(x) = \sum_{j=5}^{x-1} j$. Therefore we obtain $f_2^{\mathcal{A}'} = f_2^{\mathcal{A}} + \sum_{i\geq 6} \Delta_i t_i^{\mathcal{A}}$. The assumptions imply $f_2^{\mathcal{A}'} \leq \frac{(n+1)^2}{4} + \sum_{i\geq 6} \Delta_i \alpha_i$ and by construction we have $t_i^{\mathcal{A}'} = 0$ for $i > 5$. Hence we may write $f_2^{\mathcal{A}'} = 1 + f_1^{\mathcal{A}'} - f_0^{\mathcal{A}'} = 1 + t_2^{\mathcal{A}'} + 2t_3^{\mathcal{A}'} + 3t_4^{\mathcal{A}'} + 4t_5^{\mathcal{A}'} = \frac{n^2-n+3}{3} + \frac{t_2^{\mathcal{A}'}}{3} - t_4^{\mathcal{A}'} - \frac{8t_5^{\mathcal{A}'}}{3}$. Now we use Melchior's inequality to conclude that the estimate

$$\frac{n^2}{4} - \frac{n}{2} + \frac{1}{4} + \sum_{i\geq 6} \Delta_i \alpha_i \geq f_2^{\mathcal{A}'} \geq \frac{n^2-n+6}{3} - \frac{2t_4^{\mathcal{A}'}}{3} - 2t_5^{\mathcal{A}'}$$

holds. From this we deduce that $\frac{t_4^{\mathcal{A}'}}{3} + t_5^{\mathcal{A}'} \geq \frac{n^2}{24} - \frac{5n}{12} + \frac{7}{8} - \frac{1}{2}\sum_{i\geq 6} \Delta_i \alpha_i$. Moreover, by Theorem 2.1, relation (2.3) we have $t_2^{\mathcal{A}'} + \frac{3t_3^{\mathcal{A}'}}{2} \geq 8 + \frac{t_4^{\mathcal{A}'}}{2} + \frac{5t_5^{\mathcal{A}'}}{2}$. As $3t_3^{\mathcal{A}'} = \binom{n}{2} - t_2^{\mathcal{A}'} - 6t_4^{\mathcal{A}'} - 10t_5^{\mathcal{A}'}$ we may rewrite the last inequality as $\frac{t_2^{\mathcal{A}'}}{2} + \frac{n^2-n}{4} - 3t_4^{\mathcal{A}'} - 5t_5^{\mathcal{A}'} \geq 8 + \frac{t_4^{\mathcal{A}'}}{2} + \frac{5t_5^{\mathcal{A}'}}{2}$.
It follows $t_5^{\mathcal{A}'} \leq \frac{n^2-n}{30} + \frac{t_2^{\mathcal{A}'}}{15} - \frac{7t_4^{\mathcal{A}'}}{15} - \frac{16}{15} \leq \frac{n^2-n}{30} + \frac{t_2^{\mathcal{A}'}}{15} - \frac{t_4^{\mathcal{A}'}}{3} - \frac{16}{15}$.
From this, we obtain the chain of inequalities

$$\frac{n^2-n}{30} + \frac{t_2^{\mathcal{A}'}}{15} - \frac{16}{15} \geq \frac{t_4^{\mathcal{A}'}}{3} + t_5^{\mathcal{A}'} \geq \frac{n^2}{24} - \frac{5n}{12} + \frac{7}{8} - \frac{1}{2}\sum_{i\geq 6} \Delta_i \alpha_i,$$

from which we conclude that $t_2^{\mathcal{A}'} \geq \frac{n^2-46n+233}{8} - \frac{15}{2}\sum_{i\geq 6} \Delta_i \alpha_i$. Moreover, $f_2^{\mathcal{A}'} = 1 + t_2^{\mathcal{A}'} + 2t_3^{\mathcal{A}'} + 4(\frac{t_4^{\mathcal{A}'}}{3} + t_5^{\mathcal{A}'}) + \frac{5t_4^{\mathcal{A}'}}{3} \geq 1 + t_2^{\mathcal{A}'} + 4 \cdot \frac{n^2-10n+21}{24} - 2\sum_{i\geq 6} \Delta_i \alpha_i$, implying the estimate $\frac{(n+1)^2}{4} - \frac{n^2-10n+21}{6} - 3\sum_{i\geq 6} \Delta_i \alpha_i - 1 \geq t_2^{\mathcal{A}'}$. Thus we have established the following chain of inequalities:

$$\frac{n^2+26n-51}{12} + 3\sum_{i\geq 6} \Delta_i \alpha_i \geq t_2^{\mathcal{A}'} \geq \frac{n^2-46n+233}{8} - \frac{15}{2}\sum_{i\geq 6} \Delta_i \alpha_i.$$

This is possible only when $n \leq 95 + 2\sqrt{2056 + 63\sum_{i\geq 6} \Delta_i \alpha_i}$. It only remains to prove the last assertion. So assume that $t_i^{\mathcal{A}} = 0$ for $i \geq 5$. Then by [31, Theorem 1] we have $2\frac{n^2-n+8}{7} \leq f_2^{\mathcal{A}} \leq \frac{(n+1)^2}{4}$. This implies $n \leq 19$. $\square$

If $\mathcal{A}$ is a line arrangement in $\mathbb{P}^2(\mathbb{R})$, then we write $\overline{\mathcal{A}}$ for the (combinatorial) isomorphism class of $\mathcal{A}$. Using this notation, the last theorem immediately yields the following corollary, which closes this section.

Corollary 5.6. *Let $\epsilon > 0$ be a real parameter and let $3 \leq x \in \mathbb{N}$. Define the set $\mathfrak{A}_\epsilon^x := \{\overline{\mathcal{A}} \mid \max_{i\geq 6} t_i^{\mathcal{A}} \leq |\mathcal{A}|^{2-\epsilon}, t_i^{\mathcal{A}} = 0 \text{ for } i > x, \chi(\mathcal{A}, t) \text{ splits over } \mathbb{R}\}$. Then $\mathfrak{A}_\epsilon^x$ is finite for any choice of x and ϵ.*

5.2 Simplicial and free line arrangements on algebraic curves

In this section, we will be interested in the combinatorics of free and simplicial line arrangements in $\mathbb{P}^2(\mathbb{R})$ having the property that their dual point sets in $(\mathbb{P}^2(\mathbb{R}))^*$ are contained in the locus of some homogeneous polynomial of bounded degree. This is motivated by the observation that all dual point sets corresponding to simplicial arrangements from the infinite series $\mathcal{R}(1)$ (see Chapter 2, Section 4) are contained in the locus of a cubic polynomial (see also Chapter 4, where affine simplicial arrangements and their dual point sets are studied).

Using results obtained in the previous section, we prove that for any fixed d there are only finitely many combinatorial isomorphism classes of simplicial line arrangements whose dual point sets are contained in the locus of an irreducible homogeneous polynomial of degree at most d (see Theorem 5.13). We then proceed to show that a similar statement holds true for free line arrangements, which are not necessarily simplicial (see Theorem 5.14 and Corollary 5.7).

Before starting with the results, we give a quick reminder on the duality between points and lines in the real projective plane.

Remark 5.8. If $\mathcal{A}$ is an arrangement of lines in $\mathbb{P}^2(\mathbb{R})$, then $\mathcal{A}$ defines a dual set of points $\mathcal{A}^* \subset (\mathbb{P}^2(\mathbb{R}))^*$ in a natural way: if $\ell \in \mathcal{A}$ is a line in $\mathbb{P}^2(\mathbb{R})$, then the corresponding dual point $\ell^* = (X : Y : Z) \in (\mathbb{P}^2(\mathbb{R}))^*$ is characterized by the condition $\ell = \big\{(v_x : v_y : v_z) \in \mathbb{P}^2(\mathbb{R}) \mid Xv_x + Yv_y + Zv_z = 0\big\}$.

Similarly, if $v = (v_x : v_y : v_z) \in \mathbb{P}^2(\mathbb{R})$, then the corresponding dual line $\mathfrak{l} := v^*$ is given by $\mathfrak{l} = \big\{(X : Y : Z) \in (\mathbb{P}^2(\mathbb{R}))^* \mid v_xX + v_yY + v_zZ = 0\big\}$.

We start with the following Lemma, which explains how to use Bézout's theorem in order to give bounds on the multiplicities of vertices of $\mathcal{A}$ if we know that the dual point set $\mathcal{A}^*$ is contained in the locus of some homogeneous polynomial.

Lemma 5.8. *Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$. Assume that $\mathcal{A}^* \subset V(P)$ for some homogeneous polynomial $P \in \mathbb{R}[x, y, z]$ of degree d. Write $P = Q \cdot \prod_{1\le i\le s} l_i$, where the $l_1, ..., l_s \in \mathbb{R}[x, y, z]$ are (not necessarily distinct) linear forms and Q has no linear factors. Then $\mathcal{A}$ contains at most s vertices of multiplicity greater than d.*

Proof. Vertices v of $\mathcal{A}$ are points in $\mathbb{P}^2(\mathbb{R})$. Therefore, the dual $\mathfrak{l} := v^*$ is a line in $(\mathbb{P}^2(\mathbb{R}))^*$. By Bézout's theorem we know that $|\mathcal{A}^* \cap \mathfrak{l}| \le |V(P) \cap \mathfrak{l}| \le d$, unless $\mathfrak{l}$ is a component of $V(P)$. As by assumption $V(P)$ contains at most s linear components, this proves the claim. $\square$

The following lemma will be the key to our main theorem of this section. It gives bounds on the maximal number of vertices with high multiplicity in a line arrangement whose dual point set is contained in a projective algebraic curve of bounded degree. For this, we use the work in [29].

Lemma 5.9. *Let $d \in \mathbb{N}$ be a positive integer and assume that $\mathcal{A}$ is an arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$ such that $\mathcal{A}^* \subset V(F) \subset (\mathbb{P}^2(\mathbb{R}))^*$ for some homogeneous polynomial $F \in \mathbb{R}[x, y, z]$ of degree d. Then the following statements hold:*
a) If F has no linear factors and if $d \geq 2$, then there exist constants $N_d, \lambda_d \in \mathbb{R}$ such that for $n \geq N_d$ we have the estimate $\max_{i\geq 4} t_i^{\mathcal{A}} \leq \sum_{i\geq 4} t_i^{\mathcal{A}} \leq \lambda_d n^{\frac{11}{6}}$.
b) If F is irreducible and if $d \geq 4$, then there are constants $M_d, \mu_d \in \mathbb{R}$ such that for $n \geq M_d$ we have the estimate $\max_{i\geq 3} t_i^{\mathcal{A}} \leq \sum_{i\geq 3} t_i^{\mathcal{A}} \leq \mu_d n^{\frac{11}{6}}$.

Proof. a) Choose a line $\mathfrak{l} \subset \mathbb{P}^2(\mathbb{R}))^*$ such that for any $p \in \mathcal{A}^*$ we have $p \notin \mathfrak{l}$. After a change of coordinates we may assume that the line $\mathfrak{l}$ is given by the equation $z = 0$. Now consider the affine space $\mathbb{E} := (\mathbb{P}^2(\mathbb{R}))^* \setminus \mathfrak{l}$. Then $\mathcal{A}^* \subset \mathbb{E}$ and moreover we have $\mathcal{A}^* \subset V(G)$, where $G := F(x, y, 1) \in \mathbb{R}[x, y]$. Observe that because F has no linear factors, the polynomial $G \in \mathbb{R}[x, y]$ does not have any linear factors as well. Indeed, if F has prime factorization given by $F = \prod_{1\leq i\leq r} F_i^{e_i}$, then, as z does not divide F, the polynomial G has prime factorization $G = F(x, y, 1) = \prod_{1\leq i\leq r} F(x, y, 1)_i^{e_i}$. In particular, G has no linear factors if F has none.

In particular, the curve $V(G) \subset \mathbb{E}$ does not contain a line. By [29, Corollary 6.3] we know that there are $\lambda_d, N_d \in \mathbb{R}$ such that for $n = |\mathcal{A}^*| \geq N_d$ the points in $\mathcal{A}^*$ determine at most $\lambda_d n^{\frac{11}{6}}$ proper collinear quadruples. As every vertex of weight at least four of $\mathcal{A}$ yields a proper collinear quadruple of $\mathcal{A}^*$, this proves the claim.
b) As in part a) we may assume that for any $p \in \mathcal{A}^*$ we have $p \notin \mathfrak{l}$, where the line $\mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$ is given by the equation $z = 0$. Then $\mathcal{A}^*$ is contained in the affine space $\mathbb{E} := (\mathbb{P}^2(\mathbb{R}))^* \setminus \mathfrak{l}$ and as above we have $\mathcal{A}^* \subset V(G)$, where $G := F(x, y, 1) \in \mathbb{R}[x, y]$ is irreducible of degree $d \geq 4$. In particular, the curve $V(G)$ is neither a line nor a cubic. Thus, by [29, Corollary 6.2] we find constants $M_d, \mu_d \in \mathbb{R}$ such that for all $n \geq M_d$ the points in $\mathcal{A}^*$ determine at most $\mu_d n^{\frac{11}{6}}$ proper collinear triples. Because every vertex of $\mathcal{A}$ of weight at least three yields a proper collinear triple of $\mathcal{A}^*$, this completes the proof. □

Now we are ready to prove the first theorem of this section:

Theorem 5.13. *Let d be a positive integer and let $\mathfrak{A}_d$ denote the set of all combinatorial isomorphism classes of simplicial line arrangements $\mathcal{A}$ in $\mathbb{P}^2(\mathbb{R})$ such that there exists $F \in \mathbb{R}[x, y, z]$, homogeneous and irreducible, with $\deg(F) \leq d$ and $\mathcal{A}^* \subset V(F)$. Then the set $\mathfrak{A}_d$ is finite for any choice of d.*

Proof. We write $n := |\mathcal{A}|$. If $d \leq 2$ then the statement is trivially true. For $d = 3$ the statement follows from Lemma 5.3. Hence we may suppose that $d \geq 4$. Then, because F is irreducible, by part b) of Lemma 5.9 we know that there are $\mu_d, M_d \in \mathbb{N}$ such that $\max_{i\geq 4} t_i^{\mathcal{A}} \leq \max_{i\geq 3} t_i^{\mathcal{A}} \leq \mu_d n^{\frac{11}{6}}$ for $n > M_d$. As $\mathcal{A}$ is simplicial we have equality in Melchior's inequality, hence $t_2^{\mathcal{A}} = 3 + \sum_{i\geq 4}(i-3)t_i^{\mathcal{A}}$ and therefore $t_2^{\mathcal{A}} \leq 3 + \mu_d(d-3)^2 n^{\frac{11}{6}}$, by Lemma 5.8. Hence, using the identity $\binom{n}{2} = \sum_{i=2}^{d}\binom{i}{2}t_i^{\mathcal{A}}$, we arrive at the estimate

$$\mu_d n^{\frac{11}{6}} \geq t_3^{\mathcal{A}} \geq \frac{n^2-n}{6} - 1 - \frac{1}{3}\mu_d(d-3)^2 n^{\frac{11}{6}} - \frac{1}{3}\sum_{i=4}^{d}\binom{i}{2}\mu_d n^{\frac{11}{6}}.$$

Thus, for $n > M_d$ we have the following inequality:

$$\frac{n^2-n-6}{2} \leq \left(3 + (d-3)^2 + \sum_{i=4}^{d}\binom{i}{2}\right)\mu_d n^{\frac{11}{6}}. \tag{5.18}$$

Clearly, for fixed d inequality (5.18) can hold only for finitely many values of n. Denote the maximal such value by n_d. Then we have $|\mathcal{A}| \leq \max\{M_d, n_d\}$ for any $\mathcal{A}$ whose isomorphism class belongs to $\mathfrak{A}_d$. In particular, the set $\mathfrak{A}_d$ is finite. $\square$

Using Corollary 5.6, we observe that the requirement on the appearing polynomials F in the last theorem to be irreducible can be relaxed: if we assume that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$, then we only have to require that F does not have any linear factors. Moreover, we can then drop the condition on $\mathcal{A}$ to be simplicial. This leads to the following theorem:

Theorem 5.14. *Let d be a positive integer and denote by $\mathfrak{B}_d$ the set of combinatorial isomorphism classes of line arrangements $\mathcal{A}$ in $\mathbb{P}^2(\mathbb{R})$ satisfying the following two properties:*
1) There exists a homogeneous polynomial $F \in \mathbb{R}[x,y,z]$ without any linear factors such that $\deg(F) \leq d$, $\mathcal{A}^* \subset V(F)$.
2) $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$.
Then the set $\mathfrak{B}_d$ is finite for any choice of d.

Proof. Let $\mathcal{A}$ be as required and write $n := |\mathcal{A}|$. We may assume that $d \geq 2$ and we denote the combinatorial isomorphism class of $\mathcal{A}$ by $\overline{\mathcal{A}}$.

By part a) of Lemma 5.9 there are $\lambda_d, N_d \in \mathbb{R}$ such that for $n \geq N_d$ we have the inequality $\max_{i\geq 4} t_i^{\mathcal{A}} \leq \lambda_d n^{\frac{11}{6}}$. Now choose $M_d \geq \max\{N_d, \lambda_d^{12}\}$. Then for $n \geq M_d$ we have $n \geq \max\{N_d, \lambda_d^{12}\} \geq \lambda_d^{12}$. This gives $\lambda_d \leq n^{\frac{1}{12}}$. Thus, for $n \geq M_d$ we have $\max_{i\geq 4} t_i^{\mathcal{A}} \leq \lambda_d n^{\frac{11}{6}} \leq n^{\frac{23}{12}}$. Now define the set $S := \left\{\overline{\mathcal{A}} \mid \max_{i\geq 6} t_i^{\mathcal{A}} \leq |\mathcal{A}|^{\frac{23}{12}}, t_i^{\mathcal{A}} = 0 \text{ for } i > d, \chi(\mathcal{A}, t) \text{ splits over } \mathbb{R}\right\}$.
Thus, if $\mathcal{A}$ satisfies the conditions 1) and 2) of the theorem and if $|\mathcal{A}| \geq M_d$,

then $\overline{\mathcal{A}} \in S$. Moreover, by Corollary 5.6 we know that the set S is finite (just put $x = d$ and $\epsilon = \frac{1}{12}$ in the cited result).

On the other hand, there are only finitely many combinatorial isomorphism classes of line arrangements consisting of at most M_d lines. This proves the claim. □

Remark 5.9. We remark that for every even $n = 2m$ with $m \geq 3$ there is a simplicial arrangement of n lines, denoted $\mathcal{A}_m$, belonging to the infinite series $\mathcal{R}(1)$ and there is a reducible cubic C, consisting of a conic and a line, such that $\mathcal{A}_m^* \subset C$. Moreover, the characteristic polynomials of all these arrangements split over $\mathbb{R}$. Therefore, the requirement on F to not have any linear factors in Theorem 5.14 is necessary.

The last theorem immediately yields the following corollary, achieving the announced result for free line arrangements and closing this section.

Corollary 5.7. *Let d be a positive integer and denote by $\mathfrak{C}_d$ the set of combinatorial isomorphism classes of free line arrangements $\mathcal{A}$ in $\mathbb{P}^2(\mathbb{R})$ such that there exists $F \in \mathbb{R}[x, y, z]$, homogeneous and without any linear factors, with $\deg(F) \leq d$ and $\mathcal{A}^* \subset V(F)$. Then the set $\mathfrak{C}_d$ is finite for any choice of d.*

Proof. This follows from Theorem 5.14 because the roots of the characteristic polynomial of a free line arrangement are integral. □

5.3 Open problems and related questions

In Theorem 5.13 of the last section we have shown that there are only finitely many combinatorial isomorphism classes of simplicial line arrangements whose corresponding dual point sets are contained in the locus of an irreducible homogeneous polynomial of some fixed degree d. Thus, we ask if it is possible to give a classification result instead.

For instance, it is easy to see that for $d = 2$ the only possible candidates are near pencil arrangements consisting of three lines. Furthermore, if $d = 3$ then by Lemma 5.3 we know that a possible candidate consists of six or seven lines. Thus, the cases $d \leq 3$ are easy. But what about $d \geq 4$?

Problem 6. *Let $d \geq 4$ be a natural number. We ask for a classification up to combinatorial isomorphism of all simplicial line arrangements $\mathcal{A}$ whose corresponding dual point sets $\mathcal{A}^*$ are contained in the locus of an irreducible homogeneous polynomial of degree d.*

Observe that for $d = 4$ we can solve the problem if we additionally require that the arrangements in question are free. For this, observe that if $\mathcal{A}^*$ is contained in the locus of an irreducible polynomial of degree four, then by Lemma 5.8 we know that $t_i^{\mathcal{A}} = 0$ for $i > 4$. But then by Theorem

5.5 it follows that $|\mathcal{A}| \leq 16$. Now one may use Theorem 2.5 to find all desired arrangements. We remark that one may choose coordinates in such a way that the dual point sets of these arrangements are all contained in the irreducible curve C which may be defined by the polynomial

$$P := x^3y - x^2y^2 + 3xy^3 - \frac{x^3z}{3} - 2x^2yz - \frac{7xy^2z}{2} - \frac{2y^3z}{3} + x^2z^2 + \frac{5xyz^2}{2} + y^2z^2 - \frac{2xz^3}{3} - \frac{yz^3}{3}.$$

This shows that it might be easier to investigate the following problem first:

Problem 7. *Let $d > 4$ be a natural number. We ask for a classification up to combinatorial isomorphism of all free simplicial line arrangements $\mathcal{A}$ whose corresponding dual point sets $\mathcal{A}^*$ are contained in the locus of an irreducible homogeneous polynomial of degree d.*

Furthermore, it is unclear whether or not there is a maximal d' such that the dual point set of any simplicial arrangement which is not combinatorially isomorphic to an arrangement of the known infinite series $\mathcal{R}(1), \mathcal{R}(2)$, is contained in the locus of an irreducible homogeneous polynomial of degree at most d'. If this was true, then our results imply that (up to combinatorial isomorphism) there are only finitely many examples of simplicial line arrangements besides the two infinite series. In other words, up to finitely many corrections, the catalogue of Grünbaum presented in [19] would be complete. Therefore, the following problem seems relevant:

Problem 8. *Let $\mathcal{A}$ be a simplicial line arrangement which is not combinatorially isomorphic to an arrangement of the infinite series. Does there exist $d \in \mathbb{N}$, independent of $\mathcal{A}$, such that $\mathcal{A}^*$ is contained in the locus of some irreducible homogeneous polynomial of degree at most d?*

As above, it may be easier to restrict attention to free simplicial arrangements and attack the following problem first:

Problem 9. *Let $\mathcal{A}$ be a free simplicial line arrangement which is not combinatorially isomorphic to an arrangement of the infinite series. Does there exist $d \in \mathbb{N}$, independent of $\mathcal{A}$, such that $\mathcal{A}^*$ is contained in the locus of some irreducible homogeneous polynomial of degree at most d?*

We have the impression that a solution to any one of these problems would mean a considerable step towards a solution to the question of completeness of Grünbaum's catalogue.

We continue with a somewhat different problem related to the local combinatorial structure of a simplicial line arrangement. We are interested in the maximal number of vertices having multiplicity at least five, which can be contained in a single chamber of an irreducible simplicial line arrangement. More precisely, we ask the following:

Problem 10. *Let $\mathcal{A}$ be an irreducible simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$. Is it true that every chamber of $\mathcal{A}$ contains at most one vertex of multiplicity bigger than four?*

Observe that the answer to the above problem is “no” for simplicial arrangements of pseudolines (counterexamples can be generated by using the algorithm in [6]). Thus, a potential proof cannot be purely combinatorial. However, if the answer for straight line arrangements was “yes”, then we could for instance derive a classification of simplicial line arrangements whose characteristic polynomials split over $\mathbb{R}$ and which have only vertices of weight two, three and five: this follows from the proof of Corollary 5.5.

We close this section by posing a problem on free line arrangements: motivated by Theorem 5.11, we ask if for $x \geq 6$ there always exist absolute bounds N_x such that whenever $\mathcal{A}$ is a free line arrangement having only vertices of multiplicity bounded by x, then $|\mathcal{A}| \leq N_x$:

Problem 11. *For each $x \geq 6$, prove or disprove that there are only finitely many combinatorial isomorphism classes of free line arrangements in $\mathbb{P}^2(\mathbb{R})$ whose vertices all have weight bounded by x.*

Observe that in order to solve the last problem, it would be enough to prove a subquadratic upper bound for the values $t_i^{\mathcal{A}}$, $6 \leq i \leq x$, for any free line arrangement $\mathcal{A}$ such that $|\mathcal{A}| \geq m_x$ is sufficiently large. This follows from Corollary 5.6.

Chapter 6

Simplicial arrangements and duality

In this chapter, we investigate simplicial arrangements $\mathcal{A}$ in $V := \mathbb{P}^{r-1}(\mathbb{R})$ and associated arrangements in V^* defined by certain subsets of the set of vertices of $\mathcal{A}$. This may be motivated by the following observation. If one is interested in simplicial arrangements in V whose hyperplanes are permuted by some reflection group G, then one needs to find finite subsets of V^* which are stable under the action of G. Clearly, this amounts to taking orbits of certain *well chosen* vectors. We observe that one way to find such "nice" vectors is to consider the reflection arrangement $\mathcal{A}$ of type G, viewed as projective arrangement in V^*. Then G will act on the set $V_i(\mathcal{A})$ of vertices of $\mathcal{A}$ having multiplicity i, i.e. those points in V^* which are contained in exactly i hyperplanes of $\mathcal{A}$. Consequently, G will also act on unions of such subsets. Thus, defining a map Φ^r_j by $\Phi^r_j(\mathcal{A}) := \bigcup_{i \geq j} V_i(\mathcal{A})$, we can associate to $\mathcal{A}$ in a natural way certain finite subsets of V^* of the above type. These subsets then define hyperplane arrangements in V and in many cases the obtained arrangements are simplicial with their hyperplanes being permuted by G (see Proposition 6.3 for some instances of this phenomenon).

Clearly, the map Φ^r_j can be defined for any type of arrangement. We observe that if one considers simplicial arrangements which are not necessarily of type $\mathcal{A}(G)$ for some reflection group G, then the phenomenon described above occurs in quite a few cases as well (again, see Proposition 6.3 for some instances). Moreover, there are examples of simplicial arrangements $\mathcal{A}$ in V having the remarkable property that for suitable j the arrangement $\Phi^r_j(\mathcal{A})$ in V^* is combinatorially isomorphic to the arrangement $\mathcal{A}$ itself: for instance, if $r \geq 6$ then the reflection arrangements of type C_r have this property (see Theorem 6.5). We study this remarkable property in some detail for reflection arrangements and in the case of the real projective plane and its dual. More precisely, we classify simplicial line arrangements which are "fixed" in the above sense under the maps Φ^3_2, Φ^3_3 (see Proposition 6.1, Corollary 6.1) and we provide some upper bounds for the size of a simplicial line arrangement "fixed" under the map Φ_4^3 (see for instance Theorem 6.2, Theorem 6.3 and Corollary 6.4). Writing $t_i^{\mathcal{A}}$ for the number of vertices having multiplicity i, we observe that in all known examples we always have $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$. However, we can prove this only in some special cases (see Problem 14). Moreover, the characteristic polynomials of all known (simplicial) examples always have only real roots (see also Problem 13).

The outline of this chapter is as follows. In Section 1 we introduce some notation and we give precise definitions of the map Φ_j^r (and some variant) introduced above. Section 2 then contains our results in the case of the real projective plane. In Section 3 we collect some interesting experimental results, in particular we give an example of an arrangement "fixed" under Φ_4^3 which is not simplicial and whose characteristic polynomial has a non-real root. Moreover, we prove some related results for the reflection arrangements of type $C_r(r \geq 6), D_r(r \geq 5), F_4, H_4, E_6, E_7, E_8$. Finally, in Section 4 we point out some possibly interesting related problems.

6.1 The maps Φ_j^r and Ψ_j^r

In this section we give precise definitions of our objects of interest and we fix some notation.

First, we recall the following principle of duality between projective points and hyperplanes in $\mathbb{P}^{r-1}(\mathbb{R})$ and fix the corresponding notations:

Notation. If $\mathcal{A}$ is a projective arrangement of hyperplanes in $V := \mathbb{P}^{r-1}(\mathbb{R})$, then $\mathcal{A}$ defines a dual set of points in a natural way: if $H \in \mathcal{A}$ is a hyperplane, then the corresponding dual point $H^* \in V^*$ is determined by the condition $H = \{x \in V \mid H^*(x) = 0\}$ and vice versa.

Thus, every finite set of points in V^* (not all contained in a hyperplane) yields an arrangement of hyperplanes in V and vice versa.

Moreover, if $\mathcal{A}$ is an arrangement of hyperplanes in V or if $\mathcal{P}$ is a finite pointset in V^*, then we always denote the corresponding dual objects by $\mathcal{A}^*$ and $\mathcal{P}^*$.

Now we are ready for the central definition of this chapter.

Definition 6.1. Let $\mathcal{A}$ be an arrangement of n hyperplanes in $V := \mathbb{P}^{r-1}(\mathbb{R})$. For a natural number $i \geq r-1$ we denote by $V_i^r(\mathcal{A})$ the set of all points in V which are contained in at least i hyperplanes of $\mathcal{A}$. Then for each $r-1 \leq i \leq n$ we define the arrangement

$$\Phi_i^r(\mathcal{A}) := (V_i^r(\mathcal{A}))^*$$

of hyperplanes in the dual space V^*. Similarly, we define

$$\Psi_i^r(\mathcal{A}) := \left(V_i^r(\mathcal{A}) \setminus V_{i+1}^r(\mathcal{A})\right)^*.$$

Remark 6.1. If $\mathcal{A}, \mathcal{B}$ are arrangements in V such that the arrangement $\Phi_i^r(\mathcal{A})$ in V^* is combinatorially isomorphic to the arrangement $\mathcal{B}$, then by abuse of notation we write $\Phi_i^r(\mathcal{A}) = \mathcal{B}$. We apply the same convention for the maps Ψ_i^r.

6.2 Simplicial arrangements in $\mathbb{P}^2(\mathbb{R})$ fixed under Φ_j^3 for $j \leq 4$

In this section we investigate simplicial line arrangements $\mathcal{A}$ in the real projective plane having the property that $\Phi_j^3(\mathcal{A}) = \mathcal{A}$, for some $2 \leq j \leq 4$. The reason for considering only these small values of j comes from the fact that we do not know of any example having the above property for some $j > 4$. In the cases $j = 2, 3$, we can provide a complete classification of simplicial arrangements satisfying $\Phi_j^3(\mathcal{A}) = \mathcal{A}$ (see Proposition 6.1 and Corollary 6.1). The case $j = 4$ is more difficult but also more interesting: some of the largest known examples are fixed under the map Φ_4^3 (see also Proposition 6.3). Unfortunately, in this case we can provide only upper bounds for the number of lines, provided that some extra conditions are satisfied. Moreover, in most cases these bounds will also depend on the multiplicity of the arrangement $\mathcal{A}$ in consideration (see Theorem 6.2, Theorem 6.3, Corollary 6.4, and Theorem 6.4).

Before starting with our main discussion, we collect some already established facts in the following lemma. As we shall refer to these facts many times in this chapter, this will allow us to express ourselves conveniently.

Lemma 6.1. *Let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$. Then the following statements hold:*

a) $t_2^{\mathcal{A}} \leq \frac{\binom{n}{2}+6}{7}$;

b) $t_k^{\mathcal{A}} = 0$ *for* $k > \frac{n}{2}$;

c) $\min(t_2^{\mathcal{A}}, t_3^{\mathcal{A}}) \leq \frac{\binom{n}{2}+15}{9}$ *and* $\max(t_2^{\mathcal{A}}, t_3^{\mathcal{A}}) > \frac{n^2+3n}{27}$;

d) $2t_2^{\mathcal{A}} \leq \frac{f_2^{\mathcal{A}}}{2} < 2t_2^{\mathcal{A}} + t_3^{\mathcal{A}} \leq 2t_2^{\mathcal{A}} + t_3^{\mathcal{A}} + \frac{t_4^{\mathcal{A}}}{3} \leq \frac{\binom{n}{2}}{3} + 5$;

e) If $t_i^{\mathcal{A}} = 0$ *for* $i > 6$, *then* $2t_2^{\mathcal{A}} + t_3^{\mathcal{A}} + \frac{t_4^{\mathcal{A}}}{3} = \frac{\binom{n}{2}}{3} + 5$;

f) If $\chi(\mathcal{A}, t)$ *has only real roots, then* $f_2^{\mathcal{A}} \leq \frac{(n+1)^2}{4}$;

g) $\frac{2f_1^{\mathcal{A}}}{3} = 2(f_0^{\mathcal{A}} - 1) = f_2^{\mathcal{A}} \geq \frac{2}{9}\left(n^2 + 3n\right).$

Proof. This follows from Proposition 5.1, Proposition 2.1, Corollary 5.2, Theorem 5.2, Lemma 5.4 and Lemma 5.5. $\square$

We start our discussion of simplicial arrangements fixed under some map Φ_j^k by considering the easiest case first. Namely, we set $k = 3$ and $j = 2$. We obtain the following result:

Proposition 6.1. *Let $\mathcal{A}$ be a simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$. Then $\mathcal{A}$ is a near-pencil arrangement if and only if $\Phi_2^3(\mathcal{A}) = \mathcal{A}$.*

Proof. Assume first that $\mathcal{A}$ is a near-pencil arrangement and write $n := |\mathcal{A}|$. Then $t_2^{\mathcal{A}} = n-1, t_{n-1}^{\mathcal{A}} = 1$ and $t_i^{\mathcal{A}} = 0$ for $i \notin \{2, n-1\}$. Moreover, all vertices of weight two are contained in a single line $\ell \in \mathcal{A}$. Considering the arrangement $\Phi_2^3(\mathcal{A})$, we observe that ℓ corresponds to a vertex of weight $n-1$ of $\Phi_2^3(\mathcal{A})$ while the remaining $n-1$ lines determined by the points of $V_2^3(\mathcal{A})$ correspond to vertices of weight two of $\Phi_2^3(\mathcal{A})$. It follows that $\Phi_2^3(\mathcal{A})$ is a near-pencil as well.

Now assume that $\Phi_2^3(\mathcal{A}) = \mathcal{A}$ and suppose that $\mathcal{A}$ is not a near-pencil. Then by part g) of Lemma 6.1 we obtain

$$\frac{(n-3)^2}{3} = \frac{n^2+3n}{3} - 3n + 3 \leq f_1^{\mathcal{A}} - 3n + 3 = 3f_0^{\mathcal{A}} - 3n.$$

Moreover, $\Phi_2^3(\mathcal{A}) = \mathcal{A}$ implies that $f_0^{\mathcal{A}} = n$. It follows $(n-3)^2 \leq 0$. But then $n = 3$ and $\mathcal{A}$ is a near-pencil, contradicting our assumption. $\square$

Naturally, the next case to consider is $k = 3$ and $j = 3$. Also in this case we can give a complete answer:

Theorem 6.1. *Let $\mathcal{A}$ be an irreducible simplicial arrangement in $\mathbb{P}^2(\mathbb{R})$. Assume that we have $\sum_{i\geq 3} t_i^{\mathcal{A}} \leq |\mathcal{A}|$. Then $|\mathcal{A}| \leq 14$.*

Proof. We write $n := |\mathcal{A}|$. Then by part g) of Lemma 6.1 we obtain the following inequality:

$$f_1^{\mathcal{A}} = 3(f_0^{\mathcal{A}} - 1) = 3t_2^{\mathcal{A}} - 3 + 3\sum_{i\geq 3} t_i^{\mathcal{A}} \leq 3t_2^{\mathcal{A}} - 3 + 3n.$$

Applying Lemma 6.1, part g) once more, we conclude that the estimate

$$\frac{(n-3)^2}{3} = \frac{n^2+3n}{3} - 3n + 3 \leq f_1^{\mathcal{A}} - 3n + 3 \leq 3t_2^{\mathcal{A}}$$

holds. But then by part a) of Lemma 6.1 we obtain the inequality

$$\frac{(n-3)^2}{9} \leq t_2^{\mathcal{A}} \leq \frac{\binom{n}{2}+6}{7}. \tag{6.1}$$

It is now easy to see that (6.1) can hold only when $n \leq 14$. $\square$

Corollary 6.1. *Let $\mathcal{A}$ be a simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$ such that $\Phi_3^3(\mathcal{A}) = \mathcal{A}$. Then $\mathcal{A}$ is combinatorially isomorphic to one of the arrangements $A(10,2), A(10,3), A(13,2)$ (as denoted in [19]).*

Proof. Clearly, the condition $\Phi_3^3(\mathcal{A}) = \mathcal{A}$ implies that $\sum_{i\geq 3} t_i^{\mathcal{A}} = |\mathcal{A}|$. The result now follows from Theorem 6.1 together with Theorem 2.5. $\square$

We now come to our main object of study in this section. Namely, we are interested in simplicial arrangements $\mathcal{A}$ in $\mathbb{P}^2(\mathbb{R})$ satisfying the condition $\Phi_4^3(\mathcal{A}) = \mathcal{A}$. We start with the following lemma:

Lemma 6.2. *Let $\mathcal{A}$ be an irreducible simplicial line arrangement in $\mathbb{P}^2(\mathbb{R})$ and assume that $\Phi_4^3(\mathcal{A}) = \mathcal{A}$. If $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$, then there exists $j \geq 5$ such that $t_j^{\mathcal{A}} > 0$.*

Proof. We write $n := |\mathcal{A}|$. Assume that $t_i^{\mathcal{A}} = 0$ for $i > 4$. Using this together with part g) of Lemma 6.1, we conclude that $t_2^{\mathcal{A}} = 3 + t_4^{\mathcal{A}}$. By assumption $\Phi_4^3(\mathcal{A}) = \mathcal{A}$ and therefore $t_2^{\mathcal{A}} = 3 + n$.

As $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$, we may use Lemma 6.1, part c) to obtain the lower bound $t_2^{\mathcal{A}} > \frac{f_2^{\mathcal{A}}}{6}$. Now we use [31, Theorem 1] to conclude that

$$n + 3 = t_2^{\mathcal{A}} > \frac{f_2^{\mathcal{A}}}{6} > \frac{2(n^2 - n + 8)}{6 \cdot 7} = \frac{n^2 - n + 8}{21}.$$

The last inequality implies $n \leq 24$. Using Theorem 2.5, we see that there is no irreducible simplicial arrangement $\mathcal{A}$ of multiplicity at most four such that $\Phi_4^3(\mathcal{A}) = \mathcal{A}$ and $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$. This proves the claim. $\square$

Corollary 6.2. *Let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$ of multiplicity at most four. Assume that $\Phi_4^3(\mathcal{A}) = \mathcal{A}$. Then $t^{\mathcal{A}} = \left(3 + n, \frac{n^2 - 15n - 6}{6}, n\right)$ and $\chi(\mathcal{A}, t)$ has a non-real root.*

Proof. By assumption we have $t_4^{\mathcal{A}} = n$ and $t_2^{\mathcal{A}} = 3 + n$. By part e) of Lemma 6.1 we obtain the corresponding value for $t_3^{\mathcal{A}}$. The claim about $\chi(\mathcal{A}, t)$ follows from Theorem 5.5 (the $t-$vectors of all simplicial arrangements $\mathcal{A}'$ with multiplicity four for which $\chi(\mathcal{A}', t)$ splits over $\mathbb{R}$ differ from the one given above). $\square$

Remark 6.2. i) Let $\mathcal{A}$ be projective line arrangement of multiplicity four such that $\Phi_4^3(\mathcal{A}) = \mathcal{A}$. Then every line of $\mathcal{A}$ contains exactly four vertices of weight four. By assumption, we also have $t_4^{\mathcal{A}} = |\mathcal{A}|$. So, if we write $n := |\mathcal{A}|$, then $\mathcal{A}$ defines n points and n lines in the real projective plane such that every point is incident with exactly four of the lines and such that every line contains exactly four of the points. Such a pair of points and lines is called a n_4-configuration (see [20] for more on this and related topics).
ii) Corollary 6.2 leaves us with the impression that the existence of a simplicial arrangement $\mathcal{A}$ with multiplicity four which is fixed under Φ_4^3 seems to be unlikely.

We observe that all currently known examples of simplicial arrangements in $\mathbb{P}^2(\mathbb{R})$ which are fixed under Φ_4^3 share the property $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$. Moreover, in these examples, the corresponding characteristic polynomials always have only real roots.

In the next theorem we give a quite satisfactory bound on the size of a simplicial arrangement $\mathcal{A}$ whose characteristic polynomial has a non-real root and which satisfies both $\Phi_4^3(\mathcal{A}) = \mathcal{A}$ and $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$.

Theorem 6.2. *Let $\mathcal{A}$ be a simplicial arrangement in $\mathbb{P}^2(\mathbb{R})$ and assume that $\chi(\mathcal{A}, t)$ has a non-real root. If $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$ and $\sum_{i \geq 4} t_i^{\mathcal{A}} \leq |\mathcal{A}|$, then $|\mathcal{A}| \leq 49$. In particular, if $\Phi_4^3(\mathcal{A}) = \mathcal{A}$ and $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$, then $|\mathcal{A}| \leq 49$.*

Proof. We write $n := |\mathcal{A}|$. As $\mathcal{A}$ is simplicial we have

$$f_2^{\mathcal{A}} + 2 = 2t_2^{\mathcal{A}} + 2t_3^{\mathcal{A}} + 2\sum_{i \geq 4} t_i^{\mathcal{A}}.$$

Using part d) of Lemma 6.1 and the fact that $\chi(\mathcal{A}, t)$ has a non-real root, we derive the following inequality:

$$\frac{\binom{n}{2}}{3} + 5 + t_3^{\mathcal{A}} + \frac{5}{3}t_4^{\mathcal{A}} + 2\sum_{i \geq 5} t_i^{\mathcal{A}} > \frac{(n+1)^2}{4} + 2.$$

It follows $t_3 > \frac{n^2 - 16n - 33 + 4t_4^{\mathcal{A}}}{12}$. Moreover, by part c) of Lemma 6.1 we have $t_3^{\mathcal{A}} \leq \frac{n^2 - n + 30}{18}$. Combining these inequalities we obtain

$$\frac{n^2 - n + 30}{18} \geq \frac{n^2 - 16n - 33 + 4t_4^{\mathcal{A}}}{12}.$$

The last estimate forces $n \leq 49$. □

Remark 6.3. We remark that to our knowledge there are no known simplicial arrangements in $\mathbb{P}^2(\mathbb{R})$ whose characteristic polynomials have a non-real root and which are fixed under Φ_4^3. However, there are non-simplicial examples of such arrangements. For instance the arrangement $\mathcal{A}_{\zeta_5}$ defined in Example 6.1 in the following section. Even though the arrangement $\mathcal{A}_{\zeta_5}$ itself is not simplicial, it is closely connected to the set of simplicial arrangements defined over $\mathbb{Q}(\zeta_5) \cap \mathbb{R} = \mathbb{Q}(\sqrt{5})$ (see Proposition 6.3). We remark also that $\mathcal{A}_{\zeta_5}$ satisfies the condition $t_2^{\mathcal{A}_{\zeta_5}} \geq t_3^{\mathcal{A}_{\zeta_5}}$.

If we assume that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$, then we can drop the condition $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$ and still get bounds on the size of $\mathcal{A}$ in case $\Phi_4^3(\mathcal{A}) = \mathcal{A}$. However, to do so we need to fix (or at least bound) the multiplicity of $\mathcal{A}$ first. Consequently, the obtained bounds will depend explicitly on the given multiplicities.

Theorem 6.3. *Let $x \in \mathbb{N}$ and let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$. Assume that the multiplicity of $\mathcal{A}$ is at most x and suppose that the characteristic polynomial $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$.*

If we have $\sum_{4\leq i\leq x} t_i^{\mathcal{A}} \leq n$, then the number of lines of $\mathcal{A}$ is bounded by

$$n \leq 2x^2 - 10x + 17 + 2\sqrt{x^4 - 10x^3 + 42x^2 - 85x + 67}.$$

In particular, the given bound holds if $\Phi_4^3(\mathcal{A}) = \mathcal{A}$.

Proof. According to part f) of Lemma 6.1, if the characteristic polynomial $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$, then we have the following inequality:

$$\frac{(n+1)^2}{4} \geq f_2^{\mathcal{A}}. \tag{6.2}$$

As $\mathcal{A}$ is assumed to be simplicial, we may rewrite (6.2) as

$$\frac{2}{3}\left(3 + \binom{n}{2} + \sum_{4\leq i\leq x}\left(2i - 3 - \binom{i}{2}\right) t_i^{\mathcal{A}}\right) \leq \frac{(n+1)^2}{4}. \tag{6.3}$$

By rearranging terms, we see that (6.3) is equivalent to

$$2 + \frac{n^2 - n}{3} - \frac{(n+1)^2}{4} \leq \frac{2}{3}\sum_{4\leq i\leq x}\left(\binom{i}{2} - 2i + 3\right) t_i^{\mathcal{A}}. \tag{6.4}$$

Since by assumption we have $\sum_{4\leq i\leq x} t_i^{\mathcal{A}} \leq n$, we conclude that (6.4) implies

$$2 + \frac{n^2 - n}{3} - \frac{(n+1)^2}{4} \leq \frac{2}{3}\left(\binom{x}{2} - 2x + 3\right) n.$$

The last inequality now yields the claim. $\square$

If we focus on (simplicial) arrangements with multiplicity at most five whose characteristic polynomials have only real roots, then we have the following useful lemma whose proof was given in Chapter 5 (see Lemma 5.7).

Lemma. *Let $\mathcal{A}$ be an arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$ having multiplicity at most five. Assume that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$. Then the following estimates hold:*

$$\begin{aligned}\frac{t_4^{\mathcal{A}}}{3} + t_5^{\mathcal{A}} &\geq \frac{n^2 - 10n + 21}{24};\\ t_2^{\mathcal{A}} &\geq \frac{n^2 - 46n + 233}{8} + 2t_4^{\mathcal{A}};\\ \max(t_4^{\mathcal{A}}, t_5^{\mathcal{A}}) &\geq \frac{n^2 - 10n + 21}{32}.\end{aligned}$$

Theorem 6.3 together with Lemma 5.7 is almost good enough to give a complete classification of simplicial arrangements fixed under Φ_4^3 whose characteristic polynomials have only real roots and whose vertices have weight bounded by five. We give the following corollary:

Corollary 6.3. *Let $\mathcal{A}$ be a simplicial arrangement in $\mathbb{P}^2(\mathbb{R})$ of multiplicity five such that $\chi(\mathcal{A}, t)$ splits over $\mathbb{R}$. If $\Phi_4^3(\mathcal{A}) = \mathcal{A}$, then $|\mathcal{A}| \leq 33$ and $t^{\mathcal{A}} \in$ $\{(30, 10, 15, 6), (50, 28, 9, 19), (54, 30, 7, 22), (57, 34, 6, 24), (62, 35, 3, 28),$ $(66, 38, 1, 31), (69, 43, 0, 33)\}$.*

Proof. This follows from Theorem 6.3 and Lemma 5.7 by examining possible $t-$vectors. □

Remark 6.4. i) We observe that for any $n \in \{28, 29, 30, 31, 32, 33\}$ the t-vector of an arrangement satisfying the conditions in Corollary 6.3 is uniquely determined. Thus, in order to obtain a complete classification of such arrangements, it would be enough to show that the $t-$vectors in question cannot correspond to real line arrangements. However, this seems to be a quite challenging problem. Moreover, we observe that the condition $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$ is automatic in the situation of Corollary 6.3.
ii) The t-vector of the arrangement $\mathcal{A} := A(21, 2)$ (as denoted in [19]) is given by $t^{\mathcal{A}} = (30, 10, 15, 6)$. Moreover, we indeed have $\Phi_4^3(\mathcal{A}) = \mathcal{A}$. We conjecture that this arrangement is the only simplicial example with multiplicity at most five which is fixed under Φ_4^3.

The bound obtained in Theorem 6.3 is quadratic in the multiplicity x of the given arrangement. In the next proposition we want to improve this by giving a linear bound. However, for this we need the assumption $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$. Moreover, the obtained linear bounds will be stronger only for sufficiently large values of x (see Remark 6.5).

Proposition 6.2. *Let $x \in \mathbb{N}$ and let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$ of multiplicity at most x. Assume that $\sum_{i \geq 4} t_i^{\mathcal{A}} \leq n$. If $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$ then $n < 27x - 57$.*

Proof. By assumption we obtain the following chain of inequalities:

$$t_2^{\mathcal{A}} = 3 + \sum_{i=4}^{x} (i-3) t_i^{\mathcal{A}} \leq 3 + (x-3) \sum_{i=4}^{x} t_i^{\mathcal{A}} \leq 3 + (x-3)n \leq (x-2)n.$$

Applying part c) of Lemma 6.1 we conclude that

$$\frac{n^2 + 3n}{27} < t_2^{\mathcal{A}} \leq n(x-2).$$

Dividing by n it follows $n < 27x - 57$. □

Remark 6.5. The bound given in Proposition 6.2 is stronger than the bound given in Theorem 6.3 for all $x \geq 10$.

Combining Theorem 6.2, Theorem 6.3 and Proposition 6.2, we obtain the following corollary:

Corollary 6.4. *Let $6 \leq x \in \mathbb{N}$ and assume that $\mathcal{A}$ is a simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$ having multiplicity x. If $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$ and $\Phi_4^3(\mathcal{A}) = \mathcal{A}$ then we have the upper bound*

$$n \leq \min\left(2x^2 - 10x + 17 + 2\sqrt{x^4 - 10x^3 + 42x^2 - 85x + 67}, 27x - 57\right).$$

Proof. If $\chi(\mathcal{A}, t)$ has a non-real root, then $|\mathcal{A}| \leq 49$, by Theorem 6.2. If $x \geq 6$ then both upper bounds given in Theorem 6.3, Proposition 6.2 are bigger than 49. This proves the claim. $\square$

Using the work in [17], we can relax the condition $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$ considerably. However, there is also a price that we have to pay: namely, we cannot provide explicit bounds any longer.

Theorem 6.4. *Let $\mathcal{A}$ be an irreducible simplicial arrangement of n lines in $\mathbb{P}^2(\mathbb{R})$. Let $4 \leq x \in \mathbb{N}$ and assume that the multiplicity of $\mathcal{A}$ is at most x. If $\sum_{i=4}^{x} t_i^{\mathcal{A}} \leq n$ and $t_3^{\mathcal{A}} \leq \frac{n}{6x} t_2^{\mathcal{A}}$, then there exists $N_x \in \mathbb{N}$ such that $n \leq N_x$. In particular, if $\Phi_4^3(\mathcal{A}) = \mathcal{A}$ and $t_3^{\mathcal{A}} \leq \frac{n}{6x} t_2^{\mathcal{A}}$, then $n \leq N_x$.*

Proof. As $\mathcal{A}$ is simplicial with multiplicity at most x, we obtain the estimate

$$t_2^{\mathcal{A}} \leq (x-2)n$$

in the same way as in the proof of Proposition 6.2. By assumption this implies the following inequality:

$$t_3^{\mathcal{A}} \leq \frac{n}{6x} t_2^{\mathcal{A}} \leq \frac{n}{6x}(x-2)n = \frac{n^2(x-2)}{6x} = \left(\frac{1}{6} - \frac{1}{3x}\right) n^2. \tag{6.5}$$

Now, as $t_2^{\mathcal{A}} \leq (x-2)n$, by [17, Theorem 1.5] it follows that for sufficiently large $n > N_0$ the arrangement $\mathcal{A}$ differs by at most $O(x-2)$ lines from one of the following three alternatives:

- $n - O(x-2)$ lines through a single point ;
- The arrangement $\mathcal{A}_m$ of the infinite series $\mathcal{R}(1)$ for some $m = \frac{n}{2} + O(x-2)$;
- An arrangement dual to a coset of a finite subgroup H of the real non-singular points of an irreducible cubic curve with $|H| = n + O(x-2)$.

We can exclude the first two possibilities for sufficiently large $n > N_1 \geq N_0$ because in these cases we would have a vertex of multiplicity bigger than x. But because of (6.5) we can exclude the third possibility for sufficiently large $n > N_2 \geq N_1$ as well: observe that in this case the number of triple points has magnitude $\frac{n^2}{6} + O(n)$, contradicting (6.5). This shows that there must exist some absolute bound $N_x \leq N_2$ such that $n \leq N_x$.

$\square$

6.3 Reflection arrangements and some experimental results

In this section we want to give some experimental results related to the maps Φ_j^k which seem interesting in the context of simplicial arrangements, see Proposition 6.3. Moreover, we establish related results for the reflection arrangements of type F_4, H_4, E_6, E_7, E_8, C_r $(r \geq 6)$ and D_r $(r \geq 5)$. For this, see Theorem 6.5, Theorem 6.6 and Theorem 6.7.

We start by giving the following example of a non-simplicial arrangement in $\mathbb{P}^2(\mathbb{R})$ which is fixed under the map Φ_4^3.

Example 6.1. *Write $\zeta_5 := \exp\left(\frac{2\pi i}{5}\right)$ and consider the following set Q of points in the dual projective plane $(\mathbb{P}^2(\mathbb{R}))^*$:*

$$Q := \{\left(-2\zeta_5 - 6{\zeta_5}^2 - 6{\zeta_5}^3 - 2{\zeta_5}^4 : -\zeta_5 - 3{\zeta_5}^2 - 3{\zeta_5}^3 - {\zeta_5}^4 : 0\right),$$
$$\left(-6\zeta_5 - 16{\zeta_5}^2 - 16{\zeta_5}^3 - 6{\zeta_5}^4 : -4\zeta_5 - 10{\zeta_5}^2 - 10{\zeta_5}^3 - 4{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4\right),$$
$$\left(-6\zeta_5 - 15{\zeta_5}^2 - 15{\zeta_5}^3 - 6{\zeta_5}^4 : -4\zeta_5 - 10{\zeta_5}^2 - 10{\zeta_5}^3 - 4{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4\right),$$
$$\left(-4\zeta_5 - 10{\zeta_5}^2 - 10{\zeta_5}^3 - 4{\zeta_5}^4 : -3\zeta_5 - 7{\zeta_5}^2 - 7{\zeta_5}^3 - 3{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4\right),$$
$$\left(-6\zeta_5 - 16{\zeta_5}^2 - 16{\zeta_5}^3 - 6{\zeta_5}^4 : -4\zeta_5 - 11{\zeta_5}^2 - 11{\zeta_5}^3 - 4{\zeta_5}^4 : -2\zeta_5 - 6{\zeta_5}^2 - 6{\zeta_5}^3 - 2{\zeta_5}^4\right),$$
$$\left(-4\zeta_5 - 11{\zeta_5}^2 - 11{\zeta_5}^3 - 4{\zeta_5}^4 : -3\zeta_5 - 8{\zeta_5}^2 - 8{\zeta_5}^3 - 3{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4\right),$$
$$\left(-2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4 : -2\zeta_5 - 4{\zeta_5}^2 - 4{\zeta_5}^3 - 2{\zeta_5}^4 : -\zeta_5 - 2{\zeta_5}^2 - 2{\zeta_5}^3 - {\zeta_5}^4\right),$$
$$\left(6\zeta_5 + 14{\zeta_5}^2 + 14{\zeta_5}^3 + 6{\zeta_5}^4 : 4\zeta_5 + 10{\zeta_5}^2 + 10{\zeta_5}^3 + 4{\zeta_5}^4 : 2\zeta_5 + 5{\zeta_5}^2 + 5{\zeta_5}^3 + 2{\zeta_5}^4\right),$$
$$\left(8\zeta_5 + 20{\zeta_5}^2 + 20{\zeta_5}^3 + 8{\zeta_5}^4 : 5\zeta_5 + 13{\zeta_5}^2 + 13{\zeta_5}^3 + 5{\zeta_5}^4 : 2\zeta_5 + 5{\zeta_5}^2 + 5{\zeta_5}^3 + 2{\zeta_5}^4\right),$$
$$\left(4\zeta_5 + 12{\zeta_5}^2 + 12{\zeta_5}^3 + 4{\zeta_5}^4 : 4\zeta_5 + 10{\zeta_5}^2 + 10{\zeta_5}^3 + 4{\zeta_5}^4 : 2\zeta_5 + 6{\zeta_5}^2 + 6{\zeta_5}^3 + 2{\zeta_5}^4\right),$$
$$\left(8\zeta_5 + 20{\zeta_5}^2 + 20{\zeta_5}^3 + 8{\zeta_5}^4 : 6\zeta_5 + 14{\zeta_5}^2 + 14{\zeta_5}^3 + 6{\zeta_5}^4 : 2\zeta_5 + 6{\zeta_5}^2 + 6{\zeta_5}^3 + 2{\zeta_5}^4\right),$$
$$\left(2\zeta_5 + 6{\zeta_5}^2 + 6{\zeta_5}^3 + 2{\zeta_5}^4 : 2\zeta_5 + 6{\zeta_5}^2 + 6{\zeta_5}^3 + 2{\zeta_5}^4 : 2\zeta_5 + 5{\zeta_5}^2 + 5{\zeta_5}^3 + 2{\zeta_5}^4\right),$$
$$\left(4\zeta_5 + 12{\zeta_5}^2 + 12{\zeta_5}^3 + 4{\zeta_5}^4 : 3\zeta_5 + 9{\zeta_5}^2 + 9{\zeta_5}^3 + 3{\zeta_5}^4 : 2\zeta_5 + 5{\zeta_5}^2 + 5{\zeta_5}^3 + 2{\zeta_5}^4\right),$$
$$\left(2\zeta_5 + 6{\zeta_5}^2 + 6{\zeta_5}^3 + 2{\zeta_5}^4 : 2\zeta_5 + 5{\zeta_5}^2 + 5{\zeta_5}^3 + 2{\zeta_5}^4 : \zeta_5 + 2{\zeta_5}^2 + 2{\zeta_5}^3 + {\zeta_5}^4\right),$$
$$\left(-6\zeta_5 - 14{\zeta_5}^2 - 14{\zeta_5}^3 - 6{\zeta_5}^4 : -4\zeta_5 - 9{\zeta_5}^2 - 9{\zeta_5}^3 - 4{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4\right),$$
$$(-1 : -1 : 0),$$
$$\left(-2\zeta_5 - 6{\zeta_5}^2 - 6{\zeta_5}^3 - 2{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4\right),$$
$$\left(-\zeta_5 - 3{\zeta_5}^2 - 3{\zeta_5}^3 - {\zeta_5}^4 : -\zeta_5 - 2{\zeta_5}^2 - 2{\zeta_5}^3 - {\zeta_5}^4 : 0\right),$$
$$\left(-{\zeta_5}^2 - {\zeta_5}^3 : 0 : 0\right),$$
$$\left(-2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4 : -\zeta_5 - 2{\zeta_5}^2 - 2{\zeta_5}^3 - {\zeta_5}^4 : 0\right),$$

$\left(0 : {\zeta_5}^2 + {\zeta_5}^3 : 0\right),$

$\left(-8\zeta_5 - 21{\zeta_5}^2 - 21{\zeta_5}^3 - 8{\zeta_5}^4 : -5\zeta_5 - 13{\zeta_5}^2 - 13{\zeta_5}^3 - 5{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4\right),$

$\left(0 : 0 : -\zeta_5 - 2{\zeta_5}^2 - 2{\zeta_5}^3 - {\zeta_5}^4\right),$

$\left(-2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4 : -\zeta_5 - 3{\zeta_5}^2 - 3{\zeta_5}^3 - {\zeta_5}^4 : -\zeta_5 - 3{\zeta_5}^2 - 3{\zeta_5}^3 - {\zeta_5}^4\right),$

$\left(-10\zeta_5 - 26{\zeta_5}^2 - 26{\zeta_5}^3 - 10{\zeta_5}^4 : -6\zeta_5 - 16{\zeta_5}^2 - 16{\zeta_5}^3 - 6{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4\right),$

$\left(-3\zeta_5 - 8{\zeta_5}^2 - 8{\zeta_5}^3 - 3{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4 : -\zeta_5 - 3{\zeta_5}^2 - 3{\zeta_5}^3 - {\zeta_5}^4\right),$

$\left(2\zeta_5 + 4{\zeta_5}^2 + 4{\zeta_5}^3 + 2{\zeta_5}^4 : 2\zeta_5 + 4{\zeta_5}^2 + 4{\zeta_5}^3 + 2{\zeta_5}^4 : \zeta_5 + 2{\zeta_5}^2 + 2{\zeta_5}^3 + {\zeta_5}^4\right),$

$\left(0 : 2{\zeta_5}^2 + 2{\zeta_5}^3 : 2{\zeta_5}^2 + 2{\zeta_5}^3\right),$

$\left(4\zeta_5 + 12{\zeta_5}^2 + 12{\zeta_5}^3 + 4{\zeta_5}^4 : 2\zeta_5 + 8{\zeta_5}^2 + 8{\zeta_5}^3 + 2{\zeta_5}^4 : 2{\zeta_5}^2 + 2{\zeta_5}^3\right),$

$\left(-14\zeta_5 - 36{\zeta_5}^2 - 36{\zeta_5}^3 - 14{\zeta_5}^4 : -9\zeta_5 - 23{\zeta_5}^2 - 23{\zeta_5}^3 - 9{\zeta_5}^4 : -4\zeta_5 - 10{\zeta_5}^2 - 10{\zeta_5}^3 - 4{\zeta_5}^4\right),$

$\left(0 : -1 : -{\zeta_5}^2 - {\zeta_5}^3\right),$

$\left(-2{\zeta_5}^2 - 2{\zeta_5}^3 : -{\zeta_5}^2 - {\zeta_5}^3 : -{\zeta_5}^2 - {\zeta_5}^3\right),$

$\left(-2\zeta_5 - 4{\zeta_5}^2 - 4{\zeta_5}^3 - 2{\zeta_5}^4 : -\zeta_5 - 3{\zeta_5}^2 - 3{\zeta_5}^3 - {\zeta_5}^4 : 0\right),$

$\left(-8\zeta_5 - 20{\zeta_5}^2 - 20{\zeta_5}^3 - 8{\zeta_5}^4 : -5\zeta_5 - 13{\zeta_5}^2 - 13{\zeta_5}^3 - 5{\zeta_5}^4 : -3\zeta_5 - 8{\zeta_5}^2 - 8{\zeta_5}^3 - 3{\zeta_5}^4\right),$

$\left(0 : -1 : \zeta_5 + 2{\zeta_5}^2 + 2{\zeta_5}^3 + {\zeta_5}^4\right),$

$\left(\zeta_5 + 2{\zeta_5}^2 + 2{\zeta_5}^3 + {\zeta_5}^4 : \zeta_5 + 2{\zeta_5}^2 + 2{\zeta_5}^3 + {\zeta_5}^4 : \zeta_5 + 2{\zeta_5}^2 + 2{\zeta_5}^3 + {\zeta_5}^4\right),$

$\left(\zeta_5 + 4{\zeta_5}^2 + 4{\zeta_5}^3 + {\zeta_5}^4 : \zeta_5 + 3{\zeta_5}^2 + 3{\zeta_5}^3 + {\zeta_5}^4 : {\zeta_5}^2 + {\zeta_5}^3\right),$

$\left(-8\zeta_5 - 20{\zeta_5}^2 - 20{\zeta_5}^3 - 8{\zeta_5}^4 : -5\zeta_5 - 12{\zeta_5}^2 - 12{\zeta_5}^3 - 5{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4\right),$

$\left(2\zeta_5 + 6{\zeta_5}^2 + 6{\zeta_5}^3 + 2{\zeta_5}^4 : \zeta_5 + 3{\zeta_5}^2 + 3{\zeta_5}^3 + {\zeta_5}^4 : \zeta_5 + 2{\zeta_5}^2 + 2{\zeta_5}^3 + {\zeta_5}^4\right),$

$\left(-2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4 : -\zeta_5 - 3{\zeta_5}^2 - 3{\zeta_5}^3 - {\zeta_5}^4\right),$

$\left(8\zeta_5 + 20{\zeta_5}^2 + 20{\zeta_5}^3 + 8{\zeta_5}^4 : 5\zeta_5 + 12{\zeta_5}^2 + 12{\zeta_5}^3 + 5{\zeta_5}^4 : 3\zeta_5 + 7{\zeta_5}^2 + 7{\zeta_5}^3 + 3{\zeta_5}^4\right),$

$\left(\zeta_5 + 3{\zeta_5}^2 + 3{\zeta_5}^3 + {\zeta_5}^4 : {\zeta_5}^2 + {\zeta_5}^3 : {\zeta_5}^2 + {\zeta_5}^3\right),$

$\left(10\zeta_5 + 26{\zeta_5}^2 + 26{\zeta_5}^3 + 10{\zeta_5}^4 : 6\zeta_5 + 16{\zeta_5}^2 + 16{\zeta_5}^3 + 6{\zeta_5}^4 : 4\zeta_5 + 11{\zeta_5}^2 + 11{\zeta_5}^3 + 4{\zeta_5}^4\right),$

$\left(-4\zeta_5 - 12{\zeta_5}^2 - 12{\zeta_5}^3 - 4{\zeta_5}^4 : -3\zeta_5 - 9{\zeta_5}^2 - 9{\zeta_5}^3 - 3{\zeta_5}^4 : -\zeta_5 - 4{\zeta_5}^2 - 4{\zeta_5}^3 - {\zeta_5}^4\right),$

$\left(-2\zeta_5 - 6{\zeta_5}^2 - 6{\zeta_5}^3 - 2{\zeta_5}^4 : -2\zeta_5 - 6{\zeta_5}^2 - 6{\zeta_5}^3 - 2{\zeta_5}^4 : -{\zeta_5}^2 - {\zeta_5}^3\right),$

$\left(0 : \zeta_5 + 2{\zeta_5}^2 + 2{\zeta_5}^3 + {\zeta_5}^4 : \zeta_5 + 3{\zeta_5}^2 + 3{\zeta_5}^3 + {\zeta_5}^4\right),$

$\left(-2\zeta_5 - 6{\zeta_5}^2 - 6{\zeta_5}^3 - 2{\zeta_5}^4 : -\zeta_5 - 3{\zeta_5}^2 - 3{\zeta_5}^3 - {\zeta_5}^4 : -{\zeta_5}^2 - {\zeta_5}^3\right),$

$\left(-6\zeta_5 - 16{\zeta_5}^2 - 16{\zeta_5}^3 - 6{\zeta_5}^4 : -5\zeta_5 - 13{\zeta_5}^2 - 13{\zeta_5}^3 - 5{\zeta_5}^4 : -4\zeta_5 - 10{\zeta_5}^2 - 10{\zeta_5}^3 - 4{\zeta_5}^4\right),$

$\left(-2\zeta_5 - 6{\zeta_5}^2 - 6{\zeta_5}^3 - 2{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4 : -{\zeta_5}^2 - {\zeta_5}^3\right),$

$\left(-6\zeta_5 - 16{\zeta_5}^2 - 16{\zeta_5}^3 - 6{\zeta_5}^4 : -5\zeta_5 - 13{\zeta_5}^2 - 13{\zeta_5}^3 - 5{\zeta_5}^4 : -2\zeta_5 - 6{\zeta_5}^2 - 6{\zeta_5}^3 - 2{\zeta_5}^4\right),$

$\left(0 : -\zeta_5 - 2{\zeta_5}^2 - 2{\zeta_5}^3 - {\zeta_5}^4 : 1\right),$

$\left(\zeta_5 + 4{\zeta_5}^2 + 4{\zeta_5}^3 + {\zeta_5}^4 : 2{\zeta_5}^2 + 2{\zeta_5}^3 : {\zeta_5}^2 + {\zeta_5}^3\right),$

$\left(2\zeta_5 + 4{\zeta_5}^2 + 4{\zeta_5}^3 + 2{\zeta_5}^4 : -1 : 0\right),$

$\left(-10\zeta_5 - 26{\zeta_5}^2 - 26{\zeta_5}^3 - 10{\zeta_5}^4 : -6\zeta_5 - 15{\zeta_5}^2 - 15{\zeta_5}^3 - 6{\zeta_5}^4 : -2\zeta_5 - 5{\zeta_5}^2 - 5{\zeta_5}^3 - 2{\zeta_5}^4\right),$

$\left(10\zeta_5 + 26{\zeta_5}^2 + 26{\zeta_5}^3 + 10{\zeta_5}^4 : 8\zeta_5 + 21{\zeta_5}^2 + 21{\zeta_5}^3 + 8{\zeta_5}^4 : 4\zeta_5 + 11{\zeta_5}^2 + 11{\zeta_5}^3 + 4{\zeta_5}^4\right),$

$\left(0 : \zeta_5 + 2{\zeta_5}^2 + 2{\zeta_5}^3 + {\zeta_5}^4 : {\zeta_5}^2 + {\zeta_5}^3\right),$

$\left(-\zeta_5 - 2{\zeta_5}^2 - 2{\zeta_5}^3 - {\zeta_5}^4 : -\zeta_5 - 2{\zeta_5}^2 - 2{\zeta_5}^3 - {\zeta_5}^4 : 1\right),$

$\left(10\zeta_5 + 26{\zeta_5}^2 + 26{\zeta_5}^3 + 10{\zeta_5}^4 : 6\zeta_5 + 15{\zeta_5}^2 + 15{\zeta_5}^3 + 6{\zeta_5}^4 : 4\zeta_5 + 10{\zeta_5}^2 + 10{\zeta_5}^3 + 4{\zeta_5}^4\right),$

$\left(-2\zeta_5 - 6{\zeta_5}^2 - 6{\zeta_5}^3 - 2{\zeta_5}^4 : -{\zeta_5}^2 - {\zeta_5}^3 : -{\zeta_5}^2 - {\zeta_5}^3\right),$

$\left(0 : -\zeta_5 - 2{\zeta_5}^2 - 2{\zeta_5}^3 - {\zeta_5}^4 : {\zeta_5}^2 + {\zeta_5}^3\right),$

$\left(-2\zeta_5 - 6{\zeta_5}^2 - 6{\zeta_5}^3 - 2{\zeta_5}^4 : -2{\zeta_5}^2 - 2{\zeta_5}^3 : -{\zeta_5}^2 - {\zeta_5}^3\right)\}.$

Then Q defines an arrangement of 61 *projective lines in* $\mathbb{P}^2(\mathbb{R})$*. We call* *this arrangement $\mathcal{A}_{\zeta_5}$. The corresponding t-vector is given by*

$$t^{\mathcal{A}_{\zeta_5}} = (540, 90, 30, 6, 0, 0, 15, 10).$$

In particular, the arrangement $\mathcal{A}_{\zeta_5}$ is not simplicial and its characteristic polynomial has a non-real root. Still, one computes that $\Phi_4^3(\mathcal{A}_{\zeta_5}) = \mathcal{A}_{\zeta_5}$.

In the following, we say that an arrangement $\mathcal{A}$ is *defined* over some algebraic number field $\mathbb{K}$, if $\mathbb{K}$ is the minimal field of definition for $\mathcal{A}$ in the sense of [4]. In the next proposition we use this terminology and collect some of the most interesting experimental results concerning simplicial arrangements and the maps Φ_j^k (including the case $k > 3$). In particular, we establish a connection between the arrangement $\mathcal{A}_{\zeta_5}$ defined above and a large subset of the set of currently known simplicial arrangements defined over the field $\mathbb{Q}(\zeta_5) \cap \mathbb{R} = \mathbb{Q}(\sqrt{5})$.

Proposition 6.3. *i) Let $\mathcal{A} := \mathcal{A}_{\zeta_5}$ be the arrangement from the above example. Then we have $\Phi_5^3(\mathcal{A}) = A(31,1)$ and $\Phi_8^3(\mathcal{A}) = A(25,3)$, using the notation in [19]. Moreover, we have $\Phi_2^3(\mathcal{A}(H_3)) = A(31,1)$, where $\mathcal{A}(H_3)$ denotes the reflection arrangement of type H_3. Observe also that $A(31,1)$ is the largest currently known sporadic simplicial line arrangement defined over $\mathbb{Q}(\sqrt{5})$.*
ii) Consider the arrangements $A(21,2), A(37,2)$ and $A(37,3)$ (as denoted in [19]). Then all three of them are fixed under Φ_4^3. Note that $A(37,2)$ and $A(37,3)$ are the largest currently known sporadic simplicial line arrangements defined over the number fields $\mathbb{Q}(\sqrt{3}), \mathbb{Q}$ respectively.
iii) Let $\mathcal{A}(C_3)$ be the reflection arrangement of type C_3. Then we have $\Phi_2^3(\mathcal{A}(C_3)) = A(13,2)$ (again as denoted in [19]).
iv) Let $\mathcal{A}(C_4), \mathcal{A}(F_4)$ be the reflection arrangements of type C_4, F_4 respectively. Then we have $\Phi_5^4(\mathcal{A}(C_4)) = \mathcal{A}(F_4)$.
v) Let $\mathcal{A}(C_5)$ be the reflection arrangement of type C_5. Then we have $\Phi_8^5(\mathcal{A}(C_5)) = A_4^5$. Here A_4^5 denotes the arrangement Nr.4 in the classification of crystallographic simplicial arrangements of rank five given in [8].
vi) Let $\mathcal{A}(C_6)$ be the reflection arrangement of type C_6. Then we have $\Phi_{13}^6(\mathcal{A}(C_6)) = A_4^6$. Here A_4^6 denotes the arrangement Nr.4 in the classification of crystallographic simplicial arrangements of rank six given in [8]. Observe also that A_4^6 is the maximal sporadic crystallographic arrangement in rank six.
vii) Let $\mathcal{A}(E_6)$ be the reflection arrangement of type E_6. Then we have $\Phi_{15}^6(\mathcal{A}(E_6)) = A_3^6$. As above, A_3^6 denotes the arrangement Nr.3 in the classification of crystallographic simplicial arrangements of rank six given in [8].

Proof. This can be verified by computer calculations. □

In the next theorem we want to prove that for reflection arrangements of type C_r, we always find some j such that $\Phi_j^r(\mathcal{A}_r) = \mathcal{A}_r$, at least for $r \geq 6$.

Theorem 6.5. *Let $\mathcal{A}_r := \mathcal{A}(C_r)$ be the reflection arrangement of type C_r for $r \geq 6$. Then we have $\Phi^r_{1+(r-2)^2}(\mathcal{A}_r) = \mathcal{A}_r$.*

Proof. For $1 \leq j \leq r$ we denote by α_j the j-th standard basis vector in $\mathbb{R}^r$, also denote by $\pi : \mathbb{R}^r \setminus \{0\} \longrightarrow \mathbb{P}^{r-1}(\mathbb{R})$ the natural projection. We choose the crystallographic root system of type C_r for $\mathcal{A}_r$, i.e. $\mathcal{A}_r$ consists of the hyperplanes defined by the projective images of vectors contained in the orbits $\mathcal{O}_W(\alpha_j)$ for $1 \leq j \leq r$. Here $W \subset \mathrm{GL}(\mathbb{R}^r)$ is the matrix group generated by $\sigma_1, ..., \sigma_r$, where

$$\sigma_i(\alpha_j) := \alpha_j - c_{i,j}\alpha_i$$

for $1 \leq i \leq r$ and

$$c_{i,j} := \left\{ \begin{array}{ll} 2, & \text{for } i = j \\ 0, & \text{for } |i-j| > 1 \\ -1, & \text{for } |i-j| = 1 \text{ and } (i,j) \neq (r-1, r) \\ -2, & \text{for } (i,j) = (r-1, r) \end{array} \right\}.$$

From this we deduce that the vertices of $\mathcal{A}_r$ all have multiplicity given by $\frac{l^2+l}{2} + (r-1-l)^2$ for $0 \leq l \leq r-1$. Moreover, there are r vertices of multiplicity $(r-1)^2$ and there are $r^2 - r$ vertices of multiplicity $1 + (r-2)^2$. Furthermore, as $\frac{l^2+l}{2} + (r-1-l)^2 < 1 + (r-2)^2$ for $r \geq 6$ and $2 \leq l \leq r-1$, we conclude that for $r \geq 6$ we have $|\Phi^r_{1+(r-2)^2}(\mathcal{A}_r)| = r^2$.

Now let $W^\vee$ be the group generated by the transposed matrices $\sigma_1^\vee, ..., \sigma_r^\vee$ and observe that

$$\begin{aligned} \left(\pi\left(\mathcal{O}_{W^\vee}(\alpha_1) \cup \mathcal{O}_{W^\vee}(\alpha_2)\right)\right)^* &= \Phi^r_{1+(r-2)^2}(\mathcal{A}_r), \\ |\mathcal{O}_{W^\vee}(\alpha_1) \cup \mathcal{O}_{W^\vee}(\alpha_2)| &= 2|\Phi^r_{1+(r-2)^2}(\mathcal{A}_r)| = 2r^2. \end{aligned}$$

Next we define

$$v_i := \left\{ \begin{array}{ll} 2\alpha_1 - \alpha_2, & \text{for } i = 1 \\ -\alpha_{i-1} + 2\alpha_i - \alpha_{i+1}, & \text{for } 2 \leq i \leq r-2 \\ -\alpha_{r-2} + 2\alpha_{r-1} - 2\alpha_r, & \text{for } i = r-1 \\ -\alpha_{r-1} + 2\alpha_r, & \text{for } i = r \end{array} \right\}.$$

One checks that $v_1, ..., v_r$ are all contained in $\mathcal{O}_{W^\vee}(\alpha_1) \cup \mathcal{O}_{W^\vee}(\alpha_2)$. Moreover, a quick calculation shows that for $1 \leq i \leq r$ we have

$$\sigma_i^\vee(v_j) = v_j - c_{i,j}^\vee v_i,$$

where

$$c_{i,j}^\vee := \left\{ \begin{array}{ll} 2, & \text{for } i = j \\ 0, & \text{for } |i-j| > 1 \\ -1, & \text{for } |i-j| = 1 \text{ and } (i,j) \neq (r, r-1) \\ -2, & \text{for } (i,j) = (r, r-1) \end{array} \right\}.$$

This shows that up to a change of basis, the set $\mathcal{O}_{W^\vee}(\alpha_1) \cup \mathcal{O}_{W^\vee}(\alpha_2)$ is the root system of type B_r. As the corresponding Weyl groups are isomorphic, we see that $\Phi^r_{1+(r-2)^2}(\mathcal{A}_r) = (\pi(\mathcal{O}_{W^\vee}(\alpha_1) \cup \mathcal{O}_{W^\vee}(\alpha_2)))^*$ and $\mathcal{A}_r$ are combinatorially isomorphic. This completes the proof. □

Remark 6.6. i) We note that $\Phi^r_{1+(r-2)^2}(\mathcal{A}_r)$ and $\mathcal{A}_r$ are *not* equivalent as crystallographic arrangements. See also [8] for background and for a classification of (finite) crystallographic arrangements.
ii) We remark that the arrangement defined by $\pi(\mathcal{O}_{W^\vee}(\alpha_1) \cup \mathcal{O}_{W^\vee}(\alpha_2))$ is combinatorially isomorphic to $\mathcal{A}_r$ also for $2 \leq r \leq 5$. However, the inequality $\frac{l^2+l}{2} + (r-1-l)^2 < 1 + (r-2)^2$ does not hold any longer for every $2 \leq l \leq r-1$. This explains why the theorem is true only for $r \geq 6$.

If we consider the reflection groups of type D_r, then we observe a similar behaviour. However, this time we make use of the maps Ψ^r_j to describe the situation.

Theorem 6.6. *Let $\mathcal{A}_r := \mathcal{A}(D_r)$ be the reflection arrangement of type D_r for $r \geq 5$. Then we have $\Psi^r_{3-r+(r-2)^2}(\mathcal{A}_r) = \mathcal{A}_r$.*

Proof. The proof is analogous to the proof of Theorem 6.5. For $1 \leq j \leq r$ we denote by α_j the j-th standard basis vector in $\mathbb{R}^r$, also let us denote by $\pi : \mathbb{R}^r \setminus \{0\} \longrightarrow \mathbb{P}^{r-1}(\mathbb{R})$ the natural projection. We choose the crystallographic root system of type D_r for $\mathcal{A}_r$, i.e. $\mathcal{A}_r$ consists of the hyperplanes defined by the projective images of vectors contained in the orbits $\mathcal{O}_W(\alpha_j)$ for $1 \leq j \leq r$. Here $W \subset \mathrm{GL}(\mathbb{R}^r)$ is the matrix group generated by $\sigma_1, ..., \sigma_r$, where

$$\sigma_i(\alpha_j) := \alpha_j - c_{i,j}\alpha_i$$

for $1 \leq i \leq r$ and

$$c_{i,j} := \left\{ \begin{array}{ll} 2, & \text{for } i = j \\ 0, & \text{for } |i-j| > 1 \text{ and } \{i,j\} \neq \{r-2, r\} \\ -1, & \text{for } |i-j| = 1 \text{ and } \{i,j\} \neq \{r-1, r\} \\ -1, & \text{for } \{i,j\} = \{r-2, r\} \end{array} \right\}.$$

This implies that the multiplicity of an arbitrary vertex of $\mathcal{A}_r$ is given by one of the following numbers:

$$\frac{(r-1)^2 + r - 1}{2},$$
$$\frac{l^2+l}{2} + (r-1-l)^2 - r + l + 1, \text{ for } 0 \leq l \leq r-3.$$

Moreover, we observe that the multiplicity $3-r+(r-2)^2$ occurs in total r^2-r many times. We conclude that for $r \geq 5$ we have $|\Psi^r_{3-r+(r-2)^2}(\mathcal{A}_r)| = r^2 - r$.

Now let $W^\vee$ be the group generated by the transposed matrices $\sigma_1^\vee, ..., \sigma_r^\vee$ and observe that

$$\begin{aligned}(\pi(\mathcal{O}_{W^\vee}(\alpha_2)))^* &= \Psi^r_{3-r+(r-2)^2}(\mathcal{A}_r),\\ |\mathcal{O}_{W^\vee}(\alpha_2)| = 2|\Psi^r_{3-r+(r-2)^2}(\mathcal{A}_r)| &= 2(r^2-r).\end{aligned}$$

Next we define

$$v_i := \left\{\begin{array}{ll} 2\alpha_1 - \alpha_2, & \text{for } i = 1 \\ -\alpha_{i-1} + 2\alpha_i - \alpha_{i+1}, & \text{for } 2 \leq i \leq r-3 \\ -\alpha_{r-3} + 2\alpha_{r-2} - \alpha_{r-1} - \alpha_r, & \text{for } i = r-2 \\ -\alpha_{r-2} + 2\alpha_{r-1}, & \text{for } i = r-1 \\ -\alpha_{r-2} + 2\alpha_r, & \text{for } i = r \end{array}\right\}.$$

One checks that $v_1, ..., v_r$ are all contained in $\mathcal{O}_{W^\vee}(\alpha_2)$.
Moreover, one verifies that for $1 \leq i \leq r$ we have

$$\sigma_i^\vee(v_j) = v_j - c_{i,j} v_i,$$

where $c_{i,j}$ is defined as above. This shows that up to a change of basis, the set $\mathcal{O}_{W^\vee}(\alpha_2)$ is the root system of type D_r. From this we see that $\Psi^r_{3-r+(r-2)^2}(\mathcal{A}_r) = (\pi(\mathcal{O}_{W^\vee}(\alpha_2)))^*$ and $\mathcal{A}_r$ are combinatorially isomorphic. This completes the proof. $\square$

For the exceptional groups of type F_4, H_4, E_6, E_7, E_8 we have the following result:

Theorem 6.7. *i) Let $\mathcal{A} := \mathcal{A}(F_4)$ be the reflection arrangement of type F_4. Then we have $\Phi^4_9(\mathcal{A}) = \Psi^4_9(\mathcal{A}) = \mathcal{A}$.*
ii) Let $\mathcal{A} := \mathcal{A}(H_4)$ be the reflection arrangement of type H_4. Then we have $\Phi^4_{15}(\mathcal{A}) = \Psi^4_{15}(\mathcal{A}) = \mathcal{A}$.
iii) Let $\mathcal{A} := \mathcal{A}(E_6)$ be the reflection arrangement of type E_6. Then we have $\Psi^6_{15}(\mathcal{A}) = \mathcal{A}$.
iv) Let $\mathcal{A} := \mathcal{A}(E_7)$ be the reflection arrangement of type E_7. Then we have $\Psi^7_{30}(\mathcal{A}) = \mathcal{A}$.
v) Let $\mathcal{A} := \mathcal{A}(E_8)$ be the reflection arrangement of type E_8. Then we have $\Phi^8_{63}(\mathcal{A}) = \Psi^8_{63}(\mathcal{A}) = \mathcal{A}$.

Proof. This can be verified by machine computations. $\square$

Remark 6.7. It appears that for the groups of type A_r we do not find any j such that one of the arrangements $\Phi^r_j(\mathcal{A}(A_r))$, $\Psi^r_j(\mathcal{A}(A_r))$ is combinatorially isomorphic to $\mathcal{A}(A_r)$. However, there are sporadic simplicial *line* arrangements that may be obtained for instance from the (non-simplicial) arrangement $\Psi^6_9(\mathcal{A}(A_6))$ by taking suitable restrictions. In particular, using the notation in [19], the arrangement $A(16, 7)$ arises in this way. This seems particularly interesting as the mentioned arrangement is *not* crystallographic.

6.4 Open problems and related questions

In this section we collect some interesting open problems related to the considerations of the current chapter. All problems deal with the case of the real projective plane. We start with the following one, which asks for realizability of the $t-$vectors obtained in Corollary 6.3.

Problem 12. *Prove or disprove that none of the elements in* $\{(50, 28, 9, 19), (54, 30, 7, 22), (57, 34, 6, 24), (62, 35, 3, 28), (66, 38, 1, 31), (69, 43, 0, 33)\}$ *is realizable as t-vector of a real projective line arrangement.*

A solution to the above problem would either give us a certain classification result (compare Remark 6.4), or it would yield new examples of simplicial arrangements with nice properties.

We continue with the following question which arises from experimental data gathered so far:

Problem 13. *Let* $\mathcal{A}$ *be a simplicial arrangement in* $\mathbb{P}^2(\mathbb{R})$*. Is it true that* $\Phi_4^3(\mathcal{A}) = \mathcal{A}$ *implies that* $\chi(\mathcal{A}, t)$ *has only real roots?*

Observe that the answer to the above question is "no", if we do not require the arrangement in consideration to be simplicial: the arrangement $\mathcal{A}_{\zeta_5}$ defined in Example 6.1 is a counterexample in this case.

The next problem we want to pose concerns the number of double and triple points determined by a simplicial arrangement $\mathcal{A}$ in the real projective plane such that $\Phi_4^3(\mathcal{A}) = \mathcal{A}$. All currently known examples share the property $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$. In Corollary 6.3 we have seen that this property holds true at least if the multiplicity of $\mathcal{A}$ is at most five and if $\chi(\mathcal{A}, t)$ has only real roots. We ask if it holds in general:

Problem 14. *Let* $\mathcal{A}$ *be a simplicial arrangement in* $\mathbb{P}^2(\mathbb{R})$*. Does* $\Phi_4^3(\mathcal{A}) = \mathcal{A}$ *always imply that* $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$ *?*

Note that if the answer to the above question was "yes", then by Theorem 6.2 we could conclude that at least for $|\mathcal{A}| \geq 50$ the property $\Phi_4^3(\mathcal{A}) = \mathcal{A}$ implies that all roots of $\chi(\mathcal{A}, t)$ are real. Thus, a positive solution to Problem 14 automatically gives a positive solution to Problem 13, at least for sufficiently large arrangements.

Moreover, we observe that despite being non-simplicial, the arrangement $\mathcal{A}_{\zeta_5}$ also does satisfy the condition $t_2^{\mathcal{A}} \geq t_3^{\mathcal{A}}$. Thus, compared to Problem 13, it seems that there might be a chance that the answer to Problem 14 is always "yes", even if the arrangement in consideration is non-simplicial.

We now turn to our final problem. A closer examination of (almost) all arguments given in Section 6.2 shows that we actually only used the condition

$$\sum_{i\geq 4} t_i^{\mathcal{A}} \leq |\mathcal{A}|, \tag{6.6}$$

which is easily implied by the stronger condition $\Phi_4^3(\mathcal{A}) = \mathcal{A}$.
Moreover, we observe that all currently known examples of simplicial arrangements in the real projective plane satisfy condition (6.6).

Thus, we ask if (6.6) is true for any simplicial arrangement in $\mathbb{P}^2(\mathbb{R})$:

Problem 15. *Let $\mathcal{A}$ be a simplicial arrangement in $\mathbb{P}^2(\mathbb{R})$. Is it always true that $\sum_{i\geq 4} t_i^{\mathcal{A}} \leq |\mathcal{A}|$?*

Denote by S_x the set of combinatorial isomorphism classes of simplicial line arrangements in $\mathbb{P}^2(\mathbb{R})$ with multiplicity at most x whose characteristic polynomials split over $\mathbb{R}$. We close this chapter with the following interesting observation: if the answer to Problem 15 was "yes", then by Theorem 6.3 we could conclude that for any $x \in \mathbb{N}$ the set S_x is finite. Note that this was proved to be true at least for $3 \leq x \leq 5$ in Chapter 5.

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