Asymptotics for Subcritical Fully Nonlinear Equations with Isolated Singularities

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Asymptotik für subkritische voll nichtlineare Gleichungen mit isolierten Singularitäten

ZUSAMMENFASSUNG

In dieser Dissertation betrachten wir die Gleichung

$$\sigma_k(A^u) = u^{\left(p-\frac{n+2}{n-2}\right)k},$$

wobei $n \geq 3$ und $p \in \left(\frac{n}{n-2}, \frac{n+2}{n-2}\right)$. Dabei ist σ_k das k-te elementarsymmetrische Polynom in den Eigenwerten von A^u und

$$A^u = -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}D^2u + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}\nabla u \otimes \nabla u - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|\nabla u|^2 I,$$

wobei ∇u den Gradienten von u und D^2u die Hessesche Matrix bezeichnen. Diese Gleichung ergibt sich in natürlicher Weise aus dem σ_k-Yamabe-Problem. Für $k = 1$ erhalten wir

$$-\Delta u = u^p;$$

dies ist einfach eine klassische subkritische semilinear-elliptische Gleichung.

Für $1 \leq k < \frac{n}{2}$ zeigen wir, dass eine zulässige Lösung dieser Gleichung mit nicht-hebbarer isolierter Singularität asymptotisch gleich einer radialen Lösung ist. Mit Hilfe einer genauen Analyse der linearisierten Gleichung sind wir dann in der Lage, asymptotische Entwicklungen höherer Ordnung für die Lösungen zu zeigen. Diese Resultate verallgemeinern die früheren bahnbrechenden Arbeiten von Caffarelli, Gidas und Spruck.

Als Beiprodukt erhalten wir Schoens Harnack-Ungleichung in Euklidischen Kugeln, das asymptotische Verhalten ganzer Lösungen. Basierend auf dem asymptotischen Verhalten erhalten wir einen weiteren Beweis des Liouville-Satz von Li und Li.

Schlüsselwörter: asymptotisches Verhalten, vollständig nichtlineare Gleichungen, isolierte Singularitäten

Asymptotics for Subcritical Fully Nonlinear Equations with Isolated Singularities

ABSTRACT

In this thesis, we consider this equation

$$\sigma_k(A^u) = u^{(p-\frac{n+2}{n-2})k},$$

where $n \geq 3$ and $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$. Here σ_k denotes the kth elementary symmetric function of the eigenvalues of A^u, and

$$A^u = -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}D^2u + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}\nabla u \otimes \nabla u - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|\nabla u|^2 I,$$

where ∇u denotes the gradient of u and D^2u denotes the Hessian of u. This equation arises naturally from the σ_k Yamabe equation. When $k = 1$, it amounts to

$$-\Delta u = u^p,$$

which is simply a classical subcritical semilinear elliptic equation.

We study the asymptotic behavior of solutions in a punctured ball. For $1 \leq k < \frac{n}{2}$, we prove that an admissible solution to this equation with a non-removable isolated singular point is asymptotic to a radial solution. Then we are able to obtain higher order expansion of solutions using analysis of the linearized operators. These results generalize earlier pioneering work of Caffarelli, Gidas and Spruck.

As a side effect, we also obtain Schoen's Harnack type inequality in Euclidean balls, asymptotic behavior of an entire solution. Based on the asymptotic behavior, we are able to give another proof of the Liouville type theorem obtained by Li and Li.

Key words: asymptotic behavior, fully nonlinear equations, isolated singularities

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Chapter 1

Introduction

1.1 Yamabe problem

Let (M^n, g_0) be an n-dimensional, compact, smooth Riemannian manifold without boundary. For $n = 2$, we see from the uniformization theorem of Poincaré that there exist metrics that are pointwise conformal to g and have constant Gauß curvature. For $n \geq 3$, the well-known Yamabe problem is to determine whether there exist metrics with constant scalar curvature that are pointwise conformal to g_0. The answer to the Yamabe problem is proved to be affirmative through Yamabe [87], Trudinger [82], Aubin [6] and Schoen [74]. See Lee and Parker [47] for a survey. See also Bahri [9] and Brezis and Bahri [10] for works on the Yamabe problem and related ones. Let $n \geq 3$ and $g = u^{\frac{4}{n-2}} g_0$ for some positive function u. The scalar curvature R_g of g can be calculated as

$$R_g = u^{-\frac{n+2}{n-2}}(R_{g_0} u - \frac{4(n-1)}{n-2}\Delta_{g_0} u),$$

where R_{g_0} denotes the scalar curvature of g_0 and Δ_{g_0} is the Laplace-Beltrami operator. Therefore the Yamabe problem is equivalent to the existence of a solution to

$$-\Delta_{g_0} u + \frac{n-2}{4(n-1)} R_{g_0} u = \frac{n-2}{4(n-1)} R_g u^{\frac{n+2}{n-2}}, \tag{1.1.1}$$

where $R_g \equiv c$ for some constant c.

The first two terms of the operator on the left in (1.1.1), that is,

$$\mathcal{L}_{g_0} := \Delta_{g_0} - \frac{n-2}{4(n-1)} R_{g_0}$$

give a second order linear elliptic differential operator known as the conformal Laplacian of the metric g_0.

Consider

$$Q(\varphi) = \frac{\int_{M^n}(|\nabla_{g_0}\varphi|^2 + \frac{n-2}{4(n-1)} R_{g_0}\varphi^2)}{(\int_{M^n}|\varphi|^{\frac{2n}{n-2}})^{\frac{n-2}{n}}},$$

for $\varphi \in H^1(M^n)\backslash\{0\}$. It is easy to see that a positive critical point of the functional Q is a solution to (1.1.1). The Sobolev quotient is given by

$$Q(M^n, g_0) = \inf\{Q(\varphi)|\varphi \in H^1(M^n)\backslash\{0\}\}.$$

Yamabe [87] attempted to prove that $Q(M^n, g_0)$ is always achieved. However, Trudinger [82] pointed out that Yamabe's proof is wrong and also corrected Yamabe's proof in the case $Q(M^n, g_0) \le 0$.

It was proved by Aubin [6] that $Q(M^n, g_0)$ is attained if

$$Q(M^n, g_0) < Q(\mathbb{S}^n, g_c), \tag{1.1.2}$$

where $(\mathbb{S}^n, g_c)$ denotes the standard n sphere. Aubin also verified the above inequality for $n \ge 6$ and M^n not locally conformally flat. The remaining cases are much more difficult since the local geometry does not contain sufficient information to conclude (1.1.2). In [74], Schoen established (1.1.2) by construcing global test functions in the remaining cases based on the positive mass theorem of Schoen and Yau [78].

In [77], Schoen obtained compactness results for the Yamabe problem. He proved that if (M^n, g_0) is locally conformally flat but not conformally diffeomorphic to the standard sphere, then all solutions to (1.1.1) stay in a compact set of $C^2(M^n)$. When (M^n, g_0) is not locally conformally flat, the same conclusion was proved by Li and Zhang [57] and Marques [62] independently for $n \le 7$. For $8 \le n \le 24$, it was proved that this compactness result is still true under the assumption that the positive mass theorem holds in these dimensions, see Li and Zhang for $8 \le n \le 11$ [57, 58], and Khuri, Marques and Schoen [45] for $12 \le n \le 24$. However, there are counterexamples in dimensions $n \ge 25$, see Brendle [14] for $n \ge 52$, and Brendle and Marques [15] for $25 \le n \le 51$.

1.2 σ_k Yamabe problem

Recently, there is a lot of attention focusing on the Yamabe problem for the σ_k curvature, briefly the σ_k Yamabe problem. First we recall the Schouten tensor

$$A_g = \frac{1}{n-2}(Ric_g - \frac{R_g}{2(n-1)}g),$$

where Ric_g is the Ricci tensor of g. Then we can decompose the Riemannian curvature tensor, Rm, into two parts

$$Rm = W_g + A_g \owedge g,$$

where W_g is the Weyl tensor and $\owedge$ denotes the Kulkari-Nomizu product, see for instance [12]. The main property of the Weyl tensor is its conformal invariance. Therefore the behavior of Riemannian curvature tensor under a conformal transformation of the metric is totally determined by the Schouten tensor.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be the set of eigenvalues of a symmetric $n \times n$ matrix A and for $1 \le k \le n$, σ_k denote the kth elementary symmetric function of the eigenvalues

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{1.2.1}$$

So the σ_k curvature of (M^n, g) is defined as $\sigma_k(g^{-1}A_g)$. The σ_k Yamabe problem on (M^n, g) consists in finding metrics with constant σ_k curvature in the same conformal class of g, namely,

$$\sigma_k(g^{-1}A_g) = constant. \tag{1.2.2}$$

When $k = 1$, it is the well known Yamabe problem. When $k \geq 2$, the equation becomes fully nonlinear PDE, and we need to recall the following notion: a metric g on M is said to be k admissible or in the Γ_k^+ class if it belongs to the kth positive cone Γ_k^+, where

$$g \in \Gamma_k^+ \iff \sigma_j(g^{-1}A_g) > 0, \;\; \forall\, 1 \leq j \leq k.$$

If g is in Γ_k^+ class, then the fully nonlinear equation is elliptic. In [39], Guan, Viaclovsky and Wang assert that if $g \in \Gamma_k^+$, then

$$Ric_g \geq \frac{2k-n}{2n(k-1)} R_g g.$$

Thus it is easy to see that $Ric_g > 0$ when $2k > n$.

When $k \neq \frac{n}{2}$ and (M^n, g) is locally conformally flat, (1.2.2) is the Euler-Lagrange equation of the functional

$$\mathcal{F}_k(g) = \frac{1}{(vol_g(M^n))^{\frac{n-2k}{n}}} \int_{M^n} \sigma_k(g^{-1}A_g) dvol_g,$$

see Viaclovsky [83]. In the case $k = \frac{n}{2}$, Brendle and Viaclovsky [16] present a variational characterization for (1.2.2). Under the assumption that g is in the Γ_k^+ class, the σ_k Yamabe problem for locally conformally flat manifolds has been solved by Li and Li [48] and Guan and Wang [37] independently. This result is also extended to much more general symmetric functions of $\lambda(g^{-1}A_g)$ by Li and Li [49]. In addition, Guan and Wang [38] applied the gradient flow method to derive the conformally invariant Sobolev inequality for locally conformally flat manifolds. In the case of general manifolds, the solution to the σ_k Yamabe problem has been obtained by Chang, Gursky and Yang [21] first for $k = 2$ and $n = 4$, by Ge and Wang [29] for $k = 2$ and $n > 8$, by Li and Nguyen [53] for $k = \frac{n}{2}$, by Gursky and Viaclovsky [40, 41] for $2k > n$. For $2 \leq 2k \leq n$ this problem has been solved by Sheng, Trudinger and Wang [81] under the extra hypothesis that the operator is variational. We should point out that this hypothesis always holds for $k = 1, 2$, while it is shown in [13] that this extra assumption is equivalent to the locally conformally flatness. Hence, the σ_k Yamabe problem is still open for $3 \leq k < n/2$ with (M^n, g) not locally conformally flat.

1.3 Singular σ_k Yamabe problem

Given $(\mathbb{S}^n, g_c)$, the singular σ_k Yamabe problem is to construct a new metric g with constant σ_k curvature conformal to g_c and complete on $\Omega \subset \mathbb{S}^n$, where Ω is a domain in $\mathbb{S}^n$. This problem can be transformed to a problem in $\tilde{\Omega} \subset \mathbb{R}^n$ with a conformally flat metric. In this setting, if we consider the metric on $\tilde{\Omega}$ as $\tilde{g} = u^{\frac{4}{n-2}}|dx|^2$, where $|dx|^2$ is the usual Euclidean metric, then we will solve the equation

$$\sigma_k(A^u) = R \;\text{ in } \tilde{\Omega} \tag{1.3.1}$$

with singular boundary behavior. Here and throughout the thesis we use this notation:

$$A^u = -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}D^2u + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}\nabla u \otimes \nabla u - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|\nabla u|^2 I,$$

where ∇u denotes the gradient of u and D^2u denotes the Hessian of u. Taking $k = 1$, we see that (1.3.1) becomes

$$-\Delta u = Ru^{\frac{n+2}{n-2}}.$$

The singular σ_k Yamabe problem has been extensively studied in recent years, also in the case when the ambient manifold is more general than the sphere. When $k = 1$ and $R_g > 0$, Schoen and Yau [79] proved that if a complete conformal metric g exists on a domain $\Omega \subset \mathbb{S}^n$ with $\sigma_1(g^{-1}A_g)$ bounded away from blew by a positive constant, then the Hausdorff dimension of $\mathbb{S}^n\backslash\Omega$, $dim_{\mathcal{H}}(\mathbb{S}^n\backslash\Omega) \leq \frac{n-2}{2}$. If in addition, $|R_g|+|\nabla_g R_g|$ are bounded and there exists a constant c_0 such that $Ric_g \geq -c_0 g$, then $dim_{\mathcal{H}}(\mathbb{S}^n\backslash\Omega) < \frac{n-2}{2}$. In [75] Schoen constructed complete conformal metrics on $\mathbb{S}^n\backslash\Lambda$ when Λ is either a finite discrete set on $\mathbb{S}^n$ containing at least two points or a set arising essentially as the limit set of a Kleinian group. Later Mazzeo and Pacard gave another proof of the result in [64]. They also proved in [63] that if $\Omega \subset \mathbb{S}^n$ is domain such that $\mathbb{S}^n\backslash\Omega$ consists of a finite number of disjoint smooth submanifolds of dimension $1 \leq k \leq \frac{n-2}{2}$, then there exists a complete metric on $\mathbb{S}^n\backslash\Omega$ with its scalar curvature identical to $n(n-1)$. See [69] for the earlier results in this direction. For the negative scalar curvature case, the results of Loewner and Nirenberg [59], Aviles [7], and Veron [85] imply that if $\Omega \subset \mathbb{S}^n$ admits a complete conformal metric with negative constant scalar curvature, then $dim_{\mathcal{H}}(\mathbb{S}^n\backslash\Omega) > \frac{n-2}{2}$. Loewer and Nirenberg [59] also proved that if $\Omega \subset \mathbb{S}^n$ is a domain with smooth boundary, then there exists a complete conformal metric on Ω with its scalar curvature identical to -1. Later this result was generalized by Finn [28] to the case of $\partial\Omega$ consisting of smooth submanifolds of dimension greater than $\frac{n-2}{2}$ and with boundary. For other development related to the negative scalar curvature case, see [46, 61, 67] and the references therein.

When $2 \leq k < \frac{n}{2}$, the singular σ_k Yamabe problem has been solved by Mazzieri and Ndiaye [65]. They proved that for a given finite set Λ of more than one point in $\mathbb{S}^n$, satisfying some additional assumptions involving their positions in the case $card(\Lambda) \geq 5$, there are complete metrics on $\mathbb{S}^n\backslash\Lambda$, conformal to the standard metric g_c and having positive constant σ_k curvature. See [19, 66] for connected sum construction for σ_k curvature. In [24] Chang, Hang and Yang proved that if $\Omega \subset \mathbb{S}^n$ ($n \geq 5$) admits a complete, conformal metric g with

$$\sigma_1(g^{-1}A_g) \geq c_0 > 0, \;\; \sigma_2(g^{-1}A_g) \geq 0, \;\; \text{and} \;\; |R_g| + |\nabla_g R_g| \leq c_1,$$

then $dim_{\mathcal{H}}(\mathbb{S}^n\backslash\Omega) < \frac{n-4}{2}$. This result was generalized by González [32] and Guan, Lin and Wang [35] to the case of $2 < k < \frac{n}{2}$: if $\Omega \subset \mathbb{S}^n$ admits a complete, conformal metric g with

$$\sigma_1(g^{-1}A_g) \geq c_0 > 0, \;\; \sigma_2(g^{-1}A_g), \cdots, \sigma_k(g^{-1}A_g) \geq 0, \;\; \text{and} \;\; |R_g| + |\nabla_g R_g| \leq c_1,$$

then $dim_{\mathcal{H}}(\mathbb{S}^n\backslash\Omega) < \frac{n-2k}{2}$. González [33] also showed that isolated singularities of C^3 solutions to (1.3.1) with finite volume are bounded.

1.4 The object of study and main results

We restrict our attention in this thesis to study the asymptotic behavior of singular solutions to the equation

$$\sigma_k^{1/k}(A^u) = cu^{p-\frac{n+2}{n-2}} \tag{1.4.1}$$

on the punctured ball, $B_2(0) \setminus \{0\}$, where $n \geq 3$, $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ and c is normalized to be $\binom{n}{k}/2^k$.

There are some relevant study on the singular solutions to (1.3.1), namely, the above equation with $p = \frac{n+2}{n-2}$. When $k = 1$ and the right hand side of (1.3.1) $R = 0$, (1.3.1) is the Laplace equation. A classical theorem of Bôcher asserts that any positive harmonic function in the punctured ball $B_1(0)\backslash\{0\}$ can be expressed as the sum of a multiple of the fundamental solution to the Laplace equation and a harmonic function in the unit ball $B_1(0)$. When $2 \leq k \leq \frac{n}{2}$ and $R = 0$, Li and Nguyen [54] obtained the classification of the positive solutions to (1.3.1). Li also [52] proved that a locally Lipschitz viscosity solution in $\mathbb{R}^n\backslash\{0\}$ must be radially symmetric about 0. In the case $k = 1$ and $R > 0$, Caffarelli, Gidas and Spruck [17] proved the asymptotic radial symmetry of positive singular solutions to (1.3.1) on a punctured ball, and further proved that such solutions are asymptotic to radial singular solutions to (1.3.1) on $\mathbb{R}^n\backslash\{0\}$. More precisely, for any singular solution $u(x)$ to (1.3.1) in $B_1(0)\backslash\{0\}$, there exists a radial singular solution $u^*(|x|)$ to (1.3.1) on $\mathbb{R}^n\backslash\{0\}$ such that

$$u(x) = u^*(|x|)(1 + o(1)), \text{ as } |x| \to 0.$$

A key ingredient in the proof of the above asymptotic behavior near 0 is a "measure theoretic" variation of the moving plane technique, which had been developed by Alexandrov [1, 2, 3, 4, 5], Serrin [73], Gidas, Ni and Nirenberg [30] to prove symmetries of solutions to certain elliptic PDEs. Later, Korevaar, Mazzeo, Pacard and Schoen [44] improved the $o(1)$ remainder term to $O(|x|^\alpha)$ for some α. They also provided an expansion of u after the order $|x|^\alpha$ using rescaling analysis, classification of global singular solutions and analysis of linearized operators at these global singular solutions. When $2 \leq k \leq n$ and $R > 0$, Chang, Han and Yang [23] classified all possible radial solutions to (1.3.1) in Γ_k^+ class on an annular domain including punctured ball and punctured Euclidean space. In [51], Li proved that an admissible solution with an isolated singularity at $0 \in \mathbb{R}^n$ to (1.3.1) is asymptotically symmetric. Later, Han, Li and Teixeira [42] studied the singular solution to (1.3.1) on a punctured ball when $2 \leq k \leq n$. Using the polar coordinate $x = (r, \theta)$ with $r = |x|$ and $\theta \in \mathbb{S}^{n-1}$, we introduce cylindrical variable $t = -\ln r$, so that

$$g = u^{\frac{4}{n-2}}(x)|dx|^2 = e^{-2w(t,\theta)}(dt^2 + d\theta^2).$$

They proved that

$$|w(t,\theta) - w^*(t)| \leq Ce^{-\alpha t} \quad \text{as } t \to \infty,$$

where $w^*(t)$ is a radial solution to (1.3.1). They also had the higher order expansion of w when $2 \leq k \leq \frac{n}{2}$. In 2013, A similar result was obtained by Wang [86] for conformal quotient equation.

When $1 \leq k < \frac{n}{2}$, for some technical reasons, we replace u in (1.4.1) with $u^{\frac{k(n-2)}{n-2k}}$, then obtain

$$\sigma_k^{1/k}(B^u) = cu^{p-\frac{n+2k}{n-2k}} \text{ in } B_2(0) \setminus \{0\}, \tag{1.4.2}$$

with $\frac{n}{n-2k} < p < \frac{n+2k}{n-2k}$, where

$$B^u = -\frac{2k}{n-2k} u^{-\frac{n+2k}{n-2k}} D^2 u + \frac{2kn}{(n-2k)^2} u^{-\frac{2n}{n-2k}} \nabla u \otimes \nabla u - \frac{2k^2}{(n-2k)^2} u^{-\frac{2n}{n-2k}} |\nabla u|^2 I.$$

Taking $k = 1$, (1.4.1) or (1.4.2) amounts to, modulo a harmless positive constant,

$$-\Delta u = u^p.$$

$\sigma_1(A^u) = u^{p-\frac{n+2}{n-2}}$ is simply a classical subcritical semilinear elliptic equation.

This subcritical equation arises naturally from the σ_k Yamabe equation, at least for the case $1 \le k < \frac{n}{2}$. Let us take $k = 1$ as an example. Suppose that $u > 0$ is a solution to

$$-\Delta_m u = u^{\frac{m+2}{m-2}} \ \text{ in } \ \mathbb{R}^m \backslash \Lambda$$

with $\Lambda = \mathbb{R}^{m-n} \subset \mathbb{R}^m$, $m > n$, where Δ_m is the Laplace operator in m dimensions. Let u depend only on the first n variables. Then u is also a solution to

$$-\Delta_n u = u^p \ \text{ in } \ \mathbb{R}^n \backslash \{0\},$$

where Δ_n is the Laplace operator in n dimensions, and $p = \frac{m+2}{m-2}$. If the dimension of Λ is less than $\frac{m-2}{2}$, then we have that $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$. When the dimension of Λ is equal to $\frac{m-2}{2}$, we see that $p = \frac{n}{n-2}$. Each of these cases near isolated singular point has been well studied. For $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, Gidas and Spruck [31] proved that if the singularity at 0 is non-removable, then

$$u(x) = \frac{c_0}{|x|^{\frac{2}{p-1}}}(1 + o(1)), \ \text{ as } \ |x| \to 0,$$

where $c_0 = [\frac{2(n-2)}{(p-1)^2}(p - \frac{n}{n-2})]^{\frac{1}{p-1}}$. For $p = \frac{n}{n-2}$, Aviles [8] obtained that if the singularity at 0 is non-removable, then

$$u(x) = [\frac{(n-2)^2}{2|x|^2 \ln(1/|x|)}]^{\frac{n-2}{2}}(1 + o(1)), \ \text{ as } \ |x| \to 0.$$

Our first theorem is a complete characterization for the solutions near isolated singularities.

Theorem 1.4.1. *Assume that $u \in C^2(B_2(0) \setminus \{0\})$ is a positive solution to (1.4.1) in $B_2(0)\backslash\{0\}$ in the Γ_k^+ class. Then either there exist two constants C_1 and C_2 such that*

$$\frac{C_1}{|x|^{\frac{2}{p-1}}} \le u(x) \le \frac{C_2}{|x|^{\frac{2}{p-1}}}, \tag{1.4.3}$$

or u can be extended as a Hölder continuous function on $B_2(0)$; when $k = 1$, u can actually be extended as a smooth solution to all of $B_2(0)$.

This theorem was obtained by González in [34] for $1 \le k < \frac{n}{2} - 1$. The main ingredient in her proof is the divergence structure of σ_k together with an Obata type argument. When $k = 1$ and $\frac{n}{n-2} < p \le \frac{n+2}{n-2}$, Caffarelli, Gidas and Spruck [17] proved this theorem. For $k = 1$ and

$\frac{n}{n-2} < p \le \frac{n+2}{n-2}$, this result was established by Gidas and Spruck [31]. When $1 \le k \le n$ and $p = \frac{n+2}{n-2}$, it was obtained by Han, Li and Teixeira [42]. We should note that this theorem is also valid for equation (1.4.2). To get the statement of the theorem for (1.4.2), we can replace (1.4.1) in Theorem 1.4.1 with (1.4.2).

In order to get the asymptotic behavior of a positive solution to (1.4.2) in $B_2(0)\backslash\{0\}$, we set $v(t,\theta) = |x|^{\frac{2}{p-1}}u(x)$ with $t = -\ln|x|$ and $\theta = \frac{x}{|x|}$. Since $u(x)$ is a solution to (1.4.2) in $B_2(0)\backslash\{0\}$, we see that $v(t,\theta)$ is a solution to

$$\sigma_k(B^v) = cv^{(p+1)k} \ \text{ in } \{t > -\ln 2\} \times \mathbb{S}^{n-1}, \tag{1.4.4}$$

with $\frac{n}{n-2k} < p < \frac{n+2k}{n-2k}$. Here

$$B^v := \begin{pmatrix} B^v_{11} & B^v_{1j} \\ B^v_{i1} & B^v_{ij} \end{pmatrix}$$

is a block matrix, where

$$B^v_{11} = (a + \frac{a^2}{2})v_t^2 - \frac{a^2}{2}v_\theta^2 - av_{tt}v + a(ab-1)v_tv - \frac{ab}{2}(2-ab)v^2,$$

$$B^v_{1j} = -avv_{t\theta_j} + a(1+a)v_tv_{\theta_j} + a(ab-1)vv_{\theta_j},$$

$$B^v_{i1} = -avv_{\theta_i t} + a(1+a)v_tv_{\theta_i} + a(ab-1)vv_{\theta_i},$$

and

$$B^v_{ij} = a(1+a)v_{\theta_i}v_{\theta_j} - avv_{\theta_i\theta_j} + [\frac{ab}{2}(2-ab)v^2 - a(ab-1)vv_t - \frac{a^2}{2}v_t^2 - \frac{a^2}{2}v_\theta^2]\delta_{ij}$$

with $a = \frac{2k}{n-2k}$ and $b = \frac{2}{p-1}$. Thanks to the asymptotically radially symmetric properties (Theorem 2.1.4) and some a priori estimates by Guan and Wang [36], we can find that any admissible solution u to (1.4.2) with a non-removable singularity at 0 is asymptotic to any radial solution to (1.4.2) satisfying (1.4.3). In terms of $v(t,\theta)$, we have

Theorem 1.4.2. *Let $v(t,\theta)$ be a smooth solution to (1.4.4) in $\{t > -\ln 2\} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class, where $n \ge 3$, $1 \le k < \frac{n}{2}$. Then for any radial solution $\xi(t)$ to (1.4.2) in $\mathbb{R} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class satisfying $C_1 \le \xi(t) \le C_2$, there exist constants $\alpha > 0$, $C > 0$ and t_0 such that in the case $k(ab-1)^2 \ne 2a(2-ab)$,*

$$|v(t,\theta) - \xi(t)| \le C\max\{e^{-t}, e^{-Re(\alpha_1)t}\} \ for \ t > t_0; \tag{1.4.5}$$

in the case $k(ab-1)^2 = 2a(2-ab)$,

$$|v(t,\theta) - \xi(t)| \le C\max\{e^{-t}, te^{-\alpha_0 t}\} \ for \ t > t_0, \tag{1.4.6}$$

where $\alpha_1 = \frac{k}{a}(ab-1) - \sqrt{\frac{k^2}{a^2}(ab-1)^2 - \frac{2k}{a}(2-ab)}$ and $\alpha_0 = \frac{k}{a}(ab-1)$. In particular, $v(t,\theta) - c^*$ also satisfies (1.4.5) or (1.4.6), where $c^* = (\frac{(n-2k)^{1/k}ab(2-ab)}{n^{1/k}})^{\frac{1}{p-1}}$ is a solution to (1.4.4).

A linearization procedure and some integral estimates show that the radial average of $v(t,\theta)$,

$\beta(t)$, solves some perturbation form of (1.4.4). By exploiting the perturbed ODE satisfied by $\beta(t)$, we prove that the average $\beta(t)$ is approximated by any radial solution $\xi(t)$ to (1.4.4) satisfying $C_1 \le \xi(t) \le C_2$. Combining Theorem 2.1.4, we arrive at the above theorem.

For the sake of simplicity, we set $w(t,\theta) = -a \ln v(t,\theta)$ with $a = \frac{2k}{n-2k}$. Since $v(t,\theta)$ is a solution to (1.4.4), we have that $w(t,\theta)$ is a solution to

$$\sigma_k(B^w) = ce^{-\frac{(p-1)k}{a}w} \text{ in } \{t > -\ln 2\} \times \mathbb{S}^{n-1}. \tag{1.4.7}$$

Here

$$B^w = \begin{pmatrix} B^w_{11} & B^w_{1j} \\ B^w_{i1} & B^w_{ij} \end{pmatrix}$$

is a block matrix, where

$$B^w_{11} = w_{tt} + \frac{1}{2}(w_t^2 - 2(ab-1)w_t - ab(2-ab)) - \frac{1}{2}w_\theta^2,$$

$$B^w_{1j} = w_{t\theta_j} + w_t w_{\theta_j} - (ab-1)w_{\theta_j},$$

$$B^w_{i1} = w_{\theta_i t} + w_t w_{\theta_i} - (ab-1)w_{\theta_i},$$

and

$$B^w_{ij} = w_{\theta_i\theta_j} + w_{\theta_i}w_{\theta_j} + \frac{1}{2}(-w_t^2 + 2(ab-1)w_t + ab(2-ab) - w_\theta^2)\delta_{ij}$$

with $a = \frac{2k}{n-2k}$ and $b = \frac{2}{p-1}$.

Inspired by the work of Korevaar, Mazzeo, Pacard and Schoen [44] and Han, Li, Teixeira [42], we obtain higher order expansions for solutions to (1.4.7):

Theorem 1.4.3. *Let $w(t,\theta)$ be a solution to (1.4.7) in $\{t > -\ln 2\} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class, and let $\varphi(t)$ be the radial solution to (1.4.7) in $\mathbb{R} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class. Then in the case $\varphi_t \equiv 0$, for any $Re(a_{N2})$, $N \ge 1$, there is a constant m_N that satisfies $m_N Re(a_{01}) \le Re(a_{N2}) < (m_N+1)Re(a_{01})$, some functions $\varphi_i(t,\theta)$, $1 \le i \le m_N - 1$,*

$$f_0(t) = c_{01}e^{-a_{01}t} + c_{02}e^{-a_{02}t}, \quad f_j(t,\theta) = c_{j2}e^{-a_{j2}t}Y_j(\theta), \quad 1 \le j \le N,$$

which are solutions to the linearized equation of (1.4.7) at $\varphi(t)$, such that for large t and small $\varepsilon_0 > 0$,

$$|w(t,\theta) - \varphi(t) - f_0(t) - \sum_{i=1}^{m_N-1}\varphi_i(t,\theta) - \sum_{j=1}^{N} f_j(t,\theta)| \le Ce^{-(m_N+1)Re(a_{01})t+\varepsilon_0 t}, \tag{1.4.8}$$

where c_{01}, c_{j2} are constants, $a_{j2} = \frac{(ab-1)(n-2k)+\sqrt{(ab-1)^2(n-2k)^2-4((2-ab)(n-2k)-\lambda_j)}}{2}$, $Re(a_{01}) = Re\left(\frac{(ab-1)(n-2k)-\sqrt{(ab-1)^2(n-2k)^2-4(2-ab)(n-2k)}}{2}\right) > 0$ and $(\lambda_j, Y_j(\theta))$ is the eigendata of $-\Delta_{\mathbb{S}^{n-1}}$; in the case $\varphi_t \not\equiv 0$, under the assumption $Re(a_{02}) \le 2Re(a_{01}) - \varepsilon_0$, there is a function

$$f_0(t) = c_0(t),$$

which is a solution to the linearized equation of (1.4.7) at $\varphi(t)$, such that for large t and small $\varepsilon_0 > 0$,

$$|w(t,\theta) - \varphi(t) - f_0(t)| \leq Ce^{-2Re(a_{01})t+\varepsilon_0 t}, \tag{1.4.9}$$

where $|c_0(t)| \leq Ce^{-Re(a_{01})t+\frac{\varepsilon_0}{2}t}$.

This theorem requires some knowledge on the spectrum of the linearized operator of (1.4.7). We first obtain this linearized operator. Then after a long computation we proved that the indicial root of the linearized operator $\rho_j > \frac{\sqrt{13}-1}{2}$ for $\lambda_j \geq 2n$ in Lemma 4.2.2. Next for a nonhomogeneous linearized equation, we apply a decomposition of the solutions with Wronskian function and the maximum principle to get the higher order estimates, then an iteration argument leads to the above theorem.

The analysis of linearized operator should be useful in constructing solutions to (1.3.1) on $\mathbb{S}^n \backslash \Lambda$, and in analyzing the moduli space of solutions to (1.3.1) on $\mathbb{S}^n \backslash \Lambda$, when Λ is a submanifold. Actually Mazzeo and Pacard [63] proved that when $k = 1$, there is a family of positive solutions to (1.4.1). Moreover, the solution space is locally a real analytic variety. Therefore along the line of the approach in [63] or following the way in the work of Roidos and Schrohe [70, 71, 72], we expect to obtain the same result for $2 \leq k < \frac{n}{2}$ in our future work.

As a side effect, we apply the moving spheres method to obtain the Harnack type inequality in Euclidean balls, asymptotic behavior of an entire solution. Based on the asymptotic behavior, we are able to give another proof of the remarkable Liouville type theorem obtained by Li and Li [49]. Recently, using the method of moving spheres and other approaches, Li and Nguyen [55] established blow-up profiles for any blowing-up sequence of solutions to general conformally invariant fully nonlinear elliptic equations on Euclidean domains.

Our next result concerns Schoen's Harnack type inequality without using the Liouville type theorem.

Theorem 1.4.4. *Suppose that $u \in C^2(B_{3R}(0))$ is a positive solution to*

$$\sigma_k^{1/k}(A^u) = u^{p-\frac{n+2}{n-2}} \ \ in \ \ B_{3R}(0) \tag{1.4.10}$$

for some $R > 0$. Then

$$\left(\max_{B_R(0)} u\right)\left(\min_{B_{2R}(0)} u\right)^{\alpha} \leq CR^{(2-n)\alpha},$$

where C depends only on n, and $\alpha = \frac{2}{(n-2)(p-1)-2} > 0$.

When $k = 1$ and $p = \frac{n+2}{n-2}$, the above theorem was proved by Schoen [76] based on the Liouville type theorem of Caffarelli, Gidas and Spruck [17]. In the case $1 \leq k \leq n$ and $p = \frac{n+2}{n-2}$, Li and Li [48] obtained the result by the method of moving spheres, a variant of the method of moving planes. When $k = 1$, $\alpha = 1$ and $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$, this theorem was proved by Li and Zhang [56] under an additional hypothesis that $\max_{\bar{B}_R} u \geq 1$. We note that our conclusion is invariant under the scaling $u(x) \to R^{\frac{2}{p-1}} u(Rx)$. The Harnack type inequality yields the following consequence as established by Schoen in [76] for $k = 1$, $p = \frac{n+2}{n-2}$, by Li and Li in [48] for $1 \leq k \leq n$ and $p = \frac{n+2}{n-2}$.

Corollary 1.4.5. *Let u be as in Theorem 1.4.4. Then*

$$\int_{B_R} u^{\frac{n(p-1)}{2}} \leq C(n). \tag{1.4.11}$$

Owing to the Harnack type inequality, we are able to get the asymptotic behavior of an entire solution.

Theorem 1.4.6. *Let $u \in C^2(\mathbb{R}^n)$ be a positive solution to*

$$\sigma_k^{1/k}(A^u) = u^{p-\frac{n+2}{n-2}} \ \ in \ \ \mathbb{R}^n$$

in the Γ_k^+ class, where $\frac{n}{n-2} < p < \frac{n+2}{n-2}$. Then

$$0 < \liminf_{|x|\to\infty} \left(|x|^{n-2}u(x)\right) \leq \limsup_{|x|\to\infty} \left(|x|^{n-2}u(x)\right) < \infty, \tag{1.4.12}$$

and

$$\limsup_{|x|\to\infty} \left(|x|^{n-1}|\nabla u(x)| + |x|^n|\nabla^2 u(x)|\right) < \infty. \tag{1.4.13}$$

In the case $p = \frac{n+2}{n-2}$, the above theorem was proved by Li and Li [48]. Next we recall the remarkable Liouville type theorem obtained by Li and Li [49].

Theorem 1.4.7. *For $n \geq 3$, assume that $u \in C^2(\mathbb{R}^n)$ is a positive solution to $\sigma_k^{1/k}(A^u) = u^{p-\frac{n+2}{n-2}}$ in $\mathbb{R}^n$ in the Γ_k^+ class for some p, $-\infty < p \leq \frac{n+2}{n-2}$. Then either $u \equiv$ constant or $p = \frac{n+2}{n-2}$ and, for some $\bar{x} \in \mathbb{R}^n$ and some positive constants a_1 and b_1 satisfying $2b_1^2a_1^{-2}I$ in the Γ_k^+ class and $\sigma_k(2b_1^2a_1^{-2}I) = 1$,*

$$u(x) \equiv \left(\frac{a_1}{1 + b_1^2|x - \bar{x}|^2}\right)^{(n-2)/2}, \quad x \in \mathbb{R}^n.$$

When $k = 1$ and $p = \frac{n+2}{n-2}$, this theorem was established by Caffarelli, Gidas and Spruck [17], while under some additional hypothesis, it was proved by Obata [68] and Gidas, Ni and Nirenberg [30]. Somewhat different proofs of the result of Caffarelli, Gidas and Spruck were given in [26], [56]. For $1 \leq k \leq n$ and $p = \frac{n+2}{n-2}$, under some hypothesis on u near infinity, the result was proved by Viaclovsky [83], [84]. For $k = 2$, $n = 4$ and $p = \frac{n+2}{n-2}$, the result is due to Chang, Gursky and Yang [21]. For $1 \leq k \leq n$, the result was established Li and Li [48]. For $k = 2$ and $p = \frac{n+2}{n-2}$ in dimension $n = 5$, as well as for the same case in dimension $n \geq 6$ under the additional hypothesis $\int_{\mathbb{R}^n} u^{2n/(n-2)} < \infty$, the result was established by Chang, Gursky and Yang [22]. When $k = 1$ and $1 < p < \frac{n+2}{n-2}$, this result was obtained by Gidas and Spruck [31]. The proof of this theorem bases on an observation on the behavior of isolated singularities, which avoids using global information of the entire solution. Based on the asymptotic behavior of u near infinity (Theorem 1.4.6), we are able to give an another proof of the above theorem, Liouville type theorem, with $\frac{n}{n-2} < p < \frac{n+2}{n-2}$.

Corollary 1.4.8. *For $n \geq 3$, let $u \in C^2(\mathbb{R}^n)$ be a positive solution to*

$$\sigma_k^{1/k}(A^u) = u^{p-\frac{n+2}{n-2}} \ \ in \ \mathbb{R}^n$$

in the Γ_k^+ class, where $\frac{n}{n-2} < p < \frac{n+2}{n-2}$. Then $u \equiv$ constant.

It is not hard to see that this result is also available for $p = \frac{n+2}{n-2}$. But it can not cover the case $-\infty < p \leq \frac{n}{n-2}$, because it heavily depends on the positivity of α (Theorem 1.4.4), where $\alpha = \frac{2}{(n-2)(p-1)-2}$.

By the way, it is sometimes more convenient to use different forms of (1.4.1). Let $u^{\frac{4}{n-2}} = e^{-2w_0} = v_0^{-2}$. From the definition of A^u, it is easy to see that (1.4.1) with $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ is equivalent to

$$\sigma_k^{1/k}(A^{w_0}) = e^{-\beta_0 w_0}, \tag{1.4.14}$$

where $1 < \beta_0 < 2$ and

$$A^{w_0} = D^2 w_0 + \nabla w_0 \otimes \nabla w_0 - \frac{|\nabla w_0|^2}{2} I,$$

or

$$\sigma_k^{1/k}(A^{v_0}) = v_0^{\beta_1},$$

where $0 < \beta_1 < 1$ and

$$A^{v_0} = v_0 D^2 v_0 - \frac{|\nabla v_0|^2}{2} I.$$

This thesis is organized as follows. In Chapter 2, we establish the classification of singularities, Theorem 1.4.1. In Chapter 3, we prove the asymptotic behavior, Theorem 1.4.2, by exploiting the ODE satisfied by the radial average. In Chapter 4, we give a proof of Theorem 1.4.3 by an analysis of the linearized operator. Theorems 1.4.4-1.4.6 are carried out in the last chapter, Chapter 5.

Chapter 2

Classification of singularities

In this chapter, we will establish the classification of solutions to (1.4.1) near an isolated singularity.

2.1 Preliminary

In this section, we list some preliminary facts which we will use later.

The following fact follows from a classical result of G. C. Evans [27]: Let E be a closed subset of $B_2(0)$ of capacity 0-the standard capacity with respect to the Dirichlet integral, and let $u \in C^2(B_2(0) \setminus E)$ and $v \in C^2(B_2(0))$ satisfy

$$u > v \text{ and } \Delta u \leq 0 \leq \Delta v \text{ in } B_2(0) \setminus E. \tag{2.1.1}$$

Then

$$\liminf_{dist(x,E)\to 0}[u(x) - v(x)] > 0. \tag{2.1.2}$$

Let $S \in C^1(\mathbb{R}^n \times \mathcal{S}^{n\times n})$ satisfy

$$-\frac{\partial S}{\partial M_{ij}}(p, M) > 0, \quad \forall (p, M) \in \mathbb{R}^n \times \mathcal{S}^{n\times n},$$

and let, for $\beta \in \mathbb{R}\backslash\{0\}$,

$$T(t, p, M) := S(t^{-\frac{1+\beta}{\beta}} p, t^{-\frac{2+\beta}{\beta}} M), \quad (t, p, M) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^{n\times n},$$

where $\mathcal{S}^{n\times n}$ denotes the set of $n \times n$ real symmetric matrices, $\mathcal{S}^{n\times n}_+$ denotes the subset of $\mathcal{S}^{n\times n}$ consisting of positive definite matrices. ($O(n)$ denotes the set of $n \times n$ real orthogonal matrices.)

Theorem 2.1.1. *(Corollary 1.5 in [51]) For $n \geq 2$, let S, β and T be as above. If $-1 < \beta < 0$, we further require that*

$$S(p, 0) \geq 0 \;\; \forall p \in \mathbb{R}^n.$$

Assume that $u \in C^2(B_2(0)\backslash\{0\})$ and $v \in C^2(B_2(0))$ satisfy

$$v > 0 \;\; \text{in } B_2(0)$$

$$u > v \ \ in\ B_2(0)\backslash\{0\}$$

$$\Delta u \le 0 \ \ in\ B_2(0)\backslash\{0\}$$

$$T(u, \nabla u, D^2 u) \ge 0 \ge T(v, \nabla v, D^2 v) \ \ in\ B_2(0)\backslash\{0\}.$$

Then

$$\liminf_{|x|\to 0}[u(x) - v(x)] > 0.$$

Next we will list some theorems that are useful in our proofs. The main ingredient in the proof of these theorems is a blow up argument together with the moving sphere technique. The first theorem is a global result for solutions in $\mathbb{R}^n$.

Theorem 2.1.2. *Assume that $u \in C^2(\mathbb{R}^n \setminus \{0\})$ is a positive solution to (1.4.1) in $\mathbb{R}^n \setminus \{0\}$ in the Γ_k^+ class with $-\infty < p < \frac{n+2}{n-2}$. Then u is radially symmetric about the origin and $u'(r) < 0$.*

Based on the observation in [49] and Theorem 2.1.1, the proof of the above theorem is along the proof of Theorem 1.2 in [51]. When $k = 1$ and $\frac{n}{n-2} \le p \le \frac{n+2}{n-2}$, the above theorem is obtained by Caffarelli, Gidas and Spruck [17]. When $p = \frac{n+2}{n-2}$, Li [51] proved this theorem.

The second result is the fastest blow up rate of solutions near a singular point.

Theorem 2.1.3. *Assume that $u \in C^2(B_2(0) \setminus \{0\})$ is a positive solution to (1.4.1) in $B_2(0)\backslash\{0\}$ in the Γ_k^+ class with $1 < p < \frac{n+2}{n-2}$. Then*

$$\limsup_{|x|\to 0} |x|^{\frac{2}{p-1}} u(x) < \infty. \tag{2.1.3}$$

The exponent $\frac{2}{p-1}$ in (2.1.3) with $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ is sharp for $1 \le k < \frac{n}{2}$, see Section 2.3 for details. There are two ways to prove this theorem. The first one is from the Liouville type theorem obtained by Li and Li [49], which Prof. Dr. YanYan Li told me. The second is following the proof of Theorem 1.1' in [51] together with Theorem 2.1.1. When $k = 1$ and $\frac{n}{n-2} < p \le \frac{n+2}{n-2}$, this theorem is proved by Caffarelli, Gidas and Spruck [17]. When $p = \frac{n+2}{n-2}$, Li [51] showed it.

The third theorem states that the solutions are asymptotically radially symmetric.

Theorem 2.1.4. *Assume that $u \in C^2(B_2 \setminus \{0\})$ is a positive solution to (1.4.1) in the Γ_k^+ class. Then*

$$u(x) = \bar{u}(|x|)(1 + O(|x|)) \ as\ x \to 0, \tag{2.1.4}$$

where $\bar{u}(|x|) = |\mathbb{S}^{n-1}|^{-1} \int_{\mathbb{S}^{n-1}} u(|x|, \theta) d\theta$ is the spherical average of u.

The arguments of Theorem 1.3 in [51] and Theorem 2.1.1 yield the above theorem. When $k = 1$, these results were proved by Caffarelli, Gidas and Spruck in [17]. When $1 \le k \le n$ and $p = \frac{n+2}{n-2}$, Li [51] obtained the similar results.

For $x \in \mathbb{R}^n$ and $\lambda > 0$, consider the Kelvin transformation of u:

$$u_{x,\lambda} = \frac{\lambda^{n-2}}{|y-x|^{n-2}} u(x + \frac{\lambda^2 (y-x)}{|y-x|^2}), \ \ y \in \mathbb{R}^n\backslash\{x\}.$$

For the reader's convenience, some calculus lemmas are taken from [56].

Lemma 2.1.5. *(Lemma 11.2 in [56]) Let $f \in C^1(\mathbb{R}^n)$, $n \geq 1$, $\nu > 0$. Assume that*

$$(\frac{\lambda}{|y-x|})^\nu f(x + \frac{\lambda^2(y-x)}{|y-x|^2}) \leq f(y), \ \forall \lambda > 0, \quad x \in \mathbb{R}^n, \quad |y-x| \geq \lambda.$$

Then $f \equiv constant$.

Lemma 2.1.6. *(Lemma 11.1 in [56]) Let $f \in C^1(\mathbb{R}^n)$, $n \geq 1$, $\nu > 0$. Suppose that for every $x \in \mathbb{R}^n$, there exists $\lambda(x) > 0$ such that*

$$(\frac{\lambda}{|y-x|})^\nu f(x + \frac{\lambda^2(y-x)}{|y-x|^2}) = f(y), \quad y \in \mathbb{R}^n \backslash \{x\}.$$

Then for some $a \geq 0$, $d > 0$, $\bar{x} \in \mathbb{R}^n$,

$$f(x) = \pm \left(\frac{a}{d + |x-\bar{x}|^2} \right)^{\frac{\nu}{2}}.$$

2.2 Classification of singularities

In this section, we will prove Theorem 1.4.1.

Lemma 2.2.1. *Suppose u is a positive solution to (1.4.1) in the Γ_k^+ class. Then for all $0 < r < \frac{1}{4}$, we have*

$$\sup_{B_{2r}(0)\backslash \bar{B}_{r/2}(0)} u \leq C \inf_{B_{2r}(0)\backslash \bar{B}_{r/2}(0)} u, \tag{2.2.1}$$

where C is a positive constant independent of r.

Proof. Let

$$v(x) = r^{\frac{2}{p-1}} u(rx).$$

It follows from Theorem 2.1.3 that

$$0 < v(x) \leq C, \ \forall \ |x| \in [\frac{1}{4}, 4],$$

where C is a positive constant independent of r. Moreover, v satisfies (1.4.2) as well. By Harnack inequality in [36], we get

$$\sup_{\frac{1}{2} \leq |x| \leq 2} v(x) \leq C \inf_{\frac{1}{2} \leq |x| \leq 2} v(x),$$

where C is independent of r. Then (2.2.1) follows. □

By Harnack inequality, we claim

$$\liminf_{|x| \to 0} u(x) = \infty, \tag{2.2.2}$$

if 0 is a non-removable singularity of u. In fact, there exists a sequence x_j such that

$$r_j = |x_j| \to 0 \text{ and } u(x_j) \to \infty \text{ as } j \to \infty. \tag{2.2.3}$$

It follows from (2.2.1) that

$$\inf_{|x|=r_j} u \geq \frac{1}{C} u(x_j).$$

By the maximum principle,

$$\inf_{r_{j+1}\leq|x|\leq r_j} u(x) = \inf_{|x|=r_j, r_{j+1}} u(x) \geq \frac{1}{C}\min\{u(x_j), u(x_{j+1})\} \to \infty,$$

as $j \to \infty$. The claim is proved.

Proposition 2.2.2. *Let u be a positive solution to (1.4.1) in the Γ_k^+ class. If*

$$\liminf_{|x|\to 0} |x|^{\frac{2}{p-1}} u(x) = 0,$$

then u can be extended as a Hölder continuous function near the origin 0*. When $k = 1$, u can actually be extend as a smooth solution to all of $B_2(0)$.*

Proof. By Theorem 2.1.3,

$$\sup_{0<|x|\leq 1} |x|^{\frac{2}{p-1}} u(x) < \infty. \tag{2.2.4}$$

Since $\Delta u \leq 0$ in $B_2(0)\backslash\{0\}$, we have

$$u(x) \geq \min_{\partial B_1(0)} u > 0, \ \forall\, 0 < |x| \leq 1. \tag{2.2.5}$$

In fact, let v be the solution to $\Delta v = 0$ in $B_1(0)$ with $v = u$ on $\partial B_1(0)$. From the maximum principle, we have $\min_{\partial B_1(0)} u \leq v(x) \leq \max_{\partial B_1(0)} u$ for all $x \in \bar{B}_1(0)$. Since $v(x) - u(x) \leq \max_{\partial B_r(0)}(v-u)\frac{r^{n-2}}{|x|^{n-2}}$ for $x \in \partial B_r(0)$, $v(x) - u(x) = 0$ for $x \in \partial B_1(0)$ and $\Delta(v - u) \geq 0$, by the maximum principle again, we see

$$v(x) - u(x) \leq \max_{\partial B_r(0)}(v-u)\frac{r^{n-2}}{|x|^{n-2}} \leq \frac{(\max_{\partial B_r(0)} v + \max_{\partial B_r(0)} u) r^{n-2}}{|x|^{n-2}},$$

for $x \in B_1(0)\backslash B_r(0)$. Sending $r \to 0$, we obtain $v(x) \leq u(x)$ for $x \in \bar{B}_1(0)\backslash\{0\}$. Thus $u(x) \geq v(x) \geq \min_{\partial B_1(0)} u$ for $x \in \bar{B}_1(0)\backslash\{0\}$.

Since $\liminf_{|x|\to 0} |x|^{\frac{2}{p-1}} u(x) = 0$, by (2.2.1), we obtain

$$\lim_{|x|\to 0} |x|^{\frac{2}{p-1}} u(x) = 0.$$

Then we have that $u(x) \leq |x|^{\frac{2}{1-p}+c_0}$ for some constant $c_0 > 0$. It follows that

$$\int_{B_\varepsilon(0)} u^{\frac{n(p-1)}{2}} dx \leq \varepsilon.$$

Because the above inequality is invariant under the scaling $u(x) \to \varepsilon^{\frac{2}{p-1}} u(\varepsilon x)$, we get

$$\int_{B_1(0)} u^{\frac{n(p-1)}{2}} dx \leq \varepsilon,$$

for small $\varepsilon > 0$, that is, the volume smallness condition. Following the argument similar to that in [33], we have u is bounded, then u is a Hölder continuous function.

When $k = 1$, we will obtain much stronger conclusion, that is, $u \in C^\infty(B_2(0))$. This part is inspired by the work [44]. First, we claim that $u \in L^q(B_2(0))$ for some $q > \frac{n(p-1)}{2}$. Let $v(t, \theta) = |x|^{\frac{2}{p-1}} u(x)$, where $t = -\ln |x|$, $\theta = x/|x|$. Since u is a solution to $-\Delta u = u^p$, we see that v is a solution to

$$v_{tt} + \Delta_{\mathbb{S}^{n-1}} v + \frac{2}{a}(ab-1)v_t - \frac{b}{a}(2-ab)v + v^p = 0 \ \text{ in } [-\ln 2, \infty) \times \mathbb{S}^{n-1},$$

where $a = \frac{2}{n-2}$ and $b = \frac{2}{p-1}$. The hypothesis $\lim_{x\to 0} |x|^{\frac{2}{p-1}} u(x) = 0$ implies that

$$v(t, \theta) \to 0 \text{ uniformally as } t \to \infty.$$

Therefore we see that

$$\Delta v \geq -\frac{2}{a}(ab-1)v_t + C_0 v$$

for some $C_0 > 0$ and $t \geq t_0$ for sufficiently large t_0. Now we consider the function $\hat{v} = C_1 e^{-\alpha_1 t} + \varepsilon e^{\alpha_2 t}$ where $\alpha_1 > 0$, $\alpha_2 > 0$ and C_1 is chosen large enough such that $C_1 e^{-\alpha_1 t_0} > v(t_0, \theta)$ for all $\theta \in \mathbb{S}^{n-1}$. Note that $\hat{v}$ is a solution to $\Delta\hat{v} = -\frac{2}{a}(ab-1)\hat{v}_t + C_0\hat{v}$. By the maximum principle, we have

$$v(t, \theta) \leq C_1 e^{-\alpha_1 t} + \varepsilon e^{\alpha_2 t}$$

for all θ. Since C_1 is independent of ε, we may let ε go to zero. Then we get $e^{\alpha_1 t} v(t, \theta) \leq C_1$ for $t \geq t_0$. Writing this in terms of u, we have

$$u(x) \leq C_1 |x|^{q_1} \text{ for } q_1 = \frac{2}{1-p} + \alpha_1.$$

This implies that $u \in L^q(B_2(0))$ for some $q > \frac{n(p-1)}{2}$. Next from Lemma 2.1 in [17], we see that $u \in L^p(B_2(0))$ and u is a distribution solution to $-\Delta u = u^p$ in $B_2(0)$. Then by the estimates in [88], we obtained

$$||u||_{W^{2,q/p}(\Omega)} \leq C||u^p||_{L^{q/p}(B_2(0))} \leq C||u||^p_{L^q(B_2(0))} \leq C,$$

where $\Omega \subset\subset B_2(0)$. Using the Sobolev embedding theorem, we have that $u \in L^{\frac{nq}{np-2q}}(\Omega) \subset L^p(\Omega)$. After finite steps, we get $u \in C^\alpha(B_2(0))$. Then by the Schauder theory, $u \in C^{2,\alpha}(B_2(0))$. Finally we have $u \in C^\infty(B_2(0))$.

Now we have established this proposition. □

Proof of Theorem 1.4.1. By the above proposition and Theorem 2.1.3, we have that either there exist two constants C_1 and C_2 such that

$$\frac{C_1}{|x|^{\frac{2}{p-1}}} \leq u(x) \leq \frac{C_2}{|x|^{\frac{2}{p-1}}},$$

or u can be extended as a Hölder continuous function on $B_2(0)$, when $k = 1$, u can actually be extended as a smooth solution to all of $B_2(0)$. □

2.3 Sharpness of Theorem 2.1.3

The two lemmas in this section give the sharpness of Theorem 2.1.3.

Lemma 2.3.1. *For $n \geq 3$, let $u(x) = |x|^{\frac{2}{1-p}}$, $x \in \mathbb{R}^n \backslash \{0\}$, where $\frac{n}{n-2} < p < \frac{n+2}{n-2}$. Then*

$$\lambda(A^u) = \{-1, 1, \ldots, 1\} \frac{4}{(n-2)(p-1)^2}(p - \frac{n}{n-2})u^{p-\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n \backslash \{0\}. \tag{2.3.1}$$

Proof. We write $u(x)$ as $u(r)$ with $r = |x|$, and only need to verify (2.3.1) at $x = (r, 0, \ldots, 0)$, $r > 0$. At this point,

$$\nabla u = (u'(r), 0, \ldots, 0), \; D^2 u = diag(u'', \frac{u'(r)}{r}, \ldots, \frac{u'(r)}{r}),$$

and

$$A^u(x) = diag(\lambda_1^u(r), \lambda_2^u(r), \ldots, \lambda_n^u(r)),$$

where

$$\lambda_1^u(r) = -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}u'' + \frac{2(n-1)}{(n-2)^2}u^{-\frac{2n}{n-2}}(u')^2,$$

and

$$\lambda_2^u(r) = \cdots = \lambda_n^u(r) = -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}\frac{u'}{r} - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}(u')^2.$$

With this we compute

$$u' = \frac{2}{1-p}r^{\frac{p+1}{1-p}} = \frac{2}{1-p}u^{\frac{p+1}{2}}, \; u'' = \frac{2}{1-p}\frac{p+1}{2}u^{\frac{p-1}{2}}u' = \frac{2(p+1)}{(p-1)^2}u^p,$$

$$\begin{aligned}
\lambda_1^u(r) &= -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}\frac{2(p+1)}{(p-1)^2}u^p + \frac{2(n-1)}{(n-2)^2}u^{-\frac{2n}{n-2}}\frac{4}{(p-1)^2}u^{p+1} \\
&= -\frac{4(p+1)}{(n-2)(p-1)^2}u^{p-\frac{n+2}{n-2}} + \frac{8(n-1)}{(n-2)^2(p-1)^2}u^{p-\frac{n+2}{n-2}} \\
&= -\frac{4}{(n-2)(p-1)^2}(p - \frac{n}{n-2})u^{p-\frac{n+2}{n-2}},
\end{aligned}$$

and

$$\begin{aligned}
\lambda_2^u(r) = \cdots = \lambda_n^u(r) &= -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}\frac{2}{1-p}r^{\frac{2p}{1-p}} - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}\frac{4}{(p-1)^2}u^{p+1} \\
&= \frac{4}{(n-2)(p-1)}u^{p-\frac{n+2}{n-2}} - \frac{8}{(n-2)^2(p-1)^2}u^{p-\frac{n+2}{n-2}} \\
&= \frac{4}{(n-2)(p-1)^2}(p - \frac{n}{n-2})u^{p-\frac{n+2}{n-2}}.
\end{aligned}$$

This lemma is established. □

Lemma 2.3.2. *For $\lambda_0 = (-1, 1, \dots, 1) \in \mathbb{R}^n$, $n \geq 2$,*

$$\sigma_k(\lambda_0) > 0, \ \textit{for}\ 1 \leq k < \frac{n}{2},$$
$$\sigma_k(\lambda_0) = 0, \ \textit{for}\ k = \frac{n}{2},$$
$$\sigma_k(\lambda_0) < 0, \ \textit{for}\ \frac{n}{2} < k \leq n.$$

It follows that $\frac{4}{(n-2)(p-1)^2}(p - \frac{n}{n-2})u^{p-\frac{n+2}{n-2}}(-1, 1, \dots, 1)$ belongs to Γ_k^+, $\forall\ 1 \leq k < \frac{n}{2}$, and this vector does not belongs to Γ_k^+, $\forall\ k \geq \frac{n}{2}$.

Proof. It follows from Lemma 8.2 in [51]. □

Chapter 3

Asymptotic behaviors of singular solutions

A remarkable result of Caffarelli, Gidas and Spruck [17] says that if u is a positive solution to

$$-\Delta u = u^p \quad \text{in } B_2(0)\backslash\{0\}$$

with a non-removable isolated singularity, then for $\frac{n}{n-2} < p < \frac{n+2}{n-2}$,

$$u(x) = \frac{c_0}{|x|^{2/(p-1)}}(1 + o(1)), \quad \text{as } x \to 0,$$

where $c_0 = [\frac{2(n-2)}{(p-1)^2}(p - \frac{n}{n-2})]^{1/(p-1)}$.

Our objective in this chapter is to study similar problems for singular solutions to (1.4.2) with $1 \le k < \frac{n}{2}$. In order to get the asymptotic behavior of a positive solution to (1.4.2) in $B_2(0)\backslash\{0\}$, we set $v(t,\theta) = |x|^{\frac{2}{p-1}}u(x)$ with $t = -\ln|x|$ and $\theta = \frac{x}{|x|}$. Since $u(x)$ is a solution to (1.4.2) in $B_2(0)\backslash\{0\}$, we see that $v(t,\theta)$ is a solution to

$$\sigma_k(B^v) = cv^{(p+1)k} \quad \text{in } \{t > -\ln 2\} \times \mathbb{S}^{n-1}, \tag{3.0.1}$$

with $\frac{n}{n-2k} < p < \frac{n+2k}{n-2k}$. Here

$$B^v := \begin{pmatrix} B^v_{11} & B^v_{1j} \\ B^v_{i1} & B^v_{ij} \end{pmatrix}$$

is a block matrix, where

$$B^v_{11} = (a + \frac{a^2}{2})v_t^2 - \frac{a^2}{2}v_\theta^2 - av_{tt}v + a(ab-1)v_t v - \frac{ab}{2}(2-ab)v^2,$$

$$B^v_{1j} = -avv_{t\theta_j} + a(1+a)v_t v_{\theta_j} + a(ab-1)vv_{\theta_j},$$

$$B^v_{i1} = -avv_{\theta_i t} + a(1+a)v_t v_{\theta_i} + a(ab-1)vv_{\theta_i},$$

and

$$B^v_{ij} = a(1+a)v_{\theta_i}v_{\theta_j} - avv_{\theta_i\theta_j} + [\frac{ab}{2}(2-ab)v^2 - a(ab-1)vv_t - \frac{a^2}{2}v_t^2 - \frac{a^2}{2}v_\theta^2]\delta_{ij}$$

with $a = \frac{2k}{n-2k}$ and $b = \frac{2}{p-1}$. From $\frac{n}{n-2k} < p < \frac{n+2k}{n-2k}$, it is easy to see that $1 < ab < 2$.

3.1 Classification of radial solutions

In this section, we will show the asymptotic behaviors of positive radial solutions to (3.0.1) on $\mathbb{R} \times \mathbb{S}^{n-1}$.

When $v(t,\theta) := \xi(t)$ is a function of t, B^v becomes a block diagonal matrix

$$B^\xi = \begin{pmatrix} A - \frac{B}{2} & 0 \\ 0 & \frac{B}{2}\delta_{ij} \end{pmatrix},$$

where

$$A = a\xi^2 - a\xi\xi_{tt},$$

and

$$B = a[-a\xi_t^2 - 2(ab-1)\xi_t\xi + b(2-ab)\xi^2].$$

Therefore we have

$$\sigma_k(B^\xi) = \binom{n-1}{k-1}(A - \frac{B}{2})(\frac{B}{2})^{k-1} + \binom{n-1}{k}(\frac{B}{2})^k = \binom{n}{k}(\frac{B}{2})^{k-1}[\frac{k}{n}(A - \frac{B}{2}) + \frac{n-k}{2n}B].$$

Meanwhile from definition of A and B, we get

$$\begin{aligned}\frac{k}{n}(A - \frac{B}{2}) + \frac{n-k}{2n}B &= \frac{k}{n}A + \frac{n-2k}{2n}B \\ &= \frac{k}{n}a\xi_t^2 - \frac{k}{n}a\xi\xi_{tt} + \frac{k}{n}[-a\xi_t^2 - 2(ab-1)\xi_t\xi + b(2-ab)\xi^2] \\ &= \frac{n-2k}{2n}a[-a\xi_{tt}\xi - 2(ab-1)\xi_t\xi + b(2-ab)\xi^2].\end{aligned}$$

It follows from $\sigma_k(B^\xi) = c\xi^{(p+1)k}$ with $c = \binom{n}{k}/2^k$ that

$$[-a\xi_t^2 - 2(ab-1)\xi_t\xi + b(2-ab)\xi^2]^{k-1}[-a\xi_{tt}\xi - 2(ab-1)\xi_t\xi + b(2-ab)\xi^2] = c_1\xi^{(p+1)k}, \quad (3.1.1)$$

where $c_1 = \frac{n}{a^k(n-2k)}$. Now we denote above equation as $\sigma_k(B^\xi) = c_1\xi^{(p+1)k}$.

Since σ_k has a fixed sign on $\mathbb{R}^n \times \mathbb{S}^{n-1}$, then either $B > 0$ or $B < 0$ for all $t \in \mathbb{R}$. And we claim $B > 0$, i.e.,

$$-a\xi_t^2 - 2(ab-1)\xi_t\xi + b(2-ab)\xi^2 > 0$$

by a contradiction argument. Indeed $\xi(t) \in \Gamma_k^+$ and $k \geq 2$ imply $\sigma_1, \sigma_2 > 0$. If $B < 0$, then

$$\sigma_1 > 0 \Rightarrow \frac{1}{n}A + \frac{n-2}{2n}B > 0 \Rightarrow A > (1 - \frac{n}{2})B,$$

and

$$\sigma_2 > 0 \Rightarrow \frac{2}{n}A + \frac{n-4}{2n}B > 0 \Rightarrow A < (1 - \frac{n}{4})B.$$

Combining these two inequalities, we have

$$(1 - \frac{n}{4})B > A > (1 - \frac{n}{2})B \Leftrightarrow 1 - \frac{n}{4} < 1 - \frac{n}{2} \Leftrightarrow \frac{1}{2} > 1.$$

This is a contradiction.

Next we claim that if $B > 0$ and $\sigma_k > 0$, then it is also satisfies $\sigma_l > 0$ for any $1 \le l < k$. In fact, $\sigma_k > 0$ and $B > 0$ imply that

$$\frac{k}{n}A + \frac{n-2k}{2n}B > 0.$$

Then it follows, for $1 \le l < k$, that

$$\begin{aligned}\sigma_l &= \binom{n}{l}(\frac{B}{2})^{l-1}[\frac{l}{n}(A - \frac{B}{2}) + \frac{n-l}{2n}B] = \frac{l}{k}\binom{n}{l}(\frac{B}{2})^{l-1}[\frac{k}{n}A + (\frac{k}{l} - \frac{2k}{n})\frac{B}{2}] \\ &= \frac{l}{k}\binom{n}{l}[\frac{k}{n}A + (1 - \frac{2k}{n})\frac{B}{2} + (\frac{k}{l} - 1)\frac{B}{2}] > 0.\end{aligned}$$

We now get an upper bound on the function $\xi(t)$. It is easy to see that the trajectory $(\xi(t), \xi_t(t))$ is contained within the homoclinic orbit of the Hamiltonian system

$$[-a\eta_t^2 + b(2-ab)\eta^2]^{k-1}[-a\eta_{tt}\eta + b(2-ab)\eta^2] = c_1\eta^{(p+1)k} \tag{3.1.2}$$

which tends to $(0, 0)$ as t tends both to $+\infty$ and $-\infty$. Let $(\hat{\eta}(t), \hat{\eta}_t(t))$ parameterize this orbit. Then we conclude that

$$\sup \xi \le \sup \hat{\eta}.$$

The conservation of Hamiltonian energy for (3.1.2) now shows that

$$[-a\hat{\eta}_t^2 + b(2-ab)\hat{\eta}^2]^k = \frac{2c_1}{p+1}\hat{\eta}^{(p+1)k}.$$

$\hat{\eta}$ attains its supremum then $\hat{\eta}_t = 0$, so we get the upper bound

$$\xi^{p-1}(t) < (\sup \hat{\eta})^{p-1} = b(2-ab)(\frac{p+1}{2c_1})^{1/k}.$$

Let $y_1 = \xi$ and $y_2 = \xi_t$. Then (3.1.1) is equivalent to the dynamical system

$$\begin{cases}\frac{dy_1}{dt} = y_2, \\ \frac{dy_2}{dt} = -\frac{c_1 y_1^{(p+1)k-1}}{aB^{k-1}} - \frac{2}{a}(ab-1)y_2 + \frac{b}{a}(2-ab)y_1.\end{cases} \tag{3.1.3}$$

Now we find the equilibrium points of (3.1.3) by solving the system:

$$\begin{cases}y_2 = 0, \\ -\frac{c_1 y_1^{(p+1)k-1}}{aB^{k-1}} + \frac{b}{a}(2-ab)y_1 = 0.\end{cases} \tag{3.1.4}$$

Since $y_2 = 0$, the second equation has the solution $y_{1,1} = 0$ for all values of the parameters and, in addition, the solution

$$y_{1,2} = (\frac{b(2-ab)}{c_1^{1/k}})^{\frac{1}{p-1}} > 0 \tag{3.1.5}$$

Thus the system (3.1.3) has one equilibrium point $Y_1 = (0,0)$, and an additional one, $Y_2 = (y_{1,2}, 0)$ with $y_{1,2}$ is given by (3.1.5).

To study the behavior of (3.1.3) near $(0,0)$, we analyze its linear approximation:

$$\begin{cases} \frac{dy_1}{dt} = y_2, \\ \frac{dy_2}{dt} = \frac{b}{a}(2-ab)y_1 - \frac{2}{a}(ab-1)y_2. \end{cases}$$

Thus, we have the characteristic equation

$$\lambda^2 + \frac{2}{a}(ab-1)\lambda - \frac{b}{a}(2-ab) = 0,$$

with the roots

$$\lambda_1 = \frac{1}{a}(2-ab) > 0 \text{ and } \lambda_2 = -b = -\frac{2}{p-1} < 0.$$

The corresponding eigenvectors $\overrightarrow{v}_{1(2)}$ are given by

$$\overrightarrow{v}_{1(2)} = \begin{pmatrix} 1 \\ \lambda_{1(2)} \end{pmatrix},$$

such that $A_{(0,0)}\overrightarrow{v}_{1(2)} = \lambda_{1(2)}\overrightarrow{v}_{1(2)}$, where

$$A_{(0,0)} = \begin{pmatrix} 0 & 1 \\ \frac{b}{a}(2-ab) & -\frac{2}{a}(ab-1) \end{pmatrix}.$$

Therefore we see that the point $(0,0)$ is a saddle point.

Now we study the second equilibrium point Y_2:

$$Y_2 = (y_{1,2}, 0) \text{ with } y_{1,2} = (\frac{b(2-ab)}{c_1^{1/k}})^{\frac{1}{p-1}} > 0$$

Here we have the following linear approximation of (3.1.3) near the point Y_2:

$$A_{Y_2} = \begin{pmatrix} 0 & 1 \\ -\frac{2k}{a}(2-ab) & -\frac{2k}{a}(ab-1) \end{pmatrix},$$

with the characteristic equation

$$r^2 + \frac{2k}{a}(ab-1)r + \frac{2k}{a}(2-ab) = 0$$

We have the following eigenvalues

$$r_1 = (-\frac{2}{a}(ab-1)k + \sqrt{(\frac{2}{a}(ab-1)k)^2 - \frac{8k}{a}(2-ab)})/2,$$

and

$$r_2 = (-\frac{2}{a}(ab-1)k - \sqrt{(\frac{2}{a}(ab-1)k)^2 - \frac{8k}{a}(2-ab)})/2,$$

and the eigenvectors $\overrightarrow{z}_{1(2)} := (1, r_{1(2)})^T$ such that $A_{Y_2}\overrightarrow{z}_{1(2)} = r_{1(2)}\overrightarrow{z}_{1(2)}$. Thus, we have the following alternatives for the equilibrium point Y_2:

1.If $(\frac{2}{a}(ab-1)k)^2 - \frac{8k}{a}(2-ab) \geq 0$, then Y_2 is a stable node.

2.If $(\frac{2}{a}(ab-1)k)^2 - \frac{8k}{a}(2-ab) < 0$, then Y_2 is a stable focus.

Now we study the critical points of the system (3.1.3). We determine asymptotic behavior of solutions to system (3.1.3) by comparing them with the corresponding solutions of the linearized system at the points Y_1 and Y_2. there are solutions $Y(t)$ that tend to Y_1 or Y_2 as $t \to \infty$. Locally, the nonlinear system may be thought of as a perturbation of the linear one. The results from Hartman (Chapter X, Theorem 13.1, Corollary 16.3 in [43]) guarantee that the principal term of asymptotic behavior is the same for a solution tending to the origin for the system (3.1.3) and its linearization.

Next we will investigate the first integral of φ, where $\xi(t) = e^{-\frac{1}{a}\varphi(t)}$. From (3.1.1), we see that φ satisfies

$$B(\varphi)^{k-1}(a\varphi_{tt} + B(\varphi)) = c_2 e^{-\frac{(p-1)k}{a}\varphi}, \tag{3.1.6}$$

where $c_2 = \frac{n}{n-2k}$ and

$$B(\varphi) = -\varphi_t^2 + 2(ab-1)\varphi_t + ab(2-ab).$$

Proposition 3.1.1. *If φ is a bounded solution to (3.1.6) in the Γ_k^+ class, then*

$$B(\varphi)^k = \frac{2c_2}{p+1} e^{-\frac{(p-1)k}{a}\varphi} + e^{\frac{2k}{a}\varphi} h(t,\varphi).$$

where $h(t,\varphi) > 0$ for large t

Proof. Since

$$\begin{aligned}
(e^{-\frac{2k}{a}\varphi}B(\varphi)^k)_t &= -\frac{2k}{a}e^{-\frac{2k}{a}\varphi}\varphi_t B(\varphi)^k + e^{-\frac{2k}{a}\varphi}kB(\varphi)^{k-1}[-2\varphi_t\varphi_{tt} + 2(ab-1)\varphi_{tt}] \\
&= -\frac{2k}{a}e^{-\frac{2k}{a}\varphi}\varphi_t B(\varphi)^k - 2ke^{-\frac{2k}{a}\varphi}B(\varphi)^{k-1}\varphi_{tt}(\varphi_t - ab + 1) \\
&= -\frac{2k}{a}e^{-\frac{2k}{a}\varphi}(\varphi_t - ab + 1)B(\varphi)^{k-1}(a\varphi_{tt} + B(\varphi)) - \frac{2k}{a}(ab-1)e^{-\frac{2k}{a}\varphi}B(\varphi)^k \\
&= -\frac{2kc_2}{a}e^{-\frac{k(p+1)\varphi}{a}}\varphi_t + \frac{2kc_2}{a}(ab-1)e^{-\frac{(p+1)k}{a}\varphi} - \frac{2k}{a}(ab-1)e^{-\frac{2k}{a}\varphi}B(\varphi)^k \\
&= \frac{2c_2}{p+1}(e^{-\frac{(p+1)k}{a}\varphi})_t + \frac{2kc_2}{a}(ab-1)e^{-\frac{(p+1)k}{a}\varphi} - \frac{2k}{a}(ab-1)e^{-\frac{2k}{a}\varphi}B(\varphi)^k,
\end{aligned}$$

we have

$$
\begin{aligned}
(e^{-\frac{2k}{a}\varphi}B(\varphi)^k - \frac{2c_2}{p+1}e^{-\frac{(p+1)k}{a}\varphi})_t = & -\frac{2k}{a}(ab-1)[e^{-\frac{2k}{a}\varphi}B(\varphi)^k - \frac{2c_2}{p+1}e^{-\frac{(p+1)k}{a}\varphi}] \\
& + \frac{2kc_2}{a}(ab-1)(1-\frac{2}{p+1})e^{-\frac{(p+1)k}{a}\varphi}.
\end{aligned}
$$

Using variation of constant, we obtain

$$
\begin{aligned}
e^{-\frac{2k}{a}\varphi}B(\varphi)^k - \frac{2c_2}{p+1}e^{-\frac{(p+1)k}{a}\varphi} = & \frac{2kc_2}{a}(ab-1)(1-\frac{2}{p+1})e^{-\frac{2k}{a}(ab-1)t}\int_0^t e^{-\frac{k(p+1)}{a}\varphi+\frac{2k}{a}(ab-1)s}ds \\
& + Ce^{-\frac{2k}{a}(ab-1)t} =: h(t,\varphi).
\end{aligned}
$$

where C is a constant.

It follows that

$$
B(\varphi)^k = \frac{2c_2}{p+1}e^{-\frac{(p-1)k}{a}\varphi} + e^{\frac{2k}{a}\varphi}h(t,\varphi).
$$

Now we show that $h(t,\varphi) > 0$ for large t. In fact, by the boundedness of φ, we have

$$
\begin{aligned}
e^{-\frac{2k}{a}(ab-1)t}\int_0^t e^{-\frac{k(p+1)}{a}\varphi+\frac{2k}{a}(ab-1)s}ds \geq & C_1 e^{-\frac{2k}{a}(ab-1)t}\int_0^t e^{\frac{2k}{a}(ab-1)s}ds \\
= & \frac{aC_1}{2k(ab-1)}e^{-\frac{2k}{a}(ab-1)t}(e^{\frac{2k}{a}(ab-1)t}-1) \\
= & \frac{aC_1}{2k(ab-1)}(1-e^{-\frac{2k}{a}(ab-1)t}) \\
\geq & \frac{aC_1}{4k(ab-1)},
\end{aligned}
$$

for large t. Then it is easy to see that $h(t,\varphi) > 0$ for large t. So this proposition is established. □

3.2 Perturbed ODE satisfied by the radial average

As in the previous section, in terms of $t = -\ln r = -\ln|x|$,

$$
v(t,\theta) = r^{\frac{2}{p-1}}u(r,\theta) = e^{-\frac{1}{a}w(t,\theta)},
$$

the spherical average of $v(t,\theta)$

$$
\beta(t) := \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} v(t,\theta)d\theta,
$$

and the spherical average of $w(t,\theta)$

$$
\gamma(t) := \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} w(t,\theta)d\theta,
$$

(2.1.3) is reformulated as

$$v(t,\theta) \le C \ \text{ and } e^{-\frac{1}{a}w(t,\theta)} \le C \tag{3.2.1}$$

We also derive from (2.1.4) that

$$|v(t,\theta) - \beta(t)| \le C\beta(t)e^{-t}, \tag{3.2.2}$$

and

$$|w(t,\theta) - \gamma(\theta)| \le Ce^{-t} \tag{3.2.3}$$

Equation (3.2.2) is simply a reformulation of (2.1.4) in terms of $v(t,\theta)$ and $\beta(t)$. In terms of $w(t,\theta)$, (3.2.2) is

$$|e^{-\frac{1}{a}w(t,\theta)-\ln\beta(t)} - 1| \le Ce^{-t},$$

from which it follows that, for some $\hat{C} > 0$,

$$|w(t,\theta) + a\ln\beta(t)| \le \hat{C}e^{-t}. \tag{3.2.4}$$

Integrating over $\mathbb{S}^{n-1}$, we obtain

$$|\gamma(t) + a\ln\beta(t)| \le \hat{C}e^{-t}. \tag{3.2.5}$$

Equations (3.2.4) and (3.2.5) imply (3.2.3).

We have the gradient estimates for positive singular solutions $u(x)$ in the Γ_k^+ class to (1.4.2) in $B_2(0)\backslash\{0\}$.

Proposition 3.2.1. *Let $u(x)$ be a positive singular solution to (1.4.2) in $B_2(0)\backslash\{0\}$ in the Γ_k^+ class, $v(t,\theta)$, $\beta(t)$, $w(t,\theta)$ and $\gamma(t)$ be defined above. Then for any $\delta > 0$ small, there exists a constant $C > 0$ depending on δ such that*

$$|\nabla_{t,\theta}^j(v(t,\theta) - \beta(t))| \le C\beta(t)e^{-(1-\delta)t}, \tag{3.2.6}$$

and

$$|\nabla_{t,\theta}^j(w(t,\theta) - \gamma(t))| \le Ce^{-(1-\delta)t} \tag{3.2.7}$$

for all $t \ge 0$ and $j = 1, 2$.

Now we provide an argument for (3.2.7). First, (3.2.1) and the gradient estimates for solutions to (1.4.2), see [36], give a bound $\hat{B} > 0$ depending on $j > 0$ and C in (3.2.1), such that

$$|\nabla_{t,\theta}^j w(t,\theta)| \le \hat{B}. \tag{3.2.8}$$

This obviously leads to

$$|\nabla_{t,\theta}^j \gamma(t)| \le \hat{B}, \tag{3.2.9}$$

which, together with (3.2.8), implies that

$$|\nabla_{t,\theta}^j(w(t,\theta) - \gamma(t))| \le 2\hat{B}.$$

This estimate, together with (3.2.3) and interpolation, proves (3.2.7). Similarly, we have (3.2.6).

Let $v(t,\theta)$ be a positive solution to (1.4.4) in $\{t > t_0\} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class, and $\beta(t)$ is defined in the beginning of this section. We first make a proposition,

Proposition 3.2.2. *If $2k < n$, then $\beta(t)$ satisfies*

$$[-a\beta_t^2 - 2(ab-1)\beta_t\beta + b(2-ab)\beta^2]^{k-1}[-a\beta_{tt}\beta - 2(ab-1)\beta_t\beta + b(2-ab)\beta^2] = c_1\beta^{(p+1)k}(1+\eta_1(t)), \tag{3.2.10}$$

where $\eta_1(t)$ has the decay rate $\eta_1(t) = O(e^{-t})$ as $t \to \infty$.

Proof. Let $\sigma_k(A^v)$ be a functional of $v(t,\theta)$, then with

$$\hat{v}(t,\theta) = v(t,\theta) - \beta(t),$$

we have the following expansion

$$\sigma_k(A^v) = \sigma_k(A^\beta) + L_{\beta(t)}[\hat{v}(t,\theta)] + \hat{\eta}_1(t,\theta),$$

where $L_{\beta(t)}$ denotes the linearized operator for $\sigma_k(A^{v(t,\theta)})$ at $\beta(t)$, and $\hat{\eta}_1(t,\theta)$ satisfies

$$|\hat{\eta}_1(t,\theta)| = O(\beta^2(t)e^{-2(1-\delta)t}) \ \text{ as } \ t \to \infty,$$

by (3.2.6) and $|\nabla_t^j\beta(t)| \le C$. Next,

$$v(t,\theta)^{(p+1)k} = \beta(t)^{(p+1)k}(1 + \frac{\hat{v}(t,\theta)}{\beta(t)})^{(p+1)k} = \beta(t)^{(p+1)k} + (p+1)k\beta(t)^{(p+1)k-1}\hat{v}(t,\theta) + \beta(t)^{(p+1)k}\hat{\eta}_2(t,\theta),$$

where $|\hat{\eta}_2(t,\theta)| = O(\beta(t)^2e^{-2t})$ as $t \to \infty$ from (3.2.2). Integrating over $\theta \in \mathbb{S}^{n-1}$, we have

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \sigma_k(A^{\beta(t)})d\theta = [-a\beta_t^2 - 2(ab-1)\beta_t\beta + b(2-ab)\beta^2]^{k-1}[-a\beta_{tt}\beta - 2(ab-1)\beta_t\beta + b(2-ab)\beta^2],$$

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} L_{\beta(t)}[\hat{v}(t,\theta)]d\theta = 0, \quad \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \beta(t)^{(p+1)k}d\theta = \beta(t)^{(p+1)k},$$

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \hat{\eta}_1(t,\theta) = O(e^{-2(1-\delta)t}) \ \text{ as } \ t \to \infty,$$

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \beta(t)^{(p+1)k-1}\hat{v}(t,\theta)d\theta = O(\beta^{(p+1)k}e^{-t}) \quad \text{as} \quad t \to \infty,$$

and

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \beta(t)^{(p+1)k}\hat{\eta}_2(t,\theta) = O(\beta^{(p+1)k+2}e^{-2t}) \quad \text{as} \quad t \to \infty.$$

Thus we obtain (3.2.10). □

Next we make another proposition relating the asymptotic behavior of u with that of $\beta(t)$.

Lemma 3.2.3. *If*

$$\liminf_{x\to 0} |x|^{\frac{2}{p-1}}u(x) > 0 \tag{3.2.11}$$

Then for some $\varepsilon > 0$,

$$B(\beta) := -a\beta_t^2 - 2(ab-1)\beta_t\beta + b(2-ab)\beta^2 \ge \varepsilon \tag{3.2.12}$$

for all sufficiently large t.

Proof. Inequality (3.2.12) is proved by noting that (3.2.11) and (3.2.1) imply that

$$\frac{1}{C} \leq \beta(t) \leq C \tag{3.2.13}$$

for some C, which, together with (3.2.10), implies that, for large t, $B(\beta)$ never changes sign, which, in turn with (3.2.13), $|\nabla^j_{t,\theta} v(t,\theta)| \leq C$ and $|\nabla^j_{t,\theta}\beta(t)| \leq C$, implies that, for some $\varepsilon > 0$, $B(\beta) \geq \varepsilon$ for all sufficiently large t. ◻

Let $w(t,\theta)$ be a bounded solution to (1.4.7) in $\{t > t_0\} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class, and $\gamma(t)$ is defined in the beginning of this section. We make the proposition

Proposition 3.2.4. *If $2k < n$, then $\gamma(t)$ satisfies*

$$e^{\frac{(p-1)k}{a}\gamma}\{[-\gamma_t^2 + 2(ab-1)\gamma_t + ab(2-ab)]^{k-1}[a\gamma_{tt} - \gamma_t^2 + 2(ab-1)\gamma_t + ab(2-ab)] + \eta_2\} = c_2(1+\eta_3(t)), \tag{3.2.14}$$

where $\eta_2(t) = O(e^{-2(1-\delta)t})$, $\eta_3 = O(e^{-2t})$ as $t \to \infty$, for arbitrarily small $\delta > 0$ as in (3.2.7), and $c_2 = \frac{n}{n-2k}$.

Proof. Let $\sigma_k(A^w)$ be a functional of $w(t,\theta)$, then with $\hat{w}(t,\theta) := w(t,\theta) - \gamma(t)$, we have the following expansion

$$\sigma_k(A^w) = \sigma_k(A^{\gamma(t)}) + L_{\gamma(t)}[\hat{w}(t,\theta)] + \hat{\eta}_1(t,\theta),$$

where $L_{\gamma(t)}$ denotes the linearized operator for $\sigma_k(A^w)$ at $\gamma(t)$, and $\hat{\eta}_1(t,\theta)$ satisfies $|\hat{\eta}_1(t,\theta)| = O(e^{-2(1-\delta)t})$ as $t \to \infty$ by (3.2.7) and (3.2.9). Next,

$$e^{\frac{(p-1)k}{a}w(t,\theta)} = e^{\frac{(p-1)k}{a}\gamma(t)} \cdot e^{\frac{(p-1)k}{a}\hat{w}(t,\theta)},$$

and

$$e^{-\frac{(p-1)k}{a}\hat{w}(t,\theta)} = 1 - \frac{(p-1)k}{a}\hat{w}(t,\theta) + \hat{\eta}_2(t,\theta),$$

where $|\hat{\eta}_2(t,\theta)| = O(e^{-2t})$ as $t \to \infty$ from (3.2.3). Integrating over $\theta \in \mathbb{S}^{n-1}$, we get

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \sigma_k(A^{\gamma(t)})d\theta = [-\gamma_t^2 + 2(ab-1)\gamma_t + ab(2-ab)]^{k-1}[a\gamma_{tt} - \gamma_t^2 + 2(ab-1)\gamma_t + ab(2-ab)],$$

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \hat{w}(t,\theta)d\theta = 0$$

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} L_{\gamma(t)}[\hat{w}(t)]d\theta = 0$$

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \hat{\eta}_1(t,\theta)d\theta = O(e^{-2(1-\delta)t}) \text{ as } t \to \infty,$$

and

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \hat{\eta}_2(t,\theta)d\theta = O(e^{-2t}) \text{ as } t \to \infty.$$

Finally, we obtain (3.2.14). ◻

Lemma 3.2.5. *If (3.2.11) holds, then for some $\varepsilon > 0$,*

$$B(\gamma) := -\gamma_t^2 + 2(ab-1)\gamma_t + ab(2-ab) \geq \varepsilon,$$

for all sufficiently large t.

Proof. If (3.2.11) holds, then from (3.2.1), we have

$$-C \leq \gamma(t) \leq C \tag{3.2.15}$$

for some C. Together with (3.2.14), we see that, for large t, $B(\gamma)$ never changes sign. Furthermore, it follows from (3.2.9) and (3.2.15) that for some $\varepsilon > 0$, $B(\gamma) \geq \varepsilon$ for all sufficiently large t. □

3.3 Asymptotic to a radial solution

Based on Theorem 2.1.4, we give a proof of Theorem 1.4.2.

Proof of Theorem 1.4.2. Recall that $v(t,\theta)$ is a solution to

$$\sigma_k(A^v) = c_1 v^{(p+1)k} \ \text{ in } \ \{t > t_0\} \times \mathbb{S}^{n-1}.$$

From Theorem 2.1.4, we have (3.2.2), that is,

$$|v(t,\theta) - \beta(t)| \leq C\beta(t)e^{-t}.$$

We claim that for any solution $\xi(t)$ to $\sigma_k(A^\xi) = c_1\xi^{(p+1)k}$ in $\mathbb{R} \times \mathbb{S}^{n-1}$ with $C_1 \leq \xi \leq C_2$,

$$|\beta(t) - \xi(t)| \leq Ce^{-\alpha t}.$$

Combining above two inequalities, we obtain

$$|v(t,\theta) - \xi(t)| \leq C\max\{e^{-\alpha t}, e^{-t}\}. \tag{3.3.1}$$

In particular, we have

$$|v(t,\theta) - y_{1,2}| \leq C\max\{e^{-\alpha t}, e^{-t}\},$$

where

$$y_{1,2} = \left(\frac{b(2-ab)}{c_1^{1/k}}\right)^{\frac{1}{p-1}} > 0.$$

Recall that $\beta(t)$ is a solution to

$$\beta_{tt} + \frac{2}{a}(ab-1)\beta_t - \frac{b}{a}(2-ab)\beta = -\frac{c_1\beta^{(p+1)k-1}}{aB^{k-1}(\beta)}(1+\eta_1(t)). \tag{3.3.2}$$

Then in the case $k(ab-1)^2 \neq 2a(2-ab)$, we have

$$|\beta(t) - y_{1,2}| \leq Ce^{-Re(\alpha_1)t}; \tag{3.3.3}$$

in the case $k(ab-1)^2 = 2a(2-ab)$, we have

$$|\beta(t) - y_{1,2}| \leq Cte^{-\alpha_0 t}, \tag{3.3.4}$$

where $\alpha_1 = \frac{k}{a}(ab-1) - \sqrt{\frac{k^2}{a^2}(ab-1)^2 - \frac{2k}{a}(2-ab)}$ and $\alpha_0 = \frac{k}{a}(ab-1)$.

For the sake of clarity, we divide the proof into three steps.

Step 1, we will show that $\hat{\beta}(t) := \beta(t) - y_{1,2} \to 0$ and $\hat{\beta}_t(t) \to 0$ as $t \to \infty$. In fact, inspired by the work of Caffarelli, Gidas and Spruck [17], we multiply (3.3.2) by β_t and integrate by parts, then we have

$$\frac{1}{2}\beta_t^2|_s^t + \frac{2}{a}(ab-1)\int_s^t \beta_t^2 - \frac{b(2-ab)}{2a}\beta^2|_s^t = O(\beta^{(p-1)k+2}|_s^t) + O(e^{-s}).$$

It follows that $\int_s^\infty \beta_t^2 < C$ and therefore $\beta_t \to 0$ as $t \to \infty$. Multiplying (3.3.2) by β_{tt} and integrating by parts, we also find $\int_s^\infty \beta_{tt}^2 < C$ and so $\beta_{tt} \to 0$ as $t \to \infty$. Passing to the limit as $t \to \infty$ in (3.3.2), we conclude that

$$\lim_{t\to\infty} \beta(t) = (\frac{b(2-ab)}{c_1^{1/k}})^{\frac{1}{p-1}} = y_{1,2}.$$

Step 2, we claim that $|\hat{\beta}(t)| = |\beta(t) - y_{1,2}| \leq Ce^{-Re(\alpha_1)t + \varepsilon t}$ for any $\varepsilon > 0$ and $t \geq t_0$ for sufficiently large t_0. From (3.3.2), we see that $\hat{\beta}(t)$ satisfies

$$\hat{\beta}_{tt} + \frac{2k}{a}(ab-1)\hat{\beta}_t + \frac{2k}{a}(2-ab)\hat{\beta} = \hat{\eta}(t, \hat{\beta}, \hat{\beta}_t),$$

where $\hat{\beta} \to 0$, $\hat{\beta}_t \to 0$ and $|\hat{\eta}(t, \hat{\beta}, \hat{\beta}_t)| = o(\sqrt{|\hat{\beta}|^2 + |\hat{\beta}_t|^2})$ as $t \to \infty$. It follows that

$$Y(t) = e^{tA}Y(t_0) + \int_{t_0}^t e^{(t-s)A} f(s, Y(s))ds.$$

where

$$Y(t) = (\hat{\beta}(t), \hat{\beta}_t(t))^T, \quad f(s, Y(s)) = (0, \hat{\eta}(s, \hat{\beta}, \hat{\beta}_s))^T$$

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{2k}{a}(2-ab) & -\frac{2k}{a}(ab-1) \end{pmatrix},$$

Because the real parts of the characteristic roots of A are negative, there exist positive constants C and $Re(\alpha_1) > 0$ such that

$$|e^{tA}| \leq Ce^{-Re(\alpha_1)t}, \quad \text{for } t \geq 0.$$

Then we have

$$|Y(t)| \leq C|Y(t_0)|e^{-Re(\alpha_1)t} + C\int_{t_0}^t e^{-(t-s)Re(\alpha_1)}|f(s, Y(s))|ds.$$

For $\varepsilon < Re(\alpha_1)$, there exists $t_0 > 0$ such that $|f(t, Y(t))| \leq \frac{\varepsilon|Y(t)|}{C}$ for any $t \geq t_0$. Then it follows

$$e^{Re(\alpha_1)t}|Y(t)| \leq C|Y(t_0)| + \varepsilon \int_{t_0}^{t} e^{Re(\alpha_1)s}|Y(s)|ds.$$

The Gronwall inequality yields

$$e^{Re(\alpha_1)t}|Y(t)| \leq C|Y(t_0)|e^{\varepsilon t},$$

that is,

$$|Y(t)| \leq C|Y(t_0)|e^{-Re(\alpha_1)t+\varepsilon t}.$$

Now we have $|\hat{\beta}(t)| + |\hat{\beta}_t(t)| \leq Ce^{-Re(\alpha_1)t+\varepsilon t}$ for sufficiently large t.

Step 3, we will prove that in the case $k(ab-1)^2 \neq 2a(2-ab)$,

$$|\beta(t) - y_{1,2}| \leq Ce^{-Re(\alpha_1)t};$$

in the case $k(ab-1)^2 = 2a(2-ab)$,

$$|\beta(t) - y_{1,2}| \leq Cte^{-\alpha_0 t}.$$

Now $\hat{\beta}$ satisfies this equation

$$\hat{\beta}_{tt} + \frac{2k}{a}(ab-1)\hat{\beta}_t + \frac{2k}{a}(2-ab)\hat{\beta} = \hat{\eta}(t, \hat{\beta}, \hat{\beta}_t),$$

where $|\hat{\beta}(t)| \leq Ce^{-Re(\alpha_1)t+\varepsilon t}$, $|\hat{\beta}_t(t)| \leq Ce^{-Re(\alpha_1)t+\varepsilon t}$ and $|\hat{\eta}(t, \hat{\beta}, \hat{\beta}_t)| = O(|\hat{\beta}|^2 + |\hat{\beta}_t|^2) \leq Ce^{-2Re(\alpha_1)t+2\varepsilon t}$. For any small $\varepsilon_1 > 0$, we can choose $\varepsilon > 0$ small enough such that $|\hat{\eta}| \leq Ce^{-Re(\alpha_1)t-\varepsilon_1 t}$.

When $k(ab-1)^2 \neq 2a(2-ab)$, by the variation of constant formula, we see that

$$\hat{\beta}(t) = c_1 e^{-\alpha_1 t} + c_2 e^{-\alpha_2 t} + e^{-\alpha_2 t} \int_{t_0}^{t} \frac{e^{-\alpha_1 s}\hat{\eta}}{e^{-\frac{2k}{a}(ab-1)s}} ds + e^{-\alpha_1 t} \int_{t}^{\infty} \frac{e^{-\alpha_2 s}\hat{\eta}}{e^{-\frac{2k}{a}(ab-1)s}} ds,$$

where $\alpha_2 = \frac{k}{a}(ab-1) + \sqrt{\frac{k^2}{a^2}(ab-1)^2 - \frac{2k}{a}(2-ab)}$, c_1 and c_2 are constants. Note that $Re(\alpha_2) \geq Re(\alpha_1)$. From the decay rate of $\hat{\eta}$, we have

$$|e^{-\alpha_2 t} \int_{t_0}^{t} \frac{e^{-\alpha_1 s}\hat{\eta}}{e^{-\frac{2k}{a}(ab-1)s}} ds| \leq Ce^{-Re(\alpha_1)t},$$

and

$$|e^{-\alpha_1 t} \int_{t}^{\infty} \frac{e^{-\alpha_2 s}\hat{\eta}}{e^{-\frac{2k}{a}(ab-1)s}} ds| \leq Ce^{-Re(\alpha_1)t},$$

Combining the above inequalities, we obtain

$$|\hat{\beta}(t)| \leq Ce^{-Re(\alpha_1)t}.$$

When $k(ab-1)^2 = 2a(2-ab)$, by the variation of constant formula, we have that

$$\hat{\beta}(t) = c_1 t e^{-\alpha_0 t} + c_2 e^{-\alpha_0 t} + e^{-\alpha_0 t} \int_{t_0}^{t} \frac{s e^{-\alpha_0 s} \hat{\eta}}{e^{-\frac{2k}{a}(ab-1)s}} ds + t e^{-\alpha_0 t} \int_{t}^{\infty} \frac{e^{-\alpha_0 s} \hat{\eta}}{e^{-\frac{2k}{a}(ab-1)s}} ds.$$

From the decay of $\hat{\eta}$, we see that

$$|e^{-\alpha_0 t} \int_{t_0}^{t} \frac{s e^{-\alpha_0 s} \hat{\eta}}{e^{-\frac{2k}{a}(ab-1)s}} ds| \leq C t e^{-\alpha_0 t},$$

and

$$|t e^{-\alpha_0 t} \int_{t}^{\infty} \frac{e^{-\alpha_0 s} \hat{\eta}}{e^{-\frac{2k}{a}(ab-1)s}} ds| \leq C t e^{-\alpha_0 t}.$$

By the above inequalities, we obtain

$$|\hat{\beta}(t)| \leq C t e^{-\alpha_0 t}.$$

Similarly, ξ is any solution to

$$\xi_{tt} + \frac{2}{a}(ab-1)\xi_t - \frac{b}{a}(2-ab)\xi = -\frac{c_1 \xi^{(p+1)k-1}}{a B^{k-1}(\xi)}$$

with $C_1 \leq \xi \leq C_2$. Then in the case $k(ab-1)^2 \neq 2a(2-ab)$, we have

$$|\xi(t) - y_{1,2}| \leq C e^{-Re(\alpha_1)t}; \tag{3.3.5}$$

in the case $k(ab-1)^2 = 2a(2-ab)$, we have

$$|\xi(t) - y_{1,2}| \leq C t e^{-\alpha_0 t}. \tag{3.3.6}$$

When $k(ab-1)^2 \neq 2a(2-ab)$, from (3.3.3) and (3.3.5), we obtain

$$|\beta(t) - \xi(t)| = |(\beta(t) - y_{1,2}) - (\xi(t) - y_{1,2})| \leq |\beta(t) - y_{1,2}| + |\xi(t) - y_{1,2}| \leq C e^{-Re(\alpha_1)t}.$$

When $k(ab-1)^2 = 2a(2-ab)$, by (3.3.4) and (3.3.6), we see

$$|\beta(t) - \xi(t)| = |(\beta(t) - y_{1,2}) - (\xi(t) - y_{1,2})| \leq |\beta(t) - y_{1,2}| + |\xi(t) - y_{1,2}| \leq C t e^{-\alpha_0 t}.$$

Now we proved the claim. □

Chapter 4

Higher order asymptotic behaviors of the singular solutions

In this chapter, we will obtain the higher order asymptotic behaviors of the singular solutions to (1.4.2) by an analysis of the linearized operator. Let $u(x)$ be a solution to $\sigma_k(B^u) = cu^{(p-\frac{n+2k}{n-2k})k}$, $\frac{n}{n-2k} < p < \frac{n+2k}{n-2k}$, for x over the punctured ball $B_2(0)\backslash\{0\}$, where c is normalized to be $\binom{n}{k}/2^k$. It is assumed that B^u is in the Γ_k^+ class over $B_2(0)\backslash\{0\}$.

For the sake of simplicity, we set $w(t,\theta) = -a\ln(|x|^{\frac{2}{p-1}}u(x))$, where $t = -\ln|x|$, $\theta = \frac{x}{|x|}$ and $a = \frac{2k}{n-2k} > 0$. Since $u(x)$ is a solution to (1.4.2), we have that $w(t,\theta)$ is a solution to

$$\sigma_k(B^w) = ce^{-\frac{(p-1)k}{a}w} \ \text{ in } \ \{t > -\ln 2\} \times \mathbb{S}^{n-1}. \tag{4.0.1}$$

Here

$$B^w = \begin{pmatrix} B^w_{11} & B^w_{1j} \\ B^w_{i1} & B^w_{ij} \end{pmatrix}$$

is a block matrix, where

$$B^w_{11} = w_{tt} + \frac{1}{2}(w_t^2 - 2(ab-1)w_t - ab(2-ab)) - \frac{1}{2}w_\theta^2,$$

$$B^w_{1j} = w_{t\theta_j} + w_t w_{\theta_j} - (ab-1)w_{\theta_j},$$

$$B^w_{i1} = w_{\theta_i t} + w_t w_{\theta_i} - (ab-1)w_{\theta_i},$$

and

$$B^w_{ij} = w_{\theta_i\theta_j} + w_{\theta_i}w_{\theta_j} + \frac{1}{2}(-w_t^2 + 2(ab-1)w_t + ab(2-ab) - w_\theta^2)\delta_{ij}$$

with $a = \frac{2k}{n-2k}$ and $b = \frac{2}{p-1}$.

It follows from Theorem 2.1.3 that (3.2.1) holds, i.e., for some constant $C_2 > 0$,

$$w(t,\theta) \le C_2$$

for all $(t,\theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$. It follows from our discussion in the beginning of the previous section that u with isolated singularities implies, for some constant $C_1 > 0$,

$$w(t,\theta) \geq -C_1$$

for $(t,\theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$, namely, $w(t,\theta)$ is bounded over $(t,\theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$.

We make the following assertions about the behavior of $w(t,\theta)$ as $t \to \infty$.

• Let $t_j \to \infty$ be any sequence tending to ∞, then $\{w_j(t,\theta) := w(t+t_j,\theta)\}$ has a subsequence converging to any bounded limiting solution $\varphi(t)$ of (4.0.1) defined for $(t,\theta) \in \mathbb{R} \times \mathbb{S}^{n-1}$. The convergence is uniform on any compact subset of $\mathbb{R}^+ \times \mathbb{S}^{n-1}$.

• Any angular derivative of w, $\partial_\theta w(t,\theta)$ converges to 0 as $t \to \infty$.

• $\partial_\theta w(t,\theta)$ converges to 0 at an exponential rate as $t \to \infty$, and

$$|w(t,\theta) - \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} w(t,\omega)d\omega|$$

converges to 0 at an exponential rate as $t \to \infty$.

• For any bounded solution $\varphi(t)$ of (4.0.1) such that $w(t,\theta)$ converges to $\varphi(t)$ at an exponential rate as $t \to \infty$.

By the result of last section we know that $-C_1 \leq w(t,\theta) \leq C_2$ for all $t \geq 0$. By local estimates for (4.0.1) [36], we also get the uniform boundedness of any derivative $|\nabla^j_{t,\theta} w(t,\theta)| \leq C$ for all $t \geq 0$. Let $\{\tau_j\}$ be any sequence of numbers converging to ∞, and define $w_j(t,\theta) = w(t+\tau_j,\theta)$. Then $w_j(t,\theta)$ defined on $[-\tau_j,\infty) \times \mathbb{S}^{n-1}$ and satisfies (4.0.1) there. Using the uniform bounds on any derivative of w_j, we may choose a subsequence of the v_j converging in the C^2 topology on any compact subset of $\mathbb{R} \times \mathbb{S}^{n-1}$. The limit function, w_∞, still satisfies (4.0.1), does not go to ∞ because of the lower bound for w and is defined on the whole cylinder. By Theorem 2.1.2, we deduce that $v_\infty(t,\theta) = v_\infty(t)$.

4.1 Linearization of the subcritical σ_k Yamabe equation

To compute the linearized operator of (4.0.1) at a bounded radial solution $\varphi(t)$, we use

$$B^w = \begin{pmatrix} B^w_{11} & B^w_{1j} \\ B^w_{i1} & B^w_{ij} \end{pmatrix}.$$

When $w(t,\theta) = \varphi(t)$, B^w becomes a block diagonal matrix

$$\begin{pmatrix} A - \frac{B}{2} & 0 \\ 0 & \frac{B}{2}\delta_{ij} \end{pmatrix},$$

where

$$A := \varphi_{tt}$$

and

$$B := -\varphi_t^2 + 2(ab-1)\varphi_t + ab(2-ab).$$

When we linearized $\sigma_k(B^w)$ at such a block diagonal matrix, the coefficient matrix consisting of the coefficients of the Newton tensor

$$T_{ij} = \frac{1}{(k-1)!}\delta^{i_1\ldots i_{k-1}i}_{j_1\ldots j_{k-1}j}A_{i_1j_1}\cdots A_{i_{k-1}j_{k-1}}$$

is also diagonal

$$T_{11} = \binom{n-1}{k-1}\frac{B^{k-1}}{2^{k-1}},$$

while for $i \geq 2$,

$$\begin{aligned}
T_{ii} &= \binom{n-1}{k-1}\frac{n-k}{n-1}\frac{B^{k-1}}{2^{k-1}} + \frac{n-2}{k-2}\frac{B^{k-2}}{2^{k-2}}(A-\frac{B}{2}) \\
&= \binom{n-1}{k-1}\frac{B^{k-2}}{2^{k-2}}(\frac{n-k}{n-1}\frac{B}{2}) + \binom{n-1}{k-1}\frac{k-1}{n-1}\frac{B^{k-2}}{2^{k-2}}(A-\frac{B}{2}) \\
&= \binom{n-1}{k-1}\frac{B^{k-2}}{2^{k-2}}[\frac{k-1}{n-1}A + \frac{n-2k+1}{n-1}\frac{B}{2}].
\end{aligned}$$

So the linearization of $\sigma_k(B^w)$ at $\varphi(t)$ is

$$\begin{aligned}
L_\varphi[f] &= T_{11}[f_{tt} + \varphi_t f_t - (ab-1)f_t] + \sum_{i=2}^{n}[f_{\theta_i\theta_i} - \varphi_t f_t + (ab-1)f_t] \\
&= T_{11}f_{tt} + [T_{11} - (n-1)T_{22}](\varphi_t - (ab-1))f_t + T_{22}\Delta_\theta f \\
&= \binom{n-1}{k-1}\frac{B^{k-1}}{2^{k-1}}f_{tt} + \binom{n-1}{k-1}\frac{B^{k-2}}{2^{k-2}}[\frac{k-1}{n-1}A + \frac{n-2k+1}{n-1}\frac{B}{2}]\Delta_\theta f \\
&\quad + \{\binom{n-1}{k-1}\frac{B^{k-1}}{2^{k-1}} - \binom{n-1}{k-1}\frac{B^{k-2}}{2^{k-2}}[(k-1)A + (n-2k+1)\frac{B}{2}]\}(\varphi_t - (ab-1))f_t \\
&= \binom{n-1}{k-1}\frac{B^{k-2}}{2^{k-2}}[\frac{B}{2}f_{tt} + (\frac{k-1}{n-1}A + \frac{n-2k+1}{n-1}\frac{B}{2})\Delta_\theta f \\
&\quad - ((k-1)A + (n-2k)\frac{B}{2})(\varphi_t - (ab-1))f_t] \\
&= \binom{n-1}{k-1}\frac{B^{k-2}}{2^{k-2}}[D(t)f_{tt} + E(t)f_t + F(t)\Delta_\theta f],
\end{aligned}$$

where

$$D(t) := \frac{B}{2},$$

$$E(t) := -(\varphi_t - (ab-1))((k-1)A + (n-2k)\frac{B}{2}),$$

and

$$F(t) := \frac{k-1}{n-1}A + \frac{n-2k+1}{n-1}\frac{B}{2}.$$

Since $\varphi(t)$ is a solution to
$$\sigma_k(B^\varphi) = ce^{-\frac{(p-1)k}{a}\varphi},$$
where c is normalized to be $\binom{n}{k}/2^k$, the linearization of the nonlinear partial differential equation $\sigma_k(B^w) = ce^{-\frac{(p-1)k}{a}w}$ at $\varphi(t)$ is then
$$L_\varphi f + \frac{c(p-1)k}{a}e^{-\frac{(p-1)k}{a}\varphi} f = 0.$$
If we take the projections of $f(t,\cdot)$ into spherical harmonics:
$$f(t,\theta) = \sum_j f_j(t)Y_j(\theta),$$
where $Y_j(\theta)$ are the normalized eigenfunctions of Δ_θ on $L^2(\mathbb{S}^{n-1})$, then f_j satisfies ordinary differential equation
$$L_j[f_j] := f_j''(t) + \frac{E(t)}{D(t)}f_j'(t) + \{-\lambda_j\frac{F(t)}{D(t)} + \frac{n(p-1)e^{-\frac{(p-1)k}{a}\varphi}}{2aB^{k-1}}\}f_j(t) = 0, \tag{4.1.1}$$
where λ_j are the eigenvalues of $-\Delta_\theta$ on $L^2(\mathbb{S}^{n-1})$ associated with $Y_j(\theta)$, thus
$$\lambda_0 = 0,\ \ \lambda_1 = \lambda_2 = \cdots = \lambda_n = n-1,\ \ \lambda_j \geq 2n,\ \ \forall\ j > n.$$
Note that $Y_0(\theta) = constant$.

4.2 Comparison theorems

Similar to properties of the linearized operator to the scalar curvature operator used in [44], and the σ_k curvature operator used in [42], we have the following properties for the L_j's.

Proposition 4.2.1. *For all solutions $\varphi(t)$ to (4.0.1) with $h > 0$, $k < \frac{n}{2}$ and $j \geq 1$, the following holds:*

(a) $L_j[f] = 0$ has a pair of linearly independent solution basis in $\mathbb{R}$, one of which grows unbounded and the other one decays exponentially as $t \to \infty$;

(b) Any solution of $L_j[f] = 0$ which is bounded for $\mathbb{R}^+$ must decay exponentially;

(c) Any solution of $L_j[f] = 0$ which is bounded for all of $\mathbb{R}$ must be identically 0;

(d) Any non-zero solution of $L_j[f] = 0$ which is bounded for all of $\mathbb{R}^+$ must be unbounded in $\mathbb{R}^-$.

Following the proof of Proposition 2 in [42], for $1 \leq j \leq n$, (a)-(d) of this proposition follow from an explicit solution basis to (4.1.1); for $j \geq n+1$, the argument relies on the sign of the coefficient of the term of zero order to be negative.

Since $\varphi(t)$ satisfies (4.0.1), with $\sigma_k(B^\varphi) = \binom{n}{k}e^{-\frac{(p-1)k}{a}\varphi}/2^k$, (4.0.1) becomes
$$B^{k-1}(a\varphi_{tt} + B) = c_2 e^{-\frac{(p-1)k}{a}\varphi}, \tag{4.2.1}$$

where $c_2 = \frac{n}{n-2k}$, $a = \frac{2k}{n-2k}$ and $B = -\varphi_t^2 + 2(ab-1)\varphi_t + ab(2-ab)$. Let f_0 is a solution to (4.1.1) for $\lambda_0 = 0$, namely,

$$f_0''(t) + \frac{E(t)}{D(t)} f_0'(t) + \frac{n(p-1)e^{-\frac{(p-1)k}{a}\varphi}}{2aB^{k-1}} f_0(t) = 0.$$

It follows from Theorem 1.4.2 that

$$|\varphi_t| \le Ce^{-\alpha t},$$

and

$$|\varphi + \frac{a}{(p-1)k} \ln[\frac{n-2k}{n}(ab(2-ab))^k]| \le Ce^{-\alpha t}.$$

Then using the same ODE technique in Theorem 1.4.2, we are able to get

$$|f_0(t)| \le Ce^{-Re(a_{01})t},$$

where

$$a_{01} = \frac{(ab-1)(n-2k) - \sqrt{(ab-1)^2(n-2k)^2 - 4(n-2k)(2-ab)}}{2},$$

or

$$|f_0(t)| \le Cte^{-\frac{(ab-1)(n-2k)}{2}t}.$$

When $\varphi \equiv -\frac{a}{(p-1)k} \ln[\frac{n-2k}{n}(ab(2-ab))^k]$, (4.1.1) for $\lambda_0 = 0$ is a constant coefficients ODE, then we see that

$$f_0(t) = c_{01}e^{-a_{01}t} + c_{02}e^{-a_{02}t},$$

where

$$a_{02} = \frac{(ab-1)(n-2k) + \sqrt{(ab-1)^2(n-2k)^2 - 4(n-2k)(2-ab)}}{2},$$

and c_{01}, c_{02} are arbitrary constants.

Proof of Proposition 4.2.1. For $\lim_{t\to\infty} h(t,\varphi) > 0$ and $\lambda_j = n-1$, which corresponds to $Y_j(\theta) = \theta_j$, we claim that

$$f_1^+ := (ab - \varphi_t)e^t$$

is a solution to (4.1.1) for $\lambda_j = n-1$. This is due the translation invariance of (1.4.2): if $u(x)$ is a solution to (1.4.2), so is $u(x+\hat{a})$ for any $\hat{a} \in \mathbb{R}^n$. In terms of $w(t,\theta)$, this means that

$$w_{\hat{a}}(t,\theta) := -ab\ln|x| - a\ln u(x+\hat{a})$$

is a solution to (4.0.1). Thus $\partial_{\hat{a}_j}|_{\hat{a}=0} w_{\hat{a}}(t,\theta)$ is a solution to the linearized equation of (4.0.1). But when $w(t,\theta) = \varphi(t)$, we have Since

$$\begin{aligned}\partial_{\hat{a}_j}|_{\hat{a}=0} w_{\hat{a}}(t,\theta) &= -a\partial_{x_j}\ln u(x) = \partial_{x_j}(ab\ln|x| + w(t,\theta)) \\ &= \frac{ab}{|x|}\theta_j + \varphi_t(-\frac{1}{|x|})\theta_j = (ab-\varphi_t)e^t\theta_j.\end{aligned}$$

Therefore $(ab-\varphi_t)e^t$ is a solution to (4.1.1) with $\lambda_j = n-1$. Another solution to (4.1.1) with

$\lambda_j = n-1$ is

$$f_1^- = (ab-\varphi_t)e^t \int_t^\infty \frac{1}{(ab-\varphi_t)^2 e^{2t}} \exp(-\int_0^t \frac{E(s)}{D(s)} ds) dt$$

When $\lim_{t\to\infty} h(t,\varphi) > 0$, and $2k < n$, the solution $\varphi(t)$ has the bound

$$\frac{1}{C} \le ab - \varphi_t \le C$$

for some $C > 0$. So $\{(ab-\varphi_t)e^t, (ab-\varphi_t)\int_t^\infty \frac{1}{(ab-\varphi_t)^2 e^{2t}} \exp(-\int_0^t \frac{E(s)}{D(s)} ds)dt\}$ forms a solution basis for (4.1.1) with $\lambda_j = n-1$, with one exponentially decaying and the other one exponentially growing, and the conclusion of the proposition in the case $\lambda_j = n-1$ follows from the explicit basis.
For $\lambda_j \ge 2n$, we will verify that

$$\text{the coefficient of } f_j \text{ in (4.1.1) has a negative upper bound.} \tag{4.2.2}$$

Assuming (4.2.2), we sketch the proof for properties (a)-(d) of L_j for the case $\lambda_j \ge 2n$. The key is to check that for $\lambda > 0$ small, $e^{\pm\lambda t}$ are supersolutions of $L_j[f] = 0$. This is because

$$\begin{aligned} L_j[e^{\pm\lambda}] &= e^{\pm\lambda}(\lambda^2 \pm \lambda \frac{E(t)}{D(t)} - \lambda_j \frac{F(t)}{D(t)} + \frac{n(p-1)}{2aB^{k-1}} e^{-\frac{(p-1)k}{a}\varphi}) \\ &= e^{\pm\lambda}(\lambda^2 \pm \lambda[1-(n-1)\frac{F(t)}{D(t)}](\varphi_t - (ab-1)) - \lambda_j \frac{F(t)}{D(t)} + \frac{n(p-1)}{2aB^{k-1}} e^{-\frac{(p-1)k}{a}\varphi}), \end{aligned}$$

and it follows from (4.2.2) that for $\lambda_j \ge 2n$,

$$-\lambda_j \frac{F(t)}{D(t)} + \frac{n(p-1)}{2aB^{k-1}} e^{-\frac{(p-1)k}{a}\varphi}$$

has a negative upper bound. Furthermore, from (4.2.1), we get

$$\begin{aligned} F(t) &= \frac{k-1}{n-1}A + \frac{n-2k+1}{n-1}\frac{B}{2} \\ &= \frac{k-1}{n-1}\frac{1}{a}(\frac{n}{n-2k}\frac{e^{-\frac{(p-1)k}{a}\varphi}}{B^{k-1}} - B) + \frac{n-2k+1}{k-1}\frac{B}{2} \\ &= \frac{k-1}{n-1}\frac{n-2k}{2k}\frac{n}{n-2k}\frac{e^{-\frac{(p-1)k}{a}\varphi}}{B^{k-1}} - \frac{k-1}{n-1}\frac{n-2k}{2k}B + \frac{n-2k+1}{2(n-1)}B \\ &= \frac{n(k-1)}{2k(n-1)}\frac{e^{-\frac{(p-1)k}{a}\varphi}}{B^{k-1}} + \frac{n-k}{2k(n-1)}B. \end{aligned}$$

Therefore we have

$$\begin{aligned}
\frac{F(t)}{D(t)} &= \frac{n(k-1)}{k(n-1)} \frac{e^{-\frac{(p-1)k}{a}\varphi}}{B^k} + \frac{n-k}{k(n-1)} \\
&= \frac{n(k-1)}{k(n-1)} \frac{e^{-\frac{(p-1)k}{a}\varphi}}{\frac{2n}{(p+1)(n-2k)} e^{-\frac{(p-1)k}{a}\varphi} + e^{\frac{2k}{a}\varphi} h(t,\varphi)} + \frac{n-k}{k(n-1)} \\
&= \frac{n(k-1)}{k(n-1)} \frac{(p+1)(n-2k)}{2n} \frac{e^{-\frac{(p-1)k}{a}\varphi}}{e^{-\frac{(p-1)k}{a}\varphi} + \hat{h}(t,\varphi)} + \frac{n-k}{k(n-1)},
\end{aligned}$$

where $\hat{h}(t,\varphi) = \frac{(p+1)(n-2k)}{2n} e^{\frac{2k}{a}\varphi} h(t,\varphi)$. Here we used the fact that

$$B^k = \frac{2n}{(p+1)(n-2k)} e^{-\frac{(p-1)k}{a}\varphi} + e^{\frac{2k}{a}\varphi} h(t,\varphi), \quad \lim_{t\to\infty} h(t,\varphi) > 0.$$

By (3.2.1), we see that

$$\frac{n-k}{k(n-1)} \leq \frac{F(t)}{D(t)} \leq \frac{(p+1)(k-1)(n-2k)}{2k(n-1)} + \frac{n-k}{k(n-1)}.$$

It is clear that we can choose $\lambda > 0$ small to make

$$L_j[e^{\pm\lambda t}] < 0$$

for all $t \in \mathbb{R}$.

Now fix such a $\lambda > 0$. We claim that if $f(t)$ is a bounded solution to $L_j[f] = 0$ in $\mathbb{R}^{\pm}$, then

$$|f(t)| \leq |f(0)| e^{\mp\lambda t}$$

for all $t \in \mathbb{R}^{\pm}$, which then implies (b). This is because for any $\epsilon > 0$, $|f(0)|e^{\mp\lambda t} + \epsilon e^{\pm\lambda t}$ is a supersolution to L_j in $\mathbb{R}^{\pm}$. Therefore if $f(t)$ is a bounded solution to $L_j[f] = 0$ in $\mathbb{R}^{\pm}$, then by the maximum principle, we have

$$|f(t)| \leq |f(0)| e^{\mp\lambda t} + \epsilon e^{\pm\lambda t}$$

for all $t \in \mathbb{R}^{\pm}$. For any fixed $t \in \mathbb{R}^{\pm}$, since the above estimate holds for all $\epsilon > 0$, we can send ϵ to 0 to verify our claim. (c) follows from (b) and the maximum principle, and (d) obviously is a direct corollary of (c).

Next, any L_j has a pair of linearly independent basis $\{f_1, f_2\}$ in $\mathbb{R}$. If both are bounded in $\mathbb{R}^+$, and $\hat{c}_1$ and $\hat{c}_2$ are such that

$$\hat{c}_1 f_1(0) + \hat{c}_2 f_2(0) = 0,$$

then our claim implies that

$$\hat{c}_1 f_1(t) + \hat{c}_2 f_2(t) \equiv 0 \text{ in } \mathbb{R}^+,$$

contradicting their choice.

It remains to establish that there is a nontrivial solution to $L_j[f] = 0$ bounded in $\mathbb{R}^+$. Since we have verified that L_j is uniformly elliptic in $\mathbb{R}^+$ and satisfies the maximum principle, we can establish the desired existence by a convergence argument for solutions which are constructed on a sequence of finite intervals that exhaust $\mathbb{R}^+$.

Finally we will verify (4.2.2). Since $1 \geq B > 0$, we see that, when $\lambda_j \geq 2n$, the coefficient of $f_j(t)$ in (4.1.1) is bounded from above by

$$\begin{aligned}
&-\lambda_j \frac{F(t)}{D(t)} + \frac{n(p-1)}{2aB^{k-1}} e^{-\frac{(p-1)k}{a}\varphi} \\
=& -\lambda_j \Big[\frac{n(k-1)}{k(n-1)} \frac{e^{-\frac{(p-1)k}{a}\varphi}}{B^k} + \frac{n-k}{k(n-1)}\Big] + \frac{n(p-1)}{2aB^k} e^{-\frac{(p-1)k}{a}\varphi} B \\
\leq& -\frac{2ne^{-\frac{(p-1)k}{a}\varphi}}{(n-1)B^k}\Big(\frac{n(k-1)}{k} - \frac{(p-1)(n-1)}{4a} B\Big) - \frac{2n(n-k)}{k(n-1)} \\
<& -\frac{2ne^{-\frac{(p-1)k}{a}\varphi}}{(n-1)B^k}\Big(n - \frac{n}{k} - \frac{4k}{n-2k}\frac{n-2k}{2k}\frac{n-1}{4}\Big) - \frac{2n(n-k)}{k(n-1)} \\
=& -\frac{2ne^{-\frac{(p-1)k}{a}\varphi}}{(n-1)B^k}\Big(n - \frac{n}{k} - \frac{n}{2} + \frac{1}{2}\Big) - \frac{2n(n-k)}{k(n-1)} \\
<& -\frac{2n(n-k)}{k(n-1)} < 0,
\end{aligned}$$

when $n > k \geq 2$. When $k = 1$, the above estimate gives

$$\begin{aligned}
&-\lambda_j \frac{F(t)}{D(t)} + \frac{n(p-1)}{2aB^{k-1}} e^{-\frac{(p-1)k}{a}\varphi} \\
\leq& -\frac{2ne^{-\frac{(p-1)k}{a}\varphi}}{(n-1)B^k}\Big(\frac{n(k-1)}{k} - \frac{(p-1)(n-1)}{4a} B\Big) - \frac{2n(n-k)}{k(n-1)} \\
=& \, n(e^{-\frac{(p-1)k}{a}\varphi} - 2) \\
\leq& -n,
\end{aligned}$$

as $\varphi(t) \geq 0$, which follows from

$$\frac{2c_2}{p+1} e^{-\frac{(p-1)k}{a}\varphi} + e^{\frac{2k}{a}\varphi} h(t,\varphi) = B \leq 1,$$

with $\lim_{t\to\infty} h(t,\varphi) > 0$. □

This proposition can be considered as comparison theorem for $L_j[f] = 0$. However Theorem 1.4.5 requires some more detailed knowledge about the linearized operator to $\sigma_k(B^w) = ce^{-\frac{p-1}{a}kw}$. To be more precise, the decay rates of bounded solutions to $L_j[f] = 0$ on $\mathbb{R}^+$ need to be faster that e^{-t} when $\lambda_j \geq 2n$. In the case $2k < n$ and $h > 0$, L_j is an ordinary differential operator without period coefficients, so we can not apply the Floquet theory. Now we can formulate and prove a version that does not need L_j to have the structure to apply the Floquet theory.

Lemma 4.2.2. *When $2k < n$ and $h > 0$, for any $\beta \leq \beta_*$ with $\beta_* \geq (\sqrt{13} - 1)/2$, and for all $\lambda_j \geq 2n$, any bounded solution f of $L_j[f] = 0$ in $\mathbb{R}^+$ satisfies $|f(t)| \leq Ce^{-\beta t}$.*

Remark 4.2.1. In the case $2k < n$ and $h > 0$, above lemma allows a direct construction of a bounded (in fact, decaying) fundamental solution to $L_j[f] = 0$ in $\mathbb{R}^+$ which is positive. Therefore in such cases comparison theorems show that the characteristic roots ρ_j of L_j are monotone increasing as λ_j increases.

Proof. When $\lambda_j \geq 2n$, $\lambda \leq 2$, we can find $\lambda \geq \frac{\sqrt{13}-1}{2}$. In fact, we have

$$\begin{aligned}
L_j[e^{-\lambda t}] &= e^{-\lambda t}(\lambda^2 - \lambda[1-(n-1)\frac{F(t)}{D(t)}](\varphi_t - (ab-1)) - \lambda_j \frac{F(t)}{D(t)} + \frac{n(p-1)}{2aB^k} e^{-\frac{(p-1)k}{a}\varphi} B) \\
&\leq e^{-\lambda t}(\lambda^2 - \lambda + \lambda(n-1)\frac{F(t)}{D(t)} - 2n\frac{F(t)}{D(t)} + \frac{n(p-1)}{2a}\frac{e^{-\frac{(p-1)k}{a}\varphi}}{B^k}) \\
&\leq e^{-\lambda t}(\lambda^2 - \lambda + (n(\lambda-2)-\lambda)\frac{F(t)}{D(t)} + \frac{n(p-1)}{2a}\frac{e^{-\frac{(p-1)k}{a}\varphi}}{B^k}) \\
&\leq e^{-\lambda t}(\lambda^2 - \lambda - [(2-\lambda)n+\lambda]\frac{e^{-\frac{(p-1)k}{a}\varphi}}{B^k} + + \frac{n(p-a)}{2a}\frac{e^{-\frac{(p-1)k}{a}\varphi}}{B^k} \\
&\quad -[(2-\lambda)n+\lambda]\frac{n-k}{k(n-1)}).
\end{aligned}$$

Here we use the fact that

$$1-(n-1)\frac{F(t)}{D(t)} \leq 1 - \frac{n-k}{k} = \frac{2k-n}{k} < 0.$$

Since

$$B^k = \frac{2n}{(p+1)(n-2k)} e^{-\frac{(p-1)k}{a}\varphi} + \hat{h}(t,\varphi), \quad h(t,\varphi) > 0,$$

we get

$$\begin{aligned}
L_j[e^{-\lambda t}] &\leq e^{-\lambda t}(\lambda^2 - \lambda + [\frac{n(p-1)}{2a} - [(2-\lambda)n+\lambda]\frac{n(k-1)}{k(n-1)}]\frac{e^{-\frac{(p-1)k}{a}\varphi}}{\frac{2n}{(p+1)(n-2k)}e^{-\frac{(p-1)k}{a}\varphi} + \hat{h}(t,\varphi)} \\
&\quad -[(2-\lambda)n+\lambda]\frac{n-k}{k(n-1)}) \\
&< e^{-\lambda t}(\lambda^2 - \lambda + \frac{(P+1)(n-2k)}{2}[\frac{p-1}{2a} - [(2-\lambda)n+\lambda]\frac{k-1}{k(n-1)}] \\
&\quad -[(2-\lambda)n+\lambda]\frac{n-k}{k(n-1)}) \\
&\leq e^{-\lambda t}(\lambda(\lambda-1) + \frac{(p+1)(n-2k)}{2}[1 - [(2-\lambda)n+\lambda]\frac{k-1}{k(n-1)}] \\
&\quad -[(2-\lambda)n+\lambda]\frac{n-k}{k(n-1)})
\end{aligned}$$

Let $\lambda = 1+\varepsilon$. Then we have $2-\lambda = 1-\varepsilon$, $\lambda - 1 = \varepsilon$ and

$$\begin{aligned}
L_j[e^{-\lambda t}] &< e^{-\lambda t}((1+\varepsilon)\varepsilon + \frac{(p+1)(n-2k)}{2}(1 - [(n+1)-(n-1)\varepsilon]\frac{k-1}{k(n-1)}) \\
&\quad -[(n+1)-(n-1)\varepsilon]\frac{n-k}{k(n-1)})
\end{aligned}$$

Next we obtain

$$
\begin{aligned}
L_j[e^{-\lambda t}] < & e^{-\lambda t}((1+\varepsilon)\varepsilon + n[1 - \frac{(n+1)(k-1)}{k(n-1)} + (n-1)\varepsilon\frac{k-1}{k(n-1)}] \\
& -\frac{(n+1)(n-k)}{k(n-1)} + \varepsilon(n-1)\frac{n-k}{k(n-1)}) \\
= & e^{-\lambda t}((1+\varepsilon)\varepsilon + \frac{nk(n-1) - n(n+1)(k-1) - (n+1)(n-k)}{k(n-1)}) \\
& +\varepsilon n(1-\frac{1}{k}) + \varepsilon(\frac{n}{k} - 1)) \\
= & e^{-\lambda t}(\varepsilon^2 + n\varepsilon - 1).
\end{aligned}
$$

Now we can choose $\varepsilon = \frac{\sqrt{n^2+4}-n}{2}$, that is,

$$
\lambda = \frac{\sqrt{n^2+4} - n + 2}{2} \geq \frac{\sqrt{13}-1}{2}
$$

such that $L_j[e^{-\lambda t}] \leq 0$. Then by the maximum principle, we have

$$
|f(t)| \leq Ce^{-\beta t}.
$$

□

Note that the coefficient of f_j' in $L_j[f_j]$ is

$$
\frac{E(t)}{D(t)} = \frac{-(\varphi_t - (ab-1))(2(k-1)\varphi_{tt} + (n-2k)(-\varphi_t^2 + 2(ab-1)\varphi_t + ab(2-ab)))}{-\varphi_t^2 + 2(ab-1)\varphi_t + ab(2-ab)},
$$

which may alter its sign as $t \to \infty$. Therefore it is hard to be controlled in computation. In [42], they introduced a function $V(t)$ to remove this term involving f_j'. But in our case, it is not easy to recover the results by their method.

4.3 Existence of a parametrix

In this section, we will give some existence results for the linearized equation in weighted Sobolev spaces, which inspired by [44, 63, 64]

Let $M := \mathbb{R} \times \mathbb{S}^{n-1}$ and $M^+ := [0, +\infty) \times \mathbb{S}^{n-1}$. For $\delta \in \mathbb{R}$, we define the weighted L^2 space

$$
L^2_\delta(M) := e^{\delta t} L^2(M)
$$

endowed with the norm

$$
\|u\|_{L^2_\delta(M)} := \|e^{-\delta t} u\|_{L^2(M)}.
$$

More generally, for $k \in \mathbb{N}$, we define the weighted Sobolev space

$$
\|u\|_{W^{k,2}_\delta(M)} := e^{\delta t} W^{k,2}(M)
$$

endowed with the norm $\|u\|_{W^{k,2}_\delta(M)} := \|e^{-\delta t}u\|_{W^{k,2}(M)}$. It is easy to check that $(W^{k,2}_\delta(M), \|\cdot\|_{W^{k,2}_\delta(M)})$ is a Banach space. We also define the weighted Hölder space $C^{k,\alpha}_\delta(M)$ as

$$\{u: \|u\|_{C^{k,\alpha}_\delta(M)} := \|e^{-\delta t}u\|_{C^{k,\alpha}(M)} < \infty\}.$$

For the more information on the weighted Sobolev spaces and Hölder spaces, see the references ([11], [18]).

We next define

$$Lu := (\partial_{tt} + \bar{d}\partial_t + \bar{e}\Delta_\theta + \bar{g})u,$$

where $\bar{d}$, $\bar{e} > 0$ and $\bar{g}$ are constants. We also use $\mathtt{Ind}$ to denote the set of the indicial roots of the operator L.

$u \in W^{1,2}_\delta(M)$ is said to be a solution to $Lu = f$ for $f \in L^2_\delta(M)$, if u satisfies

$$\int_M (-\partial_t u\partial_t v - \bar{e}\partial_\theta u\partial_\theta v + \bar{d}\partial_t uv + \bar{g}uv)dtd\omega = \int_M fvdtd\omega,$$

for all $v \in C^\infty_c(M)$.

Lemma 4.3.1. *Assume that $f \in L^2_\delta(M)$, where $\delta \in \mathbb{R}$ and $\delta \notin \mathtt{Ind}$. If $u \in W^{1,2}_\delta(M)$ is a solution to $Lu = f$ in M. Then we have*

$$\|u\|_{W^{2,2}_\delta(M)} \le C\|f\|_{L^2_\delta(M)}.$$

Proof. Let $U = e^{-\delta t}u$ and $F = e^{-\delta t}f$. It follows that U and F satisfy the following equation

$$e^{-\delta t}L(e^{\delta t}U) = F,$$

that is,

$$\partial_{tt}U + \bar{e}\Delta_\theta U + (2\delta + \bar{d})\partial_t U + (\delta^2 + \delta\bar{d} + \bar{g})U = F \ \text{ in } M.$$

For $t_0 \in \mathbb{Z}$, applying the interior estimates with $\Omega' = [t_0-1, t_0+1]\times\mathbb{S}^{n-1}$ and $\Omega = [t_0-2, t_0+2]\times\mathbb{S}^{n-1}$, we have

$$\|U\|_{W^{2,2}(\Omega')} \le C(\|U\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)}).$$

Then we sum the above inequality over t_0, and obtain

$$\|U\|_{W^{2,2}(M)} \le C(\|U\|_{L^2(M)} + \|F\|_{L^2(M)}). \tag{4.3.1}$$

Now we claim that

$$\|U\|_{L^2(M)} \le C\|F\|_{L^2(M)}. \tag{4.3.2}$$

In fact, we employ the Fourier transform of U and F in t, that is,

$$\hat{U}(\xi,\theta) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} U(t,\theta)e^{-i\xi t}dt \text{ and } \hat{F}(\xi,\theta) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} F(t,\theta)e^{-i\xi t}dt.$$

Using the eigenfunction decomposition of the function F as $F = \sum_{j\in\mathbb{N}} F_j$, where, for all $t \in [0,\infty)$,

$F_j(t,\cdot) \in E_j$ the j-th eigenspace of $-\Delta_{\mathbb{S}^{n-1}}$ associated to the eigenvalue λ_j, we see that

$$\hat{U} = \sum_{j\in\mathbb{N}} \frac{1}{(\delta + \frac{\bar{d}}{2} + i\xi)^2 - \frac{\bar{d}^2}{4} + \bar{g} - \bar{e}\lambda_j} \hat{F}_j. \tag{4.3.3}$$

By Plancheral formula, we obtain

$$\|U\|^2_{L^2(M)} = \|\hat{U}\|^2_{L^2(M)} \leq C\|\hat{F}\|^2_{L^2(M)} = \|F\|^2_{L^2(M)},$$

where C depends on δ.

Combining (4.3.1) and (4.3.2), we have

$$\|U\|_{W^{2,2}(M)} \leq C\|F\|_{L^2(M)}.$$

This completes the proof of this lemma.

□

Remark 4.3.1. In [11], Bartnik obtained the similar L^p estimates for Δ operator in Euclidean spaces, where $1 < p < \infty$. The proof relies on a sharp estimate for the flat Laplacian based on an explicit expression for the integral kernel of Δ^{-1} on weighted Sobolev spaces.

Corollary 4.3.2. *Assume that $f \in L^2_\delta(M^+)$, where $\delta \in \mathbb{R}$ and $\delta \notin \mathtt{Ind}$. If $u \in W^{1,2}_\delta(M^+)$ is a solution to $Lu = f$ in M^+ with $u = 0$ on $\{0\} \times \mathbb{S}^{n-1}$. Then we have*

$$\|u\|_{W^{2,2}_\delta(M^+)} \leq C\|f\|_{L^2_\delta(M^+)}.$$

Proof. It can be proved by a standard method. For the reader's convenience, we include the proof. We now extend u and f to $t < 0$ by odd extension, i.e.,

$$\tilde{u}(t,\theta) = \begin{cases} u(t,\theta), & t \geq 0, \\ -u(-t,\theta), & t < 0, \end{cases}$$

and

$$\tilde{f}(t,\theta) = \begin{cases} f(t,\theta), & t \geq 0, \\ -f(-t,\theta), & t < 0, \end{cases}$$

It is not difficult to verify that $\tilde{u} \in W^{2,2}_\delta(M^+)$, $\tilde{f} \in L^2_\delta(M^+)$. And $\tilde{u}$ and $\tilde{f}$ satisfy the following equation

$$L\tilde{u} = \tilde{f} \ \text{ in } M$$

almost everywhere. Then by the above lemma, we see that

$$\|\tilde{u}\|_{W^{2,2}_\delta(M)} \leq C\|\tilde{f}\|_{L^2_\delta(M)} \leq 2C\|f\|_{L^2_\delta(M^+)}.$$

It follows that

$$\|u\|_{W^{2,2}_\delta(M^+)} \leq C\|f\|_{L^2_\delta(M^+)}.$$

□

Proposition 4.3.3. *Assume that $f \in L^2_\delta(M^+)$, where $\delta \in \mathbb{R}$ and $\delta \notin$ `Ind`. Then there exists a solution $u \in W^{1,2}_\delta(M^+)$ to $Lu = f$ in M^+ with $u = 0$ on $\{0\} \times \mathbb{S}^{n-1}$. Moreover, we have*

$$\|u\|_{W^{2,2}_\delta(M^+)} \leq C\|f\|_{L^2_\delta(M^+)}.$$

Proof. We employ the eigenfunction decomposition of the function f as $f(t,\theta) = \sum_{j\in\mathbb{N}} f_j(t,\theta)$, where, for all $t \in [0,+\infty)$, $f_j(t,\cdot) \in E_j$ the j-th eigenspace of $-\Delta_{\mathbb{S}^{n-1}}$ associated to the eigenvalue λ_j. Denote $j_0 \in \mathbb{N}$ be the least index such that

$$(\delta + \frac{\bar{d}}{2})^2 - \frac{\bar{d}^2}{4} + \bar{g} < \bar{e}\lambda_{j_0}.$$

Let $\tilde{f} = \sum_{j\geq j_0} f$. It is easy to see that $\tilde{f} \in L^2_\delta(M^+)$. For any $t_0 > 0$, there exists a unique solution u_{t_0} to

$$\begin{cases} Lu_{t_0} = \tilde{f}, & \text{in } (0,t_0) \times \mathbb{S}^{n-1}, \\ u_{t_0} = 0, & \text{on } \{0\} \times \mathbb{S}^{n-1} \text{ and } \{t_0\} \times \mathbb{S}^{n-1}. \end{cases}$$

In fact, the existence of u_{t_0} is from the Fredholm alternative theorem, since Lu_{t_0} satisfies the maximum principle when $j \geq j_0$, that is, 0 is the unique solution to $Lu_{t_0} = 0$ with vanishing boundary condition.

Now we claim that

$$\|u_{t_0}\|_{W^{2,2}_\delta((0,t_0)\times\mathbb{S}^{n-1})} \leq C\|\tilde{f}\|_{L^2_\delta((0,t_0)\times\mathbb{S}^{n-1})},$$

where C does not dependent on t_0. In particular, we have

$$\|u_{t_0}\|_{W^{1,2}_\delta((0,t_0)\times\mathbb{S}^{n-1})} \leq C\|\tilde{f}\|_{L^2_\delta((0,t_0)\times\mathbb{S}^{n-1})}. \tag{4.3.4}$$

In fact, using the similar argument in Lemma 4.3.1, we are able to get the estimates.

By the interior estimates, we obtain

$$\|u_m\|_{W^{2,2}_\delta((0,1)\times\mathbb{S}^{n-1})} \leq C\|\tilde{f}\|_{L^2_\delta((0,m)\times\mathbb{S}^{n-1})} \leq C, \quad \forall m > 1,$$

where C does not depend on m. Therefore there is a subsequence $\{u^{(1)}_m\}$, such that

$$u^{(1)}_m \to u^{(1)} \quad \text{in} \quad W^{1,2}_\delta((0,1)\times\mathbb{S}^{n-1}), \quad \text{as } m \to \infty.$$

Similarly there is a subsequence $\{u^{(k)}_m\}$, such that

$$u^{(k)}_m \to u^{(k)} \quad \text{in} \quad W^{1,2}_\delta((0,k)\times\mathbb{S}^{n-1}), \quad \text{as } m \to \infty,$$

and $u^{(k)} = u^{(j)}$ in $(0,j) \times \mathbb{S}^{n-1}$, $j = 1, 2, \cdots, k-1$.

Define $\tilde{u}(t,\theta) = u^{(k)}(t,\theta)$, if $(t,\theta) \in (0,k) \times \mathbb{S}^{n-1}$, then $\tilde{u}(t,\theta)$ is defined on $(0,\infty) \times \mathbb{S}^{n-1}$. Consider sequence $\{u^{(m)}_m\}$ in diagram, for any $(0,k) \times \mathbb{S}^{n-1}$,

$$u^{(m)}_m \to \tilde{u} \quad \text{in} \quad W^{1,2}_\delta((0,k)\times\mathbb{S}^{n-1}), \quad \text{as } m \to \infty \quad (\{u^{(m)}_m\} \subset \{u^{(k)}_m\}, \text{ if } m > k),$$

Since

$$Lu_m^{(m)} = \tilde{f} \quad \text{in } (0,k)\times\mathbb{S}^{n-1}, \quad \forall\, m > k,$$

we then find $\tilde{u} \in W^{1,2}_{\delta,loc}(M^+)$ is the solution to

$$L\tilde{u} = \tilde{f}.$$

Let $m \to \infty$ in (4.3.4), we see that

$$\|\tilde{u}\|_{W^{1,2}_\delta(M^+)} \le C\|\tilde{f}\|_{L^2_\delta(M^+)}.$$

It follows that

$$\|\tilde{u}\|_{W^{2,2}_\delta(M^+)} \le C\|\tilde{f}\|_{L^2_\delta(M^+)}. \tag{4.3.5}$$

When $j \le j_0 - 1$, by the variation of constant formula, we see that

$$u_j(t,\theta) = u_j^-(t)\int_{m^+}^t \frac{u_j^+(s)f_j(t,\theta)}{W_0(s)}ds - u_j^+(t)\int_{m^-}^t \frac{u_j^-(s)f_j(s,\theta)}{W_0(s)}ds,$$

where the Wronskian of $\{u_j^+(s) := \exp(\frac{-\bar{d}+\sqrt{\bar{d}^2-4(\bar{g}-\bar{e}\lambda_j)}}{2}s), u_j^-(s) = \exp(\frac{-\bar{d}-\sqrt{\bar{d}^2-4(\bar{g}-\bar{e}\lambda_j)}}{2}s)\}$, $W_0(s) = e^{-\bar{d}s}$, and $m^+ = 0$ or ∞, $m^- = 0$ or ∞. It is not difficult to see that

$$\|e^{-\delta t}u_j\|_{L^2(M^+)} \le C\|e^{-\delta t}f_j\|_{L^2(M^+)},$$

where C depends on δ and j. Then we see that

$$\|u_j\|_{W^{2,2}_\delta(M^+)} \le C\|f_j\|_{L^2_\delta(M^+)}. \tag{4.3.6}$$

Therefore

$$u = \sum_{j=0}^{j_0-1} u_j + \tilde{u}$$

is the solution to $Lu = f$ in M^+ with $u = 0$ on $\{0\}\times\mathbb{S}^{n-1}$ and $\lim_{t\to\infty} e^{-\delta t}u = 0$. Moreover, combining (4.3.5) and (4.3.6), we have

$$\|u\|_{W^{2,2}_\delta(M^+)} \le C\|f\|_{L^2_\delta(M^+)}.$$

□

Inspired by the work of R. Lockhart and R. McOwen [60], we will get

Proposition 4.3.4. *Assume that $f \in L^2_\delta(M^+)$, where $\delta \in \mathbb{R}$ and $\delta \notin \mathtt{Ind}$. Then there exists a solution $u \in W^{1,2}_\delta(M^+)$ to $L_\varphi u = f$ in M^+ with $u = 0$ on $\{0\}\times\mathbb{S}^{n-1}$. Moreover, we have*

$$\|u\|_{W^{2,2}_\delta(M^+)} \le C\|f\|_{L^2_\delta(M^+)}.$$

Proof. From the above proposition, we have that, for given $f \in L^2_\delta(M^+)$, there exists a solution

$u^* \in W^{2,2}_\delta(M^+)$ such that

$$Lu^* = f \ \text{ in } M^+.$$

Then by Theorem 1.4.2, we conclude that

$$\|Lu^* - L_\varphi u^*\|_{L^2_\delta(M^+)} \le Ce^{-\alpha t}\|u^*\|_{W^{2,2}_\delta(M^+)}.$$

It follows that

$$\|f - L_\varphi L^{-1} f\|_{L^2_\delta(M^+)} \le Ce^{-\alpha t}\|f\|_{L^2_\delta(M^+)},$$

where C does not depend on t. Let t large enough such that $Ce^{-\alpha t} < 1$. Then we see that

$$\|I - L_\varphi L^{-1}\| < 1.$$

By Neumann series, we get $L_\varphi L^{-1}$ is invertible and bounded. It follows that the inverse of L_φ is $L_\varphi^{-1} = L(L_\varphi L^{-1})^{-1}$. Then we obtain

$$\|u\|_{L^2_\delta(M^+)} \le C\|f\|_{L^2_\delta(M^+)}.$$

Finally, by the estimates in [88], we have

$$\|u\|_{W^{2,2}_\delta(M^+)} \le C\|f\|_{L^2_\delta(M^+)}.$$

□

4.4 Expansion in terms of Wronskian

With the knowledge in the previous sections, following the same line of the proof of Proposition 3 in [42], we can establish

Proposition 4.4.1. *Suppose that $f(t,\theta) \to 0$ as $t \to \infty$, uniformly in $\theta \in \mathbb{S}^{n-1}$, and satisfies*

$$L_\varphi(f) + \frac{(p-1)k}{2^k a}\binom{n}{k} e^{-\frac{(p-1)k}{a}\varphi} f = r(t,\theta), \tag{4.4.1}$$

for all $t > t_0$ and $\theta \in \mathbb{S}^{n-1}$. Suppose that for some $0 < \beta < \min\{\beta_*, (ab-1)(n-2k)+1\}$ and $0 < \varepsilon_0 \ll \beta$, $|r(t,\theta)| \le Ce^{-\beta t}$. When $\varphi_t \equiv 0$, $\beta \ne Re(a_{01})$ and $Re(a_{02})$, there exist constants c_{01} and c_{02}, such that

$$|f(t,\theta) - (c_{01}e^{-a_{01}t} + c_{02}e^{-a_{02}t})Y_0(\theta)| \le Ce^{-\beta t}, \tag{4.4.2}$$

where

$$a_{01} = \frac{(ab-1)(n-2k) - \sqrt{(ab-1)^2(n-2k)^2 - 4(n-2k)(2-ab)}}{2},$$

and

$$a_{02} = \frac{(ab-1)(n-2k) + \sqrt{(ab-1)^2(n-2k)^2 - 4(n-2k)(2-ab)}}{2}.$$

In fact, if $\min\{\beta_*, (ab-1)(n-2k)\} \le \beta < (ab-1)(n-2k)+1$*, then it continues to hold, and when*

$\beta \geq (ab-1)(n-2k)+1$, we have

$$|f(t,\theta) - (c_{01}e^{-a_{01}t} + c_{02}e^{-a_{02}t})Y_0(\theta)| \leq Ce^{-((ab-1)(n-2k)+1)t}. \tag{4.4.3}$$

When $\beta = Re(a_{01})$ or $\beta = Re(a_{02})$, we get

$$|f(t,\theta) - (c_{01}e^{-a_{01}t} + c_{02}e^{-a_{02}t})Y_0(\theta)| \leq Cte^{-\beta t}. \tag{4.4.4}$$

When $\varphi \neq 0$, $Re(a_{02}) \leq 2Re(a_{01}) - \varepsilon_0$ and $0 < \beta < \min\{\beta_*, (ab-1)(n-2k)+1+\varepsilon_0\}$, there exists a function $c_0(t)$ such that

$$|f(t,\theta) - c_0(t)Y_0(\theta)| \leq Ce^{-\beta t + 2\varepsilon_0 t}. \tag{4.4.5}$$

Indeed, if $\min\{\beta_*, ab-1)(n-2k)+1+\varepsilon_0\} \leq \beta < (ab-1)(n-2k)+1+\varepsilon_0$, then it continues to hold, and when $\beta \geq (ab-1)(n-2k)+1+\varepsilon_0$, we have

$$|f(t,\theta) - c_0(t)Y_0(\theta)| \leq Ce^{-((ab-1)(n-2k)+1)+2\varepsilon_0 t}. \tag{4.4.6}$$

Proof. Define

$$\hat{f}(t,\theta) = f(t,\theta) - \sum_{j=0}^{n} \pi_j[f(t,\theta)]Y_j(\theta),$$

where $f_j(t) := \pi_j[f(t,\theta)]$ is the L^2 orthogonal projection of $f(t,\theta)$ onto span $\{Y_j(\theta)\}$. Then

$$\int_{\mathbb{S}^{n-1}} \hat{f}(t,\theta)Y_j(\theta)d\theta = \int_{\mathbb{S}^{n-1}} \nabla\hat{f}(t,\theta)\cdot\nabla Y_j(\theta)d\theta = \int_{\mathbb{S}^{n-1}} \Delta_\theta Y_j(\theta)\hat{f}(t,\theta)d\theta = 0. \tag{4.4.7}$$

for $j = 0, \ldots, n$. As a consequence

$$\int_{\mathbb{S}^{n-1}} \Delta_\theta f(t,\theta)\hat{f}(t,\theta)d\theta = -\int_{\mathbb{S}^{n-1}} |\nabla_\theta \hat{f}(t,\theta)|^2 d\theta, \tag{4.4.8}$$

and

$$\int_{\mathbb{S}^{n-1}} f_t(t,\theta)\hat{f}(t,\theta)d\theta = \int_{\mathbb{S}^{n-1}} \hat{f}_t(t,\theta)\hat{f}(t,\theta)d\theta = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}^{n-1}} |\hat{f}(t,\theta)|^2 d\theta. \tag{4.4.9}$$

In the following we will prove separately the expected decays for $\hat{f}(t,\theta)$ and $f_j(t) = \pi_j[f(t,\theta)]$, for $j = 0,1,\ldots,n$. We first estimate $f_j(t) = \pi_j[f(t,\theta)]$ for $j = 0,1,\ldots,n$. Multiplying both sides of (4.4.1) by

$$(\frac{B^{k-2}}{2^{k-2}}\binom{n-1}{k-1}D(t))^{-1}Y_j(\theta)$$

and integrating over $\theta \in \mathbb{S}^{n-1}$, we have

$$f_j''(t) + \frac{E(t)}{D(t)}f_j'(t) + [\frac{n(p-1)}{2aB^{k-1}}e^{-\frac{(p-1)k}{a}\varphi} - \lambda_j\frac{F(t)}{D(t)}]f_j(t) = r_j(t), \tag{4.4.10}$$

where

$$r_j(t) = \int_{\mathbb{S}^{n-1}} R(t,\theta)Y_j(\theta)d\theta$$

with

$$R(t,\theta) = (\frac{B^{k-2}}{2^{k-2}}\binom{n-1}{k-1}D(t))^{-1}r(t,\theta).$$

First we treat the case $\varphi_t \not\equiv 0$. We will write out the details for the $h(t,\varphi) > 0$. For $j = 1, 2, \ldots, n$, $\lambda_j = n-1$, and $f_1^+(t) = (ab-\varphi_t)e^t$ and

$$f_1^- = (ab-\varphi_t)e^t \int_t^\infty \frac{1}{(ab-\varphi_t)^2 e^{2t}} \exp(-\int_0^t \frac{E(s)}{D(s)}ds)dt$$

form a solution basis to the homogeneous equation

$$f''(t) + \frac{E(t)}{D(t)}f'(t) + [\frac{n(p-1)}{2aB^{k-1}}e^{-\frac{(p-1)k}{a}\varphi} - (n-1)\frac{F(t)}{D(t)}]f(t) = 0.$$

Since $f_j(t)$ is a solution to (4.4.10) and $f_j(t) \to 0$ as $t \to \infty$, by the variation of constant formula, we have

$$f_j(t) = cf_1^-(t) + f_1^-(t)\int_0^t \frac{f_1^+(s)r_j(s)}{W_1(s)}ds + f_1^+(t)\int_t^\infty \frac{f_1^-(s)r_j(s)}{W_1(s)} \qquad (4.4.11)$$

for some constant c, where

$$W_1(s) = f_1^+(s)(f_1^-(s))' - f_1^-(s)(f_1^+(s))'$$

is the Wronskian of $\{f_1^+, f_1^-\}$, and satisfies

$$(W_1(s))' = -\frac{E(s)}{D(s)}W_1(s).$$

From

$$\begin{aligned}
\frac{E(t)}{D(t)} &= [1-(n-1)\frac{F(t)}{D(t)}](\varphi_t - (ab-1)) \\
&= [1 - \frac{(k-1)(p+1)(n-2k)}{2k}\frac{e^{-\frac{(p-1)k}{a}\varphi}}{e^{-\frac{(p-1)k}{a}\varphi} + \hat{h}(t,\varphi)} - \frac{n-k}{k}](\varphi_t - (ab-1)) \\
&= [\frac{2k-n}{k} - \frac{(k-1)(p+1)(n-2k)}{2k}\frac{e^{-\frac{(p-1)k}{a}\varphi}}{e^{-\frac{(p-1)k}{a}\varphi} + \hat{h}(t,\varphi)}](\varphi_t - (ab-1))
\end{aligned}$$

we get

$$\frac{1}{C}e^{-(n-2k)(ab-1)s-\varepsilon_0 s} \le W_1(s) \le Ce^{-(n-2k)(ab-1)s+\varepsilon_0 s}$$

for $\varepsilon_0 \ll \beta$. According to our assumption on the decay rate of $r(t,\theta)$, we have

$$|r_j(s)| \le Ce^{-\beta s}.$$

Therefore we obtain

$$|\int_t^\infty \frac{f_1^-(s)r_j(s)}{W_1(s)}ds| \le C\int_t^\infty e^{-(1+\beta-2\varepsilon_0)s}ds \le Ce^{-(1+\beta-2\varepsilon_0)t},$$

from which we deduce that

$$|f_1^+(t)\int_t^\infty \frac{f_1^-(s)r_j(s)}{W_1(s)}ds| \le Ce^{-(\beta-2\varepsilon_0)t}.$$

When $\beta \neq (n-2k)(ab-1)+1+\varepsilon_0$, we also have

$$|\int_0^t \frac{f_1^+(s)r_j(s)}{W_1(s)}ds| \le C\int_0^t e^{((n-2k)(ab-1)+1+\varepsilon_0-\beta)s}ds \le Ce^{((n-2k)(ab-1)+1+\varepsilon_0-\beta)t},$$

from which we conclude that

$$|f_1^-(t)\int_0^t \frac{f_1^+(s)r_j(s)}{W_1(s)}ds| \le Ce^{-(\beta-2\varepsilon_0)t}.$$

Putting these estimates into (4.4.11), we get

$$|f_j - c(ab-\varphi_t)e^t\int_t^\infty \frac{1}{(ab-\varphi_t)^2e^{2t}}\exp(-\int_0^t \frac{E(s)}{D(s)}ds)dt| \le Ce^{-(\beta-2\varepsilon_0)t}.$$

When $\beta = (n-2k)(ab-1)+1+\varepsilon_0$, we see that

$$|\int_0^t \frac{f_1^+(s)r_j(s)}{W_1(s)}ds| \le C\int_0^t ds \le Ct,$$

from which we deduce that

$$|f_1^-(t)\int_0^t \frac{f_1^+(s)r_j(s)}{W_1(s)}ds| \le Cte^{-(\beta-2\varepsilon_0)t}.$$

Therefore from (4.4.11), we see

$$|f_j - c(ab-\varphi_t)e^t\int_t^\infty \frac{1}{(ab-\varphi_t)^2e^{2t}}\exp(-\int_0^t \frac{E(s)}{D(s)}ds)dt| \le Cte^{-(\beta-2\varepsilon_0)t}.$$

For $j=0$, $f_0(t)$ is a solution to

$$f_0''(t) + \frac{E(t)}{D(t)}f_0'(t) + \frac{n(p-1)}{2aB^{k-1}}e^{-\frac{(p-1)k}{a}\varphi}f_0(t) = r_0(t). \tag{4.4.12}$$

It follows that

$$Y(t) = c\Phi(t) + \Phi(t)\int_{t_0}^t \Phi^{-1}(s)F(s)ds,$$

where $\Phi(t)$ is a fundamental solution matrix of the system

$$\frac{dY(t)}{dt} = A(t)Y(t)$$

with

$$A(t) = \begin{pmatrix} 0 & 0 \\ -\frac{n(p-1)}{2aB^{k-1}}e^{-\frac{(p-1)k}{a}\varphi} & -\frac{E(t)}{D(t)} \end{pmatrix},$$

$Y(t) = (f_0(t), f_0'(t))^T$ and $F(s) = (0, r_0(s))^T$. Employing the same argument in Theorem 1.4.2, we get that, for any small ε_0,

$$|\Phi(t)| \leq Ce^{-Re(a_{01})t+\frac{\varepsilon_0}{2}t},$$

where

$$a_{01} = \frac{(ab-1)(n-2k) - \sqrt{(ab-1)^2(n-2k)^2 - 4(n-2k)(2-ab)}}{2}.$$

Then by the definition of the Wronskian, we see

$$|\Phi(t)| \geq Ce^{-Re(a_{02})t-\frac{\varepsilon_0}{2}t},$$

where

$$a_{02} = \frac{(ab-1)(n-2k) + \sqrt{(ab-1)^2(n-2k)^2 - 4(n-2k)(2-ab)}}{2}.$$

When $\beta < Re(a_{01})$, for some $t_0 \in (0, \infty)$, from the decay rate of $r_0(s)$, we have

$$|Y(t) - c\Phi(t)| \leq C\int_{t_0}^{t} e^{(-Re(a_{01})+\frac{\varepsilon_0}{2})(t-s)}e^{-\beta s}ds = Ce^{-\beta t}.$$

When $Re(a_{02}) \leq 2Re(a_{01}) - \varepsilon_0 < \beta$, $t_0 = \infty$, from the decay rate of $r_0(s)$, we obtain

$$|Y(t) - c\Phi(t)| \leq C\int_{t}^{\infty} e^{(-Re(a_{02})-\frac{\varepsilon_0}{2})(t-s)}e^{-\beta s}ds = Ce^{-\beta t}.$$

Combining above estimates, we have

$$|f_0(t) - c_0(t)| \leq Ce^{-\beta t},$$

where $|c_0(t)| \leq Ce^{-Re(a_{01})t+\frac{\varepsilon_0}{2}t}$.

Now we estimate the decay rate of $\hat{f}(t, \theta)$. This part is analogous to an approach in [80]. Multiplying both sides of (4.4.10) by

$$(\frac{B^{k-2}}{2^{k-2}}\binom{n-1}{k-1}D(t))^{-1}\hat{f}(t,\theta),$$

integrating over $\theta \in \mathbb{S}^{n-1}$ and from (4.4.8) and (4.4.9), we get

$$
\begin{aligned}
&\int_{\mathbb{S}^{n-1}} (\hat{f}_{tt}(t,\theta)\hat{f}(t,\theta) + \frac{E(t)}{D(t)}\hat{f}_t(t,\theta)\hat{f}(t,\theta) + \frac{n(p-1)}{2aB^{k-1}}e^{-\frac{(p-1)k}{a}\varphi}|\hat{f}(t,\theta)|^2)d\theta \\
&-\frac{F(t)}{D(t)}\int_{\mathbb{S}^{n-1}} |\nabla_\theta \hat{f}(t,\theta)|^2 d\theta \\
= &\int_{\mathbb{S}^{n-1}} R(t,\theta)\hat{f}(t,\theta)d\theta,
\end{aligned}
$$

where

$$\frac{1}{C}r(t,\theta) \le R(t,\theta) = (\frac{B^{k-2}}{2^{k-2}}\binom{n-1}{k-1}D(t))^{-1}r(t,\theta) \le Cr(t,\theta).$$

Defining

$$y(t) = \sqrt{\int_{\mathbb{S}^{n-1}} |\hat{f}(t,\theta)|^2 d\theta},$$

then

$$y(t)y'(t) = \int_{\mathbb{S}^{n-1}} \hat{f}_t(t,\theta)\hat{f}(t,\theta)d\theta,$$

and

$$y(t)y''(t) = \int_{\mathbb{S}^{n-1}} (\hat{f}_{tt}(t,\theta)\hat{f}(t,\theta) + |\hat{f}_t(t,\theta)|^2)d\theta - |y'(t)|^2.$$

By Cauchy-Schwarz inequality

$$y(t)y'(t) \le (\int_{\mathbb{S}^{n-1}} |\hat{f}_t(t,\theta)|^2 d\theta)^{\frac{1}{2}} y(t),$$

which implies that

$$|y'(t)|^2 \le \int_{\mathbb{S}^{n-1}} |\hat{f}_t(t,\theta)|^2 d\theta.$$

Since $\lambda_j = 2n$, we have

$$\int_{\mathbb{S}^{n-1}} |\nabla_\theta \hat{f}(t,\theta)|^2 d\theta \ge 2n \int_{\mathbb{S}^{n-1}} |\hat{f}(t,\theta)|^2 d\theta$$

Combining these inequalities, we obtain

$$y(t)y''(t) + \frac{E(t)}{D(t)}y(t)y'(t) + [\frac{n(p-1)}{2aB^{k-1}}e^{-\frac{(p-1)k}{a}\varphi} - 2n\frac{F(t)}{D(t)}]y^2(t) \ge -||R(t,\cdot)||_{L^2(\mathbb{S}^{n-1})}y(t)$$

whenever $y(t) > 0$, from which we deduce that

$$y''(t) + \frac{E(t)}{D(t)}y'(t) + [\frac{n(p-1)}{2aB^{k-1}}e^{-\frac{(p-1)k}{a}\varphi} - 2n\frac{F(t)}{D(t)}]y(t) \ge -||R(t,\cdot)||_{L^2(\mathbb{S}^{n-1})}$$

whenever $y(t) > 0$. According to our assumption on $r(t,\theta)$, we have

$$||R(t,\cdot)||_{L^2(\mathbb{S}^{n-1})} \le Ce^{-\beta t}$$

for some constant $C > 0$. By Lemma 4.2.2,

$$(\partial_{tt} + \frac{E(t)}{D(t)}\partial_t + [\frac{n(p-1)}{2aB^{k-1}}e^{-\frac{(p-1)k}{a}\varphi} - 2n\frac{F(t)}{D(t)}])e^{-\beta t} \leq -\epsilon e^{-\beta t}$$

for some $\epsilon > 0$ when $\beta < \beta_*$. Therefore $z(t) := \frac{C}{\epsilon}e^{-\beta t}$ satisfies

$$(\partial_{tt} + \frac{E(t)}{D(t)}\partial_t + [\frac{n(p-1)}{2aB^{k-1}}e^{-\frac{(p-1)k}{a}\varphi} - 2n\frac{F(t)}{D(t)}])(z(t) - y(t)) \leq 0 \tag{4.4.13}$$

whenever $y(t) > 0$ We also know that $y(t) \to 0$ as $t \to \infty$. We may choose $C > 0$ large so that $z(0) \geq y(0)$. Then we claim that

$$y(t) \leq z(t) \;\; \forall\, t \geq 0.$$

Indeed, if not, $\min(z(t) - y(t)) < 0$ is finite, and is attained at some t_* and $\partial_t(z(t) - y(t))_{t=t_*} = 0$, as well as $\partial_{tt}(z(t) - y(t))|_{t=t_*} \geq 0$. This contradicts (4.4.13). Thus we conclude that

$$y(t) = \sqrt{\int_{\mathbb{S}^{n-1}} |\hat{f}(t,\theta)|^2 d\theta} \leq \frac{C}{\epsilon}e^{-\beta t}.$$

We can now bootstrap this integral estimate to obtain a pointwise decay estimate

$$|\hat{f}(t,\theta)| \leq Ce^{-\beta t}.$$

When $\beta \geq \beta_*$, we can simply split those components f_j of f with $\lambda_j = 2n$ from $\hat{f}(t,\theta)$, and estimate them as we did for f_j, $j = 0, 1, \ldots, n$, and estimate $\hat{f}(t,\theta)$ with an improved exponential decay rate.

When $\varphi_t \equiv 0$, that is, $\varphi \equiv -\frac{a}{(p-1)k}\ln[\frac{n-2k}{n}(ab(2-ab))^k]$, along the above steps, we can get (4.4.2) and (4.4.3) easily. We only write out the detail for $j = 0$.

For $j = 0$, $f_0^+ = e^{-a_{01}t}$ and $f_0^- = e^{-a_{02}t}$ form a solution basis to the homogeneous equation

$$f_0''(t) + (ab-1)(n-2k)f_0'(t) + (2-ab)(n-2k)f_0(t) = 0, \tag{4.4.14}$$

where

$$a_{01} = \frac{(ab-1)(n-2k) - \sqrt{(ab-1)^2(n-2k)^2 - 4(n-2k)(2-ab)}}{2},$$

and

$$a_{02} = \frac{(ab-1)(n-2k) + \sqrt{(ab-1)^2(n-2k)^2 - 4(n-2k)(2-ab)}}{2}.$$

Since f_0 is a solution to (4.4.10) with $\lambda_0 = 0$ and $f_0 \to 0$ as $t \to \infty$, by the variation of constant formula, we see

$$f_0(t) = c_{01}f_0^+(t) + c_{02}f_0^-(t) + f_0^-(t)\int_{m^+}^t \frac{f_0^+(s)r_0(s)}{W_0(s)}ds - f_0^+(t)\int_{m^-}^t \frac{f_0^-(s)r_0(s)}{W_0(s)}ds, \tag{4.4.15}$$

where the Wronskian of $\{f_0^+(s), f_0^-(s)\}$, $W_0(s) = e^{-(n-2k)(ab-1)s}$, and $m^+ = 0$ or ∞, $m^- = 0$ or ∞.

When $a_{01} \neq \beta$, from the decay rate of $r(t,\theta)$, we get

$$|\int_{m^-}^t \frac{f_0^-(s) r_0(s)}{W_0(s)} ds| \leq Ce^{Re(a_{01})t - \beta t},$$

from which we deduce that

$$|f_0^+(t) \int_{m^-}^t \frac{f_0^-(s) r_0(s)}{W_0(s)} ds| \leq Ce^{-\beta t}.$$

Meanwhile we have

$$|\int_{m^+}^t \frac{f_0^+(s) r_0(s)}{W_0(s)} ds| \leq Ce^{Re(a_{02})t - \beta t},$$

from which we conclude that

$$|f_0^-(t) \int_{m^+}^t \frac{f_0^+(s) r_0(s)}{W_0(s)} ds| \leq Ce^{-\beta t}.$$

Combining above estimates and (4.4.15), we obtain

$$|f_0(t) - c_{01} f_0^+(t) - c_{02} f_0^-(t)| \leq Ce^{-\beta t}.$$

When $a_{01} = \beta$ or $a_{02} = \beta$, we have

$$|f_0(t) - c_{01} f_0^+(t) - c_{02} f_0^-(t)| \leq Cte^{-\beta t}.$$

□

Remark 4.4.1. In fact, if $|f(t,\theta)| \leq Ce^{-\alpha t}$, $\alpha > Re(a_{N2})$ and $|r(t,\theta)| \leq Ce^{-\beta t}$, $\alpha < \beta < Re(a_{(N+1)2})$, $N \geq 1$, then we still have

$$|f(t,\theta)| \leq Ce^{-\beta t},$$

where

$$a_{N2} = \frac{(ab-1)(n-2k) + \sqrt{(ab-1)^2(n-2k)^2 - 4((2-ab)(n-2k) - \lambda_N)}}{2}.$$

Now we provide a proof of Theorem 1.4.3. The proof is similar to the one for singular Yamabe problem [44] and the other one for singular σ_k Yamabe problem [42], once we have got the needed linear analysis.

Proof of Theorem 1.4.3. Our starting point is

$$L_\varphi w_1 + E(w_1) + \frac{(p-1)k}{a} e^{-\frac{(p-1)k}{a}\varphi} w_1 = 0,$$

and we have

$$|E(w_1)| \leq Ce^{-2\alpha t}, \quad \text{whenever we have } |w_1, \partial w_1, \partial^2 w_2| \leq Ce^{-\alpha t}, \tag{4.4.16}$$

where $w_1(t,\theta) = w(t,\theta) - \varphi(t)$. We only give a proof for $\varphi_t \equiv 0$. When $\varphi_t \not\equiv 0$, it is not hard to obtain (1.4.9) under the assumption $Re(a_{02}) \leq 2Re(a_{01}) - \varepsilon_0$. In Theorem 1.4.2 we have established the following:

Step 1. For some $\alpha_0 > 0$, $|w_1, \partial w_1, \partial^2 w_1| \leq Ce^{-\alpha_0 t}$.

If $\alpha_0 \geq 2Re(a_{01})$, we stop and have proved

$$|w(t,\theta) - \varphi(t)| = |w_1(t,\theta)| \leq Ce^{-2Re(a_{01})t}.$$

If $Re(a_{01}) < \alpha_0 < 2Re(a_{01})$, we jump to Step 3; if $\alpha_0 \leq Re(a_{01})$, we move to

Step 2. Recall that we now have

$$|E(w_1)| \leq Ce^{-2\alpha_0 t}.$$

If $Re(a_{01}) < 2\alpha_0 \leq 2Re(a_{01})$, then we still have

$$|E(w_1)| \leq Ce^{-2\alpha t}$$

for some $Re(a_{01}) < 2\alpha < 2Re(a_{01})$ and can apply Proposition 4.4.1 to imply that

$$|w(t,\theta) - \varphi(t) - f_0(t)| \leq Ce^{-2\alpha t}, \tag{4.4.17}$$

where $f_0 := c_{01}e^{-a_{01}t} + c_{02}e^{-a_{02}t}$ for some constants a_{01} and a_{02}, then jump to Step 3; if $2\alpha_0 \leq Re(a_{01})$, we may take α_0 to satisfy $2\alpha_0 < Re(a_{01})$ and apply Proposition 4.4.1 to imply that

$$|w_1(t,\theta) - f_0(t)| \leq Ce^{-2\alpha_0 t}.$$

This certainly implies that

$$|w_1(t,\theta)| \leq Ce^{-2\alpha_0 t}. \tag{4.4.18}$$

Next we use higher derivative estimates for $w(t,\theta)$ and $\varphi(t)$ and interpolation with (4.4.18) to obtain

$$|w_1, \partial w_1, \partial^2 w_1| \leq Ce^{-2\alpha' t}$$

for any $\alpha' < \alpha_0$. Now we go back to the beginning of Step 2 and repeat the process with a new $\alpha_1 > \alpha_0$ to replace the α_0 there, say, $\alpha_1 = 1.9\alpha_0$. After a finite number of steps, we will reach a stage where $2\alpha > Re(a_{01})$ and ready to move onto

Step 3. At this stage, we have

$$|w_1(t,\theta)| \leq Ce^{-Re(a_{01})t}.$$

Repeating the last part of Step 2 involving the derivative estimates for $w(t,\theta)$ and $\varphi(t)$ to bootstrap the estimates for $E(w_1)$ to

$$|E(w_1)| \leq Ce^{-\alpha t}$$

with α can be as close to $2Re(a_{01})$ as one needs. Then we can apply Proposition 4.4.1 to obtain (1.4.8). We should note that $|E(w_1)| \leq Ce^{-2Re(a_{01})}$ when $k = 1$.

Step 4. Case 1: $k = 1$.

From the previous steps, we see that

$$|w_1(t,\theta) - f_0(t)| \leq Ce^{-\beta_1 t},$$

where $\beta_1 = \min\{2Re(a_{01}), Re(a_{12})\}$. We also have that w_1 satisfies

$$L_\varphi w_1 + \frac{(p-1)}{a} e^{-\frac{(p-1)}{a}\varphi} w_1 = \frac{(p-1)^2}{a^2} e^{-\frac{(p-1)}{a}\varphi} w_1^2 + E_1(w_1),$$

where $E_1(w_1) = O(w_1^3)$.

There exists a constant $m_1 \geq 2$ such that $m_1 Re(a_{01}) \leq Re(a_{12}) < (m_1+1)Re(a_{01})$. Let $w_2(t,\theta) := w_1(t,\theta) - f_0(t)$. It follows that

$$|w_2(t,\theta)| \leq Ce^{-2Re(a_{01})t}$$

and

$$L_\varphi w_2 + \frac{(p-1)}{a} e^{-\frac{(p-1)}{a}\varphi} w_2 = \frac{(p-1)^2}{a^2} e^{-\frac{(p-1)}{a}\varphi} (w_2^2 + 2w_2 f_0 + f_0^2) + E_1(w_2 + f_0).$$

By Proposition 4.3.3, there exists a solution $\varphi_1(t,\theta)$ to $L_\varphi \varphi_1 + \frac{(p-1)}{a} e^{-\frac{(p-1)}{a}\varphi} \varphi_1 = \frac{(p-1)^2}{a^2} e^{-\frac{(p-1)}{a}\varphi} f_0^2$ such that $|\varphi_1(t,\theta)| \leq Ce^{-2Re(a_{01})t}$. Then we obtain that

$$L_\varphi(w_2 - \varphi_1) + \frac{(p-1)}{a} e^{-\frac{(p-1)}{a}\varphi}(w_2 - \varphi_1) = \frac{(p-1)^2}{a^2} e^{-\frac{(p-1)}{a}\varphi}(w_2^2 + 2w_2 f_0) + E_1(w_2 + f_0),$$

where $|\frac{(p-1)^2}{a^2} e^{-\frac{(p-1)}{a}\varphi}(w_2^2 + 2w_2 f_0) + E_1(w_2 + f_0)| \leq Ce^{-3Re(a_{01})t}$. It follows that

$$|w_3(t,\theta) := w_2(t,\theta) - \varphi_1(t,\theta)| \leq Ce^{-3Re(a_{01})t},$$

and

$$L_\varphi w_3 + \frac{(p-1)}{a} e^{-\frac{(p-1)}{a}\varphi} w_3 = E_{21}(w_2, f_0) + E_{22}(w_2, f_0),$$

where $|E_{21}(w_2, f_0)| \leq Ce^{-3Re(a_{01})t}$ and $|E_{22}(w_2, f_0)| \leq Ce^{-4Re(a_{01})t}$. Let $\varphi_2(t,\theta)$ be a solution to

$$L_\varphi \varphi_2 + \frac{(p-1)}{a} e^{-\frac{(p-1)}{a}\varphi} \varphi_2 = E_{21}(w_2, f_0).$$

Then we obtain

$$|\varphi_2(t,\theta)| \leq Ce^{-3Re(a_{01})t}.$$

It follows that

$$|w_3(t,\theta) - \varphi_2(t,\theta)| \leq Ce^{-4Re(a_{01})t}.$$

Similarly, we have

$$|w_1(t,\theta) - f_0(t) - \sum_{j=1}^{m_1-1} \varphi_j(t,\theta) - c_{12} e^{-a_{12}t} Y_1(\theta)| \leq Ce^{-(m_1+1)Re(a_{01})t}.$$

For any $Re(a_{N2})$, there exists a constant m_N such that $m_N Re(a_{01}) \leq Re(a_{N2}) < (m_N+1)Re(a_{01})$. Employing the same procedure as above, we get

$$|w_1(t,\theta) - f_0(t) - \sum_{j=1}^{m_N-1} \varphi_j(t,\theta) - \sum_{j=1}^{N} c_{j2} e^{-a_{j2}t} Y_j(\theta)| \leq Ce^{-(m_N+1)Re(a_{01})t},$$

where $c_{j2},\ j=1,\dots,N$, are constants and

$$a_{j2} = \frac{(ab-1)(n-2) + \sqrt{(ab-1)^2(n-2)^2 - 4((2-ab)(n-2)-\lambda_j)}}{2}.$$

Case 2: $k \geq 2$.

We have that w_1 satisfies

$$L_\varphi w_1 + \frac{(p-1)}{a} e^{-\frac{(p-1)}{a}\varphi} w_1 = E_{11}(w_1, \partial w_1, \partial^2 w_1) + E_{12}(w_1, \partial w_1, \partial^2 w_2),$$

where $|E_{11}(w_1, \partial w_1, \partial^2 w_2)| \leq Ce^{-2(Re(a_{01})-\varepsilon)t}$, $E_{12}(w_1, \partial w_1, \partial^2 w_2) \leq Ce^{-3(Re(a_{01})-\varepsilon)t}$ for any small constant $\varepsilon > 0$.

For any $Re(a_{N2})$, there exists a constant m_N such that $m_N(Re(a_{01}) - \varepsilon) \leq Re(a_{N2}) < (m_N + 1)(Re(a_{01}) - \varepsilon)$. Following the same line of the proof of Case 1, we obtain

$$|w_1(t,\theta) - f_0(t) - \sum_{j=1}^{m_N-1} \varphi_j(t,\theta) - \sum_{j=1}^{N} c_{j2} e^{-a_{j2}t} Y_j(\theta)| \leq Ce^{-(m_N+1)(Re(a_{01})-\varepsilon)t},$$

where $c_{j2},\ j=1,\dots,N$, are constants and

$$a_{j2} = \frac{(ab-1)(n-2k) + \sqrt{(ab-1)^2(n-2k)^2 - 4((2-ab)(n-2k)-\lambda_j)}}{2}.$$

□

Remark 4.4.2. The much higher order expansion of solutions to σ_k Yamabe equation is also available, once we have the same estimate in Lemma 4.3.1 for second order elliptic equations with periodical coefficients. It is not difficult to get the similar estimate by integration by part and some Hardy type inequality.

Chapter 5

Harnack type inequality and Liouville type theorem

In this chapter, we apply the method of moving spheres to obtain Harnack type inequality in Euclidean balls, asymptotic behavior of an entire solution. Based on the asymptotic behavior, we are able to give another proof of the remarkable Liouville type theorem obtained by Li and Li [49]. For the method of moving spheres, there are, roughly speaking, three steps: one is to get started with the procedure, second is to prove that the function and the reflected one coincide if the procedure stops, and the third is to handle the case when the procedure never stops.

5.1 Harnack type inequality for Euclidean balls

In this section, we will establish Harnack type inequality for Euclidean balls. Our proof makes use of ideas in the proof of Theorem 1.5 in [56].

Proof of Theorem 1.4.4. We argue by contradiction. Suppose the contrary, then there would exist solutions u_j to (1.4.10), $j = 1, 2, \ldots$, such that

$$u_j(\bar{x}_j)\left(\min_{\overline{B_{2R_j}(0)}} u_j\right)^{\alpha} > \frac{j}{R_j^{(n-2)\alpha}}, \tag{5.1.1}$$

where $u_j(\bar{x}_j) = \max_{\overline{B_{R_j}(0)}} u_j$.

Consider

$$v_j(x) := (R_j - |x - \bar{x}_j|)^{\frac{2}{p-1}} u_j(x), \quad x \in B_{R_j}(\bar{x}_j).$$

Let $|x_j - \bar{x}_j| < R_j$ satisfy

$$v_j(x_j) = \max_{\overline{B_{R_j}(\bar{x}_j)}} v_j(x),$$

and let

$$\sigma_j = \frac{1}{2}(R_j - |x_j - \bar{x}_j|) \le \frac{R_j}{2}$$

Then we have

$$R_j - |x - \bar{x}_j| \ge \sigma_j, \quad x \in \overline{B_{\sigma_j}(x_j)}.$$

By the definition of v_j, we get

$$(2\sigma_j)^{\frac{2}{p-1}} u_j(x_j) = v_j(x_j) \geq v_j(x) \geq (\sigma_j)^{\frac{2}{p-1}} u_j(x), \quad x \in \overline{B_{\sigma_j}(x_j)}.$$

It follows that

$$2^{\frac{2}{p-1}} u_j(x_j) \geq \max_{B_{\sigma_j}(x_j)} u_j,$$

We also obtain

$$(2\sigma_j)^{\frac{2}{p-1}} u_j(x_j) = v_j(x_j) \geq v_j(\bar{x}_j) = R_j^{\frac{2}{p-1}} u_j(\bar{x}_j),$$

that is,

$$(\sigma_j)^{\frac{2}{p-1}} u_j(x_j) \geq \left(\frac{R_j}{2}\right)^{\frac{2}{p-1}} u_j(\bar{x}_j).$$

It follows that

$$u_j(x_j) \geq u_j(\bar{x}_j), \tag{5.1.2}$$

and from (5.1.1),

$$\gamma_j := u_j(x_j)^{\frac{p-1}{2}} \sigma_j \geq \frac{R_j}{2} (u_j(\bar{x}_j))^{\frac{p-1}{2}} \geq \frac{R_j}{2} [u_j(\bar{x}_j)(\min_{B_{2R_j}(0)} u_j)^{\alpha}]^{\frac{1}{(n-2)\alpha}} \geq \frac{j^{\frac{1}{(n-2)\alpha}}}{2} \to \infty. \tag{5.1.3}$$

Set

$$w_j(y) = \frac{1}{u_j(x_j)} u_j(x_j + \frac{y}{u_j(x_j)^{\frac{p-1}{2}}}), \quad |y| < \Gamma_j,$$

where $\Gamma_j = u_j(x_j)^{\frac{p-1}{2}} R_j$. Then we see that

$$\sigma_k^{1/k}(A^{w_j}) = w_j^{p - \frac{n+2}{n-2}}, \quad w_j > 0, \text{ in } B_{\Gamma_j}(0), \tag{5.1.4}$$

and

$$1 = w_j(0) \geq 2^{\frac{2}{1-p}} \max_{B_{\gamma_j}(0)} w_j. \tag{5.1.5}$$

Since w_j is superharmonic function, we have, by (5.1.1) and (5.1.2),

$$\min_{\partial B_{\Gamma_j}(0)} w_j = \min_{B_{\Gamma_j}(0)} w_j \geq \frac{\min_{\overline{B_{2R_j}(0)}} u_j}{u_j(x_j)} = \frac{u_j(\bar{x}_j)^{1/\alpha} \min_{\overline{B_{2R_j}(0)}} u_j}{u_j(x_j) u_j(\bar{x}_j)^{1/\alpha}} > \frac{j^{1/\alpha}}{\Gamma_j^{n-2}}. \tag{5.1.6}$$

For every $x \in \mathbb{R}^n$ satisfying $|x| < \frac{1}{2}\gamma_j$, we can find, as in [56], $0 < \lambda_{x,j} < 1$ such that for all $0 < \lambda < \lambda_{x,j}$ and $y \in B_{\Gamma_j}(0) \backslash B_\lambda(x)$, we have

$$w_{j,x,\lambda}(y) := (\frac{\lambda}{|y-x|})^{n-2} w_j(x + \frac{\lambda^2 (y-x)}{|y-x|^2}) \leq w_j(y). \tag{5.1.7}$$

Because of (5.1.7), we can define

$$\bar{\lambda}_j(x) = \sup\{0 < \mu < \Gamma_j - |x|,\ w_{j,x,\lambda}(y) \le w_j(y),\ \ y \in B_{\Gamma_j}(0) \backslash B_\lambda(x),\ \ 0 < \lambda < \mu\}.$$

Now we claim that for every $m > 0$,

$$\lim_{j\to\infty} \inf_{|x|\le m} \bar{\lambda}_j(x) = \infty.$$

The proof of this claim is almost the same to the proof of Lemma 5.1 in [48]. For the reader's convenience, we include the proof. For simplicity, we only prove that $\bar{\lambda}_j := \bar{\lambda}_j(0) \to \infty$, since the general case is essentially the same. Suppose the contrary, then (along a subsequence),

$$\bar{\lambda}_j \le C < \gamma_j \tag{5.1.8}$$

for some constant C independent of j. Here we have used the fact that $\gamma_j \to \infty$. By the definition of $\bar{\lambda}_j$,

$$w_j - w_{j,\bar{\lambda}_j} \ge 0 \ \text{ in } \ \Sigma_j := \{y : \ \ \bar{\lambda}_j < |y| < \Gamma_j\}.$$

From (5.1.5), (5.1.6) and (5.1.8), we get

$$\max_{\partial B_{\Gamma_j}(0)} w_{j,\bar{\lambda}_j} \le C\Gamma_j^{2-n} < j^{1/\alpha}\Gamma_j^{2-n} < \min_{\partial B_{\Gamma_j}(0)} w_j,$$

that is,

$$\min_{\partial B_{\Gamma_j}(0)} (w_j - w_{j,\bar{\lambda}_j}) > 0.$$

Recall that

$$w_j - w_{j,\bar{\lambda}_j} = 0 \ \text{ on } \ \partial B_{\bar{\lambda}_j}(0),$$

and

$$\sigma_k^{1/k}(A^{w_{j,\bar{\lambda}_j}}) - \sigma_k^{1/k}(A^{w_j}) - w_{j,\bar{\lambda}_j}^{p-\frac{n+2}{n-2}} + w_j^{p-\frac{n+2}{n-2}} < 0 \ \text{ in } \Sigma_j.$$

An application of the Hopf lemma and the strong maximum principle yields

$$(w_j - w_{j,\bar{\lambda}_j})(y) > 0, \ \ \bar{\lambda}_j < |y| \le \Gamma_j,$$

and

$$\frac{\partial(w_j - w_{j,\bar{\lambda}_j})}{\partial r}\Big|_{\partial B_{\bar{\lambda}_j}(0)} > 0.$$

Then, for some ε_j, we have

$$w_{j,\lambda}(y) \le w_j(y), \ \ \text{for } \ \bar{\lambda}_j \le \lambda \le \bar{\lambda}_j + \varepsilon_j, \ \ \lambda \le |y| \le \Gamma_j.$$

This violates the definition of $\bar{\lambda}_j$.

Since $\gamma_j \to \infty$, $w_j(0) = 1$ and $\{w_j\}$ is bounded on any compact subset of $\mathbb{R}^n$, we have that (along a subsequence),

$$w_j \to w \ \text{ in } C^2_{loc}(\mathbb{R}^n)$$

for some positive solution to

$$\sigma_k^{1/k}(A^w) = w^{p-\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n, \quad w > 0. \tag{5.1.9}$$

By above claim and the convergence of w_j to w, we obtain

$$w_{x,\lambda}(y) \le w(y), \quad \forall\, |y-x| \ge \lambda > 0.$$

It follows that $w \equiv$ *constant*, which violates (5.1.9). Theorem 1.4.4 is established. $\square$

The Harnack-type inequality yields the following consequence as established by Schoen in [76] for $k = 1$ and $p = \frac{n+2}{n-2}$. See also an alternative proof of Chen and Lin [25], which we adapt for our case.

Proof of Corollary 1.4.5. By scaling, we only need prove for the case $R = 1$. Let $u(y) = \min_{\overline{B_2(0)}} u$ for some $y \in B_2(0)$, and let $G(x, y)$ be the Green's function of $-\Delta$ in $B_{\frac{5}{2}}(0)$. Then, from

$$-\Delta u \ge u^p, \tag{5.1.10}$$

we see

$$\begin{aligned} u(y) &= \int_{B_{5/2}(0)} G(y,z)(-\Delta u(z))dz - \int_{\partial B_{5/2}(0)} \frac{\partial G(y,s)}{\partial \nu} u(s)ds \\ &\ge \int_{B_{5/2}(0)} G(y,z)u(z)^p \ge \frac{1}{C}\int_{B_1(0)} u^p. \end{aligned}$$

By Theorem 1.4.4 and the above inequality, we have

$$\int_{B_1(0)} u^{\frac{n(p-1)}{2}} = \int_{B_1(0)} u^{\frac{1}{\alpha}+p} \le (\max_{\bar{B}_1(0)} u)^{\frac{1}{\alpha}} \int_{B_1(0)} u^p \le C(\max_{\bar{B}_1(0)} u)^{\frac{1}{\alpha}} (\min_{\bar{B}_2(0)} u) \le C.$$

$\square$

5.2 Asymptotics for entire solutions

A consequence of local estimates in [36] is the following lemma:

Lemma 5.2.1. *Let $u \in C^2(B_3(0))$ be a positive solution to (1.4.10), and*

$$\int_{B_3(0)} u^{\frac{n(p-1)}{2}} \le \delta_0.$$

Then

$$\sup_{B_2(0)} u \le \frac{1}{\delta_0}.$$

Proof. Our proof is an adaption of the proof of Proposition 2.1 in [50]. Suppose the contrary, there exists a sequence of solutions $\{u_j\}$ such that

$$\int_{B_3(0)} u_j^{\frac{n(p-1)}{2}} \to 0$$

and

$$d(y_j)^{\frac{2}{p-1}} u_j(u_j) = \max_{y \in B_{2.8}(0)} d(y)^{\frac{2}{p-1}} u_j(y) \to \infty,$$

where $d(y) := \operatorname{dist}(y, \partial B_{2.8}(0)) = (2.8 - |y|)$. Let $\sigma_j = \frac{1}{2} d(y_j) > 0$, and set

$$v_j(z) = \frac{1}{u_j(y_j)} u_j(y_j + \frac{z}{u_j(y_j)^{\frac{p-1}{2}}}),$$

where $|z| \le r_j := u_j(y_j)^{\frac{p-1}{2}} \sigma_j \to \infty$. Then we obtain

$$\sigma_k^{1/k}(A^{v_j}) = v_j^{p - \frac{n+2}{n-2}} \text{ in } B_{r_j}(0)$$

and

$$\int_{B_{r_j}(0)} v_j^{\frac{n(p-1)}{2}} \to 0. \tag{5.2.1}$$

It is easy to see that $v_j(0) = 1$ and $\sup_{B_{r_j}(0)} v_j \le 2^{\frac{2}{p-1}}$. Therefore, from the estimates in [36], v_j converges uniformly in B_1 along a subsequence, violating (5.2.1). □

Now we will show the asymptotic behavior of an entire solution. Our proof follows the line of the proof of Theorem 1.28 in [48].

Proof of Theorem 1.4.6. By the maximum principle, using the positivity and the superharmonicity of u, we have

$$u(x) \ge \frac{\min_{\partial B_1(0)} u}{|x|^{n-2}} \quad \forall |x| \ge 1.$$

By Theorem 1.4.4 and the above inequality, for some positive universal constant C, we see

$$R^{2-n} \min_{\partial B_1(0)} u \le \min_{\partial B_R(0)} u \le C u(0)^{-\frac{1}{\alpha}} R^{2-n}. \tag{5.2.2}$$

For $R > 1$, let

$$u_R(x) := R^{\frac{2}{p-1}} u(Rx), \quad 1 \le |x| \le 9.$$

Then we get

$$\sigma_k^{1/k}(A^{u_R}) = u_R^{p - \frac{n+2}{n-2}}, \quad 1 \le |x| \le 9.$$

By above lemma,

$$\max_{|x|=4} \int_{B_3(x)} (u_R)^{\frac{n(p-1)}{2}} \le \int_{|y| \ge R} u^{\frac{n(p-1)}{2}} \to 0 \text{ as } R \to \infty.$$

Fix some positive number $R_0 \geq 1$ such that

$$\max_{|x|=4} \int_{B_3(x)} (u_R)^{\frac{n(p-1)}{2}} \leq \int_{|y|\geq R_0} u^{\frac{n(p-1)}{2}} \leq \delta_0, \ \ \forall\, R \geq R_0.$$

Thus, from the above lemma, we see

$$\max_{\bar{B}_2(x)} u_R \leq \frac{1}{\delta_0}, \ \ \forall\, |x| = 4.$$

By Harnack inequality,

$$\max_{\bar{B}_1(x)} u_R \leq C \min_{\bar{B}_1(x)} u_R, \ \ \forall\, |x| = 4.$$

It follows from (5.2.2) and the above inequality that

$$C^{-1}(\min_{\partial B_1(0)} u) R^{-\frac{2}{\alpha(p-1)}} \leq \min_{\bar{B}_{1/2}(x)} u_R \leq \max_{\bar{B}_{1/2}(x)} u_R \leq C u(0)^{-\frac{1}{\alpha}} R^{-\frac{2}{\alpha(p-1)}} \ \ \forall\, |x| = 4.$$

For $R \geq \max\{R_0, u(0)^{-\frac{p-1}{2}}\}$,

$$\sigma_k^{1/k}(A^{R^{\frac{2}{\alpha(p-1)}} u_R}) = R^{-((n-2)p-n)} (R^{\frac{2}{\alpha(p-1)}} u_R)^{p-\frac{n+2}{n-2}}.$$

Now we have

$$C^{-1} \leq R^{\frac{2}{\alpha(p-1)}} u_R \leq C \ \text{ in } \ B_{\frac{1}{2}}(x).$$

Applying local estimates in [36] to $R^{\frac{2}{\alpha(p-1)}} u_R$, we see

$$\sum_{i=0}^{2} |\nabla^i (R^{\frac{2}{\alpha(p-1)}} u_R)(x)| \leq C, \ \ \forall\, |x| \geq R_0,$$

where C is some positive constant independent of R. Therefore we have

$$0 < \liminf_{|x|\to\infty} (|x|^{n-2} u(x)) \leq \limsup_{|x|\to\infty} (|x|^{n-2} u(x)) < \infty$$

and

$$\limsup_{|x|\to\infty} (|x|^{n-1} |\nabla u(x)| + |x|^n |\nabla^2 u(x)|) < \infty.$$

Integrating equation $-\Delta u \geq u^p$ on $B_r(0)$ $(r > 1)$ leads to

$$\int_{B_r(0)} u^p dx \leq |\int_{\partial B_r(0)} \frac{\partial u}{\partial \nu}| \leq (\max_{|x|=r} |\nabla u(x)|)(|\partial B_r(0)|).$$

From the above inequality, we see

$$\int_{B_R(0)} u^p \leq C, \ \ \forall\, R > 0.$$

□

5.3 A Liouville type theorem

In this section, we will give another proof of the Liouville type theorem obtained by Li and Li [49].

First, we need a lemma:

Lemma 5.3.1. *If u is a positive solution to $\sigma_k^{1/k}(A^u) = u^{p-\frac{n+2}{n-2}}$ in $\mathbb{R}^n$ in the Γ_k^+ class and $\lim_{x\to\infty}|x|^{n-2}u(x) \le C_1$, where $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, then*

$$\max_{\bar{B}_R(0)} u \le c_0 \min_{\bar{B}_R(0)} u,$$

where c_0 is a positive constant independent of R.

Proof. Let

$$v(x) = R^{\frac{2}{p-1}}u(Rx), \quad x \in B_1(0).$$

We see that v is a solution to $\sigma_k^{1/k}(A^v) = v^{(p-\frac{n+2}{n-2})}$ in $B_1(0)$. Since $\lim_{x\to\infty}|x|^{n-2}u(x) \le C_1$, there exists a constant $M > 1$ such that

$$u(x) \le \frac{C_1}{|x|^{n-2}}, \quad \forall\, |x| \ge M.$$

It follows that

$$v(x) = R^{\frac{2}{p-1}}u(Rx) \le C_2 := \max\{M^{\frac{2}{p-1}} \max_{B_M(0)} u, C_1\}.$$

By Harnack inequality in [36], we see that

$$\max_{\bar{B}_{1/2}(0)} v \le c_0 \min_{\bar{B}_{1/2}(0)} v,$$

where c_0 is independent of R. Then we get

$$\max_{\bar{B}_{R/2}(0)} u \le c_0 \min_{\bar{B}_{R/2}(0)} u.$$

□

Proof of Corollary 1.4.8. We will prove this corollary by the method of moving spheres.

For $x \in \mathbb{R}^n$ and $\lambda > 0$, consider the Kelvin transformation of u:

$$u_{x,\lambda}(y) = \frac{\lambda^{n-2}}{|y-x|^{n-2}}u(x + \frac{\lambda^2(y-x)}{|y-x|^2}), \quad y \in \mathbb{R}^n\backslash\{x\}.$$

We will show that

$$u_{x,\lambda}(y) \le u(y),\ \forall \lambda > 0, \quad x \in \mathbb{R}^n, \quad |y-x| \ge \lambda.$$

Then by a calculus lemma [56], we get the conclusion

$$u \equiv constant.$$

Without loss of generality, take $x = 0$, and use u_λ to denote $u_{0,\lambda}$. From Lemma 2.1 in [56], there exists $\lambda_0 > 0$ such that

$$u_\lambda(y) \le u(y), |y| \ge \lambda, \quad \forall \lambda \in (0, \lambda_0].$$

Then we define

$$\bar{\lambda} := \sup\{\mu > 0 | u_\lambda \le u \text{ in } \mathbb{R}^n \backslash B_\lambda(0), \quad \forall 0 < \lambda < \mu\}.$$

We claim that $\bar{\lambda} = \infty$. We prove it by contradiction. Suppose $\bar{\lambda} < \infty$, we will get that there exists $\varepsilon_0 \in (0, 1)$ such that

$$u_\lambda(y) \le u(y), \quad |y| \ge \lambda, \quad \forall \lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon_0].$$

This is a contradiction to the definition of $\bar{\lambda}$. By the maximum principle, we see that

$$u_{\bar{\lambda}}(y) < u(y), \quad |y| > \bar{\lambda}$$

In particular, we have

$$u_{\bar{\lambda}}(y) < u(y), \quad R_0 \ge |y| > \bar{\lambda},$$

where $R_0 = c_0^{\frac{1}{n-2}}(\bar{\lambda} + 1)$ (c_0 is the same one in the above lemma). By the uniform continuity of u, there exists a small constant $\varepsilon_0 \in (0, 1)$ such that

$$u_\lambda(y) \le u(y), \quad R_0 \ge |y| > \lambda, \ \lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon_0]. \tag{5.3.1}$$

By the superharmonicity of u, we see that

$$u(y) \ge \frac{R_0^{n-2}}{|y|^{n-2}} \min_{\partial B_{R_0}(0)} u, \quad |y| \ge R_0,$$

Meanwhile we have

$$u_\lambda(y) = \frac{\lambda^{n-2}}{|y|^{n-2}} u(\frac{\lambda^2 y}{|y|^2}) \le \frac{(\bar{\lambda} + 1)^{n-2}}{|y|^{n-2}} \max_{\bar{B}_{R_0}(0)} u \le \frac{(\bar{\lambda} + 1)^{n-2}}{|y|^{n-2}} c_0 \min_{\bar{B}_{R_0}(0)} u \le \frac{R_0^{n-2}}{|y|^{n-2}} \min_{\partial B_{R_0}(0)} u$$

for $|y| \ge R_0$ and $\lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon_0]$. Combining the above inequalities, we see

$$u_\lambda(y) \le u(y), \quad |y| \ge R_0, \quad \lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon_0]. \tag{5.3.2}$$

From (5.3.1) and (5.3.2), we get

$$u_\lambda(y) \le u(y), \quad |y| \ge \lambda, \quad \lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon_0].$$

□

REFERENCES

[1] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large I*, Vestn. Leningr. Univ. **11** (1956), 5-17. English transl. in Am. Math. Soc. Transl. **21** (1962), 341-354.

[2] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large II*, Vestn. Leningr. Univ. **12** (1957), 15-44. English transl. in Am. Math. Soc. Transl. **21** (1962), 354-388.

[3] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large III*, Vestn. Leningr. Univ. **13** (1958), 14-26. English transl. in Am. Math. Soc. Transl. **21** (1962), 389-403.

[4] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large IV*, Vestn. Leningr. Univ. **13** (1958), 27-34. English transl. in Am. Math. Soc. Transl. **21** (1962), 403-411.

[5] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large V*, Vestn. Leningr. Univ. **13** (1958), 5-8. English transl. in Am. Math. Soc. Transl. **21** (1962), 412-416.

[6] T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*. J. Math. Pures Appl. (9) **55** (1976), 269-296.

[7] P. Aviles, *A study of the singularities of solutions of a class of nonlinear elliptic partial differential equations*. Commun. Partial Differ. Equ. **7** (1982), 609-643.

[8] P. Aviles, *Local behavior of solutions of some elliptic equations*. Comm. Math. Phys. **108** (1987), 177-192.

[9] A. Bahri, "Proof of the Yamabe conjecture without the positive mass conjecture for locally conformally flat manifolds" in *Nonlinear Variational Problems and Partial Differential Equations (Isola d'Elba, 1990)*, Pitman Res. Notes Math. Ser. 320, Longman Sci. Tech., Harlow, 1995, 13-43.

[10] A. Bahri and H. Brezis, "Non-linear elliptic equations on Riemannian manifolds with the Sobolev critical exponent" in *Topics in Geometry*. Progr. Nonlinear Differential Equations Appl. **20**, Birkhäuser, Boston, 1996, 1-100.

[11] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **34** (1986), 661-693.

[12] A. L. Besse, *Einstein Manifolds*. Springer-Verlag, Berlin, 1987

[13] T. Branson and A. Gover, *Variational status of a class of fully nonlinear curvature prescription problems*. Calc. Var. Part. Diff. Eq. **32** (2008), 253-262.

[14] S. Brendle, *Blow-up phenomena for the Yamabe equation I*. J. Amer. Math. Soc. **21** (2008), 951-979.

[15] S. Brendle and F. C. Marques, *Blow-up phenomena for the Yamabe equation II*. J. Differ. Geom. **81** (2009), 225-250.

[16] S. Brendle and J. Viaclovsky, *A variational characterization for $\sigma_{n/2}$*. Calc. Var. Partial Differential Equations **20** (2004), 399-402.

[17] L. A. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*. Commun. Pure Appl. Math. **42** (1989), 271-297.

[18] M. Cantor, *Elliptic operators and the decomposition of tensor fields*, Bull. Amer. Math. Soc. (N.S.) **5** (1981), 235-262.

[19] G. Catino and L. Mazzieri, *Connected sum construction for σ_k-Yamabe metrics*. J. Geom. Anal. **23** (2013), 812-854.

[20] S.-Y. A. Chang, M. Gursky and P. C. Yang, *An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature*. Ann. of Math. (2) **155** (2002), 709-787.

[21] S.-Y. A. Chang, M. Gursky and P. C. Yang, *An a prior estimate for a fully nonlinear equation on four-manifolds*. J. D'Analyse Math. **87** (2002), 151-186.

[22] S.-Y. A. Chang, M. Gursky and P. C. Yang, *Entire solutions of a fully nonlinear equation*, in Lectures on Partial Differential Equations, pp. 43-60. New Stud. Adv. Math., 2. Int. Press, Sommerville, MA, 2003.

[23] S.-Y. A. Chang, Z. Han and P. C. Yang, *Classification of singular radial solutions to the σ_k Yamabe equation on annular domains*. J. Differ. Equ. **216** (2005), 482-501.

[24] S.-Y. A. Chang, F. Hang and P. C. Yang, *On a class of locally conformally flat manifolds*. Int. Math. Res. Not. **3** (2004), 185-209.

[25] C. C. Chen and C. S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes*. Comm. Pure Appl. Math. **50** (1997), 971-1017.

[26] W. X. Chen and C. Li, *Classification of solutions of some nonlinear elliptic equations*. Duke Math. J., **63** (1991), 615-622.

[27] G. C. Evans, *Potentials and positively infinite singularities of harmonic functions*. Monatsh. Math. **43** (1936) 419-424.

[28] D. Finn, *Positive solutions to nonlinear elliptic equations with prescribed singularities*. Ph.D. thesis, Northeastern University, 1995.

[29] Y. Ge and G. Wang, *On a fully nonlinear Yamabe problem*. Ann. Sci. École Norm. Sup. (4) **39** (2006), 569-598.

[30] B. Gidas, W. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*. Commun. Math. Phys. **68** (1979), 209 – 243.

[31] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations. Comm. Pure Appl. Math. **34** (1981), 525-598.

[32] M. González, *Singular sets of a class of locally conformally flat manifolds*. Duke Math. J. **129** (2005), 551-572.

[33] M. González, *Removability of singularities for a class of fully non-linear elliptic equations*. Calc. Var. Partial Differential Equations. **27** (2006), 439-466.

[34] M. González, *Classification of singularities for a subcritical fully nonlinear problem*. Pacific J. Math. **226** (2006), 83-102.

[35] P. Guan, C. S. Lin and G. Wang, *Schouten tensor and some topological properties*. Commun. Anal. Geom. **13** (2005), 887-902.

[36] P. Guan and G. Wang, *Local estimates for a class of fully nonlinear equations arising from conformal geometry*. Int. Math. Res. Not. 2003, 1413-1432.

[37] P. Guan and G. Wang, *A fully nonlinear conformal flow on locally conformally flat manifolds*. J. Reine Angew. Math. **557** (2003) 219-238.

[38] P. Guan and G. Wang, *Geometric inequalities on locally conformally flat manifolds*. Duke Math. J. **124** (2004), 177-212.

[39] P. Guan, J. Viaclovsky and G. Wang, *Some properties of the Schouten tensor and applications to conformal geometry*. Trans. Amer. Math. Soc. **355** (2003), 925-933.

[40] M. Gursky and J. Viaclovsky, *Volume comparison and the σ_k-Yamabe problem*. Adv. Math. **187** (2004), 447-487.

[41] M. Gursky and J. Viaclovsky, *Prescribing symmetric functions of the eigenvalues of the Ricci tensor*. Ann. of Math. (2) **166** (2007), 475-531.

[42] Z. Han, Y. Y. Li and E. Teixeira, *Asymptotic behavior of solutions to the σ_k-Yamabe equation near isolated singularities*. Invent math. **182** (2010), 635-684.

[43] P. Hartman, *Ordinary differential equations*. Reprint of the second edition. Birkhuser, Boston, Mass., 1982.

[44] N. Korevaar, R. Mazzeo, F. Pacard and R. Schoen, *Refined asymptotics for constant scalar curvature metrics with isolated singularities*. Invent. Math. **135** (1999), 233-272.

[45] M. Khuri, F. C. Marques and R. Schoen, *A compactness theorem for the Yamabe problem*. J. Differ., Geom. **81** (2009) 143-196.

[46] D. Labutin, *Wiener regularity for large solutions of nonlinear equations*. Ark. Mat. **41** (2003), 307-339.

[47] J. Lee and T. Parker, *The Yamabe problem*. Bull. Amer. Math. Soc. (N.S.) **17** (1987), 37-91.

[48] A. Li and Y. Y. Li, *On some conformally invariant fully nonlinear equations*. Comm. Pure Appl. Math. **56** (2003), 1414-1464.

[49] A. Li and Y. Y. Li, *On some conformally invariant fully nonlinear equations, Part II: Liouville, Harnack and Yamabe*. Acta Math. **195** (2005), 117-154.

[50] Y. Y. Li, *On* $-\Delta u = K(x)u^5$ *in* $\mathbb{R}^3$. Comm. Pure Appl. Math. **46** (1993), 303-340.

[51] Y. Y. Li, *Conformally invariant fully nonlinear elliptic equations and isolated singularities*. Journal of Functional Analysis **233** (2006), 380-425.

[52] Y. Y. Li, *Local gradient estimates of solutions to some conformally invariant fully nonlinear equations*. Comm. Pure Appl. Math., **62** (2009), 1293-1326.

[53] Y. Y. Li and L. Nguyen, *A compactness theorem for a fully nonlinear Yamabe problem under a lower Ricci curvature bound*. Journal of Functional Analysis **266** (2014), 3741-3771.

[54] Y. Y. Li and L. Nguyen, *Harnack inequalities and Bôcher-type theorems for conformally invariant fully nonlinear degenerate elliptic equations*. Comm. Pure Appl. Math. **67** (2014), 1843-1876.

[55] Y. Y. Li and L. Nguyen, *Symmetry, quantitative Liouville theorems and analysis of large solutions of conformally invariant fully nonlinear elliptic equations*. arXiv:1604.06039v1. To appear in Calc. Var. Partial Differential Equations.

[56] Y. Y. Li and L. Zhang, *Liouville type theorems and Harnack type inequalities for semilinear elliptic equations*. Journal d'Analyse Mathematique, **90** (2003), 27-87.

[57] Y. Y. Li and L. Zhang, *Compactness of solutions to the Yamabe problem. II*. Calc. Var. Partial Differential Equations **24** (2005), 185-237.

[58] Y. Y. Li and L. Zhang, *Compactness of solutions to the Yamabe problem. III*. Journal of Functional Analysis **245** (2007), 478-474.

[59] C. Loewner and L. Nirenberg, *Partial differential equations invariant under conformal and projective transformation*. In: Contributions to Analysis. Academic Press, New York, 1995, 245-275.

[60] R. Lockhart and R. McOwen, *Elliptic differential operators on noncompact manifolds*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), 409-447.

[61] M. Marcus and L. Veron, *Boundary trace of positive solutions of supercritical semilinear elliptic equations in dihedral domains*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **15** (2016), 501-542.

[62] F. C. Marques, A priori estimates for the Yamabe problem in the nonlocally conformally flat cases. J. Differ. Geom. **71** (2005), 315-346.

[63] R. Mazzeo and F. Pacard, *A construction of singular solutions for a semilinear elliptic equation using asymptotic analysis*. J. Differ. Geom. **44** (1996), 331-370.

[64] R. Mazzeo and F. Pacard, *Constant scalar curvature metrics with isolated singularities*. Duke Math. J. **99** (1999), 353-418.

[65] L. Mazzieri and C. Ndiaye, *Existence of solutions for the singular σ_k Yamabe problem*. Preprint.

[66] L. Mazzieri and A. Segatti, *Constant σ_k-curvature metrics with Delaunay type ends*. Adv. Math. **229** (2012), 3147-3191,

[67] R. McOwen, “Singularities and the conformal scalar curvature equation”. In: *Geometric Analysis and Nonlinear Partial Differential Equations, Denton, TX, 1990*. Lecture Notes in Pure and Appl. Math., 144, Dekker, New York, 1993, 221-233.

[68] M. Obata, *The conjectures on conformal transformations of Riemannian manifolds*. J. Differential Geom., **6** (1971), 247-258.

[69] F. Pacard, *The Yamabe problem on subdomains of even-dimensional spheres*. Topol. Methods Nonlinear Anal. **6** (1995), 137-150.

[70] N. Roidos and E. Schrohe, *Bounded imaginary powers of cone differential operators on higher order Mellin-Sobolev spaces and applications to the Cahn-Hilliard equation*. J. Differential Equations **257** (2014), 611-637.

[71] N. Roidos and E. Schrohe, *Existence and maximal L^p-regularity of solutions for the porous medium equation on manifolds with conical singularities*. Comm. Partial Differential Equations **41** (2016), 1441-1471.

[72] N. Roidos and E. Schrohe, *Smoothness and long time existence for solutions of the porous medium equation on manifolds with conical singularities*. arXiv:1708.07542.

[73] J. Serrin, *A symmetry problem in potential theory*. Arch. Ration. Mech. Anal. **43** (1971), 304-318.

[74] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*. J. Differential Geom. **20** (1984), 479-495.

[75] R. Schoen, *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation*. Commun. Pure Appl. Math. **41** (1988), 317-392.

[76] R. Schoen, Courses at Stanford University, 1988, and New York University, 1989.

[77] R. Schoen, ”On the number of constant scalar curvature metrics in a conformal class” in *Differential Geometry*, Pitman Monogr. Surveys Pure Appl. Math. 52, Longman Sci. Tech., Harlow, 1991, 311-320.

[78] R. Schoen and S. T. Yau, *On the proof of the positive mass conjecture in general relativity*. Comm. Math. Phys. **65** (1979), 45-76.

[79] R. Schoen and S. T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*. Invent. Math. **92** (1988), 47-71.

[80] S. D. Taliaferro and L. Zhang, *Asymptotic symmetries for conformal scalar curvature equations with singularity*. Calc. Var. Partial Differ. Equ. **26** (2006), 401-428.

[81] W. Sheng, N. Trudinger and X. J. Wang, *The Yamabe problem for higher order curvatures*. J. Differ. Geom. **77** (2007), 515-553.

[82] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*. Ann. Scuola Norm. Sup. Pisa **22** (1968), 265-274.

[83] J. Viaclovsky, *Conformal geometry, contact geometry, and the calculus of variations*. Duke Math. J. **101** (2000), 283-316.

[84] J. Viaclovsky, *Conformally invariant Monge-Ampère equations: global solutions*. Trans. Amer. Math. Soc., **352** (2000), 4371-4379.

[85] L. Veron, *Singularités éliminables d'équations elliptiques non linéaires*. J. Differ. Equ. **41** (1981), 225-242.

[86] Y. P. Wang, *Asymptotic behavior of solutions to the conformal quotient equation*. Thesis (Ph.D.) – Rutgers The State University of New Jersey - New Brunswick. 2013. 78 pp.

[87] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*. Osaka Math. J. **12** (1960), 21-37.

[88] W. Zhang and J. Bao, *Regularity of very weak solutions for nonhomogeneous elliptic equation*. Commun. Contemp. Math. **15** (2013), 19 pp.

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