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**Research Article**

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# Toric extremal Kähler-Ricci solitons are Kähler-Einstein

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**Abstract:** In this short note, we prove that a Calabi extremal Kähler-Ricci soliton on a compact toric Kähler manifold is Einstein. This settles for the class of toric manifolds a general problem stated by the authors that they solved only under some curvature assumptions.

**Keywords:** Extremal Kähler metrics, Kähler-Ricci solitons, Einstein manifolds, Toric manifolds

**MSC:** 53C25, 53C55, 58D19

## Introduction

Let $M^{2n}$ be a compact Kähler manifold and let $\Omega \in H^{1,1}(M)$ be a Kähler class. In the attempt to identify "special" representatives of $\Omega$, several notions of "canonical" Kähler metrics have been introduced. A natural choice are of course Kähler-Einstein metrics, generalized by *extremal* metrics and *Kähler-Ricci solitons (KRS)*. Extremal metrics are defined to be critical points of the *Calabi functional*

$$\omega \mapsto \int_M s_\omega^2 \omega^n$$

that maps the Kähler metric $\omega$ to the $L^2$-norm of its scalar curvature. The Euler-Lagrange equation of the Calabi functional is

$$\mathrm{grad}_\omega(s_\omega) \text{ is holomorphic.} \tag{1}$$

Kähler-Ricci solitons are Kähler metrics that satisfy the relation

$$\rho + c\omega = \mathcal{L}_X \omega \tag{2}$$

with their Ricci form $\rho$, for some vector field $X$ that is holomorphic and, in the compact case, is the gradient of a smooth function $f\colon M \to \mathbb{R}$. The KRS equation forces $\omega$ to lie in the class $2\pi c_1(M)$.

In [2] we addressed the problem whether the same $\omega \in 2\pi c_1(M)$ can be extremal and a KRS without being Einstein and we proved the following.

**Theorem 1** ([2]). *A compact extremal KRS with positive holomorphic sectional curvature is Kähler-Einstein.*

Toric manifolds are compact Kähler $2n$-manifolds admitting an effective Hamiltonian action of an $n$-torus $\mathbb{T}$ by Kähler automorphism. Although in an algebraic geometric context, Fulton calls them a "remarkably

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fertile testing ground for general theories" and, also from the Kähler geometric point of view, their richness of symmetries makes them a large park of examples.

As compact symplectic manifolds, they are characterized by the image of their moment map, that is a *Delzant polytope*, i.e. a convex polytope $\Delta \subset \mathbb{R}^n$ with certain combinatoric properties. Given a compact symplectic toric manifold with moment image $\Delta$, all possible compatible complex structure are described by a single function, as we explain below.

The $\mathbb{T}$-invariant Kähler geometry on a dense subset is well described in the coordinates given by the moment map itself. In these coordinates, the extremal condition (1) has a particularly simple description, see e.g. [1].

Separately, it is known that every toric Fano manifolds admits a KRS, see e.g. [4] and references therein where, in addition, Donaldson explains also the relation between the soliton field $X$ and the Delzant polytope. The existence of extremal metrics in the toric setting is discussed in [1].

The purpose of this note is to prove the following result.

**Theorem 2.** *A compact toric Calabi-extremal Kähler-Ricci soliton is Kähler-Einstein.*

This solves the problem stated in [2] for the class of toric Kähler metrics, that can have holomorphic sectional curvature of any sign and so are not included in Theorem 1.

The proof of Theorem 2 is based on the combinatoric properties of Delzant polytopes and the boundary behavior of the Abreu potential. The problem in its full generality remains open.

**Problem.** *Prove that every extremal Kähler-Ricci soliton is Einstein or find a counterexample.*

Another class of manifold related to toric Kähler manifolds is given by toric bundles, where the existence of KRS has been studied in [5]. It would be interesting to apply the techniques of toric geometry from [1, 4] to study the existence of extremal or constant scalar curvature Kähler metrics in this class of manifolds and establish an analogue of Theorem 2.

# 1 Proof of Theorem 2

Let $(M, g, \omega)$ be a toric Kähler manifold, with moment map $\mu\colon M \to \Delta = \mu(M) \subset \mathbb{R}^n$. The moment image can be written as

$$\Delta = \{x \in \mathbb{R}^n : \ell_k(x) \geq b_k,\ 1 \leq k \leq d\} \tag{3}$$

as intersection of the $d$ half-spaces $\{x \in \mathbb{R}^n : \ell_k(x) - b_k \geq 0\}$.

The linear functions $\ell_k$ are defined by $\ell_k(x) = \langle u_k, x\rangle$, where $u_k$ is the normal to the *facet* $\{\ell_k(x) = 0\} \cap \Delta$ and $b_k \in \mathbb{R}$. The combinatoric property of being Delzant implies the following.

**Lemma 1.1.** *Let $\Delta$ be a Delzant polytope in $\mathbb{R}^n$. Then the vertices of $\Delta$ cannot lie on any affine hyperplane.*

*Proof.* Let $P$ be a vertex of $\Delta$. By definition of Delzant polytope, the exactly $n$ edges meeting at $P$ are of the form $tv_i$ for $t \in [0, a_i]$ and the $v_i$ can be taken to be a basis of $\mathbb{Z}^n$. Further $n$ vertices are of the form $P_i = a_i v_i$ and they cannot lie on the same affine hyperplane of $\mathbb{R}^n$ as the $v_i$ are linearly independent over $\mathbb{R}$. □

Given a compact toric symplectic manifold $(M, \omega)$ with Delzant polytope $\Delta$, consider the dense subset

$$M^0 = \{p \in M : \text{ the } \mathbb{T}\text{-action is free at } p\} \simeq \Delta^0 \times \mathbb{T},$$

where $\Delta^0$ is the interior of $\Delta$ and $(x, y) \in \Delta^0 \times \mathbb{T}$ are the *symplectic coordinates*. In these coordinates, the $\mathbb{T}$-action is just the group multiplication on the second component. In particular, $\mathbb{T}$-invariant tensor fields on $M^0$ depend only on $x \in \Delta^0$.

All $\mathbb{T}$-invariant complex structures compatible with $\omega$ are determined by the *Abreu potential*, a function $g\colon \Delta^0 \to \mathbb{R}$ given by

$$2g(x) = \sum \ell_k(x) \log \ell_k(x) + h(x), \tag{4}$$

on the interior of $\Delta$, where the $\ell_k$ are from (3) and $h$ is a smooth function on $\Delta$, see [1].

In the $(x, y)$-coordinates, the symplectic form is the canonical $\omega = dx_i \wedge dy_i$ and the Kähler metric corresponding to $g$ as in (4) is $g_{ij}(x)dx_i \cdot dx_j + g^{ij}(x)dy_i \cdot dy_j$, where $G = (g_{ij})$ is the (Euclidean) Hessian of $g$. The matrix $G$ has to be singular on the boundary of $\Delta$ in order for the metric to extend smoothly on the whole $M$. However, it is possible to describe the behavior of $G$ on the vertices of $\Delta$.

**Lemma 1.2.** *The inverse of the Hessian matrix $G$ vanishes at the vertices of $\Delta$.*

*Proof.* Without loss of generality, up to translations and to a transformation of $SL(n, \mathbb{Z})$, we can assume that 0 is a vertex and that the edges meeting there are the coordinate axes $x_1, \dots, x_n$.

The transformed polytope is then given by

$$\Delta = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^n : x_i \geq 0\} \cap \bigcap_{i=n+1}^{d} \{x \in \mathbb{R}^n : \ell_k(x) \geq 0\}$$

and the linear functions $\ell_k$ do not vanish at zero.

The Abreu potential $g$ is given by

$$2g(x) = \sum_{i=1}^{n} x_i \log x_i + \underbrace{\sum_{i=n+1}^{d} \ell_i(x) \log \ell_i(x) + h(x)}_{=:\tilde{h}(x)}$$

and its Hessian matrix is

$$G_{ij} = \frac{\delta_{ij}}{x_j} + \tilde{h}_{,ij}(x) \tag{5}$$

where the function $\tilde{h}_{,ij}$ is given by

$$\tilde{h}_{,ij} = \sum_{k=n+1}^{d} \frac{\ell_{k,i}(x)\ell_{k,j}(x)}{\ell_k(x)} + h_{,ij}. \tag{6}$$

From [1, Thm. 2.8], the determinant of $G$ is given by

$$\frac{1}{\det G} = \delta(x) x_1 \cdots x_n \cdot \ell_{n+1}(x) \cdots \ell_d(x)$$

for some function $\delta$ *strictly positive and smooth on the whole $\Delta$.*

The entry $g^{ij}$ of $G^{-1}$ is given by

$$g^{ij}(x) = \frac{1}{\det G} \operatorname{cof}(G)_{ij}$$

where $\operatorname{cof}(G)$ is the cofactor matrix of $G$.

The conclusion follows from the claim that

$$\operatorname{cof}(G)_{ij} = o\left(\frac{1}{x_1 \cdots x_n}\right).$$

From (5) one can see that, after eliminating the $i$-th row and the $j$-th column, the variables $x_i$ and $x_j$ can appear at the denominator only in the derivatives of $\tilde{h}$, but from (6) we see that their limit for $x \to 0$ is finite, so the claim is true. □

Abreu's characterization (see [1] and references therein) of toric extremal metrics relies on the fact that a $\mathbb{T}$-invariant function $f$ has a holomorphic gradient if, and only if, it is an affine function in the symplectic coordinates. This follows from an explicit computation in complex coordinates of the $(1, 0)$-part of the gradient of $f$ and the relation between the symplectic coordinates and the *complex coordinates* on $M^0$. We use this on the Riemannian scalar curvature $s$ and on the potential $f$ of $X$.

Finally, we make use of the fact that a certain quantity is preserved on a gradient Ricci soliton, see e.g. [3, Prop. 1.15]. This follows from the differentiation of the Ricci soliton equation and some curvature identities.

*Proof of Theorem 2.* We have that $(g, \operatorname{grad} f)$ is a gradient Ricci soliton, so the preserved quantity mentioned above reads, in our notation,

$$s + |\nabla f|^2 + 2f = \text{const}\,.$$

Using with the extremal assumption, it follows that both $f$ and $|\nabla f|^2$ are affine functions in the interior of $\Delta$.

If $f = a \cdot x$, then one has that

$$|\nabla f|^2 = a^T G^{-1}(x) a$$

is an affine function as well. If we consider its extension to the whole $\mathbb{R}^n$, it is zero in all the vertices of $\Delta$ by Lemma 1.2. On the other hand, the zeros of a nonzero affine function is a proper affine hyperplane, so by Lemma 1.1 we can conclude that the length of $X$ must be the zero function. So $X = 0$ and the metric is Einstein. □

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