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Non-Abelian sigma models from Yang–Mills theory compactified on a circle

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ABSTRACT

We consider SU(N) Yang–Mills theory on $\mathbb{R}^{2,1} \times S^1$, where S^1 is a spatial circle. In the infrared limit of a small-circle radius the Yang–Mills action reduces to the action of a sigma model on $\mathbb{R}^{2,1}$ whose target space is a $2(N-1)$-dimensional torus modulo the Weyl-group action. We argue that there is freedom in the choice of the framing of the gauge bundles, which leads to more general options. In particular, we show that this low-energy limit can give rise to a target space $\mathrm{SU}(N)\times\mathrm{SU}(N)/\mathbb{Z}_N$. The latter is the direct product of SU(N) and its Langlands dual $\mathrm{SU}(N)/\mathbb{Z}_N$, and it contains the above-mentioned torus as its maximal Abelian subgroup. An analogous result is obtained for any non-Abelian gauge group.

1. Introduction and summary

Pure Yang–Mills or QCD-like theories in four spacetime dimensions are strongly coupled in the infrared limit. It is known that one can partially overcome this difficulty by compactifying Yang–Mills theory on a circle S^1_R with small radius R (see e.g. [1,2] and references therein). In the adiabatic limit, when the metric on S^1_R is scaled down, the $d=4$ SU(N) Yang–Mills action can be reduced (already on the classical level) to the action of a $d=3$ sigma model whose target space is $T\times T^\vee/W$. Here, $T=\mathrm{U}(1)^{N-1}$ is the Cartan torus in SU(N) corresponding to Wilson loops around S^1_R, and $T^\vee$ is the Cartan torus in the Langlands dual $\mathrm{SU}(N)/\mathbb{Z}_N$. The torus $T^\vee$ parametrizes the dual (magnetic) photons on $\mathbb{R}^{2,1}$ and corresponds to 't Hooft loops around S^1_R [3,4]. Finally, W is the Weyl group, which for SU(N) is the finite permutation group S_N.

The above-mentioned action on $\mathbb{R}^{2,1}$ may be augmented by an effective potential for the sigma-model scalar fields, which appears from an additional centre-stabilizing term breaking SU(N) to $\mathrm{U}(1)^{N-1}$ and from quantum loop corrections, as discussed e.g. in [5,6]. In our paper we focus on the derivation of kinetic terms in the low-energy limit of pure Yang–Mills theory. Therefore, for the time being, we ignore a possible symmetry-breaking potential.

The main message of the paper is that there is important freedom in the choice of the framing[1] of the gauge bundle, and that this leads to the option of enlarging the sigma-model target space from $T\times T^\vee/W$ to a non-Abelian group, up to the maximal space $\mathcal{M}=\mathrm{SU}(N)\times\mathrm{SU}(N)/\mathbb{Z}_N$. In other words, we shall show how the classical Yang–Mills model on $\mathbb{R}^{2,1}\times S^1_R$ can be reduced to a sigma model on $\mathbb{R}^{2,1}$ with non-Abelian target space $\mathcal{M}$ or a subgroup thereof including the torus $T\times T^\vee \cong \mathrm{U}(1)^{2(N-1)}$. For a general gauge group G with weight lattice Γ_w, the sigma-model target space will be $\mathcal{M}=G\times G^\vee$, where $G^\vee$ denotes the Langlands dual group, whose weight lattice $\Gamma^\vee_w$ is dual to Γ_w. Thus, the target-space geometry $\mathcal{M}$ of our sigma models obtained from Yang–Mills theory on $\mathbb{R}^{2,1}\times S^1_R$ in the small-R limit essentially depends on conditions imposed on the gauge potential $\mathcal{A}$ and the gauge transformations along S^1_R.

The Yang–Mills reduction to the Abelian sigma model on $\mathbb{R}^{2,1}$ (where $\mathcal{M}$ is toroidal) points at an Abelian confinement mechanism based on Dirac monopoles, Abelian vortices and the dual Meissner effect. The Abelian dual superconductor approach has various limitations, like any other confinement mechanism (see e.g. [1,10]). For this reason there have been efforts to extend the dual superconductor mechanism to models with non-Abelian

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[1] A bundle over a manifold M is called framed over a submanifold $N\subset M$ if its fibres over N are fixed. Framed bundles are often used in discussions of instantons and monopoles as well as on manifolds with boundaries, marked points, punctures etc. (see e.g. [7–9]).

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monopoles and non-Abelian vortices (see e.g. [1,10,11] and references therein). The suggestion of this paper aims in the same direction.

2. Action functional

Space $\mathbb{R}^{2,1} \times S^1_R$. We consider Yang–Mills theory on the direct product manifold $\mathbb{R}^{2,1} \times S^1_R$ with coordinates $(x^\mu) = (x^a, x^3)$, where $x^a \in \mathbb{R}^{2,1}$ and $x^3 \in [0, 2\pi]$, in which the metric reads

$$\mathrm{d}s_R^2 = g^R_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{ab}\,\mathrm{d}x^a \mathrm{d}x^b + R^2(\mathrm{d}x^3)^2 , \tag{2.1}$$

where $(\eta_{ab}) = \mathrm{diag}(-1, 1, 1)$ with $a, b = 0, 1, 2$, and the angular coordinate obeys $x^3 \sim x^3 + 2\pi$. The dimensionful coordinate $\tilde{x}^3 = Rx^3 \sim \tilde{x}^3 + 2\pi R$ parametrizes the circle S^1_R of radius R.

As Yang–Mills structure group we consider mainly $G = \mathrm{SU}(N)$, however an arbitrary semisimple compact Lie group G will also be discussed. Let I_i with $i = 1, \dots, N^2 - 1$ be a basis of the Lie algebra $su(N)$ realized as $N \times N$ matrices (fundamental representation). We use the normalization condition

$$\mathrm{tr}\, I_i I_j = -\tfrac{1}{2}\delta_{ij} . \tag{2.2}$$

For generators I_i in the adjoint representation of G we will use the same normalization (2.2) but with $i = 1, \dots, \dim G$.

Gauge fields. Let us consider the principal $\mathrm{SU}(N)$-bundle P over $\mathbb{R}^{2,1} \times S^1_R$ and the associated complex vector bundle $E \to \mathbb{R}^{2,1} \times S^1_R$ with fibres $V = \mathbb{C}^N$. Let $\mathcal{A}$ be a gauge potential (a connection on P and E) with values in $su(N)$, so that

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = \tfrac{1}{2}\mathcal{F}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \qquad \text{with}$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{2.3}$$

is the $su(N)$-valued gauge field (curvature of $\mathcal{A}$). On $\mathbb{R}^{2,1} \times S^1_R$ we have the obvious splitting

$$\mathcal{A} = \mathcal{A}_\mu\,\mathrm{d}x^\mu = \mathcal{A}_a\,\mathrm{d}x^a + \mathcal{A}_3\,\mathrm{d}x^3 ,$$

$$\mathcal{F} = \tfrac{1}{2}\mathcal{F}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \tfrac{1}{2}\mathcal{F}_{ab}\,\mathrm{d}x^a \wedge \mathrm{d}x^b + \mathcal{F}_{a3}\,\mathrm{d}x^a \wedge \mathrm{d}x^3 . \tag{2.4}$$

For unit radius $R = 1$, indices of $\mathcal{F}_{\mu\nu}$ are raised by the non-deformed inverse metric tensor $(g^{\mu\nu}) = (\eta^{ab}, 1)$. On the deformed space $\mathbb{R}^{2,1} \times S^1_R$, however, we must raise them with the metric (2.1), and thus the contravariant field components are

$$\mathcal{F}^{ab}_R = g^{ac}_R g^{bd}_R \mathcal{F}_{cd} = \eta^{ac}\eta^{bd}\mathcal{F}_{cd} = \mathcal{F}^{ab} \qquad \text{and}$$

$$\mathcal{F}^{a3}_R = g^{ac}_R g^{33}_R \mathcal{F}_{c3} = \eta^{ac} R^{-2}\mathcal{F}_{c3} = R^{-2}\mathcal{F}^{a3} . \tag{2.5}$$

Action. We consider the standard Yang–Mills action functional

$$S = -\frac{1}{2e^2}\int_{\mathbb{R}^{2,1}\times S^1} \mathrm{d}^4x\,\sqrt{|\det g^R|}\,\mathrm{tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}_R$$

$$= -\frac{1}{2e^2 R}\int_{\mathbb{R}^{2,1}\times S^1} \mathrm{d}^4x\,\mathrm{tr}\big(R^2\,\mathcal{F}_{ab}\mathcal{F}^{ab} + 2\mathcal{F}_{a3}\mathcal{F}^{a3}\big) , \tag{2.6}$$

where e is the gauge coupling constant. Here we used (2.5) as well as $\det(g^R_{\mu\nu}) = -R^2$. We do not consider the topological θ-term since finally it will only change the metric on the moduli space.

3. Adiabatic approach

"Slow" and "fast" variables. The adiabatic approach to differential equations, based on the introduction of "slow" and "fast" variables, exists for more than 90 years and is used in many areas of physics. Briefly, if "slow" variables parametrize a space X and "fast" variables parametrize a space Y (of dimensions p and q, respectively) then on the direct product manifold $Z = X \times Y$ one should consider a metric

$$g_\varepsilon = g_X + \varepsilon^2 g_Y , \tag{3.1}$$

where g_X is a metric on X, g_Y is a metric on Y and $\varepsilon \in [0, \infty)$ is a real parameter. The adiabatic limit refers to the geometric process of shrinking a metric in some directions while leaving it fixed in the others, g_X in the case (3.1). That is, one studies differential equations on $Z = X \times Y$ with the metric (3.1) and the small-ε limit in these equations. More generally, the adiabatic method applies to a fibration $Z \to X$ or if X is a calibrated submanifold of Z (see e.g. [12]).

Slow soliton dynamics. In the simplest case $X = \mathbb{R}$ with $g_X = -1$ (time axis) one looks at solutions of differential equations on Y ("static" solutions) and then switches on a "slow" dependence on time. By using this approach, Manton has shown [13] that, in the "slow-motion limit", monopole dynamics in Minkowski space $\mathbb{R}^{3,1} = \mathbb{R}^{0,1} \times \mathbb{R}^{3,0} = X \times Y$ can be described by geodesics in the moduli space $\mathcal{M}^n_Y$ of static n-monopole solutions. In other words, it was shown [13,7,14] that the Yang–Mills–Higgs model on $\mathbb{R}^{3,1}$ for slow motion reduces to a sigma model in one dimension whose target space is the n-monopole moduli space $\mathcal{M}^n_Y$ of solutions to the Yang–Mills–Higgs equations on $Y = \mathbb{R}^3$.

On three-dimensional manifolds Y with a boundary ∂Y, instead of monopoles one may consider nontrivial flat connections and the slow dynamics of Chern–Simons "solitons"; this was done in [15,16]. The adiabatic approach was also extended to vortices in $1+2$ dimensions, to Seiberg–Witten equations in $d = 4$ Euclidean dimensions, and to instantons viewed as moving solitons in $d = q+1 \geq 5$ dimensions (see e.g. [14,17–19] for reviews and references).

Sigma models on the space X of slow variables. As far as we know, the adiabatic reduction of (super-)Yang–Mills theory in $p+q \geq 4$ dimensions with $q \geq 2$ to sigma models in $p \geq 2$ dimensions has been investigated for the first time in the physics literature in [20–22] and in the mathematical literature in [9,23,24]. The case $q = 1$ with $Y = S^1$ was studied in [3,4], where $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric gauge theories on $\mathbb{R}^{2,1} \times S^1_R$ were reduced to sigma models on $\mathbb{R}^{2,1}$.

General scheme of adiabatic reduction. For Yang–Mills equations on a $(p+q)$-dimensional manifold $X \times Y$ with a metric (3.1), adiabatic reduction implies the following sequential steps:
1) One classifies the Yang–Mills solutions on Y not depending on the coordinates on X and describes the moduli space $\mathcal{M}_Y$ of such solutions. For $q = 1$ one should consider flat connections on Y.
2) One assumes that the gauge potential $\mathcal{A} = \mathcal{A}_X + \mathcal{A}_Y$ has $\mathcal{A}_X \neq 0$ and that $\mathcal{A}$ depends on the coordinates x^a of X only via the moduli-space $\mathcal{M}_Y$ coordinates X^i, i.e. by allowing for $X^i = X^i(x^a)$ in $\mathcal{A}(X^i)$.
3) One substitutes $\mathcal{A} = \mathcal{A}_X + \mathcal{A}_Y$ into the Yang–Mills action functional on $X \times Y$ with the metric (3.1) and performs its small-ε limit. Then one shows that Yang–Mills theory on $X \times Y$ reduces to a sigma model describing maps from X into the moduli space $\mathcal{M}_Y$.

For $p = 3$ and $q = 1$, the target space is enhanced to $\mathcal{M}_X \times \mathcal{M}_Y$, where $\mathcal{M}_X$ denotes the moduli space of dual gauge fields on X, to be discussed later.

We emphasize that the geometry of the moduli space $\mathcal{M}_Y$ depends essentially on the details of the bundles and connections involved. For instance, for flat connections on two-dimensional manifolds, the geometry of the moduli space $\mathcal{M}_Y$ depends on a boundary (if any) of Y, on the number of marked points and punctures, on what kind of bundles is considered (irreducible or reducible, framed or unframed, with holomorphic or parabolic structure, etc.) and so on (see e.g. [8,9,25–27]). As far as we know, for $Y = S^1$ only reducible bundles with tori as moduli spaces were considered (see e.g. [3,4,2,6]). However, the case of flat connections on two-dimensional spaces Y mentioned above shows that one can have more than one possibility, that more general cases may be considered. This is what we want to discuss below.

4. Connections on S^1 and their holonomy

G-bundles over S^1. Let G be a semisimple compact Lie group,[2] $\mathfrak{g}$ its Lie algebra, $P = S^1 \times G \to S^1$ be a trivial principal G-bundle over $Y = S^1$ and $\mathcal{A}_{S^1}$ a connection one-form on P. It will be convenient to parametrize the unit circle S^1 by $\exp(2\pi \mathrm{i}\varphi) \in S^1$ with $\varphi = x^3/2\pi \in [0, 1]$. The connection $\mathcal{A}_{S^1}$ belongs to the space $\mathcal{N}_{S^1} := \Omega^1(S^1, \mathfrak{g})$ of one-forms on S^1 with values in $\mathfrak{g}$.[3] The loop group $LG = C^\infty(S^1, G)$ of gauge transformations in P acts on $\mathcal{N}_{S^1}$ by the formula

$$f \in LG : \mathcal{A}_{S^1} \mapsto \mathcal{A}^f_{S^1} = f^{-1}\mathcal{A}_{S^1} f + f^{-1}\check{\mathrm{d}} f \qquad \text{with}$$
$$\check{\mathrm{d}} = \mathrm{d}x^3\partial_3 = \mathrm{d}\varphi\partial_\varphi \ . \tag{4.1}$$

Note that $f(\varphi+1) = f(\varphi)$ for $f(\varphi) \in LG$ (periodicity).

Holonomy map. For any $\varphi \in [0, 1]$ we introduce the map

$$h_\varphi : \quad \mathcal{N}_{S^1} \ni \mathcal{A}_{S^1} \quad \mapsto \quad h_\varphi(\mathcal{A}_{S^1}) \in G \tag{4.2}$$

which is defined as the unique solution to the differential equation [27]

$$h_\varphi^{-1}(\mathcal{A}_{S^1})\,\check{\mathrm{d}} h_\varphi(\mathcal{A}_{S^1}) = \mathcal{A}_{S^1} \qquad \text{with} \qquad h_0(\mathcal{A}_{S^1}) = \mathrm{Id}\ . \tag{4.3}$$

For this map we have the invariance condition

$$h_\varphi(\mathcal{A}^f_{S^1}) = f^{-1}(1) h_\varphi(\mathcal{A}_{S^1}) f(\varphi)\ , \tag{4.4}$$

where $\mathcal{A}^f_{S^1}$ is given in (4.1). The map (4.2) assigns to any $\mathcal{A}_{S^1} \in \mathcal{N}_{S^1}$ a section $(\varphi, h_\varphi) \in S^1 \times G$ of the G-bundle $P = S^1 \times G$ over S^1. Note that h_φ *is not periodic* in φ, i.e. $h_0 \neq h_1$, since h_φ defines a line segment in the group G which covers S^1 in the base of fibration $P \to S^1$. In fact, h_1 is the holonomy of $\mathcal{A}_{S^1}$ defining a Wilson loop around S^1.

Recall that the *based* loop group $\Omega G \subset LG = \Omega G \rtimes G$ is defined as the kernel of the evaluation map $LG \to G$, $f(\varphi) \mapsto f(1)$, and therefore $\Omega G = LG/G$. At the endpoint $\varphi = 1$ we get the *holonomy map*

$$h_1 : \quad \mathcal{N}_{S^1} \to G\ , \tag{4.5}$$

where h_φ is defined by (4.3). From (4.3) and (4.4) one sees that the action of ΩG on $\mathcal{N}_{S^1}$ is free,[4]

$$h_1(\mathcal{A}^f_{S^1}) = h_1(\mathcal{A}_{S^1}) \quad \Leftrightarrow \quad \mathcal{A}^f_{S^1}(1) = \mathcal{A}_{S^1}(1) \quad \text{for} \quad f \in \Omega G\ , \tag{4.6}$$

and the holonomy map (4.5) is injective on the quotient of $\mathcal{N}_{S^1} = \Omega^1(S^1, \mathfrak{g})$ by ΩG. Thus, (4.5) is the projection in the principal ΩG-bundle over G, and $\mathcal{N}_{S^1}$ is the total space of this bundle [27].

Abelianization. Consider now the holonomy element $h_1(\mathcal{A}_{S^1}) \in G$ which parametrizes a flat connection $\mathcal{A}_{S^1}$. From (4.4) we see that under gauge transformations $f \in LG$ it is transformed as

$$h_1(\mathcal{A}^f_{S^1}) = f^{-1}(1) h_1(\mathcal{A}_{S^1}) f(1)\ , \tag{4.7}$$

i.e. only global gauge transformations defined by constant matrices $f(1) \in G$ act on $h_1(\mathcal{A}_{S^1})$. It is known that by a suitable choice of $f(1) \in G$ one can transform any element $h_1(\mathcal{A}_{S^1}) \in G$ to an element in $T/W \subset G$, where T is a maximal torus (Cartan torus) in G and W is the Weyl group of G. The moduli space of $\mathcal{A}_{S^1}$ is defined as the quotient of $\mathcal{N}_{S^1}$ under the action of the group LG of gauge transformations. In this case, the gauge orbits are parametrized by the orbifold T/W. This is usually meant by "Abelianization".

Although the reduction of the group G to its maximal Abelian subgroup is quite popular, many papers claim that it is not natural and not even obligatory (see e.g. [1,10,11] and references therein). We join these arguments by suggesting to control Abelianization through the framing of bundles.[5]

Framed bundles and moduli of $\mathcal{A}_{S^1}$. Recall that a bundle E over a manifold Y is called framed over a point $p \in Y$ if its fibre E_p over this point is fixed and therefore cannot be transformed by gauge transformations. This means that matrices f of gauge transformations at this point are restricted to the identity, $f(p) = \mathrm{Id}$, i.e. the group $\mathcal{G}$ of gauge transformations in the bundle E reduces to the subgroup $\mathcal{G}_0$ which keeps E_p unchanged. Framing a principal G-bundle P over Y at a point $p \in Y$ is achieved by simply fixing a point h_p in the fibre G_p over p. For instantons on $Y = \mathbb{R}^4$, bundles are framed at infinity in $\mathbb{R}^4$, which forbids global gauge transformations and renders the instanton moduli space hyper-Kähler. Similarly, for monopoles on $\mathbb{R}^3$, bundles are framed at infinity in $\mathbb{R}^3$, which prevents global Abelian gauge transformations generated by the Cartan torus T in the gauge group G. Only after this framing one obtains a hyper-Kähler structure on the moduli space of monopoles. In both cases of instantons and monopoles, the use of unframed bundles is not natural since the hyper-Kähler structure on their moduli spaces is important for various calculations and theoretical predictions. In the same spirit, we suggest to frame our G-bundles over S^1 at the point $\varphi = 1$ on S^1. Then (4.1)–(4.7) imply that after framing one cannot transform via (4.7) the holonomy element $h_1(\mathcal{A}_{S^1})$ to the Cartan subgroup $T \subset G$, since global gauge transformations are no longer allowed. The admissible gauge transformations now belong to the *based* loop group ΩG, and the moduli space $\mathcal{M}_{S^1}$ of flat connections $\mathcal{A}_{S^1}$ on $P \to S^1$ is therefore the entire group manifold G.

Dependence on R. The radius R of the circle S^1_R is a *free* external parameter. Hence, one can in principle introduce a dependence on R in the coordinates X^i on $\mathcal{M}_{S^1} \cong G$ in such a way that for $R < R_0$

[2] Here we consider instead of $\mathrm{SU}(N)$ a semisimple compact Lie group G since this does not change the discussion.

[3] We identify $\mathfrak{g}$ and its standard dual $\mathfrak{g}^*$ using the Killing–Cartan form on $\mathfrak{g}$ which is proportional to (2.2) for $\mathfrak{g} = su(N)$.

[4] Recall that $f(1) = \mathrm{Id}$ for $f \in \Omega G$.

[5] It is possible to make the reduction from G to $T \subset G$ dependent on extra conditions or parameters (see below).

the group G is reduced to some closed subgroup $H \subset G$ (contraction) containing T and for $R \geq R_0$ one has the whole group G. Here R_0 is some fixed scale parameter. By engineering a suitable dependence $X^i = X^i(R)$, scenarios may be envisioned which are more refined than those in the literature for gauge models compactified on S^1_R.

5. Sigma-model effective action

In this final section we consider a gauge group G having in mind $G = \mathrm{SU}(N)$ with generators I_i and trace (2.2). However, one can easily generalize all formulæ to a compact semisimple Lie group G by introducing a proper trace normalized as (2.2). Then in $\mathcal{A} = \mathcal{A}^i I_i$ and $\mathcal{F} = \mathcal{F}^i I_i$ one can take I_i as generators of SU(N) or as generators of G. Thus, we discuss the generic case and keep $G = \mathrm{SU}(N)$ as an illustration.

Dependence on $x^a \in \mathbb{R}^{2,1}$. In Section 4 we have executed step (1) of the adiabatic approach algorithm and described the moduli space $\mathcal{M}_{S^1}$ of connections $\mathcal{A}_{S^1}$ on S^1_R. Now we return to Yang–Mills theory on $\mathbb{R}^{2,1} \times S^1_R$ as discussed in Section 2 and assume, according to step (2), that the gauge potential $\mathcal{A} = \mathcal{A}_{\mathbb{R}^{2,1}} + \mathcal{A}_{S^1}$ depends on $x^a \in \mathbb{R}^{2,1}$ only via the coordinates X^i on the moduli space $\mathcal{M}_{S^1}$, i.e.

$$X^i = X^i(x^a) \qquad \text{and} \qquad \mathcal{A}_\mu = \mathcal{A}_\mu(X^i(x^a), x^3)\ . \tag{5.1}$$

These moduli parameters $X = \{X^i\}$ define a map[6]

$$X: \quad \mathbb{R}^{2,1} \to G \tag{5.2}$$

from $\mathbb{R}^{2,1}$ to the moduli space $\mathcal{M}_{S^1} \cong G$.

Infinitesimal change of $\mathcal{A}_3$. For any fixed $x^a \in \mathbb{R}^{2,1}$, the part $\mathcal{A}_{S^1} = \mathcal{A}_{S^1}(X^i(x^a), x^3)$ of the gauge potential $\mathcal{A}$ belongs to the space $\mathcal{N}_{S^1}$, which is fibred (see (4.5)) over the moduli space G parametrized by coordinates X^i. We introduce the tangent bundle $T\mathcal{N}_{S^1}$ of $\mathcal{N}_{S^1}$ as a fibration

$$h_{1*}: \quad T\mathcal{N}_{S^1} \to TG \tag{5.3}$$

with fibres $T_{\mathcal{A}_{S^1}}\Omega G \cong \Omega\mathfrak{g}$ at any point $\mathcal{A}_{S^1} \in G$. For any given point $\mathcal{A}_{S^1} \in G$ we have $T_{\mathcal{A}_{S^1}} G \cong \mathfrak{g}$ and therefore

$$T_{\mathcal{A}_{S^1}}\mathcal{N}_{S^1} = h_1^* T_{\mathcal{A}_{S^1}} G \oplus T_{\mathcal{A}_{S^1}}\Omega G \ \cong\ \mathfrak{g} \oplus \Omega\mathfrak{g}\ . \tag{5.4}$$

Note that x^a is an "external" parameter for $\mathcal{A}_{S^1}$ in (5.3) and (5.4), and the derivatives

$$\partial_a \mathcal{A}_{S^1} = \frac{\partial X^i}{\partial x^a}\, \partial_i \mathcal{A}_{S^1} \qquad \text{with} \qquad \partial_i = \frac{\partial}{\partial X^i} \tag{5.5}$$

belong to the space $T_{\mathcal{A}_{S^1}}\mathcal{N}_{S^1}$ for any $x^a \in \mathbb{R}^{2,1}$. According to (5.4), one can decompose the derivatives (5.5) into two parts,

$$\partial_a \mathcal{A}_3 = (\partial_a X^i)\xi_{i3} + D_3(\epsilon_i \partial_a X^i)\ , \tag{5.6}$$

where

$$\xi_{i3} \equiv \delta_i \mathcal{A}_3 \tag{5.7}$$

belongs to $T_{\mathcal{A}_{S^1}} G \cong \mathfrak{g}$ and ϵ_i belongs to $T_{\mathcal{A}_{S^1}}\Omega G \cong \Omega\mathfrak{g}$, $i = 1, \dots, \dim G$. These ϵ_i are arbitrary $\mathfrak{g}$-valued gauge parameters, and

$$\epsilon_a := \epsilon_i \partial_a X^i \tag{5.8}$$

are their pull-back to $\mathbb{R}^{2,1}$.

It is natural to fix ϵ_i by requiring

$$D_3 \xi_{i3} = 0 \quad \Leftrightarrow \quad D_3^2 \epsilon_i = D_3 \partial_i \mathcal{A}_3 \tag{5.9}$$

so that ξ_{i3} are orthogonal to infinitesimal gauge transformations of $\mathcal{A}_{S^1}$ generated by ϵ_i. Note that these $\mathfrak{g}$-valued gauge parameters ϵ_i define a connection $\epsilon_i \mathrm{d}X^i$ on the moduli space $\mathcal{M}_{S^1}$ (cf. [20,22]), and ϵ_a from (5.8) define a connection $\epsilon_a \mathrm{d}x^a$ on a G-bundle over $\mathbb{R}^{2,1}$ pulled back from the connection $\epsilon_i \mathrm{d}X^i$ on $\mathcal{M}_{S^1}$.

"Electric" part of effective action. We discussed in detail the $\mathcal{A}_{S^1}$-part of the connection $\mathcal{A} = \mathcal{A}_{\mathbb{R}^{2,1}} + \mathcal{A}_{S^1}$ on $\mathbb{R}^{2,1} \times S^1_R$. On the other hand, the components $\mathcal{A}_a$ for $\mathcal{A}_{\mathbb{R}^{2,1}} = \mathcal{A}_a \mathrm{d}x^a$ are yet not fixed. It is natural to identify them with the sum of ϵ_a from (5.8) [22] and arbitrary $\mathfrak{g}$-valued functions $\chi_a = \chi_a^i(X^j)\xi_{i3}$ which belong to the kernel of D_3 due to (5.9),

$$\begin{aligned} &\mathcal{A}_a = \chi_a + \epsilon_a \qquad \Rightarrow \\ &\mathcal{F}_{a3} = \partial_a \mathcal{A}_3 - D_3 \mathcal{A}_a = (\partial_a X^i)\xi_{i3} \ \in T_{\mathcal{A}_{S^1}} G \cong \mathfrak{g}\ . \end{aligned} \tag{5.10}$$

Both $\mathcal{A}_a \mathrm{d}x^a$ and $(\chi_a + \epsilon_a)\mathrm{d}x^a$ can be considered as gauge potentials on $\mathbb{R}^{2,1}$ with values in the loop algebra $L\mathfrak{g} = \mathfrak{g} \oplus \Omega\mathfrak{g}$. Substituting (5.10) into (2.6), we obtain the term

$$-\frac{1}{e^2 R} \int\limits_{\mathbb{R}^{2,1}\times S^1} \mathrm{d}^4 x\, \eta^{ab}\, \mathrm{tr}\mathcal{F}_{a3}\mathcal{F}_{b3} = \frac{1}{e^2 R} \int\limits_{\mathbb{R}^{2,1}} \mathrm{d}^3 x\, \eta^{ab}\, g_{ij}\, \partial_a X^i \partial_b X^j\ , \tag{5.11}$$

where

$$g_{ij} = -\int\limits_{S^1} \mathrm{d}x^3\ \mathrm{tr}(\delta_i \mathcal{A}_3\, \delta_j \mathcal{A}_3) \tag{5.12}$$

is a metric on the group G in the holonomic basis. Thus, this part of the action (2.6) reduces to the action of a sigma model on $\mathbb{R}^{2,1}$ with target $\mathcal{M}_{S^1} \cong G$.

"Magnetic" part of effective action. Concerning the first term in the action (2.6), the logic is as follows [3,4,6]. If the components $\mathcal{F}_{ab}$ are nonsingular for $R \to 0$ then this term is negligible for small R in comparison with the term (5.11), so it can be discarded. On the other hand, if we allow $R\mathcal{F}_{ab}$ to remain finite for $R \to 0$, then for the Abelian case $\mathcal{M}_{S^1} \cong T/W$ one can dualize to a "magnetic" photon [3,4,6]. In particular, the dual Abelian potential $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_\mu \mathrm{d}x^\mu$ on $\mathbb{R}^{2,1} \times S^1$ is subject to

$$\frac{1}{e}\, \mathcal{F}_{ab} = \frac{e}{R}\, \varepsilon^c_{ab}\, \partial_c \tilde{\mathcal{A}}_3\ , \tag{5.13}$$

where the moduli space of $\tilde{\mathcal{A}}_3$ (the component of $\tilde{\mathcal{A}}$ along S^1) is parametrized by a dual torus $T^\vee$, which is a maximal torus in the Langlands dual group $G^\vee$. Substituting (5.13) into the action (2.6) one generates the term

$$\frac{e^2}{R} \int\limits_{\mathbb{R}^{2,1}} \mathrm{d}^3 x\, \eta^{ab}\, \tilde{g}_{\alpha\beta}\, \partial_a \tilde{X}^\alpha \partial_b \tilde{X}^\beta\ , \tag{5.14}$$

where $\tilde{X}^\alpha, \alpha = 1, \dots, r$, are coordinates on $T^\vee \subset G^\vee$ and $\tilde{g}_{\alpha\beta}$ is a metric on $T^\vee$. Of course, in this Abelian case in (5.11) one should keep only $X^\alpha \in T$.

[6] Not to be confused with the space X.

The action (5.14) can be generalized to the non-Abelian case. For this, let us admit a dual gauge potential $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_a \mathrm{d}x^a + \tilde{\mathcal{A}}_3 \mathrm{d}x^3$ taking values in the dual Lie algebra Lie$G^\vee$ and the corresponding dual gauge field

$$\tilde{\mathcal{F}} = \tfrac{1}{2}\tilde{\mathcal{F}}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \qquad \text{with}$$
$$\tilde{\mathcal{F}}_{\mu\nu} = \partial_\mu \tilde{\mathcal{A}}_\nu - \partial_\nu \tilde{\mathcal{A}}_\mu + [\tilde{\mathcal{A}}_\mu, \tilde{\mathcal{A}}_\nu]\,, \tag{5.15}$$

where $\tilde{\mathcal{F}}_{\mu\nu} = \tilde{\mathcal{F}}^i_{\mu\nu}\tilde{I}_i$ with generators $\tilde{I}_i$ of $G^\vee$. Then to $\tilde{\mathcal{A}}_3$ and $\tilde{\mathcal{F}}_{a3}$ one can apply the same logic as to $\mathcal{A}_3$ and $\mathcal{F}_{a3}$. We conclude that the moduli space of $\tilde{\mathcal{A}}_3$ living on S^1 is the dual Lie group $G^\vee$ and

$$\tilde{\mathcal{F}}_{a3} = (\partial_a \tilde{X}^i)\delta_i \tilde{\mathcal{A}}_3\,, \tag{5.16}$$

where $\tilde{X}^i$ are local coordinates on $G^\vee$. The duality between $\mathcal{F}_{\mu\nu}$ and $\tilde{\mathcal{F}}_{\mu\nu}$ on $\mathbb{R}^{2,1}\times S^1$ is given by

$$\frac{1}{e}\,\mathcal{F}^i_{\mu\nu} = \frac{e}{2}\sqrt{|\det g^R|}\,\varepsilon_{\mu\nu\lambda\sigma}\tilde{\mathcal{F}}_R^{i\lambda\sigma}\,, \tag{5.17}$$

where $g_R^{\mu\nu}$ from (2.5) is used for raising indices of $\tilde{\mathcal{F}}^i_{\mu\nu}$. It follows from (5.17) that

$$\frac{1}{e}\,\mathcal{F}^i_{ab} = \frac{e}{R}\,\varepsilon^c_{ab}\tilde{\mathcal{F}}^i_{c3}\,. \tag{5.18}$$

Using (5.16) and (5.18), we obtain

$$\begin{aligned} -\frac{1}{2e^2R}\int\limits_{\mathbb{R}^{2,1}\times S^1} \mathrm{d}^4x\, R^2\,\mathrm{tr}\mathcal{F}_{ab}\mathcal{F}^{ab} &= \frac{1}{4e^2R}\int\limits_{\mathbb{R}^{2,1}\times S^1} \mathrm{d}^4x\, R^2\,\delta_{kl}\mathcal{F}^k_{ab}\mathcal{F}^{lab} \\ &= \frac{e^2}{R}\int\limits_{\mathbb{R}^{2,1}} \mathrm{d}^3x\,\eta^{ab}\,\tilde{g}_{ij}\partial_a\tilde{X}^i\partial_b\tilde{X}^j\,, \end{aligned} \tag{5.19}$$

where

$$\tilde{g}_{ij} = \frac{1}{2}\int\limits_{S^1}\mathrm{d}x^3\,\delta_{kl}\,(\delta_i\tilde{\mathcal{A}}_3^k\,\delta_j\tilde{\mathcal{A}}_3^l) \tag{5.20}$$

is a metric on the group $G^\vee$ in the holonomic basis. Thus, for small radius R of the circle S^1_R the Yang–Mills action on $\mathbb{R}^{2,1}\times S^1_R$ can be reduced to the effective action of a sigma model on $\mathbb{R}^{2,1}$ with target $G\times G^\vee$,

$$S_{\text{eff}} = \frac{1}{R}\int\limits_{\mathbb{R}^{2,1}}\mathrm{d}^3x\left(\frac{1}{e^2}\,\eta^{ab}\,g_{ij}\partial_aX^i\partial_bX^j + e^2\eta^{ab}\,\tilde{g}_{ij}\partial_a\tilde{X}^i\partial_b\tilde{X}^j\right). \tag{5.21}$$

For $G = \mathrm{SU}(N)$, this is the group $\mathrm{SU}(N)\times\mathrm{SU}(N)/\mathbb{Z}_N$. For the Abelian case this action agrees with those considered in the literature.

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