DE GRUYTER Forum Math. 2016; 28 (6):1159–1166

Research Article

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When are Zariski chambers numerically determined?

DOI: 10.1515/forum-2015-0087
Received May 12, 2015; revised November 28, 2015

Abstract: The big cone of every smooth projective surface X admits a natural decomposition into Zariski chambers. The purpose of this note is to give a simple criterion for the interiors of all Zariski chambers on X to be numerically determined Weyl chambers. Such a criterion generalizes the results of Bauer–Funke [4] on K3 surfaces to arbitrary smooth projective surfaces. In the last section, we study the relation between decompositions of the big cone and elliptic fibrations on some surfaces.

Keywords: Zariski decomposition, big cone, elliptic fibration

MSC 2010: Primary 14C20; secondary 14J28

Communicated by: Karl Strambach

1 Introduction

The main purpose of this note is to study numerical properties of the decomposition of the big cone of smooth projective surfaces into Zariski chambers, i.e. the decomposition induced by the variation of the Zariski decomposition of divisors over the big cone. Recall that, given a pseudo-effective $\mathbb{R}$-divisor D on a smooth projective surface X, there exist effective $\mathbb{R}$-divisors P_D and N_D such that

$$D = P_D + N_D \tag{1.1}$$

and the following conditions are satisfied

(Z1) the divisor P_D is nef,

(Z2) either $N_D = 0$ or $N_D = \sum_{i=1}^{s} \alpha_i C_i$, where $\alpha_i > 0$, and the intersection matrix $[C_i.C_j]_{i,j=1,\dots,n}$ is negative-definite,

(Z3) one has $P_D.C_i = 0$ for all $i = 1, \dots, s$.

The divisor P_D (resp. N_D) in (1.1) is called the *positive* (resp. the *negative*) part of D. One can show (see [16] or [3] for a short proof in modern language) that the *Zariski decomposition* (1.1) of the divisor D is uniquely determined by conditions (Z1), (Z2) and (Z3). Moreover, all sections of D come in effect from P_D, which can be expressed in terms of the *volume* $\mathrm{vol}(D) = \mathrm{vol}(P_D)$ (see [10] for details).

Given an algebraic surface X, by [5, Theorem 1.2], the variation of the Zariski decomposition over the big cone $\mathrm{Big}(X)$ leads to the *Zariski decomposition* of the cone $\mathrm{Big}(X)$. Indeed, suppose that P is a big and nef divisor. Recall the following definition (see [5, p. 214]).

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Definition 1 (Zariski chamber). The *Zariski chamber* $\sum_P$ associated to P is defined as

$$\Sigma_P := \{B \in \mathrm{Big}(X) : \text{irreducible components of } N_B \text{ are the only irreducible curves on } X \text{ that intersect } P \text{ with multiplicity } 0\}.$$

By [5, Theorem 1.2], Zariski chambers yield a locally finite decomposition of the cone Big(X) into locally polyhedral subcones such that the support of the negative part of the Zariski decomposition of all divisors in the subcone is constant.

On the other hand, it follows immediately from property (Z2) (see (1.1)) that the negative part N_D of the Zariski decomposition is either trivial or its support consists of *negative* curves, i.e. curves with negative self-intersection. One can use such curves to define another decomposition of the big cone. Let $\mathcal{N}(X)$ be the set of all *irreducible* negative curves on X. Each curve $C \in \mathcal{N}(X)$ defines the hyperplane in the Néron–Severi space $\mathrm{NS}_{\mathbb{R}}(X)$ of X

$$C^{\perp} = \{D : D.C = 0\} \subset \mathrm{NS}_{\mathbb{R}}(X),$$

and the decomposition of the set

$$\mathrm{Big}(X) \setminus \bigcup_{C \in \mathcal{N}(X)} C^{\perp} \tag{1.2}$$

into connected components yields a decomposition of (an open and dense subset of) the cone Big(X) into subcones.

Definition 2 (Simple Weyl chamber). Connected components of the set (1.2) are called *simple Weyl chambers* of X.

Traditionally the (simple) Weyl chambers are defined if X is a surface carrying only (-2)-curves as negative curves, see e.g. [4]. By a slight abuse of terminology we extend this definition to arbitrary surfaces and arbitrary negative curves.

It is natural to compare the two chamber decompositions. Since the Zariski chambers need not in general be either open or closed, whereas Weyl chambers are by definition open, it is natural to ask *under which condition the interior of each Zariski chamber is a simple Weyl chamber?*

If it happens that all interiors of Zariski chambers coincide with simple Weyl chambers, then we say that *Zariski chambers are numerically determined* (by the sign of the intersection product with negative curves). A condition for Zariski chambers on K3 surfaces to be numerically determined was given in [4, Theorem 1.2]. Here we prove the following criterion, which is valid for all smooth projective surfaces.

Theorem 3 (A criterion for Zariski chambers to be numerically determined). *Let X be a smooth projective surface. The following conditions are equivalent:*
(a) *the interior of each Zariski chamber on X is a simple Weyl chamber,*
(b) *if two irreducible negative curves $C_1 \neq C_2$ on X meet (i.e. $C_1.C_2 > 0$), then*

$$C_1.C_2 \geq \sqrt{C_1^2 \cdot C_2^2}.$$

Remark 4. In practice condition (b) in Theorem 3 means that the support of the (non-trivial) negative part of the Zariski decomposition of every big divisor on X consists of pairwise disjoint curves. Indeed, the condition in question implies that if the intersection matrix of two irreducible negative curves $C_1, C_2 \subset X$ is negative-definite, then it is diagonal.

After proving Theorem 3 in Section 2, we study the relation between elliptic fibrations and Zariski chambers on Enriques and K3 surfaces in Section 3. It should be mentioned, that this note was motivated and inspired by the earlier results of Bauer and Funke [4] on the K3 case.

Convention. In this note we work over the base field $\mathbb{C}$. Surfaces over algebraically closed field of arbitrary characteristic are considered only in Lemma 14.

Elliptic fibrations are not assumed to have a section. For basic facts on various types of divisors and cones associated to a smooth complex variety (resp. on elliptic fibrations) the reader should consult [10] (resp. [13]).

2 Proof of Theorem 3

Proof of Theorem 3. The implication (a) ⇒ (b): We argue by contraposition. Let C_1, C_2 be negative curves on X such that $C_1.C_2 \neq 0$ and the matrix $[C_i.C_j]_{i,j=1,2}$ is negative definite. To simplify our notation we put

$$a := -C_1^2, \quad b := -C_2^2 \quad \text{and} \quad c := C_1.C_2.$$

Then we have $a, b, c > 0$ and $ab > c^2$.

We will construct two big divisors D_1 and D_2 such that

$$\operatorname{supp}(N_{D_1}) = \operatorname{supp}(N_{D_2}) = \{C_1, C_2\}, \tag{2.1}$$

the curves C_1, C_2 are the only irreducible curves on X that meet the positive part P_{D_1} (resp. P_{D_2}) with multiplicity 0, and

$$D_1.C_1 < 0, \quad D_1.C_2 < 0 \quad \text{but} \quad D_2.C_1 > 0, \quad D_2.C_2 < 0. \tag{2.2}$$

Let H be an ample divisor on X. For $k \in \mathbb{N}$ we define the following divisors:

$$T_k = (ab - c^2)H + k[(b(H.C_1) + c(H.C_2))C_1 + (a(H.C_2) + c(H.C_1))C_2].$$

Then, by direct computation, we have

$$T_1.C_1 = T_1.C_2 = 0 \qquad \text{and} \qquad T_k.C_1 < 0, \quad T_k.C_2 < 0 \quad \text{for } k \geq 2. \tag{2.3}$$

In particular, T_1 is a nef divisor. Moreover, by definition

$$\text{for all irreducible curves } C \subset X \text{ such that } C \neq C_1, C_2 \text{ we have } T_1.C > 0. \tag{2.4}$$

Let $D_1 = T_2$. Then

$$D_1 = T_1 + [(b(H.C_1) + c(H.C_2))C_1 + (a(H.C_2) + c(H.C_1))C_2]$$

is the Zariski decomposition of D_1. Indeed, by (2.4) and (2.3) the divisor T_1 satisfies conditions (Z1) and (Z3). The choice of the curves C_1, C_2 implies that condition (Z2) is satisfied. Since the Zariski decomposition of D_1 is uniquely determined by (Z1)–(Z3), the claim follows.

Finally, we define

$$D_2 := (ab - c^2)H + (b(H.C_1) + c(H.C_2) + c)C_1 + (a(H.C_2) + c(H.C_1) + 2a)C_2.$$

Then, by direct computation one gets $D_2.C_1 = ac > 0$ and $D_2.C_2 = c^2 - 2ab < 0$. Moreover, the choice of the curves C_1, C_2 combined with (2.4), (2.3) implies that $D_2 = T_1 + (cC_1 + 2aC_2)$ is the Zariski decomposition of D_2, so that $\operatorname{supp}(N_{D_2}) = \{C_1, C_2\}$.

To complete the proof, observe that (2.1) and (2.4) yield that $D_1, D_2 \in \sum_{T_1}$ (cf. Definition 1). On the other hand (2.2) and (2.4) yield that D_1 (resp. D_2) belongs to a Weyl chamber (i.e. it does not belong to $C^\perp$ for an irreducible curve $C \subset X$). The two Weyl chambers in question do not coincide by (2.2).

The implication (b) ⇒ (a): The opposite implication is elementary. Let D be a big divisor with Zariski decomposition (1.1) that belongs to the interior of the chamber Σ_P for a big and nef P.

By [5, Proposition 1.8] the only irreducible curves $C \subset X$ such that $C.P_D = 0$ are the components of the negative part N_D. In particular, one has

$$D.C \geq P_D.C > 0 \quad \text{for every irreducible curve } C \subset X \text{ such that } C \notin \operatorname{supp}(N_D).$$

Moreover, [5, Proposition 1.8] yields that the components of $\operatorname{supp}(N_D)$ are precisely the irreducible curves that meet P with multiplicity zero.

Recall that the support of N_D consists of mutually *disjoint* curves $C_1, \ldots, C_s$ (see Remark 4). Consequently, one obtains

$$D.C_i = (P_D + N_D).C_i = \alpha_i C_i^2 < 0,$$

which completes the proof. □

As an immediate consequence one obtains the following corollary, that is a direct generalization of [4, Theorem 1.2].

Corollary 5. *Let X be a smooth projective surface with* $\mathrm{kod}(X) = 0$. *Then the following conditions are equivalent:*
(a) *the simple Weyl chambers are the interiors of Zariski chambers on X,*
(b) *there is no pair* C_1, C_2 *of smooth rational curves on X such that* $C_1.C_2 = 1$.

Proof. Recall that the only irreducible curves on X with negative self-intersection are (-2)-curves. The latter are smooth and rational. Theorem 3 immediately yields the claim. □

In fact, a stronger statement holds for Enriques surfaces. In order to prove it we need the following observation that is generally attributed to E. Looijenga.

Observation 6. *Let r be class of a nodal curve on an Enriques surface X and let* $x \in \mathrm{Num}(X)$ *satisfy the conditions* $x^2 = -2$ *and* $(x - r) \in 2\,\mathrm{Num}(X)$. *Then* $(\pm x)$ *is effective.*

Proof. Let $\pi : Y \to X$ be the universal K3-cover of X and let $x = r + 2y$ for a class $y \in \mathrm{Num}(X)$. Since π is étale, we have $\pi^*(r) = r' + r''$, where r', r'' are classes of smooth rational curves. Therefore, we have

$$\pi^*(x) = (r' + \pi^*(y)) + (r'' + \pi^*(y)).$$

Observe that $(r' + \pi^*(y))^2 = -2$. Indeed, we have

$$-2 = x^2 = (r + 2y)^2 = r^2 + 4r.y + y^2,$$

which yields $r.y + y^2 = 0$. The claim then follows from an elementary computation.

Thus, by Riemann–Roch, the class $(r' + \pi^*(y))$ is either effective or anti-effective. The assertion of the observation follows from the equality $x = \pi_*(r' + \pi^*(y))$. □

After these preparations we can prove the following stronger version of Corollary 5 for Enriques surfaces.

Proposition 7. *Let X be an Enriques surface. Then the following conditions are equivalent:*
(a) *Zariski chambers on X are numerically determined,*
(b) *for all nodal curves* $C_1, C_2 \subset X$, *the intersection number* $C_1.C_2$ *is even.*

Proof. The implication (a) ⇒ (b): We assume to the contrary that X contains two nodal curves C_1 and C_2 with $C_1.C_2$ odd. Since we have assumed (a) to hold, we infer from Corollary 5 that $C_1.C_2 \geq 3$.

By [7, Remark 4.3] (see also [6]) there exists a birational surjective morphism $\varphi : X \to S$, where $S \subset \mathbb{P}^5$ is a degree-10 surface with at most rational double points as singularities (a Fano model of X). Furthermore, it follows from [7, Proposition 4.2] and [7, Remark 4.7] that there exists an isometry mapping $\mathrm{Num}(X)$ onto the Enriques lattice $T_{2,3,7} = E_8 \oplus U$ such that the classes of C_1, C_2 are equivalent modulo $2\,\mathrm{Num}(X)$ to classes that belong to the 496 roots listed in [7, Remark 4.7]. By Observation 6, we can assume both classes to be effective. Moreover, Corollary 5 implies that the surface X contains no pairs of smooth rational curves that meet transversally in exactly one point, so both classes in question are represented by irreducible rational curves C_1', C_2'.

On one hand, by construction the intersection number $C_1'.C_2'$ is odd, so we have $C_1'.C_2' \geq 3$. On the other hand, direct computation of intersection numbers of classes listed in [7, Remark 4.7] yields $C_1'.C_2' \leq 2$. (To give a more geometric justification of the last inequality, observe that we compute intersection numbers of smooth rational degree-d curves on the Fano model $S \subset \mathbb{P}^5$, where $d \leq 4$ – see [7, Remark 4.7]). In this way we obtain a contradiction and the proof is complete.

The implication (b) ⇒ (a) follows immediately from Corollary 5. □

We end this section with two examples: degree-d hypersurfaces in $\mathbb{P}^3$ for $d = 3$ and $d \geq 4$.

Example 8. Let X_3 be a smooth cubic surface in $\mathbb{P}^3$. Obviously, the only negative curves on X_3 are the 27 lines L and one has $L^2 = (-1)$. If two lines $L_1 \neq L_2$ on X_3 meet, then one has

$$L_1.L_2 = 1 = \sqrt{L_1^2 \cdot L_2^2}.$$

Thus Zariski chambers on X_3 are numerically determined by Theorem 3. This follows also from [5, Proposition 3.4].

The next example generalizes [4, Proposition 3.1].

Example 9. Let $X_d \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 4$ which contains two intersecting lines L_1, L_2 (e.g. the degree-d Fermat surface, see [14]). By adjunction, we have

$$L_1^2 = L_2^2 = (-d + 2),$$

so condition (b) in Theorem 3 is not satisfied for the intersection matrix of the lines in question. Hence, Zariski chambers on X_d are not numerically determined.

3 Zariski chambers and elliptic fibrations

In this section we characterize the Enriques surfaces for which Zariski chambers are numerically determined in terms of elliptic fibrations, and explain why no such characterization is possible for K3 surfaces.

Let X be an Enriques surface. Recall that for such surfaces the Weyl decomposition is given by irreducible (-2)-curves, i.e. *simple roots*. A general Enriques surface carries no (-2)-curves, so both decompositions of $\mathrm{Big}(X)$ become trivial. An Enriques surfaces is called *nodal* if and only if it contains a smooth rational curve. Moreover, a nodal Enriques surface X is called *general nodal* if and only if any two (-2)-curves on X are congruent modulo $(2\,\mathrm{Num}(X))$ (see e.g. [9, Section 5], [11]).

After these preparations we show the following proposition.

Proposition 10. *Let X be an Enriques surface. Then, the following conditions are equivalent:*
(a) *the simple Weyl chambers are the interiors of Zariski chambers on X,*
(b) *every fiber of every elliptic fibration on X has at most two components.*

Proof. The implication (a) ⇒ (b): By the Kodaira classification of singular fibers (see e.g. [13, Section 4]), if a fiber of an elliptic fibration on X has at least three components, then the fiber in question contains two smooth rational curves C_1, C_2 that meet transversally in exactly one point. Corollary 5 completes the proof.

The implication (b) ⇒ (a): Suppose that a Weyl chamber on X is not the interior of any Zariski chamber. Let C_1, C_2 be two (-2)-curves on X such that $C_1.C_2 = 1$ (see Corollary 5) and let M be the orthogonal complement of $\mathrm{span}(C_1, C_2)$ in the lattice $\mathrm{Num}(X) = E_8 \oplus U$, where U stands for the hyperbolic plane.

By definition and the Hodge Index Theorem, M is a rank-8 lattice of index $(1, 7)$, so we can apply the Meyer Theorem (see e.g. [15, Corollary 2 on p. 43]) and [2, Proposition 16.1 (ii)] to find a primitive class

$$D \in M \text{ such that } D^2 = 0 \text{ and } |D| \neq \emptyset. \tag{3.1}$$

Let $\pi : Y \to X$ be the universal K3-cover of X. Recall that every smooth rational curve E on X defines the Picard–Lefschetz reflection:

$$\mathfrak{s}_E : \mathrm{H}^2(X, \mathbb{Z}) \ni D \mapsto D + (D.E)E \in \mathrm{H}^2(X, \mathbb{Z}).$$

Moreover, the counterimage of E under π decomposes into two disjoint smooth rational curves E^+, E^-. Analogously, we have the Picard–Lefschetz reflection $\mathfrak{s}_{E^+}$ defined by E^+ on $\mathrm{H}^2(Y, \mathbb{Z})$ (see [2, Section VIII.1]).

By [2, Lemma VIII.17.4], there exist smooth rational curves $E_1, \ldots, E_k$ on X such that for the composition of Picard–Lefschetz reflections $\mathfrak{p}_X := (\mathfrak{s}_{E_1} \circ \cdots \circ \mathfrak{s}_{E_k})$ we have

$$\mathfrak{p}_X(D) \text{ is a half-pencil of an elliptic fibration on } X. \tag{3.2}$$

We put $\mathfrak{p}_Y := (\mathfrak{s}_{E_1^+} \circ \mathfrak{s}_{E_1^-} \circ \ldots \circ \mathfrak{s}_{E_k^+} \circ \mathfrak{s}_{E_k^-})$. As one can check (see e.g. [12, Section 2.3]) we have

$$\mathfrak{p}_Y \circ \pi^* = \pi^* \circ \mathfrak{p}_X \qquad \text{and} \qquad \pi_*(\mathfrak{p}_Y(C_i^+)) = \mathfrak{p}_X(C_i) \quad \text{for } i = 1, 2. \tag{3.3}$$

To simplify our notation, we label the four curves $C_1^\pm, C_2^\pm$ on the K3 surface Y in such way that $C_1^+.C_2^+ = 1$. Since $(\mathfrak{p}_Y(C_i^+))^2 = -2$ for $i = 1, 2$, and $(\mathfrak{p}_Y(C_1^+) + \mathfrak{p}_Y(C_2^+))^2 = -2$, we can always assume that $|\mathfrak{p}_Y(C_1^+)| \neq \emptyset$

and $|\mathfrak{p}_Y(C_1^+ + C_2^+)| \neq \emptyset$. Recall that Picard–Lefschetz reflections are isometries. Thus, from (3.3) and (3.2) we infer that $\mathfrak{p}_X(C_1 + C_2)$, $\mathfrak{p}_X(C_1)$ are effective divisors on X, and their supports are contained in a fiber of the elliptic fibration given by $|2\mathfrak{p}_X(D)|$. Finally, the equality $\mathfrak{p}_X(C_1).\mathfrak{p}_X(C_1 + C_2) = -1$ implies that the fiber in question is reducible, but it cannot be of Kodaira type I_2. The Kodaira classification ([13, Section 4]) completes the proof. □

As a direct consequence of the above proposition we obtain the following corollary.

Corollary 11. *On every general nodal Enriques surface X the simple Weyl chambers are the interiors of Zariski chambers.*

Proof. Assume that a Weyl chamber on X is not the interior of any Zariski chamber. Let C_1 and C_2 be two (-2)-curves on X such that $C_1.C_2 = 1$. From $C_1.(C_2 - C_1) = 3$ we infer that C_1 and C_2 are not congruent modulo $(2\,\mathrm{Num}(X))$ and the proof is complete. □

It should be emphasized, that a statement analogous to Proposition 10 does not hold for elliptic K3 surfaces, as the following example shows.

Example 12. (cf. [4, Section 3]) Let $Y_4 \subset \mathbb{P}_3(\mathbb{C})$ be a smooth quartic surface, such that
(i) a plane cuts the quartic Y_4 along a conic C and two lines l', l'',
(ii) the Picard group $\mathrm{Pic}(Y_4)$ is generated by C, l', l''.
Obviously, the line l' (resp. l'') defines the elliptic fibration $|\mathcal{O}_{Y_4}(1) - l'|$ (resp. $|\mathcal{O}_{Y_4}(1) - l''|$), but such a fibration has a unique reducible fiber and the latter is of Kodaira type I_2.

Moreover, by [4, Proposition 3.1 (ii)] the curves C, l', l'' are the only (-2)-curves on Y_4. Since the conic C meets each line with multiplicity two, *no fiber of an elliptic fibration on Y_4 has more than two components*. On the other hand, the *Zariski and Weyl decompositions on Y_4 do not coincide* by [4, Proposition 3.1 (iv)].

Remark 13. (i) Recall that, by [9, Theorem 2], a nodal Enriques surfaces is general nodal if and only if every elliptic fibration on X has at most one reducible fiber that consists of two irreducible components. Thus, Corollary 11 follows also from Proposition 10 and [9, Theorem 2].

(ii) It is well known that, if $\pi : Y \to X$ is the K3-cover of a general nodal Enriques surface X, then we have $\rho(Y) = 11$, where $\rho(Y)$ stands for the Picard number of Y. Standard arguments yield that if none of the conditions of Proposition 10 is satisfied, then we have $\rho(Y) \geq 12$.

(iii) The K3 surface of Example 12 satisfies the condition $\rho(Y_4) = 3$. An analysis of the proof of Proposition 10 (see (3.1)) shows that no similar example with a K3 surface of Picard number ≥ 7 can be constructed. Indeed, one can use the Meyer Theorem and Picard–Lefschetz reflections again.

(iv) Obviously, given an elliptic K3 surface with a section and a reducible fiber of the elliptic fibration in question, [4, Theorem 1.3] implies that Zariski chambers are not numerically determined.

Recall that the results of [5] and [4] were formulated only for complex varieties. On the other hand, the Zariski decomposition can be defined for any pseudo-effective $\mathbb{Q}$-divisor on a non-singular projective surface over an algebraically closed field of an arbitrary characteristic (see e.g. [1, Theorem 14.14]), so it seems very natural to ask whether our results remain valid for surfaces defined over an algebraically closed field of positive characteristic.

One can easily see that the constructions of divisors in the proof of Theorem 3 do not require the assumption that the ground field is $\mathbb{C}$. On the other hand, the proof of Proposition 10 relies on the study of some divisors on the K3-cover, so it cannot be repeated for all Enriques surfaces over algebraically closed fields. Below, we follow the advice of an anonymous referee and give a characteristic-free proof of Proposition 10. Obviously, since we deal with Enriques surfaces over fields of any characteristic, we have to consider quasi-elliptic fibrations. More precisely, we prove the following lemma (for the discussion of genus-1 pencils on Enriques surfaces the reader can consult [8, p. 172]).

Lemma 14. *Let X be an Enriques surface over an algebraically closed $\mathbb{K}$. Assume that X contains a pair of nodal curves C_1 and C_2 such that $C_1.C_2 = 1$. Then there exists a genus-1 pencil on X with a singular member that consists of at least three components.*

Proof. We put $R := C_1 + C_2$ and define (cf. [6, p. 589])

$$\Phi(R) := \mathrm{Min}_E\{E.R : |2E| \text{ is a genus-1 pencil}\}.$$

We repeat verbatim the first part of the proof of [6, Lemma 4.1.1] for the nodal divisor R to obtain the inequality

$$\Phi(R) < 2.$$

If $\Phi(R) = 0$, the claim of the lemma follows directly from the classification of singular fibers of genus-1 fibrations (see e.g. [8, p. 288]).

Therefore, (interchanging the curves C_1 and C_2 when necessary) we can assume that $\Phi(R) = 1$ and there exists a half-fiber E such that

$$E.C_1 = 1 \quad \text{and} \quad E.C_2 = 0.$$

If we put $(e_1, e_2, e_3) := (E, E + C_1, E + C_1 + C_2)$, then we have $e_i.e_j = (1 - \delta_{i,j})$, so e_1, e_2, e_3 form an isotropic 3-sequence. It follows from [8, Corollary 2.5.6] that such a sequence can be extended to an isotropic 10-sequence $(e_1, \dots, e_{10})$.

We claim that none of $e_3, \dots, e_{10}$ are nef. Indeed, suppose that e_j is nef for some $j > 3$. Then by the definition of an isotropic sequence we have

$$e_j.E = e_j.(E + C_1) = e_j.(E + C_1 + C_2) = 1.$$

The latter implies $e_j.(C_1 + C_2) = 0$ and $\Phi(R) = 0$. A contradiction.

Thus, by [8, Lemma 3.3.1], we can assume that

$$e_4 = E + C_1 + C_2 + C_3,$$

where C_3 is a nodal curve such that $C_2.C_3 = 1$. Recall that the sequence $(e_1, \dots, e_{10})$ is isotropic, so we have

$$E.C_2 = E.C_3 = 0.$$

Thus $|2E|$ is a genus-1 pencil with a singular member that contains the A_2-configuration $C_2 + C_3$. The claim of the lemma follows from the classification of singular fibers of genus-1 fibrations. □

Observe that the proof of the above lemma cannot be repeated for K3 surfaces, so it does not yield Remark 13 (iii).

Acknowledgment: The first author would like to thank T. Bauer, M. Joumaah and M. Schütt for useful discussions. We thank I. Dolgachev for answering our questions on the *r*-invariant of Enriques surfaces and anonymous referees for their inspiring remarks (in particular, for the idea of the proof of Lemma 14).

Funding: The work was partially supported by National Science Centre, Poland, grant no. N N201 608040 (first author) and grant no. 2014/15/B/ST1/02197 (second author).

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