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Prime power divisors of binomial coefficients

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1. Introduction

Around 1850, Chebyshev was the first mathematician who proved any worthwhile results on the prime counting function $\pi(x)$, namely that it is bounded from above and below by $\frac{x}{\log x}$. This can be obtained by looking at the prime factors of the "middle" binomial coefficients $\binom{2n}{n}$, because these coefficients have enormously many distinct prime factors. On the other hand, in 1975, Erdös [5] conjectured that for $n > 4$, the binomial coefficient $\binom{2n}{n}$ is never square-free. In a subsequent article, Erdös and Graham [3] asked more general questions (see also [4] and [6]), namely: Given a positive integer a, does for sufficiently large n, there exist a prime p such that $p^a \mid \binom{2n}{n}$? Does p tend to infinity for increasing n? Do these properties also hold for binomial coefficients $\binom{2n \pm d}{n}$, if d is "not too large"?

The original conjecture of Erdös was settled by Sárközy [14] in 1985 for sufficiently large n. In [10], [11] and [13], the author gave answers to the second and third question mentioned above, using a new exponential sum estimate [12]. In this paper, we will finally give an answer to all three questions by proving the following

Theorem 1. *Let* $0 < \varepsilon < 1$, *and* $a \in \mathbb{N}$. *Then there exists* $m_0 = m_0(\varepsilon, a)$ *such that for all* $m \geqq m_0$ *and all* $0 \leqq k \leqq m$ *satisfying*

$$(1) \qquad |m - 2k| < m^{1-\varepsilon},$$

we have $p^a \mid \binom{m}{k}$ *for some prime* $p > \frac{1}{2} m^{\frac{1}{a+1}}$.

The main tool for the proof of Theorem 1 will be an upper bound of the exponential sum

$$\sum_{p \leqq P} e\left(x\left(\frac{h_1}{p^{j_1}} + \dots + \frac{h_r}{p^{j_r}}\right)\right),$$

where $e(x) = e^{2\pi i x}$ for real x, as usual. This generalizes the corresponding results obtained by Jutila [7] and the author [12]. As a corollary, we get an asymptotic formula for

$$\operatorname{card}\left\{p \leqq P : \left\{\frac{x}{p^j}\right\} < \sigma_j (1 \leqq j \leqq J)\right\},$$

where $0 < \sigma_j \leqq 1$.

In a forthcoming paper, we will apply these exponential sum estimates to the method of Sárközy [14] in order to obtain upper and lower bounds for the highest a-th power dividing binomial coefficients.

2. Preliminaries

In the sequel, let r be a positive integer and let real numbers h_i $(1 \leqq i \leqq r)$ and positive integers j_i $(1 \leqq i \leqq r)$ be given such that

$$h = h_1 \geqq 1, \tag{2}$$

$$H = \max\{|h_i| : 1 \leqq i \leqq r\}$$

and

$$1 \leqq j = j_1 < j_2 < \dots < j_r \leqq J, \tag{3}$$

where J is a positive real number. We define

$$\Lambda(X, Y) = \left(\frac{\log X}{\log Y}\right)^2.$$

All the explicit and implicit constants may only depend on J. We adopt the convention that the constant c, which always is assumed to be positive, may change its value within equations and inequalities. This enables us to write

$$x^{1-c} \log x \ll x^{1-c},$$

for instance.

We will make use of methods due to Vinogradov-Karacuba, van der Corput and Jutila [7].

Lemma 1 [8]. *Let N, P and $n \geqq 2$ be positive integers, let $f(x)$ be a real function having continuous $(n+1)$th derivative in the interval $N \leqq x \leqq N + P$. Let c_0, c_1, c_2, c_3 and c_4 be positive constants satisfying $c_0 < 1$, $c_1 < 1$, $c_2 + c_4 < c_1$, let t be an integer with $c_0 n \leqq t \leqq n$, and integers s_i $(1 \leqq i \leqq t)$, $2 \leqq s_i \leqq n$, such that for $N \leqq x \leqq N + P$ the following inequalities hold*:

(4) $$\left|\frac{1}{(n+1)!} f^{(n+1)}(x)\right| \leqq P^{-c_1(n+1)},$$

(5) $$P^{-c_2 s_i} \leqq \left|\frac{1}{s_i!} f^{(s_i)}(x)\right| \leqq P^{-c_3 s_i} \quad (1 \leqq i \leqq t).$$

Then for each positive integer $P_1 \leqq P$ *and*

$$S = \sum_{x=N}^{N+P_1-1} e\big(f(x)\big),$$

we have for some positive constants A *and* γ

$$|S| \leqq A P^{1-\gamma/n^2}.$$

Lemma 2. *Let* $T' > 2$, $T = T'h$ *such that*

(6) $$0 \leqq P' < P < T^{\frac{1}{j+1} + \frac{1}{100 j^3}}$$

and

(7) $$P(\log P)^J > JH(3\log T)^J.$$

Then

$$\left|\sum_{x=P}^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}} + \ldots + \frac{h_r}{x^{j_r}}\right)\right)\right| \ll P^{1-cA(P,T)}.$$

Proof. Without loss of generality we may assume

(8) $$J > 8$$

and

(9) $$P > (2J)^{150 J^2}.$$

We intend to apply Lemma 1 and put $N = P$,

$$f(x) = T'\left(\frac{h_1}{x^{j_1}} + \ldots + \frac{h_r}{x^{j_r}}\right),$$

(10) $$n = \left[100 j^2 \frac{\log T}{\log P}\right],$$

$$c_0 = \frac{1}{500 j^5}, \quad c_1 = 1 - \frac{1}{50 j^2}, \quad c_2 = 1 - \frac{1}{25 j^2}, \quad c_3 = \frac{1}{20 j^2}, \quad c_4 = \frac{1}{60 j^2}$$

and

(11) $$\{s_i : 1 \leqq i \leqq t\} = \left\{s \in \mathbb{N} : \left(1 + \frac{1}{5j^2}\right)\frac{\log T}{\log P} \leqq s \leqq \left(1 + \frac{1}{j}\right)\frac{\log T}{\log P}\right\}.$$

By (6), we have

$$\frac{\log T}{\log P} > \left(\frac{1}{j+1} + \frac{1}{100 j^3}\right)^{-1}. \tag{12}$$

Therefore

$$\left(\left(1+\frac{1}{j}\right) - \left(1+\frac{1}{5j^2}\right)\right)\frac{\log T}{\log P} > 1 + \frac{1}{3j} > 1,$$

thus $\{s_i\} \neq \emptyset$. Moreover, (12) yields

$$t \geqq \left(\left(1+\frac{1}{j}\right) - \left(1+\frac{1}{5j^2}\right)\right)\frac{\log T}{\log P} - 1 > c_0 100 j^2 \frac{\log T}{\log P} \geqq c_0 n.$$

Obviously, $2 \leqq s_i \leqq n$. It remains to check (4) and (5) in Lemma 1. For $x > 0$ and $m \in \mathbb{N}$, we have

$$\frac{1}{m!} f^{(m)}(x) = (-1)^m \frac{T'}{m!} \sum_{i=1}^{r} \frac{j_i (j_i+1)\cdots(j_i+m-1) h_i}{x^{j_i+m}}. \tag{13}$$

(7) and (6) give $P > H$. By (2) and (7), we thus get

$$\frac{h}{P^j} \geqq \frac{|h_i|}{P^{j_i}} \quad (1 \leqq i \leqq r), \tag{14}$$

which implies for $P = N \leqq x \leqq N + P = 2P$ with (9)

$$\left|\frac{1}{(n+1)!} f^{(n+1)}(x)\right| \leqq J^{n+1} T' \left(\frac{|h_1|}{P^{j_1+n+1}} + \cdots + \frac{|h_r|}{P^{j_r+n+1}}\right) \leqq \frac{J^{n+1} T'}{P^{n+1}} r \frac{h}{P^j}$$
$$\leqq \frac{J^{n+2} T}{P^{j+n+1}} = P^{(n+2)\frac{\log J}{\log P} + \frac{\log T}{\log P} - (j+n+1)} = P^{-c_1'(n+1)},$$

where

$$c_1' = 1 + \frac{j}{n+1} - \frac{(n+2)\log J}{(n+1)\log P} - \frac{\log T}{(n+1)\log P} > c_1$$

by (9) and (10). Hence (4) holds.

In the same way, we get for $P \leqq x \leqq 2P$ and $s \in \{s_i\}$ by (11)

$$\left|\frac{1}{s!} f^{(s)}(x)\right| \leqq \frac{J^{s+1} T}{P^{j+s}} = P^{-c_3' s},$$

where

$$c_3' = 1 + \frac{j}{s} - \frac{(s+1)\log J}{s \log P} - \frac{\log T}{s \log P} > c_3.$$

This proves the upper bound in (5). By (13), we have for $P \leqq x \leqq 2P$ and $s \in \{s_i\}$ using (2)

$$\left|\frac{1}{s!} f^{(s)}(x)\right| \geqq T'\left(\frac{h}{x^{j+s}} - (r-1)\binom{J-1+s}{s}\frac{H}{x^{j_2+s}}\right). \tag{15}$$

We apply a weak form of Stirling's formula, namely

$$\left(n+\frac{1}{2}\right)\log n - n + \frac{3}{2}\left(1 - \log\frac{3}{2}\right) \leqq \log n! \leqq \left(n+\frac{1}{2}\right)\log n - n + 1\,.$$

For $J-1 < s$, this implies together with (8)

$$\binom{J-1+s}{s} < s^{J-1}\,. \tag{16}$$

For $J-1 \geqq s$, we have trivially

$$\binom{J-1+s}{s} < 4^{J-1}\,. \tag{17}$$

By (15), (16), (17), (2) and (3), we get for $s_0 = \max(s, 4)$

$$\begin{aligned}\left|\frac{1}{s!} f^{(s)}(x)\right| &\geqq \frac{T'}{x^{j+s}}\left(h - \frac{J s_0^{J-1} H}{x^{j_2-j}}\right)\\ &\geqq \frac{T'}{(2P)^{j+s}}\left(h - \frac{J s_0^{J-1} H}{P^{j_2-j}}\right)\\ &\geqq \frac{T'}{(2P)^{j+s}}\left(h - \frac{J s_0^{J-1} H}{P}\right).\end{aligned} \tag{18}$$

By (7), (11) and (12)

$$P > 2 J s_0^J H\,.$$

Therefore, (18) yields

$$\left|\frac{1}{s!} f^{(s)}(x)\right| \geqq \frac{T'h}{2(2P)^{j+s}} = P^{-c_2' s}\,,$$

where

$$c_2' = \left(1 + \frac{j}{s}\right)\left(1 + \frac{\log 2}{\log P}\right) + \frac{\log 2}{s \log P} - \frac{\log T}{s \log P} \leqq c_2$$

using (11), (9) and (12). Hence the lower bound in (5) also holds. This proves Lemma 2.

Lemma 3. *Let $T' > 2$, $T = T'h$ and $0 \leqq P' < P$ such that*

$$P \geqq T^{\frac{1}{j+1} + \frac{1}{100 j^3}} \tag{19}$$

and

$$P > 2J^3 H. \tag{20}$$

Then

$$\left| \sum_{x=P}^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}} + \cdots + \frac{h_r}{x^{j_r}} \right)\right) \right| \ll \frac{P^{j+1}}{T}.$$

Proof. We use van der Corput's well-known method. Again let

$$f(x) = T'\left(\frac{h_1}{x^{j_1}} + \cdots + \frac{h_r}{x^{j_r}} \right).$$

We assume without loss of generality

$$P > (2J)^{200 J^2}. \tag{21}$$

For $x > 0$,

$$f'(x) = -T'\left(\frac{j_1 h_1}{x^{j_1+1}} + \cdots + \frac{j_r h_r}{x^{j_r+1}} \right).$$

Since by (2), (3) and (20) for $P \leqq x \leqq 2P$

$$\begin{aligned} f''(x) &= T'\left(\frac{j_1(j_1+1)h_1}{x^{j_1+2}} + \cdots + \frac{j_r(j_r+1)h_r}{x^{j_r+2}} \right) \\ &\geqq \frac{T'}{x^{j_2+2}} \left(j(j+1) h x^{j_2 - j} - (r-1) J(J+1) H \right) \\ &\geqq \frac{T'}{(2P)^{j_2+2}} (2hP - J^3 H) > 0, \end{aligned}$$

$f'(x)$ obviously is an increasing function. Because of (14), we get for $P \leqq x \leqq P + P'$ by (19) and (21)

$$\begin{aligned} |f'(x)| &\leqq \frac{T'J}{P}\left(\frac{h_1}{P^{j_1}} + \cdots + \frac{h_r}{P^{j_r}} \right) \leqq \frac{T'J}{P} r \frac{h}{P^j} \leqq \frac{J^2 T}{P^{j+1}} \\ &< J^2 P^{-\frac{1}{100 j}} < J^2 (2J)^{-\frac{2J}{j}} \leqq \frac{1}{4} < 1. \end{aligned}$$

Thus Lemma 4.8 in [15] implies

$$\sum_{x=P}^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}} + \cdots + \frac{h_r}{x^{j_r}}\right)\right) = \int_P^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}} + \cdots + \frac{h_r}{x^{j_r}}\right)\right) dx + O(1)\,. \tag{22}$$

The function $f'(x)$ is increasing in $[P, P+P']$ and, by (3) and (20), we have in this interval

$$\begin{aligned} -f'(x) &\geqq T'\left(\frac{jh}{x^{j+1}} - (r-1)\frac{JH}{x^{j_2+1}}\right) \\ &\geqq \frac{T'}{(2P)^{j+1}}\left(jh - \frac{J^2 H}{P^{j_2-j}}\right) \\ &\geqq \frac{1}{2}\frac{T'h}{(2P)^{j+1}} \geqq \frac{1}{2}\frac{T}{(2P)^{j+1}}\,. \end{aligned}$$

Hence Lemma 4.2 in [15] gives

$$\left|\int_P^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}} + \cdots + \frac{h_r}{x^{j_r}}\right)\right) dx\right| \ll \frac{P^{j+1}}{T}\,.$$

By (22), the desired result follows.

Lemma 4. *Let* $T' > 2$, $0 \leqq P' < P$ *and*

$$P > JH(3\log T'H)^J\,. \tag{23}$$

Then

$$\left|\sum_{x=P}^{P+P'} e\left(T'\left(\frac{h_1}{x^{j_1}} + \cdots + \frac{h_r}{x^{j_r}}\right)\right)\right| \ll P^{1-cA(P,T'H)} + \frac{P^{j+1}}{T'}\,.$$

Proof. For $P \geqq T'$, the lemma obviously holds. In case $P < T'$, (23) implies (7) and (20). Hence the proof is completed by Lemma 2 and Lemma 3.

Lemma 5. *Let* $x > 2$. *Then*

$$\left|\sum_{m \leqq M} e\left(x\left(\frac{h_1}{m^{j_1}} \cdots + \frac{h_r}{m^{j_r}}\right)\right)\right| \ll M^{1-cA(M,xH)} + M^{j+1}x^{-1} + H^2(\log xH)^{2J}\,.$$

Proof. For $M \leqq J^2H^2(3\log xH)^{2J}$, the lemma is obvious. Thus we assume

$$M > J^2H^2(3\log xH)^{2J}\,. \tag{24}$$

Let $\frac{1}{2} \leqq \kappa < 1$. Then

$$\left| \sum_{m \leqq M} e\left(x\left(\frac{h_1}{m^{j_j}} + \cdots + \frac{h_r}{m^{j_r}}\right)\right)\right| \leqq \left|\sum_{m \leqq M^\kappa}\right| + \sum_{\substack{v \geqq 0 \\ M^\kappa 2^v \leqq M}} \left| \sum_{M^\kappa 2^v \leqq m < \min(M^\kappa 2^{v+1}, M)} \right|$$

$$\leqq M^\kappa + \sum_v R_v ,$$

where

$$R_v = \left| \sum_{M^\kappa 2^v \leqq m < \min(M^\kappa 2^{v+1}, M)} e\left(x\left(\frac{h_1}{m^{j_1}} + \cdots + \frac{h_r}{m^{j_r}}\right)\right)\right| .$$

By Lemma 4, (24), and since $\kappa \geqq \frac{1}{2}$, we get

$$R_v \ll (M^\kappa 2^v)^{1 - c\Lambda(M^\kappa 2^v, xH)} + (M^\kappa 2^v)^{j+1} x^{-1}$$
$$\ll 2^v M^{\kappa - c\kappa^3 \Lambda(M, xH)} + 2^{(j+1)v} M^{(j+1)\kappa} x^{-1} .$$

In $\sum R_v$, the variable v runs through the interval $0 \leqq v \leqq \frac{(1-\kappa)\log M}{\log 2}$, therefore

$$\sum 2^v \leqq 2 M^{1-\kappa}$$

and

$$\sum 2^{(j+1)v} \leqq 2^{j+1} M^{(j+1)(1-\kappa)} .$$

For large $\kappa < 1$, the lemma follows.

We would like to mention that Lemma 5 is non-trivial only for $x \geqq M^j$.

Lemma 6. *Let* $2 \leqq M \leqq M' \leqq \min(2M, N) \leqq N \leqq x$ *and* $B \geqq 2$. *Then*

$$T := \sum_{M < m \leqq M'} \left| \sum_{B < n \leqq \frac{N}{m}} \Lambda(n) e\left(x\left(\frac{h_1}{(mn)^{j_1}} + \cdots + \frac{h_r}{(mn)^{j_r}}\right)\right)\right|^2$$
$$\ll (N^2 M^{-1 - c\Lambda(M, xH)} + N^{j+2}(Mx)^{-1} + N + N^2 M^{-2} H)(\log xH)^{J+2} ,$$

where $\Lambda(n)$ *denotes von Mangoldt's function.*

Proof. For $M \leqq JH(3\log xH)^J$, we obviously have

$$T \leqq JH(3\log xH)^J \left(\frac{N}{M}\log N\right)^2$$
$$\ll N^2 M^{-2} H (\log xH)^{J+2} ,$$

which proves the lemma in this case. Hence, let

$$M > JH(3\log xH)^J. \tag{25}$$

Clearly

$$T = \sum_{M < m \leqq M'} \sum_{B < n_1 \leqq \frac{N}{m}} \sum_{B < n_2 \leqq \frac{N}{m}} \Lambda(n_1)\Lambda(n_2)\, e\left(x \sum_{i=1}^{r} \left(\frac{h_i}{(mn_1)^{j_i}} - \frac{h_i}{(mn_2)^{j_i}}\right)\right)$$

$$= \sum_{B < n_1 \leqq \frac{N}{M}} \sum_{B < n_2 \leqq \frac{N}{M}} \Lambda(n_1)\Lambda(n_2) \sum_{\substack{M < m \leqq M' \\ m \leqq \frac{N}{n_1}, m \leqq \frac{N}{n_2}}} e\left(x\left(\frac{h_1 \Delta_1}{m^{j_1}} + \cdots + \frac{h_r \Delta_r}{m^{j_r}}\right)\right)$$

with

$$\Delta_i = \left(\frac{1}{n_1^{j_i}} - \frac{1}{n_2^{j_i}}\right) \quad (1 \leqq i \leqq r).$$

Thus

$$T \ll (\log N)^2 \sum_{n_1 \leqq \frac{N}{M}} \sum_{n_2 \leqq \frac{N}{M}} \left| \sum_{\substack{M < m \leqq M' \\ m \leqq \frac{N}{n_1}, m \leqq \frac{N}{n_2}}} e\left(x\left(\frac{h_1 \Delta_1}{m^{j_1}} + \cdots + \frac{h_r \Delta_r}{m^{j_r}}\right)\right)\right| \tag{26}$$

$$\ll (\log N)^2 \left(N + \sum\sum_{0 < n_1 < n_2 \leqq \frac{N}{M}} \left| \sum_{M < n \leqq M'} e\left(x\left(\frac{h_1 \Delta_1}{m^{j_1}} + \cdots + \frac{h_r \Delta_r}{m^{j_r}}\right)\right)\right|\right)$$

$$= (\log N)^2 (N + T_1),$$

say. By the mean value theorem, we have for fixed $1 < n_1 < n_2 \leqq \frac{N}{M}$

$$0 < \Delta_r < \cdots < \Delta_1 \leqq 1.$$

By this and (25), Lemma 4 implies

$$T_1 = \sum\sum_{1 < n_1 < n_2 \leqq \frac{N}{M}} \left| \sum_{M < m \leqq M'} e\left(x\Delta_1\left(\frac{h_1}{m^{j_1}} + \frac{h_2 \Delta_2/\Delta_1}{m^{j_2}} + \cdots + \frac{h_r \Delta_r/\Delta_1}{m^{j_r}}\right)\right)\right| \tag{27}$$

$$\ll \sum\sum_{1 < n_1 < n_2 \leqq \frac{N}{M}} \left(M^{1 - c\Lambda(M, xH)} + M^{j+1}(x\Delta_1)^{-1}\right)$$

$$\ll N^2 M^{-1 - c\Lambda(M, xH)} + M^{j+1} x^{-1} \sum\sum_{1 < n_1 < n_2 \leqq \frac{N}{M}} \frac{1}{\Delta_1}.$$

We set $\Delta_0 = n_2 - n_1$. For $1 < n_1 < n_2$ and $1 \leqq i \leqq r$, we get

$$\Delta_i = \frac{n_2^{j_i} - n_1^{j_i}}{(n_1 n_2)^{j_i}} = \Delta_0 \frac{n_2^{j_i - 1} + n_2^{j_i - 2} n_1 + \cdots + n_2 n_1^{j_i - 2} + n_1^{j_i - 1}}{(n_1 n_2)^{j_i}} \geqq \frac{\Delta_0 j_i}{n_2^{j_i + 1}}.$$

Therefore,

$$\sum\sum_{1<n_1<n_2\leqq\frac{N}{M}}\frac{1}{\Delta_1}=\sum_{0<\Delta_0<\frac{N}{M}}\ \sum\sum_{\substack{1<n_1<n_2\leqq\frac{N}{M}\\ n_2-n_1=\Delta_0}}\frac{1}{\Delta_1}$$
$$\leqq\frac{1}{j_1}\sum_{0<\Delta_0<\frac{N}{M}}\frac{1}{\Delta_0}\sum_{0<n_2\leqq\frac{N}{M}}n_2^{j+1}$$
$$\ll\left(\frac{N}{M}\right)^{j+2}\log N\,.$$

By (27),

$$T_1\ll N^2M^{-1-cA(M,xH)}+N^{j+2}(Mx)^{-1}\log N\,.$$

This and (26) yield the desired result.

3. Vaughan's identity

The application of Vaughan's identity instead of Vinogradov's rather complicated combinatorial argument is by now a well-known technique in analytic number theory. In our case, it simplifies the method of Jutila [7].

As a corollary to Vaughan's identity (see for instance [16], [17], or [2], p. 138–140), we have

Lemma 7. *Let $U\geqq 2$, $V\geqq 2$, $UV\leqq N$, and let $f(x)$ be a complex-valued function satisfying $|f(x)|=1$ for real x. Then*

$$\sum_{n\leqq N}\Lambda(n)f(n)\ll V+(\log N)S_1+S_2\,,$$

where $\Lambda(n)$ denotes von Mangoldt's function, and

$$S_1=\sum_{t\leqq UV}\max_{w>0}\left|\sum_{w\leqq s\leqq\frac{N}{t}}f(st)\right|,$$

$$S_2=\sum_{U<m<\frac{N}{V}}\ \sum_{V<n\leqq\frac{N}{m}}\ \sum_{\substack{d\leqq U\\ d|m}}\mu(d)\Lambda(n)f(mn)\,.$$

Lemma 8. *Let $x>0$ and $2\leqq N\leqq x^{1/j}$. Then*

$$\sum_{n\leqq N}\Lambda(n)e\left(x\left(\frac{h_1}{n^{j_1}}+\cdots+\frac{h_r}{n^{j_r}}\right)\right)\ll(N^{1-cA(N,xH)}+N^{\frac{j+2}{2}}x^{-\frac{1}{2}}+N^{\frac{5}{6}}H^2)(\log xH)^{4J}\,.$$

Proof. We apply Lemma 7 with

$$f(n) = e\left(x\left(\frac{h_1}{n^{j_1}} + \cdots + \frac{h_r}{n^{j_r}}\right)\right).$$

First consider S_2. By splitting up the sum $\sum_m$ into intervals $M \leqq m < 2M$, we get

$$S_2 \ll (\log N) \max_{U < M < M' \leqq \min\left(2M, \frac{N}{V}\right)} \left| \sum_{M \leqq m < M'} \left(\sum_{V < n \leqq \frac{N}{m}} \Lambda(n) f(mn) \right) \left(\sum_{\substack{d \leqq U \\ d \mid m}} \mu(d) \right) \right|.$$

Cauchy's inequality implies

$$\begin{aligned} \left| \sum_{M \leqq m < M'} \right| &\leqq \left(\sum_m \left| \sum_{V < n \leqq \frac{N}{m}} \Lambda(n) f(mn) \right|^2 \right)^{1/2} \left(\sum_m \left(\sum_{\substack{d \leqq U \\ d \mid m}} \mu(d) \right)^2 \right)^{1/2} \\ &= T_1^{1/2} T_2^{1/2}, \end{aligned}$$

say. By Lemma 6,

$$T_1 \ll \left(N^2 M^{-1-cA(M,xH)} + N^{j+2}(Mx)^{-1} + N + N^2 M^{-2} H\right)(\log xH)^{J+2}.$$

Moreover,

$$\begin{aligned} T_2 &\leqq \sum_{M \leqq m < M'} \left(\sum_{\substack{d \leqq U \\ d \mid m}} 1 \right)^2 = \sum_{d_1 \leqq U} \sum_{d_2 \leqq U} \sum_{\substack{M \leqq m < M' \\ m \equiv 0 \bmod d_1,\, m \equiv 0 \bmod d_2}} 1 \\ &\leqq 2M \sum_{d_1 \leqq U} \sum_{d_2 \leqq U} \frac{(d_1, d_2)}{d_1 d_2} \leqq 2M \sum_{b \leqq U} b \sum_{\substack{d_1 \leqq U \\ d_1 \equiv 0 \bmod b}} \sum_{\substack{d_2 \leqq U \\ d_2 \equiv 0 \bmod b}} \frac{1}{d_1 d_2} \\ &\ll M(\log N)^3. \end{aligned}$$

Together, we get

$$\begin{aligned} (28) \qquad S_2 &\ll (\log N) \max_{U < M \leqq \frac{N}{V}} (T_1 T_2)^{1/2} \\ &\ll (\log xH)^{\frac{J+7}{2}} \max_{U < M \leqq \frac{N}{V}} \left(NM^{-cA(M,xH)} + N^{\frac{j+2}{2}} x^{-\frac{1}{2}} + (NM)^{\frac{1}{2}} + NM^{-\frac{1}{2}} H^{\frac{1}{2}}\right) \\ &\ll \left(NU^{-cA(U,xH)} + N^{\frac{j+2}{2}} x^{-\frac{1}{2}} + NV^{-\frac{1}{2}} + NU^{-\frac{1}{2}} H^{\frac{1}{2}}\right)(\log xH)^{4J}. \end{aligned}$$

It remains to bound S_1. For $1 \leqq w \leqq \frac{N}{t}$, we have by Lemma 5

$$\sum_{w \leqq s \leqq \frac{N}{t}} e\left(x\left(\frac{h_1}{(st)^{j_1}} + \cdots + \frac{h_r}{(st)^{j_r}}\right)\right) = \sum_{w \leqq s \leqq \frac{N}{t}} e\left(\frac{x}{t^{j_1}}\left(\frac{h_1}{s^{j_1}} + \frac{h_2 t^{j_1 - j_2}}{s^{j_2}} + \cdots + \frac{h_r t^{j_1 - j_r}}{s^{j_r}}\right)\right)$$

$$\ll \left(\frac{N}{t}\right)^{1 - c\Lambda\left(\frac{N}{t}, xH\right)} + \left(\frac{N}{t}\right)^{j+1}\left(\frac{x}{t^j}\right)^{-1} \tag{29}$$

$$+ w^{1 - c\Lambda(w, xH)} + H^2 (\log xH)^{2J}.$$

The function

$$g(y) = y^{1 - c'(\log y)^2}$$

is increasing for $1 \leqq y < \exp\left((3c')^{-1/2}\right)$. Without loss of generality we may assume that for c in (29), we have $c \leqq \frac{j^2}{3}$. Then for $N \leqq x^{1/j}$

$$\frac{N}{t} \leqq (xH)^{1/j} = \exp\left(\frac{\log xH}{j}\right) \leqq \exp\left(\left(\frac{3c}{(\log xH)^2}\right)^{-1/2}\right).$$

Hence, for $N \leqq x^{1/j}$, we get by (29)

$$\sum_{w \leqq s \leqq \frac{N}{t}} e\left(x\left(\frac{h_1}{(st)^{j_1}} + \cdots + \frac{h_r}{(st)^{j_r}}\right)\right) \ll \left(\frac{N}{t}\right)^{1 - c\Lambda\left(\frac{N}{t}, xH\right)} + \left(\frac{N}{t}\right)^{j+1} x^{-1} t^j + H^2 (\log xH)^{2J}$$

$$= \left(\frac{N}{t}\right)^{1 - c\Lambda\left(\frac{N}{t}, xH\right)} + N^{j+1}(xt)^{-1} + H^2 (\log xH)^{2J},$$

thus

$$\max_{w > 0} \left| \sum_{w \leqq s \leqq \frac{N}{t}} e\left(x\left(\frac{h_1}{(st)^{j_1}} + \cdots + \frac{h_r}{(st)^{j_r}}\right)\right)\right| \ll \left(\frac{N}{t}\right)^{1 - c\Lambda\left(\frac{N}{t}, xH\right)}$$

$$+ N^{j+1}(xt)^{-1} + H^2 (\log xH)^{2J}.$$

Therefore,

$$S_1 \ll \sum_{t \leqq UV}\left(\left(\frac{N}{t}\right)^{1 - c\Lambda\left(\frac{N}{t}, xH\right)} + N^{j+1}(xt)^{-1} + H^2 (\log xH)^{2J}\right)$$

$$\ll N^{1 - c\Lambda\left(\frac{N}{UV}, xH\right)} \sum_{t \leqq UV} t^{-1 + c\Lambda\left(\frac{N}{UV}, xH\right)} + N^{j+1} x^{-1} \sum_{t \leqq UV} \frac{1}{t} + UVH^2 (\log xH)^{2J}$$

$$\ll N^{1 - c\Lambda\left(\frac{N}{UV}, xH\right)} \Lambda\left(\frac{N}{UV}, xH\right)^{-1} (UV)^{c\Lambda\left(\frac{N}{UV}, xH\right)} + N^{j+1} x^{-1} \log UV$$

$$+ UVH^2 (\log xH)^{2J}.$$

For $U = V = N^{1/3}$ and $N \leqq x^{1/j}$, this together with (28) implies according to Lemma 7

$$\sum_{n \leqq N} \Lambda(n) e\left(x\left(\frac{h_1}{n^{j_1}} + \cdots + \frac{h_r}{n^{j_r}}\right)\right)$$
$$\ll (N^{1-c\Lambda(N,xH)}\Lambda(N,xH)^{-1} + N^{j+1}x^{-1} + N^{\frac{2}{3}}H^2)(\log xH)^{2J+1}$$
$$+ N^{1/3} + (N^{1-c\Lambda(N,xH)} + N^{\frac{j+2}{2}}x^{-\frac{1}{2}} + N^{5/6}H^{\frac{1}{2}})(\log xH)^{4J}$$
$$\ll (N^{1-c\Lambda(N,xH)} + N^{\frac{j+2}{2}}x^{-\frac{1}{2}} + N^{\frac{5}{6}}H^2)(\log xH)^{4J}.$$

Theorem 2. *Let* $2 \leqq N \leqq x^{1/j}$. *Then*

$$\sum_{p \leqq N} e\left(x\left(\frac{h_1}{p^{j_1}} + \cdots + \frac{h_r}{p^{j_r}}\right)\right) \ll (N^{1-c\Lambda(N,xH)} + N^{\frac{j+2}{2}}x^{-\frac{1}{2}} + N^{\frac{5}{6}}H^2)(\log xH)^{4J}.$$

Proof. By Chebyshev's theorem ([2], p. 55),

$$\sum_{n \leqq N} \Lambda(n) e\left(x\left(\frac{h_1}{n^{j_1}} + \cdots + \frac{h_r}{n^{j_r}}\right)\right) = \sum_{\substack{p \quad a \\ p^a \leqq N}}\sum \log p \; e\left(x\left(\frac{h_1}{p^{aj_1}} + \cdots + \frac{h_r}{p^{aj_r}}\right)\right)$$

$$(30) \qquad = \sum_{p \leqq N} \log p \; e\left(x\left(\frac{h_1}{p^{j_1}} + \cdots + \frac{h_r}{p^{j_r}}\right)\right) + O\left(\log N \pi(\sqrt{N})\right)$$

$$= \sum_{p \leqq N} \log p \; e\left(x\left(\frac{h_1}{p^{j_1}} + \cdots + \frac{h_r}{p^{j_r}}\right)\right) + O(\sqrt{N}).$$

Put

$$h(N) = (N^{1-c\Lambda(N,xH)} + N^{\frac{j+2}{2}}x^{-\frac{1}{2}} + N^{\frac{5}{6}}H^2)(\log xH)^{4J}.$$

By partial summation, Lemma 8 and (30) give

$$\left|\sum_{p \leqq N} e\left(x\left(\frac{h_1}{p^{j_1}} + \cdots + \frac{h_r}{p^{j_r}}\right)\right)\right| \leqq \left|\sum_{p \leqq N} \log p \; e\left(x\left(\frac{h_1}{p^{j_1}} + \cdots + \frac{h_r}{p^{j_r}}\right)\right)\right| \frac{1}{\log N}$$
$$+ \left|\int_2^N \sum_{p \leqq t} \log p \; e\left(x\left(\frac{h_1}{p^{j_1}} + \cdots + \frac{h_r}{p^{j_r}}\right)\right) \frac{dt}{t(\log t)^2}\right|$$
$$\ll \frac{h(N)}{\log N} + \int_2^N \frac{h(t)}{t(\log t)^2}\,dt + \frac{\sqrt{N}}{\log N}$$
$$\ll h(N) + (\log xH)^{4J}\int_2^N t^{-c\Lambda(t,xH)}\,dt$$
$$+ (\log xH)^{4J}x^{-1/2}\int_2^N t^{j/2}\,dt + H^2(\log xH)^{4J}\int_2^N t^{-\frac{1}{6}}\,dt$$

$$\ll h(N) + (\log xH)^{4J}\Big(\sqrt{N} + \int_{\sqrt{N}}^{N} t^{-cA(t,xH)}\,dt\Big)$$

$$+ \big(N^{\frac{j+2}{2}} x^{-\frac{1}{2}} + N^{\frac{5}{6}} H^2\big)(\log xH)^{4J}$$

$$\ll h(N) + (\log xH)^{4J}\big(\sqrt{N} + N^{-\frac{1}{2}cA(\sqrt{N},xH)} N\big)$$

$$\ll h(N).$$

This completes the proof of Theorem 2.

4. Vinogradov's Fourier series method

The following method may be found in [18], p. 32, or [1], Lemma 2.1.

Let $0 < \Delta < \frac{1}{4}$. For $J \in \mathbb{N}$ and real numbers A_j and B_j $(1 \leqq j \leqq J)$ with

$$0 \leqq B_j - A_j \leqq 1 - 2\Delta,$$

there are 1-periodic functions $\psi_j(z)$, satisfying

$$\psi_j(z) = \begin{cases} 1 & \text{for } A_j \leqq z \leqq B_j, \\ 0 & \text{for } B_j + \Delta \leqq z \leqq 1 + A_j - \Delta, \end{cases}$$

and $0 \leqq \psi_j(z) \leqq 1$ for all z, such that

(31) $$\psi_j(z) = B_j - A_j + \Delta + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} a_{m,j}\, e(mz),$$

where $a_{m,j} \in \mathbb{C}$ and for $|m| > 0$ and $1 \leqq j \leqq J$

(32) $$|a_{m,j}| \ll \frac{1}{m^2 \Delta}.$$

Theorem 3. *Let* $2 \leqq P \leqq x^{1/J}$, $\underline{\sigma} = (\sigma_1, \ldots, \sigma_J)$ *with* $0 < \sigma_j \leqq 1$ *for* $1 \leqq j \leqq J$ *and*

$$D(\underline{\sigma}) = D(\underline{\sigma}; P, x) = \operatorname{card}\left\{p \leqq P : \left\{\frac{x}{p^j}\right\} < \sigma_j\ (1 \leqq j \leqq J)\right\}.$$

Then we have for arbitrary $\varepsilon > 0$

$$D(\underline{\sigma}) = \sigma_1 \cdots \sigma_J \pi(P) + O\big(P^{1-cA(P,x)} + P^{\frac{J+2}{2}+\varepsilon} x^{-\frac{1}{2}}\big)(\log x)^{4J}.$$

Proof. For $\underline{A} = \{A_1, \ldots, A_J\}$, $\underline{B} = \{B_1, \ldots, B_J\}$, let

$$T(\underline{A}, \underline{B}) = \operatorname{card}\left\{p \leqq P : A_j \leqq \left\{\frac{x}{p^j}\right\} \leqq B_j \, (1 \leqq j \leqq J)\right\}.$$

Then

$$T(\underline{A}, \underline{B}) \leqq \sum_{p \leqq P} \left(\sum_{j=1}^{J} \psi_j\left(\frac{x}{p^j}\right)\right) \leqq T(\underline{A} - \underline{\Delta}, \underline{B} + \underline{\Delta}), \tag{33}$$

where $\underline{\Delta} = (\Delta, \ldots, \Delta)$. By (31),

$$\prod_{j=1}^{J} \psi_j\left(\frac{x}{p^j}\right) = \prod_{j=1}^{J} (B_j - A_j + \Delta) + O\left(\sum_{\emptyset \neq \Gamma \subseteqq \{1, \ldots, J\}} \prod_{j \in \Gamma} \left(\sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} a_{m,j}\, e\left(\frac{mx}{p^j}\right)\right)\right). \tag{34}$$

By (32),

$$\begin{aligned} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} a_{m,j}\, e\left(\frac{mx}{p^j}\right) &= \sum_{0 < |m| < \Delta^{-2}} a_{m,j}\, e\left(\frac{mx}{p^j}\right) + O\left(\sum_{|m| \geqq \Delta^{-2}} a_{m,j}\, e\left(\frac{mx}{p^j}\right)\right) \\ &= \sum_{0 < |m| < \Delta^{-2}} a_{m,j}\, e\left(\frac{mn}{p^j}\right) + O\left(\sum_{|m| \geqq \Delta^{-2}} \frac{1}{m^2 \Delta}\right) \\ &= \sum_{0 < |m| < \Delta^{-2}} a_{m,j}\, e\left(\frac{mx}{p^j}\right) + O(\Delta). \end{aligned} \tag{35}$$

Define L to be the $O(\,)$ term in (34). The preceding equation yields

$$\begin{aligned} L &= \sum_{r=1}^{J} \sum_{1 \leqq j_1 < \ldots < j_r \leqq J} \prod_{i=1}^{r} \left(\sum_{0 < |m_i| < \Delta^{-2}} a_{m_i, j_i}\, e\left(\frac{m_i x}{p^{j_i}}\right) + O(\Delta)\right) \\ &\ll \sum_{r=1}^{J} \sum_{1 \leqq j_1 < \ldots < j_r \leqq J} \sum_{i=1}^{r} \left(\sum_{0 < |m_i| < \Delta^{-2}} a_{m_i, j_i}\, e\left(\frac{m_i x}{p^{j_i}}\right)\right) + O(\Delta). \end{aligned}$$

Since $P \leqq x^{1/J}$, we get by (32), (34) and Theorem 2

$$\begin{aligned} \sum_{p \leqq P} \left(\prod_{j=1}^{J} \psi_j\left(\frac{x}{p^j}\right)\right) &= \sum_{p \leqq P} \left(\prod_{j=1}^{J} (B_j - A_j + \Delta)\right) \\ &\quad + O\left(\max_{1 \leqq j_1 < \ldots < j_r \leqq J} \left| \sum_{0 < |m_1| < \Delta^{-2}} \cdots \sum_{0 < |m_r| < \Delta^{-2}} a_{m_1, j_1} \cdots a_{m_r, j_r} \right.\right. \\ &\quad \left.\left. \times \sum_{p \leqq P} e\left(x\left(\frac{m_1}{p^{j_1}} + \cdots + \frac{m_r}{p^{j_r}}\right)\right)\right|\right) + O(P\Delta) \\ &= \sum_{p \leqq P} \left(\prod_{j=1}^{J} (B_j - A_j + \Delta)\right) \\ &\quad + O\left((P^{1 - c\Lambda(P, x\Delta^{-2})} + P^{\frac{J+2}{2}} x^{-\frac{1}{2}} + P^{\frac{5}{6}} \Delta^{-4})(\log x \Delta^{-2})^{4J} \Delta^{-J}\right), \end{aligned} \tag{36}$$

if we choose for some suitable $\gamma = \gamma(J) > 0$

$$\Delta = P^{-\gamma A(P,x)}.$$

At this point, we have to check that $\Delta < \frac{1}{4}$ as required by the initial conditions. This can be done by distinguishing two cases. If $(\log P)^3 \ll (\log x)^2$, we have

$$P^{1-cA(P,x)}(\log x)^{4J} \gg P,$$

hence the theorem obviously holds. Therefore, we may assume $(\log P)^3 \gg (\log x)^2$, which yields the desired inequality for suitable γ. Then for arbitrarily small $\varepsilon > 0$

$$\begin{aligned}\sum_{p \leqq P} \left(\prod_{j=1}^{J} \psi_j \left(\frac{x}{p^i} \right) \right) &= \left(\prod_{j=1}^{J} (B_j - A_j + \Delta) \right) \pi(P) \\ &\quad + O\left((P^{1-cA(P,x)} + P^{\frac{J+2}{2}+\varepsilon} x^{-\frac{1}{2}} + P^{\frac{5}{6}+\varepsilon}) (\log x)^{4J} \right) \\ &= \left(\prod_{j=1}^{J} (B_j - A_j + \Delta) \right) \pi(P) \\ &\quad + O\left((P^{1-cA(P,x)} + P^{\frac{J+2}{2}+\varepsilon} x^{-\frac{1}{2}}) (\log x)^{4J} \right).\end{aligned}$$

Let R denote the error term of the last equality. Then, by (33),

$$T(\underline{A}, \underline{B}) \leqq \pi(P) \prod_{j=1}^{J} (B_j - A_j + \Delta) + |R| \tag{37}$$

and

$$T(\underline{A} - \underline{\Delta}, \underline{B} + \underline{\Delta}) \geqq \pi(P) \prod_{j=1}^{J} (B_j - A_j + \Delta) - |R|. \tag{38}$$

Replacing $\underline{A}, \underline{B}$ by $\underline{A} - \underline{\Delta}, \underline{A}$ resp. $\underline{B}, \underline{B} + \underline{\Delta}$, (37) implies

$$T(\underline{A} - \underline{\Delta}, \underline{A}) \leqq (2\Delta)^J \pi(P) + |R|$$

respectively

$$T(\underline{B}, \underline{B} + \underline{\Delta}) \leqq (2\Delta)^J \pi(P) + |R|.$$

Thus, by (38),

$$\begin{aligned}T(\underline{A}, \underline{B}) &= T(\underline{A} - \underline{\Delta}, \underline{B} + \underline{\Delta}) - T(\underline{A} - \underline{\Delta}, \underline{A}) - T(\underline{B}, \underline{B} + \underline{\Delta}) \\ &\geqq \pi(P) \prod_{j=1}^{J} (B_j - A_j + \Delta) - 3|R| - 2(2\Delta)^J \pi(P) \\ &= \pi(P) \prod_{j=1}^{J} (B_j - A_j) + O(P^{1-cA(P,x)}) + O(R) \\ &= \pi(P) \prod_{j=1}^{J} (B_j - A_j) + O(R).\end{aligned}$$

Similarly, we get by (37)

$$T(\underline{A}, \underline{B}) \leqq \pi(P) \prod_{j=1}^{J} (B_j - A_j) + O(R).$$

Together, we have

$$T(\underline{A}, \underline{B}) = \pi(P) \prod_{j=1}^{J} (B_j - A_j) + O(R).$$

Setting $A_j = 0$, $B_j = \sigma_j$ $(1 \leqq j \leqq J)$, the desired result follows by observing that

$$D(\underline{\sigma}; P, x) = T(\underline{A}, \underline{B}).$$

5. Proof of Theorem 1

Let m and n be positive integers, and p a prime. We define $U_p(m, n)$ to be the number of "carries" which occur when adding m and n in p-adic notation. Moreover, let $e(n; p) = \max\{a \geqq 0 : p^a | n\}$. An old result of Kummer is the following

Lemma 9 ([9], p. 116).

$$e\left(\binom{m+n}{m}; p\right) = U_p(m, n).$$

Now we are in the position to give the proof of Theorem 1. By (1),

$$-\frac{m}{2} < -m^{1-\varepsilon} < m - 2k < m^{1-\varepsilon} < \frac{m}{2}$$

for sufficiently large m. Hence

$$m < \min(4k, 4(m-k)). \tag{39}$$

Thus

$$|m - 2k| < (\min(k, m-k))^{1-\varepsilon'} \tag{40}$$

for a suitable $\varepsilon' > 0$. Set

$$n = \min(k, m-k), \quad d = |m - 2k|.$$

Then

$$\binom{2n+d}{n} = \binom{m}{k}. \tag{41}$$

Let $\delta = \frac{\varepsilon'}{3}$; let $J \in \mathbb{N}$ with $J > \frac{12a}{\varepsilon'}$. For sufficiently large n and $K = \left[\frac{J}{1+\delta}\right]$, we get by Theorem 3 and the prime number theorem that

$$\left\{n^{1/(J+1)} < p < (2n)^{1/(J+1)} : \frac{2}{3} < \left\{\frac{n}{p^j}\right\} (K < j \leqq J), \left\{\frac{n}{p^K}\right\} < \frac{1}{2}\right\}$$

is not empty, i.e. there is a prime p satisfying

$$\frac{1}{2} p^{J+1} < n < p^{J+1}, \tag{42}$$

$$\left\{\frac{n}{p^j}\right\} > \frac{2}{3} \quad (K < j \leqq J) \tag{43}$$

and

$$\left\{\frac{n}{p^K}\right\} < \frac{1}{2}. \tag{44}$$

Write n in p-adic notation, namely

$$n = n_J p^J + n_{J-1} p^{J-1} + \cdots + n_1 p + n_0 \quad (0 \leqq n_j < p).$$

For $K < j \leqq J$, we have by (43)

$$\frac{2}{3} < \left\{\frac{n}{p^j}\right\} = \frac{n_{j-1} p^{j-1} + \cdots + n_0}{p^j},$$

thus

$$\frac{n_{j-1}}{p} > \frac{2}{3} - (p-1)\left(\frac{1}{p^2} + \cdots + \frac{1}{p^j}\right) > \frac{2}{3} - \frac{1}{p}.$$

This implies for $p \geqq 7$

$$n_{j-1} > \frac{1}{2} p,$$

i.e.

$$n_j > \frac{1}{2} p \quad (K \leqq j < J). \tag{45}$$

By (44), we get in a similar fashion

$$n_{K-1} < \frac{1}{2} p. \tag{46}$$

By (40) and (42), as well as the choice of d, n and J,

$$0 \leqq d = |m - 2k| < n^{1-\varepsilon'} < p^{(J+1)(1-\varepsilon')} < p^{K-1}.$$

Writing d in p-adic notation, too, we therefore get

$$d = d_{K-2} p^{K-2} + \cdots + d_0$$

with integers d_j, $0 \leqq d_j < p$. Thus we have by (42), (45) and (46)

$$n = n_J p^J + \cdots + n_{K-1} p^{K-1} + \cdots + n_0 ,$$

$$n + d = n'_J p^J + \cdots + n'_{K-1} p^{K-1} + \cdots + n'_0 ,$$

where

$$n_j = n'_j > \frac{1}{2} p \quad (K \leqq j \leqq J) .$$

By Lemma 9,

$$e\left(\binom{2n+d}{n}; p\right) \geqq J - K \geqq a ,$$

which means that there is a p satisfying $p^a | \binom{2n+d}{n}$. By the definition of n and (39) we have

$$n > \frac{1}{4} m ,$$

thus by (42)

$$p > \left(\frac{m}{4}\right)^{\frac{1}{J+1}} \geqq \frac{1}{2} m^{\frac{1}{J+1}} .$$

By (41), the proof of Theorem 1 is completed.

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