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Deformation quantization of endomorphism bundles

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1. Introduction

A deformation quantization of a Poisson manifold $(M, \{\cdot\,,\cdot\})$ is an associative product $*$ on $C^\infty(M)[[\hbar]]$ so that

$$f * g = fg + \hbar P_1(f, g) + \hbar^2 P_2(f, g) + \cdots,$$

$*$ being $\hbar$ linear, P_i bidifferential, and

$$[f, g] = i\hbar\{f, g\} + O(\hbar),$$

where $[\cdot\,,\cdot]$ is the commutator with respect to $*$.

This notion was first introduced in [1].

The question of existence and classification of deformation quantizations on general Poisson manifolds was solved in 1997 by Kontsevich in [9].

The simpler case of existence of deformation quantizations of the canonical Poisson structure on symplectic manifolds was solved already in [16]. A simple geometric construction of deformation quantizations of symplectic manifolds was given by Fedosov in [6]. The advantage of Fedosov's construction compared to the ones in [9] and [16] is that it is easy to handle and also suitably generalizable. The most general setting of the Fedosov construction is probably given in [15], where deformation quantizations of symplectic Lie algebroids is done. Also the classification of deformation quantizations becomes amenable in view of the Fedosov construction. In the case of symplectic manifolds this was done in [11], and the classification of deformation quantizations on a symplectic manifold (M, ω) is given by the points (characteristic classes) θ in the space

$$\frac{\omega}{i\hbar} + H^2(M, \mathbb{C}[[\hbar]]).$$

One of the main classes of examples of deformation quantizations of symplectic manifolds are those coming from the asymptotic calculus of pseudodifferential operators on manifolds, see for example [13]. If we consider the asymptotic calculus of pseudodifferential operators on a manifold M, we will get a deformation quantization of the cotangent bundle T^*M of M, where T^*M is equipped with the canonical symplectic structure.

This example gives the connection to index theory. On a deformation quantization of any symplectic $2n$ dimensional manifold there is a canonical trace, unique up to multiplication by a scalar, of the form

$$\mathrm{Tr}(a) = \frac{1}{n!(i\hbar)^n}\Big(\int_M a\omega^n + O(\hbar)\Big). \tag{1.1}$$

By an appropriate choice of the representation of the quantization, i.e. after applying a linear isomorphism of $C^\infty(M)[[\hbar]]$ of the form

$$f \to f + \hbar D_1(f) + \cdots$$

one can assure that Tr has the form

$$\mathrm{Tr}(a) = \frac{1}{n!(i\hbar)^n}\int_M a\omega^n,$$

which fixes it uniquely.

In most proofs of the Atiyah-Singer index theorem and related "local" index theorems, one of the main difficulties is to compute the trace of a certain operator on a Hilbert space, usually $L^2(M)$ as above. In order to compute this trace, a scaling $\hbar$ in $\mathbb{R}_+$ of the operator is introduced, and the asymptotic expansion of the trace as $\hbar \to 0$ becomes computable, at least the constant term in the expansion. The computations coming out of this are computations like (1.1). This is why computing the canonical trace on deformation quantizations is called algebraic index theory. Actually, according to [13], computing the trace on deformation quantizations in a way that will be described now, implies the Atiyah-Singer index theorem.

Many elements, though not all, on which computing the trace is interesting, are first components of classes in cyclic periodic homology. The cyclic periodic homology or rather cohomology was invented by Connes in [5]. It is the noncommutative analog of de Rahm cohomology and was already at the beginning intimately connected to index theory. A complex computing the cyclic periodic homology of a unital algebra A over a field k is given by

$$CC^{\mathrm{per}}_{\mathrm{even}}(A) = \prod_i A \otimes \bar{A}^{\otimes 2i}; \quad CC^{\mathrm{per}}_{\mathrm{odd}}(A) = \prod_i A \otimes \bar{A}^{\otimes 2i+1}$$

where $\bar{A} = A/k \cdot 1$ and the differential

$$CC^{\mathrm{per}}_{\mathrm{even}}(A) \overset{b+B}{\longleftrightarrow} CC^{\mathrm{per}}_{\mathrm{odd}}(A)$$

is given by

$$b(a_0 \otimes \cdots \otimes a_n)$$
$$= \sum_{k=0}^{n-1} (-1)^k a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

and

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n} (-1)^k 1 \otimes a_k \otimes a_{k+1} \cdots \otimes a_n \otimes a_0 \dots a_{k-1}.$$

If for example $p \in M_n(A)$ is a projection, a class in $HC^{\mathrm{per}}_{\mathrm{even}}(A)$ is given by the formula

$$\mathrm{tr}\left(p + \sum_{k \geqq 1} \frac{(2k)!}{k!} (p - 1/2) \otimes p^{\otimes 2k} \right),$$

where

$$\mathrm{tr} : M_n(A)^{\otimes k} \to A^{\otimes k}$$

is the map given by

$$(M_1 \otimes a_1) \otimes \cdots \otimes (M_k \otimes a_k) \mapsto \mathrm{tr}(M_1 \dots M_k) a_1 \otimes \cdots \otimes a_k.$$

Therefore $\mathrm{tr}(p)$ can be regarded as the first component of a class in cyclic periodic homology. This class is also called the Chern character of p.

Evaluating Tr on the first component gives a morphism of complexes

$$\mathrm{Tr} : CC^{\mathrm{per}}_{*}(A^{\hbar}_{c}) \to \mathbb{C}[[\hbar, \hbar^{-1}], \tag{1.2}$$

where $A^{\hbar}_{c}$ is the algebra of compactly supported elements $C^{\infty}_{\mathrm{c}}(M)[[\hbar]])$ in a deformation quantization $A^{\hbar}$, and $\mathbb{C}[[\hbar, \hbar^{-1}]$ is considered as a complex concentrated at degree zero with the trivial boundary map.

Computing the trace on elements that are first components of a class in cyclic periodic homology is therefore the same as computing (1.2) at the level of homology.

In [11] it is proved that

$$\mathrm{Tr}(\cdot) \sim (-1)^n \int_M \hat{A}(TM) e^{\theta} \tilde{\mu}(\cdot)$$

where $\sim$ means that the two sides define the same morphism at the level of homology. Here θ is the characteristic class of the deformation and $\tilde{\mu}$ is the map $CC^{\mathrm{per}}_{*}(A^{\hbar}) \to \Omega^{*}(M)$ given by

$$\tilde{\mu}(a_0 \otimes \cdots \otimes a_k) = \frac{1}{k!} \tilde{a}_0 d\tilde{a}_1 \cdots d\tilde{a}_k, \quad \tilde{a}_i = a_i \bmod \hbar.$$

This settles the problem of computing the trace at the level of homology for deformation quantizations of symplectic manifolds.

1.1. Contents of the paper. Below I propose a definition of deformation quantization of endomorphism bundles over a symplectic manifold. The motivation is clear: Deformation quantizations of the trivial line bundle are the algebraic analogs of pseudodifferential operators in line bundles. Therefore deformation quantizations of an endomorphism bundle $\mathrm{End}(E)$, E vector bundle over M, should be the algebraic analogs of pseudodifferential operators in any vector bundle having $\mathrm{End}(E)$ as endomorphism bundle.

The definition proposed requires a product $*$ on

$$\Gamma\big(\mathrm{End}(E)\big)[[\hbar]]$$

so the algebra $\big(\Gamma\big(\mathrm{End}(E)\big)[[\hbar]], *\big)$ is locally isomorphic to $M_N(\mathscr{W}_n)$; here $\mathscr{W}_n$ is the Weyl algebra, the canonical deformation quantization of the standard symplectic structure on $\mathbb{R}^{2n}$.

It turns out that the Fedosov construction also works in this case. Thus let $M_N(\mathbb{A}^\hbar)$ be the algebra of jets at zero of elements in $M_N(\mathscr{W}_n)$. Associated to $\big(M, \omega, \mathrm{End}(E)\big)$ there is an algebra bundle $\mathbb{W}$ with fiber $M_N(\mathbb{A}^\hbar)$. Put $\mathfrak{g} = \mathrm{Der}\big(M_N(\mathbb{A}^\hbar)\big)$. There is now a short exact sequence

$$0 \to \frac{1}{\hbar}\mathbb{C}[[\hbar]] \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

of Lie algebras. The Fedosov construction then consists, for a given element θ in

$$\frac{\omega}{i\hbar} + H^2(M, \mathbb{C}[[\hbar]]),$$

in constructing a flat connection ∇ in $\mathbb{W}$ with values in $\mathfrak{g}$, such that $\ker\nabla \simeq \Gamma\big(\mathrm{End}(E)\big)[[\hbar]]$ linearly and ∇ admits a lift $\tilde{\nabla}$ to a connection with values in $\tilde{\mathfrak{g}}$ and curvature θ. The product on $\Gamma(\mathbb{W})$ induces a product on $\ker\nabla = \Gamma\big(\mathrm{End}(E)\big)[[\hbar]]$. This product gives a deformation quantization of $\mathrm{End}(E)$, and θ will be an isomorphism invariant of the deformation quantization.

This construction is done in section 3. In this section the following is also shown.

Theorem 1. *A deformation quantization of* $\mathrm{End}(E)$ *is isomorphic to the algebra of flat sections of a Fedosov connection, and the isomorphism classes of deformation quantizations of* $\mathrm{End}(E)$ *are classified by the points in*

$$\frac{\omega}{i\hbar} + H^2(M, \mathbb{C}[[\hbar]]).$$

The principle that a deformation quantization comes as flat sections in a certain infinite dimensional vector bundle is not special to deformation quantizations. In section 2 it is shown that sections of $\Gamma\big(\mathrm{End}(E)\big)$ are flat sections in an algebra bundle with fibre

$$M_N(\mathbb{C}[[\hat{x}_1, \ldots, \hat{x}_n]]).$$

The reason for redoing this construction for $\mathrm{End}(E)$ is that it is notationally simpler, and, hopefully, clarifies the construction. Therefore in section 3 only the differences in the construction for $\mathrm{End}(E)$ and deformation quantizations of endomorpism bundles are spelled out.

Like in the scalar case there are canonical traces on deformation quantizations of endomorphism bundles. The rest of the paper is devoted to an index theory for these traces. The methods used for this have been developed by Nest and Tsygan in [11], [12], [2] and [15]. These methods are based on the following:

(1) The action of the reduced cyclic complex $\overline{C}^{\lambda}_{*}(A)$ on $CC^{\mathrm{per}}_{*}(A)$:

$$\overline{C}^{\lambda}_{*}(A) \times CC^{\mathrm{per}}_{*}(A) \to CC^{\mathrm{per}}_{*}(A).$$

(2) The construction of the fundamental class, a special class in $\overline{C}^{\lambda}_{*}(A^{\hbar}_{E})$, where $A^{\hbar}_{E}$ is the deformation quantization of $\mathrm{End}(E)$. Or rather the construction of a class in the Čech complex, $\check{C}^{*}\big(M, \overline{C}^{\lambda}_{*}(A^{\hbar}_{E})\big)$, with values in the presheaf $V \to \overline{C}^{\lambda}_{*}(A^{\hbar}_{E|V})$.

(3) Computations in Lie algebra cohomology in order to identify the fundamental class at the level of cohomology.

The fundamental class U lives in $\overline{C}^{\lambda}_{2n-1}\big(M_N(\mathscr{W}_n)\big)$. Its role is that it relates Tr to $\tilde{\mu}$ when evaluated at classes that are scalar $\bmod \hbar$. This has the effect that

$$\mathrm{Tr}(U \cdot a) = (-1)^n \int \tilde{\mu}(a), \quad a \in CC^{\mathrm{per}}_{*}\big(M_N(\mathscr{W}_{n,\mathrm{c}})\big).$$

Here, as before, the subscript "c" denotes the ideal of compactly supported elements in the deformation algebra in question.

The following plays the major role.

Theorem 2. *In cohomology, the class U has a unique extension to a class in $\check{C}^{*}\big(M, \overline{C}^{\lambda}_{*}(A^{\hbar}_{E})\big)$, also denoted by U. On classes of the form $a_0 \otimes \cdots \otimes a_k \in CC^{\mathrm{per}}_{*}(A^{\hbar}_{E,\mathrm{c}})$, where a_i is a scalar* $\bmod \hbar$*, the following holds*:

$$\mathrm{Tr}(U \cdot a) = (-1)^n \int \tilde{\mu}(a) \bmod \hbar.$$

It is not difficult to see that this implies for general classes $a_0 \otimes \cdots \otimes a_k$ in $CC^{\mathrm{per}}_{*}(A^{\hbar}_{E,\mathrm{c}})$ that

$$\mathrm{Tr}(U \cdot a_0 \otimes \cdots \otimes a_k) = (-1)^n \int \mathrm{ch}\big(\mathrm{End}(E)\big)^{-1} \mathrm{ch}(\nabla)(\tilde{a}_0 \otimes \cdots \otimes \tilde{a}_k) \bmod \hbar$$

where $\mathrm{ch}\big(\mathrm{End}(E)\big)$ is the usual Chern character of $\mathrm{End}(E)$ as a vector bundle, ∇ is a connection in $\mathrm{End}(E)$, and

$$\begin{aligned} &\mathrm{ch}(\nabla)(\tilde{a}_0 \otimes \cdots \otimes \tilde{a}_k) \\ &\quad = \int_{\Delta_k} \mathrm{tr}\big(\tilde{a}_0 e^{-t_0\nabla^2} \nabla(\tilde{a}_i) e^{-t_1\nabla^2} \cdots \nabla(\tilde{a}_k) e^{-t_k\nabla^2}\big)\, dt_0 \cdots dt_{k-1} \end{aligned}$$

the J.L.O. cocycle associated to ∇.

To finish the index theory, the fundamental class has to be identified. This is done via Lie algebra cohomology. We have the Gelfand-Fuks morphism of complexes

$$C^*\big(\mathfrak{g}, \mathfrak{su}(N) + \mathfrak{u}(n); \bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar)\big)\big) \to \check{C}^*\big(M, \Omega^*\big(M, \bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar)\big)\big)\big),$$

the latter complex being quasi-isomorphic to $\check{C}^*\big(M, \bar{C}_*^\lambda(A_E^\hbar)\big)$. As in the case of $M_N(\mathscr{W}_n)$, there is a fundamental class U in $\bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar)\big)$ extending uniquely in cohomology to a class in Lie algebra cohomology, also denoted by U. It turns out that $GF(U)$ is equivalent to U in $\check{C}^*\big(M, \bar{C}_*^\lambda(A_E^\hbar)\big)$ via the quasi-isomorphism between $\check{C}^*\big(M, \bar{C}_*^\lambda(A_E^\hbar)\big)$ and $\check{C}^*\big(M, \Omega^*\big(M, \bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar)\big)\big)\big)$. Hence the question of computing or identifying U is now a question of computations in

$$C^*\big(\mathfrak{g}, \mathfrak{su}(N) + \mathfrak{u}(n); \bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar)\big)\big).$$

It turns out to be useful to work with the differential graded algebra $M_N(\mathbb{A}^\hbar)[\eta]$, where η is a formal variable, $\eta^2 = 0$ and the differential is given by $\dfrac{\partial}{\partial \eta}$.

The reason for doing this is to include the identity operation on $CC_*^{\mathrm{per}}\big(M_N(\mathbb{A}^\hbar)\big)$. Thus the action of $\bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar)\big)$ on $CC_*^{\mathrm{per}}\big(M_N(\mathbb{A}^\hbar)\big)$ extends to an action of $\bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar)[\eta]\big)$ on $CC_*^{\mathrm{per}}\big(M_N(\mathbb{A}^\hbar)\big)$. With this action, the classes $\eta^{(k+1)} = k!\eta^{\otimes k+1}$ become the identity operations. The main technical theorem of this paper, Theorem 6.0.3, states the following:

Theorem 3. *U is equivalent to*

$$\sum_{m \geqq 0} \big(\hat{A} \cdot e^\theta \cdot (\mathrm{ch})^{-1}\big)_{2m}^{-1} \cdot \eta^{(m)}$$

in $C^*\big(\mathfrak{g}, \mathfrak{su}(N) + \mathfrak{u}(n); \bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar)[\eta]\big)\big)$, *i.e. it defines the same cohomology class.*

Here $\hat{A}$ is the Lie algebraic class $\hat{A}$ coming from $\mathfrak{u}(n)$, ch *is the Lie algebraic Chern character coming from $\mathfrak{su}(N)$ and θ is the Lie algebraic class of deformation.*

From this follows the index theorem:

Theorem 4. *Let $A_E^\hbar$ be a deformation quantization of* $\mathrm{End}(E)$ *and let θ be the associated characteristic class. Then the identity*

$$\mathrm{Tr}(a) = (-1)^n \int \hat{A} \cdot e^\theta \cdot \mathrm{ch}(\nabla)(\tilde{a})$$

holds when a is a cycle in $CC_*^{\mathrm{per},\mathbb{C}}(A_{E,\mathrm{c}}^\hbar)$.

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Similar notions of deformation quantization of endomorphism bundles have been studied by Fedosov [7] and by Bursztyn and Waldmann [4].

2. Sections of endomorphism bundles as flat sections in a profinite bundle

Let $\mathbb{O}_n = \mathbb{C}[[\hat{x}_1, \dots, \hat{x}_n]]$ and $M_N(\mathbb{O}_n) = M_N \otimes \mathbb{O}_n$, where M_N is the algebra of $N \times N$ complex matrices. We give $M_N(\mathbb{O}_n)$ a grading by setting

$$\deg(b \otimes \hat{x}_i) = 1, \quad b \in M_N.$$

Furthermore we give $M_N(\mathbb{O}_n)$ the I-adic topology, where I is the ideal generated by elements of degree $\geqq 1$.

Definition 2.0.1. Let G be the group of continuous automorphisms of $M_N(\mathbb{O}_n)$ such that the induced automorphism on the centre $\mathbb{O}_n$ is an automorphism induced by an automorphism on $\mathbb{R}[[\hat{x}_1, \dots, \hat{x}_n]]$.

Lemma 2.0.2. *An automorphism in G is the composition of an automorphism induced by an automorphism of* $\mathbb{R}[[\hat{x}_1, \dots, \hat{x}_n]]$ *and an inner automorphism.*

Proof. Given an automorphism Φ in G let φ be the induced automorphism on $\mathbb{O}_n$. Considering $\chi = \Phi \circ (\varphi \otimes \mathrm{id})^{-1}$ we have that χ is an $\mathbb{O}_n$-module map.

For A in $M_N(\mathbb{C})$ we have

$$\chi(A) = D_0(A) + \text{higher order terms}, \quad D_0(A) \in M_N.$$

Since D_0 is an automorphism of M_N, it is inner, and hence it extends to $M_N(\mathbb{O}_n)$. Let $\chi_1 = \chi \circ D_0^{-1}$. Then

$$\chi_1(A) = A + \sum_i \hat{x}_i D_i(A) + \text{higher order terms}.$$

Each D_i is a derivation of M_N and hence given by a commutator by an element B_i in M_N. Therefore

$$\chi_1 \circ \exp\Big(\sum_i \mathrm{ad}(\hat{x}_i B_i)\Big)(A) = A + \text{terms of order} \geqq 2.$$

Continuing by induction, we get a sequence of elements C_k in $M_N(\mathbb{O}_n)$, where $\deg(C_k) = k$, with

$$\big(\chi \circ \exp\big(\mathrm{ad}(C_0)\big) \circ \cdots \circ \exp\big(\mathrm{ad}(C_k)\big)\big)(A) = A + \text{terms of order} \geqq k+1.$$

Since the product

$$\exp\big(\mathrm{ad}(C_0)\big) \circ \cdots \circ \exp\big(\mathrm{ad}(C_k)\big) \circ \cdots$$

converges, we see by the Hausdorff-Campbell formula that χ is inner.

From Lemma 2.0.2 we get:

Proposition 2.0.3. *The Lie algebra of the Lie group G is*

$$\mathbb{W}_n^0 \ltimes \big(\mathfrak{gl}_N(\mathbb{O}_n)/\text{centre}\big),$$

where $\mathbb{W}_n^0$ *is the Lie algebra of formal vector fields vanishing at zero.*

We note that $\mathrm{Der}\big(M_N(\mathbb{O}_n)\big)$ is larger than $\mathbb{W}_n^0 \ltimes \big(\mathfrak{gl}(\mathbb{O}_n)/\text{centre}\big)$, namely

$$\mathbb{W}_n \ltimes \big(\mathfrak{gl}_N(\mathbb{O}_n)/\text{centre}\big),$$

where $\mathbb{W}_n$ are all formal vector fields on $\mathbb{R}^n$.

Lemma 2.0.4. *Let* X *be a contractible open subset of* $\mathbb{R}^k$, *let* $\mathrm{Aut}(M_n)$ *be the automorphism group of* M_n, *and let* $\mathrm{Der}(M_n)$ *be the derivations of* M_n. *Any smooth map* $\varphi_1 : X \to \mathrm{Aut}(M_n)$ *(resp.* $\varphi_2 : X \to \mathrm{Der}(M_n)$*) lifts to a smooth map* $\tilde{\varphi}_1 : X \to \mathrm{Gl}_n$ *(resp.* $\tilde{\varphi}_2 : X \to M_n$*).*

Proof. First the case of φ_1:

Let $\{e_{ij}\}$ be the standard matrix units. The families

$$x \to \varphi_1(x)(e_{ii})$$

of projections over X give rise to a family of line bundles $\{l_i\}$ over X by

$$l_i = \big(\varphi_1(x)(e_{ii})\big)(\mathbb{R}^n).$$

Since X is contractible, these line bundles are trivial. Let v_1 be a smooth nowhere vanishing section of l_1, and let

$$v_i(x) = \big(\varphi_1(x)(e_{i1})\big)\big(v_1(x)\big).$$

Put

$$\big(\tilde{\varphi}_1(x)\big)(e_i) = v_i(x),$$

where e_i is the vector $(a_1, \ldots, a_n) \in \mathbb{R}^n$ with $a_j = 0,\ j \neq i$, and $a_i = 1$. Since

$$\big(\tilde{\varphi}_1(x)e_{ij}\big(\tilde{\varphi}_1(x)\big)^{-1}\big)v_k(x) = \tilde{\varphi}_1(x)e_{ij}e_k = \tilde{\varphi}_1(x)e_i\delta_{jk} = \delta_{jk}v_i(x),$$

and

$$\begin{aligned}\varphi_1(x)(e_{ij})v_k(x) &= \big(\varphi_1(x)(e_{ij})\big)\big(\varphi_1(x)(e_{k1})\big)v_1(x)\\ &= \delta_{kj}\varphi_1(x)(e_{i1})\big(v_1(x)\big) = \delta_{kj}v_i(x),\end{aligned}$$

we have that $\tilde{\varphi}_1 : X \to \mathrm{Gl}_n$ is smooth, and that

$$\tilde{\varphi}_1(x)A\big(\tilde{\varphi}_1(x)\big)^{-1} = \varphi_1(x)(A), \quad A \in M_n.$$

In the case of φ_2 let $\varphi : X \times \mathbb{R} \to \mathrm{Aut}(M_n)$ be defined by

$$\varphi(x,t) = \exp\big(t\varphi_2(x)\big).$$

This is clearly a smooth map, and hence by the first part of the lemma we get a smooth lifting of φ to $\tilde{\varphi} : X \times \mathbb{R} \to \mathrm{Gl}_n$.

Defining

$$\tilde{\varphi}_2(x) = \frac{\partial}{\partial t}(\tilde{\varphi})(x,0),$$

the lemma follows.

In view of Lemma 2.0.4, the proof of Lemma 2.0.2 actually shows

Lemma 2.0.5. *A smooth family of elements in G over a contractible open subset X of $\mathbb{R}^k$ lifts to a smooth family over X of elements in the group of invertible elements in $M_n(\mathbb{O}_n)$.*

2.1. Jets of sections of endomorphism bundles. Let A be an algebra bundle over a manifold M. In this case we let

$$I_m = \{a \in \Gamma(A) \mid a(m) = 0\}.$$

Given this, we define

$$J_m A = \varprojlim_k \Gamma(A)/I_m^k.$$

Denote by J_m the quotient map from $\Gamma(A)$ into $J_m A$. In the following we are only interested in the case, where $A = \mathrm{End}(E)$, E a vector bundle. Note that $J_m A \simeq M_N(\mathbb{O}_n)$ by choosing a trivialization; and that any other trivialization leads to an automorphism of $M_N(\mathbb{O}_n)$ belonging to G.

Suppose we are given a smooth path of automorphisms Φ_t in G, according to Lemma 2.0.5 we can write it as $\Phi_t = \varphi_t \circ \chi_t$, where φ_t is a smooth path of automorphisms induced by automorphisms of $\mathbb{R}[[\hat{x}_1, \dots, \hat{x}_n]]$ and χ_t is a smooth path of inner automorphisms of $M_N(\mathbb{O}_n)$. It is well known that φ_t lifts to a smooth path of local diffeomorphisms $\tilde{\Phi}_t$ of $\mathbb{R}^n$ preserving zero, and since χ_t lifts to a smooth path of invertible elements in $M_N(\mathbb{O}_n)$, it lifts, by the Borel lemma, to a smooth path of invertible elements in $M_N\big(C^\infty(U)\big)$, where U is an open subset of $\mathbb{R}^n$ containing zero. We thus get

Proposition 2.1.1. *Any smooth path of automorphisms in G lifts to a smooth path of local bundle automorphisms of $M_N\big(C^\infty(\mathbb{R}^n)\big)$ preserving zero.*

2.2. The frame bundle. For a manifold M with a vector bundle E we define the following:

Definition 2.2.1. The frame bundle $\tilde{M}_E$ is given by

$$\tilde{M}_E = \{(m,\Phi) \mid m \in M, \Phi : M_N(\mathbb{O}_n) \xrightarrow{\sim} J_m \,\mathrm{End}(E)\}.$$

We note that $\tilde{M}_E$ is a profinite manifold and in fact a principal bundle over M with fibre G.

Proposition 2.2.2. *For all* $(m, \Phi) \in \tilde{M}_E$ *there is a canonical isomorphism*

$$\omega_{(m,\Phi)} : T_{(m,\Phi)}\tilde{M}_E \to \operatorname{Der}\big(M_N(\mathbb{O}_n)\big).$$

The map ω *these isomorphisms define from* $T\tilde{M}_E$ *to* $\operatorname{Der}\big(M_N(\mathbb{O}_n)\big)$ *satisfies*

(i) $\omega(A^*) = A,\qquad A \in \mathbb{W}_n^0 \ltimes \big(\mathfrak{gl}_N(\mathbb{O}_n)/\text{centre}\big),$

(ii) $\varphi^*\omega = ad\varphi^{-1}\omega\qquad$ *for* $\varphi \in G$,

(iii) $d\omega + \frac{1}{2}[\omega, \omega] = 0,\quad x$

where A^* *is the fundamental vector field corresponding to* A.

In other words, ω *is a flat connection in* $\tilde{M}_E$ *with values in* $\operatorname{Der}\big(M_N(\mathbb{O}_n)\big)$.

Proof. Suppose we are given a path in $\tilde{M}_E$ so $\gamma(t) \in \tilde{M}_E$ with $\dot{\gamma}(0) = v$, $\gamma(0) = (m, \Phi)$, and $\gamma(t) = (m_t, \Phi_t)$. This lifts to a path of trivializations $\tilde{\gamma} = (m_t, \tilde{\Phi}_t)$, $\tilde{\Phi}_t : M_N\big(C^\infty(U)\big) \to \Gamma\big(\operatorname{End}(E)\big)$ that map 0 to m_t.

Define $\omega(v)$ to be the derivation

$$J_0(a) \to J_0\left(\frac{d}{dt}\big(\tilde{\Phi}^{-1} \circ \tilde{\Phi}_t(a)\big)\right), \quad a \in M_N\big(\mathbb{C}^\infty(U)\big).$$

This does not depend on the choice of γ or the lifting to $\tilde{\gamma}$ and will be an isomorphism.

Identity (i) follows, since ω is the canonical one form on the fibres of $\tilde{M}_E$.

For identity (ii), we have to compute $\omega\big(\varphi_*(v)\big)$, $\varphi \in G$, but

$$\omega\big(\varphi_*(v)\big) = \left(J_0(a) \to J_0\left(\frac{d}{dt}\big(\tilde{\varphi}^{-1} \circ \tilde{\Phi}^{-1} \circ \tilde{\Phi}_t \circ \tilde{\varphi}(a)\big)\right)\right) = \operatorname{ad}(\varphi^{-1})\omega(v).$$

Identity (iii) is equivalent to

$$\omega\big([\omega^{-1}(X), \omega^{-1}(Y)]\big) = [X, Y], \quad X, Y \in \operatorname{Der}\big(M_N(\mathbb{O}_n)\big).$$

This is actually a consequence of identity (i), because the statement is obvious for X, Y of the form $\frac{\partial}{\partial \hat{x}_i} \in \operatorname{Der}\big(M_N(\mathbb{O}_n)\big)$. For X, Y in $\mathbb{W}_n^0 \ltimes \big(\mathfrak{gl}_N(\mathbb{O}_n)/\text{centre}\big)$ it follows, because ω is the canonical one form on the fibres of $\tilde{M}_E$. Hence it suffices to check the case when X is of the form $\frac{\partial}{\partial \hat{x}_i}$ and $Y \in \mathbb{W}_n^0 \ltimes \big(\mathfrak{gl}_N(\mathbb{O}_n)/\text{centre}\big)$. Therefore let φ_t be the one parameter group for Y on G. We have

$$\omega\big([\omega^{-1}(X),\omega^{-1}(Y)]\big) = \omega\bigg(\lim_{t\to 0}\frac{1}{t}\big((\varphi_t)_*\omega^{-1}(X) - \omega^{-1}(X)\big)\bigg)$$

$$= \lim_{t\to 0}\frac{1}{t}\big(\mathrm{ad}(\varphi_t^{-1})(X) - X\big) = [X, Y],$$

from which the proposition follows.

Given $\tilde{M}_E$, we define the jet bundle of $\mathrm{End}(E)$ by

$$JE = \tilde{M}_E \times_G M_N(\mathbb{O}_n).$$

The flat connection on $\tilde{M}_E$ gives a flat connection in JE in the following way: Choose a trivialization of $JE_{|U} \xrightarrow{\sim} U \times M_N(\mathbb{O}_n)$. This corresponds to a lift $\sigma : U \to \tilde{M}_{E|_U}$ of the projection $P : \tilde{M}_E \to M$. In this trivialization the connection ∇ is given by $\nabla = d - \sigma^*(\omega)$, where ω is the connection described in Proposition 2.1.1.

Proposition 2.2.3. *The complex $\big(\Omega^*(M, JE), \nabla\big)$ is acyclic, and the cohomology is isomorphic to $\Gamma\big(\mathrm{End}(E)\big)$.*

Proof. There is an injective map j from $\Gamma\big(\mathrm{End}(E)\big)$ into JE given by

$$j(\gamma) = \big((m,\Phi), \Phi^{-1}(J_m\gamma)\big), \quad \gamma \in \Gamma\big(\mathrm{End}(E)\big).$$

To see that the image of j belongs to the kernel of ∇, choose a trivialization in the sense of a local bundle map $\Phi : M_N\big(C^\infty(U)\big) \to \Gamma\big(\mathrm{End}(E)\big)$, U open subset of $\mathbb{R}^n$. Denote the induced map from U to M by Φ. From this trivialization we get a special trivialization of $\tilde{M}_E$ by letting $\tilde{\Phi}_u$ denote the map $M_N(\mathbb{O}_n) = J_u\big(M_N\big(C^\infty(U)\big)\big) \to J_{\Phi(u)}\big(\Gamma\big(\mathrm{End}(E)\big)\big)$ induced from Φ and then define

$$U \times G \ni (u, g) \to \big(\Phi(u), g\tilde{\Phi}_u\big).$$

Using this trivialization, we get a local bundle isomorphism

$$C^\infty(U) \otimes M_N(\mathbb{O}_n) \to JE,$$

and in this trivialization it is not difficult to see that

$$\nabla = d - \sum_i dx_i \otimes \frac{\partial}{\partial \hat{x}_i}. \tag{2.1}$$

If $\gamma \in M_N\big(C^\infty(U)\big)$, we have that $j(\gamma)$ is just given by the Taylor expansion in each point, i.e.

$$j(\gamma)(u) = \sum_I \frac{\partial^{|I|}\gamma}{\partial x^I}\hat{x}^I,$$

where I runs through all multi-indices. Hence $j(\gamma) \in \ker(\nabla)$.

A computation in $\big(\Omega^*\big(U, M_N(\mathbb{O}_n)\big), \nabla\big)$, where ∇ is as in (2.1), gives that $\big(\Omega^*\big(U, M_N(\mathbb{O}_n)\big), \nabla\big)$ is acyclic, and the cohomology is $j\big(M_N\big(C^\infty(U)\big)\big)$.

We have thus seen that $(\Omega^*(M, JE), \nabla)$ is locally acyclic and the cohomology is locally isomorphic to $\Gamma(\mathrm{End}(E))$. Since we have also seen that $\Gamma(\mathrm{End}(E))$ is globally contained in $\ker\nabla$, i.e. the cohomology of $(\Omega^*(M, JE), \nabla)$ is a module over $\Gamma(\mathrm{End}(E))$, the statement follows.

3. Deformation quantization of endomorphism bundles

We start with looking at $\mathbb{R}^{2n}$ with the standard symplectic structure, and denote the coordinates by $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$. On $C^\infty(\mathbb{R}^{2n})[[\hbar]]$ we consider the Weyl quantization given by the product

$$(f * g)(x, \xi) = \exp\left(\frac{i\hbar}{2}\sum_{k=1}^{n}(\partial_{x_k}\partial_{\eta_k} - \partial_{\xi_k}\partial_{y_k})\right) f(x, \xi) g(y, \eta)|_{(x=y,\xi=\eta)}.$$

We will denote the Weyl quantization by $\mathscr{W}_n$.

Since the definition of the product in the Weyl quantization of two functions f, g only uses derivatives of f and g, the Weyl quantization makes sense over any open subset U of $\mathbb{R}^{2n}$. We will in this case talk about the Weyl quantization over U.

Let (M, ω) be a symplectic manifold, and let E be a vector bundle over M.

Definition 3.0.4. A deformation quantization of $\mathrm{End}(E)$ is a $\hbar$-linear associative product $*$ on $\Gamma(\mathrm{End}(E))[[\hbar]]$, continuous in the $\hbar$-adic topology and satisfying

$$f * g = fg + \hbar B_i(f, g) + \cdots,$$

where $f, g \in \Gamma(\mathrm{End}(E))$, and the B_i are bidifferential expressions. Furthermore we require that $(\Gamma(\mathrm{End}(E))[[\hbar]], *)$ is locally isomorphic to $M_N(\mathscr{W}_n)$, where $\mathscr{W}_n$ is the Weyl algebra on some open subset of $\mathbb{R}^{2n}$.

In this case locally isomorphic means that we are given a local bundle isomorphism $\Phi : M_N(C^\infty(U))[[\hbar]] \to \Gamma(\mathrm{End}(E))[[\hbar]]|_U$ over a local symplectomorphism such that the product $*'$ induced by $*$ on $M_N(C^\infty(U))[[\hbar]]$ is isomorphic to $M_N(\mathscr{W}_n)$. Isomorphic means that there exist differential operators $\{D_i\}$ such that the map

$$\varphi : (M_N(C^\infty(U))[[\hbar]], *') \to M_N(\mathscr{W}_n)$$

given by

$$\varphi(a) = a + \hbar D_i(a) + \cdots, \quad a \in M_N(C^\infty(U))$$

is an isomorphism of algebras.

We want to do the same construction for deformation quantizations as we did for endomorphism bundles. We therefore need the infinitesimal version of $M_N(\mathscr{W}_n)$. This is just given by considering $\mathbb{O}_{2n}[[\hbar]]$, $\mathbb{O}_{2n} = \mathbb{C}[[\hat{x}_1, \ldots, \hat{x}_n, \hat{\xi}_1, \ldots, \hat{\xi}_n]]$, with the same product as in the Weyl quantization. With this product we denote the algebra by $\mathbb{A}^\hbar$. The infinitesimal structure of $M_N(\mathcal{W}_n)$ will then be $M_N(\mathbb{A}^\hbar)$.

Definition 3.0.5. A formal symplectomorphism of $\mathbb{O}_{2n}$ is a continuous automorphism of $\mathbb{O}_{2n}$ induced from an automorphism of

$$\mathbb{R}[[\hat{x}_1, \ldots, \hat{x}_n, \hat{\xi}_1, \ldots, \hat{\xi}_n]]$$

that preserves the formal standard Poisson bracket $\{\cdot, \cdot\}$ on $\mathbb{O}_{2n}$.

Let G be the subgroup of automorphisms of $M_N(\mathbb{A}^\hbar)$ such that $\Phi \in G$ if Φ is $\hbar$ linear and continuous. Moreover, $\Phi \bmod \hbar$ becomes an automorphism Φ_0 of $M_N(\mathbb{O}_{2n})$, and Φ_0 induces a formal symplectomorphism on $\mathbb{O}_{2n}$.

If $\Phi \in G$, we let φ denote the induced symplectomorphism on $\mathbb{O}_{2n}$. In this case we will say that Φ is an automorphism over φ.

Lemma 3.0.6. *Every automorphism of $M_N(\mathbb{A}^\hbar)$ over the identity symplectomorphism is inner.*

Proof. Let Φ be such an automorphism. Since it is an automorphism $\bmod \hbar$ over the identity, it is inner $\bmod \hbar$, and we can hence assume that Φ is the identity $\bmod \hbar$. In other words,

$$\Phi(a) = a + \hbar D_1(a) + \cdots, \quad a \in M_N(\mathbb{O}_{2n}).$$

Since Φ is an automorphism, D_1 is a derivation of $M_N(\mathbb{O}_{2n})$ and hence of the form $X + [A, \cdot]$, where X is a formal vector field and $A \in M_N(\mathbb{O}_{2n})$. If we assume that $a, b \in \mathbb{O}_{2n}$, we have that

$$\{a, D_1(b)\} + \{D_1(a), b\} = D_1(\{a, b\}),$$

since Φ is an automorphism. This means that X is a formal hamiltonian vector field. Therefore there exists an element x in $\mathbb{O}_{2n}$ such that $D_1(a) = \{x, a\}$. Hence we have

$$\Phi \circ \exp\bigl(-\mathrm{ad}(x + A)\bigr) = \mathrm{id} \bmod \hbar^2.$$

Continuing in this way, the result follows.

It is well known from [11] that the Lie algebra of G in the case where $N = 1$, is given by

$$\mathfrak{g}_0 = \left\{ \frac{ia}{\hbar} \,\middle|\, a \in \mathbb{A}^\hbar, a \text{ real} \bmod \hbar, a \in (\hat{x}_1, \ldots, \hat{\xi}_1, \ldots)^2 \bmod \hbar \right\} \Big/ \frac{\mathbb{C}[[\hbar]]}{\hbar},$$

and that any element of G is of the form $\exp(g)$, $g \in \mathfrak{g}_0$.

We therefore see that the Lie algebra $\mathfrak{g}_0$ of G for arbitrary N is given by

$$\mathfrak{g}_0 = \left\{ \frac{ia}{\hbar} + b \,\middle|\, b \in M_N(\mathbb{A}^\hbar), a \in \mathbb{A}^\hbar, a \text{ real} \bmod \hbar, \right.$$

$$\left. a \in (\hat{x}_1, \ldots, \hat{\xi}_1, \ldots)^2 \bmod \hbar \right\} \Big/ \frac{\mathbb{C}[[\hbar]]}{\hbar}.$$

To this Lie algebra we add the derivations $\partial_{\hat{x}_1}, \ldots, \partial_{\hat{\xi}_1}, \ldots$ and call the enlarged Lie algebra $\mathfrak{g}$.

Let us suppose that we are given a deformation quantization $A_E^\hbar$ of $\mathrm{End}(E)$. We define

$$I_m = \{a \in A_E^\hbar \,|\, a(m) = 0\},$$

and we let I_m^n denote the nth power of the ideal I_m in the undeformed product. The jet of $A_E^\hbar$ in m is defined by

$$J_m A_E^\hbar = \varprojlim_k A_E^\hbar / I_m^k.$$

Since the value of the product in $A_E^\hbar$ in a point only depends on the derivatives in that point, the product descends to $J_m A_E^\hbar$.

If we choose a trivialization of $A_E^\hbar$ around m, we get an isomorphism $J_m A_E^\hbar \xrightarrow{\sim} M_N(\mathbb{A}^\hbar)$. Any other trivialization will give an automorphism of $M_N(\mathbb{A}^\hbar)$ in G.

As in the case of $\mathrm{End}(E)$ we do the following:

Definition 3.0.7. The frame bundle $\tilde{M}_{A_E^\hbar}$ is given by

$$\tilde{M}_{A_E^\hbar} = \{(m, \Phi) \,|\, m \in M, \Phi : M_N(\mathbb{A}^\hbar) \xrightarrow{\sim} J_m \mathbb{A}_E^\hbar\}.$$

As before $\tilde{M}_{A_E^\hbar}$ is a principal bundle with fibre G.

Proposition 3.0.8. *For all $(m, \Phi) \in \tilde{M}_{A_E^\hbar}$ there is a canonical isomorphism*

$$\omega_{(m,\Phi)} : T_{(m,\Phi)} \tilde{M}_{A_E^\hbar} \to \mathfrak{g}.$$

The map ω these isomorphisms define from $T\tilde{M}_E$ to $\mathfrak{g}$ satisfies

(i) $\omega(A^*) = A, \qquad A \in \mathfrak{g}_0,$

(ii) $\varphi^* \omega = \mathrm{ad}\, \varphi^{-1} \omega, \qquad \varphi \in G,$

(iii) $d\omega + \frac{1}{2}[\omega, \omega] = 0,$

where A is the fundamental vector field corresponding to A.

In other words, ω is a flat connection with values in $\mathfrak{g}$.

Proof. The same as the case of endomorphism bundles.

As in the case of endomorphism bundles, we get a flat connection ∇ in the bundle

$$JA_E^\hbar = \tilde{M}_{A_E^\hbar} \times_G M_N(\mathbb{A}^\hbar).$$

Proposition 3.0.9. *The complex* $(\Omega^*(M, JA_E^\hbar), \nabla)$ *is acyclic, and*

$$\ker \nabla \xrightarrow{\sim} A_E^\hbar.$$

Proof. The same as the case of endomorphism bundles.

We see that $H' = \mathrm{U}(n) \times \mathrm{PU}(N)$ is a maximal compact subgroup of G. The $\mathrm{U}(n)$-component comes from a maximal compact subgroup of the symplectic group, $\mathrm{SP}(2n)$, and $\mathrm{PU}(N)$ (the projective unitary group) comes from the maximal compact subgroup of the action of Gl_N on $M_N(\mathbb{C})$.

Since H' is a maximal compact subgroup of G, we can reduce the bundle $\tilde{M}_{A^\hbar}$ to an H' bundle P', which is easily seen to be a reduction of the principal bundle consisting of dual symplectic frames and frames of $\mathrm{End}(E)$. We thus see that $JA_E^\hbar$ is, in fact, isomorphic to $P' \times_H \mathbb{A}^\hbar$. We will denote this bundle by $\mathbb{W}$.

Because we are working with an endomorphism bundle, the bundle P' admits a lift to an $H = \mathrm{U}(n) \times \mathrm{SU}(N)$ principal bundle P.

We introduce a grading on $\mathbb{A}^\hbar$ in which $\hat{x}_1, \dots, \hat{x}_n, \hat{\xi}_1, \dots, \hat{\xi}_n$ has degree 1 and $\hbar$ has degree 2. This also gives a grading on $M_N(\mathbb{A}^\hbar)$. Furthermore we see that the action of $\mathrm{U}(n)$ on $\mathbb{A}^\hbar$ preserves the grading and we hence get a grading on $\mathbb{W}$.

We note that we have an extension of Lie algebras

$$0 \to \frac{1}{\hbar} i\mathbb{R} + \mathbb{C}[[\hbar]] \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0, \tag{3.1}$$

where

$$\tilde{\mathfrak{g}} = \left\{ \frac{ia}{\hbar} + b \,\middle|\, a \in \mathbb{A}^\hbar, a \text{ real} \bmod \hbar, b \in M_N(\mathbb{A}^\hbar) \right\}$$

with bracket given by commutators. The grading $M_N(\mathbb{A}^\hbar)$ also gives a grading on $\tilde{\mathfrak{g}}$ and therefore also on the subbundle of $\mathbb{W}/\hbar$ with fiber $\tilde{\mathfrak{g}}$, denoted by $\tilde{\mathfrak{g}}_M$. Let $\mathfrak{h} = \mathfrak{u}(n) + \mathfrak{su}(N)$. There is an embedding of $\mathfrak{h}$ in $\tilde{\mathfrak{g}}$ compatible with the quotient map $\tilde{\mathfrak{g}} \to \mathfrak{g}$ and the embedding $\mathfrak{h} \to \mathfrak{g}$. This embedding is obvious in the case of $\mathfrak{su}(N)$. Clearly $\mathfrak{u}(n)$ embeds in $\mathfrak{sp}(2n)$, the Lie algebra of the symplectic group. Moreover $\mathfrak{sp}(2n)$ embeds in $\mathfrak{g}$ as the Lie sub algebra generated by elements on the form

$$i\hbar^{-1} v_1 v_2, \quad v_1, v_2 \in \{\hat{x}_1, \dots, \hat{\xi}_1, \dots\}.$$

The embedding of $\mathfrak{u}(n)$ in $\tilde{\mathfrak{g}}$ is the restriction of the embedding of $\mathfrak{sp}(2n)$ in $\tilde{\mathfrak{g}}$ given by

$$i\hbar^{-1} v_1 v_2 \to i\hbar^{-1} v_1 * v_2 + \frac{i}{2\hbar} \omega_{st}(v_1, v_2), \quad v_1, v_2 \in \{\hat{x}_1, \dots, \hat{\xi}_1, \dots\}.$$

Since H acts semi-simple on $\mathfrak{g}$, we get an H-equivariant lift of the quotient map $\tilde{\mathfrak{g}} \to \mathfrak{g}$. We can therefore lift the connection ∇ to a connection $\tilde{\nabla}$ taking values in $\tilde{\mathfrak{g}}$. This means that we have a collection of local $\tilde{\mathfrak{g}}$-one forms $\{A_i\}$, i being labels of trivializations of $\mathbb{W}$, satisfying

$$A_i = g_{ij}\, dg_{ji} + g_{ij} A_j g_{ji},$$

where g_{ij} are the transition functions.

This connection, however, is not flat. But because of the extension (3.1), the curvature $dA_i + \frac{1}{2}[A_i, A_i]$ is in $\Omega^2\left(M, \frac{1}{\hbar} i\mathbb{R} + \mathbb{C}[[\hbar]]\right)$. Clearly the associated cohomology class is independent of the choice of lifting of ∇. By checking the definition of ∇, one sees that the $\frac{1}{\hbar}$ component of $\tilde{\nabla}^2$ is $\frac{\omega}{i\hbar}$, where ω is the symplectic structure.

So for each deformation quantization of $\mathrm{End}(E)$ we have a connection ∇ in $\mathbb{W}$ such that $\ker \nabla$ is isomorphic to the deformation quantization, and the lifting of ∇ to a $\tilde{\mathfrak{g}}$-valued connection gives an element in

$$\frac{\omega}{i\hbar} + H^2(M, \mathbb{C}[[\hbar]]).$$

Proposition 3.0.10. *Let $A^\hbar_{1,E}$ and $A^\hbar_{2,E}$ be deformation quantizations with characteristic classes θ_1 and θ_2. Then $A^\hbar_{1,E} \simeq A^\hbar_{2,E}$ if and only if $\theta_1 = \theta_2$ in $\frac{\omega}{i\hbar} + H^2(M, \mathbb{C}[[\hbar]])$.*

The proof of the above proposition relies on the following

Lemma 3.0.11. *Let $I_\omega : TM \to T^*M$ be the bundle isomorphism induced by ω. Since $T^*M \subset \mathbb{W}$, the isomorphism I_ω induces an element A in $\Omega^1(M, \mathbb{W})$. Put*

$$A_{-1} = \frac{A}{\hbar} \in \Omega^1(M, \tilde{\mathfrak{g}}_M).$$

Then the complex

$$\left(\Omega^*(M, \tilde{\mathfrak{g}}_M), \mathrm{Ad}\, A_{-1}\right)$$

is acyclic.

Proof. Since the action of $\mathrm{Ad}\, A_{-1}$ commutes with the action of $C^\infty(M)$ on $\Omega^*(M, \tilde{\mathfrak{g}}_M)$, it is enough to prove the statement locally. Locally the complex is just

$$\left(C^\infty(\mathbb{R}^{2n}) \otimes \left(i/\hbar \hat{\Omega}^* + M_N(\hat{\Omega}^*)[[\hbar]]\right), \mathrm{id} \otimes \hat{d}\right),$$

where $(\hat{\Omega}^*, \hat{d})$ is the complex of formal differential forms. From this the lemma follows.

Proof of Proposition 3.0.10. Let $\tilde{\nabla}_1$ and $\tilde{\nabla}_2$ be the two connections with

$\tilde{\nabla}_1^2 = \tilde{\nabla}_2^2 = \theta$. Note that we can actually assume that $\tilde{\nabla}_1^2 = \tilde{\nabla}_2^2$ in $\frac{\omega}{i\hbar} + \Omega^2(M, \mathbb{C}[[\hbar]])$. We have

$$\tilde{\nabla}_1 - \tilde{\nabla}_2 = R_0 + R_1 + \cdots, \quad R_i \in \Omega^1(M, \tilde{\mathfrak{g}}_M^i).$$

From the equality of the curvatures we get $[A_{-1}, R_0] = 0$, and hence by Lemma 3.0.11 we have an element $g_1 \in \Gamma(\tilde{\mathfrak{g}}_M^1)$ such that $[g_1, A_{-1}] = R_0$. Considering the connection $\nabla_{2,0} = \mathrm{ad}(\exp g_1)\nabla_2$, we therefore have

$$\tilde{\nabla}_1 - \tilde{\nabla}_{2,0} = R_1' + \cdots, \quad R_i' \in \Omega^1(M, \tilde{\mathfrak{g}}_M^i).$$

Continuing by induction and using the Hausdorff-Campbell formula we get an element g in $\Gamma\big(\mathrm{Aut}(\tilde{\mathfrak{g}}_M)\big)$ conjugating ∇_1 into ∇_2, and hence the deformations will be isomorphic.

Let us now assume that $A_{1,E}^{\hbar}$ and $A_{2,E}^{\hbar}$ are isomorphic. This induces an isomorphism between $\tilde{M}_{A_{1,E}^{\hbar}}$ and $\tilde{M}_{A_{2,E}^{\hbar}}$ compatible with the connections on $\tilde{M}_{A_{1,E}^{\hbar}}$ and $\tilde{M}_{A_{2,E}^{\hbar}}$. In particular we get an automorphism of $\mathbb{W}$ mapping ∇_1 to ∇_2, from which we see that $A_{1,E}^{\hbar}$ and $A_{2,E}^{\hbar}$ have the same characteristic class.

Theorem 3.0.12. *The deformation quantizations of an endomorphism bundle are classified by the affine space*

$$\frac{\omega}{i\hbar} + H^2(M, \mathbb{C}[[\hbar]]).$$

Proof. We only need to prove that for a class θ in $\frac{\omega}{i\hbar} + \Omega^2(M, \mathbb{C}[[\hbar]])$, we have a deformation quantization with characteristic class θ. To do this we start with a connection in P and thus get an H-connection ∇ in $\mathbb{W}$. We have that

$$[\nabla + A_{-1}, \nabla + A_{-1}] = \frac{\omega}{i\hbar} + 2[\nabla, A_{-1}] + [\nabla, \nabla].$$

One checks that $[A_{-1}, [\nabla, A_{-1}]] = 0$, and according to Lemma 3.0.11 we get an element $A_0 \in \Omega^1(M, \tilde{\mathfrak{g}}_M)$ such that $[A_{-1}, A_0] = [\nabla, A_{-1}]$. If we put $\nabla_0 = \nabla + A_{-1} + A_0$, we have

$$[\nabla_0, \nabla_0] - \theta = 0 \bmod \Omega^2(M, \tilde{\mathfrak{g}}_M^{i \geqq 0}).$$

Let us now assume that we have constructed ∇_n with $\nabla_n^2 - \theta = 0 \bmod \Omega^2(M, \tilde{\mathfrak{g}}_M^{i \geqq n})$. By the Bianchi identity we have $[\nabla_n, [\nabla_n, \nabla_n]] = 0$, and therefore $[A_{-1}, ([\nabla_n, \nabla_n] - \theta)_n] = 0$, where $([\nabla_n, \nabla_n] - \theta)_n$ is the n-th component of $[\nabla_n, \nabla_n] - \theta$. As before, we get an element A_n with $[A_{-1}, A_n] = ([\nabla_n, \nabla_n] - \theta)_n$, and considering $\nabla_{n+1} = \nabla_n + A_n$ we have

$$[\nabla_{n+1}, \nabla_{n+1}] - \theta = 0 \bmod \Omega^2(M, \tilde{\mathfrak{g}}_M^{i \geqq n+1}).$$

For each class θ in $\frac{\omega}{i\hbar} + \Omega^2(M, \mathbb{C}[[\hbar]])$ we can thus construct a connection ∇_F with values in $\tilde{\mathfrak{g}}$ and curvature θ. We therefore only need to check that the complex

$(\Omega^*(M,\mathbb{W}),\nabla_F)$ is acyclic, that the kernel is isomorphic to $\Gamma(\mathrm{End}(E)[[\hbar]])$, and that the product induced by the product on $\mathbb{W}$ gives a deformation quantization of $\mathrm{End}(E)$. These results, however, follow from the proof of the Proposition 3.0.10, since we locally can conjugate ∇_F to a connection on the form

$$d - \sum_{i=1}^{n}(\partial_{\tilde{x}_i}\otimes dx_i + \partial_{\tilde{\xi}_i}\otimes d\xi_i).$$

4. Lie algebra cohomology

Definition 4.0.13. A differential graded Lie algebra $(\mathfrak{g},d)$ over a commutative unital ring k is a ($\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$) graded k-module $\mathfrak{g}$ with a bracket operation $[\cdot\,,\cdot]:\mathfrak{g}^j\times\mathfrak{g}^i\to\mathfrak{g}^{i+j}$ and a differential $\partial:\mathfrak{g}^i\to\mathfrak{g}^{i-1}$ satisfying:

(i) $\partial[g_1,g_2]=[\partial g_1,g_2]+(-1)^{|g_1|}[g_1,\partial g_2]$,

(ii) $[g_1,g_2]=-(-1)^{|g_1|\,|g_2|}[g_2,g_1]$,

(iii) $[g_1,[g_2,g_3]]+(-1)^{|g_3|(|g_1|+|g_2|)}[g_3,[g_1,g_2]]+(-1)^{|g_1|(|g_2|+|g_3|)}[g_2,[g_3,g_1]]=0$,

where $|\cdot|$ is the degree.

A $\mathfrak{g}$ module $\mathbb{L}^*$ is a complex $\mathbb{L}^*$ with an action of $\mathfrak{g}$, i.e. we have a map $\mathfrak{g}^i\times\mathbb{L}^j\to\mathbb{L}^{i+j}$ satisfying

$$g_1g_2l-(-1)^{|g_1|\,|g_2|}g_2g_1l=[g_1,g_2]l$$

and

$$\partial_{\mathbb{L}^*}(gl)=(\partial_{\mathfrak{g}}g)l+(-1)^{|g|}g(\partial_{\mathbb{L}^*}l).$$

Given a differential graded Lie algebra $\mathfrak{g}$, we can define a differential graded Lie algebra $\mathfrak{g}[\varepsilon]$ as follows:

- $\mathfrak{g}[\varepsilon]=\mathfrak{g}+\varepsilon\mathfrak{g}$, where $|\varepsilon|=1$,
- $[g_1,\varepsilon g_2]=\varepsilon[g_1,g_2]$,
- $[\varepsilon g_1,\varepsilon g_2]=0$,
- $\partial(g_1+\varepsilon g_2)=\partial_{\mathfrak{g}}g_1+g_2-\varepsilon\partial_{\mathfrak{g}}g_2$.

Also one can construct the enveloping algebra by setting

$$U(\mathfrak{g})=T(\mathfrak{g})/(g_1\otimes g_2-(-1)^{|g_1|\,|g_2|}g_2\otimes g_1=[g_1,g_2]),$$

where $T(\mathfrak{g})$ is the tensor algebra. Furthermore $U(\mathfrak{g})$ has a differential induced by the differential on $\mathfrak{g}$ by the graded Leibniz rule and a grading.

We note that $\big(U(\mathfrak{g}[\varepsilon]),\partial\big)$ is a $\mathfrak{g}$ module. Let $\mathfrak{h}$ denote a Lie sub algebra of $\mathfrak{g}$. For a $\mathfrak{g}$ module $\mathbb{L}^*$ we define

$$C^*(\mathfrak{g},\mathfrak{h};\mathbb{L}^*) = \mathrm{Hom}_{U(\mathfrak{g})}\big(U(\mathfrak{g}[\varepsilon]) \otimes_{U(\mathfrak{h}[\varepsilon])} k, \mathbb{L}^*\big).$$

This will have a differential induced by the differential on $U(\mathfrak{g}[\varepsilon])$ and the differential on $\mathbb{L}^*$. The homology of this complex will be denoted by $H^*(\mathfrak{g},\mathfrak{h};\mathbb{L}^*)$.

We are now going to give a construction of classes in $C^*(\mathfrak{g},\mathfrak{h};\mathbb{L}^*)$ in special cases. First we assume that $\mathbb{L}^*$ is homotopically constant in the sense defined below.

Definition 4.0.14. A $\mathfrak{g}$-module $\mathbb{L}^*$ is called homotopically constant if there exist operations

$$\iota_g : \mathbb{L}^* \to \mathbb{L}^{*-1}, \quad g \in \mathfrak{g}$$

satisfying

$$[\partial,\iota_g] = L_g, \qquad [\partial, L_g] = 0,$$
$$[L_{g_1},\iota_{g_2}] = \iota_{[g_1,g_2]}, \quad [\iota_{g_1},\iota_{g_2}] = 0,$$

where we have denoted the action of $\mathfrak{g}$ by L_g.

In other words, we have an action of the differential graded algebra $U(\mathfrak{g}[\varepsilon])$ on $\mathbb{L}^*$.

If we furthermore assume that there is an $\mathfrak{h}$-equivariant projection $\nabla : \mathfrak{g} \to \mathfrak{h}$ of the embedding $\mathfrak{h} \to \mathfrak{g}$, we get the usual Chern-Weil homomorphism, i.e. a map of complexes

$$\mathrm{CW} : C^*(\mathfrak{h}[\varepsilon],\mathfrak{h};\mathbb{L}^*) \to C^*(\mathfrak{g},\mathfrak{h};\mathbb{L}^*)$$

given in the following way:

For an element $g_1 \wedge g_2$ define $R(g_1 \wedge g_2) = [\nabla(g_1),\nabla(g_2)] - \nabla([g_1,g_2])$. Taking cup product gives $R^n : \wedge^{2n}\mathfrak{g} \to \wedge^n \varepsilon\mathfrak{h}$. By composition this gives a map

$$\varphi : C^*(\mathfrak{h}[\varepsilon],\mathfrak{h};\mathbb{L}^*) \to C^*(\mathfrak{g},\mathbb{L}^*).$$

There are operations λ^n on $\mathbb{L}^*$ given by

$$g_1 \wedge \cdots \wedge g_n \to \iota_{g_1-\nabla g_1} \cdots \iota_{g_n-\nabla g_n} l,$$

where $l \in \mathbb{L}^*$. Finally we set

$$\mathrm{CW}(a) = \sum_n \lambda^n \cup \varphi(a),$$

where $\cup$ is the cup product. One checks that this gives a morphism of complexes.

Next we will give a construction of classes in $C^*(\mathfrak{h}[\varepsilon],\mathfrak{h};\mathbb{L}^*)$ for special cases of $\mathbb{L}^*$. To this end we need the following

Definition 4.0.15. A $\mathfrak{g}$-module $\mathbb{L}^*$ is called very homotopically constant, if $\mathbb{L}^*$ is homotopically constant and we have operations

$$L_{\underline{g}} : \mathbb{L}^* \to \mathbb{L}^{*+1}, \quad g \in \mathfrak{g},$$

$$\iota_{\underline{g}} : \mathbb{L}^* \to \mathbb{L}^*, \qquad g \in \mathfrak{g},$$

satisfying the eight conditions

$$[\partial, \iota_{\underline{g}}] = L_{\underline{g}} - \iota_g, \quad [\partial, L_{\underline{g}}] = L_g, \quad [\iota_{\underline{g_1}}, \iota_{\underline{g_2}}] = 0, \quad [\iota_{g_1}, \iota_{\underline{g_2}}] = 0,$$

$$[\iota_{\underline{g_1}}, L_{g_2}] = \iota_{\underline{[g_1, g_2]}}, \quad [\iota_{\underline{g_1}}, L_{\underline{g_2}}] = 0, \quad [L_{g_1}, L_{\underline{g_2}}] = L_{\underline{[g_1, g_2]}}, \quad [L_{\underline{g_2}}, L_{\underline{g_2}}] = 0.$$

In other words, we have an action of the differential graded algebra $U(\mathfrak{g}[\varepsilon, \eta]) = U(\mathfrak{g}[\varepsilon][\eta])$ on $\mathbb{L}^*$.

We now assume that $\mathbb{L}^*$ is a very homotopically constant $\mathfrak{h}$-module. We denote by $\mathbb{L}^*_{\mathfrak{h}+\underline{\mathfrak{h}}}$ the elements $l \in \mathbb{L}^*$ with $L_h l = 0$ and $L_{\underline{h}} l = 0$. We note that this is a complex. We get a morphism of complexes $\mathbb{L}^*_{\mathfrak{h}+\underline{\mathfrak{h}}} \to C^*(\mathfrak{h}[\varepsilon], \mathfrak{h}; \mathbb{L}^*)$ by:

$$\mathbb{L}^*_{\mathfrak{h}+\underline{\mathfrak{h}}} \ni l \to (h_1 \cdots h_n \to \iota_{\underline{h_1}} \cdots \iota_{\underline{h_n}} l).$$

4.1. Examples. We are going to give some examples of relative classes in Lie algebra cohomology.

Example 1. Consider the extension (3.1) and choose an $\mathfrak{h}$-equivariant lift $\nabla : \mathfrak{g} \to \tilde{\mathfrak{g}}$ of the quotient map $\tilde{\mathfrak{g}} \to \mathfrak{g}$, where $\mathfrak{h}$ is $\mathfrak{u}(n) + \mathfrak{su}(N)$. We then define the class θ in $C^*\left(\mathfrak{g}, \mathfrak{h}; \frac{1}{\hbar}\mathbb{C}[[\hbar]]\right)$ by

$$\theta(g_1, g_2) = [\nabla g_1, \nabla g_2] - \nabla([g_1, g_2]). \tag{4.1}$$

We want to show that θ actually comes from a sort of Chern-Weil map. Let $k : \tilde{\mathfrak{g}} \to \mathfrak{g}$ denote the quotient map, and let $\tilde{\mathfrak{h}} = k^{-1}(\mathfrak{h})$, i.e.

$$\tilde{\mathfrak{h}} = \mathfrak{h} + \frac{i}{\hbar}\mathbb{R} + \mathbb{C}[[\hbar]].$$

Note that $C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*)$ and $C^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}; \mathbb{L}^*)$ are quasi-isomorphic, when $\mathbb{L}^*$ is a $\mathfrak{g}$ module. We have a Chern-Weil homomorphism

$$\mathrm{CW} : C^*(\tilde{\mathfrak{h}}[\varepsilon], \tilde{\mathfrak{h}}; \mathbb{L}^*) \to C^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}; \mathbb{L}^*)$$

as before. A choice of an $\tilde{\mathfrak{h}}$ equivariant split $\nabla' : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{h}}$ is given by $\nabla' = \nabla'' \circ k + \mathrm{id} - \nabla \circ k$, where $\nabla'' : \mathfrak{g} \to \mathfrak{h}$ is an $\mathfrak{h}$-equivariant splitting of the embedding $\mathfrak{h} \to \mathfrak{g}$. Let θ be the projection of $\tilde{\mathfrak{h}}$ on $\frac{i}{\hbar}\mathbb{R} + \mathbb{C}[[\hbar]]$. It is now easy to see that under the quasi-isomorphism between $C^*\left(\mathfrak{g}, \mathfrak{h}; \frac{1}{\hbar}\mathbb{C}[[\hbar]]\right)$ and $C^*\left(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}; \frac{1}{\hbar}\mathbb{C}[[\hbar]]\right)$, the class $\mathrm{CW}(-\theta)$ is the same as the θ we defined in the start of this example.

Example 2. Some other classes in $C^*(\mathfrak{g}, \mathfrak{h})$ we need, are also coming from a Chern-Weil construction. We consider an $\mathfrak{h}$-equivariant splitting ∇' of the embedding $\mathfrak{h} \to \mathfrak{g}$. Composed with the projection $\mathfrak{u}(n) + \mathfrak{su}(N) \to \mathfrak{su}(N)$, we get an $\mathfrak{h}$ equivariant map $\nabla : \mathfrak{g} \to \mathfrak{su}(N)$. Using this we therefore get a Chern-Weil homomorphism

$$\mathrm{CW} : C^*\big(\mathfrak{su}(N)[\varepsilon], \mathfrak{su}(N)\big) \to C^*\big(\mathfrak{g}, \mathfrak{su}(N)\big).$$

It is clear that this homomorphism in fact maps into $C^*(\mathfrak{g}, \mathfrak{h})$. We therefore get the usual classes; for example the usual chern character ch, which is

$$\exp(R) = \sum \frac{1}{n!} \operatorname{Tr}(R^n),$$

where

$$R(g_1, g_2) = [\nabla g_1, \nabla g_2] - \nabla [g_1, g_2].$$

Here Tr denotes the usual normalized trace on $\mathfrak{su}(N)$. This class is of course the Chern-Weil map on the symmetric polynomial ch on $\mathfrak{su}(N)$ given by

$$\operatorname{ch}(h_1, \ldots, h_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{n!} \operatorname{Tr}(h_{\sigma(1)} \cdots h_{\sigma(n)}).$$

Since $C^*\big(\mathfrak{su}(N)[\varepsilon], \mathfrak{su}(N)\big)$ embeds in $C^*(\tilde{\mathfrak{h}}[\varepsilon], \tilde{\mathfrak{h}})$, the Chern-Weil construction given in this example is just a particular case of the Chern-Weil homomorphism

$$\mathrm{CW} : C^*(\tilde{\mathfrak{h}}[\varepsilon], \tilde{\mathfrak{h}}) \to C^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}).$$

Example 3. We can of course do the construction from Example 2 for $\mathfrak{u}(n)$ instead of $\mathfrak{su}(N)$, and in this way get a Chern-Weil homomorphism

$$C^*\big(\mathfrak{u}(n)[\varepsilon], \mathfrak{u}(n)\big) \to C^*(\mathfrak{g}, \mathfrak{h}).$$

This again can also be viewed as the composition

$$C^*\big(\mathfrak{u}(n)[\varepsilon], \mathfrak{u}(n)\big) \to C^*(\tilde{\mathfrak{h}}[\varepsilon], \tilde{\mathfrak{h}}) \to C^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}).$$

We will in particular be interested in the symmetric polynomial $\hat{A}$ coming from the map

$$h \to \det\left(\frac{h/2}{\sinh(h/2)}\right).$$

5. Cyclic homology

We consider a differential graded unital algebra (A, δ) over a commutative ring k containing $\mathbb{Q}$, i.e. an algebra A that can be written as $A = \bigoplus_n A^n$, where A^n's are k-submodules of A and $A^n A^m \subset A^{n+m}$. Elements in A^n are said to have degree n and we will denote the degree of an element a by $|a|$. Furthermore $\delta : A^* \to A^{*-1}$ has to be a differential and satisfy $\delta(ab) = \delta(a)b + (-1)^{|a|}a\delta(b)$.

Define an operator τ on $A^{\otimes(n+1)}$ by

$$\tau(a_0 \otimes \cdots \otimes a_n) = (-1)^{(|a_n|+1)\sum_{i=0}^{n-1}(|a_i|+1)} a_n \otimes a_0 \otimes \cdots \otimes a_{n-1},$$

and consider the complex

$$\cdots \xleftarrow{b+\delta} C^n(A)/\mathrm{Im}(1-\tau) \xleftarrow{b+\delta} C^{n+1}(A)/\mathrm{Im}(1-\tau) \xleftarrow{b+\delta} \cdots$$

where $C^{n+1}(A)$ denotes the set of elements of the form $a_0 \otimes \cdots \otimes a_k$ with $k + \sum |a_i| = n$,

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n-1} (-1)^{k+\sum_{i=0}^{k}|a_i|} a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n$$
$$+ (-1)^{(|a_n|+1)\sum_{i<n}(|a_i|+1)+|a_n|} a_n a_0 \otimes \cdots \otimes a_{n-1}$$

and

$$\delta(a_0 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n} (-1)^{\sum_{i=1}^{k-1}(|a_i|+1)} a_0 \otimes \cdots \otimes \delta(a_k) \otimes \cdots \otimes a_n.$$

The complex is denoted by $C_*^\lambda(A)$, and the homology is the cyclic homology of A denoted by $HC_*(A)$.

The reduced cyclic homology is given by the homology of the complex

$$\cdots \xleftarrow{b+\delta} \bar{C}^n(A)/\mathrm{Im}(1-\tau) \xleftarrow{b+\delta} \bar{C}^{n+1}(A)/\mathrm{Im}(1-\tau) \xleftarrow{b+\delta} \cdots,$$

where $\bar{C}^*(A)$ comes from considering $\bar{A}^{\otimes *}$ instead of $A^{\otimes *}$, where $\bar{A} = A/k \cdot 1$.

The reduced cyclic homology of A is denoted by $\overline{HC}_*(A)$, and the complex above, computing the reduced cyclic homology, is denoted by $\bar{C}_*^\lambda(A)$.

It is well known, see [10] and [2], that there is an exact sequence

$$(5.1) \qquad \cdots HC_n(k) \to HC_n(A) \to \overline{HC}_n(A) \to HC_{n-1}(k) \to \cdots.$$

We will briefly give a construction, due to Brodzki, of the connecting morphism $\overline{HC}_*(A) \to HC_{*-1}(k)$ at the level of complexes, see [3] and [2]; i.e. a morphism of complexes

$$\mathrm{Br} : \bar{C}_*^\lambda(A) \to C_{*-1}^\lambda(k)$$

giving the connecting homomorphism at the level of homology. Let $l : A \to k$ be a k-linear map with $l(1) = 1$. Put

$$\rho(a) = l\big(\delta(a)\big), \quad a \in A,$$

$$\rho(a_1 \otimes a_2) = l(a_1)l(a_2) - l(a_1 a_2), \quad a_1 \otimes a_2 \in A^{\otimes 2},$$

$$\rho = 0 \quad \text{on } A^{\otimes m}, \quad m \geqq 3,$$

and define $\mathrm{Br} : \bar{C}_*^\lambda(A) \to C_{*-1}^\lambda(k)$ by setting

$$\mathrm{Br}(a_0 \otimes \cdots \otimes a_m) = \sum_{i=0}^{m} (-1)^{\sum\limits_{k<i}(|a_k|+1) \sum\limits_{k \geqq i}(|a_k|+1)} (\rho \otimes \cdots \otimes \rho)$$

$$(a_i \otimes \cdots \otimes a_0 \otimes a_m \otimes \cdots \otimes a_{i-1})(n+1)! \cdot 1^{\otimes 2n+1}$$

on $\bar{C}_{2n+1}^\lambda$, and letting Br be zero on $\bar{C}_{2n}^\lambda$ Br.

We now consider the differential graded algebra $k[\eta]$, where η has degree one, $\eta^2 = 0$, and the differential is given by ∂_η. For a differential graded algebra A we define $A[\eta]$ to be $A \otimes k[\eta]$, where $\otimes$ is the tensor product of differential graded algebras. It is not difficult to see that $HC_*(A[\eta]) = 0$, and we therefore have

Proposition 5.0.1. *The morphism*

$$\mathrm{Br} : \bar{C}_*^\lambda(A[\eta]) \to C_{*-1}^\lambda(k)$$

is a quasi-isomorphism.

Since it is not standard, we mention that the reduced cyclic homology is Morita invariant; at least in the case of algebras and matrices over these algebras. To see this, let A be an algebra, and let $l : A \to k$ be a map needed in the construction of Br. Let tr denote the normalized trace $\mathrm{tr} : M_n(A) \to A$. We have a commutative diagram

$$\begin{array}{ccccccccc}
\longrightarrow & HC^*\big(M_N(A)\big) & \longrightarrow & \overline{HC}^*\big(M_N(A)\big) & \xrightarrow{\mathrm{Br}} & HC^{*-1}(k) & \longrightarrow \\
& \big\downarrow \mathrm{tr} & & \big\downarrow \mathrm{tr} & & \big\| & \\
\longrightarrow & HC^*(A) & \longrightarrow & \overline{HC}^*(A) & \xrightarrow{\mathrm{Br}} & HC^{*-1}(k) & \longrightarrow
\end{array}$$

where $\mathrm{Br} : \overline{HC}^*\big(M_N(A)\big) \to HC^{*-1}(k)$ is induced by $l \circ \mathrm{tr}$. According to [10], $\mathrm{tr} : HC^*\big(M_n(A)\big) \to HC^*(A)$ is an isomorphism. The result therefore follows from the exact sequence (5.1).

5.1. Operations on the periodic complex. For the periodic cyclic complex we consider $\prod_n A \otimes \bar{A}^{\otimes n}$. We give this a $\mathbb{Z}/2\mathbb{Z}$ grading by

$$|a_0 \otimes \cdots \otimes a_n| = n + \sum_i |a_i| \bmod 2.$$

On this complex we consider the differential $b + B + \delta$, where b and δ are given as before, and

$$B(a_0 \otimes \cdots \otimes a_n)$$
$$= \sum_{i=0}^{n} (-1)^{\sum_{j \leqq i}(|a_j|+1) \sum_{j \geqq i+1}(|a_j|+1)} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

We will denote this complex by $CC_*^{\mathrm{per}}(A)$.

The main feature about cyclic periodic homology that we are going to need, is the following (see [14] for complete formulas):

Theorem 5.1.1. *There is a morphism of complexes*

$$\bar{C}_*^{\lambda}(A[\eta]) \otimes CC_*^{\mathrm{per}}(A) \to CC_*^{\mathrm{per}}(A)$$

satisfying the following:

- $n!\eta^{\otimes n+1} \cdot a = a$ *for* $a \in CC_*^{\mathrm{per}}(A)$.
- *The component in* A *of* $(b_1 \otimes \cdots \otimes b_n) \cdot (a_0 \otimes \cdots \otimes a_m)$, *where*

$$b_1 \otimes \cdots \otimes b_n \in \bar{C}_*^{\lambda}(A),$$

is zero when $m \neq n$ *and equal to*

$$\sum_i \frac{1}{n!} (-1)^{i(n-1)} a_0 [b_{i+1}, a_1] \cdots [b_n, a_{n-i}][b_1, a_{n-i+1}] \cdots [b_i, a_n]$$

when $m = n$.

The framework underlying Theorem 5.1.1 also gives other operations on $CC_*^{\mathrm{per}}(A)$, see [14] for details. Let $\big(C^*(A,A), b\big)$ be the Hochschild cohomological complex, i.e. $C^*(A,A) = \mathrm{Hom}_k(\bar{A}^{\otimes *}, A)$, and

$$b\varphi(a_1, \otimes, a_{n+1}) = (-1)^n a_1 \varphi(a_2, \ldots, a_{n+1})$$
$$+ \sum_{j=1}^{n} (-1)^{n+j} \varphi(a_1, \ldots, a_j, a_{j+1}, \ldots, a_{n+1}) - \varphi(a_1, \ldots, a_n) a_{n+1}.$$

Given two elements, φ in $C^n(A,A)$ and ψ in $C^m(A,A)$, define

$$\varphi \circ \psi(a_1, \ldots, a_{n+m-1})$$
$$= \sum_{j \geqq 0} (-1)^{(n-1)j} \varphi\big(a_1, \ldots, a_j, \psi(a_{j+1}, \ldots, a_{j+m}), \ldots\big).$$

Set

$$[\varphi, \psi] = \varphi \circ \psi - (-1)^{(n+1)(m+1)} \psi \circ \varphi.$$

With this bracket and a suitably defined grading, $C^*(A,A)$ actually becomes a differential graded Lie algebra.

For $\varphi \in C^n(A,A)$ one can construct operations

$$L_\varphi : CC_*^{\mathrm{per}}(A) \to CC_{*-n+1}^{\mathrm{per}}(A),$$
$$I_\varphi : CC_*^{\mathrm{per}}(A) \to CC_{*-n}^{\mathrm{per}}(A)$$

such that

$$[L_\varphi, L_\psi] = L_{[\varphi,\psi]},$$
$$[I_\varphi, L_\psi] = I_{[\varphi,\psi]},$$
$$[B+b, I_\varphi] = I_{b\varphi} + L_\varphi.$$

6. The fundamental class

We consider the reduced cyclic homology complex of $M_N(\mathbb{A}^\hbar)[\hbar^{-1}]$. According to Lemma 5.1.1 in [2] and the Morita invariance of reduced the cyclic homology, the homology is given in the following way:

$$\overline{HC}_i\big(M_N(\mathbb{A}^\hbar)[\hbar^{-1}]\big) = \mathbb{C}[[\hbar,\hbar^{-1}], \quad i = 1,3,\dots,2n-1,$$
$$\overline{HC}_i\big(M_N(\mathbb{A}^\hbar)[\hbar^{-1}]\big) = 0 \qquad \text{otherwise.}$$

A concrete generator for the homology in dimension $2n-1$ is given by

$$U_0 = \frac{1}{2n(i\hbar)^n}\sum_{\sigma\in S_{2n}}\big(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(2n)}\big),$$

where $(v_1,\dots,v_{2n}) = (\hat{x}_1,\hat{\xi}_1,\dots,\hat{x}_n,\hat{\xi}_n)$.

Let $\tilde{\mathfrak{h}}$ denote the inverse image of $\mathfrak{h}$ under the map $\tilde{\mathfrak{g}} \to \mathfrak{g}$. Note that U_0 is invariant under the action of $\tilde{\mathfrak{h}}$ and therefore, by the result on $\overline{HC}_*\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])\big)$, extends uniquely in homology to a class

$$U \in C^*\big(\tilde{\mathfrak{h}}[\varepsilon],\tilde{\mathfrak{h}};\overline{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])\big)\big).$$

We wish to work with the differential graded algebra $M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]$ instead of $M_N(\mathbb{A}^\hbar[\hbar^{-1}])$. In the complex

$$\overline{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]\big)$$

we define $\eta^{(k)} = k!\eta^{\otimes k+1}$.

We also define operations on $\overline{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]\big)$ by

$$\iota_g(a_0 \otimes \cdots \otimes a_p) = \sum_{i=0}^{p} (-1)^{\sum_{k \leq i} (|a_k|+1)(|g|+1)} a_0 \otimes \cdots \otimes a_i \otimes g \otimes \cdots \otimes a_p,$$

where $g \in M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]$. Put $\iota_{\underline{g}} = \iota_{\eta g}$ and $L_{\underline{g}} = L_{\eta g}$, $g \in M_N(\mathbb{A}^\hbar[\hbar^{-1}])$. Here L denotes the usual L operation on the reduced cyclic complex.

With these operations $\bar{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta])$ becomes almost very homotopically constant over the commutator Lie algebra of $M_N(\mathbb{A}^\hbar[\hbar^{-1}])$. (Instead of the relations $[\partial, \iota_{\underline{g}}] = L_{\underline{g}} - \iota_g$ and $[\partial, L_{\underline{g}}] = L_g$ we have the relations $[\partial, \iota_{\underline{g}}] = L_{\underline{g}} + \iota_g$ and $[\partial, L_{\underline{g}}] = -L_g$. Phrased differently: $\bar{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta])$ is very homotopically constant if we replace $\iota_{\underline{g}}$ and $L_{\underline{g}}$ by $\iota_{-\underline{g}}$ and $L_{-\underline{g}}$.)

As in the case of very homotopically constant modules, we get a morphism of complexes

$$\bar{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta])_{\mathfrak{h}+\underline{\mathfrak{h}}} \to C^*(\tilde{\mathfrak{h}}[\varepsilon], \tilde{\mathfrak{h}}; \bar{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]))$$

given by

$$l \to ((h_1\varepsilon, \ldots, h_p\varepsilon) \to (-1)^p \iota_{\underline{h}_1} \cdots \iota_{\underline{h}_p} l).$$

Note that $\eta^{(k)} \in \bar{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta])_{\mathfrak{h}+\underline{\mathfrak{h}}}$, and we therefore get classes

$$\eta^{[k]} \quad \text{in } C^*(\tilde{\mathfrak{h}}[\varepsilon], \tilde{\mathfrak{h}}; \bar{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]))$$

given by

$$(h_1\varepsilon, \ldots, h_p\varepsilon) \to (-1)^p \iota_{h_1\eta} \cdots \iota_{h_p\eta} \eta^{(k)}.$$

Lemma 6.0.2. *The formula*

$$U = \sum_{m \geqq 0} (\hat{A} \cdot e^{-\theta} \cdot \mathrm{ch}^{-1})^{-1}_{2m} \cdot \eta^{[m]}$$

holds in $H^*(\tilde{\mathfrak{h}}[\varepsilon], \tilde{\mathfrak{h}}; \bar{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]))$.

Proof. It is well known from [2] that there is a splitting principle, i.e. the inclusion morphism

$$H^*(\tilde{\mathfrak{h}}[\varepsilon], \tilde{\mathfrak{h}}; \bar{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta])) \to H^*((\tilde{\mathfrak{d}}_n + \mathfrak{su}(N))[\varepsilon], (\tilde{\mathfrak{d}}_n + \mathfrak{su}(N)); \bar{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]))$$

where $\tilde{\mathfrak{d}}_n = \mathfrak{d}_n + \hbar^{-1}\mathbb{C}[[\hbar]]$ and $\mathfrak{d}_n$ is the set of $n \times n$ diagonal matrices, is injective.

Therefore we only have to identify the two classes in

$$H^*((\tilde{\mathfrak{d}}_n + \mathfrak{su}(N))[\varepsilon], (\tilde{\mathfrak{d}}_n + \mathfrak{su}(N)); \bar{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta])).$$

We note that we can factor the classes at hand in the following way: Write

$$M_N(\mathbb{A}^\hbar[\hbar^{-1}]) = \mathbb{A}_1^\hbar[\hbar^{-1}] \otimes \cdots \otimes \mathbb{A}_1^\hbar[\hbar^{-1}] \otimes M_N(\mathbb{A}_1^\hbar)[\hbar^{-1}],$$

where $\mathbb{A}_1^\hbar$ is the formal Weyl algebra in one variable. We can write $U = U_1 \times \cdots \times U_1 \times U_1'$, where U_1 is the extension of the fundamental class in $C^*\big(\mathfrak{d}_1[\varepsilon], \mathfrak{d}_1; \bar{C}_*^\lambda(\mathbb{A}_1^\hbar[\hbar^{-1}, \eta])\big)$, and U_1' is the extension of the fundamental class in

$$C^*\big(\big(\tilde{\mathfrak{d}}_1 + \mathfrak{su}(N)\big)[\varepsilon], \tilde{\mathfrak{d}}_1 + \mathfrak{su}(N); \bar{C}_*^\lambda\big(M_N(\mathbb{A}_1^\hbar[\hbar^{-1}])[\eta]\big)\big).$$

We thus need to identify U_1 and U_1'.

In the first case we can represent U_1 by

$$U_1 = \sum_{m=1}^{\infty} \frac{1}{m} \big((i\hbar)^{-1}\hat{\xi} \otimes \hat{x}\big)^{\otimes m} c_1^{m-1}, \tag{6.1}$$

where c_1 is the first Chern class.

Recall that the definition of $\mathbb{A}_1^\hbar[\hbar^{-1}]$ is $\mathbb{C}[[\hat{x}, \hat{\xi}]][[\hbar, \hbar^{-1}]$ with a product $*$. Given an element f in $\mathbb{A}_1^\hbar[\hbar^{-1}]$, we can regard f as a function in the variables $\hat{x}, \hat{\xi}$ with values in $\mathbb{C}[[\hbar, \hbar^{-1}]$. Hence, given f in $\mathbb{A}_1^\hbar[\hbar^{-1}]$, we can define $l(f) = f(0,0)$. With this l we get, according to section 5, a quasi-isomorphism of complexes

$$\mathrm{Br} : \bar{C}_*^\lambda(\mathbb{A}^\hbar[\hbar^{-1}, \eta]) \to C_*^\lambda(\mathbb{C}[[\hbar, \hbar^{-1}]).$$

One checks that this gives a morphism of complexes

$$C^*\big(\mathfrak{d}_1[\varepsilon], \mathfrak{d}_1; \bar{C}_*^\lambda(\mathbb{A}_1^\hbar[\hbar^{-1}, \eta])\big) \xrightarrow{\mathrm{Br}} C^*\big(\mathfrak{d}_1[\varepsilon], \mathfrak{d}_1; C_*^\lambda(\mathbb{C}[[\hbar, \hbar^{-1}])\big)$$

and therefore a quasi-isomorphism of complexes.

A computation shows that

$$\mathrm{Br}(U_1) = \sum_{m=0}^{\infty} 1^{(m)} \hat{A}_{2m}^{-1},$$

where $1^{(m)} = m!(m+1)!1^{\otimes(2m+1)}$, and $\hat{A}$ is as in example 3 in section 4. On the other hand we have $\mathrm{Br}(\eta^{[m]}) = 1^{(m)}$, where $\eta^{[m]}$ is the class

$$\varepsilon d_1, \ldots, \varepsilon d_k \to (-1)^k \iota_{d_1\eta} \cdots \iota_{d_k\eta} \eta^{(m)}, \quad d_i \in \mathfrak{d}_1.$$

Since Br is a quasi-isomorphism, we have

$$U_1 = \sum_{m=0}^{\infty} \eta^{[m]} \hat{A}_{2m}^{-1}$$

in $H^*\big(\mathfrak{d}_1[\varepsilon], \mathfrak{d}_1; \hat{C}_*^\lambda(\mathbb{A}_1^\hbar[\hbar^{-1}, \eta])\big)$.

We note that we have a morphism of complexes

$$\begin{array}{c} C^*\big(\big(\tilde{\mathfrak{d}}_1 + \mathfrak{su}(N)\big)[\varepsilon], \tilde{\mathfrak{d}}_1 + \mathfrak{su}(N); \overline{C}^\lambda_*\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])\big)\big) \\ \Big\downarrow \mathrm{Tr} \\ C^*\big(\big(\tilde{\mathfrak{d}}_1 + \mathfrak{su}(N)\big)[\varepsilon], \tilde{\mathfrak{d}}_1 + \mathfrak{su}(N); \overline{C}^\lambda_*(\mathbb{A}^\hbar[\hbar^{-1}])\big), \end{array}$$

where in the bottom row $\mathfrak{su}(N)$ acts trivially, and Tr denotes the morphism of complexes

$$\overline{C}^\lambda_*\big(M_N(\mathbb{A}^\hbar_1)[\hbar^{-1}]\big) \to \overline{C}^\lambda_*(\mathbb{A}^\hbar_1[\hbar^{-1}])$$

induced by the normalized trace Tr.

In the top row we have the class U'_1 that in homology is the unique extension of the fundamental class. In the bottom row we have a class U' given by the same formula as in 6.1. Also this is, in homology, a unique extension of the fundamental class. Therefore $\mathrm{Tr}(U) = U'$ in

$$H^*\big(\big(\tilde{\mathfrak{d}}_1 + \mathfrak{su}(N)\big)[\varepsilon], \tilde{\mathfrak{d}}_1 + \mathfrak{su}(N); \overline{C}^\lambda_*(\mathbb{A}^\hbar[\hbar^{-1}])\big).$$

We further note that there are morphisms of complexes

$$\begin{array}{c} C^*\big(\big(\tilde{\mathfrak{d}}_1 + \mathfrak{su}(N)\big)[\varepsilon], \tilde{\mathfrak{d}}_1 + \mathfrak{su}(N); \overline{C}^\lambda_*\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]\big)\big) \\ \Big\downarrow \mathrm{Tr} \\ C^*\big(\big(\tilde{\mathfrak{d}}_1 + \mathfrak{su}(N)\big)[\varepsilon], \tilde{\mathfrak{d}}_1 + \mathfrak{su}(N); \overline{C}^\lambda_*(\mathbb{A}^\hbar[\hbar^{-1}][\eta])\big) \\ \Big\downarrow \mathrm{Br} \\ C^*\big(\big(\tilde{\mathfrak{d}}_1 + \mathfrak{su}(N)\big)[\varepsilon], \tilde{\mathfrak{d}}_1 + \mathfrak{su}(N); C^\lambda_*(\mathbb{C}[[\hbar, \hbar^{-1}]))\big), \end{array}$$

where Br is the Brodzki map as before. We thus have

$$\mathrm{Br}\big(\mathrm{Tr}(U'_1)\big) = \sum_{m \leqq 0} \hat{A}^{-1}_{2m} 1^{(m)}.$$

Note that

$$\mathrm{Br}\big(\mathrm{Tr}(\eta^{[m]})\big) = \sum_{l=0}^{\infty} 1^{(m+l)} (e^\theta \, \mathrm{ch})_{2l},$$

where θ is given in example 1 in section 4 and ch is given in example 2 in section 4.

Since $\mathrm{Br} \circ \mathrm{Tr}$ is a quasi-isomorphism, we get

$$U'_1 = \sum_{m=0}^{\infty} \big(\hat{A} \cdot e^{-\theta} \cdot (\mathrm{ch})^{-1}\big)^{-1}_{2m} \eta^{[m]}$$

in $H^*\big(\big(\tilde{\mathfrak{d}}_1 + \mathfrak{su}(N)\big)[\varepsilon], \tilde{\mathfrak{d}}_1 + \mathfrak{su}(N); \overline{C}^\lambda_*\big(M_N(\mathbb{A}^\hbar_1[\hbar^{-1}])[\eta]\big)\big)$.

The lemma now follows, since $\hat{A}$ is multiplicative.

It is well known, see [2], that the restriction homomorphism

$$C^*\big(\tilde{\mathfrak{g}}[\varepsilon],\tilde{\mathfrak{h}};\bar{C}^\lambda_*\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]\big)\big)\to C^*\big(\tilde{\mathfrak{h}}[\varepsilon],\tilde{\mathfrak{h}};\bar{C}^\lambda_*\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]\big)\big)$$

is a quasi-isomorphism. But in $C^*\big(\tilde{\mathfrak{g}}[\varepsilon],\tilde{\mathfrak{h}};\bar{C}^\lambda_*\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]\big)\big)$ there are two classes that map to $\eta^{[m]}$ under the restriction homomorphism, namely $\mathrm{CW}(\eta^{[m]})$ and the class

$$(g_1\varepsilon,\dots,g_p\varepsilon)\to(-1)^p\iota_{g_1\eta}\cdots\iota_{g_p\eta}\eta^{(m)}. \tag{6.2}$$

Therefore $\mathrm{CW}(\eta^{[m]})$ and the class (6.2) are equivalent in

$$C^*\big(\tilde{\mathfrak{g}}[\varepsilon],\tilde{\mathfrak{h}};\bar{C}^\lambda_*\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]\big)\big).$$

Considering the restriction homomorphism

$$C^*\big(\tilde{\mathfrak{g}}[\varepsilon],\tilde{\mathfrak{h}};\bar{C}^\lambda_*\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]\big)\big)\to C^*\big(\tilde{\mathfrak{g}},\tilde{\mathfrak{h}};\bar{C}^\lambda_*\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]\big)\big)$$

we get that in the righthand side $\mathrm{CW}(\eta^{[m]})$ is equivalent to $\eta^{(m)}$. For $\mathfrak{g}$-modules $\mathbb{L}^*$ we have that $C^*(\tilde{\mathfrak{g}},\tilde{\mathfrak{h}};\mathbb{L}^*)$ is quasi-isomorphic to $C^*(\mathfrak{g},\mathfrak{h};\mathbb{L}^*)$. Combining these observations we have:

Theorem 6.0.3. *Let U be an extension of U_0 to a class in*

$$C^*\big(\mathfrak{g},\mathfrak{h};\bar{C}^\lambda_*\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]\big)\big).$$

In $H^*\big(\mathfrak{g},\mathfrak{h};\bar{C}^\lambda_*\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]\big)\big)$ we have the following equality:

$$U=\sum_{m\geqq 0}\big(\hat{A}\cdot e^\theta\cdot(\mathrm{ch})^{-1}\big)^{-1}_{2m}\cdot\eta^{(m)}.$$

7. The Gelfand-Fuks construction

We consider a $\mathfrak{g}$-module $\mathbb{L}^*$, where $\mathfrak{g}$ is as in section 3. Given a deformation quantization $A^\hbar_E$, we can consider the bundle

$$\tilde{M}_{A^\hbar_E}\times_G\mathbb{L}^*$$

and also consider the differential forms with values in this bundle. We will denote this by $\Omega^*(M,\mathbb{L}^*)$. Furthermore we get a flat connection ∇ induced from the connection on $\tilde{M}_{A^\hbar_E}$. Using this connection and the differential on $\mathbb{L}^*$, we get a complex $\Omega(M,\mathbb{L}^*)$. The Gelfand-Fuks construction gives a morphism of complexes

$$\mathrm{GF}:C^*(\mathfrak{g},\mathfrak{h};\mathbb{L}^*)\to\Omega^*(M,\mathbb{L}^*)$$

defined in the following way:

Choose a $\mathrm{U}(n) \times \mathrm{SU}(N)$ trivialization of $\tilde{M}_{A_E^\hbar} \times_G \mathbb{L}^*$. In a given trivialization write $\nabla = d + A$, where A is the connection one form. Given vector fields $X_1, \dots, X_p$ and l in $C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*)$, define

$$\mathrm{GF}(l)(X_1, \dots, X_p) = l(X_1, \dots, X_p).$$

The Gelfand-Fuks construction applied to the examples of classes in $C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*)$ constructed in section 4 gives the following:

Example 1. $\mathrm{GF}(\theta)$ is the characteristic class of the deformation quantization.

Example 2. $\mathrm{GF}(\mathrm{ch})$ is the Chern character of $\mathrm{End}(E)$.

Example 3. $\mathrm{GF}(\hat{A})$ is the $\hat{A}$ class of TM.

The main example we are going to look at, is the case where $\mathbb{L}^*$ is $\bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar)[\hbar^{-1}]\big)$. We first note that

Lemma 7.0.4. *Let $A_E^\hbar$ be a deformation quantization over $\mathbb{R}^{2n}$. The complex $\big(\Omega^*\big(\mathbb{R}^{2n}, \bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar)\big)\big), \nabla\big)$ is acyclic, and the cohomology consists of jets on the diagonal of elements in $\bar{C}_*^\lambda\big(M_N(\mathscr{W}_n)\big)$.*

Proof. We can assume that ∇ is of the form $d - \sum_i (\partial_{\hat{x}_i} \otimes dx_i + \partial_{\hat{\xi}_i} \otimes d\xi_i)$. We have a short exact sequence of complexes

$$0 \to \big(\Omega^*\big(\mathbb{R}^{2kn}/\Delta, M_N(\mathbb{A}^\hbar)^{\otimes k}\big), \nabla\big) \to \big(\Omega^*\big(\mathbb{R}^{2kn}, M_N(\mathbb{A}^\hbar)^{\otimes k}\big), \nabla\big)$$
$$\stackrel{\varphi^*}{\to} \big(\Omega^*\big(\mathbb{R}^{2k}, M_N(\mathbb{A}^\hbar)^{\otimes k}\big), \nabla\big) \to 0,$$

where $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2kn}$ is the map onto the diagonal δ. From the associated long exact sequence, we see that $\big(\Omega^*\big(\mathbb{R}^{2k}, M_N(\mathbb{A}^\hbar)^{\otimes k}\big), \nabla\big)$ is acyclic and that the cohomology consists of jets on the diagonal of elements of $M_N(\mathscr{W}_n)^{\otimes k}$. By considering the following short exact sequence of complexes

$$0 \to \big(\Omega^*\big(\mathbb{R}^{2k}, 1 \otimes M_N(\mathbb{A}^\hbar) \otimes \cdots \otimes M_N(\mathbb{A}^\hbar) + \cdots$$
$$+ M_N(\mathbb{A}^\hbar) \otimes \cdots \otimes M_N(\mathbb{A}^\hbar) \otimes 1\big), \nabla\big)$$
$$\to \big(\Omega^*\big(\mathbb{R}^{2k}, M_N(\mathbb{A}^\hbar)^{\otimes k}\big), \nabla\big) \to \big(\Omega^*\big(\mathbb{R}^{2k}, \overline{M_N(\mathbb{A}^\hbar)}^{\otimes k}\big), \nabla\big) \to 0$$

we see that $\big(\Omega^*\big(\mathbb{R}^{2k}, \overline{M_N(\mathbb{A}^\hbar)}^{\otimes k}\big), \nabla\big)$ is acyclic and that the cohomology consists of jets on the diagonal of elements in $\overline{M_N(\mathscr{W}_n)}^{\otimes k}$.

Let a be an element in $\Omega^*\big(M, \overline{M_N(\mathbb{A}^\hbar)}^{\otimes k}/\mathrm{Im}(1-\tau)\big)$ with $\nabla(a) = 0$. We can then lift a to an element $\tilde{a} \in \Omega^*\big(\mathbb{R}^{2k}, \overline{M_N(\mathbb{A}^\hbar)}^{\otimes k}\big)$, where $\nabla(\tilde{a}) \in \Omega^*\big(\mathbb{R}^{2k}, \mathrm{Im}(1-\tau)\big)$. However, $b = \frac{1}{k}\sum_{i=0}^{k-1} \tau^i(\tilde{a})$ is also a lift of a, and $\nabla(b) = \sum_{i=0}^{k-1} \tau^i \nabla(\tilde{a}) = 0$. The lemma follows from this.

8. Traces on deformation quantizations and index theory

We consider a deformation quantization $A_E^\hbar$ of an endomorphism bundle $\mathrm{End}(E)$ over a symplectic manifold M of dimension n. Let $A_{E,\mathrm{c}}^\hbar$ be the algebra of elements in $A_E^\hbar$ with compact support. This algebra has a canonical $\mathbb{C}[[\hbar,\hbar^{-1}]$ valued trace defined in the following way:

Let (V_i,Φ_i) be a cover of M and $\Phi_i^{-1}: M_N(\mathscr{W}_n)\to A_E^\hbar$ local isomorphisms over V_i. Let ρ_{V_i} be a partition of unity with respect to the covering. For an element $a\in A_{E,\mathrm{c}}^\hbar$ we define

$$\mathrm{Tr}(a)=\sum_i\int\frac{1}{n!(i\hbar)^n}\,\mathrm{tr}\big(\Phi_i(\rho_i * a)\big)\omega_{st}^n,$$

where ω_{st} is the standard symplectic form on $\mathbb{R}^{2n}$, and tr denotes the normalized trace on M_N. That Tr is independent of the choices made, and that it is a trace, hinges on the following two propositions.

Proposition 8.0.5. *Let E be the trivial line bundle. Then* Tr *is a trace and independent of the choices made.*

Proof. See [7].

Proposition 8.0.6. *Suppose $\mathscr{W}_n$ is the Weyl algebra over some contractible open subset U of $\mathbb{R}^{2n}$, then every automorphism over the identity map of $M_N(\mathscr{W}_n)$ is inner.*

Proof. More or less the same as Lemma 3.0.6, see also Lemma 2.0.4.

We consider the trace as a functional on $CC_*^{\mathrm{per}}(A_{E,\mathrm{c}}^\hbar)$, and we want to compute the trace at the level of homology. To this end we consider the following situation:

Given an element b in $\bar{C}_*^\lambda(A_E^\hbar[\hbar^{-1},\eta])$, we define $\chi_{\mathrm{Tr}}(b)$ in $CC^*_{\mathrm{per}}(A_{E,\mathrm{c}}^\hbar)$ in the following way:

$$\chi_{\mathrm{Tr}}(b)(a)=\mathrm{Tr}(b\cdot a),$$

where $\cdot$ means the action of $\bar{C}_*^\lambda(A_E^\hbar[\hbar^{-1},\eta])$ on $CC_*^{\mathrm{per}}(A_{E,\mathrm{c}}^\hbar[\hbar^{-1}])$, see Theorem 5.1.1. We will now extend this to elements in the Čech complex $\check{C}^*\big(M,\bar{C}_*^\lambda(A_E^\hbar[\hbar^{-1},\eta])\big)$ with values in the presheaf $V\to\bar{C}_*^\lambda(A_{E|V}^\hbar[\hbar^{-1},\eta])$. This is done in the following proposition.

Proposition 8.0.7. *Take $\{b_{V_0\dots V_p}\}$ in $\check{C}^*\big(M,\bar{C}_*^\lambda(A_E^\hbar[\hbar^{-1},\eta])\big)$ and a in*

$$CC_*^{\mathrm{per}}(A_{E,\mathrm{c}}^\hbar[\hbar^{-1}]).$$

Define

$$\chi_{\mathrm{Tr}}(\{b_{V_0\dots V_p}\})(a)=\sum_{V_0,\dots,V_p}\chi_{\mathrm{Tr}}(b_{V_0,\dots,v_p})(I_{\rho_{V_0}}[B+b,I_{\rho_{V_1}}]\dots[B+b,I_{\rho_{V_p}}]a).$$

This gives a morphism of complexes

$$\check{C}^*\big(M,\bar{C}_*^\lambda(A_E^\hbar[\hbar^{-1},\eta])\big)\otimes CC_*^{\mathrm{per}}(A_{E,\mathrm{c}}^\hbar[\hbar^{-1}])\to\mathbb{C}[[\hbar,\hbar^{-1}].$$

Proof. See [11].

8.1. The fundamental class in the Čech complex. Recall that we have the canonical coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ on $\mathbb{R}^{2n}$. We will also use this notation for the associated coordinate functions and consider these coordinate functions as elements in $\mathcal{W}_n$.

The fundamental class U_0 in $\bar{C}_*^\lambda\big(M_N(\mathcal{W}_n)[\hbar^{-1}]\big)$ is given by

$$U_0 = \frac{1}{2n(i\hbar)^n} \sum_{\sigma \in S_{2n}} (v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_{2n}}),$$

where $(v_1, \dots, v_{2n}) = (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$.

By the same argument as in the section on the fundamental class in Lie algebra cohomology, this class extends uniquely in cohomology to a class U in $\check{C}^*\big(M, \bar{C}_*^\lambda(A_E^\hbar[\hbar^{-1}])\big)$. In order to connect this class to the fundamental class defined in Lie algebra cohomology, we introduce the complex

$$\check{C}^*\Big(M, \Omega^*\big(M, \bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])\big)\big)\Big).$$

According to Lemma 7.0.4, this complex is quasi-isomorphic to the complex $\check{C}^*\big(M, \bar{C}_*^\lambda(A_E^\hbar[\hbar^{-1}])\big)$. Furthermore we have the Gelfand-Fuks morphism

$$\mathrm{GF} : C^*\big(\mathfrak{g}, \mathfrak{h}; \bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])\big)\big) \to \check{C}^*\Big(M, \Omega^*\big(M, \bar{C}_*^\lambda\big(M_N(\mathbb{A}^\hbar[\hbar^{-1}])\big)\big)\Big).$$

Because of uniqueness we get that $\mathrm{GF}(U) = U$ in cohomology. By Theorem 6.0.3 and the splitting principle, we therefore get

Theorem 8.1.1. *In the complex* $\check{C}^*\big(M, \bar{C}_*^\lambda(A_E^\hbar[\hbar^{-1}, \eta])\big)$ *the two classes*

$$U \quad and \quad \sum_{m \geqq 0} (\hat{A} \cdot e^\theta \cdot \mathrm{ch})_{2m}^{-1} \cdot \eta^{(m)}$$

are equivalent.

With this we are now in position to prove:

Theorem 8.1.2. $\chi_{\mathrm{Tr}}(U)(a_0 \otimes \cdots \otimes a_k)$ *has no singularities in* $\hbar$ *and*

$$\chi_{\mathrm{Tr}}(U)(a_0 \otimes \cdots \otimes a_k) = (-1)^n \int \mathrm{ch}^{-1}\big(\mathrm{End}(E)\big)\, \mathrm{ch}(\nabla)(\tilde{a}_0 \otimes \cdots \otimes \tilde{a}_k) \bmod \hbar.$$

Here $\tilde{a}_i$ *is* $a_i \bmod \hbar$, *and* $\mathrm{ch}(\nabla)$ *is the* J.L.O. *cocycle associated to* ∇ (*see also* [8]), *i.e.*

$$\begin{aligned} &\mathrm{ch}(\nabla)(\tilde{a}_0 \otimes \cdots \otimes \tilde{a}_k) \\ &\quad = \int_{\Delta_k} \mathrm{tr}\big(\tilde{a}_0 e^{-t_0\nabla^2} \nabla(\tilde{a}_i) e^{-t_1\nabla^2} \cdots \nabla(\tilde{a}_k) e^{-t_k\nabla^2}\big)\, dt_0 \cdots dt_{k-1}, \end{aligned}$$

where tr *is the normalized trace on* $\mathrm{End}(E)$.

According to Theorem 8.1.1, *we therefore have that*

$$\operatorname{Tr}(a_0 \otimes \cdots \otimes a_k \cdot e^{-\theta}) = (-1)^n \int \hat{A} \cdot \operatorname{ch}(\nabla)(\tilde{a}_0 \otimes \cdots \otimes \tilde{a}_k) \bmod \hbar.$$

Proof. Because of Morita equivalence, it is enough to look at the case where $\tilde{a}_i$ is scalar for all i. We have that

$$\chi_{\operatorname{Tr}}(U)(a_0 \otimes \cdots \otimes a_k) = \sum_{V_0} \chi_{\operatorname{tr}}(U_0)\big(I_{\rho V_0}(a_0 \otimes \cdots \otimes a_k)\big) + \cdots,$$

and it is not difficult to see that ... is zero modulo $\hbar$. The explicit formula for $\chi_{\operatorname{Tr}}(U_0)$ gives

$$\sum_{V_0} \chi_{\operatorname{Tr}}(U_0)\big(I_{\rho V_0}(a_0 \otimes \cdots \otimes a_{2n})\big) = (-1)^n \frac{1}{2n!} \int \tilde{a}_0\, d\tilde{a}_1 \cdots d\tilde{a}_{2n} \bmod \hbar.$$

The result follows from this.

By adopting the arguments in [12] one sees that

$$\frac{d}{d\hbar} \operatorname{Tr}(a \cdot e^{-\theta}) = 0$$

when a is a cycle in $CC_*^{\mathrm{per},\mathbb{C}}(A_{E,\mathrm{c}}^{\hbar})$. The notation $CC_*^{\mathrm{per},\mathbb{C}}(A_{E,\mathrm{c}}^{\hbar})$ means cyclic periodic homology of $A_{E,\mathrm{c}}^{\hbar}$ as a $\mathbb{C}$-algebra. Together with Theorem 8.1.2 we get

Theorem 8.1.3. *The identity*

$$\operatorname{Tr}(a) = (-1)^n \int \hat{A} \cdot e^{\theta} \cdot \operatorname{ch}(\nabla)(\tilde{a})$$

holds when a is a cycle in $CC_*^{\mathrm{per},\mathbb{C}}(A_{E,\mathrm{c}}^{\hbar})$.

References

[1] *F. Bayen*, *M. Flato*, *C. Fronsdal*, *A. Lichnerowicz*, and *D. Sternheimer*, Deformation theory and quantization, 1, 2, Ann. Phys. **111** (1978), 61–151.
[2] *P. Bressler*, *R. Nest*, and *B. Tsygan*, Riemann-Roch theorems via deformation quantization, I, II, Adv. Math. **167**(3) (2002), 1–25, 26–73.
[3] *J. Brodzki*, Semi-direct products, derivations and reduced cyclic homology, J. reine angew. Math. **436** (1993), 177–195.
[4] *H. Bursztyn* and *S. Waldmann*, Deformation quantization of hermitian vector bundles, Lett. Math. Phys. **53**(4) (2000), 349–365.
[5] *A. Connes*, Noncommutative differential geometry, I.H.E.S. Publ. Math. **62** (1985), 257–360.
[6] *B. Fedosov*, A simple geometric construction of deformation quantization, J. Diff. Geom. **40** (1994), 213–238.
[7] *B. Fedosov*, Deformation quantization and index theory, Math. Topics **9**, Akademie Verlag, 1996.
[8] *A. Gorokhovsky*, Characters of cycles, equivariant characteristic classes and Fredholm modules, Comm. Math. Phys. **200**(1) (1999), 1–23.
[9] *M. Kontsevich*, Deformation quantization of Poisson manifolds, q-alg/9709040, 1997.
[10] *J. L. Loday*, Cyclic Homology, Springer, 1998.
[11] *R. Nest* and *B. Tsygan*, Algebraic index theorem, Comm. Math. Phys. **127** (1995), 223–262.
[12] *R. Nest* and *B. Tsygan*, Algebraic index theorem for families, Adv. Math. **113** (1995), 161–205.

[13] *R. Nest* and *B. Tsygan*, Formal versus analytic index theorems, IMRN **11** (1996), 557–564.
[14] *R. Nest* and *B. Tsygan*, On the cohomology ring of an algebra, in: Advances in Geometry, J.-L. Brylinski, R. Brylinski, V. Nistor, B. Tsygan, and P. Xu, eds., Progr. Math. **172** (1998), 337–370.
[15] *R. Nest* and *B. Tsygan*, Deformations of symplectic Lie algebroids, deformations of holomorphic symplectic structures, and index theorems, Asian J. Math. **5**(4) (2001), 599–636.
[16] *M. De Wilde* and *P. B. A. Lecomte*, Existence of star products and of formal deformations in Poisson Lie algebra of arbitrary symplectic manifolds, Lett. Math. Phys. **7** (1983), 487–496.

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