An overlapping additive Schwarz preconditioner for boundary element approximations to the Laplace screen and Lamé crack problems

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Dedicated to Professor Erwin Stein on the occasion of his 70[h] birthday.

Abstract — We study a two-level overlapping additive Schwarz preconditioner for the *h*-version of the Galerkin boundary element method when used to solve hypersingular integral equations of the first kind on an open surface in $\mathbb{R}^3$. These integral equations result from Neumann problems for the Laplace and Lamé equations in the exterior of the surface. We prove that the condition number of the preconditioned system is bounded by $O(1 + \log^2(H/\delta))$, where H denotes the diameter of the subdomains and δ the size of the overlap.

Keywords: Galerkin boundary element method, *h* version, additive Schwarz, overlapping, preconditioned conjugate gradient

1. INTRODUCTION

In this paper we consider an overlapping additive Schwarz method for the Galerkin boundary element approximation of the Laplace screen and Lamé crack problems in three dimensions with Neumann boundary conditions. In the context of boundary element methods, these problems are reformulated into hypersingular integral equations on the screen, the solutions of which are then approximated with spline functions by the *h*-version of the Galerkin method (see details in Section 2).

It is well known that the linear algebraic systems arising from these approximations can be efficiently solved with preconditioning techniques using domain decompositions. Substructuring techniques have been studied [1,2,8] for hypersingular integral equations on surfaces in $\mathbb{R}^3$. For equations on curves in $\mathbb{R}^2$, overlapping additive Schwarz methods have been discussed in [15,17] for both the *h*- and *p*-

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versions. For the case of surfaces in $\mathbb{R}^3$ where the *p*-version is used on anisotropic elements, this type of preconditioners has been studied in [9]. However, they have not been thoroughly addressed in case the *h*-version is used for these integral equations on surfaces. The purpose of the work in this paper is to fill this gap.

We prove that the condition number of the preconditioned system is bounded by $O(1 + \log^2(H/\delta))$, where H is the diameter of the subdomains, and δ is the size of the overlaps. We note that the overlapping subdomains are *not* required to be quadrilaterals, in fact their boundaries can be zigzag closed lines (see Fig. 1).

In the finite element literature, overlapping additive Schwarz methods have been discussed in e.g. [6,12]. The algorithm in [12] consists of mainly two steps. The first step involves the elimination of interior unknowns by using Schur complements. The second step involves subspace decomposition with subspaces being vertex spaces. The algorithm is different from the one analysed in the present paper.

In [6] there is a very brief sketch of the proof in the 3-dimensional case of differential operators, which corresponds to our situation of hypersingular integral operators on a surface in $\mathbb{R}^3$ with the $H_{00}^{1/2}$ or $\widetilde{H}^{1/2}$ norm. A detailed analysis of the situation is not given there. We believe that our analysis, going over several pages, is by no means trivial.

The paper is organised as follows. In Section 2 we introduce the model problems and the resulting hypersingular integral equations, together with their Galerkin approximations. The abstract framework of the additive Schwarz method is presented in Section 3. In Section 4 we define the overlapping subdomains and the resulting subspace decomposition, and state the main result of the paper. The proof is carried out by proving different lemmas in Section 5. Numerical experiments for both the Laplace and Lamé operators are presented in Section 6.

2. MODEL PROBLEMS AND BOUNDARY-INTEGRAL EQUATIONS

Let $\Gamma \subset \mathbb{R}^3$ be an open two dimensional surface with a polygonal boundary, and let **n** be a normal vector to Γ, which defines two sides Γ_1 and Γ_2 of Γ. In the unbounded domain $\Omega = \mathbb{R}^3 \setminus \overline{\Gamma}$, we consider the following Neumann problem, deferring the definitions of the usual Sobolev spaces to a later paragraph in the section.

Given two functions $f_1, f_2 \in (H^{-1/2}(\Gamma))^d$, $d = 1$ or $d = 3$, such that $f := f_1 - f_2 \in (\widetilde{H}^{-1/2}(\Gamma))^d$, find $U \in (H^1_{\text{loc}}(\Omega))^d$ satisfying

$$\begin{aligned}
\mathcal{L}U &= 0 &&\text{in } \Omega \\
\mathcal{B}_i U &= f_i &&\text{on } \Gamma_i, \quad i = 1,2
\end{aligned} \tag{2.1}$$

where

$$\mathcal{L}U = \begin{cases} \Delta U, & \text{(Laplace operator)} & d = 1 \\ \mu \Delta U + (\lambda + \mu)\,\text{grad div}\,U & \text{(Lamé operator)} & d = 3 \end{cases}$$

and

$$\mathcal{B}_i U = \begin{cases} \dfrac{\partial U}{\partial \mathbf{n}}\Big|_{\Gamma_i}, & d = 1 \\[2mm] TU|_{\Gamma_i}, & d = 3 \end{cases}$$

with $TU \in \mathbb{R}^3$ being the traction defined by

$$TU = \lambda \,(\operatorname{div} U)\mathbf{n} + \mu \frac{\partial U}{\partial \mathbf{n}} + \mu \mathbf{n} \times \operatorname{curl} U.$$

Here λ and μ are the Lamé constants satisfying $\mu > 0$ and $2\lambda + \mu > 0$. Additionally, the vanishing condition $U(x) = O(1/|x|)$ is required at infinity.

The problem (2.1) can be reformulated into hypersingular integral equations of the first kind on the two dimensional open surface Γ. In order to do so, we first note that the fundamental solution of the differential operator $\mathscr{L}$ is given by

$$G(x,y) = \begin{cases} \dfrac{1}{2\pi} \dfrac{1}{|x-y|}, & d = 1 \\[3mm] \dfrac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \left\{ \dfrac{1}{|x-y|} I_{3\times 3} + \dfrac{\lambda + \mu}{\lambda + 3\mu} \dfrac{(x-y)(x-y)^t}{|x-y|^3} \right\}, & d = 3 \end{cases}$$

where $x, y \in \mathbb{R}^3$, $I_{3\times 3}$ is the identity matrix of size 3×3, and t denotes the transpose of the matrix. Let $u = [U]|_\Gamma$ be the jump of U across Γ. Then the problem (2.1) is equivalent to (see [13,14])

$$Du = g \tag{2.2}$$

where $D : (\widetilde{H}^{1/2}(\Gamma))^d \to (H^{-1/2}(\Gamma))^d$ is the hypersingular integral operator defined as

$$Du(x) := \begin{cases} -\dfrac{\partial}{\partial \mathbf{n}_x} \displaystyle\int_\Gamma \dfrac{\partial}{\partial \mathbf{n}_y} G(x,y) u(y) \, ds_y, & d = 1 \\[3mm] -T_x \displaystyle\int_\Gamma (T_y G(x,y))^t u(y) \, ds_y, & d = 3 \end{cases} \tag{2.3}$$

and

$$g(x) := \begin{cases} f_1(x) + f_2(x) - \displaystyle\int_\Gamma (f_1(x) - f_2(x)) \dfrac{\partial}{\partial \mathbf{n}_y} G(x,y) \, ds_y, & d = 1 \\[3mm] f_1(x) + f_2(x) - \displaystyle\int_\Gamma (f_1(x) - f_2(x)) T_y G(x,y) \, ds_y, & d = 3. \end{cases}$$

In order to describe the operator properties of D, we now give the definitions for the function spaces mentioned above. Let $S \subset \mathbb{R}^n$, $n = 1$ or 2, be a Lipschitz domain. We denote by $L_2(S)$ the usual space of Lebesgue square integrable functions equipped with the standard norm $\|\cdot\|_{L_2(S)}$, and by $H^1(S)$ the usual Sobolev space equipped with the norm

$$\|v\|_{H^1(S)}^2 = |v|_{H^1(S)}^2 + \|v\|_{L_2(S)}^2$$

where $|v|_{H^1(S)} = \|\nabla v\|_{L_2(S)}$. The space $H^{1/2}(S)$ is defined by Hilbert space interpolation [3] so that

$$H^{1/2}(S) := [L_2(S), H^1(S)]_{1/2}$$

with the norm

$$\|u\|_{H^{1/2}(S)}^2 = \int_0^\infty K(t,u)^2 \frac{dt}{t^2}$$

where the K-functional is defined, for $u \in L_2(S) + H^1(S)$, by

$$K(t,u)^2 = \inf_{u=u_0+u_1} \left(\|u_0\|_{L_2(S)}^2 + t^2 \|u_1\|_{H^1(S)}^2 \right).$$

Similarly, we define the subspace $\widetilde{H}^{1/2}(S)$ by

$$\widetilde{H}^{1/2}(S) := [L_2(S), H_0^1(S)]_{1/2}$$

with the norm

$$\|u\|_{\widetilde{H}^{1/2}(S)}^2 = \int_0^\infty K(t,u)^2 \frac{dt}{t^2}$$

where the K-functional is defined, for $u \in L_2(S) + H_0^1(S)$, by

$$K(t,u)^2 = \inf_{u=u_0+u_1} \left(\|u_0\|_{L_2(S)}^2 + t^2 \|u_1\|_{H^1(S)}^2 \right).$$

The spaces $H^{-1/2}(S)$ and $\widetilde{H}^{-1/2}(S)$ are defined as the dual spaces of $\widetilde{H}^{1/2}(S)$ and $H^{1/2}(S)$, respectively, with respect to the usual extension of the L_2 inner product on S.

The spaces $(\widetilde{H}^{1/2}(S))^d$ and $(H^{1/2}(S))^d$ for $d = 3$ are defined as spaces of functions in $\widetilde{H}^{1/2}(S)$ and $H^{1/2}(S)$ componentwise.

It was shown in [13,14] that the operator D defined in (2.3) is a bijective mapping from $(\widetilde{H}^{1/2}(\Gamma))^d$ onto $(H^{-1/2}(\Gamma))^d$. Moreover (see also [7]), if

$$a(v,w) := \langle Dv, w \rangle \quad \forall v, w \in (\widetilde{H}^{1/2}(\Gamma))^d$$

(where $\langle Dv, w \rangle$ denotes the duality pairing which coincides with the L_2 inner product on Γ if $Dv, w \in (L_2(\Gamma))^d$) then $a(\cdot, \cdot)$ is a positive-definite and symmetric bilinear form on $(\widetilde{H}^{1/2}(\Gamma))^d$ satisfying

$$a(v,v) \simeq \|v\|_{(\widetilde{H}^{1/2}(\Gamma))^d}^2 \quad \forall v \in (\widetilde{H}^{1/2}(\Gamma))^d. \tag{2.4}$$

Here, and in what follows, the notation $a \preceq b$ is used to indicate that $a \leqslant cb$ for some positive constant c that is independent of the main quantities of interest, while $a \simeq b$ is equivalent to $a \preceq b$ and $b \preceq a$.

A weak form of equation (2.2) is the problem of finding

$$u \in (\widetilde{H}^{1/2}(\Gamma))^d : \quad a(u,v) = \langle g, v \rangle \quad \forall v \in (\widetilde{H}^{1/2}(\Gamma))^d \tag{2.5}$$

The problem (2.5) will be approximated by first constructing a finite-dimensional subspace $\mathscr{S} \subset (\widetilde{H}^{1/2}(\Gamma))^d$, and then finding

$$u_{\mathscr{S}} \in \mathscr{S} : \quad a(u_{\mathscr{S}}, v) = \langle g, v \rangle \quad \forall v \in \mathscr{S}. \tag{2.6}$$

In the remainder, for notational convenience we will consider only the case $d = 1$. The case $d = 3$ should be treated similarly.

3. ABSTRACT FRAMEWORK OF ADDITIVE SCHWARZ METHODS

Additive Schwarz methods provide fast solutions to equation (2.6) by solving (at the same time) problems of smaller size. Let $\mathscr{S}$ be decomposed as

$$\mathscr{S} = \mathscr{S}_0' + \cdots + \mathscr{S}_J' \tag{3.1}$$

where $\mathscr{S}_i'$, $i = 0, \dots, J$, are subspaces of $\mathscr{S}$. Let $b_i(\cdot, \cdot) : \mathscr{S}_i' \times \mathscr{S}_i' \to \mathbb{R}$ be a bilinear form defined on $\mathscr{S}_i'$ and $P_i : \mathscr{S} \to \mathscr{S}_i'$ be a projection defined by

$$b_i(P_i v, w) = a(v, w) \quad \forall v \in \mathscr{S}, \quad \forall w \in \mathscr{S}_i'.$$

Then the additive Schwarz method for equation (2.6) consists in solving, by an iterative method, the equation

$$P u_{\mathscr{S}} = g \tag{3.2}$$

where $P := P_0 + \cdots + P_J$ is the additive Schwarz operator and g is given by $g = \sum_{i=0}^{J} g_i$, with $g_i \in \mathscr{S}_i'$ being solutions of

$$b_i(g_i, w) = \langle g, w \rangle \quad \forall w \in \mathscr{S}_i'. \tag{3.3}$$

The equivalence of (2.6) and (3.2) was discussed in [16]. The operator P can be considered as the preconditioned version of D, i.e., $P = BD$ with some preconditioner B. In the implementation of the method, e.g., with the conjugate gradient method, one is interested in computing $B^{-1}r$ for a residual $r \in \mathscr{S}$:

Additive Schwarz Algorithm:

1. Subspace corrections: For $j = 0, \dots, J$, find $u_j \in \mathscr{S}_j'$ such that

$$b_j(u_j, v_j) = \langle r, v_j \rangle \quad \forall v_j \in \mathscr{S}_j'. \tag{3.4}$$

2. Preconditioned residual:

$$B^{-1}r = \sum_{j=0}^{J} u_j. \tag{3.5}$$

Bounds for the minimum and maximum eigenvalues of the additive Schwarz operator P can be obtained by using the following lemma (see [10,18]).

Lemma 3.1. *(i) If there exists a positive constant C_0 such that for any $u \in \mathscr{S}$ and $u_i \in \mathscr{S}_i'$ satisfying $u = \sum_{i=0}^{J} u_i$ there holds*

$$a(u,u) \leqslant C_0 \sum_{i=0}^{J} b_i(u_i, u_i) \tag{3.6}$$

then

$$\lambda_{\max}(P) \leqslant C_0.$$

(ii) If there exists a positive constant C_1 such that any $u \in \mathscr{S}$ has a decomposition $u = \sum_{i=0}^{J} u_i$ satisfying

$$\sum_{i=0}^{J} b_i(u_i, u_i) \leqslant C_1 a(u,u) \tag{3.7}$$

then

$$\lambda_{\min}(P) \geqslant C_1^{-1}.$$

4. OVERLAPPING ADDITIVE SCHWARZ PRECONDITIONER

4.1. Boundary element subspace

In this subsection we define the finite-dimensional subspace $\mathscr{S}$ in (2.6) on a two-level grid.

The coarse grid: Assume that Γ is partitioned into subdomains Γ_i, $i = 1, \ldots, J$, where each subdomain Γ_i is the image of the reference square $\hat{R} = (-1,1)^2$ under a smooth bijective mapping $\mathscr{F}_i : \hat{R} \to \Gamma_i$. Denoting by H the diameter of the subdomains, we assume that

$$\| J_{\mathscr{F}_i} \|_{L_\infty(\hat{R})} \preceq H, \qquad \| J_{\mathscr{F}_i^{-1}} \|_{L_\infty(\hat{R})} \preceq H^{-1}$$

where $J_{\mathscr{F}_i}$ denotes the Jacobian matrix of the transformation and the norm is a matrix norm. The partition is assumed to be conforming in the sense that the non-empty intersection of a pair of distinct subdomains is a single common vertex or edge of both subdomains, and that each vertex of the domain Γ coincides with at least one subdomain vertex.

We define on this coarse grid the space $\mathscr{S}_0$ of continuous piecewise bilinear functions, vanishing on the boundary of Γ.

The fine grid: Each subdomain Γ_i is further divided into disjoint quadrilateral or triangular elements, giving a locally uniform mesh of element of size h_i in Γ_i. We denote by h the maximum value of h_i, $i = 1, \ldots, J$.

The finite-dimensional space $\mathscr{S}$ is defined as the space of continuous piecewise-bilinear functions (in the case of quadrilateral elements) or piecewise-linear functions (in the case of triangular elements) on the fine grid, vanishing on the boundary of Γ.
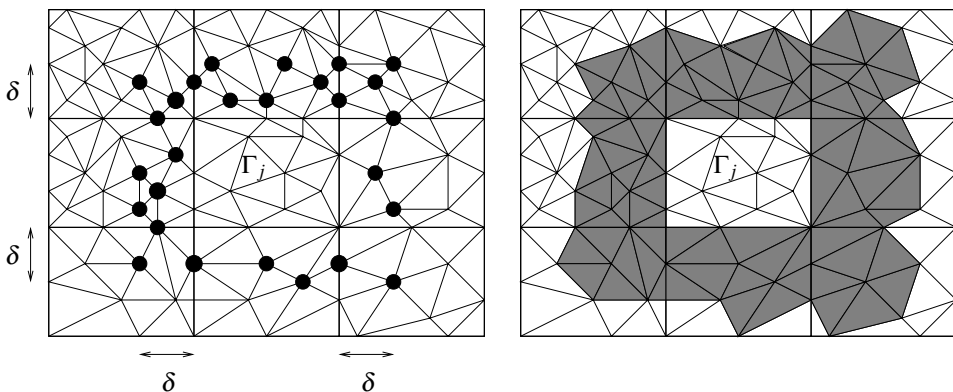
Figure 1. • vertex at a distance δ to $\overline{\Gamma}_j$, $\tilde{\Gamma}_j$: shaded region, $\Gamma'_j = \Gamma_j \cup \tilde{\Gamma}_j$: overlapping subdomain, R_j: (rectangle in shaded region) support of cutoff function ϑ_j.

We denote by $\{\mathbf{x}_k : k \in \mathcal{N}\}$ the set of all vertices of elements in the fine grid which are not on the boundary of Γ, and by $\{\varphi_k : k \in \mathcal{N}\}$ the set of nodal basis functions in $\mathcal{S}$, i.e., $\varphi_k(\mathbf{x}_l) = \delta_{kl}$. We also define subspaces $\mathcal{S}_j = \mathcal{S} \cap \tilde{H}^{1/2}(\Gamma_j)$ of functions in $\mathcal{S}$ supported in $\overline{\Gamma}_j$.

4.2. Overlapping subdomains

We extend each subdomain Γ_j in the following way. First we define, for some $\delta > 0$ called the overlap size,

$$\tilde{\mathcal{S}}_j = \text{span}\{\varphi_k : \mathbf{x}_k \notin \overline{\Gamma}_j, \ \text{dist}(\mathbf{x}_k, \partial \Gamma_j) \leqslant \delta\}$$

and denote

$$\tilde{\Gamma}_j = \text{supp}\{\varphi_k : \varphi_k \in \tilde{\mathcal{S}}_j\}$$

which is the shaded area in Fig. 1. (Here the distance is defined with the max norm $\|\mathbf{x}\| = \max\{|x_1|, |x_2|\}$ where $\mathbf{x} = (x_1, x_2)$.) The extended subdomain Γ'_j is then defined as $\Gamma'_j = \overline{\Gamma}_j \cup \tilde{\Gamma}_j$. We note that Γ'_j need not be a quadrilateral domain. Also, if δ is chosen such that $\delta \in (0, H]$, then

$$\text{diam}(\Gamma'_i) \simeq H. \tag{4.1}$$

4.3. Subspace decomposition

The decomposition (3.1) is performed with subspaces $\mathcal{S}'_j$, $j = 0, 1, \ldots, J$, defined as

$$\mathcal{S}'_0 = \Pi_F \mathcal{S}_0, \quad \mathcal{S}'_j = \mathcal{S}_j \cup \tilde{\mathcal{S}}_j = \mathcal{S} \cap \tilde{H}^{1/2}(\Gamma'_j), \quad j = 1, \ldots, J,$$

where Π_F is the interpolation operator which interpolates continuous functions into functions in $\mathscr{S}$.

The bilinear forms $b_i(\cdot, \cdot)$ associated with the subspaces $\mathscr{S}_i'$ (see Section 3) are defined as

$$b_0(v, w) = a(\Pi_C v, \Pi_C w) \qquad \forall v, w \in \mathscr{S}_0' \tag{4.2}$$

and

$$b_j(v, w) = a(v, w) \qquad \forall v, w \in \mathscr{S}_j', \quad j = 1, \dots, J \tag{4.3}$$

where Π_C be the interpolation operator which interpolates continuous functions into functions in $\mathscr{S}_0$.

We note that if $v_0 \in \mathscr{S}_0$ and $v_0' \in \mathscr{S}_0'$ then

$$v_0' = \Pi_F v_0 \iff v_0 = \Pi_C v_0'. \tag{4.4}$$

This subspace decomposition completely defines the additive Schwarz operator P introduced in Section 3.

4.4. Matrix form of additive Schwarz algorithm

In this subsection we give the matrix formulation of the algorithm described in Section 3.

Coarse space correction: Let $\{\varphi_k : k \in \mathscr{N}_C\}$ be the set of nodal basis functions for $\mathscr{S}_0$ consisting of bilinear functions on the coarse grid. Then a basis for $\mathscr{S}_0'$ is $\{\tilde{\varphi}_k : k \in \mathscr{N}_C\}$, where $\tilde{\varphi}_k = \Pi_F \varphi_k$. Recall the basis $\{\varphi_l : l \in \mathscr{N}\}$ for $\mathscr{S}$ (see Subsection 4.1). Each $\tilde{\varphi}_k$ can be represented as

$$\tilde{\varphi}_k = \sum_{l \in \mathscr{N}} \varepsilon_{k,l} \varphi_l$$

where $\varepsilon_{k,l} = \varphi_k(\mathbf{x}_l)$. Let

$$\mathbf{R}_0 = [\varepsilon_{k,l}]_{k \in \mathscr{N}_C, \, l \in \mathscr{N}}.$$

Then $\mathbf{R}_0$ is the matrix that transforms the basis of $\mathscr{S}$ into the basis of $\mathscr{S}_0'$.

Equation (3.4) for $j = 0$ reads as: Find $u_0 = \sum_{k \in \mathscr{N}_C} \alpha_k \tilde{\varphi}_k \in \mathscr{S}_0'$ such that

$$\sum_{k \in \mathscr{N}_C} \alpha_k b_0(\tilde{\varphi}_k, \tilde{\varphi}_j) = \langle r, \tilde{\varphi}_j \rangle, \quad j \in \mathscr{N}_C$$

which is equivalent to, noting (4.2) and (4.4),

$$\sum_{k \in \mathscr{N}_C} \alpha_k a(\varphi_k, \varphi_j) = \sum_{l \in \mathscr{N}} \varepsilon_{j,l} \langle r, \varphi_l \rangle, \quad j \in \mathscr{N}_C. \tag{4.5}$$

For any $v = \sum_{k \in \mathscr{N}} v_k \varphi_k \in \mathscr{S}$, let $\mathbf{v} = (v_k)_{k \in \mathscr{N}}$. Then

$$\langle v, \varphi_j \rangle = \mathbf{M} \mathbf{v}$$

where $\mathbf{M}$ is the mass matrix defined as $\mathbf{M} = [(\varphi_k, \varphi_l)]_{k,l \in \mathcal{N}}$. Denoting by $\mathbf{A}_0$ the stiffness matrix $[a(\varphi_k, \varphi_l)]_{k,l \in \mathcal{N}_C}$ computed on the coarse grid (with bilinear functions) and by α the vector $(\alpha_k)_{k \in \mathcal{N}_C}$, we can rewrite (4.5) as

$$\mathbf{A}_0 \alpha = \mathbf{R}_0 \mathbf{M} \mathbf{r}$$

implying

$$\alpha = \mathbf{A}_0^{-1} \mathbf{R}_0 \mathbf{M} \mathbf{r}$$

so that

$$\mathbf{u}_0 = \mathbf{R}_0^t \alpha = \mathbf{R}_0^t \mathbf{A}_0^{-1} \mathbf{R}_0 \mathbf{M} \mathbf{r}.$$

Corrections on $\mathscr{S}_i'$, $i = 1, \ldots, J$: A basis for $\mathscr{S}_i'$ with $i = 1, \ldots, J$ is more straightforward: it is extracted from $\{\varphi_l : l \in \mathcal{N}\}$ by a Boolean matrix $\mathbf{R}_i$. Using the same argument as above we obtain

$$\mathbf{u}_i = \mathbf{R}_i^t \mathbf{A}_i^{-1} \mathbf{M}_i \mathbf{R}_i \mathbf{r}, \quad i = 1, \ldots, J$$

where

$$\mathbf{A}_i = \mathbf{R}_i \mathbf{A} \mathbf{R}_i^t, \qquad \mathbf{M}_i = \mathbf{R}_i \mathbf{M} \mathbf{R}_i^t.$$

By noting that $\mathbf{M}_i \mathbf{R}_i = \mathbf{R}_i \mathbf{M}$ we deduce

$$\mathbf{u}_i = \mathbf{R}_i^t \mathbf{A}_i^{-1} \mathbf{R}_i \mathbf{M} \mathbf{r}, \quad i = 1, \ldots, J.$$

Preconditioned residual: The preconditioned residual $u = B^{-1} r$ is computed by

$$\mathbf{u} = \sum_{i=0}^{J} \mathbf{R}_i^t \mathbf{A}_i^{-1} \mathbf{R}_i \mathbf{M} \mathbf{r}.$$

4.5. The main result

Theorem 4.1. *The condition number of the additive Schwarz operator P is bounded as*

$$\varkappa(P) \preceq 1 + \log^2(H/\delta).$$

Proof. The above estimate is a result of Lemma 3.1 and Lemmas 5.3 and 5.4 proved in the next section. □

5. TECHNICAL LEMMAS

For the analysis, if $R \subset \mathbb{R}^n$, $n = 1, 2$, is a Lipschitz domain, we also define the following norms:

$$\|v\|_{H^{1/2}(R)}^2 := \frac{1}{\operatorname{diam}(R)} \|v\|_{L_2(R)}^2 + |v|_{H^{1/2}(R)}^2 \tag{5.1}$$

and

$$\|v\|_{\widetilde{H}^{1/2}(R)}^2 := |v|_{H^{1/2}(R)}^2 + \int_R \frac{v^2(\mathbf{x})}{\operatorname{dist}(\mathbf{x}, \partial R)} \, d\mathbf{x} \tag{5.2}$$

where

$$|v|_{H^{1/2}(R)}^2 := \int_R \int_R \frac{|v(\mathbf{x}) - v(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{n+1}} \, d\mathbf{x} \, d\mathbf{y}. \tag{5.3}$$

It was proved in Lemma 2 in [2] that if $R = [0, H] \times [0, H]$, then

$$\|v\|_{\widetilde{H}^{1/2}(R)} \simeq \|v\|_{\widetilde{H}^{1/2}(R)} \quad \forall v \in \widetilde{H}^{1/2}(R) \tag{5.4}$$

where the constants are independent of H. In the same manner one can prove that if $R = [0, H]$, then

$$\|v\|_{H^{1/2}(R)} \simeq \|v\|_{H^{1/2}(R)} \quad \forall v \in H^{1/2}(R). \tag{5.5}$$

For the semi-norm $|\cdot|_{H^{1/2}(R)}$ with $R = I \times J$, where I and J are intervals, it is useful to use the following equivalent form, which was proved in Lemma 5.3 in [11] (see also Exercise 5.1 following that lemma)

$$|v|_{H^{1/2}(R)}^2 \simeq \int_I \int_I \frac{\|v(x, \cdot) - v(x', \cdot)\|_{L_2(J)}^2}{|x - x'|^2} \, dx \, dx' + \int_J \int_J \frac{\|v(\cdot, y) - v(\cdot, y')\|_{L_2(I)}^2}{|y - y'|^2} \, dy \, dy' \tag{5.6}$$

where the constants in the equivalence are independent of the sizes of I, J, and R.

The following results were proved in Lemma 3.5 in [6]; however, for the convenience of the reader we present the detailed proof to justify that the results still hold with the norm defined in (5.1).

Lemma 5.1. *Let* $0 < \delta < \beta$.

1. *If* $v \in H^{1/2}(0, \beta)$, *then*

$$\int_\delta^\beta \frac{|v(x)|^2}{x} \, dx \preceq \left(1 + \log^2 \frac{\beta}{\delta}\right) \|v\|_{H^{1/2}(0,\beta)}^2. \tag{5.7}$$

2. *If* $v \in H^{1/2}((0, \beta)^2)$, *then*

$$\int_0^\beta \left(\int_\delta^\beta \frac{|v(x,y)|^2}{x} \, dx\right) dy \preceq \left(1 + \log^2 \frac{\beta}{\delta}\right) \|v\|_{H^{1/2}((0,\beta)^2)}^2. \tag{5.8}$$

Proof. Let $\mathscr{S}_\delta$ be the space of continuous piecewise-linear functions on a uniform mesh of size δ on $(0, \beta)$. We first prove (5.7) for $v_\delta \in \mathscr{S}_\delta$. By noting that (see Lemma 3.3 in [6])

$$\|v_\delta\|^2_{L_\infty(0,\beta)} \preceq \left(1 + \log\frac{\beta}{\delta}\right) \|v_\delta\|^2_{H^{1/2}(0,\beta)}$$

we have

$$\int_\delta^\beta \frac{|v_\delta(x)|^2}{x}\,\mathrm{d}x \leqslant \|v_\delta\|^2_{L_\infty(0,\beta)} \int_\delta^\beta \frac{\mathrm{d}x}{x} \preceq \left(1 + \log^2\frac{\beta}{\delta}\right) \|v_\delta\|^2_{H^{1/2}(0,\beta)}.$$

Now consider $v \in H^{1/2}(0, \beta)$. Let $v_\delta \in \mathscr{S}_\delta$ be the L_2 projection of v onto $\mathscr{S}_\delta$. Then

$$\|v - v_\delta\|^2_{L_2(0,\beta)} \preceq \delta\, |v|^2_{H^{1/2}(0,\beta)}.$$

Moreover, as was proved in [4] (see also [5]),

$$\|v_\delta\|_{H^{1/2}(0,\beta)} \preceq \|v\|_{H^{1/2}(0,\beta)}$$

which, together with (5.5), yields

$$\|\!|v_\delta|\!\|_{H^{1/2}(0,\beta)} \preceq \|\!|v|\!\|_{H^{1/2}(0,\beta)}.$$

Therefore,

$$\int_\delta^\beta \frac{|v(x)|^2}{x}\,\mathrm{d}x \preceq \int_\delta^\beta \frac{|v(x) - v_\delta(x)|^2}{x}\,\mathrm{d}x + \int_\delta^\beta \frac{|v_\delta(x)|^2}{x}\,\mathrm{d}x$$

$$\preceq \frac{1}{\delta}\|v - v_\delta\|^2_{L_2(0,\beta)} + \int_\delta^\beta \frac{|v_\delta(x)|^2}{x}\,\mathrm{d}x$$

$$\preceq |v|^2_{H^{1/2}(0,\beta)} + \left(1 + \log^2\frac{\beta}{\delta}\right) \|v_\delta\|^2_{H^{1/2}(0,\beta)}$$

$$\preceq \left(1 + \log^2\frac{\beta}{\delta}\right) \|\!|v|\!\|^2_{H^{1/2}(0,\beta)};$$

Inequality (5.8) is obtained by using (5.7) and (5.6)

$$
\begin{aligned}
\int_0^\beta \left(\int_\delta^\beta \frac{|v(x,y)|^2}{x}\, dx \right) dy
&\preceq \left(1 + \log^2 \frac{\beta}{\delta} \right) \int_0^\beta \||v(\cdot,y)\||^2_{H^{1/2}(0,\beta)}\, dy \\
&\preceq \left(1 + \log^2 \frac{\beta}{\delta} \right) \int_0^\beta \left[\frac{1}{\beta} \|v(\cdot,y)\|^2_{L_2(0,\beta)} \right. \\
&\qquad \left. + \int_0^\beta \int_0^\beta \frac{|v(x,y)-v(x',y)|^2}{|x-x'|^2}\, dx\, dx' \right] dy \\
&\preceq \left(1 + \log^2 \frac{\beta}{\delta} \right) \left[\frac{1}{\beta} \|v\|^2_{L_2((0,\beta)^2)} \right. \\
&\qquad \left. + \int_0^\beta \int_0^\beta \frac{\|v(x,\cdot)-v(x',\cdot)\|^2_{L_2(0,\beta)}}{|x-x'|^2}\, dx\, dx' \right] \\
&\preceq \left(1 + \log^2 \frac{\beta}{\delta} \right) \||v\||^2_{H^{1/2}((0,\beta)^2)}
\end{aligned}
$$

finishing the proof of the lemma. $\qquad\square$

The above results will be used to prove the following lemma, which is crucial to obtain the estimate for the minimum eigenvalue of P (see Lemma 3.1). Let R be the union of overlapping rectangular subdomains R_l, $l = 1,\ldots,J$, of diameters β_l. Assume that the size of the overlap is δ. Let $\{\vartheta_l : l = 1,\ldots,J\}$ be a partition of unity on R defined by piecewise bilinear functions such that supp $\vartheta_l = \overline{R}_l$.

Lemma 5.2. *If $\delta \leqslant \min_l \beta_l/2$, then for any $w \in \widetilde{H}^{1/2}(R)$ there holds*

$$
\sum_{l=1}^J \||\vartheta_l w\||^2_{\widetilde{H}^{1/2}(R_l)} \preceq \sum_{l=1}^J \left(1 + \log \frac{\beta_l}{\delta} \right)^2 \|w\|^2_{H^{1/2}(R_l)} + \int_R \frac{|w(\mathbf{x})|^2}{\mathrm{dist}(\mathbf{x},\partial R)}\, d\mathbf{x}. \quad (5.9)
$$

Proof. Without loss of generality, we can assume that $R_l = I \times I$, where $I = (0,\beta_l)$, so that ϑ_l can be defined as

$$
\vartheta_l(x,y) = \eta_l(x)\eta_l(y), \qquad
\eta_l(t) = \begin{cases}
t/\delta, & 0 \leqslant t \leqslant \delta \\
1, & \delta < t < \beta_l - \delta \\
(\beta_l - t)/\delta, & \beta_l - \delta \leqslant t \leqslant \beta_l.
\end{cases} \quad (5.10)
$$

In view of (5.2) and (5.6) in order to estimate $\|\vartheta_l w\|^2_{\widetilde{H}^{1/2}(R_l)}$ we will estimate

$$
T_1 := \int_I \int_I \frac{\|(\vartheta_l w)(x,\cdot) - (\vartheta_l w)(x',\cdot)\|^2_{L_2(I)}}{|x-x'|^2}\, dx\, dx'
$$

and

$$T_2 := \int_{R_l} \frac{|(\vartheta_l w)(\mathbf{x})|^2}{\mathrm{dist}(\mathbf{x}, \partial R_l)} \, d\mathbf{x}.$$

Splitting the integral over $I = (0, \beta_l)$ into three integrals over $I_1 := [0, \delta]$, $I_2 := [\delta, \beta_l - \delta]$, and $I_3 := [\beta_l - \delta, \beta_l]$, and denoting, for any v,

$$A_{ij}(v) := \iint_{I_i \times I_j} \frac{\|v(x, \cdot) - v(x', \cdot)\|^2_{L_2(I)}}{|x - x'|^2} \, dx \, dx'$$

we observe that, by symmetry, in order to estimate T_1 it suffices to consider $A_{11}(\vartheta_l w)$, $A_{12}(\vartheta_l w)$, $A_{13}(\vartheta_l w)$, $A_{23}(\vartheta_l w)$, and $A_{33}(\vartheta_l w)$. There is no need to consider $A_{22}(\vartheta_l w)$ because $A_{22}(\vartheta_l w) = A_{22}(w)$. The symmetric shape of ϑ_l implies the similarity of $A_{11}(\vartheta_l w)$ and $A_{33}(\vartheta_l w)$, and of $A_{12}(\vartheta_l w)$ and $A_{23}(\vartheta_l w)$. There remains three terms to be considered: $A_{11}(\vartheta_l w)$, $A_{12}(\vartheta_l w)$, and $A_{13}(\vartheta_l w)$.

Firstly, for A_{11} we have

$$
\begin{aligned}
A_{11}(\vartheta_l w) &\leqslant 2 \iint_{I_1 \times I_1} \frac{\|\vartheta_l(x, \cdot) - \vartheta_l(x', \cdot)\|^2_{L_2(I)}}{|x - x'|^2} \|w(x, \cdot)\|^2_{L_2(I)} \, dx \, dx' \\
&\quad + 2 \iint_{I_1 \times I_1} \frac{\|w(x, \cdot) - w(x', \cdot)\|^2_{L_2(I)}}{|x - x'|^2} \|\vartheta_l(x, \cdot)\|^2_{L_2(I)} \, dx \, dx' \\
&\leqslant 2 \iint_{I_1 \times I_1} \frac{\sum_{i=1}^{3} \left\| \frac{x}{\delta} \eta(\cdot) - \frac{x'}{\delta} \eta(\cdot) \right\|^2_{L_2(I_i)}}{|x - x'|^2} \|w(x, \cdot)\|^2_{L_2(I)} \, dx \, dx' + 2\beta_l A_{11}(w) \\
&\preceq \frac{1}{\delta} \|w\|^2_{L_2(I_1 \times I)} + A_{11}(w).
\end{aligned}
$$

It was proved in Lemma 3.4 in [6] that

$$\frac{1}{\delta} \|w\|^2_{L_2(I_1 \times I)} \leqslant c \left(1 + \log \frac{\beta_l}{\delta} \right) \|w\|^2_{H^{1/2}(R_l)} \tag{5.11}$$

where c is independent of w, β_l, and δ. Thus $A_{11}(\vartheta_l w)$ is bounded by the right-hand side of (5.9).

For the term $A_{12}(\vartheta_l w)$ we have

$$
\begin{aligned}
A_{12}(\vartheta_l w) &= \frac{1}{\delta^2} \int_0^\delta \int_\delta^{\beta_l - \delta} \frac{\|xw(x, \cdot) - \delta w(x', \cdot)\|^2_{L_2(I)}}{|x - x'|^2} \, dx' \, dx \\
&\leqslant \frac{2}{\delta^2} \int_0^\delta \left(\int_\delta^{\beta_l - \delta} \frac{dx'}{|x - x'|^2} \right) |x - \delta|^2 \|w(x, \cdot)\|^2_{L_2(I)} \, dx + 2 A_{12}(w) \\
&\leqslant \frac{2}{\delta} \|w\|^2_{L_2(I_1 \times I)} + 2 A_{12}(w).
\end{aligned}
$$

Using again (5.11) we deduce that $A_{12}(\vartheta_l w)$ is bounded by the right-hand side of (5.9).

Finally, for $A_{13}(\vartheta_l w)$ since the assumption $\delta \leqslant \beta_l/2$ implies that $\beta_l/2 - x \geqslant 0$ for $x \in I_1$ and $x' - \beta_l/2 \geqslant 0$ for $x' \in I_3$, we find

$$
\begin{aligned}
A_{13}(\vartheta_l w) &= \frac{1}{\delta^2} \int_0^\delta \int_{\beta_l-\delta}^{\beta_l} \frac{\|xw(x,\cdot) - (\beta_l - x')w(x',\cdot)\|_{L_2(I)}^2}{|x-x'|^2} \, \mathrm{d}x' \, \mathrm{d}x \\
&\leqslant \frac{2}{\delta^2} \int_0^\delta \int_{\beta_l-\delta}^{\beta_l} \frac{|x+x'-\beta_l|^2}{|x-x'|^2} \|w(x,\cdot)\|_{L_2(I)}^2 \, \mathrm{d}x' \, \mathrm{d}x \\
&\quad + \frac{2}{\delta^2} \int_0^\delta \int_{\beta_l-\delta}^{\beta_l} |\beta_l - x'|^2 \frac{\|w(x,\cdot) - w(x',\cdot)\|_{L_2(I)}^2}{|x-x'|^2} \, \mathrm{d}x' \, \mathrm{d}x \\
&\leqslant \frac{2}{\delta^2} \int_0^\delta \int_{\beta_l-\delta}^{\beta_l} \frac{\left|\left(x'-\frac{\beta_l}{2}\right) - \left(\frac{\beta_l}{2}-x\right)\right|^2}{\left|\left(x'-\frac{\beta_l}{2}\right) + \left(\frac{\beta_l}{2}-x\right)\right|^2} \|w(x,\cdot)\|_{L_2(I)}^2 \, \mathrm{d}x' \, \mathrm{d}x + 2A_{13}(w) \\
&\leqslant \frac{2}{\delta} \|w\|_{L_2(I_1 \times I)}^2 + 2A_{13}(w).
\end{aligned}
$$

Inequality (5.11) yields the estimate for $A_{13}(\vartheta_l w)$, which implies that T_1 is bounded by the right-hand side of (5.9).

To estimate T_2 we first assume, without loss of generality, that $\mathrm{dist}(\mathbf{x}, \partial R_l) = x$, where $\mathbf{x} = (x,y)$. Then

$$
\begin{aligned}
T_2 &\leqslant \int_0^{\beta_l} \left[\int_0^\delta \frac{x^2}{\delta^2} \frac{|w(x,y)|^2}{x} \, \mathrm{d}x + \int_\delta^{\beta_l} \frac{|w(x,y)|^2}{x} \, \mathrm{d}x \right] \mathrm{d}y \\
&\leqslant \frac{1}{\delta} \|w\|_{L_2(I_1 \times I)}^2 + \int_0^{\beta_l} \int_\delta^{\beta_l} \frac{|w(x,y)|^2}{x} \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}
$$

The desired estimate for T_2 now follows from (5.11) and (5.8), finishing the proof. $\square$

Lemma 5.3. *For any $u \in \mathscr{S}$ there exist $u_i \in \mathscr{S}_i'$ satisfying $u = \sum_{i=0}^J u_i$ such that*

$$
\sum_{i=0}^J b_i(u_i, u_i) \preceq \left(1 + \log^2 \frac{H}{\delta} \right) a(u,u), \tag{5.12}
$$

where the constant is independent of u, H, h, and δ.

Proof. To define a decomposition for $u \in \mathscr{S}$ we need a projection and a partition of unity. Since the operator $-\Delta$ with domain of definition $\widetilde{H}^1(\Gamma) = H_0^1(\Gamma)$ is positive definite and self-adjoint, we can define $\Lambda = \sqrt{-\Delta}$ which in turn is self-adjoint as an operator from $\widetilde{H}^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$. Moreover,

$$
\langle \Lambda \xi, \xi \rangle \simeq \|\xi\|_{\widetilde{H}^{1/2}(\Gamma)}^2 \quad \forall \xi \in \widetilde{H}^{1/2}(\Gamma).
$$

Let $P_H : \widetilde{H}^{1/2}(\Gamma) \longrightarrow \mathscr{S}_0$ be the projection defined by the inner product $\langle \Lambda \cdot, \cdot \rangle$, i.e.,

$$\langle \Lambda P_H v, w \rangle = \langle \Lambda v, w \rangle \quad \forall v \in \widetilde{H}^{1/2}(\Gamma), \, w \in \mathscr{S}_0.$$

Using standard arguments one can prove that for any $v \in \widetilde{H}^{1/2}(\Gamma)$ there hold

$$\|P_H v\|_{\widetilde{H}^{1/2}(\Gamma)} \preceq \|v\|_{\widetilde{H}^{1/2}(\Gamma)}, \qquad \|P_H v - v\|_{L_2(\Gamma)} \preceq H^{1/2} \|v\|_{\widetilde{H}^{1/2}(\Gamma)}. \tag{5.13}$$

We next define a partition of unity having the properties of the partition in Lemma 5.2. For $j = 1, \ldots, J$ we define

$$R_j = \Gamma_j \cup \{\mathbf{x} \in \Gamma : \text{dist}(\mathbf{x}, \partial \Gamma_j) \leqslant \delta\}$$

which are quadrilateral domains (see Fig. 1). It is crucial to define R_j as a quadrilateral domain so that Lemma 5.2 can be applied. The cutoff function ϑ_j is defined as in the proof of Lemma 5.2 so that supp $\vartheta_j = \overline{R}_j$.

We are now ready to define a decomposition of $u \in \mathscr{S}$ such that (5.12) holds. For any $u \in \mathscr{S}$ let $u_0 = \Pi_F P_H u \in \mathscr{S}_0'$ and $u_j = \Pi_F(\vartheta_j w) \in \mathscr{S}_j'$, $j = 1, \ldots, J$, where $w = u - u_0$. The fact that supp $u_j \subset \overline{\Gamma}_j'$ so that $u_j \in \mathscr{S}_j'$ for $j = 1, \ldots, J$ is clear from the definitions of R_j and Γ_j'. It is also clear that $u = \sum_{i=0}^{J} u_i$. By using (4.4), (5.13) and Lemma 7 in [2], we obtain the following estimates

$$\|\Pi_C u_0\|_{\widetilde{H}^{1/2}(\Gamma)} = \|P_H u\|_{\widetilde{H}^{1/2}(\Gamma)}$$
$$\preceq \|u\|_{\widetilde{H}^{1/2}(\Gamma)} \tag{5.14}$$

$$\|u_i\|_{\widetilde{H}^{1/2}(\Gamma)} \preceq \|\vartheta_i w\|_{\widetilde{H}^{1/2}(\Gamma)} \tag{5.15}$$

$$\|w\|_{\widetilde{H}^{1/2}(\Gamma)} = \|\Pi_F(u - P_H u)\|_{\widetilde{H}^{1/2}(\Gamma)}$$
$$\preceq \|u - P_H u\|_{\widetilde{H}^{1/2}(\Gamma)}$$
$$\preceq \|u\|_{\widetilde{H}^{1/2}(\Gamma)} \tag{5.16}$$

and

$$\|w\|_{L_2(\Gamma)} = \|\Pi_F(u - P_H u)\|_{L_2(\Gamma)}$$
$$\preceq \|u - P_H u\|_{L_2(\Gamma)}$$
$$\preceq H^{1/2} \|u\|_{\widetilde{H}^{1/2}(\Gamma)}. \tag{5.17}$$

Moreover, by Lemma 6 in [2] there holds

$$\sum_j \|\vartheta_j w\|_{\widetilde{H}^{1/2}(\Gamma)}^2 \preceq \sum_j \|w\|_{\widetilde{H}^{1/2}(R_i)}^2 \tag{5.18}$$

Hence by using successively (2.4), (5.14), (5.15), (5.4), (5.18), (5.9), (4.1), (5.16), and (5.17) we obtain

$$\sum_{i=0}^{J} b_i(u_i,u_i) \simeq \|\Pi_C u_0\|_{\widetilde{H}^{1/2}(\Gamma)}^2 + \sum_{i=1}^{J} \|u_i\|_{\widetilde{H}^{1/2}(\Gamma)}^2$$

$$\preceq \|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 + \sum_{i=1}^{J} \|\vartheta_i w\|_{\widetilde{H}^{1/2}(\Gamma)}^2$$

$$\simeq \|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 + \sum_{i=1}^{J} \|\|\vartheta_i w\|\|_{\widetilde{H}^{1/2}(\Gamma)}^2$$

$$\simeq \|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 + \sum_{i=1}^{J} \|\|\vartheta_i w\|\|_{\widetilde{H}^{1/2}(R_i)}^2$$

$$\preceq \|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 + \left(1+\log^2\frac{H}{\delta}\right) \sum_{i=1}^{J} \|\|w\|\|_{H^{1/2}(R_i)}^2 + \int_{\Gamma} \frac{|w(x)|^2}{\operatorname{dist}(x,\partial\Gamma)}\,\mathrm{d}x$$

$$\preceq \|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 + \left(1+\log^2\frac{H}{\delta}\right) \sum_{i=1}^{J} \|w\|_{H^{1/2}(R_i)}^2 + \|w\|_{\widetilde{H}^{1/2}(\Gamma)}^2$$

$$\preceq \|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 + \left(1+\log^2\frac{H}{\delta}\right) \sum_{i=1}^{J} \frac{1}{H}\|w\|_{L_2(R_i)}^2$$

$$\preceq \|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2 + \left(1+\log^2\frac{H}{\delta}\right) \frac{1}{H}\|w\|_{L_2(\Gamma)}^2$$

$$\preceq \left(1+\log^2\frac{H}{\delta}\right) \|u\|_{\widetilde{H}^{1/2}(\Gamma)}^2$$

$$\preceq \left(1+\log^2\frac{H}{\delta}\right) a(u,u)$$

completing the proof.   □

Lemma 5.4. *For any $u \in \mathscr{S}$ if $u = \sum_{i=0}^{J} u_i$ for some $u_i \in \mathscr{S}_i$, then*

$$a(u,u) \preceq \sum_{i=0}^{J} b_i(u_i,u_i)$$

where the constant is independent of u, h, δ, and H.

Proof. By construction there are at most ν subdomains $\overline{\Gamma}_i'$ to which any $x \in \Gamma$ can belong. (The utmost case happens when $\delta = H$.) A standard colouring argument

Table 1.
Condition numbers for Laplace equation, overlap size $\delta = h$.

DoF	h	Unpreconditioned	$H = 2h$	$H = 1/2$
9	1/4	1.83	4.51	1.58
49	1/8	3.49	5.12	5.12
225	1/16	6.94	5.38	4.85
961	1/32	13.84	5.46	5.16
3969	1/64	27.77	5.47	6.02
16129	1/128	55.73	5.48	7.29
65025	1/256	111.68	5.47	8.93
261121	1/512	223.60	5.54	10.92

yields, noting that $u_0 = \Pi_F \Pi_C u_0$,

$$a(u,u) \simeq \|u\|^2_{\widetilde{H}^{1/2}(\Gamma)}$$

$$\preceq \|u_0\|^2_{\widetilde{H}^{1/2}(\Gamma)} + \left\| \sum_{i=1}^{J} u_i \right\|^2_{\widetilde{H}^{1/2}(\Gamma)}$$

$$\preceq \|\Pi_C u_0\|^2_{\widetilde{H}^{1/2}(\Gamma)} + \nu \sum_{i=1}^{J} \|u_i\|^2_{\widetilde{H}^{1/2}(\Gamma)}$$

$$\simeq \sum_{i=0}^{J} b_i(u_i, u_i)$$

completing the proof.   □

6. NUMERICAL EXPERIMENTS

We solved (2.2) for both the Laplace and Lamé operators, i.e. with D defined by (2.3), and with Γ chosen to be $[-1,1]^2$. The right hand side g of (2.2) is chosen to be $g(x) \equiv 1$ in the Laplace case, and $g(x) = (-y, x, 0)$ in the Lamé case.

The experiments are performed with numerous values of H, h, and δ, but we only report a few cases here. All numbers seem to agree with our theoretical result. Tables 1 and 2 present the condition number for both cases, whereas Tables 3 and 4 present the corresponding number of iterations. We also plot in Fig. 2 the condition number versus H/δ for the Laplace case. A logarithmic growth can be observed. The authors are grateful to Dr. Matthias Maischak for carrying out the experiments.

Table 2.
Condition number for Lamé equation.

DoF	h	Unpreconditioned	$H = 2h$		$H = 4h$	
			$\delta = h$	$\delta = 2h$	$\delta = h$	$\delta = 2h$
147	1/8	4.14	2.38	4.11		
675	1/16	8.24	2.78	4.27	5.15	9.01
2883	1/32	16.50	3.88	4.31	5.43	9.01
11907	1/64	33.07	5.36	5.88	5.56	9.01

Table 3.
Number of iterations for Laplace equation, overlap size $\delta = h$.

DoF	h	Unpreconditioned	$H = 2h$	$H = 1/2$
9	1/4	4	4	4
49	1/8	11	15	15
225	1/16	16	20	17
961	1/32	23	21	18
3969	1/64	31	21	19
16129	1/128	44	21	20
65025	1/256	63	20	23
261121	1/512	89	20	24

Table 4.
Number of iterations for Lamé equation.

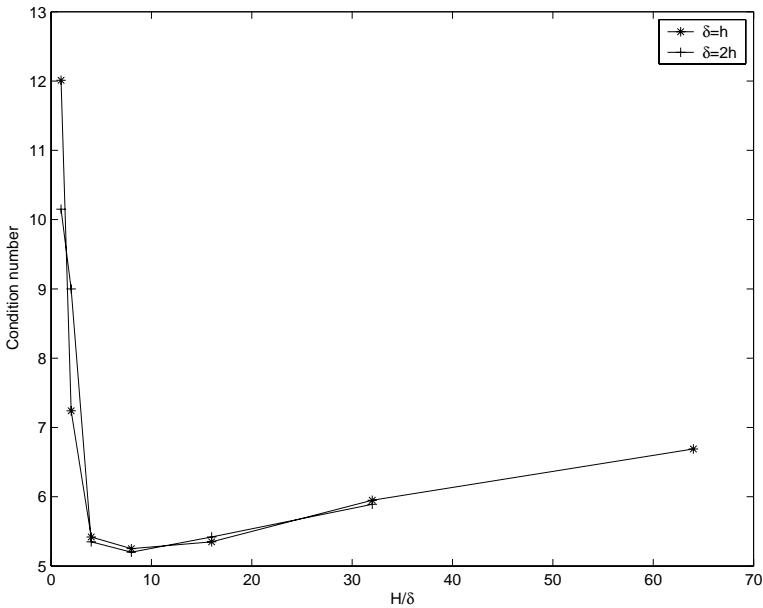| DoF	h	Unpreconditioned	$H = 2h$		$H = 4h$	
			$\delta = h$	$\delta = 2h$	$\delta = h$	$\delta = 2h$
147	1/8	24	13	16		
675	1/16	34	16	19	20	22
2883	1/32	48	19	21	22	25
11907	1/64	68	23	23	24	26

Figure 2. Condition number vs H/δ for DoF $= 16129$.

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